1 CEWS Estimator under the factor model

In this section we will develop the *Cross-Evolutionary Wavelet Estimator* by using the factor structure and we will prove the convergence of this estimator under the structure. Recall the consistent and unbiaised estimator of the CEWS :

$$\widehat{\boldsymbol{S}}_j\left(k/T\right) = \frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\boldsymbol{d}_{l,k+m}\boldsymbol{d}'_{l,k+m}$$

by multiplying both sides by N^{-1} and by using (**??**), (**??**) and (**??**) we successively obtain :

$$\begin{aligned}
N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right) &= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\boldsymbol{X_{t;T}}\psi_{j,k+s}(t)\sum_{t'=0}^{T}\boldsymbol{X'_{t';T}}\psi_{j,k+s}(t') && \text{by } (\textbf{??})\\
&= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}\sum_{m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{W}_p\left(m/T\right)\boldsymbol{\xi}_{p,m}\psi_{p,m}(t)\\
&\qquad\sum_{m'=0}^{T}\sum_{p'=-J}^{-1}\boldsymbol{\xi}'_{p',m'}\boldsymbol{W}_{p'}\left(m'/T\right)'\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t') && \text{by } (\textbf{??})\\
&= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}\sum_{m=0}^{T}\sum_{p=-J}^{-1}(\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m+\boldsymbol{\epsilon}_{p,m})\psi_{p,m}(t)\\
&\qquad\sum_{m'=0}^{T}\sum_{p'=-J}^{-1}(\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m+\boldsymbol{\epsilon}_{p,m})'\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t') && \text{by } (\textbf{??})
\end{aligned}$$

In order to ease readibility, let's define two objects :

$$C_t = \sum_{m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m\psi_{p,m}(t) \tag{1.1}$$

$$E_t = \sum_{m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{\epsilon}_{p,m}\psi_{p,m}(t) \tag{1.2}$$

Consequently,

$$\begin{aligned}
N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right) &= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}[C_t+E_t][C_t+E_t]'\psi_{j,k+s}(t)\psi_{j,k+s}(t')\\
&= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}[C_tC_{t'}+C_tE'_{t'}+E_tC'_{t'}+E_tE'_{t'}]\psi_{j,k+s}(t)\psi_{j,k+s}(t')
\end{aligned}$$