1 CEWS Estimator under the factor model

In this section we will develop the *Cross-Evolutionary Wavelet Estimator* by using the factor structure and we will prove the convergence of this estimator under the structure. Recall the consistent and unbiaised estimator of the CEWS :

$$\widehat{\boldsymbol{S}}_j\left(k/T\right) = \frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\boldsymbol{d}_{l,k+m}\boldsymbol{d}'_{l,k+m}$$

by multiplying both sides by N^{-1} and by using (**??**), (**??**) and (**??**) we successively obtain :

$$\begin{aligned}
N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right) &= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\boldsymbol{X_{t;T}}\psi_{j,k+s}(t)\sum_{t'=0}^{T}\boldsymbol{X'_{t';T}}\psi_{j,k+s}(t') && \text{by (??)}\\
&= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}\sum_{m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{W}_p\left(m/T\right)\boldsymbol{\xi}_{p,m}\psi_{p,m}(t)\\
&\qquad\sum_{m'=0}^{T}\sum_{p'=-J}^{-1}\boldsymbol{\xi}'_{p',m'}\boldsymbol{W}_{p'}\left(m'/T\right)'\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t') && \text{by (??)}\\
&= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}\sum_{m=0}^{T}\sum_{p=-J}^{-1}(\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m+\boldsymbol{\epsilon}_{p,m})\psi_{p,m}(t)\\
&\qquad\sum_{m'=0}^{T}\sum_{p'=-J}^{-1}(\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m+\boldsymbol{\epsilon}_{p,m})'\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t') && \text{by (??)}
\end{aligned}$$

In order to ease readibility, let's define two objects :

$$C_t = \sum_{m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m\psi_{p,m}(t) \tag{1.1}$$

$$E_t = \sum_{m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{\epsilon}_{p,m}\psi_{p,m}(t) \tag{1.2}$$

Consequently,

$$\begin{aligned}
N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right) &= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}[C_t+E_t][C_t+E_t]'\psi_{j,k+s}(t)\psi_{j,k+s}(t')\\
&= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}[C_tC_{t'}+C_tE'_{t'}+E_tC'_{t'}+E_tE'_{t'}]\psi_{j,k+s}(t)\psi_{j,k+s}(t')
\end{aligned}$$

To further simplify the notation we define the following,

$$K_{j,k} = N^{-1}\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}C_tC'_{t'}\psi_{j,k+s}(t)\psi_{j,k+s}(t') \tag{1.3}$$

$$\Upsilon_{j,k} = N^{-1}\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}E_tE'_{t'}\psi_{j,k+s}(t)\psi_{j,k+s}(t') \tag{1.4}$$

$$\Theta_{j,k} = N^{-1}\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\sum_{t'=0}^{T}C_tE'_{t'}\psi_{j,k+s}(t)\psi_{j,k+s}(t') \tag{1.5}$$

The estimator (**??**) can now be expressed as,

$$N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right) = K_{j,k}+\Theta_{j,k}+\Theta'_{j,k}+\Upsilon_{j,k} \tag{1.6}$$

From Park et al. (2014) this estimator converge to the true Cross-Evoluationary Wavelet Spectrum. The next development assert the same convergence with the estimator redefined by the factor structure.

First, the expectation of the estimator can be decomposed thanks to (1.6),

$$\mathrm{E}\left[\widehat{\boldsymbol{S}}_j\left(k/T\right)\right] = \mathrm{E}\left[NK_{j,k}\right]+\mathrm{E}\left[N\Theta_{j,k}\right]+\mathrm{E}\left[N\Theta_{j,k}\right]+\mathrm{E}\left[N\Upsilon_{j,k}\right]$$

Let's turn out focus on the first term. From (1.3) and (1.1) and the linearity of expectation, the latter is then,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\boldsymbol{\Lambda}_{p,m}\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}'_{m'}\right]\boldsymbol{\Lambda}'_{p',m'}\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

Given the lemma **??**,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$
$$- \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\mathrm{E}\left[\boldsymbol{\epsilon}_{p,m}\boldsymbol{\epsilon}'_{p',m'}\right]\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

which is equivalent to

$$\begin{aligned}
\mathrm{E}\left[NK_{j,k}\right] &= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t') - \mathrm{E}\left[\Upsilon jk\right] \quad \text{by (1.4)}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\sum_{t=0}^{T}\psi_{p,m}(t)\psi_{j,k+s}(t)\sum_{t'=0}^{T}\psi_{p,m}(t')\psi_{j,k+s}(t') - \mathrm{E}\left[\Upsilon jk\right]\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\left[\sum_{t=0}^{T}\psi_{p,m}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[\Upsilon jk\right]\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{n=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(n+k/T\right)\left[\sum_{t=0}^{T}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[\Upsilon jk\right] \text{ by change of variable : } m = n+k\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{n=0}^{T}\sum_{p=-J}^{-1}\left[\boldsymbol{S}_p\left(k/T\right)+O(\frac{n}{T})\right]\left[\sum_{t=0}^{T}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[\Upsilon jk\right] \text{ by Lipschitz continuity}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(k/T\right)\sum_{n=0}^{T}\left[\sum_{t=0}^{T}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[\Upsilon jk\right] + O(T^{-1})
\end{aligned}$$

Nason et al. (2000) proved that $\sum_{n=0}^{T}\left[\sum_{t=0}^{T}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 = A_{p,j}$. Consequently,

$$\begin{aligned}
\mathrm{E}\left[NK_{j,k}\right] &= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(k/T\right)A_{p,j} - \mathrm{E}\left[\Upsilon jk\right] + O(T^{-1})\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\mathrm{E}\left[\boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k}\right] - \mathrm{E}\left[\Upsilon jk\right] + O(T^{-1}) \text{ by the expectation of the raw periodogram.}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\mathrm{E}\left[\sum_{j=-J}^{-1}\bar{A}_{j,l}\boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k}\right] - \mathrm{E}\left[\Upsilon jk\right] + O(T^{-1}) \text{ by linearity of expectation}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[\Upsilon jk\right] + O(T^{-1}) \text{ by expectation of the corrected periodogram.}\\
&= \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[\Upsilon jk\right] + O(T^{-1})
\end{aligned}$$