

Lemma 1. *Given the assumptions on the LSW and factor structure,*

$$\mathbf{\Lambda}_{j,k} \mathbb{E} [\mathbf{F}_k \mathbf{F}_{k'}'] \mathbf{\Lambda}_{j',k'}' = \begin{cases} \mathbf{S}_j(k/T) - \mathbb{E} [\boldsymbol{\epsilon}_{j,k} \boldsymbol{\epsilon}_{j,k'}'] & , \text{if } j = j' \text{ and } k = k' \\ -\mathbb{E} [\boldsymbol{\epsilon}_{j,k} \boldsymbol{\epsilon}_{j',k'}'] & , \text{otherwise} \end{cases}$$

Proof. From the factor structure (??),

$$\mathbf{W}_j(k/T) \boldsymbol{\xi}_{j,k} \boldsymbol{\xi}_{j',k'}' \mathbf{W}_{j'}'(k'/T) = \mathbf{\Lambda}_{j,k} \mathbf{F}_k \mathbf{F}_{k'}' \mathbf{\Lambda}_{j',k'}' + \mathbf{\Lambda}_{j,k} \mathbf{F}_k \boldsymbol{\epsilon}_{j',k'}' + \boldsymbol{\epsilon}_{j,k} \mathbf{F}_{k'}' \mathbf{\Lambda}_{j',k'}' + \boldsymbol{\epsilon}_{j,k} \boldsymbol{\epsilon}_{j',k'}'$$

Taking expectation on both sides,

$$\mathbf{W}_j(k/T) \mathbb{E} [\boldsymbol{\xi}_{j,k} \boldsymbol{\xi}_{j',k'}'] \mathbf{W}_{j'}'(k'/T) = \mathbf{\Lambda}_{j,k} \mathbb{E} [\mathbf{F}_k \mathbf{F}_{k'}'] \mathbf{\Lambda}_{j',k'}' + \mathbf{\Lambda}_{j,k} \mathbb{E} [\mathbf{F}_k \boldsymbol{\epsilon}_{j',k'}'] + \mathbb{E} [\boldsymbol{\epsilon}_{j,k} \mathbf{F}_{k'}'] \mathbf{\Lambda}_{j',k'}' + \mathbb{E} [\boldsymbol{\epsilon}_{j,k} \boldsymbol{\epsilon}_{j',k'}']$$

The second and third term on the RHS is zero since $\mathbf{F}_k \perp \boldsymbol{\epsilon}_{j',k'}, \forall j, k, k'$ (assumption (??)).

This leaves us with

$$\mathbf{W}_j(k/T) \mathbb{E} [\boldsymbol{\xi}_{j,k} \boldsymbol{\xi}_{j',k'}'] \mathbf{W}_{j'}'(k'/T) = \mathbf{\Lambda}_{j,k} \mathbb{E} [\mathbf{F}_k \mathbf{F}_{k'}'] \mathbf{\Lambda}_{j',k'}' + \mathbb{E} [\boldsymbol{\epsilon}_{j,k} \boldsymbol{\epsilon}_{j',k'}']$$

Finally, by the assumption (??) on the increments of the LSW representation,

$$\mathbb{E} [\boldsymbol{\xi}_{j,k} \boldsymbol{\xi}_{j',k'}'] = \begin{cases} \mathbf{I}_N & , \text{if } j = j' \text{ and } k = k' \\ \mathbf{0}_N & , \text{otherwise} \end{cases}$$

,where \mathbf{I}_N is the identity matrix of rank N and $\mathbf{0}_N$ is the null matrix of rank N . This along with the definition of the CEWS we obtain the desired result. \square

Lemma 2. *Given the assumption on the LSW and factor structure and the set \mathcal{S} defined in (??), when $T \rightarrow \infty$ (rate of convergence ?)*

$$\text{Var} [N_i] \rightarrow 0 \quad \forall i \in \mathcal{S}$$

Proof. When $i = K_{j,k}$,

$$\begin{aligned} \text{Var} [N K_{j,k}] &= \text{Var} \left[\frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{t,t',m,m'=0}^{T-1} \sum_{p,p'=-J}^{-1} \mathbf{\Lambda}_{p,m} \mathbf{F}_m \mathbf{F}_{m'}' \mathbf{\Lambda}_{p',m'}' \psi_{p,m}(t) \psi_{p',m'}(t') \psi_{j,k+s}(t) \psi_{j,k+s}(t') \right] \quad \text{by (??), (?)} \\ &\leq \mathbb{E} \left[\left(\frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{t,t',m,m'=0}^{T-1} \sum_{p,p'=-J}^{-1} \mathbf{\Lambda}_{p,m} \mathbf{F}_m \mathbf{F}_{m'}' \mathbf{\Lambda}_{p',m'}' \psi_{p,m}(t) \psi_{p',m'}(t') \psi_{j,k+s}(t) \psi_{j,k+s}(t') \right)^2 \right] \\ &= \left(\frac{1}{2M+1} \sum_{l=-J}^{-1} \bar{A}_{j,l} \right)^2 \mathbb{E} \left[\sum_{s_0,s_1=-M}^M \sum_{\substack{m_0,m'_0 \\ m_1,m'_1=0}}^T \sum_{\substack{mp_0,p'_0 \\ p_1,p'_1=-J}}^{-1} \mathbf{\Lambda}_{p_0,m_0} \mathbf{F}_{m_0} \mathbf{F}_{m'_0}' \mathbf{\Lambda}_{p'_1,m'_1}' \mathbf{\Lambda}_{p_1,m_1} \mathbf{F}_{m_1} \mathbf{F}_{m'_1}' \mathbf{\Lambda}_{p'_1,m'_1}' \right. \\ &\quad \left. \sum_{\substack{t_0,t'_0 \\ t_1,t'_1}} \psi_{p_0,m_0}(t_0) \psi_{p'_0,m'_0}(t'_0) \psi_{j,k+s_0}(t_0) \psi_{j,k+s_0}(t'_0) \psi_{p_1,m_1}(t_1) \psi_{p'_1,m'_1}(t'_1) \psi_{j,k+s_1}(t_1) \psi_{j,k+s_1}(t'_1) \right] \\ &= \left(\frac{1}{2M+1} \sum_{l=-J}^{-1} \bar{A}_{j,l} \right)^2 \mathbb{E} \left[\sum_{\substack{m_0,m'_0 \\ m_1,m'_1=0}}^T \sum_{\substack{mp_0,p'_0 \\ p_1,p'_1=-J}}^{-1} \mathbf{\Lambda}_{p_0,m_0} \mathbf{F}_{m_0} \mathbf{F}_{m'_0}' \mathbf{\Lambda}_{p'_1,m'_1}' \mathbf{\Lambda}_{p_1,m_1} \mathbf{F}_{m_1} \mathbf{F}_{m'_1}' \mathbf{\Lambda}_{p'_1,m'_1}' \right. \\ &\quad \left. \sum_{s_0=-M}^M \sum_{t_0} \psi_{p_0,m_0}(t_0) \psi_{j,k+s_0}(t_0) \sum_{t'_0} \psi_{p'_0,m'_0}(t'_0) \psi_{j,k+s_0}(t'_0) \sum_{s_1=-M}^M \sum_{t_1} \psi_{p_1,m_1}(t_1) \psi_{j,k+s_1}(t_1) \sum_{t'_1} \psi_{p'_1,m'_1}(t'_1) \psi_{j,k+s_1}(t'_1) \right] \\ &= \frac{1}{(2M+1)^2} \left(\sum_{l=-J}^{-1} \bar{A}_{j,l} \right)^2 \mathbb{E} \left[\sum_{\substack{m_0,m'_0 \\ m_1,m'_1=0}}^T \sum_{\substack{mp_0,p'_0 \\ p_1,p'_1=-J}}^{-1} \mathbf{\Lambda}_{p_0,m_0} \mathbf{F}_{m_0} \mathbf{F}_{m'_0}' \mathbf{\Lambda}_{p'_1,m'_1}' \mathbf{\Lambda}_{p_1,m_1} \mathbf{F}_{m_1} \mathbf{F}_{m'_1}' \mathbf{\Lambda}_{p'_1,m'_1}' \right. \\ &\quad \left. \sum_{s_0=-M}^M \boldsymbol{\Psi}_{p_0,j}(m_0 - k - s_0) \boldsymbol{\Psi}_{p'_0,j}(m'_0 - k - s_0) \sum_{s_1=-M}^M \boldsymbol{\Psi}_{p_1,j}(m_1 - k - s_1) \boldsymbol{\Psi}_{p'_1,j}(m'_1 - k - s_1) \right] \quad \text{by def. of CCWF.} \end{aligned}$$

All sums are finite when $T \rightarrow \infty$ since the cross-correlation wavelet functions have bounded support (Proof). However we need the additional assumption that $\mathbb{E} [F_k^{(u)4}] < \infty$. The whole expression tends to zero if $M(T) \rightarrow \infty$ when $T \rightarrow \infty$ (rate of convergence ?).

The proofs when $i = \mathcal{S} \setminus \{K_{j,k}\}$ are analogous with the additional requirement that $\mathbb{E} [\epsilon_{j,k}^{(u)4}] < \infty$. \square

Theorem 1. Given the assumptions on the LSW representation, the factor structure and the definitions (??), (??) and (??),

$$\begin{aligned} \mathbb{E}[NK_{j,k}] &= \mathbf{S}_j(k/T) - \mathbb{E}[N\Upsilon jk] \\ \mathbb{E}[N\Theta_{j,k}] &= 0 \end{aligned}$$

Proof. Let's turn out focus on the first expectation. From (??) and (??) and the linearity of expectation, the latter is then,

$$\mathbb{E}[NK_{j,k}] = \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{t,t',m,m'=0}^{T-1} \sum_{p,p'=-J}^{-1} \Lambda_{p,m} \mathbb{E}[\mathbf{F}_m \mathbf{F}'_{m'}] \Lambda'_{p',m'} \psi_{p,m}(t) \psi_{p',m'}(t') \psi_{j,k+s}(t) \psi_{j,k+s}(t')$$

Given the lemma 1,

$$\begin{aligned} \mathbb{E}[NK_{j,k}] &= \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{t,t',m=0}^{T-1} \sum_{p=-J}^{-1} \mathbf{S}_p(m/T) \psi_{p,m}(t) \psi_{p,m}(t') \psi_{j,k+s}(t) \psi_{j,k+s}(t') \\ &\quad - \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{t,t',m,m'=0}^{T-1} \sum_{p,p'=-J}^{-1} \mathbb{E}[\epsilon_{p,m} \epsilon'_{p',m'}] \psi_{p,m}(t) \psi_{p,m}(t') \psi_{j,k+s}(t) \psi_{j,k+s}(t') \end{aligned}$$

which is equivalent to

$$\begin{aligned} \mathbb{E}[NK_{j,k}] &= \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{t,t',m=0}^{T-1} \sum_{p=-J}^{-1} \mathbf{S}_p(m/T) \psi_{p,m}(t) \psi_{p,m}(t') \psi_{j,k+s}(t) \psi_{j,k+s}(t') - \mathbb{E}[N\Upsilon jk] \quad \text{by (??)} \\ &= \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{m=0}^{T-1} \sum_{p=-J}^{-1} \mathbf{S}_p(m/T) \sum_{t=0}^{T-1} \psi_{p,m}(t) \psi_{j,k+s}(t) \sum_{t'=0}^{T-1} \psi_{p,m}(t') \psi_{j,k+s}(t') - \mathbb{E}[N\Upsilon jk] \\ &= \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{m=0}^{T-1} \sum_{p=-J}^{-1} \mathbf{S}_p(m/T) \left[\sum_{t=0}^{T-1} \psi_{p,m}(t) \psi_{j,k+s}(t) \right]^2 - \mathbb{E}[N\Upsilon jk] \\ &= \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{n=0}^{T-1} \sum_{p=-J}^{-1} \mathbf{S}_p(n+k/T) \left[\sum_{t=0}^{T-1} \psi_{p,n+k}(t) \psi_{j,k+s}(t) \right]^2 - \mathbb{E}[N\Upsilon jk] \quad \text{by change of variable : } m = n + k \\ &= \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{n=0}^{T-1} \sum_{p=-J}^{-1} \left[\mathbf{S}_p(k/T) + O\left(\frac{n}{T}\right) \right] \left[\sum_{t=0}^{T-1} \psi_{p,n+k}(t) \psi_{j,k+s}(t) \right]^2 - \mathbb{E}[N\Upsilon jk] \quad \text{by Lipschitz continuity} \\ &= \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{p=-J}^{-1} \mathbf{S}_p(k/T) \sum_{n=0}^{T-1} \left[\sum_{t=0}^{T-1} \psi_{p,n+k}(t) \psi_{j,k+s}(t) \right]^2 - \mathbb{E}[N\Upsilon jk] + O(T^{-1}) \end{aligned}$$

Nason et al. (2000) proved that $\sum_{n=0}^{T-1} \left[\sum_{t=0}^{T-1} \psi_{p,n+k}(t) \psi_{j,k+s}(t) \right]^2 = A_{p,j}$. Consequently,

$$\begin{aligned} \mathbb{E}[NK_{j,k}] &= \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{p=-J}^{-1} \mathbf{S}_p(k/T) A_{p,j} - \mathbb{E}[N\Upsilon jk] + O(T^{-1}) \\ &= \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \mathbb{E}[\mathbf{d}_{j,k} \mathbf{d}'_{j,k}] - \mathbb{E}[N\Upsilon jk] + O(T^{-1}) \quad \text{by the expectation of the raw periodogram.} \\ &= \frac{1}{2M+1} \sum_{s=-M}^M \mathbb{E} \left[\sum_{j=-J}^{-1} \bar{A}_{j,l} \mathbf{d}_{j,k} \mathbf{d}'_{j,k} \right] - \mathbb{E}[N\Upsilon jk] + O(T^{-1}) \quad \text{by linearity of expectation} \\ &= \frac{1}{2M+1} \sum_{s=-M}^M \mathbf{S}_j(k/T) - \mathbb{E}[N\Upsilon jk] + O(T^{-1}) \quad \text{by expectation of the corrected periodogram.} \\ &= \mathbf{S}_j(k/T) - \mathbb{E}[N\Upsilon jk] + O(T^{-1}) \end{aligned}$$

Now, the second expectation is developed analogously,

$$\mathbb{E}[N\Theta_{j,k}] = \frac{1}{2M+1} \sum_{s=-M}^M \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{t,t',m,m'=0}^{T-1} \sum_{p,p'=-J}^{-1} \Lambda_{p,m} \mathbb{E}[\mathbf{F}_m \epsilon'_{p',m'}] \psi_{p,m}(t) \psi_{p',m'}(t') \psi_{j,k+s}(t) \psi_{j,k+s}(t')$$

However, given that $\mathbf{F}_k \perp \epsilon_{j',k'}, \forall j, k, k'$, the covariance matrix is null - i.e. $\mathbb{E}[\mathbf{F}_m \epsilon'_{p',m'}] = \mathbf{0}_{(K \times N)}$. Finally, the compact support of the wavelets and the boundedness of all other terms (additional assumption on $\Lambda_{j,k}$!) provides the desired result. \square