Lemma 1. *Given the assumptions on the LSW and factor structure,*

$$\mathbf{\Lambda}_{j,k}\mathrm{E}\left[\mathbf{F}_k\mathbf{F}'_{k'}\right]\mathbf{\Lambda}'_{j',k'} = \begin{cases} \mathbf{S}_j\left(k/T\right) - \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j,k}\right] & ,\text{if } j = j' \text{ and } k = k' \\ -\mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}\right] & ,\text{otherwise} \end{cases}$$

Proof. From the factor structure (**??**),

$$\mathbf{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\mathbf{W}_{j'}\left(k'/T\right) = \mathbf{\Lambda}_{j,k}\mathbf{F}_k\mathbf{F}_{k'}\mathbf{\Lambda}_{j',k'} + \mathbf{\Lambda}_{j,k}\mathbf{F}_k\boldsymbol{\epsilon}'_{j',k'} + \boldsymbol{\epsilon}_{j,k}\mathbf{F}_{k'}\mathbf{\Lambda}'_{j',k'} + \boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}$$

Taking expectation on both sides,

$$\mathbf{W}_j\left(k/T\right)\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\right]\mathbf{W}_{j'}\left(k'/T\right) = \mathbf{\Lambda}_{j,k}\mathrm{E}\left[\mathbf{F}_k\mathbf{F}_{k'}\right]\mathbf{\Lambda}_{j',k'} + \mathbf{\Lambda}_{j,k}\mathrm{E}\left[\mathbf{F}_k\boldsymbol{\epsilon}'_{j',k'}\right] + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\mathbf{F}_{k'}\right]\mathbf{\Lambda}'_{j',k'} + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}\right]$$

The second and third term on the RHS is zero since $F_k \perp \boldsymbol{\epsilon}_{j',k'}, \forall j, k, k'$ (assumption (**??**)). This leaves us with

$$\mathbf{W}_j\left(k/T\right)\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\right]\mathbf{W}_{j'}\left(k'/T\right) = \mathbf{\Lambda}_{j,k}\mathrm{E}\left[\mathbf{F}_k\mathbf{F}_{k'}\right]\mathbf{\Lambda}_{j',k'} + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}\right]$$

Finally, by the assumption (**??**) on the increments of the LSW representation,

$$\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\right] = \begin{cases} \mathbf{I}_N & ,\text{if } j = j' \text{ and } k = k' \\ \mathbf{0}_N & ,\text{otherwise} \end{cases}$$

,where $\mathbf{I}_N$ is the identity matrix of rank N and $\mathbf{0}_N$ is the null matrix of rank N. This along with the definition of the CEWS we obtain the desired result. $\square$

Lemma 2. *Given the assumption on the LSW and factor structure,* *(rate of convergence ?)*

$$\mathrm{Var}\left[NK_{j,k}\right] \to 0 \quad \text{if } T \to \infty$$

Proof.

$$\mathrm{Var}\left[NK_{j,k}\right] = \mathrm{Var}\left[\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\mathbf{\Lambda}_{p,m}\mathbf{F}_m\mathbf{F}'_{m'}\mathbf{\Lambda}'_{p',m'}\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')\right] \quad \text{by (??), (?}$$

$$\leq \mathrm{E}\left[\left(\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\mathbf{\Lambda}_{p,m}\mathbf{F}_m\mathbf{F}'_{m'}\mathbf{\Lambda}'_{p',m'}\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')\right)^2\right]$$

$$= \left(\frac{1}{2M+1}\sum_{l=-J}^{-1}\bar{A}_{j,l}\right)^2\mathrm{E}\left[\sum_{\substack{s_0,s_1=-M \\ m_1,m'_1=0}}^{M}\sum_{\substack{m_0,m'_0 \\ m_1,m'_1=0}}^{T}\sum_{\substack{mp_0,p'_0 \\ p_1,p'_1=-J}}^{-1}\mathbf{\Lambda}_{p_0,m_0}\mathbf{F}_{m_0}\mathbf{F}'_{m_0}\mathbf{\Lambda}'_{p'_1,m'_1}\mathbf{\Lambda}_{p_1,m_1}\mathbf{F}_{m_1}\mathbf{F}'_{m_1}\mathbf{\Lambda}'_{p'_1,m'_1}\sum_{\substack{t_0,t'_0 \\ t_1,t'_1}}\psi_{p_0,m_0}(t_0)\psi_{p'_0,n}\right.$$

$\square$

Theorem 1. *Given the assumptions on the LSW representation, the factor structure and the definitions (**??**), (**??**) and (**??**),*

$$\mathrm{E}\left[NK_{j,k}\right] = \mathbf{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon jk\right]$$
$$\mathrm{E}\left[N\Theta_{j,k}\right] = 0$$

Proof. Let's turn out focus on the first expectation. From (**??**) and (**??**) and the linearity of expectation, the latter is then,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\mathbf{\Lambda}_{p,m}\mathrm{E}\left[\mathbf{F}_m\mathbf{F}'_{m'}\right]\mathbf{\Lambda}'_{p',m'}\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

Given the lemma 1,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m=0}^{T}\sum_{p=-J}^{-1}\mathbf{S}_p\left(m/T\right)\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

$$- \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\mathrm{E}\left[\boldsymbol{\epsilon}_{p,m}\boldsymbol{\epsilon}'_{p',m'}\right]\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

which is equivalent to

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1} \sum_{s=-M}^{M} \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{t,t',m=0}^{T} \sum_{p=-J}^{-1} \boldsymbol{S}_p\left(m/T\right) \psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t') - \mathrm{E}\left[N\Upsilon jk\right] \quad \text{by (\textbf{??})}$$

$$= \frac{1}{2M+1} \sum_{s=-M}^{M} \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{m=0}^{T} \sum_{p=-J}^{-1} \boldsymbol{S}_p\left(m/T\right) \sum_{t=0}^{T} \psi_{p,m}(t)\psi_{j,k+s}(t) \sum_{t'=0}^{T} \psi_{p,m}(t')\psi_{j,k+s}(t') - \mathrm{E}\left[N\Upsilon jk\right]$$

$$= \frac{1}{2M+1} \sum_{s=-M}^{M} \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{m=0}^{T} \sum_{p=-J}^{-1} \boldsymbol{S}_p\left(m/T\right) \left[\sum_{t=0}^{T} \psi_{p,m}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right]$$

$$= \frac{1}{2M+1} \sum_{s=-M}^{M} \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{n=0}^{T} \sum_{p=-J}^{-1} \boldsymbol{S}_p\left(n+k/T\right) \left[\sum_{t=0}^{T} \psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right] \text{ by change of variable : } m = n +$$

$$= \frac{1}{2M+1} \sum_{s=-M}^{M} \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{n=0}^{T} \sum_{p=-J}^{-1} \left[\boldsymbol{S}_p\left(k/T\right) + O(\frac{n}{T})\right] \left[\sum_{t=0}^{T} \psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right] \text{ by Lipschitz continuity}$$

$$= \frac{1}{2M+1} \sum_{s=-M}^{M} \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{p=-J}^{-1} \boldsymbol{S}_p\left(k/T\right) \sum_{n=0}^{T} \left[\sum_{t=0}^{T} \psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1})$$

Nason et al. (2000) proved that $\sum_{n=0}^{T} \left[\sum_{t=0}^{T} \psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 = A_{p,j}$. Consequently,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1} \sum_{s=-M}^{M} \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{p=-J}^{-1} \boldsymbol{S}_p\left(k/T\right) A_{p,j} - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1})$$

$$= \frac{1}{2M+1} \sum_{s=-M}^{M} \sum_{j=-J}^{-1} \bar{A}_{j,l}\mathrm{E}\left[\boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k}\right] - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1}) \text{ by the expectation of the raw periodogram.}$$

$$= \frac{1}{2M+1} \sum_{s=-M}^{M} \mathrm{E}\left[\sum_{j=-J}^{-1} \bar{A}_{j,l}\boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k}\right] - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1}) \text{ by linearity of expectation}$$

$$= \frac{1}{2M+1} \sum_{s=-M}^{M} \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1}) \text{ by expectation of the corrected periodogram.}$$

$$= \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1})$$

Now, the second expectation is developed analoguously,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1} \sum_{s=-M}^{M} \sum_{j=-J}^{-1} \bar{A}_{j,l} \sum_{t,t',m,m'=0}^{T} \sum_{p,p'=-J}^{-1} \boldsymbol{\Lambda}_{p,m}\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{\epsilon}'_{p',m'}\right] \psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

However, given that $\boldsymbol{F}_k \perp \boldsymbol{\epsilon}_{j',k'}, \forall j, k, k'$, the covariance matrix is null - i.e. $\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{\epsilon}'_{p',m'}\right] = \boldsymbol{0}_{(K\times N)}$. Finally, the compact support of the wavelets and the boundedness of all other terms (additional assumption on $\boldsymbol{\Lambda}_{j,k}$!) provides the desired result. $\square$