1 Eigenvalue matrix $\boldsymbol{V}_{j,k}$

This section will be dedicated to providing a charaterization of the eigenvalue matrix obtained from the eigendecomposition of the CEWS matrix (and the estimator of the CEWS) in terms of the factor model component.

1.1 Covariance matrix of $\boldsymbol{F}_k$

In this section we will analyse an estimator of the covariance matrix of $\boldsymbol{F}_k$. Define the wavelet estimator by,

$$\widehat{\boldsymbol{\Sigma}}^F_{j,k} = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{m=0}^{T-1}\boldsymbol{F}_m\boldsymbol{F}'_m\psi^2_{j,k+s}(m) \tag{1.1}$$

The estimator is an average of random variables. This allows us to use a law of large number to show the convergence. In other words, , under some conditions on $Y_{j,k+s} := \sum_{m=0}^{T-1}\boldsymbol{F}_m\boldsymbol{F}'_m\psi^2_{j,k+s}(m)$ the estimator converges to $\Sigma_{j,k}$. The expectation of the random variables $Y_{j,k+s}$ is,

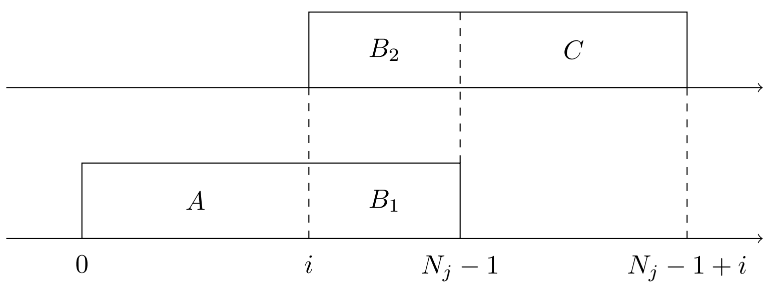
$$\begin{aligned}
\mathrm{E}\left[\sum_{m=0}^{T-1}\boldsymbol{F}_m\boldsymbol{F}'_m\psi^2_{j,k+s}(m)\right] &= \sum_{m=0}^{T-1}\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}'_m\right]\psi^2_{j,k+s}(m)\\
&= \mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}'_m\right]\sum_{m=0}^{T-1}\psi^2_{j,k+s}(m) \qquad \text{(iid factors)}\\
&= \mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}'_m\right]
\end{aligned}$$

by definition of wavelets.
The random variables $Y_{j,k+s}$ are identically distributed but not independent. Therefore, it is not sufficient to only consider their variances. By Chebyshev inequality, a sufficent condition for convergence is $|\mathrm{Cov}\left[Y_{j,k+s}, Y_{j,k+s+i}\right]| \to 0$ when $i \to \infty$. The latter condition is true for this estimator thanks to the finite support of wavelets : $N_j = (2^{-j}-1)(N_{-1}-1)+1$. From know on, we suppose that $\psi_{j,k}(t)$ lives on $[k, k+N_j-1]$.

$$\begin{aligned}
\mathrm{Cov}\left[Y_{j,k+s}, Y_{j,k+s+i}\right] &= \mathrm{E}\left[\sum_{m=0}^{T-1}\boldsymbol{F}_m\boldsymbol{F}'_m\psi^2_{j,k+s}(m)\sum_{m'=0}^{T-1}\boldsymbol{F}_{m'}\boldsymbol{F}'_{m'}\psi^2_{j,k+s+i}(m')\right] - \mathrm{E}\left[\sum_{m=0}^{T-1}\boldsymbol{F}_m\boldsymbol{F}'_m\psi^2_{j,k+s}(m)\right]^2\\
&= \mathrm{E}\left[\sum_{m=0}^{T-1}\boldsymbol{F}_m\boldsymbol{F}'_m\psi^2_{j,k+s}(m)\sum_{m'=0}^{T-1}\boldsymbol{F}_{m'}\boldsymbol{F}'_{m'}\psi^2_{j,k+s}(m'-i)\right] - (\Sigma^F_{j,k})^2\\
&= \mathrm{E}\left[\sum_{m=0}^{T-1}\boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi^2_{j,0}(m)\sum_{m'=0}^{T-1}\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi^2_{j,0}(m'-i)\right] - (\Sigma^F_{j,k})^2\\
&= \mathrm{E}\left[\sum_{m=0}^{N_j-1}\boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi^2_{j,0}(m)\sum_{m'=0}^{N_j-1}\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi^2_{j,0}(m'-i)\right] - (\Sigma^F_{j,k})^2 \qquad \text{(finite support)}
\end{aligned}$$

Let's focus the analysis in the first term. Define,

$$\alpha = min\{i-1, N_j-1\} \tag{1.2}$$
$$\gamma = max\{i-1, N_j-1\} \tag{1.3}$$



Givent the graph above, the first term can be decompose in several sums which are non-overlapping for some.

$$E\left[\left(\sum_{m=0}^{\alpha}\boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi^2_{j,0}(m) + \sum_{m=\alpha+1}^{\gamma}\boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi^2_{j,0}(m)\right)\right.$$
$$\left.\left(\sum_{m'=\alpha+1}^{\gamma}\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi^2_{j,0}(m'-i) + \sum_{m'=\gamma+1}^{i+N_j-1}\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi^2_{j,0}(m'-i)\right)\right]$$

In order to simplify notations and before distributing the product, define :

$$
\begin{aligned}
A &:= \sum_{m=0}^{\alpha} \boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi_{j,0}^2(m) \\
B_1 &:= \sum_{m=\alpha+1}^{\gamma} \boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi_{j,0}^2(m) \\
B_2 &:= \sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i) \\
C &:= \sum_{m'=\gamma+1}^{i+N_j-1} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)
\end{aligned}
$$

$$
\begin{aligned}
a &:= \sum_{m=0}^{\alpha} \psi_{j,0}^2(m) \\
b_1 &:= \sum_{m=\alpha+1}^{\gamma} \psi_{j,0}^2(m) \\
b_2 &:= \sum_{m'=\alpha+1}^{\gamma} \psi_{j,0}^2(m'-i) \\
c &:= \sum_{m'=\gamma+1}^{i+N_j-1} \psi_{j,0}^2(m'-i)
\end{aligned}
$$

Therefore, the decomposition becomes,

$$
\mathrm{E}\left[(A+B_1)(B_2+C)\right] = \mathrm{E}\left[AB_2\right] + \mathrm{E}\left[AC\right] + \mathrm{E}\left[B_1B_2\right] + \mathrm{E}\left[B_2C\right]
$$

Note that by construction all sums in the expectations except the third one are independent. Therefore,

$$
\mathrm{E}\left[A\right]\mathrm{E}\left[B_2\right] + \mathrm{E}\left[A\right]\mathrm{E}\left[A\right] + \mathrm{E}\left[B_1B_2\right] + \mathrm{E}\left[B_2\right]\mathrm{E}\left[C\right]
$$

From the iid assumption on the factors, the expectations have the same form, $\Sigma_F x$ where $x \in \{a; b_2; c\}$. Concerning the expectation $\mathrm{E}\left[B_1B_2\right]$,

$$
\begin{aligned}
\mathrm{E}\left[B_1B_2\right]^2 &= \mathrm{E}\left[\sum_{m=\alpha+1}^{\gamma} \boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi_{j,0}^2(m) \sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right]^2 \\
&\leq \mathrm{E}\left[\left(\sum_{m=\alpha+1}^{\gamma} \boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi_{j,0}^2(m)\right)^2\right] \mathrm{E}\left[\left(\sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right)^2\right]
\end{aligned}
$$

Let's analyse the second factor which "generalizes" the first one.

$$
\begin{aligned}
\mathrm{E}\left[\left(\sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right)^2\right] &= \mathrm{Var}\left[\sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right] + \Sigma_F^2 \\
&= \sum_{m'=\alpha+1}^{\gamma} \mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right]\psi_{j,0}^4(m'-i) \\
&= \mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right] \sum_{m'=\alpha+1}^{\gamma} \psi_{j,0}^4(m'-i)
\end{aligned}
$$

The last sum can further be decomposed,

$$
\begin{aligned}
\sum_{m'=\alpha+1}^{\gamma} \left[\psi_{j,0}^2(m'-i)\right]^2 &= \left(\sum_{m'=\alpha+1}^{\gamma} \psi_{j,0}^2(m'-i)\right)^2 - \sum_{p=\alpha+1}^{\gamma} \sum_{\substack{q=\alpha+1 \\ q\neq p}}^{\gamma} \psi_{j,0}^2(p-i)\psi_{j,0}^2(q-i) \\
&\leq \left(\sum_{m'=\alpha+1}^{\gamma} \psi_{j,0}^2(m'-i)\right)^2 \\
&= \begin{cases} \left(\displaystyle\sum_{m'=i}^{N_j-1} \psi_{j,0}^2(m'-i)\right)^2 & \text{if } i \in [0, N_j-1] \\ \left(\displaystyle\sum_{m'=N_j}^{i} \psi_{j,0}^2(m'-i)\right)^2 & \text{if } i \notin [0, N_j-1] \end{cases} \\
&= \begin{cases} \left(\displaystyle\sum_{m'=0}^{N_j-1-i} \psi_{j,0}^2(m')\right)^2 & \text{if } i \in [0, N_j-1] \\ \left(\displaystyle\sum_{m'=N_j-i}^{0} \psi_{j,0}^2(m')\right)^2 & \text{if } i \notin [0, N_j-1] \end{cases} \\
&\begin{cases} \leq 1 & \text{if } i \in [0, N_j-1] \\ = 0 & \text{if } i \notin [0, N_j-1] \end{cases}
\end{aligned}
$$

since $\psi_{j,0}^2(t)$ is supported on $t \in [0, N_j-1]$.
Therefore, the second factor becomes,

$$
\mathrm{E}\left[\left(\sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right)^2\right] \begin{cases} \leq \mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right] & \text{if } i \in [0, N_j-1] \\ = 0 & \text{if } i \notin [0, N_j-1] \end{cases}
$$

Finally we obtain,

$$\mathrm{E}\left[B_1B_2\right]^2 \begin{cases} \leq \mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right]^2 \sum_{m'=i}^{N_j-1} \psi_{j,0}^4(m') \sum_{m'=i}^{N_j-1} \psi_{j,0}^4(m'-i) & \text{if } i \in [0, N_j-1] \\ = 0 & \text{if } i \notin [0, N_j-1] \end{cases}$$

$$\begin{cases} \leq \mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right]^2 \sum_{m'=i}^{N_j-1} \psi_{j,0}^4(m') \sum_{m'=0}^{N_j-1-i} \psi_{j,0}^4(m') & \text{if } i \in [0, N_j-1] \\ = 0 & \text{if } i \notin [0, N_j-1] \end{cases}$$

, which is a decreasing function of i.
Equipped with those simplification we can analyse the decomposition of the covariance of $Y_{j,k+s}$,

$$\begin{aligned}
\mathrm{Cov}\left[Y_{j,k+s}, Y_{j,k+s+i}\right] &= \mathrm{E}\left[\sum_{m=0}^{N_j-1} \boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi_{j,0}^2(m) \sum_{m'=0}^{N_j-1} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right] - (\Sigma_{j,k}^F)^2 \\
&= \mathrm{E}\left[A\right]\mathrm{E}\left[B_2\right] + \mathrm{E}\left[A\right]\mathrm{E}\left[A\right] + \mathrm{E}\left[B_1B_2\right] + \mathrm{E}\left[B_2\right]\mathrm{E}\left[C\right] \\
&= (\Sigma_{j,k}^F)^2\left(ab_2 + ac + \mathrm{E}\left[B_1B_2\right] + b_1c - 1\right) \\
&= (\Sigma_{j,k}^F)^2\left(a\underbrace{(b_2 + c)}_{=1} + \mathrm{E}\left[B_1B_2\right] + b_1c - 1\right) \qquad \text{by def. of wavelets} \\
&= (\Sigma_{j,k}^F)^2\left(\sum_{m=0}^{\alpha}\psi_{j,0}^2(m) + \mathrm{E}\left[B_1B_2\right] + \sum_{m=\alpha+1}^{\gamma}\psi_{j,0}^2(m)\sum_{m'=\gamma+1}^{i+N_j-1}\psi_{j,0}^2(m'-i) - 1\right) \\
&= (\Sigma_{j,k}^F)^2\left(\sum_{m=0}^{\alpha}\psi_{j,0}^2(m) + \mathrm{E}\left[B_1B_2\right] + \left(1 - \sum_{m=0}^{\alpha}\psi_{j,0}^2(m)\right)\left(1 - \sum_{m'=0}^{N_j-1-i}\psi_{j,0}^2(m')\right) - 1\right) \\
&= (\Sigma_{j,k}^F)^2\left(\sum_{m=0}^{\alpha}\psi_{j,0}^2(m) + \mathrm{E}\left[B_1B_2\right] - \sum_{m'=0}^{N_j-1-i}\psi_{j,0}^2(m') - \sum_{m=0}^{\alpha}\psi_{j,0}^2(m) + \sum_{m=0}^{\alpha}\psi_{j,0}^2(m)\sum_{m'=0}^{N_j-1-i}\psi_{j,0}^2(m')\right) \\
&= (\Sigma_{j,k}^F)^2\left(\mathrm{E}\left[B_1B_2\right] - \sum_{m'=0}^{N_j-1-i}\psi_{j,0}^2(m') + \sum_{m=0}^{\alpha}\psi_{j,0}^2(m)\sum_{m'=0}^{N_j-1-i}\psi_{j,0}^2(m')\right) \\
&= (\Sigma_{j,k}^F)^2\left(\mathrm{E}\left[B_1B_2\right] + \sum_{m'=0}^{N_j-1-i}\psi_{j,0}^2(m')\left(\sum_{m=0}^{\alpha}\psi_{j,0}^2(m) - 1\right)\right) \\
&= (\Sigma_{j,k}^F)^2\left(\mathrm{E}\left[B_1B_2\right] - \sum_{m'=0}^{N_j-1-i}\psi_{j,0}^2(m')\sum_{m=\alpha+1}^{\gamma}\psi_{j,0}^2(m)\right)
\end{aligned}$$

By taking the absolute value and considering the above formula for lags $i \notin [0, N_j-1]$,

$$\left|\mathrm{Cov}\left[Y_{j,k+s}, Y_{j,k+s+i}\right]\right| = \left|(\Sigma_{j,k}^F)^2\right| \left|0 - 0 \cdot 0\right| = 0$$

This last relation confirms the sufficient condition to apply the needed law of large numbers. However if $i \in [0, N_j-1]$ we get the upperbound,

$$\left|\mathrm{Cov}\left[Y_{j,k+s}, Y_{j,k+s+i}\right]\right| \leq \left|(\Sigma_{j,k}^F)^2\right| \left|\mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right]^2 \sum_{m'=i}^{N_j-1}\psi_{j,0}^4(m') \sum_{m'=0}^{N_j-1-i}\psi_{j,0}^4(m') - \sum_{m'=0}^{N_j-1-i}\psi_{j,0}^2(m') \underbrace{\sum_{m=0}^{N_j-1}\psi_{j,0}^2(m)}_{=1}\right|$$

From this eigendecomposition of the CEWS estimator and (**??**) we get,

$$\left[K_{j,k} + \Theta_{j,k} + \Theta'_{j,k} + \Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k} = \widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}$$

Define,

$$Q_{j,k} = \left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}} \frac{\boldsymbol{\Lambda}'_{j,k}\widehat{\boldsymbol{\Lambda}}_{j,k}}{N} \tag{1.4}$$

$$P_{j,k} = \left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}} \frac{\boldsymbol{\Lambda}'_{j,k}\boldsymbol{\Lambda}_{j,k}}{N} \left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}} \tag{1.5}$$

Therefore we can write,

$$\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}}{N}\left[K_{j,k}+\Theta_{j,k}+\Theta'_{j,k}+\Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}=\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}}{N}\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}$$

$$\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}}{N}\left[K_{j,k}+\Theta_{j,k}+\Theta'_{j,k}+\Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}=Q_{j,k}\widehat{\boldsymbol{V}}_{j,k} \tag{1.6}$$

Let's decompose the LHS of this relation and analyze its convergence. First take,

$$\begin{aligned}\left\|\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}}{N}\left[K_{j,k}+\Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}-P_{j,k}Q_{j,k}\right\| &\leq \left\|\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}}{N}\left[K_{j,k}+\Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}-\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}}{N}N^{-1}\boldsymbol{S}_j\left(k/T\right)\widehat{\boldsymbol{\Lambda}}_{j,k}\right\| \\ &+\left\|\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}}{N}N^{-1}\boldsymbol{S}_j\left(k/T\right)\widehat{\boldsymbol{\Lambda}}_{j,k}-\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}\boldsymbol{\Lambda}_{j,k}}{N}\widehat{\boldsymbol{\Sigma}}^F_{j,k}\frac{\boldsymbol{\Lambda}'_{j,k}\widehat{\boldsymbol{\Lambda}}_{j,k}}{N}\right\|\end{aligned}$$

The first term converges asymptotically to zero. To see this note that we can write :

$$\left\|\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}}{N}\left[K_{j,k}+\Upsilon_{j,k}-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right]\widehat{\boldsymbol{\Lambda}}_{j,k}\right\|\leq\left\|\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\right\|\left\|\frac{\boldsymbol{\Lambda}'_{j,k}}{\sqrt{N}}\right\|\left\|\left[K_{j,k}+\Upsilon_{j,k}-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right]\right\|\left\|\frac{\widehat{\boldsymbol{\Lambda}}_{j,k}}{\sqrt{N}}\right\|$$

All factors except the third one are bounded by assumption. The third term converges to zero since it is sufficient in L_2 space that its expectation along its variance converges to zero :

$$\begin{aligned}\mathrm{E}\left[K_{j,k}+\Upsilon_{j,k}-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right]&=\mathrm{E}\left[K_{j,k}+\Upsilon_{j,k}\right]-N^{-1}\\&=N^{-1}\boldsymbol{S}_j\left(k/T\right)-\mathrm{E}\left[\Upsilon_{j,k}\right]+\mathrm{E}\left[\Upsilon_{j,k}\right]-N^{-1}\boldsymbol{S}_j\left(k/T\right)+O(T^{-1}) &&\text{from theorem ??}\\&=O(T^{-1})\to 0\\ \mathrm{Var}\left[K_{j,k}+\Upsilon_{j,k}-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right]&=\mathrm{Var}\left[K_{j,k}+\Upsilon_{j,k}\right]\to 0 &&\text{from lemma ??}\end{aligned}$$

It is a bit more difficult to show the convergence of the second term.

$$\left\|\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}}{N}N^{-1}\boldsymbol{S}_j\left(k/T\right)\widehat{\boldsymbol{\Lambda}}_{j,k}-\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}'_{j,k}\boldsymbol{\Lambda}_{j,k}}{N}\widehat{\boldsymbol{\Sigma}}^F_{j,k}\frac{\boldsymbol{\Lambda}'_{j,k}\widehat{\boldsymbol{\Lambda}}_{j,k}}{N}\right\|\leq\left\|\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}}\right\|\left\|\frac{\boldsymbol{\Lambda}'_{j,k}}{\sqrt{N}}\right\|\left\|N^{-1}\boldsymbol{S}_j\left(k/T\right)-\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\widehat{\boldsymbol{\Sigma}}^F_{j,k}\frac{\boldsymbol{\Lambda}'_{j,k}}{\sqrt{N}}\right\|\left\|\frac{\widehat{\boldsymbol{\Lambda}}_{j,k}}{\sqrt{N}}\right\|$$

All factors are bounded by assumption except the third one. The latter convergence to zero asymptotically. To see this, by (1.1),

$$\begin{aligned}\left\|N^{-1}\boldsymbol{S}_j\left(k/T\right)-\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\widehat{\boldsymbol{\Sigma}}^F_{j,k}\frac{\boldsymbol{\Lambda}'_{j,k}}{\sqrt{N}}\right\|&=\left\|\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\widehat{\boldsymbol{\Sigma}}^F_{j,k}\frac{\boldsymbol{\Lambda}'_{j,k}}{\sqrt{N}}-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right\|\\&=\left\|\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{m=0}^{T-1}\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\boldsymbol{F}_m\boldsymbol{F}'_m\frac{\boldsymbol{\Lambda}'_{j,k}}{\sqrt{N}}\psi^2_{j,k+s}(m)-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right\|\\&=\left\|\frac{1}{2M+1}\sum_{s=-M}^{M}\underbrace{\left[\sum_{m=0}^{T-1}\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\boldsymbol{F}_m\boldsymbol{F}'_m\frac{\boldsymbol{\Lambda}'_{j,k}}{\sqrt{N}}\psi^2_{j,k+s}(m)-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right]}_{e_{j,k+s}}\right\|\end{aligned}$$

This reorganization allows us to use a law of large number to show the convergence. The expectation of $e_{j,k+s}$ is given by :

$$\begin{aligned}\mathrm{E}\left[e_{j,k+s}\right]&=\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\left[\sum_{m=0}^{T-1}\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}'_m\right]\psi^2_{j,k+s}(m)-\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}'_m\right]\right]\frac{\boldsymbol{\Lambda}'_{j,k}}{\sqrt{N}}-N^{-1}\mathrm{E}\left[\boldsymbol{\epsilon}_{j,k+s}\boldsymbol{\epsilon}'_{j,k+s}\right]\\&=\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\left[\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}'_m\right]\sum_{m=0}^{T-1}\psi^2_{j,k+s}(m)-\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}'_m\right]\right]\frac{\boldsymbol{\Lambda}'_{j,k}}{\sqrt{N}}-N^{-1}\mathrm{E}\left[\boldsymbol{\epsilon}_{j,k+s}\boldsymbol{\epsilon}'_{j,k+s}\right]\end{aligned}$$

By definition of wavelets we know that $\sum_{m=0}^{T-1}\psi^2_{j,k+s}(m)\to 1$ when $M(T)\to\infty$. Finally since the covariance of the idiosyncratic has bounded eigenvalues by assumption on the factor model, the last term converges to zero when $N\to 0$.