1 Quantities

- $J \in \mathbb{Z}^+$ = number of scales decomposition
- $T = 2^J$ = number of time periods
- $N \in \mathbb{Z}^+$ = number of cross-section elements
- $K(\leq N)$ = number of common factors

2 Multivariate Locally Stationary Wavelet process (Park et al. (2014))

The vector $(N \times 1)$ of stochastic processes $\boldsymbol{X_{t;T}}$ follows the given decomposition :

$$\boldsymbol{X_{t;T}} = \sum_{j=-J}^{-1} \sum_{k=0}^{T-1} \boldsymbol{W}_j (k/T) \boldsymbol{\xi}_{j,k} \psi_{j,k}(t) \tag{2.1}$$

where

- $\boldsymbol{W_j(z)}$ is a lower-triangular $(N \times N)$ matrix.
  For each (m, n)-element,

$$W_j^{(m,n)}(z) \text{ is a Lipschitz continuous function on } z \in (0,1) \tag{2.2}$$

$$\sum_{j=-\infty}^{-1} \left| W_j^{(m,n)}(z) \right|^2 < \infty, \qquad \forall z \in (0,1) \qquad \text{(finite energy)} \tag{2.3}$$

$$\sum_{j=-\infty}^{-1} 2^{-j} L_j^{(m,n)} < \infty \qquad \text{(uniformly bounded Lipschitz constants } L_j\text{)} \tag{2.4}$$

- $\boldsymbol{\xi}_{j,k}$ is the vector $(N \times 1)$ of random orthonormal increments.

$$\mathrm{E}\left[\xi_{j,k}^{(u)}\right] = 0, \qquad \forall j, k, u \tag{2.5}$$

$$\mathrm{Cov}\left[\xi_{j,k}^{(u)}, \xi_{j',k'}^{(u')}\right] = \delta_{j,j'} \delta_{k,k'} \delta_{u,u'}, \qquad \forall j, j', k, k', u, u' \tag{2.6}$$

- $\psi_{j,k}(t) = \psi_{j,k-t}$ is a scalar representing a non-decimated wavelet.

We can define the *Cross-Evoluationary Wavelet Spectrum* $(N \times N)$ matrix : $\boldsymbol{S}_j(z) = \boldsymbol{W}_j(z)\boldsymbol{W}_j(z)'$. This gives us the ability to express the *local autocovariance* : $c^{(u,u')}(z,\tau) = \sum_{j=-\infty}^{-1} S_j^{(u,u')}(z)\boldsymbol{\Psi}_j(\tau)$ where $\boldsymbol{\Psi}_j(\tau) = \sum_k \psi_{j,k}(0)\psi_{j,k}(\tau)$, the *autocorrelation wavelet*. The latter also define the *inner product matrix of discrete autocorrelation wavelets* : $A_{jl} = \sum_\tau \boldsymbol{\Psi}_j(\tau)\boldsymbol{\Psi}_l(\tau)$, $A = \{A_{jl}\}_{j,l\in\mathbb{N}}$ and its inverse : $\bar{A} = A^{-1}$.A rather simple extension of the autocorrelation wavelet is the *cross-correlation wavelet* which charaterizes the dependence between two wavelets at different scales. The latter wavelet is thus defined as $\boldsymbol{\Psi}_{j,j'}(\tau) = \sum_k \psi_{j,k}(0)\psi_{j',k}(\tau)$.

Each (m, n)-element of the Cross-Evolutionary Wavelet Spectrum can be expressed as

$$S_j^{(m,n)}(k/T) = \sum_{u=1}^{N} W_j^{(m,u)}(k/T)\, W_j^{(u,n)}(k/T), \quad \forall j, k$$

From this definition it is not difficult to extend the CEWS to take into account the dependence structure between different scales and throught time :

$$S_{j,j'}^{m,n}(k/T, k'/T) = \sum_{u=1}^{N} W_j^{(m,u)}(k/T)\, W_{j'}^{(u,n)}(k'/T), \quad \forall j, j', k, k' \tag{2.7}$$

We make the following assumption regarding the latter object,

$$S_{j,j'}^{(m,n)}(k/T, k/T) = \begin{cases} S_j^{(m,n)}(k/T) & \text{if } j = j' \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

This assumption (possible improvement : condition similar to Chamberlain ?) imposes no dependence between different scales of decomposition. Notice that we don't restrict the serial dependence.

2.1 Estimation of MvLSW

- $\mathrm{E}\left[\boldsymbol{I}_{j,k}\right] = \sum_{l=-J}^{-1} A_{jl}\boldsymbol{S}_l\left(k/T\right) + O(T^{-1})$ (biaised estimator)

2.2 Estimation of MvLSW

$$\boldsymbol{d}_{j,k} = \sum_{t=0}^{T-1} \boldsymbol{X}_t \psi_{j,k}(t) \quad \text{(empirical wavelet coefficients)} \tag{2.9}$$

$$\boldsymbol{I}_{j,k} = \boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k} \quad \text{(raw wavelet periodogram)} \tag{2.10}$$

$$\bar{\boldsymbol{I}}_{j,k} = \sum_{l=-J}^{-1} \bar{A}_{j,l}\boldsymbol{I}_{l,k} \quad \text{(corrected periodogram)} \tag{2.11}$$

$$\widetilde{\boldsymbol{I}}_{j,k} = \frac{1}{2M+1}\sum_{m=-M}^{M} \boldsymbol{I}_{j,k+m} \quad \text{(smooth periodogram)} \tag{2.12}$$

$$\begin{aligned}
\widehat{\boldsymbol{S}}_j\left(k/T\right) &= \sum_{l=-J}^{-1} \bar{A}_{jl}\widetilde{\boldsymbol{I}}_{l,k} \\
&= \frac{1}{2M+1}\sum_{m=-M}^{M} \bar{\boldsymbol{I}}_{j,k+m} \\
&= \frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1} \bar{A}_{j,l}\boldsymbol{I}_{l,k+m} \quad \text{(final estimator of CEWS)}
\end{aligned} \tag{2.13}$$

2.3 Notes

- The dependence structure is entirely in $\boldsymbol{W}_j(z)$, not in $\boldsymbol{\xi}_{j,k}$.
- The lower-triangular form of $\boldsymbol{W}_j(z)$ allows us to use the Cholesky decomposition on $\boldsymbol{S}_j(z)$.

3 Factor Model

- The factor structure is imposed on the following :

$$\boldsymbol{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k} = \boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k + \boldsymbol{\epsilon}_{j,k} \tag{3.1}$$

  , not only on $\boldsymbol{\xi}_{j,k}$ since they are assumed orthonormal.

- Assumptions :
  1. $\boldsymbol{F}_k \sim (\mathbf{0}, \boldsymbol{\Sigma}_F)$, where $\boldsymbol{\Sigma}_F$ is a diagonal positive definite $(K \times K)$ matrix.
  2. $\mathrm{E}\left[F_k^{(u)^4}\right] < \infty, \forall k, u$
  3. $\boldsymbol{F}_k \perp \boldsymbol{\epsilon}_{j,k'}, \forall j, k, k'$
  4. $\boldsymbol{\epsilon}_{j,k} \sim (\mathbf{0}, \boldsymbol{\Sigma}_\epsilon)$, where $\boldsymbol{\Sigma}_\epsilon$ has bounded eigenvalues. Note : make $\boldsymbol{\Sigma}_\epsilon$ dependent on time ? what about serial dependence ?
  5. $\mathrm{E}\left[\epsilon_{j,k}^{(u)^4}\right] < \infty \quad \forall j, k, u$
  6. $\boldsymbol{\Lambda}'_{j,k}\boldsymbol{\Lambda}_{l,m} = \mathbf{0}, \forall j \neq l, \forall k \neq m.$

- We can then represent the CEWS with the factor structure :

$$\begin{aligned}
\mathrm{Var}\left[\boldsymbol{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k}\right] &= \mathrm{Var}\left[\boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k\right] + \mathrm{Var}\left[\boldsymbol{\epsilon}_{j,k}\right] \\
\boldsymbol{W}_j\left(k/T\right)\mathrm{Var}\left[\boldsymbol{\xi}_{j,k}\right]\boldsymbol{W}_j\left(k/T\right)' &= \boldsymbol{\Lambda}_{j,k}\mathrm{Var}\left[\boldsymbol{F}_k\right]\boldsymbol{\Lambda}'_{j,k} + \mathrm{Var}\left[\boldsymbol{\epsilon}_{j,k}\right] \\
\boldsymbol{W}_j\left(k/T\right)\boldsymbol{W}_j\left(k/T\right)' &= \boldsymbol{\Lambda}_{j,k}\boldsymbol{\Sigma}_F\boldsymbol{\Lambda}'_{j,k} + \boldsymbol{\Sigma}_\epsilon \qquad \text{from (2.6)} \\
\boldsymbol{S}_j\left(k/T\right) &= \boldsymbol{\Lambda}_{j,k}\boldsymbol{\Sigma}_F\boldsymbol{\Lambda}'_{j,k} + \boldsymbol{\Sigma}_\epsilon
\end{aligned}$$

3.1 Estimation

The estimation of the loadings and common factors is carried out by a non-linear least square procedure in the wavelet domain.

$$\begin{aligned}
\min_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}_{\forall j,k},\{\bar{\boldsymbol{F}}_k\}_{\forall j,k}} \quad & (NT)^{-1}\sum_t \left[\boldsymbol{X_{t;T}} - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\left(\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k\right)\psi_{j,k}(t)\right]' \left[\boldsymbol{X_{t;T}} - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\left(\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k\right)\psi_{j,k}(t)\right] \\
\text{s.t.} \quad & \frac{\bar{\boldsymbol{\Lambda}}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}}{N} = \boldsymbol{I}_K
\end{aligned} \tag{3.2}$$

After distributing the objective function becomes,

$$(NT)^{-1}\sum_t\Big[\boldsymbol{X}'_{\boldsymbol{t;T}}\boldsymbol{X_{t;T}} - \boldsymbol{X}'_{\boldsymbol{t;T}}\sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k\psi_{j,k}(t) - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\psi_{j,k}(t)\bar{\boldsymbol{F}}'_k\bar{\boldsymbol{\Lambda}}'_{j,k}\boldsymbol{X_{t;T}} + \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\sum_{l=-J}^{-1}\sum_{m=0}^{T-1}\psi_{j,k}(t)\psi_{l,m}(t)\bar{\boldsymbol{F}}'_k\bar{\boldsymbol{\Lambda}}'_{j,k}\bar{\boldsymbol{\Lambda}}_{l,m}\bar{\boldsymbol{F}}_m\Big]$$

$$(NT)^{-1}\Big[\sum_t\boldsymbol{X}'_{\boldsymbol{t;T}}\boldsymbol{X_{t;T}} - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\sum_t\boldsymbol{X}'_{\boldsymbol{t;T}}\psi_{j,k}(t)\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\bar{\boldsymbol{F}}'_k\bar{\boldsymbol{\Lambda}}'_{j,k}\sum_t\boldsymbol{X_{t;T}}\psi_{j,k}(t) + \sum_t\sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\sum_{l=-J}^{-1}\sum_{m=0}^{T-1}\psi_{j,k}(t)\psi_{l,m}(t)\bar{\boldsymbol{F}}'_k\bar{\boldsymbol{\Lambda}}'_{j,k}\bar{\boldsymbol{\Lambda}}_{l,m}\bar{\boldsymbol{F}}_m\Big]$$

By definition of the empirical wavelet coefficents,

$$(NT)^{-1}\Big[\sum_t\boldsymbol{X}'_{\boldsymbol{t;T}}\boldsymbol{X_{t;T}} - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\bar{\boldsymbol{F}}'_k\bar{\boldsymbol{\Lambda}}'_{j,k}\boldsymbol{d}_{j,k} + \sum_t\sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\sum_{l=-J}^{-1}\sum_{m=0}^{T-1}\psi_{j,k}(t)\psi_{l,m}(t)\bar{\boldsymbol{F}}'_k\bar{\boldsymbol{\Lambda}}'_{j,k}\bar{\boldsymbol{\Lambda}}_{l,m}\bar{\boldsymbol{F}}_m\Big]$$

By assumption on the loadings (possible improvement) and the fact that wavelets are normalized $\sum_t(\psi_{j,k}(t))^2 = 1, \forall j,k$,

$$(NT)^{-1}\Big[\sum_t\boldsymbol{X}'_{\boldsymbol{t;T}}\boldsymbol{X_{t;T}} - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\bar{\boldsymbol{F}}'_k\bar{\boldsymbol{\Lambda}}'_{j,k}\boldsymbol{d}_{j,k} + \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\bar{\boldsymbol{F}}'_k\bar{\boldsymbol{\Lambda}}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k\Big]$$

The First Order Conditions with respect to the factors are given by :

$$\begin{aligned}
\bar{\boldsymbol{F}}'_k\sum_{j=-J}^{-1}\bar{\boldsymbol{\Lambda}}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k} - \sum_{j=-J}^{-1}\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k} = 0 \quad ,\forall k \\
\bar{\boldsymbol{F}}'_k\sum_{j=-J}^{-1}N = \sum_{j=-J}^{-1}\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k} \quad ,\forall k & \qquad \text{from (3.2)} \\
\bar{\boldsymbol{F}}'_k = (JN)^{-1}\sum_{j=-J}^{-1}\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}
\end{aligned} \tag{3.3}$$

Replace (3.3) in the original minimization problem,

$$\begin{aligned}
\min_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}_{\forall j,k}} \quad (NT)^{-1}\Big[\sum_t\boldsymbol{X}'_{\boldsymbol{t;T}}\boldsymbol{X_{t;T}} & - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\left((JN)^{-1}\sum_{l=-J}^{-1}\bar{\boldsymbol{\Lambda}}'_{l,k}\boldsymbol{d}_{l,k}\right) \\
& - \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\left((JN)^{-1}\sum_{l=-J}^{-1}\boldsymbol{d}'_{l,k}\bar{\boldsymbol{\Lambda}}_{l,k}\right)\bar{\boldsymbol{\Lambda}}'_{j,k}\boldsymbol{d}_{j,k} \\
& + \sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\left((JN)^{-1}\sum_{l=-J}^{-1}\boldsymbol{d}'_{l,k}\bar{\boldsymbol{\Lambda}}_{l,k}\right)\bar{\boldsymbol{\Lambda}}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\left((JN)^{-1}\sum_{n=-J}^{-1}\bar{\boldsymbol{\Lambda}}'_{n,k}\boldsymbol{d}_{n,k}\right)\Big]
\end{aligned}$$

$$\begin{aligned}
\min_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}_{\forall j,k}} \quad (NT)^{-1}\Big[\sum_t\boldsymbol{X}'_{\boldsymbol{t;T}}\boldsymbol{X_{t;T}} & - (JN)^{-1}\sum_{l=-J}^{-1}\sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{\Lambda}}'_{l,k}\boldsymbol{d}_{l,k} \\
& - (JN)^{-1}\sum_{l=-J}^{-1}\sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\boldsymbol{d}'_{l,k}\bar{\boldsymbol{\Lambda}}_{l,k}\bar{\boldsymbol{\Lambda}}'_{j,k}\boldsymbol{d}_{j,k} \\
& + (JN)^{-2}\sum_{l=-J}^{-1}\sum_{n=-J}^{-1}\sum_{j=-J}^{-1}\sum_{k=0}^{T-1}\boldsymbol{d}'_{l,k}\bar{\boldsymbol{\Lambda}}_{l,k}\bar{\boldsymbol{\Lambda}}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{\Lambda}}'_{n,k}\boldsymbol{d}_{n,k}\Big]
\end{aligned}$$

$$\min_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (NT)^{-1}\Big[\sum_t \boldsymbol{X}'_{t;T}\boldsymbol{X}_{t;T} - (JN)^{-1}\sum_{j=-J}^{-1} i \sum_{j=-J}^{-1}\sum_{k=0}^{T-1} \boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{\Lambda}}'_{l,k}\boldsymbol{d}_{l,k}$$

$$- (JN)^{-1}\sum_{l=-J}^{-1}\sum_{j=-J}^{-1}\sum_{k=0}^{T-1} \boldsymbol{d}'_{l,k}\bar{\boldsymbol{\Lambda}}_{l,k}\bar{\boldsymbol{\Lambda}}'_{j,k}\boldsymbol{d}_{j,k}$$

$$+ (JN)^{-2} JN \sum_{l=-J}^{-1}\sum_{n=-J}^{-1}\sum_{j=-J}^{-1}\sum_{k=0}^{T-1} \boldsymbol{d}'_{l,k}\bar{\boldsymbol{\Lambda}}_{l,k}\bar{\boldsymbol{\Lambda}}'_{n,k}\boldsymbol{d}_{n,k}\Big] \qquad \text{from (3.2)}$$

$$\min_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (NT)^{-1}\Big[\sum_t \boldsymbol{X}'_{t;T}\boldsymbol{X}_{t;T} - (JN)^{-1}\sum_{l=-J}^{-1}\sum_{j=-J}^{-1}\sum_{k=0}^{T-1} \boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{\Lambda}}'_{l,k}\boldsymbol{d}_{l,k}\Big]$$

Minimizing the latter expression is equivalent to maximizing,

$$\max_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (JT)^{-1}N^{-2}\sum_{l=-J}^{-1}\sum_{j=-J}^{-1}\sum_{k=0}^{T-1} \boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{\Lambda}}'_{l,k}\boldsymbol{d}_{l,k}$$

Each term in the triple sum is a scalar, i.e. (1×1) matrix. Therefore we can freely take its trace,

$$\max_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (JT)^{-1}N^{-2}\sum_{l=-J}^{-1}\sum_{j=-J}^{-1}\sum_{k=0}^{T-1} \operatorname{tr}\left\{\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\bar{\boldsymbol{\Lambda}}'_{l,k}\boldsymbol{d}_{l,k}\right\}$$

From the cyclic property of the trace,

$$\max_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (JT)^{-1}N^{-2}\sum_{l=-J}^{-1}\sum_{j=-J}^{-1}\sum_{k=0}^{T-1} \operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}'_{l,k}\boldsymbol{d}_{l,k}\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\right\}$$

$$\max_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (JT)^{-1}N^{-2}\left[\sum_{k=0}^{T-1}\sum_{j=-J}^{-1} \operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}'_{j,k}\boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\right\} + 2\sum_{k=0}^{T-1}\sum_{j=-J}^{-1}\sum_{l=-J}^{j-1} \operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}'_{l,k}\boldsymbol{d}_{l,k}\boldsymbol{d}'_{j,k}\bar{\boldsymbol{\Lambda}}_{j,k}\right\}\right]$$

We recognize the raw wavelet periodogram $\boldsymbol{d}_{l,k}\boldsymbol{d}'_{j,k}$. We replace the latter with the unbiased and consistent estimator of the CEWS, i.e. $\widehat{\boldsymbol{S}}_j\,(k/T)$.
Therefore the first term becomes,

$$\sum_{k=0}^{T-1}\sum_{j=-J}^{-1} \operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}'_{j,k}\widehat{\boldsymbol{S}}_j\,(k/T)\,\bar{\boldsymbol{\Lambda}}_{j,k}\right\}$$

The second term needs a similar treatement. We replace $\boldsymbol{d}_{l,k}\boldsymbol{d}'_{j,k}$ with the unbiaised and consistent estimator of (2.7) :

$$\hat{\boldsymbol{S}}_{j,j'}(k/T,k/T) = \frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{r=-J}^{-1}\sum_{l=-J}^{-1} \bar{A}_{j,l}^{(j',r)}\boldsymbol{d}_{j,k+m}\boldsymbol{d}'_{j',k+m} \tag{3.4}$$

where $\bar{A}_{j,l}^{(j',l')} = \sum_\tau \Psi_{j,j'}(\tau)\Psi_{l,l'}(\tau)$ and $\Psi_{j,j'}(\tau) = \sum_t \psi_{j,0}(t)\psi_{j',\tau}(t)$, the inner product operator of the cross-correlation wavelet functions and the *Cross-Correlation Wavelet Function*, respectively. In his thesis Koch showed that this CCWF inherit the same properties as the autocorrelation function. One of the latter is that the functions in that family are linearly indepedent of each other. Consequently, the inner product operator of that family is invertible.

The second term on the maximization problem thus reads,

$$2\sum_{k=0}^{T-1}\sum_{j=-J}^{-1}\sum_{l=-J}^{j-1} \operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}'_{l,k}\hat{\boldsymbol{S}}_{j,l}(k/T,k/T)\bar{\boldsymbol{\Lambda}}_{j,k}\right\}$$

Consequently, the whole optimization problem is changed into

$$\max_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (JT)^{-1}N^{-2}\left[\sum_{k=0}^{T-1}\sum_{j=-J}^{-1} \operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}'_{j,k}\widehat{\boldsymbol{S}}_j\,(k/T)\,\bar{\boldsymbol{\Lambda}}_{j,k}\right\} + 2\sum_{k=0}^{T-1}\sum_{j=-J}^{-1}\sum_{l=-J}^{j-1} \operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}'_{l,k}\hat{\boldsymbol{S}}_{j,l}(k/T,k/T)\bar{\boldsymbol{\Lambda}}_{j,k}\right\}\right]$$

which is asymptotically equivalent to (see Park et al. (2014)),

$$\max_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (JT)^{-1}N^{-2}\left[\sum_{k=0}^{T-1}\sum_{j=-J}^{-1}\operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}_{j,k}'\boldsymbol{S}_j\left(k/T\right)\bar{\boldsymbol{\Lambda}}_{j,k}\right\}+2\sum_{k=0}^{T-1}\sum_{j=-J}^{-1}\sum_{l=-J}^{j-1}\operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}_{l,k}'\boldsymbol{S}_{j,l}(k/T,k/T)\bar{\boldsymbol{\Lambda}}_{j,k}\right\}\right]$$

The last term is zero by assumption (2.8),

$$\max_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (JT)^{-1}N^{-2}\sum_{k=0}^{T-1}\sum_{j=-J}^{-1}\operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}_{j,k}'\boldsymbol{S}_j\left(k/T\right)\bar{\boldsymbol{\Lambda}}_{j,k}\right\}$$

Finally, we get back the feasible problem,

$$\max_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (JT)^{-1}N^{-2}\sum_{k=0}^{T-1}\sum_{j=-J}^{-1}\operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}_{j,k}'\widehat{\boldsymbol{S}}_j\left(k/T\right)\bar{\boldsymbol{\Lambda}}_{j,k}\right\}+O(M^{-1}) \tag{3.5}$$

where $M(T)\to\infty$ when $T\to\infty$.

This final problem can be decompose into sub-problems and the latter can be solved independently. In other words, the problem (3.5) is maximized when each term in the double sum is also maximized. (Proof)

$$\max_{\{\bar{\boldsymbol{\Lambda}}_{j,k}\}\forall j,k} \quad (JT)^{-1}\sum_{k=0}^{T-1}\sum_{j=-J}^{-1}N^{-2}\operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}_{j,k}'\widehat{\boldsymbol{S}}_j\left(k/T\right)\bar{\boldsymbol{\Lambda}}_{j,k}\right\} \quad = \quad (JT)^{-1}\sum_{k=0}^{T-1}\sum_{j=-J}^{-1}\max_{\bar{\boldsymbol{\Lambda}}_{j,k}} N^{-2}\operatorname{tr}\left\{\bar{\boldsymbol{\Lambda}}_{j,k}'\widehat{\boldsymbol{S}}_j\left(k/T\right)\bar{\boldsymbol{\Lambda}}_{j,k}\right\}$$

This solution of those final optimization problems are $\sqrt{N}\tilde{\boldsymbol{\Lambda}}_{j,k}$, where $\tilde{\boldsymbol{\Lambda}}_{j,k}$ is the $(N\times K)$ matrix whose columns are the first K orthonormal eigenvectors of $\widehat{\boldsymbol{S}}_j\left(k/T\right)$.

We obtain the wanted least squares estimators of the loadings and factors as

$$\widehat{\boldsymbol{\Lambda}}_{j,k}=\sqrt{N}\tilde{\boldsymbol{\Lambda}}_{j,k} \tag{3.6}$$

$$\widehat{\boldsymbol{F}}_k=(JN)^{-1}\sum_{j=-J}^{-1}\widehat{\boldsymbol{\Lambda}}_{j,k}'\boldsymbol{d}_{j,k} \tag{3.7}$$

4 CEWS Estimator under the factor model

In this section we will develop the *Cross-Evolutionary Wavelet Estimator* by using the factor structure and we will prove the convergence of this estimator under the structure. Recall the consistent and unbiaised estimator of the CEWS :

$$\widehat{\boldsymbol{S}}_j\left(k/T\right)=\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\boldsymbol{d}_{l,k+m}\boldsymbol{d}_{l,k+m}'$$

by multiplying both sides by N^{-1} and by using (2.1), (2.9) and (3.1) we successively obtain :

$$\begin{aligned}
N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right) &= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T}\boldsymbol{X}_{t;T}\psi_{j,k+s}(t)\sum_{t'=0}^{T}\boldsymbol{X}_{t';T}'\psi_{j,k+s}(t') && \text{by (2.9)}\\
&= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T-1}\sum_{t'=0}^{T-1}\sum_{m=0}^{T-1}\sum_{p=-J}^{-1}\boldsymbol{W}_p\left(m/T\right)\boldsymbol{\xi}_{p,m}\psi_{p,m}(t)\\
&\qquad\sum_{m'=0}^{T-1}\sum_{p'=-J}^{-1}\boldsymbol{\xi}_{p',m'}'\boldsymbol{W}_{p'}\left(m'/T\right)'\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t') && \text{by (2.1)}\\
&= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T-1}\sum_{t'=0}^{T-1}\sum_{m=0}^{T-1}\sum_{p=-J}^{-1}(\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m+\boldsymbol{\epsilon}_{p,m})\psi_{p,m}(t)\\
&\qquad\sum_{m'=0}^{T-1}\sum_{p'=-J}^{-1}(\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m+\boldsymbol{\epsilon}_{p,m})'\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t') && \text{by (3.1)}
\end{aligned}$$

In order to ease readibility, let's define two objects :

$$C_t=\sum_{m=0}^{T-1}\sum_{p=-J}^{-1}\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m\psi_{p,m}(t) \tag{4.1}$$

$$E_t=\sum_{m=0}^{T-1}\sum_{p=-J}^{-1}\boldsymbol{\epsilon}_{p,m}\psi_{p,m}(t) \tag{4.2}$$

Consequently,

$$
\begin{aligned}
N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right) &= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T-1}\sum_{t'=0}^{T-1}[C_t+E_t][C_t+E_t]'\psi_{j,k+s}(t)\psi_{j,k+s}(t') \\
&= N^{-1}\frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T-1}\sum_{t'=0}^{T-1}[C_tC_{t'}+C_tE'_{t'}+E_tC'_{t'}+E_tE'_{t'}]\psi_{j,k+s}(t)\psi_{j,k+s}(t')
\end{aligned}
$$

To further simplify the notation we define the following,

$$
K_{j,k} = N^{-1}\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T-1}\sum_{t'=0}^{T-1}C_tC'_{t'}\psi_{j,k+s}(t)\psi_{j,k+s}(t') \tag{4.3}
$$

$$
\Upsilon_{j,k} = N^{-1}\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T-1}\sum_{t'=0}^{T-1}E_tE'_{t'}\psi_{j,k+s}(t)\psi_{j,k+s}(t') \tag{4.4}
$$

$$
\Theta_{j,k} = N^{-1}\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{l=-J}^{-1}\bar{A}_{j,l}\sum_{t=0}^{T-1}\sum_{t'=0}^{T-1}C_tE'_{t'}\psi_{j,k+s}(t)\psi_{j,k+s}(t') \tag{4.5}
$$

The estimator (2.13) can now be expressed as,

$$
N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right) = K_{j,k}+\Theta_{j,k}+\Theta'_{j,k}+\Upsilon_{j,k} \tag{4.6}
$$

From Park et al. (2014) this estimator converge to the true Cross-Evoluationary Wavelet Spectrum. The next development assert the same convergence with the estimator redefined by the factor structure.

First, the expectation of the estimator can be decomposed thanks to (4.6),

$$
\mathrm{E}\left[\widehat{\boldsymbol{S}}_j\left(k/T\right)\right] = \mathrm{E}\left[NK_{j,k}\right]+\mathrm{E}\left[N\Theta_{j,k}\right]+\mathrm{E}\left[N\Theta_{j,k}\right]+\mathrm{E}\left[N\Upsilon_{j,k}\right]
$$

Theorem 1 prove the asymptotic unbiaisedness of the CEWS estimator.

$$
\begin{aligned}
\mathrm{E}\left[\widehat{\boldsymbol{S}}_j\left(k/T\right)\right] &= \boldsymbol{S}_j\left(k/T\right)-\mathrm{E}\left[N\Upsilon jk\right]+O(T^{-1})+\mathbf{0}_N+\mathbf{0}_N+\mathrm{E}\left[N\Upsilon_{j,k}\right] \\
&= \boldsymbol{S}_j\left(k/T\right)+O(T^{-1})
\end{aligned}
$$

Next, to analyse the variance of the estimator, we define $\mathcal{S}=\{K_{j,k},\Theta_{j,k},\Theta'_{j,k},\Upsilon_{j,k}\}$ and we decompose the variance as follows :

$$
\mathrm{Var}\left[\widehat{\boldsymbol{S}}_j\left(k/T\right)\right] = \sum_{i\in\mathcal{S}}\mathrm{Var}\left[Ni\right]+\sum_{i\in\mathcal{S}}\sum_{\substack{j\in\mathcal{S}\\ j\neq i}}\mathrm{Cov}\left[Ni,Nj\right] \tag{4.7}
$$

5 Rotation Matrix $\boldsymbol{R}_{j,k}$

In this section we find an expression for the rotation matrix $\boldsymbol{R}_{j,k}$ (Recall the inherent indeterminacy of factor models). Imposing particular identification restrictions on the loadings and factors will specify a unique rotation matrix. In our factor model, we don' t impose any identification restrictions. Consequently, the researcher apply our model has the responsability of stating relevant restriction given the context of interest.

The rotation matrix appears in the convergence of loadings and factors.

$$
\widehat{\boldsymbol{\Lambda}}_{j,k}-\boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k} \tag{5.1}
$$

$$
\widehat{\boldsymbol{F}}_k-\boldsymbol{R}_{j,k}^{-1}\boldsymbol{F}_k \tag{5.2}
$$

Take first (5.1). Given the formula for the estimator of the loadings - i.e. (3.6), we obtain :

$$
\begin{aligned}
\widehat{\boldsymbol{\Lambda}}_{j,k} &= N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right)\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1} \\
\widehat{\boldsymbol{\Lambda}}_{j,k}-\boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k} &= N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right)\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1}-\boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k} \\
&= \left[K_{j,k}+\Theta_{j,k}+\Theta'_{j,k}+\Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1}-\boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k} \quad \text{by (4.6)}
\end{aligned}
$$

The first restriction we can impose on the rotation matrix $\boldsymbol{R}_{j,k}$ is $\boldsymbol{R}_{j,k}=\alpha\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1}$ for some α, $(K\times N)$ matrix. Such that,

$$
\widehat{\boldsymbol{\Lambda}}_{j,k}-\boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k} = \left[K_{j,k}-\boldsymbol{\Lambda}_{j,k}\alpha+\Theta_{j,k}+\Theta'_{j,k}+\Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1}
$$

In order to have unbiaised and consistent estimation of the loadings, we require respectively,

$$\mathrm{E}\left[\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\right] = 0 \tag{5.3}$$

$$\mathrm{Var}\left[\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\right] \to 0 \tag{5.4}$$

In Hilbert space, the restriction (5.4) is equivalent to the **euclidiean norm** tending to zero,

$$\left\|\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\right\| \to 0 \tag{5.5}$$

Consequently,

$$\begin{aligned}
N^{-\frac{1}{2}}\left\|\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\right\| &= N^{-\frac{1}{2}}\left\|\left[K_{j,k} - \boldsymbol{\Lambda}_{j,k}\alpha + \Theta_{j,k} + \Theta'_{j,k} + \Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1}\right\| \\
&\leq \left\|\left[K_{j,k} - \boldsymbol{\Lambda}_{j,k}\alpha + \Theta_{j,k} + \Theta'_{j,k} + \Upsilon_{j,k}\right]\right\| \left\|\frac{\widehat{\boldsymbol{\Lambda}}_{j,k}}{\sqrt{N}}\right\| \left\|\widehat{\boldsymbol{V}}_{j,k}^{-1}\right\| \\
&\leq \left[\left\|K_{j,k} - \boldsymbol{\Lambda}_{j,k}\alpha + \Upsilon_{j,k}\right\| + 2\left\|\Theta_{j,k}\right\|\right] \left\|\frac{\widehat{\boldsymbol{\Lambda}}_{j,k}}{\sqrt{N}}\right\| \left\|\widehat{\boldsymbol{V}}_{j,k}^{-1}\right\| \\
N^{\frac{1}{2}}\left\|\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\right\| &\leq \left[\left\|NK_{j,k} - N\boldsymbol{\Lambda}_{j,k}\alpha + N\Upsilon_{j,k}\right\| + 2\left\|N\Theta_{j,k}\right\|\right] \left\|\frac{\widehat{\boldsymbol{\Lambda}}_{j,k}}{\sqrt{N}}\right\| \left\|\widehat{\boldsymbol{V}}_{j,k}^{-1}\right\|
\end{aligned}$$

First, we analyse the convergence of the first term on the RHS.
Its expectation is,

$$\begin{aligned}
\mathrm{E}\left[NK_{j,k} - N\boldsymbol{\Lambda}_{j,k}\alpha + N\Upsilon_{j,k}\right] &= \mathrm{E}\left[NK_{j,k}\right] - \mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] + \mathrm{E}\left[N\Upsilon_{j,k}\right] \\
&= \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon_{j,k}\right] - \mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] + \mathrm{E}\left[N\Upsilon_{j,k}\right] \qquad \text{by theorem (1)} \\
&= \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right]
\end{aligned}$$

Therefore, given the unbiaisedness restriction (5.3) we obtain another constraint for the rotation matrix.

$$\mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] = \boldsymbol{S}_j\left(k/T\right) \tag{5.6}$$

Note that we also need $\mathrm{Var}\left[NK_{j,k} - N\boldsymbol{\Lambda}_{j,k}\alpha + N\Upsilon_{j,k}\right] \to 0$. (See next part)

$$\begin{aligned}
\mathrm{Var}\left[NK_{j,k} - N\boldsymbol{\Lambda}_{j,k}\alpha + N\Upsilon_{j,k}\right] = {} & \mathrm{Var}\left[NK_{j,k}\right] + \mathrm{Var}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] + \mathrm{Var}\left[N\Upsilon_{j,k}\right] \\
& + 2\mathrm{Cov}\left[NK_{j,k}, N\boldsymbol{\Lambda}_{j,k}\alpha\right] + 2\mathrm{Cov}\left[NK_{j,k}, N\Upsilon_{j,k}\right] + 2\mathrm{Cov}\left[N\boldsymbol{\Lambda}_{j,k}\alpha, N\Upsilon_{j,k}\right]
\end{aligned}$$

By the result of 2,

$$\mathrm{Var}\left[NK_{j,k} - N\boldsymbol{\Lambda}_{j,k}\alpha + N\Upsilon_{j,k}\right] = \mathrm{Var}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] + 2\mathrm{Cov}\left[NK_{j,k}, N\boldsymbol{\Lambda}_{j,k}\alpha\right] + 2\mathrm{Cov}\left[N\boldsymbol{\Lambda}_{j,k}\alpha, N\Upsilon_{j,k}\right] \tag{5.7}$$

5.1 Candidate Rotation matrix

This section develop a rotation matrix candidate.
Given the expectation restriction (5.6), and the fact that $\widehat{\boldsymbol{S}}_j\left(k/T\right)$ is an asymptotically unbiaised estimator of $\boldsymbol{S}_j\left(k/T\right)$,

$$\begin{aligned}
\mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] &= \mathrm{E}\left[\widehat{\boldsymbol{S}}_j\left(k/T\right)\right] \\
&= \mathrm{E}\left[NK_{j,k}\right] + \mathrm{E}\left[N\Theta_{j,k}\right] + \mathrm{E}\left[N\Theta_{j,k}\right] + \mathrm{E}\left[N\Upsilon_{j,k}\right] \qquad \text{by (4.6)} \\
&= \mathrm{E}\left[NK_{j,k}\right] + 0 + 0 + \mathrm{E}\left[N\Upsilon_{j,k}\right] \qquad \text{by theorem 1.} \\
&= \mathrm{E}\left[NK_{j,k} + N\Upsilon_{j,k}\right] \\
&= \mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}K_{j,k} + N\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}\Upsilon_{j,k}\right] \\
&= \mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\left(\boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}K_{j,k} + \boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}\Upsilon_{j,k}\right)\right]
\end{aligned}$$

Our <u>candidate</u> α is therefore $\alpha = \left(\boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}K_{j,k} + \boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}\Upsilon_{j,k}\right)$ (invertibility of matrix ?). By using the latter α we obtain the desired unbiaisedness on the loadings by construction. Do we also have the consistency of the estimator ? We answer this question by analysing the 3 terms in (5.7),

$$\begin{aligned}
\mathrm{Var}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] &= \mathrm{Var}\left[NK_{j,k} + N\Upsilon_{j,k}\right] \\
&= \mathrm{Var}\left[NK_{j,k}\right] + \mathrm{Var}\left[N\Upsilon_{j,k}\right] + 2\mathrm{Cov}\left[NK_{j,k}, N\Upsilon_{j,k}\right] \\
&= 0 + 0 + 0 \qquad \text{by lemma 1}
\end{aligned}$$

The final rotation matrix is therefore given by,

$$\widehat{\boldsymbol{R}}_{j,k} = \boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}\left[K_{j,k} + \Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1} \tag{5.8}$$

6 Eigenvalue matrix $\boldsymbol{V}_{j,k}$

This section will be dedicated to providing a charaterization of the eigenvalue matrix obtained from the eigendecomposition of the CEWS matrix (and the estimator of the CEWS) in terms of the factor model component.

6.1 Covariance matrix of $\boldsymbol{F}_k$

In this section we will analyse an estimator of the covariance matrix of $\boldsymbol{F}_k$. Define the wavelet estimator by,

$$\widehat{\boldsymbol{\Sigma}}^F_{j,k} = \frac{1}{2M+1} \sum_{s=-M}^{M} \sum_{m=0}^{T-1} \boldsymbol{F}_m \boldsymbol{F}'_m \psi^2_{j,k+s}(m) \tag{6.1}$$

The estimator is an average of random variables. This allows us to use a law of large number to show the convergence. In other words, , under some conditions on $Y_{j,k+s} := \sum_{m=0}^{T-1} \boldsymbol{F}_m \boldsymbol{F}'_m \psi^2_{j,k+s}(m)$ the estimator converges to $\Sigma_{j,k}$. The expectation of the random variables $Y_{j,k+s}$ is,

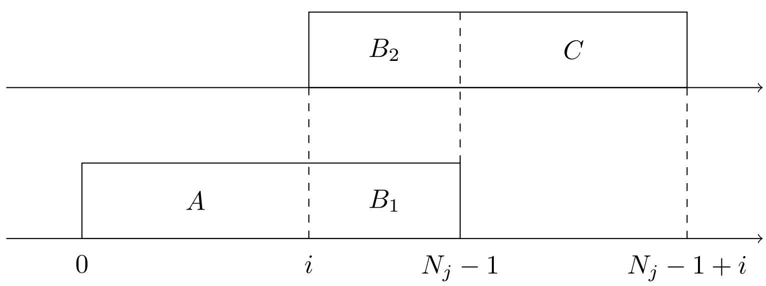
$$\begin{aligned}
\mathrm{E}\left[\sum_{m=0}^{T-1} \boldsymbol{F}_m \boldsymbol{F}'_m \psi^2_{j,k+s}(m)\right] &= \sum_{m=0}^{T-1} \mathrm{E}\left[\boldsymbol{F}_m \boldsymbol{F}'_m\right] \psi^2_{j,k+s}(m) \\
&= \mathrm{E}\left[\boldsymbol{F}_m \boldsymbol{F}'_m\right] \sum_{m=0}^{T-1} \psi^2_{j,k+s}(m) \qquad \text{(iid factors)} \\
&= \mathrm{E}\left[\boldsymbol{F}_m \boldsymbol{F}'_m\right]
\end{aligned}$$

by definition of wavelets.
The random variables $Y_{j,k+s}$ are identically distributed but not independent. Therefore, it is not sufficient to only consider their variances. By Chebyshev inequality, a sufficent condition for convergence is $|\mathrm{Cov}\left[Y_{j,k+s}, Y_{j,k+s+i}\right]| \to 0$ when $i \to \infty$. The latter condition is true for this estimator thanks to the finite support of wavelets : $N_j = (2^{-j} - 1)(N_{-1} - 1) + 1$. From know on, we suppose that $\psi_{j,k}(t)$ lives on $[k, k + N_j - 1]$.

$$\begin{aligned}
\mathrm{Cov}\left[Y_{j,k+s}, Y_{j,k+s+i}\right] &= \mathrm{E}\left[\sum_{m=0}^{T-1} \boldsymbol{F}_m \boldsymbol{F}'_m \psi^2_{j,k+s}(m) \sum_{m'=0}^{T-1} \boldsymbol{F}_{m'} \boldsymbol{F}'_{m'} \psi^2_{j,k+s+i}(m')\right] - \mathrm{E}\left[\sum_{m=0}^{T-1} \boldsymbol{F}_m \boldsymbol{F}'_m \psi^2_{j,k+s}(m)\right]^2 \\
&= \mathrm{E}\left[\sum_{m=0}^{T-1} \boldsymbol{F}_m \boldsymbol{F}'_m \psi^2_{j,k+s}(m) \sum_{m'=0}^{T-1} \boldsymbol{F}_{m'} \boldsymbol{F}'_{m'} \psi^2_{j,k+s}(m'-i)\right] - (\Sigma^F_{j,k})^2 \\
&= \mathrm{E}\left[\sum_{m=0}^{T-1} \boldsymbol{F}_{m+k+s} \boldsymbol{F}'_{m+k+s} \psi^2_{j,0}(m) \sum_{m'=0}^{T-1} \boldsymbol{F}_{m'+k+s} \boldsymbol{F}'_{m'+k+s} \psi^2_{j,0}(m'-i)\right] - (\Sigma^F_{j,k})^2 \\
&= \mathrm{E}\left[\sum_{m=0}^{N_j-1} \boldsymbol{F}_{m+k+s} \boldsymbol{F}'_{m+k+s} \psi^2_{j,0}(m) \sum_{m'=0}^{N_j-1} \boldsymbol{F}_{m'+k+s} \boldsymbol{F}'_{m'+k+s} \psi^2_{j,0}(m'-i)\right] - (\Sigma^F_{j,k})^2 \qquad \text{(finite support)}
\end{aligned}$$

Let's focus the analysis in the first term. Define,

$$\alpha = min\{i-1, N_j - 1\} \tag{6.2}$$
$$\gamma = max\{i-1, N_j - 1\} \tag{6.3}$$



Givent the graph above, the first term can be decompose in several sums which are non-overlapping for some.

$$E\left[\left(\sum_{m=0}^{\alpha} \boldsymbol{F}_{m+k+s} \boldsymbol{F}'_{m+k+s} \psi^2_{j,0}(m) + \sum_{m=\alpha+1}^{\gamma} \boldsymbol{F}_{m+k+s} \boldsymbol{F}'_{m+k+s} \psi^2_{j,0}(m)\right) \left(\sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s} \boldsymbol{F}'_{m'+k+s} \psi^2_{j,0}(m'-i) + \sum_{m'=\gamma+1}^{i+N_j-1} \boldsymbol{F}_{m'+k+s} \boldsymbol{F}'_{m'+k+s} \psi^2_{j,0}(m'-i)\right)\right]$$

In order to simplify notations and before distributing the product, define :

$$\begin{aligned}
A &:= \sum_{m=0}^{\alpha} \boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi_{j,0}^2(m) & a &:= \sum_{m=0}^{\alpha} \psi_{j,0}^2(m)\\
B_1 &:= \sum_{m=\alpha+1}^{\gamma} \boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi_{j,0}^2(m) & b_1 &:= \sum_{m=\alpha+1}^{\gamma} \psi_{j,0}^2(m)\\
B_2 &:= \sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i) & b_2 &:= \sum_{m'=\alpha+1}^{\gamma} \psi_{j,0}^2(m'-i)\\
C &:= \sum_{m'=\gamma+1}^{i+N_j-1} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i) & c &:= \sum_{m'=\gamma+1}^{i+N_j-1} \psi_{j,0}^2(m'-i)
\end{aligned}$$

Therefore, the decomposition becomes,

$$\mathrm{E}\left[(A+B_1)(B_2+C)\right] = \mathrm{E}\left[AB_2\right] + \mathrm{E}\left[AC\right] + \mathrm{E}\left[B_1B_2\right] + \mathrm{E}\left[B_2C\right]$$

Note that by construction all sums in the expectations except the third one are independent. Therefore,

$$\mathrm{E}\left[A\right]\mathrm{E}\left[B_2\right] + \mathrm{E}\left[A\right]\mathrm{E}\left[A\right] + \mathrm{E}\left[B_1B_2\right] + \mathrm{E}\left[B_2\right]\mathrm{E}\left[C\right]$$

From the iid assumption on the factors, the expectations have the same form, $\Sigma_F x$ where $x \in \{a; b_2; c\}$. Concerning the expectation $\mathrm{E}\left[B_1B_2\right]$,

$$\begin{aligned}
\mathrm{E}\left[B_1B_2\right]^2 &= \mathrm{E}\left[\sum_{m=\alpha+1}^{\gamma} \boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi_{j,0}^2(m) \sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right]^2\\
&\le \mathrm{E}\left[\left(\sum_{m=\alpha+1}^{\gamma} \boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi_{j,0}^2(m)\right)^2\right] \mathrm{E}\left[\left(\sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right)^2\right]
\end{aligned}$$

Let's analyse the second factor which "generalizes" the first one.

$$\begin{aligned}
\mathrm{E}\left[\left(\sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right)^2\right] &= \mathrm{Var}\left[\sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right] + \Sigma_F^2\\
&= \sum_{m'=\alpha+1}^{\gamma} \mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right]\psi_{j,0}^4(m'-i)\\
&= \mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right] \sum_{m'=\alpha+1}^{\gamma} \psi_{j,0}^4(m'-i)
\end{aligned}$$

The last sum can further be decomposed,

$$\begin{aligned}
\sum_{m'=\alpha+1}^{\gamma}\left[\psi_{j,0}^2(m'-i)\right]^2 &= \left(\sum_{m'=\alpha+1}^{\gamma}\psi_{j,0}^2(m'-i)\right)^2 - \sum_{p=\alpha+1}^{\gamma}\sum_{\substack{q=\alpha+1\\ q\neq p}}^{\gamma}\psi_{j,0}^2(p-i)\psi_{j,0}^2(q-i)\\
&\le \left(\sum_{m'=\alpha+1}^{\gamma}\psi_{j,0}^2(m'-i)\right)^2\\
&= \begin{cases}\left(\sum_{m'=i}^{N_j-1}\psi_{j,0}^2(m'-i)\right)^2 & \text{if } i \in [0, N_j-1]\\ \left(\sum_{m'=N_j}^{i}\psi_{j,0}^2(m'-i)\right)^2 & \text{if } i \notin [0, N_j-1]\end{cases}\\
&= \begin{cases}\left(\sum_{m'=0}^{N_j-1-i}\psi_{j,0}^2(m')\right)^2 & \text{if } i \in [0, N_j-1]\\ \left(\sum_{m'=N_j-i}^{0}\psi_{j,0}^2(m')\right)^2 & \text{if } i \notin [0, N_j-1]\end{cases}\\
&\begin{cases}\le 1 & \text{if } i \in [0, N_j-1]\\ = 0 & \text{if } i \notin [0, N_j-1]\end{cases}
\end{aligned}$$

since $\psi_{j,0}^2(t)$ is supported on $t \in [0, N_j-1]$.
Therefore, the second factor becomes,

$$\mathrm{E}\left[\left(\sum_{m'=\alpha+1}^{\gamma} \boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi_{j,0}^2(m'-i)\right)^2\right]\begin{cases}\le \mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right] & \text{if } i \in [0, N_j-1]\\ = 0 & \text{if } i \notin [0, N_j-1]\end{cases}$$

Finally we obtain,

$$\mathrm{E}\left[B_1B_2\right]^2 \begin{cases} \leq \mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right]^2 \sum_{m'=i}^{N_j-1}\psi^4_{j,0}(m') \sum_{m'=i}^{N_j-1}\psi^4_{j,0}(m'-i) & \text{if } i \in [0,N_j-1] \\ = 0 & \text{if } i \notin [0,N_j-1] \end{cases}$$

$$\begin{cases} \leq \mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right]^2 \sum_{m'=i}^{N_j-1}\psi^4_{j,0}(m') \sum_{m'=0}^{N_j-1-i}\psi^4_{j,0}(m') & \text{if } i \in [0,N_j-1] \\ = 0 & \text{if } i \notin [0,N_j-1] \end{cases}$$

, which is a decreasing function of i.
Equipped with those simplification we can analyse the decomposition of the covariance of $Y_{j,k+s}$,

$$\begin{aligned}
\mathrm{Cov}\left[Y_{j,k+s},Y_{j,k+s+i}\right] &= \mathrm{E}\left[\sum_{m=0}^{N_j-1}\boldsymbol{F}_{m+k+s}\boldsymbol{F}'_{m+k+s}\psi^2_{j,0}(m)\sum_{m'=0}^{N_j-1}\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\psi^2_{j,0}(m'-i)\right] - (\Sigma^F_{j,k})^2 \\
&= \mathrm{E}\left[A\right]\mathrm{E}\left[B_2\right] + \mathrm{E}\left[A\right]\mathrm{E}\left[A\right] + \mathrm{E}\left[B_1B_2\right] + \mathrm{E}\left[B_2\right]\mathrm{E}\left[C\right] \\
&= (\Sigma^F_{j,k})^2\left(ab_2 + ac + \mathrm{E}\left[B_1B_2\right] + b_1c - 1\right) \\
&= (\Sigma^F_{j,k})^2\left(a(\underbrace{b_2+c}_{=1}) + \mathrm{E}\left[B_1B_2\right] + b_1c - 1\right) \qquad \text{by def. of wavelets} \\
&= (\Sigma^F_{j,k})^2\left(\sum_{m=0}^{\alpha}\psi^2_{j,0}(m) + \mathrm{E}\left[B_1B_2\right] + \sum_{m=\alpha+1}^{\gamma}\psi^2_{j,0}(m)\sum_{m'=\gamma+1}^{i+N_j-1}\psi^2_{j,0}(m'-i) - 1\right) \\
&= (\Sigma^F_{j,k})^2\left(\sum_{m=0}^{\alpha}\psi^2_{j,0}(m) + \mathrm{E}\left[B_1B_2\right] + \left(1-\sum_{m=0}^{\alpha}\psi^2_{j,0}(m)\right)\left(1-\sum_{m'=0}^{N_j-1-i}\psi^2_{j,0}(m')\right) - 1\right) \\
&= (\Sigma^F_{j,k})^2\left(\sum_{m=0}^{\alpha}\psi^2_{j,0}(m) + \mathrm{E}\left[B_1B_2\right] - \sum_{m'=0}^{N_j-1-i}\psi^2_{j,0}(m') - \sum_{m=0}^{\alpha}\psi^2_{j,0}(m) + \sum_{m=0}^{\alpha}\psi^2_{j,0}(m)\sum_{m'=0}^{N_j-1-i}\psi^2_{j,0}(m')\right) \\
&= (\Sigma^F_{j,k})^2\left(\mathrm{E}\left[B_1B_2\right] - \sum_{m'=0}^{N_j-1-i}\psi^2_{j,0}(m') + \sum_{m=0}^{\alpha}\psi^2_{j,0}(m)\sum_{m'=0}^{N_j-1-i}\psi^2_{j,0}(m')\right) \\
&= (\Sigma^F_{j,k})^2\left(\mathrm{E}\left[B_1B_2\right] + \sum_{m'=0}^{N_j-1-i}\psi^2_{j,0}(m')\left(\sum_{m=0}^{\alpha}\psi^2_{j,0}(m) - 1\right)\right) \\
&= (\Sigma^F_{j,k})^2\left(\mathrm{E}\left[B_1B_2\right] - \sum_{m'=0}^{N_j-1-i}\psi^2_{j,0}(m')\sum_{m=\alpha+1}^{\gamma}\psi^2_{j,0}(m)\right)
\end{aligned}$$

By taking the absolute value and considering the above formula for lags $i \notin [0,N_j-1]$,

$$\left|\mathrm{Cov}\left[Y_{j,k+s},Y_{j,k+s+i}\right]\right| = \left|(\Sigma^F_{j,k})^2\right| \left|0 - 0\cdot 0\right| = 0$$

This last relation confirms the sufficient condition to apply the needed law of large numbers. However if $i \in [0,N_j-1]$ we get the upperbound,

$$\left|\mathrm{Cov}\left[Y_{j,k+s},Y_{j,k+s+i}\right]\right| \leq \left|(\Sigma^F_{j,k})^2\right| \left|\mathrm{Var}\left[\boldsymbol{F}_{m'+k+s}\boldsymbol{F}'_{m'+k+s}\right]^2 \sum_{m'=i}^{N_j-1}\psi^4_{j,0}(m')\sum_{m'=0}^{N_j-1-i}\psi^4_{j,0}(m') - \sum_{m'=0}^{N_j-1-i}\psi^2_{j,0}(m')\underbrace{\sum_{m=0}^{N_j-1}\psi^2_{j,0}(m)}_{=1}\right|$$

From this eigendecomposition of the CEWS estimator and (4.6) we get,

$$\left[K_{j,k} + \Theta_{j,k} + \Theta'_{j,k} + \Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k} = \widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}$$

Define,

$$Q_{j,k} = \left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}} \frac{\boldsymbol{\Lambda}'_{j,k}\widehat{\boldsymbol{\Lambda}}_{j,k}}{N} \tag{6.4}$$

$$P_{j,k} = \left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}} \frac{\boldsymbol{\Lambda}'_{j,k}\boldsymbol{\Lambda}_{j,k}}{N}\left[\widehat{\boldsymbol{\Sigma}}^F_{j,k}\right]^{\frac{1}{2}} \tag{6.5}$$

Therefore we can write,

$$\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'}{N}\left[K_{j,k}+\Theta_{j,k}+\Theta_{j,k}'+\Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}=\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'}{N}\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}$$
$$\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'}{N}\left[K_{j,k}+\Theta_{j,k}+\Theta_{j,k}'+\Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}=Q_{j,k}\widehat{\boldsymbol{V}}_{j,k} \tag{6.6}$$

Let's decompose the LHS of this relation and analyze its convergence. First take,

$$\begin{aligned}\left\|\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'}{N}\left[K_{j,k}+\Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}-P_{j,k}Q_{j,k}\right\| &\leq \left\|\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'}{N}\left[K_{j,k}+\Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}-\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'}{N}N^{-1}\boldsymbol{S}_j\left(k/T\right)\widehat{\boldsymbol{\Lambda}}_{j,k}\right\| \\ &+\left\|\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'}{N}N^{-1}\boldsymbol{S}_j\left(k/T\right)\widehat{\boldsymbol{\Lambda}}_{j,k}-\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'\boldsymbol{\Lambda}_{j,k}}{N}\widehat{\boldsymbol{\Sigma}}_{j,k}^F\frac{\boldsymbol{\Lambda}_{j,k}'\widehat{\boldsymbol{\Lambda}}_{j,k}}{N}\right\|\end{aligned}$$

The first term converges asymptotically to zero. To see this note that we can write :

$$\left\|\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'}{N}\left[K_{j,k}+\Upsilon_{j,k}-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right]\widehat{\boldsymbol{\Lambda}}_{j,k}\right\|\leq\left\|\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\right\|\left\|\frac{\boldsymbol{\Lambda}_{j,k}'}{\sqrt{N}}\right\|\left\|\left[K_{j,k}+\Upsilon_{j,k}-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right]\right\|\left\|\frac{\widehat{\boldsymbol{\Lambda}}_{j,k}}{\sqrt{N}}\right\|$$

All factors except the third one are bounded by assumption. The third term converges to zero since it is sufficient in L_2 space that its expectation along its variance converges to zero :

$$\begin{aligned}\mathrm{E}\left[K_{j,k}+\Upsilon_{j,k}-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right]&=\mathrm{E}\left[K_{j,k}+\Upsilon_{j,k}\right]-N^{-1}\\&=N^{-1}\boldsymbol{S}_j\left(k/T\right)-\mathrm{E}\left[\Upsilon_{j,k}\right]+\mathrm{E}\left[\Upsilon_{j,k}\right]-N^{-1}\boldsymbol{S}_j\left(k/T\right)+O(T^{-1}) && \text{from theorem 1}\\&=O(T^{-1})\to 0\\ \mathrm{Var}\left[K_{j,k}+\Upsilon_{j,k}-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right]&=\mathrm{Var}\left[K_{j,k}+\Upsilon_{j,k}\right]\to 0 && \text{from lemma 2}\end{aligned}$$

It is a bit more difficult to show the convergence of the second term.

$$\left\|\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'}{N}N^{-1}\boldsymbol{S}_j\left(k/T\right)\widehat{\boldsymbol{\Lambda}}_{j,k}-\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\frac{\boldsymbol{\Lambda}_{j,k}'\boldsymbol{\Lambda}_{j,k}}{N}\widehat{\boldsymbol{\Sigma}}_{j,k}^F\frac{\boldsymbol{\Lambda}_{j,k}'\widehat{\boldsymbol{\Lambda}}_{j,k}}{N}\right\|\leq\left\|\left[\widehat{\boldsymbol{\Sigma}}_{j,k}^F\right]^{\frac{1}{2}}\right\|\left\|\frac{\boldsymbol{\Lambda}_{j,k}'}{\sqrt{N}}\right\|\left\|N^{-1}\boldsymbol{S}_j\left(k/T\right)-\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\widehat{\boldsymbol{\Sigma}}_{j,k}^F\frac{\boldsymbol{\Lambda}_{j,k}'}{\sqrt{N}}\right\|\left\|\frac{\widehat{\boldsymbol{\Lambda}}_{j,k}}{\sqrt{N}}\right\|$$

All factors are bounded by assumption except the third one. The latter convergence to zero asymptotically. To see this, by (6.1),

$$\begin{aligned}\left\|N^{-1}\boldsymbol{S}_j\left(k/T\right)-\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\widehat{\boldsymbol{\Sigma}}_{j,k}^F\frac{\boldsymbol{\Lambda}_{j,k}'}{\sqrt{N}}\right\|&=\left\|\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\widehat{\boldsymbol{\Sigma}}_{j,k}^F\frac{\boldsymbol{\Lambda}_{j,k}'}{\sqrt{N}}-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right\|\\&=\left\|\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{m=0}^{T-1}\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\boldsymbol{F}_m\boldsymbol{F}_m'\frac{\boldsymbol{\Lambda}_{j,k}'}{\sqrt{N}}\psi_{j,k+s}^2(m)-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right\|\\&=\left\|\frac{1}{2M+1}\sum_{s=-M}^{M}\underbrace{\left[\sum_{m=0}^{T-1}\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\boldsymbol{F}_m\boldsymbol{F}_m'\frac{\boldsymbol{\Lambda}_{j,k}'}{\sqrt{N}}\psi_{j,k+s}^2(m)-N^{-1}\boldsymbol{S}_j\left(k/T\right)\right]}_{e_{j,k+s}}\right\|\end{aligned}$$

This reorganization allows us to use a law of large number to show the convergence. The expectation of $e_{j,k+s}$ is given by :

$$\begin{aligned}\mathrm{E}\left[e_{j,k+s}\right]&=\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\left[\sum_{m=0}^{T-1}\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}_m'\right]\psi_{j,k+s}^2(m)-\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}_m'\right]\right]\frac{\boldsymbol{\Lambda}_{j,k}'}{\sqrt{N}}-N^{-1}\mathrm{E}\left[\boldsymbol{\epsilon}_{j,k+s}\boldsymbol{\epsilon}_{j,k+s}'\right]\\&=\frac{\boldsymbol{\Lambda}_{j,k}}{\sqrt{N}}\left[\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}_m'\right]\sum_{m=0}^{T-1}\psi_{j,k+s}^2(m)-\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}_m'\right]\right]\frac{\boldsymbol{\Lambda}_{j,k}'}{\sqrt{N}}-N^{-1}\mathrm{E}\left[\boldsymbol{\epsilon}_{j,k+s}\boldsymbol{\epsilon}_{j,k+s}'\right]\end{aligned}$$

By definition of wavelets we know that $\sum_{m=0}^{T-1}\psi_{j,k+s}^2(m)\to 1$ when $M(T)\to\infty$. Finally since the covariance of the idiosyncratic has bounded eigenvalues by assumption on the factor model, the last term converges to zero when $N\to 0$.

7 Lemmas

Lemma 1. *Given the assumptions on the LSW and factor structure,*

$$\mathbf{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{F}'_{k'}\right]\mathbf{\Lambda}'_{j',k'} = \begin{cases} \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j,k}\right] & ,\text{if } j=j' \text{ and } k=k' \\ -\mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}\right] & ,\text{otherwise} \end{cases}$$

Proof. From the factor structure (3.1),

$$\boldsymbol{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\boldsymbol{W}_{j'}\left(k'/T\right) = \mathbf{\Lambda}_{j,k}\boldsymbol{F}_k\boldsymbol{F}_{k'}\mathbf{\Lambda}_{j',k'} + \mathbf{\Lambda}_{j,k}\boldsymbol{F}_k\boldsymbol{\epsilon}'_{j',k'} + \boldsymbol{\epsilon}_{j,k}\boldsymbol{F}_{k'}\mathbf{\Lambda}'_{j',k'} + \boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}$$

Taking expectation on both sides,

$$\boldsymbol{W}_j\left(k/T\right)\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\right]\boldsymbol{W}_{j'}\left(k'/T\right) = \mathbf{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{F}_{k'}\right]\mathbf{\Lambda}_{j',k'} + \mathbf{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{\epsilon}'_{j',k'}\right] + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{F}_{k'}\right]\mathbf{\Lambda}'_{j',k'} + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}\right]$$

The second and third term on the RHS is zero since $F_k \perp \boldsymbol{\epsilon}_{j',k'}, \forall j,k,k'$ (assumption (**??**)). This leaves us with

$$\boldsymbol{W}_j\left(k/T\right)\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\right]\boldsymbol{W}_{j'}\left(k'/T\right) = \mathbf{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{F}_{k'}\right]\mathbf{\Lambda}_{j',k'} + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}\right]$$

Finally, by the assumption (2.6) on the increments of the LSW representation,

$$\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\right] = \begin{cases} \boldsymbol{I}_N & ,\text{if } j=j' \text{ and } k=k' \\ \mathbf{0}_N & ,\text{otherwise} \end{cases}$$

,where $\boldsymbol{I}_N$ is the identity matrix of rank N and $\mathbf{0}_N$ is the null matrix of rank N. This along with the definition of the CEWS we obtain the desired result. □

Lemma 2. *Given the assumption on the LSW and factor structure and the set $\mathcal{S}$ defined in* (4.7)*, when $T \to \infty$ (rate of convergence ?)*

$$\mathrm{Var}\left[Ni\right] \to 0 \quad \forall i \in \mathcal{S}$$

Proof. When $i = K_{j,k}$,

$$\mathrm{Var}\left[NK_{j,k}\right] = \mathrm{Var}\left[\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T-1}\sum_{p,p'=-J}^{-1}\mathbf{\Lambda}_{p,m}\boldsymbol{F}_m\boldsymbol{F}'_{m'}\mathbf{\Lambda}'_{p',m'}\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')\right] \quad \text{by (4.3), (4}$$

$$\leq \mathrm{E}\left[\left(\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T-1}\sum_{p,p'=-J}^{-1}\mathbf{\Lambda}_{p,m}\boldsymbol{F}_m\boldsymbol{F}'_{m'}\mathbf{\Lambda}'_{p',m'}\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')\right)^2\right]$$

$$= \left(\frac{1}{2M+1}\sum_{l=-J}^{-1}\bar{A}_{j,l}\right)^2 E\left[\sum_{s_0,s_1=-M}^{M}\sum_{\substack{m_0,m'_0\\m_1,m'_1=0}}^{T}\sum_{\substack{mp_0,p'_0\\p_1,p'_1=-J}}^{-1}\mathbf{\Lambda}_{p_0,m_0}\boldsymbol{F}_{m_0}\boldsymbol{F}'_{m'_0}\mathbf{\Lambda}'_{p'_1,m'_1}\mathbf{\Lambda}_{p_1,m_1}\boldsymbol{F}_{m_1}\boldsymbol{F}'_{m'_1}\mathbf{\Lambda}'_{p'_1,m'_1}\right.$$

$$\left.\sum_{\substack{t_0,t'_0\\t_1,t'_1}}\psi_{p_0,m_0}(t_0)\psi_{p'_0,m'_0}(t'_0)\psi_{j,k+s_0}(t_0)\psi_{j,k+s_0}(t'_0)\psi_{p_1,m_1}(t_1)\psi_{p'_1,m'_1}(t'_1)\psi_{j,k+s_1}(t_1)\psi_{j,k+s_1}(t'_1)\right]$$

$$= \left(\frac{1}{2M+1}\sum_{l=-J}^{-1}\bar{A}_{j,l}\right)^2 E\left[\sum_{\substack{m_0,m'_0\\m_1,m'_1=0}}^{T}\sum_{\substack{mp_0,p'_0\\p_1,p'_1=-J}}^{-1}\mathbf{\Lambda}_{p_0,m_0}\boldsymbol{F}_{m_0}\boldsymbol{F}'_{m'_0}\mathbf{\Lambda}'_{p'_1,m'_1}\mathbf{\Lambda}_{p_1,m_1}\boldsymbol{F}_{m_1}\boldsymbol{F}'_{m'_1}\mathbf{\Lambda}'_{p'_1,m'_1}\right.$$

$$\left.\sum_{s_0=-M}^{M}\sum_{t_0}\psi_{p_0,m_0}(t_0)\psi_{j,k+s_0}(t_0)\sum_{t'_0}\psi_{p'_0,m'_0}(t'_0)\psi_{j,k+s_0}(t'_0)\sum_{s_1=-M}^{M}\sum_{t_1}\psi_{p_1,m_1}(t_1)\psi_{j,k+s_1}(t_1)\sum_{t'_1}\psi_{p'_1,m'_1}(t'_1)\psi_{j,k+s_1}(t'_1)\right]$$

$$= \frac{1}{(2M+1)^2}\left(\sum_{l=-J}^{-1}\bar{A}_{j,l}\right)^2 E\left[\sum_{\substack{m_0,m'_0\\m_1,m'_1=0}}^{T}\sum_{\substack{mp_0,p'_0\\p_1,p'_1=-J}}^{-1}\mathbf{\Lambda}_{p_0,m_0}\boldsymbol{F}_{m_0}\boldsymbol{F}'_{m'_0}\mathbf{\Lambda}'_{p'_1,m'_1}\mathbf{\Lambda}_{p_1,m_1}\boldsymbol{F}_{m_1}\boldsymbol{F}'_{m'_1}\mathbf{\Lambda}'_{p'_1,m'_1}\right.$$

$$\left.\sum_{s_0=-M}^{M}\mathbf{\Psi}_{p_0,j}(m_0-k-s_0)\mathbf{\Psi}_{p'_0,j}(m'_0-k-s_0)\sum_{s_1=-M}^{M}\mathbf{\Psi}_{p_1,j}(m_1-k-s_1)\mathbf{\Psi}_{p'_1,j}(m'_1-k-s_1)\right] \quad \text{by def. of CCWF.}$$

All sums are finite when $T \to \infty$ since the cross-correlation wavelet functions have bounded support (Proof). However we need the additional assumption that $\mathrm{E}\left[F_k^{(u)^4}\right] < \infty$. The whole expression tends to zero if $M(T) \to \infty$ when $T \to \infty$ (rate of convergence ?).

The proofs when $i = \mathcal{S} \setminus \{K_{j,k}\}$ are analoguous with the additional requirement that $\mathrm{E}\left[\epsilon_{j,k}^{(u)^4}\right] < \infty$. □

Theorem 1. *Given the assumptions on the LSW representation, the factor structure and the definitions (4.3), (4.5) and (4.4),*

$$\mathrm{E}\left[NK_{j,k}\right] = \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon jk\right]$$
$$\mathrm{E}\left[N\Theta_{j,k}\right] = 0$$

Proof. Let's turn out focus on the first expectation. From (4.3) and (4.1) and the linearity of expectation, the latter is then,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T-1}\sum_{p,p'=-J}^{-1}\boldsymbol{\Lambda}_{p,m}\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}'_{m'}\right]\boldsymbol{\Lambda}'_{p',m'}\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

Given the lemma 1,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m=0}^{T-1}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$
$$-\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T-1}\sum_{p,p'=-J}^{-1}\mathrm{E}\left[\boldsymbol{\epsilon}_{p,m}\boldsymbol{\epsilon}'_{p',m'}\right]\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

which is equivalent to

$$\begin{aligned}
\mathrm{E}\left[NK_{j,k}\right] &= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m=0}^{T-1}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t') - \mathrm{E}\left[N\Upsilon jk\right] \quad \text{by (4.4)}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{m=0}^{T-1}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\sum_{t=0}^{T-1}\psi_{p,m}(t)\psi_{j,k+s}(t)\sum_{t'=0}^{T-1}\psi_{p,m}(t')\psi_{j,k+s}(t') - \mathrm{E}\left[N\Upsilon jk\right]\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{m=0}^{T-1}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\left[\sum_{t=0}^{T-1}\psi_{p,m}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right]\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{n=0}^{T-1}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(n+k/T\right)\left[\sum_{t=0}^{T-1}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right] \quad \text{by change of variable : } m = n +\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{n=0}^{T-1}\sum_{p=-J}^{-1}\left[\boldsymbol{S}_p\left(k/T\right) + O(\frac{n}{T})\right]\left[\sum_{t=0}^{T-1}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right] \quad \text{by Lipschitz continuity}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(k/T\right)\sum_{n=0}^{T-1}\left[\sum_{t=0}^{T-1}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1})
\end{aligned}$$

Nason et al. (2000) proved that $\sum_{n=0}^{T-1}\left[\sum_{t=0}^{T-1}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 = A_{p,j}$. Consequently,

$$\begin{aligned}
\mathrm{E}\left[NK_{j,k}\right] &= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(k/T\right)A_{p,j} - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1})\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\mathrm{E}\left[\boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k}\right] - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1}) \quad \text{by the expectation of the raw periodogram.}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\mathrm{E}\left[\sum_{j=-J}^{-1}\bar{A}_{j,l}\boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k}\right] - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1}) \quad \text{by linearity of expectation}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1}) \quad \text{by expectation of the corrected periodogram.}\\
&= \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1})
\end{aligned}$$

Now, the second expectation is developed analoguously,

$$\mathrm{E}\left[N\Theta_{j,k}\right] = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T-1}\sum_{p,p'=-J}^{-1}\boldsymbol{\Lambda}_{p,m}\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{\epsilon}'_{p',m'}\right]\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

However, given that $\boldsymbol{F}_k \perp \boldsymbol{\epsilon}_{j',k'}, \forall j,k,k'$, the covariance matrix is null - i.e. $\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{\epsilon}'_{p',m'}\right] = \boldsymbol{0}_{(K\times N)}$. Finally, the compact support of the wavelets and the boundedness of all other terms (additional assumption on $\boldsymbol{\Lambda}_{j,k}$!) provides the desired result. $\square$

8 Goals

- $\|\hat{\mathbf{\Lambda}}_{j,k} - \mathbf{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\| = O(1)$
- $\|\hat{\boldsymbol{F}}_k - \boldsymbol{R}_{j,k}^{-1}\boldsymbol{F}_k\| = O(1)$