

1 CEWS Estimator under the factor model

In this section we will develop the *Cross-Evolutionary Wavelet Estimator* by using the factor structure and we will prove the convergence of this estimator under the structure. Recall the consistent and unbiased estimator of the CEWS :

$$\widehat{\mathbf{S}}_j(k/T) = \frac{1}{2M+1} \sum_{m=-M}^M \sum_{l=-J}^{-1} \bar{A}_{j,l} \mathbf{d}_{l,k+m} \mathbf{d}'_{l,k+m}$$

by multiplying both sides by N^{-1} and by using (??), (??) and (??) we successively obtain :

$$\begin{aligned} N^{-1} \widehat{\mathbf{S}}_j(k/T) &= N^{-1} \frac{1}{2M+1} \sum_{m=-M}^M \sum_{l=-J}^{-1} \bar{A}_{j,l} \sum_{t=0}^T \mathbf{X}_{t;T} \psi_{j,k+s}(t) \sum_{t'=0}^T \mathbf{X}'_{t';T} \psi_{j,k+s}(t') && \text{by (??)} \\ &= N^{-1} \frac{1}{2M+1} \sum_{m=-M}^M \sum_{l=-J}^{-1} \bar{A}_{j,l} \sum_{t=0}^T \sum_{t'=0}^T \sum_{m=0}^T \sum_{p=-J}^{-1} \mathbf{W}_p(m/T) \boldsymbol{\xi}_{p,m} \psi_{p,m}(t) \\ &\quad \sum_{m'=0}^T \sum_{p'=-J}^{-1} \boldsymbol{\xi}'_{p',m'} \mathbf{W}_{p'}(m'/T)' \psi_{p',m'}(t') \psi_{j,k+s}(t) \psi_{j,k+s}(t') && \text{by (??)} \\ &= N^{-1} \frac{1}{2M+1} \sum_{m=-M}^M \sum_{l=-J}^{-1} \bar{A}_{j,l} \sum_{t=0}^T \sum_{t'=0}^T \sum_{m=0}^T \sum_{p=-J}^{-1} (\boldsymbol{\Lambda}_{p,m} \mathbf{F}_m + \boldsymbol{\epsilon}_{p,m}) \psi_{p,m}(t) \\ &\quad \sum_{m'=0}^T \sum_{p'=-J}^{-1} (\boldsymbol{\Lambda}_{p,m} \mathbf{F}_m + \boldsymbol{\epsilon}_{p,m})' \psi_{p',m'}(t') \psi_{j,k+s}(t) \psi_{j,k+s}(t') && \text{by (??)} \end{aligned}$$

In order to ease readability, let's define two objects :

$$C_t = \sum_{m=0}^T \sum_{p=-J}^{-1} \boldsymbol{\Lambda}_{p,m} \mathbf{F}_m \psi_{p,m}(t) \quad (1.1)$$

$$E_t = \sum_{m=0}^T \sum_{p=-J}^{-1} \boldsymbol{\epsilon}_{p,m} \psi_{p,m}(t) \quad (1.2)$$

Consequently,

$$\begin{aligned} N^{-1} \widehat{\mathbf{S}}_j(k/T) &= N^{-1} \frac{1}{2M+1} \sum_{m=-M}^M \sum_{l=-J}^{-1} \bar{A}_{j,l} \sum_{t=0}^T \sum_{t'=0}^T [C_t + E_t][C_t + E_t]' \psi_{j,k+s}(t) \psi_{j,k+s}(t') \\ &= N^{-1} \frac{1}{2M+1} \sum_{m=-M}^M \sum_{l=-J}^{-1} \bar{A}_{j,l} \sum_{t=0}^T \sum_{t'=0}^T [C_t C_{t'}' + C_t E_{t'}' + E_t C_{t'}' + E_t E_{t'}'] \psi_{j,k+s}(t) \psi_{j,k+s}(t') \end{aligned}$$

To further simplify the notation we define the following,

$$K_{j,k} = N^{-1} \frac{1}{2M+1} \sum_{s=-M}^M \sum_{l=-J}^{-1} \bar{A}_{j,l} \sum_{t=0}^T \sum_{t'=0}^T C_t C_{t'}' \psi_{j,k+s}(t) \psi_{j,k+s}(t') \quad (1.3)$$

$$\Upsilon_{j,k} = N^{-1} \frac{1}{2M+1} \sum_{s=-M}^M \sum_{l=-J}^{-1} \bar{A}_{j,l} \sum_{t=0}^T \sum_{t'=0}^T E_t E_{t'}' \psi_{j,k+s}(t) \psi_{j,k+s}(t') \quad (1.4)$$

$$\Theta_{j,k} = N^{-1} \frac{1}{2M+1} \sum_{s=-M}^M \sum_{l=-J}^{-1} \bar{A}_{j,l} \sum_{t=0}^T \sum_{t'=0}^T C_t E_{t'}' \psi_{j,k+s}(t) \psi_{j,k+s}(t') \quad (1.5)$$

The estimator (??) can now be expressed as,

$$N^{-1} \widehat{\mathbf{S}}_j(k/T) = K_{j,k} + \Theta_{j,k} + \Theta'_{j,k} + \Upsilon_{j,k} \quad (1.6)$$

From Park et al. (2014) this estimator converge to the true Cross-Evolutionary Wavelet Spectrum. The next development assert the same convergence with the estimator redefined by the factor structure.

First, the expectation of the estimator can be decomposed thanks to (1.6),

$$\mathbb{E} \left[\widehat{\mathbf{S}}_j(k/T) \right] = \mathbb{E} [N K_{j,k}] + \mathbb{E} [N \Theta_{j,k}] + \mathbb{E} [N \Theta'_{j,k}] + \mathbb{E} [N \Upsilon_{j,k}]$$

Theorem ?? prove the asymptotic unbiasedness of the CEWS estimator.

$$\begin{aligned} \mathbb{E} \left[\widehat{\mathbf{S}}_j(k/T) \right] &= \mathbf{S}_j(k/T) - \mathbb{E} [N \Upsilon_{j,k}] + O(T^{-1}) + \mathbf{0}_N + \mathbf{0}_N + \mathbb{E} [N \Upsilon_{j,k}] \\ &= \mathbf{S}_j(k/T) + O(T^{-1}) \end{aligned}$$

Next, to analyse the variance of the estimator, we define $\mathcal{S} = \{K_{j,k}, \Theta_{j,k}, \Theta'_{j,k}, \Upsilon_{j,k}\}$ and we decompose the variance as follows :

$$\text{Var} \left[\widehat{\mathbf{S}}_j (k/T) \right] = \text{Var} [NK_{j,k}] + \text{Var} [N\Theta_{j,k}] + \text{Var} [N\Theta'_{j,k}] + \text{Var} [N\Upsilon_{j,k}] + \sum_{i \in \mathcal{S}} \sum_{\substack{j \in \mathcal{S} \\ j \neq i}} \text{Cov} [Ni, Nj] \quad (1.7)$$