Lemma 1. *Given the assumptions on the LSW and factor structure,*

$$\boldsymbol{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{F}'_{k'}\right]\boldsymbol{\Lambda}'_{j',k'} = \begin{cases} \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j,k}\right] & \text{,if } j = j' \text{ and } k = k' \\ -\mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}\right] & \text{,otherwise} \end{cases}$$

Proof. From the factor structure (**??**),

$$\boldsymbol{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\boldsymbol{W}_{j'}\left(k'/T\right) = \boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k\boldsymbol{F}_{k'}\boldsymbol{\Lambda}_{j',k'} + \boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k\boldsymbol{\epsilon}'_{j',k'} + \boldsymbol{\epsilon}_{j,k}\boldsymbol{F}_{k'}\boldsymbol{\Lambda}'_{j',k'} + \boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}$$

Taking expectation on both sides,

$$\boldsymbol{W}_j\left(k/T\right)\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\right]\boldsymbol{W}_{j'}\left(k'/T\right) = \boldsymbol{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{F}_{k'}\right]\boldsymbol{\Lambda}_{j',k'} + \boldsymbol{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{\epsilon}'_{j',k'}\right] + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{F}_{k'}\right]\boldsymbol{\Lambda}'_{j',k'} + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}\right]$$

The second and third term on the RHS is zero since $F_k \perp \boldsymbol{\epsilon}_{j',k'}, \forall j, k, k'$ (assumption (**??**)). This leaves us with

$$\boldsymbol{W}_j\left(k/T\right)\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\right]\boldsymbol{W}_{j'}\left(k'/T\right) = \boldsymbol{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{F}_{k'}\right]\boldsymbol{\Lambda}_{j',k'} + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}'_{j',k'}\right]$$

Finally, by the assumption (**??**) on the increments of the LSW representation,

$$\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}'_{j',k'}\right] = \begin{cases} \boldsymbol{I}_N & \text{,if } j = j' \text{ and } k = k' \\ \boldsymbol{0}_N & \text{,otherwise} \end{cases}$$

,where $\boldsymbol{I}_N$ is the identity matrix of rank N and $\boldsymbol{0}_N$ is the null matrix of rank N. This along with the definition of the CEWS we obtain the desired result. □