Lemma 1. *Given the assumptions on the LSW and factor structure,*

$$\boldsymbol{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{F}_{k'}'\right]\boldsymbol{\Lambda}_{j',k'}' = \begin{cases}\boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}_{j,k}'\right] & \textit{,if } j = j' \textit{ and } k = k' \\ -\mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}_{j',k'}'\right] & \textit{,otherwise}\end{cases}$$

Proof. From the factor structure (**??**),

$$\boldsymbol{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}_{j',k'}'\boldsymbol{W}_{j'}\left(k'/T\right) = \boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k\boldsymbol{F}_{k'}\boldsymbol{\Lambda}_{j',k'} + \boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k\boldsymbol{\epsilon}_{j',k'}' + \boldsymbol{\epsilon}_{j,k}\boldsymbol{F}_{k'}\boldsymbol{\Lambda}_{j',k'}' + \boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}_{j',k'}'$$

Taking expectation on both sides,

$$\boldsymbol{W}_j\left(k/T\right)\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}_{j',k'}'\right]\boldsymbol{W}_{j'}\left(k'/T\right) = \boldsymbol{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{F}_{k'}\right]\boldsymbol{\Lambda}_{j',k'} + \boldsymbol{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{\epsilon}_{j',k'}'\right] + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{F}_{k'}\right]\boldsymbol{\Lambda}_{j',k'}' + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}_{j',k'}'\right]$$

The second and third term on the RHS is zero since $F_k \perp \boldsymbol{\epsilon}_{j',k'}, \forall j,k,k'$ (assumption (**??**)). This leaves us with

$$\boldsymbol{W}_j\left(k/T\right)\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}_{j',k'}'\right]\boldsymbol{W}_{j'}\left(k'/T\right) = \boldsymbol{\Lambda}_{j,k}\mathrm{E}\left[\boldsymbol{F}_k\boldsymbol{F}_{k'}\right]\boldsymbol{\Lambda}_{j',k'} + \mathrm{E}\left[\boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}_{j',k'}'\right]$$

Finally, by the assumption (**??**) on the increments of the LSW representation,

$$\mathrm{E}\left[\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}_{j',k'}'\right] = \begin{cases}\boldsymbol{I}_N & \text{,if } j = j' \text{ and } k = k' \\ \boldsymbol{0}_N & \text{,otherwise}\end{cases}$$

,where $\boldsymbol{I}_N$ is the identity matrix of rank N and $\boldsymbol{0}_N$ is the null matrix of rank N. This along with the definition of the CEWS we obtain the desired result. □

Lemma 2. *Given the assumption on the LSW and factor structure and the set $\mathcal{S}$ defined in* (**??**)*, when $T \to \infty$ (rate of convergence ?)*

$$\mathrm{Var}\left[Ni\right] \to 0 \quad \forall i \in \mathcal{S}$$

Proof. When $i = K_{j,k}$,

$$\begin{aligned}
\mathrm{Var}\left[NK_{j,k}\right] &= \mathrm{Var}\left[\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m\boldsymbol{F}_{m'}'\boldsymbol{\Lambda}_{p',m'}'\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')\right] \quad \text{by (??), (?} \\
&\leq \mathrm{E}\left[\left(\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\boldsymbol{\Lambda}_{p,m}\boldsymbol{F}_m\boldsymbol{F}_{m'}'\boldsymbol{\Lambda}_{p',m'}'\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')\right)^2\right] \\
&= \left(\frac{1}{2M+1}\sum_{l=-J}^{-1}\bar{A}_{j,l}\right)^2 E\left[\sum_{s_0,s_1=-M}^{M}\sum_{\substack{m_0,m_0'\\m_1,m_1'=0}}^{T}\sum_{\substack{mp_0,p_0'\\p_1,p_1'=-J}}^{-1}\boldsymbol{\Lambda}_{p_0,m_0}\boldsymbol{F}_{m_0}\boldsymbol{F}_{m_0'}'\boldsymbol{\Lambda}_{p_1',m_1'}'\boldsymbol{\Lambda}_{p_1,m_1}\boldsymbol{F}_{m_1}\boldsymbol{F}_{m_1'}'\boldsymbol{\Lambda}_{p_1',m_1'}'\right. \\
&\qquad \left.\sum_{\substack{t_0,t_0'\\t_1,t_1'}}\psi_{p_0,m_0}(t_0)\psi_{p_0',m_0'}(t_0')\psi_{j,k+s_0}(t_0)\psi_{j,k+s_0}(t_0')\psi_{p_1,m_1}(t_1)\psi_{p_1',m_1'}(t_1')\psi_{j,k+s_1}(t_1)\psi_{j,k+s_1}(t_1')\right] \\
&= \left(\frac{1}{2M+1}\sum_{l=-J}^{-1}\bar{A}_{j,l}\right)^2 E\left[\sum_{\substack{m_0,m_0'\\m_1,m_1'=0}}^{T}\sum_{\substack{mp_0,p_0'\\p_1,p_1'=-J}}^{-1}\boldsymbol{\Lambda}_{p_0,m_0}\boldsymbol{F}_{m_0}\boldsymbol{F}_{m_0'}'\boldsymbol{\Lambda}_{p_1',m_1'}'\boldsymbol{\Lambda}_{p_1,m_1}\boldsymbol{F}_{m_1}\boldsymbol{F}_{m_1'}'\boldsymbol{\Lambda}_{p_1',m_1'}'\right. \\
&\left.\sum_{s_0=-M}^{M}\sum_{t_0}\psi_{p_0,m_0}(t_0)\psi_{j,k+s_0}(t_0)\sum_{t_0'}\psi_{p_0',m_0'}(t_0')\psi_{j,k+s_0}(t_0')\sum_{s_1=-M}^{M}\sum_{t_1}\psi_{p_1,m_1}(t_1)\psi_{j,k+s_1}(t_1)\sum_{t_1'}\psi_{p_1',m_1'}(t_1')\psi_{j,k+s_1}(t_1')\right] \\
&= \frac{1}{(2M+1)^2}\left(\sum_{l=-J}^{-1}\bar{A}_{j,l}\right)^2 E\left[\sum_{\substack{m_0,m_0'\\m_1,m_1'=0}}^{T}\sum_{\substack{mp_0,p_0'\\p_1,p_1'=-J}}^{-1}\boldsymbol{\Lambda}_{p_0,m_0}\boldsymbol{F}_{m_0}\boldsymbol{F}_{m_0'}'\boldsymbol{\Lambda}_{p_1',m_1'}'\boldsymbol{\Lambda}_{p_1,m_1}\boldsymbol{F}_{m_1}\boldsymbol{F}_{m_1'}'\boldsymbol{\Lambda}_{p_1',m_1'}'\right. \\
&\left.\sum_{s_0=-M}^{M}\boldsymbol{\Psi}_{p_0,j}(m_0-k-s_0)\boldsymbol{\Psi}_{p_0',j}(m_0'-k-s_0)\sum_{s_1=-M}^{M}\boldsymbol{\Psi}_{p_1,j}(m_1-k-s_1)\boldsymbol{\Psi}_{p_1',j}(m_1'-k-s_1)\right] \quad \text{by def. of CCWF.}
\end{aligned}$$

All sums are finite when $T \to \infty$ since the cross-correlation wavelet functions have bounded support (Proof). However we need the additional assumption that $\mathrm{E}\left[{F_k^{(u)}}^4\right] < \infty$. The whole expression tends to zero if $M(T) \to \infty$ when $T \to \infty$ (rate of convergence ?).

The proofs when $i = \mathcal{S} \setminus \{K_{j,k}\}$ are analoguous with the additional requirement that $\mathrm{E}\left[{\epsilon_{j,k}^{(u)}}^4\right] < \infty$. □

Theorem 1. *Given the assumptions on the LSW representation, the factor structure and the definitions (??), (??) and (??),*

$$\mathrm{E}\left[NK_{j,k}\right] = \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon jk\right]$$
$$\mathrm{E}\left[N\Theta_{j,k}\right] = 0$$

Proof. Let's turn out focus on the first expectation. From (??) and (??) and the linearity of expectation, the latter is then,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\boldsymbol{\Lambda}_{p,m}\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{F}'_{m'}\right]\boldsymbol{\Lambda}'_{p',m'}\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

Given the lemma 1,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$
$$-\frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\mathrm{E}\left[\boldsymbol{\epsilon}_{p,m}\boldsymbol{\epsilon}'_{p',m'}\right]\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

which is equivalent to

$$\begin{aligned}
\mathrm{E}\left[NK_{j,k}\right] &= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\psi_{p,m}(t)\psi_{p,m}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t') - \mathrm{E}\left[N\Upsilon jk\right] \quad \text{by (??)}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\sum_{t=0}^{T}\psi_{p,m}(t)\psi_{j,k+s}(t)\sum_{t'=0}^{T}\psi_{p,m}(t')\psi_{j,k+s}(t') - \mathrm{E}\left[N\Upsilon jk\right]\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{m=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(m/T\right)\left[\sum_{t=0}^{T}\psi_{p,m}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right]\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{n=0}^{T}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(n+k/T\right)\left[\sum_{t=0}^{T}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right] \text{ by change of variable : } m = n +\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{n=0}^{T}\sum_{p=-J}^{-1}\left[\boldsymbol{S}_p\left(k/T\right)+O(\frac{n}{T})\right]\left[\sum_{t=0}^{T}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right] \text{ by Lipschitz continuity}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(k/T\right)\sum_{n=0}^{T}\left[\sum_{t=0}^{T}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1})
\end{aligned}$$

Nason et al. (2000) proved that $\sum_{n=0}^{T}\left[\sum_{t=0}^{T}\psi_{p,n+k}(t)\psi_{j,k+s}(t)\right]^2 = A_{p,j}$. Consequently,

$$\begin{aligned}
\mathrm{E}\left[NK_{j,k}\right] &= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{p=-J}^{-1}\boldsymbol{S}_p\left(k/T\right)A_{p,j} - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1})\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\mathrm{E}\left[\boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k}\right] - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1}) \text{ by the expectation of the raw periodogram.}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\mathrm{E}\left[\sum_{j=-J}^{-1}\bar{A}_{j,l}\boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k}\right] - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1}) \text{ by linearity of expectation}\\
&= \frac{1}{2M+1}\sum_{s=-M}^{M}\boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1}) \text{ by expectation of the corrected periodogram.}\\
&= \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon jk\right] + O(T^{-1})
\end{aligned}$$

Now, the second expectation is developed analoguously,

$$\mathrm{E}\left[NK_{j,k}\right] = \frac{1}{2M+1}\sum_{s=-M}^{M}\sum_{j=-J}^{-1}\bar{A}_{j,l}\sum_{t,t',m,m'=0}^{T}\sum_{p,p'=-J}^{-1}\boldsymbol{\Lambda}_{p,m}\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{\epsilon}'_{p',m'}\right]\psi_{p,m}(t)\psi_{p',m'}(t')\psi_{j,k+s}(t)\psi_{j,k+s}(t')$$

However, given that $\boldsymbol{F}_k \perp \boldsymbol{\epsilon}_{j',k'}, \forall j,k,k'$, the covariance matrix is null - i.e. $\mathrm{E}\left[\boldsymbol{F}_m\boldsymbol{\epsilon}'_{p',m'}\right] = \boldsymbol{0}_{(K\times N)}$. Finally, the compact support of the wavelets and the boundedness of all other terms (additional assumption on $\boldsymbol{\Lambda}_{j,k}$!) provides the desired result. ∎