1 Quantities

- $J \in \mathbb{Z}^+$ = number of scales decomposition
- $T = 2^J$ = number of time periods
- $N \in \mathbb{Z}^+$ = number of cross-section elements
- $K(\leq N)$ = number of common factors

2 Multivariate Locally Stationary Wavelet process (Park et al. (2014))

The vector $(N \times 1)$ of stochastic processes $\boldsymbol{X_{t;T}}$ follows the given decomposition :

$$\boldsymbol{X_{t;T}} = \sum_{j=-J}^{-1} \sum_{k=0}^{T} \boldsymbol{W}_j\left(k/T\right) \boldsymbol{\xi}_{j,k} \psi_{j,k}(t) \tag{2.1}$$

where

- $\boldsymbol{W_j(z)}$ is a lower-triangular $(N \times N)$ matrix.
  For each (m,n)-element,

$$W_j^{(m,n)}(z) \text{ is a Lipschitz continuous function on } z \in (0,1) \tag{2.2}$$

$$\sum_{j=-\infty}^{-1} \left| W_j^{(m,n)}(z) \right|^2 < \infty, \qquad \forall z \in (0,1) \qquad \text{(finite energy)} \tag{2.3}$$

$$\sum_{j=-\infty}^{-1} 2^{-j} L_j^{(m,n)} < \infty \qquad \text{(uniformly bounded Lipschitz constants } L_j\text{)} \tag{2.4}$$

- $\boldsymbol{\xi}_{j,k}$ is the vector $(N \times 1)$ of random orthonormal increments.

$$\mathrm{E}\left[\xi_{j,k}^{(u)}\right] = 0, \qquad \forall j, k, u \tag{2.5}$$

$$\mathrm{Cov}\left[\xi_{j,k}^{(u)}, \xi_{j',k'}^{(u')}\right] = \delta_{j,j'}\delta_{k,k'}\delta_{u,u'}, \qquad \forall j, j', k, k', u, u' \tag{2.6}$$

- $\psi_{j,k}(t) = \psi_{j,k-t}$ is a scalar representing a non-decimated wavelet.

We can define the *Cross-Evoluationary Wavelet Spectrum* $(N \times N)$ matrix : $\boldsymbol{S}_j(z) = \boldsymbol{W}_j(z)\boldsymbol{W}_j(z)'$. This gives us the ability to express the *local autocovariance* : $c^{(u,u')}(z,\tau) = \sum_{j=-\infty}^{-1} S_j^{(u,u')}(z)\boldsymbol{\Psi}_j(\tau)$ where $\boldsymbol{\Psi}_j(\tau) = \sum_k \psi_{j,k}(0)\psi_{j,k}(\tau)$, the *autocorrelation wavelet*. The latter also define the *inner product matrix of discrete autocorrelation wavelets* : $A_{jl} = \sum_\tau \boldsymbol{\Psi}_j(\tau)\boldsymbol{\Psi}_l(\tau)$, $A = \{A_{jl}\}_{j,l\in\mathbb{N}}$ and its inverse : $\bar{A} = A^{-1}$.

2.1 Estimation of MvLSW

- $\boldsymbol{d}_{j,k} = \sum_{t=0}^{T-1} \boldsymbol{X}_t \psi_{j,k}(t)$ (empirical wavelet coefficients)
- $\boldsymbol{I}_{j,k} = \boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k}$ (raw wavelet periodogram)
  - $\mathrm{E}\left[\boldsymbol{I}_{j,k}\right] = \sum_{l=-J}^{-1} A_{jl}\boldsymbol{S}_l\left(k/T\right) + O(T^{-1})$ (biaised estimator)
- $\bar{\boldsymbol{I}}_{j,k} = \sum_{l=-J}^{-1} \bar{A}_{j,l}\boldsymbol{I}_{l,k}$ (corrected periodogram) $\Longrightarrow$ (unbiased estimator)
- $\widetilde{\boldsymbol{I}}_{j,k} = \frac{1}{2M+1}\sum_{m=-M}^{M} \boldsymbol{I}_{j,k+m}$ (smooth periodogram) $\Longrightarrow$ (consistent estimator)
- $\widehat{\boldsymbol{S}}_j\left(k/T\right) = \sum_{l=-J}^{-1} \bar{A}_{jl}\widetilde{\boldsymbol{I}}_{l,k} = \frac{1}{2M+1}\sum_{m=-M}^{M} \bar{\boldsymbol{I}}_{j,k+m} = \frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1} \bar{A}_{j,l}\boldsymbol{I}_{l,k+m}$ (final estimator of CEWS)

2.2 Notes

- The dependence structure is entirely in $\boldsymbol{W}_j(z)$, not in $\boldsymbol{\xi}_{j,k}$.
- The lower-triangular form of $\boldsymbol{W}_j(z)$ allows us to use the Cholesky decomposition on $\boldsymbol{S}_j(z)$.

3 Factor Model

- The factor structure is imposed on the following :

$$\boldsymbol{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k} = \boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k + \boldsymbol{\epsilon}_{j,k} \tag{3.1}$$

, not only on $\boldsymbol{\xi}_{j,k}$ since they are assumed orthonormal.

- Assumptions :
  1. $\boldsymbol{F}_k \sim (\boldsymbol{0}, \boldsymbol{\Sigma}_F)$, where $\boldsymbol{\Sigma}_F$ is a diagonal positive definite $(K \times K)$ matrix.
  2. $\boldsymbol{F}_k \perp \boldsymbol{\epsilon}_{j,k'}$, $\forall j, k, k'$
  3. $\boldsymbol{\epsilon}_{j,k} \sim (\boldsymbol{0}, \boldsymbol{\Sigma}_\epsilon)$, where $\boldsymbol{\Sigma}_\epsilon$ has bounded eigenvalues. Note : make $\boldsymbol{\Sigma}_\epsilon$ dependent on time ? what about serial dependence ?

- We can then represent the CEWS with the factor structure :

$$\begin{aligned}
\operatorname{Var}\left[\boldsymbol{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k}\right] &= \operatorname{Var}\left[\boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k\right] + \operatorname{Var}\left[\boldsymbol{\epsilon}_{j,k}\right] \\
\boldsymbol{W}_j\left(k/T\right)\operatorname{Var}\left[\boldsymbol{\xi}_{j,k}\right]\boldsymbol{W}_j\left(k/T\right)' &= \boldsymbol{\Lambda}_{j,k}\operatorname{Var}\left[\boldsymbol{F}_k\right]\boldsymbol{\Lambda}_{j,k}' + \operatorname{Var}\left[\boldsymbol{\epsilon}_{j,k}\right] \\
\boldsymbol{W}_j\left(k/T\right)\boldsymbol{W}_j\left(k/T\right)' &= \boldsymbol{\Lambda}_{j,k}\boldsymbol{\Sigma}_F\boldsymbol{\Lambda}_{j,k}' + \boldsymbol{\Sigma}_\epsilon \qquad \text{from (2.6)} \\
\boldsymbol{S}_j\left(k/T\right) &= \boldsymbol{\Lambda}_{j,k}\boldsymbol{\Sigma}_F\boldsymbol{\Lambda}_{j,k}' + \boldsymbol{\Sigma}_\epsilon
\end{aligned}$$

An important quantity to analyse is :

$$\boldsymbol{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k}\boldsymbol{\xi}_{j',k'}'\boldsymbol{W}_{j'}\left(k'/T\right)' = \boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k\boldsymbol{F}_{k'}'\boldsymbol{\Lambda}_{j',k'}' + \boldsymbol{\epsilon}_{j,k}\boldsymbol{F}_{k'}'\boldsymbol{\Lambda}_{j',k'}' + \boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k\boldsymbol{\epsilon}_{j',k'}' + \boldsymbol{\epsilon}_{j,k}\boldsymbol{\epsilon}_{j',k'}'$$

Each (m,n)-element of the matrix on the RHS can be written as :

$$\sum_u\sum_{u'} W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)}\xi_{j',k'}^{(u')}W_{j'}^{(u',n)}\left(k'/T\right)\left(= \sum_u W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)}\sum_{u'} W_{j'}^{(u',n)}\left(k'/T\right)\xi_{j',k'}^{(u')}\right)$$

[1].

The expectation of each term is therefore,

$$\begin{aligned}
\operatorname{E}\left[W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)}\xi_{j',k'}^{(u')}W_{j'}^{(u',n)}\left(k'/T\right)\right] &= W_j^{(m,u)}\left(k/T\right)\operatorname{E}\left[\xi_{j,k}^{(u)}\xi_{j',k'}^{(u')}\right]W_{j'}^{(u',n)}\left(k'/T\right) \\
&= \begin{cases} W_j^{(m,u)}\left(k/T\right)W_j^{(u,n)}\left(k/T\right) & \text{if } j=j', k=k', u=u' \\ 0 & \text{otherwise} \end{cases} \qquad \text{form (2.6)}
\end{aligned}$$

To apply a WLLN we need finite variances on each term :

$$\operatorname{Var}\left[W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)}\xi_{j',k'}^{(u')}W_{j'}^{(u',n)}\left(k'/T\right)\right] = \left(W_j^{(m,u)}\left(k/T\right)\right)^2\operatorname{Var}\left[\xi_{j,k}^{(u)}\xi_{j',k'}^{(u')}\right]\left(W_{j'}^{(u',n)}\left(k'/T\right)\right)^2$$

$$\operatorname{Var}\left[\xi_{j,k}^{(u)}\xi_{j',k'}^{(u')}\right] = \begin{cases} \operatorname{E}\left[\left(\xi_{j,k}^{(u)}\xi_{j',k'}^{(u')}\right)^2\right] = \begin{cases} \operatorname{E}\left[(\xi_{j,k}^{(u)})^2\right]\operatorname{E}\left[(\xi_{j',k'}^{(u')})^2\right] = 1 & \text{if } \xi_{j,k}^{(u)} \text{ is Gaussian and from(2.6)} \\ \leq \operatorname{E}\left[(\xi_{j,k}^{(u)})^4\right]^{1/2}\operatorname{E}\left[(\xi_{j',k'}^{(u')})^4\right]^{1/2} & \text{otherwise and from Cauchy.} \end{cases} & \text{if } j \neq j', k \neq k', u \neq u' \\ \operatorname{Var}\left[(\xi_{j,k}^{(u)})^2\right] \leq \operatorname{E}\left[(\xi_{j,k}^{(u)})^4\right] & \text{otherwise} \end{cases}$$

which suggests that we need finite fourth moment for the increment of the LSW process in order to have convergence in probability.
If $\operatorname{E}\left[(\xi_{j,k}^{(u)})^4\right] < \infty$, convergence ? But double sum....

Each (m,n)-element of the matrix on the RHS can be written as :

$$\sum_u\sum_{u'} W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)}\xi_{j',k'}^{(u')}W_{j'}^{(u',n)}\left(k'/T\right) = \sum_u W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)}\sum_{u'} W_{j'}^{(u',n)}\left(k'/T\right)\xi_{j',k'}^{(u')}$$

[1]Can analyse this term as $\sum_u W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)}\sum_{u'} W_{j'}^{(u',n)}\left(k'/T\right)\xi_{j',k'}^{(u')}$ and then use Slutsky ?

We can analyse the convergence of the two sum independently and then apply Slutsky. The expectation of each term in the first sum is therefore,

$$\mathrm{E}\left[W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)}\right] = W_j^{(m,u)}\left(k/T\right)\mathrm{E}\left[\xi_{j,k}^{(u)}\right] = 0$$

To apply a WLLN we need finite variances on each term :

$$\begin{aligned}\mathrm{Var}\left[W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)}\right] &= \left(W_j^{(m,u)}\left(k/T\right)\right)^2\mathrm{Var}\left[\xi_{j,k}^{(u)}\right]\\ &= \left(W_j^{(m,u)}\left(k/T\right)\right)^2 < \infty & \text{from (2.6) and (??)}\end{aligned}$$

Then,

$$\frac{1}{N}\sum_{u=1}^{N} W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)} \xrightarrow{p} 0 \quad \text{when } N \longrightarrow \infty$$

Finally by Slutsky,

$$\frac{1}{N}\sum_{u} W_j^{(m,u)}\left(k/T\right)\xi_{j,k}^{(u)}\frac{1}{N}\sum_{u'} W_{j'}^{(u',n)}\left(k'/T\right)\xi_{j',k'}^{(u')} \xrightarrow{p} 0$$

There is a problem... This result would mean that the factor structure is imposed on a null matrix asymptotically.

3.1 Estimation

The estimation of the loadings and common factors is carried out by a non-linear least square procedure in the wavelet domain.

$$\begin{aligned}\min_{\bar{\mathbf{\Lambda}}_{j,k},\bar{\boldsymbol{F}}_k} \quad & (NT)^{-1}\sum_t\left[\boldsymbol{X_{t;T}} - \sum_{j=-J}^{-1}\sum_{k=0}^{T}\left(\bar{\mathbf{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k\right)\psi_{j,k}(t)\right]'\left[\boldsymbol{X_{t;T}} - \sum_{j=-J}^{-1}\sum_{k=0}^{T}\left(\bar{\mathbf{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k\right)\psi_{j,k}(t)\right]\\ \text{s.t.} \quad & \frac{\bar{\mathbf{\Lambda}}_{j,k}'\bar{\mathbf{\Lambda}}_{j,k}}{N} = \boldsymbol{I}_K \end{aligned} \tag{3.2}$$

After distributing the objective function becomes,

$$(NT)^{-1}\sum_t\left[\boldsymbol{X}_{t;T}'\boldsymbol{X_{t;T}} - \boldsymbol{X}_{t;T}'\sum_{j=-J}^{-1}\sum_{k=0}^{T}\bar{\mathbf{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k\psi_{j,k}(t) - \sum_{j=-J}^{-1}\sum_{k=0}^{T}\psi_{j,k}(t)\bar{\boldsymbol{F}}_k'\bar{\mathbf{\Lambda}}_{j,k}'\boldsymbol{X_{t;T}} + \sum_{j=-J}^{-1}\sum_{k=0}^{T}\sum_{l=-J}^{-1}\sum_{m=0}^{T}\psi_{j,k}(t)\psi_{l,m}(t)\bar{\boldsymbol{F}}_k'\bar{\mathbf{\Lambda}}_{j,k}'\bar{\mathbf{\Lambda}}_{l,m}\bar{\boldsymbol{F}}_m\right]$$

$$(NT)^{-1}\left[\sum_t\boldsymbol{X}_{t;T}'\boldsymbol{X_{t;T}} - \sum_{j=-J}^{-1}\sum_{k=0}^{T}\sum_t\boldsymbol{X}_{t;T}'\psi_{j,k}(t)\bar{\mathbf{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k - \sum_{j=-J}^{-1}\sum_{k=0}^{T}\bar{\boldsymbol{F}}_k'\bar{\mathbf{\Lambda}}_{j,k}'\sum_t\boldsymbol{X_{t;T}}\psi_{j,k}(t) + \sum_t\sum_{j=-J}^{-1}\sum_{k=0}^{T}\sum_{l=-J}^{-1}\sum_{m=0}^{T}\psi_{j,k}(t)\psi_{l,m}(t)\bar{\boldsymbol{F}}_k'\bar{\mathbf{\Lambda}}_{j,k}'\bar{\mathbf{\Lambda}}_{l,m}\bar{\boldsymbol{F}}_m\right]$$

By definition of the empirical wavelet coefficents,

$$(NT)^{-1}\left[\sum_t\boldsymbol{X}_{t;T}'\boldsymbol{X_{t;T}} - \sum_{j=-J}^{-1}\sum_{k=0}^{T}\boldsymbol{d}_{j,k}'\bar{\mathbf{\Lambda}}_{j,k}\bar{\boldsymbol{F}}_k - \sum_{j=-J}^{-1}\sum_{k=0}^{T}\bar{\boldsymbol{F}}_k'\bar{\mathbf{\Lambda}}_{j,k}'\boldsymbol{d}_{j,k} + \sum_t\sum_{j=-J}^{-1}\sum_{k=0}^{T}\sum_{l=-J}^{-1}\sum_{m=0}^{T}\psi_{j,k}(t)\psi_{l,m}(t)\bar{\boldsymbol{F}}_k'\bar{\mathbf{\Lambda}}_{j,k}'\bar{\mathbf{\Lambda}}_{l,m}\bar{\boldsymbol{F}}_m\right]$$

The First Order Condition with respect to $\bar{\boldsymbol{F}}_k$ are : not correct

$$\bar{\boldsymbol{F}}_k'\sum_{j=-J}^{-1}\sum_{l=-J}^{-1}\sum_{m}\bar{\mathbf{\Lambda}}_{j,k}'\bar{\mathbf{\Lambda}}_{l,m}\boldsymbol{F}_m\sum_t\psi_{j,k}(t)\psi_{l,m}(t) = \sum_{j=-J}^{-1}\boldsymbol{d}_{j,k}'\bar{\mathbf{\Lambda}}_{j,k} \quad ,\forall k$$

The problem is that we cannot inverse $\sum_t\psi_{j,k}(t)\psi_{l,m}(t)$ since it could be null and that we define the common factor in term of itself. Solution : show that $\bar{\boldsymbol{F}}_k'\bar{\mathbf{\Lambda}}_{j,k}'\bar{\mathbf{\Lambda}}_{l,m}\bar{\boldsymbol{F}}_m$ is (asymptotically) zero when indices differ. With the constraint (3.2) and taking the transpose,

$$\cancel{\bar{\boldsymbol{F}}_k\sum_{j=-J}^{-1}\sum_{l=-J}^{-1}\sum_t\psi_{j,k}(t)\psi_{l,m}(t) = (N)^{-1}\sum_{j=-J}^{-1}\bar{\mathbf{\Lambda}}_{j,k}'\boldsymbol{d}_{j,k} \quad ,\forall k} \tag{3.3}$$

Replace (3.3) in the original minimization problem,

$$\min_{\bar{\mathbf{\Lambda}}_{j,k}} \quad (NT)^{-1}$$

4 Goals

- $\|\hat{\mathbf{\Lambda}}_{j,k} - \mathbf{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\| = O(1)$
- $\|\hat{\boldsymbol{F}}_k - \boldsymbol{R}_{j,k}^{-1}\boldsymbol{F}_k\| = O(1)$