1 Quantities

- $J \in \mathbb{Z}^+$ = number of scales decomposition
- $T = 2^J$ = number of time periods
- $N \in \mathbb{Z}^+$ = number of cross-section elements
- $K(\leq N)$ = number of common factors

2 Multivariate Locally Stationary Wavelet process (Park et al. (2014))

The vector $(N \times 1)$ of stochastic processes $\boldsymbol{X_{t;T}}$ follows the given decomposition :

$$\boldsymbol{X_{t;T}} = \sum_{j=-J}^{-1} \sum_{k=0}^{T} \boldsymbol{W}_j\,(k/T)\,\boldsymbol{\xi}_{j,k}\psi_{j,k}(t) \tag{2.1}$$

where

- $\boldsymbol{W_j(z)}$ is a lower-triangular $(N \times N)$ matrix.
  For each (m,n)-element,

$$W_j^{(m,n)}(z) \text{ is a Lipschitz continuous function on } z \in (0,1) \tag{2.2}$$

$$\sum_{j=-\infty}^{-1} \left| W_j^{(m,n)}(z) \right|^2 < \infty, \qquad \forall z \in (0,1) \qquad \text{(finite energy)} \tag{2.3}$$

$$\sum_{j=-\infty}^{-1} 2^{-j} L_j^{(m,n)} < \infty \qquad \text{(uniformly bounded Lipschitz constants } L_j) \tag{2.4}$$

- $\boldsymbol{\xi}_{j,k}$ is the vector $(N \times 1)$ of random orthonormal increments.

$$\mathrm{E}\left[\xi_{j,k}^{(u)}\right] = 0, \qquad \forall j,k,u \tag{2.5}$$

$$\mathrm{Cov}\left[\xi_{j,k}^{(u)}, \xi_{j',k'}^{(u')}\right] = \delta_{j,j'}\delta_{k,k'}\delta_{u,u'}, \qquad \forall j,j',k,k',u,u' \tag{2.6}$$

- $\psi_{j,k}(t) = \psi_{j,k-t}$ is a scalar representing a non-decimated wavelet.

We can define the *Cross-Evoluationary Wavelet Spectrum* $(N \times N)$ matrix : $\boldsymbol{S}_j(z) = \boldsymbol{W}_j(z)\boldsymbol{W}_j(z)'$. This gives us the ability to express the *local autocovariance* : $c^{(u,u')}(z,\tau) = \sum_{j=-\infty}^{-1} S_j^{(u,u')}(z)\boldsymbol{\Psi}_j(\tau)$ where $\boldsymbol{\Psi}_j(\tau) = \sum_k \psi_{j,k}(0)\psi_{j,k}(\tau)$, the *autocorrelation wavelet*. The latter also define the *inner product matrix of discrete autocorrelation wavelets* : $A_{jl} = \sum_\tau \boldsymbol{\Psi}_j(\tau)\boldsymbol{\Psi}_l(\tau)$, $A = \{A_{jl}\}_{j,l\in\mathbb{N}}$ and its inverse : $\bar{A} = A^{-1}$.A rather simple extension of the autocorrelation wavelet is the *cross-correlation wavelet* which charaterizes the dependence between two wavelets at different scales. The latter wavelet is thus defined as $\boldsymbol{\Psi}_{j,j'}(\tau) = \sum_k \psi_{j,k}(0)\psi_{j',k}(\tau)$.

Each (m,n)-element of the Cross-Evolutionary Wavelet Spectrum can be expressed as

$$S_j^{(m,n)}\,(k/T) = \sum_{u=1}^{N} W_j^{(m,u)}\,(k/T)\,W_j^{(u,n)}\,(k/T)\,, \quad \forall j,k$$

From this definition it is not difficult to extend the CEWS to take into account the dependence structure between different scales and throught time :

$$S_{j,j'}^{m,n}\,(k/T,k'/T) = \sum_{u=1}^{N} W_j^{(m,u)}\,(k/T)\,W_{j'}^{(u,n)}\,(k'/T)\,, \quad \forall j,j',k,k' \tag{2.7}$$

We make the following assumption regarding the latter object,

$$S_{j,j'}^{(m,n)}\,(k/T,k/T) = \begin{cases} S_j^{(m,n)}\,(k/T) & \text{if } j = j' \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

This assumption (possible improvement : condition similar to Chamberlain ?) imposes no dependence between different scales of decomposition. Notice that we don't restrict the serial dependence.

2.1 Estimation of MvLSW

- $\mathrm{E}\left[\boldsymbol{I}_{j,k}\right] = \sum_{l=-J}^{-1} A_{jl}\boldsymbol{S}_l\left(k/T\right) + O(T^{-1})$ (biaised estimator)

2.2 Estimation of MvLSW

$$\boldsymbol{d}_{j,k} = \sum_{t=0}^{T-1} \boldsymbol{X}_t\psi_{j,k}(t) \qquad \text{(empirical wavelet coefficients)} \qquad (2.9)$$

$$\boldsymbol{I}_{j,k} = \boldsymbol{d}_{j,k}\boldsymbol{d}'_{j,k} \qquad \text{(raw wavelet periodogram)} \qquad (2.10)$$

$$\bar{\boldsymbol{I}}_{j,k} = \sum_{l=-J}^{-1} \bar{A}_{j,l}\boldsymbol{I}_{l,k} \qquad \text{(corrected periodogram)} \qquad (2.11)$$

$$\widetilde{\boldsymbol{I}}_{j,k} = \frac{1}{2M+1}\sum_{m=-M}^{M} \boldsymbol{I}_{j,k+m} \qquad \text{(smooth periodogram)} \qquad (2.12)$$

$$\begin{aligned}
\widehat{\boldsymbol{S}}_j\left(k/T\right) &= \sum_{l=-J}^{-1} \bar{A}_{jl}\widetilde{\boldsymbol{I}}_{l,k} \\
&= \frac{1}{2M+1}\sum_{m=-M}^{M} \bar{\boldsymbol{I}}_{j,k+m} \\
&= \frac{1}{2M+1}\sum_{m=-M}^{M}\sum_{l=-J}^{-1} \bar{A}_{j,l}\boldsymbol{I}_{l,k+m} \qquad \text{(final estimator of CEWS)} \qquad (2.13)
\end{aligned}$$

2.3 Notes

- The dependence structure is entirely in $\boldsymbol{W}_j(z)$, not in $\boldsymbol{\xi}_{j,k}$.
- The lower-triangular form of $\boldsymbol{W}_j(z)$ allows us to use the Cholesky decomposition on $\boldsymbol{S}_j(z)$.

3 Factor Model

- The factor structure is imposed on the following :

$$\boldsymbol{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k} = \boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k + \boldsymbol{\epsilon}_{j,k} \qquad (3.1)$$

  , not only on $\boldsymbol{\xi}_{j,k}$ since they are assumed orthonormal.

- Assumptions :
  1. $\boldsymbol{F}_k \sim (\mathbf{0}, \boldsymbol{\Sigma}_F)$, where $\boldsymbol{\Sigma}_F$ is a diagonal positive definite $(K \times K)$ matrix.
  2. $\mathrm{E}\left[{F_k^{(u)}}^4\right] < \infty, \forall k, u$
  3. $\boldsymbol{F}_k \perp \boldsymbol{\epsilon}_{j,k'}, \forall j, k, k'$
  4. $\boldsymbol{\epsilon}_{j,k} \sim (\mathbf{0}, \boldsymbol{\Sigma}_\epsilon)$, where $\boldsymbol{\Sigma}_\epsilon$ has bounded eigenvalues. Note : make $\boldsymbol{\Sigma}_\epsilon$ dependent on time ? what about serial dependence ?
  5. $\boldsymbol{\Lambda}'_{j,k}\boldsymbol{\Lambda}_{l,m} = \mathbf{0}, \forall j \neq l, \forall k \neq m$.

- We can then represent the CEWS with the factor structure :

$$\begin{aligned}
\mathrm{Var}\left[\boldsymbol{W}_j\left(k/T\right)\boldsymbol{\xi}_{j,k}\right] &= \mathrm{Var}\left[\boldsymbol{\Lambda}_{j,k}\boldsymbol{F}_k\right] + \mathrm{Var}\left[\boldsymbol{\epsilon}_{j,k}\right] \\
\boldsymbol{W}_j\left(k/T\right)\mathrm{Var}\left[\boldsymbol{\xi}_{j,k}\right]\boldsymbol{W}_j\left(k/T\right)' &= \boldsymbol{\Lambda}_{j,k}\mathrm{Var}\left[\boldsymbol{F}_k\right]\boldsymbol{\Lambda}'_{j,k} + \mathrm{Var}\left[\boldsymbol{\epsilon}_{j,k}\right] \\
\boldsymbol{W}_j\left(k/T\right)\boldsymbol{W}_j\left(k/T\right)' &= \boldsymbol{\Lambda}_{j,k}\boldsymbol{\Sigma}_F\boldsymbol{\Lambda}'_{j,k} + \boldsymbol{\Sigma}_\epsilon \qquad \text{from (2.6)} \\
\boldsymbol{S}_j\left(k/T\right) &= \boldsymbol{\Lambda}_{j,k}\boldsymbol{\Sigma}_F\boldsymbol{\Lambda}'_{j,k} + \boldsymbol{\Sigma}_\epsilon
\end{aligned}$$