1 Rotation Matrix $\boldsymbol{R}_{j,k}$

In this section we find an expression for the rotation matrix $\boldsymbol{R}_{j,k}$ (Recall the inherent indeterminacy of factor models). Imposing particular identification restrictions on the loadings and factors will specify a unique rotation matrix. In our factor model, we don' t impose any identification restrictions. Consequently, the researcher apply our model has the responsability of stating relevant restriction given the context of interest.

The rotation matrix appears in the convergence of loadings and factors.

$$\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k} \tag{1.1}$$

$$\widehat{\boldsymbol{F}}_k - \boldsymbol{R}_{j,k}^{-1}\boldsymbol{F}_k \tag{1.2}$$

Take first (1.1). Given the formula for the estimator of the loadings - i.e. (**??**), we obtain :

$$\begin{aligned}
\widehat{\boldsymbol{\Lambda}}_{j,k} &= N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right)\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1} \\
\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k} &= N^{-1}\widehat{\boldsymbol{S}}_j\left(k/T\right)\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k} \\
&= \left[K_{j,k} + \Theta_{j,k} + \Theta_{j,k}' + \Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k} \qquad \text{by } (\mathbf{??})
\end{aligned}$$

The first restriction we can impose on the rotation matrix $\boldsymbol{R}_{j,k}$ is $\boldsymbol{R}_{j,k} = \alpha\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1}$ for some α, $(K \times N)$ matrix. Such that,

$$\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k} = \left[K_{j,k} - \boldsymbol{\Lambda}_{j,k}\alpha + \Theta_{j,k} + \Theta_{j,k}' + \Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1}$$

In order to have unbiaised and consistent estimation of the loadings, we require respectively,

$$\mathrm{E}\left[\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\right] = 0 \tag{1.3}$$

$$\mathrm{Var}\left[\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\right] \to 0 \tag{1.4}$$

In Hilbert space, the restriction (1.4) is equivalent to the **euclidiean norm** tending to zero,

$$\left\|\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\right\| \to 0 \tag{1.5}$$

Consequently,

$$\begin{aligned}
N^{-\frac{1}{2}}\left\|\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\right\| &= N^{-\frac{1}{2}}\left\|\left[K_{j,k} - \boldsymbol{\Lambda}_{j,k}\alpha + \Theta_{j,k} + \Theta_{j,k}' + \Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1}\right\| \\
&\le \left\|\left[K_{j,k} - \boldsymbol{\Lambda}_{j,k}\alpha + \Theta_{j,k} + \Theta_{j,k}' + \Upsilon_{j,k}\right]\right\| \left\|\frac{\widehat{\boldsymbol{\Lambda}}_{j,k}}{\sqrt{N}}\right\| \left\|\widehat{\boldsymbol{V}}_{j,k}^{-1}\right\| \\
&\le \left[\left\|K_{j,k} - \boldsymbol{\Lambda}_{j,k}\alpha + \Upsilon_{j,k}\right\| + 2\left\|\Theta_{j,k}\right\|\right] \left\|\frac{\widehat{\boldsymbol{\Lambda}}_{j,k}}{\sqrt{N}}\right\| \left\|\widehat{\boldsymbol{V}}_{j,k}^{-1}\right\| \\
N^{\frac{1}{2}}\left\|\widehat{\boldsymbol{\Lambda}}_{j,k} - \boldsymbol{\Lambda}_{j,k}\boldsymbol{R}_{j,k}\right\| &\le \left[\left\|NK_{j,k} - N\boldsymbol{\Lambda}_{j,k}\alpha + N\Upsilon_{j,k}\right\| + 2\left\|N\Theta_{j,k}\right\|\right] \left\|\frac{\widehat{\boldsymbol{\Lambda}}_{j,k}}{\sqrt{N}}\right\| \left\|\widehat{\boldsymbol{V}}_{j,k}^{-1}\right\|
\end{aligned}$$

First, we analyse the convergence of the first term on the RHS.
Its expectation is,

$$\begin{aligned}
\mathrm{E}\left[NK_{j,k} - N\boldsymbol{\Lambda}_{j,k}\alpha + N\Upsilon_{j,k}\right] &= \mathrm{E}\left[NK_{j,k}\right] - \mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] + \mathrm{E}\left[N\Upsilon_{j,k}\right] \\
&= \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\Upsilon_{j,k}\right] - \mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] + \mathrm{E}\left[N\Upsilon_{j,k}\right] \qquad \text{by theorem } (\mathbf{??}) \\
&= \boldsymbol{S}_j\left(k/T\right) - \mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right]
\end{aligned}$$

Therefore, given the unbiaisedness restriction (1.3) we obtain another constraint for the rotation matrix.

$$\mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] = \boldsymbol{S}_j\left(k/T\right) \tag{1.6}$$

Note that we also need $\mathrm{Var}\left[NK_{j,k} - N\boldsymbol{\Lambda}_{j,k}\alpha + N\Upsilon_{j,k}\right] \to 0$. (See next part)

$$\begin{aligned}
\mathrm{Var}\left[NK_{j,k} - N\boldsymbol{\Lambda}_{j,k}\alpha + N\Upsilon_{j,k}\right] &= \mathrm{Var}\left[NK_{j,k}\right] + \mathrm{Var}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] + \mathrm{Var}\left[N\Upsilon_{j,k}\right] \\
&\quad + 2\mathrm{Cov}\left[NK_{j,k}, N\boldsymbol{\Lambda}_{j,k}\alpha\right] + 2\mathrm{Cov}\left[NK_{j,k}, N\Upsilon_{j,k}\right] + 2\mathrm{Cov}\left[N\boldsymbol{\Lambda}_{j,k}\alpha, N\Upsilon_{j,k}\right]
\end{aligned}$$

By the result of **??**,

$$\mathrm{Var}\left[NK_{j,k} - N\boldsymbol{\Lambda}_{j,k}\alpha + N\Upsilon_{j,k}\right] = \mathrm{Var}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] + 2\mathrm{Cov}\left[NK_{j,k}, N\boldsymbol{\Lambda}_{j,k}\alpha\right] + 2\mathrm{Cov}\left[N\boldsymbol{\Lambda}_{j,k}\alpha, N\Upsilon_{j,k}\right] \tag{1.7}$$

1.1 Candidate Rotation matrix

This section develop a rotation matrix candidate.
Given the expectation restriction (1.6), and the fact that $\widehat{\boldsymbol{S}}_j\,(k/T)$ is an asymptotically unbiaised estimator of $\boldsymbol{S}_j\,(k/T)$,

$$\begin{aligned}
\mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] &= \mathrm{E}\left[\widehat{\boldsymbol{S}}_j\,(k/T)\right] \\
&= \mathrm{E}\left[NK_{j,k}\right] + \mathrm{E}\left[N\Theta_{j,k}\right] + \mathrm{E}\left[N\Theta_{j,k}\right] + \mathrm{E}\left[N\Upsilon_{j,k}\right] \quad \text{by } (\textbf{??}) \\
&= \mathrm{E}\left[NK_{j,k}\right] + 0 + 0 + \mathrm{E}\left[N\Upsilon_{j,k}\right] \quad \text{by theorem } \textbf{??}. \\
&= \mathrm{E}\left[NK_{j,k} + N\Upsilon_{j,k}\right] \\
&= \mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}K_{j,k} + N\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}\Upsilon_{j,k}\right] \\
&= \mathrm{E}\left[N\boldsymbol{\Lambda}_{j,k}\left(\boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}K_{j,k} + \boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}\Upsilon_{j,k}\right)\right]
\end{aligned}$$

Our candidate α is therefore $\alpha = \left(\boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}K_{j,k} + \boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}\Upsilon_{j,k}\right)$ (invertibility of matrix ?). By using the latter α we obtain the desired unbiaisedness on the loadings by construction. Do we also have the consistency of the estimator ? We answer this question by analysing the 3 terms in (1.7),

$$\begin{aligned}
\mathrm{Var}\left[N\boldsymbol{\Lambda}_{j,k}\alpha\right] &= \mathrm{Var}\left[NK_{j,k} + N\Upsilon_{j,k}\right] \\
&= \mathrm{Var}\left[NK_{j,k}\right] + \mathrm{Var}\left[N\Upsilon_{j,k}\right] + 2\mathrm{Cov}\left[NK_{j,k}, N\Upsilon_{j,k}\right] \\
&= 0 + 0 + 0 \quad \text{by lemma } \textbf{??}
\end{aligned}$$

The final rotation matrix is therefore given by,

$$\widehat{\boldsymbol{R}}_{j,k} = \boldsymbol{\Lambda}'_{j,k}(\boldsymbol{\Lambda}_{j,k}\boldsymbol{\Lambda}'_{j,k})^{-1}\left[K_{j,k} + \Upsilon_{j,k}\right]\widehat{\boldsymbol{\Lambda}}_{j,k}\widehat{\boldsymbol{V}}_{j,k}^{-1} \tag{1.8}$$