FRAMED POLYTOPES AND HIGHER CATEGORIES

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CELLULAR STRINGS ON FRAMED POLYTOPES

FRAMED POLYTOPES

Frame: ordered basis $(e_1,\dots,e_d)$

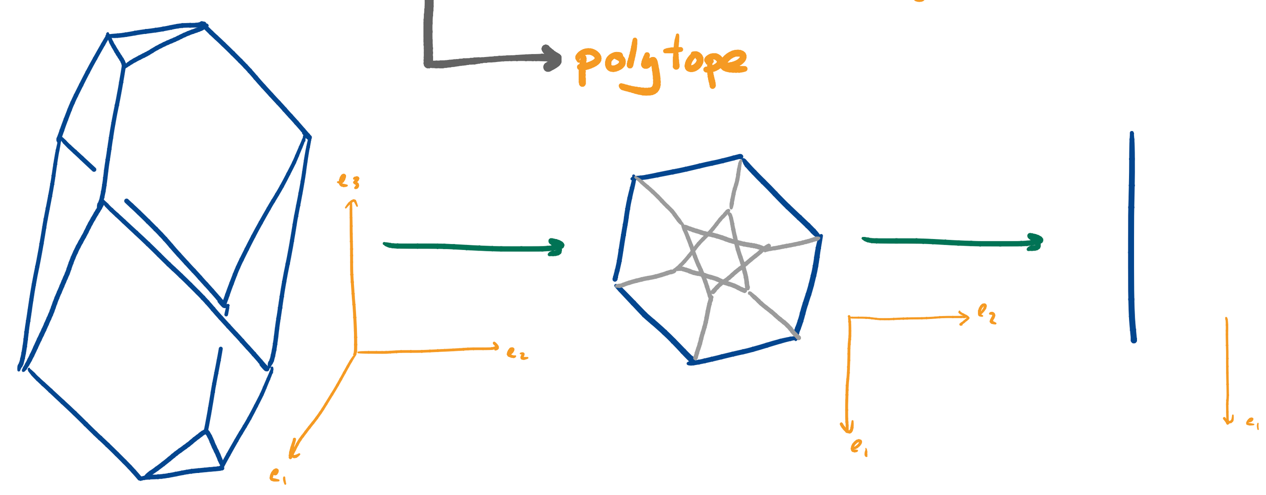
System of projections:

$$\mathbb{R}^d \xrightarrow{\pi^d_{d-1}} \mathbb{R}^{d-1} \xrightarrow{\pi^{d-1}_{d-2}} \mathbb{R}^{d-2} \longrightarrow \cdots \quad \mathbb{R}^2 \xrightarrow{\pi^2_1} \mathbb{R}$$

$\pi_k := \pi^{k+1}_k \circ \pi^{k+2}_{k+1} \circ \cdots \circ \pi^d_{d-1} : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ forgetting the last d-k coordinates

P-admissible: $F \cong \pi_k(F)$ $\forall$ k-face F of P "generic w.r.t. P"

Framed polytope: (P, F)
↳ P-admissible frame
↳ polytope

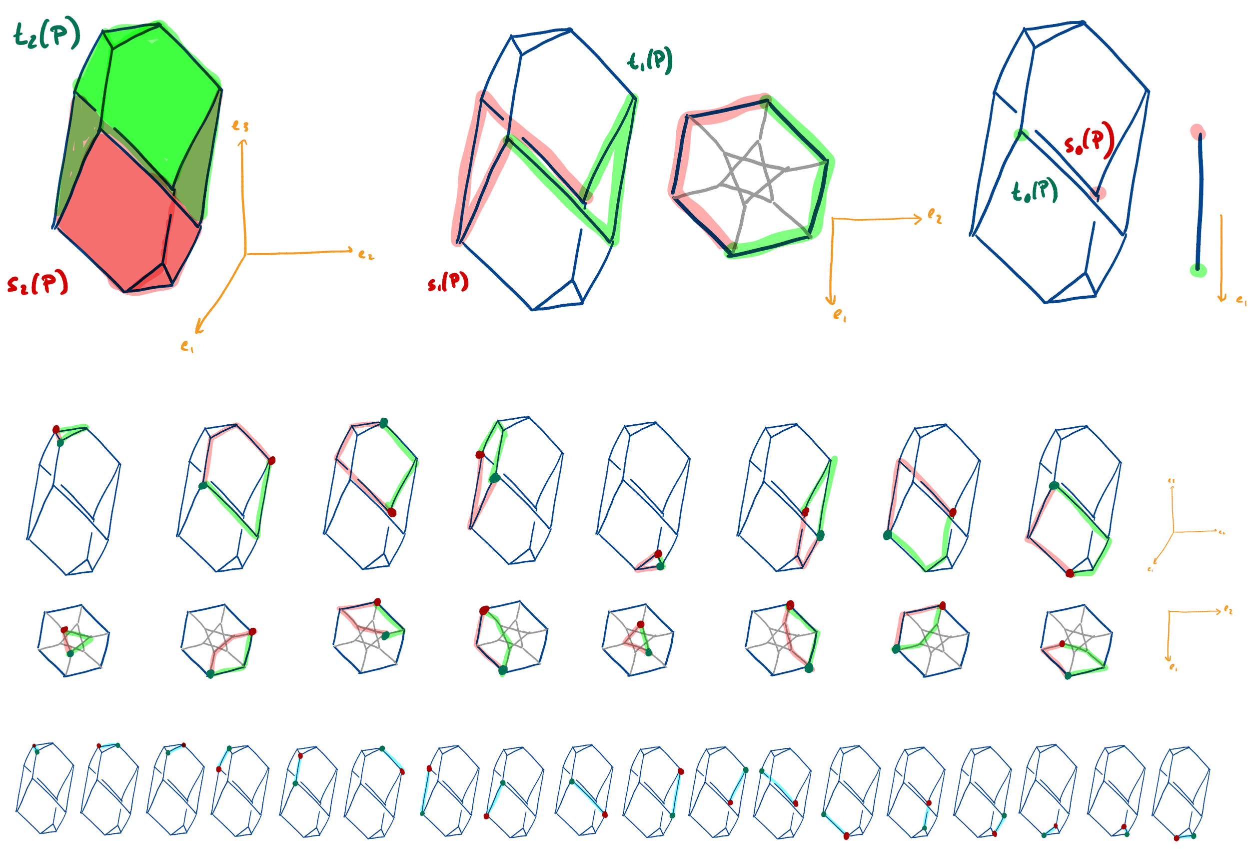


GLOBULAR STRUCTURES

$F \subseteq P$ i-face $k<i$

k-source: $s_k(F)$ = k-faces in lower envelope of $\pi_{k+1}(F)$

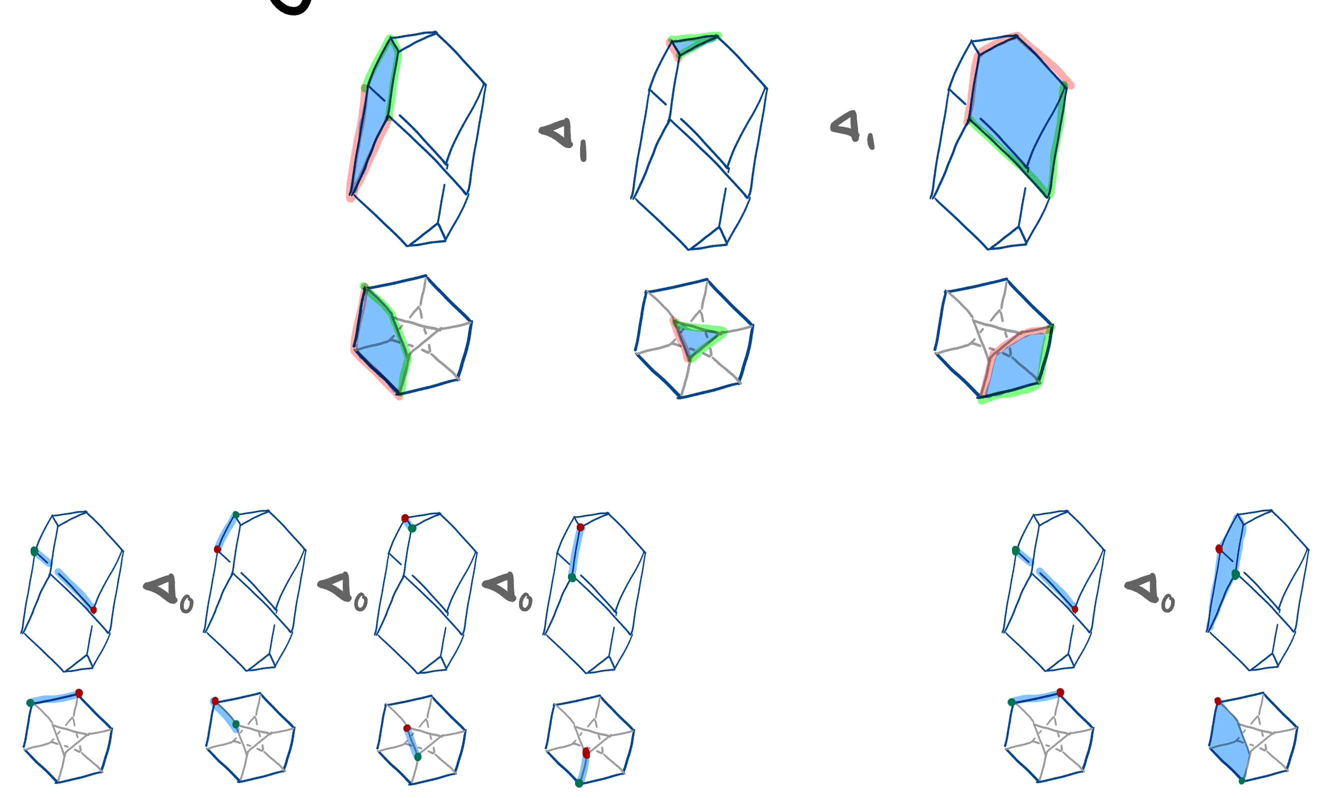
k-target: $t_k(F)$ = k-faces in upper envelope of $\pi_{k+1}(F)$



Strongly related to flag oriented matroids

CELLULAR STRINGS & LOOPS

Cellular k-string: $F_1 \lhd_k F_2 \lhd_k \cdots \lhd_k F_m$ with $t_k(F_i) \cap s_k(F_{i+1}) \neq \emptyset$



Cellular loop: $F_i = F_j$ $i \neq j$

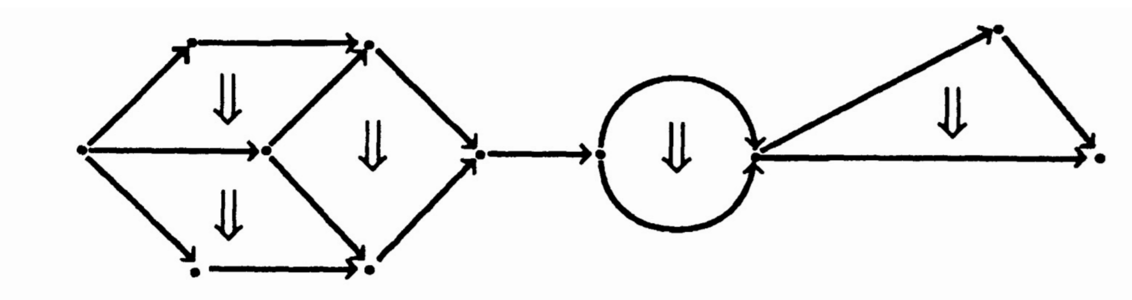
Cellular 0-strings from $s_0(P)$ to $t_0(P)$ = Billera-Kapranov-Sturmfels' cellular strings

THE CONJECTURE

KAPRANOV & VOEVODSKY

COMBINATORIAL-GEOMETRIC ASPECTS OF POLYCATEGORY THEORY

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES
VOL. XXXII-1 (1991)



Conjecture: ~~Theorem 2.3.~~ *If $M \subset \mathbb{R}^n$ is a bounded n-dimensional polytope and $p = \{\mathbb{R}^n \xrightarrow{p_{n,n-1}} \mathbb{R}^{n-1} \longrightarrow \dots \longrightarrow \mathbb{R}^2 \xrightarrow{p_{2,1}} \mathbb{R}\}$ is an admissible system of projections, then $A(M,p)$ is a composable pasting scheme.*
↳ special type of d-dimensional category

We show that a framed polytope induces a composable pasting diagram ⇔ it has no cellular loops.

CONJECTURE (revisited): Every framed polytope is loop free.

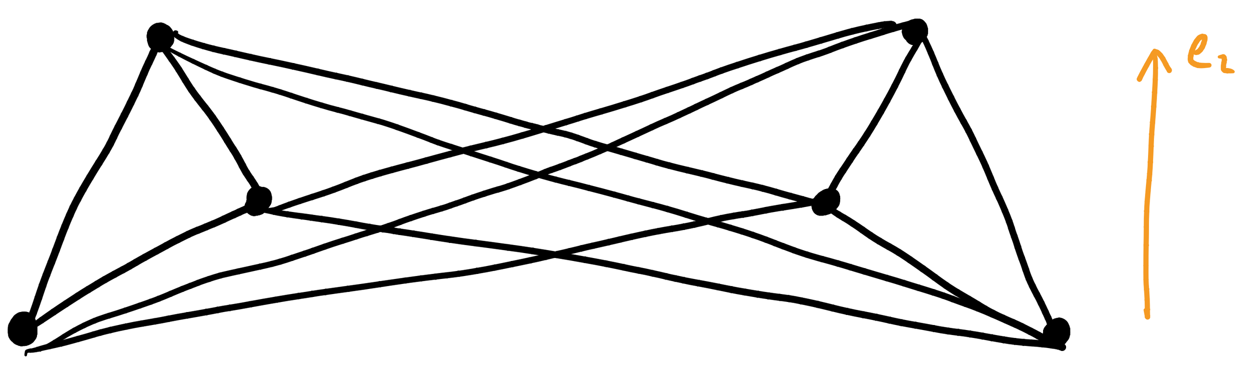
THE COUNTEREXAMPLES

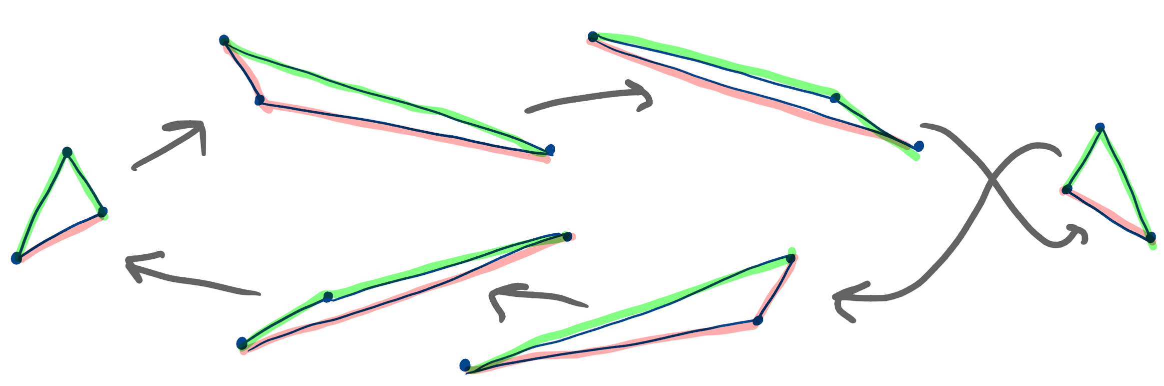
A FIRST COUNTEREXAMPLE

The 5-simplex

$$\mathrm{conv}\left(\begin{pmatrix}-3\\-1\\-1\\0\end{pmatrix}, \begin{pmatrix}-2\\1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}3\\-1\\1\\0\end{pmatrix} \right)$$

with the canonical frame has a 1-loop.





A STRONG COUNTEREXAMPLE

There is a 4-polytope with 144 vertices for which all admissible frames induce a loop.

RANDOM COUNTEREXAMPLES

A random Gaussian d-simplex with the canonical frame has a loop asymptotically almost surely when $d \to \infty$.