DEFORMATION QUANTISATION

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Abstract: In 1997, Kontsevich showed how to quantise Poisson manifolds. Our goal is to use recent work of Gaiotto-Kulp-Wu to investigate and generalise topological-holomorphic deformation quantisation through expository and working sessions.

Motivation for the workshop

- *deformation quantisation* applies to & links many areas of mathematics, among them
  - when does a Poisson manifold X admit a *quantisation* $\mathcal{O}_\hbar(X)$?
  - construction of quantum groups $U_\hbar(\mathfrak{g})$,
  - Grothendieck-Teichmuller groups and the KZ equations,
- The motivation for this workshop are the possibly tractable (holomorphic-topological) generalisations, among them
  - when does the Jet space JX of a Poisson manifold X admit a deformation quantisation $\mathcal{O}_\hbar(JX)$? The latter should be a vertex algebra.
  - construction of quantum groups $U_\hbar(\hat{\mathfrak{g}})$,
  - qKZ equations.

We will start our workshop with a review of the main technical tools for proving Kontsevich's deformation quantisation Theorem [Ko]: **formality** of the little disks operad, then prove the Theorem later.

1. Day I: Background on formality

1.1. Formality of the little T-disks.

Definition 1.1.1. The **little T-disks operad $\mathbf{E}_T$** is given in arity n as the space $\mathbf{E}_T(n)$ of inclusions of n labelled T-disks in the unit T-disk.

Operadic composition is given by rescaling a configuration of disks and embedding it into one disk of another configuration [draw a $\mathbf{E}_2$ example].

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Note that applying any symmetric monoidal functor

$$\mathrm{H}^\bullet(-),\ \mathrm{H}^\bullet(-),\ \mathrm{C}_\bullet(-)\ \simeq\ \Omega^\bullet_{dR}(-),\ \mathrm{C}^\bullet(-),\ \ldots$$

to an operad gives an operad.

For instance, we have homotopy equivalences

$$\mathbf{E}_T(1)\ \sim\ \mathrm{Conf}_1\mathbf{R}^T\ \sim\ \mathrm{pt},$$

$$\mathbf{E}_T(2)\ \sim\ \mathrm{Conf}_2\mathbf{R}^T\ =\ \mathbf{R}^T\times(\mathbf{R}^T\setminus 0)\ \sim\ S^{T-1}.$$

The homology of $\mathbf{E}_2$ is the Gerstenhaber operad, encoding Gerstenhaber or 2-Poisson algebras. More generally, the homology of $\mathbf{E}_T$ encodes T-Poisson algebras, with

$$1\ :\ (\alpha,\beta)\ \mapsto\ \alpha\cdot\beta, \qquad\qquad \mathrm{dVol}_{S^{T-1}}\ :\ (\alpha,\beta)\ \mapsto\ \{\alpha,\beta\}.$$

Definition 1.1.2. An operad is formal if it is quasi-isomorphic *as an operad* to its homology.

Theorem A. [LV, Thm. 1.1] *For any $T>1$, there is an equivalence*[1] *of dg operads*

$$\mathrm{H}_\bullet(\mathbf{E}_T)\ \xrightarrow{\sim}\ \mathrm{C}_\bullet(\mathbf{E}_T)$$

between the singular chains on $\mathbf{E}_T$ and its homology.

1.2. **Proof strategy.** The idea is to consider an operad FM_T homotopy equivalent to $\mathbf{E}_T$.

Definition 1.2.1. The arity n space $\mathrm{FM}_T(n)$ of the Fulton–MacPherson operad FM_T is defined as a closure of the embedding

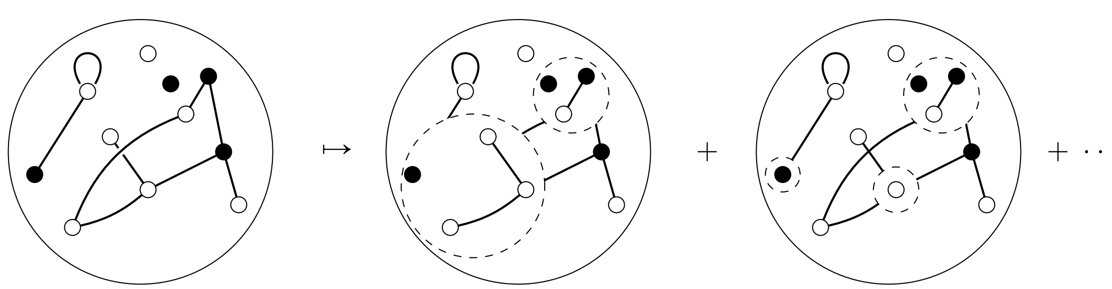
$$\mathrm{Conf}_n\mathbf{R}^T/\mathbf{R}_{>0}\ltimes\mathbf{R}^T\ \hookrightarrow\ (S^{T-1})^{\binom{n}{2}}\times[0,+\infty]^{\binom{n}{3}}.$$

There are homotopy equivalences

$$\mathbf{E}_T(n)\ \xrightarrow{\sim}\ \mathrm{Conf}_n\mathbf{R}^T\ \xrightarrow{\sim}\ \mathrm{FM}_T(n)$$

given by by "shrinking" the T-disks to points, and compactification. The composition is an equivalence of operads, so we will prove FM is formal.

The next idea is to build a graphical model, i.e. to consider a cooperad Gra of graphs.

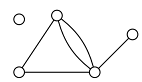


[1] i.e. zig-zag of quasiisomorphisms.

The main construction is then *Kontsevich configuration space integral*

$$\begin{array}{rcccc} I_n & : & \mathrm{Gra}_T(n) & \longrightarrow & \Omega_{PA}(\mathrm{FM}_T(n)) \\ & & \Gamma & \mapsto & \omega_\Gamma \end{array} \tag{1}$$

For any graph on n vertices



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Kontsevich defined a semi-algebraic form $\omega_\Gamma \in \Omega_{PA}(\mathrm{FM}_T(n))$ given by the pullback of the volume form on the standard sphere S^{T-1} over the edges Γ_1 of the graph

$$\omega_\Gamma = \prod_{e \in \Gamma_1} \pi_e^* \mathrm{dVol}_{S^{T-1}}.$$

Here, $\pi_e : \mathrm{FM}_T(n) \to \mathrm{FM}_T(2)$ is the forgetful map forgetting all points but the two vertices adjacent to e. The integral is defined to be zero for self-loops.

The integral (1) is *not* a quasi-isomorphism. To remedy this, one proceeds in two steps: one first *twists* the cooperad Gra via the theory of operadic twisting, obtaining a new cooperad $\mathrm{Tw}(\mathrm{Gra})$. Roughly, one formally "fills" some of the the "slots" in the cooperations, obtaining graphs with additional black vertices at which cocomposition cannot be performed. One further needs to quotient by the ideal spanned by graphs with a connected component made only of black vertices. One then gets a quotient operad Graphs_T to which the integral (1) restricts. The main result is now as follows.

Theorem 1.2.2 (Kontsevich, Lambrechts-Volić)**.** *The map*

$$\begin{array}{rcccc} I_n & : & \mathrm{Graphs}_T(n) & \longrightarrow & \Omega_{PA}(\mathrm{FM}_T(n)) \\ & & \Gamma & \mapsto & \omega_\Gamma \end{array} \tag{2}$$

is a quasi-isomorphism, which is compatible with the cooperad structures.

Sketch. Proving (2) is a map of cooperads is *fiddly* and requires showing various sums of integrals vanish.

Formality of $\mathbf{E}_T$ then follows by showing that Graphs_T is quasi-isomorphic to the cohomology of $\mathbf{E}_T$ (i.e. the linear dual of the operad of T-Poisson algebras), which is achieved by combinatorial means using Serre's spectral sequence and Cohen's description of the cohomology of configuration spaces. □

1.2.3. *Remark.* $\mathrm{FM}_2(n)$ is the topological space underlying the moduli stack $\overline{\mathcal{M}}^{\mathrm{fr}}_{0,n+1}$ of stable curves of genus 0 with $n+1$ marked points, and the graph decomposition relates to the *dual graph* of a stable curve.

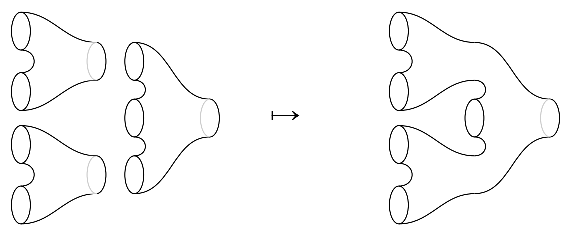
1.3. **Posing general formality results.** In a future talk, we will explain how examples of algebras over the operad $\mathbf{E}_T$ are given by *enriched* TQFTs: there is a forgetful functor

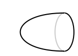
$$\mathrm{oblv} \ : \ \mathrm{TQFT}_T \ \to \ \mathbf{E}_T\text{-Alg}.$$

It is that TQFT perspective that generalises best.

To define a general formality result, we need:

(1) *Spacetime*. An operad in spaces $\mathcal{S}(n)$. A collection of spaces $\mathcal{S}(n)$ with an operad structure





$1 \ \in \ \mathcal{S}(1)$

$$\mathcal{S}(n) \times (\mathcal{S}(k_1) \times \cdots \times \mathcal{S}(k_n)) \ \to \ \mathcal{S}(k_1 + \cdots + k_n)$$

Remark. For general QFTs, we can replace $\mathcal{S}^{\otimes}$ with an arbitary category enriched in spaces.[2]

(2) *Sheaf theory*. Pick a dga $(\Omega^\bullet, \mathrm{d})$ on $\mathcal{S}(n)$, forming an operad in the category $\mathcal{C}$ of spaces equipped with a chain complex of sheaves.[3]

There should be a notion of pushforward (integral) along appropriate maps, and pullback along all maps.

We would like them to fit into a diagram

$$\Omega^\bullet_Y \xrightarrow{f^*} \Omega^\bullet_X \ \to \ \Omega^\bullet_{X/Y}$$

for any map $f : X \to Y$ of spaces. We would also like an map

$$f_*\Omega^\bullet_X \xrightarrow{\int_f} \Omega^\bullet_Y$$

for any smooth proper map f. Common examples are differential forms with de Rham/Dolbeault differentials, where $\int_f$ is integration along the fibres of f.

Examples. We will soon see the following examples:

	$\mathrm{TQFT}_{/\mathbf{R}^T}$	$\mathrm{HTFT}_{/\mathbf{R}^T \times \mathbf{C}^H}$		$\mathrm{TQFT}_{/M}$
$\mathcal{S}(n)$	$\mathrm{FM}(n)$	$\mathrm{FM}_{H,T}(n)$	$\mathcal{M}^{\mathrm{fr}}_{0,n+1}$	$\mathrm{FM}_M(n)$
$\Omega^\bullet$	$(\Omega^\bullet_{PA}, \mathrm{d}_{dR}) \simeq \mathrm{C}^\bullet_{sing}$	$(\Omega^\bullet_{PA,Dol}, \mathrm{d}_{dR} \boxplus \mathrm{d}_{Dol})$	$(\Omega^\bullet_{Dol/dR}, \mathrm{d}_{Dol/dR})$	[CW]

[2]Here for an ∞-operad $\mathcal{O}$, $\mathcal{O}^{\otimes} \to N(\mathrm{FinSet}_*)$ is the category attached to the operad.

[3]A map $(X, \mathcal{F}) \to (Y, \mathcal{G})$ is given by a map of spaces $f : X \to Y$ together with a map of sheaves $\mathcal{F} \to f^*\mathcal{G}$; the monoidal structure is the (box) product of spaces and sheaves.

In general, given a QFT with morphism spaces $\mathrm{Cob}(S_1, S_2)$ acted on by gauge group $\mathcal{G}$, we let

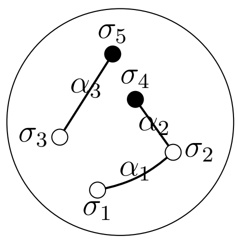
$$\mathcal{S}(n) \;\approx\; \mathrm{Cob}(S^{d-1} \sqcup \cdots \sqcup S^{d-1}, S^{d-1}), \qquad\qquad (\Omega^\bullet)^\vee \;=\; (\wedge^\bullet \mathrm{Lie}_{\mathcal{S}(n)}(\mathcal{G}), \mathrm{d}).$$

One could restate the above axioms in terms of Cob instead (which determines $\mathcal{G}$).

Feynman diagrams. We get elements $\omega_\Gamma \in \mathrm{H}^\bullet(\mathcal{S}(n), \Omega^\bullet)$ by

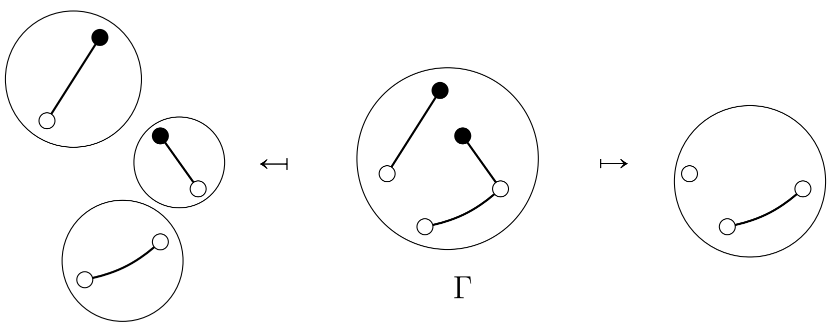
$$\begin{tikzcd} & \mathcal{S}(n_\bullet + m_\circ) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \prod_{\Gamma_0} \mathcal{S}(1) \times \prod_{\Gamma_1} \mathcal{S}(2) & & \mathcal{S}(m_\circ) \end{tikzcd} \qquad \begin{tikzcd} & q^*\beta \arrow[dr, mapsto] & \\ \beta \arrow[ur, mapsto] & & \int_p q^*\beta \end{tikzcd}$$

Here, Γ is a graph on two colours $\circ, \bullet$ without self loops and labelled by cohomology classes β (given by $\sigma_i \in \mathrm{H}^\bullet(\mathcal{S}(1), \Omega^\bullet)$ and $\alpha_e \in \mathrm{H}^\bullet(\mathcal{S}(2), \Omega^\bullet)$)

Γ 

$n_\bullet = 2,\ m_\circ = 3$

The above correspondence should be thought of as attached to restricting to subgraphs



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If we write $\mathbf{V}_i = \mathrm{H}^\bullet(\mathcal{S}(i), \Omega^\bullet)$ for the space of *propogators*, by push-pulling we get

$$\bigotimes_{i \in \Gamma_0} \mathbf{V}_1 \otimes \bigotimes_{e \in \Gamma_1} \mathbf{V}_2 \;\to\; \mathrm{H}^\bullet(\mathcal{S}(m_\circ), \Omega^\bullet)$$

and our data is in $\sigma_i \in \mathbf{V}_i$, $\alpha_e \in \mathbf{V}_2$. In the topological case $\mathbf{R}^T$,

$$\mathbf{V}_1 \;=\; \mathbf{C}, \qquad\qquad \mathbf{V}_2 \;=\; \mathbf{C}\{1, \mathrm{dVol}_{S^{T-1}}\}$$

and in the topological case over M, $\mathbf{V}_1 = \mathrm{H}_\bullet(M)$.

Definition 1.3.1. *(Feynman graphs)* There is a complex of sheaves $(\mathrm{Graphs}, \mathrm{d}_{\mathrm{Graphs}})$ on $\mathcal{S}(n)$ with a quasiisomorphism

$$(\mathrm{Graphs}, \mathrm{d}_{\mathrm{Graphs}}) \;\xrightarrow{\sim}\; (\Omega^\bullet, \mathrm{d}), \qquad\qquad \Gamma \;\mapsto\; \omega_\Gamma$$

respecting the operad structure on $\mathcal{S}(n)$.

Definition 1.3.2. *(Formality)* There is a quasi-isomorphism of complexes of sheaves on $\mathcal{S}(n)$

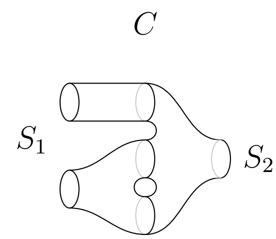
$$\mathrm{For}\ :\ \mathcal{H}^\bullet(\Omega^\bullet, \mathrm{d}) \xrightarrow{\sim} (\Omega^\bullet, \mathrm{d})$$

respecting the operad structure on $\mathcal{S}(n)$.

In practice, the formality morphism is often *defined* using Feynman graphs.

1.3.3. *Remark.* There is no reason to restrict to operads, you can work with the whole category Cob itself, and ask the same questions as above. If the above results are true, they would imply that Atiyah QFTs (the analogues of $\mathbf{E}_T$-algebras) have simple descriptions in terms of graphs.

The construction of ω_Γ on $\mathrm{Cob}(S_1, S_2)$ would be given in terms of a *pair of pants* decomposition Γ of a cobordism $C \in \mathrm{Cob}(S_1, S_2)$ then integrating a given form on the pair of pants over all possible pants decompositions.



1.4. **The holomorphic-topological case.** We now consider the holomorphic-topological case, induced by holomorphic-topological cobordisms. This is inspired by the work of Gaiotto–Kulp–Wu [GKW].

(1) The first step would be to build a suitable holomorphic-topological compactification $\mathrm{FM}_{H,T} = \widetilde{\mathrm{FM}}_{H,T}/\mathbf{R}_{>0}$ of the configuration space of points in $\mathbf{R}^T \times \mathbf{C}^H$, quotiented as in [GKW]

$$\mathrm{Conf}_n(\mathbf{R}^T \times \mathbf{C}^H)/\mathbf{R}_{>0} \rtimes (\mathbf{R}^T \times \mathbf{C}^H) \ \hookrightarrow\ \mathrm{FM}_{H,T}(n)$$

where the scaling acts by weight $(1,2,0)$ on the real manifold $\mathbf{R}^T \times \mathbf{C}^H \simeq \mathbf{R}^T \times \mathbf{R}^H_{hol} \times \mathbf{R}^H_{ahol}$. The desired definition of $\mathrm{FM}_H(n)$ should be in [Wa, §3].

We **do not** know what precise definition to use, but it should have strata labelled by dual graphs etc as in FM_T and $\mathrm{FM}_{0,1} = \overline{\mathcal{M}}^{\mathrm{fr}}$.

(2) We use the *de Rham-Dolbeault forms* on a product $X \times Y$ of a topological and complex manifold

$$\Omega^\bullet_{PA,Dol} \ :=\ (\Omega^\bullet_{sm}, \mathrm{d}_{dR} \boxplus \mathrm{d}_{Dol}),$$

given by the smooth complex-valued differential forms with differentials

$$\mathrm{d}_{dR} \ =\ \partial_{x_1} + \cdots + \partial_{x_n}, \qquad\qquad \mathrm{d}_{Dol} \ =\ \partial_{\bar{y}_1} + \cdots + \partial_{\bar{y}_n}$$

in terms of local coordinates on X and Y. See [GKW, Intro].

Note that its homology sheaf

$$\mathcal{H}^\bullet(\Omega^\bullet_{PA,Dol}) \simeq \mathcal{O}_{hol}$$

is locally constant–holomorphic.

Conjecture 1.4.1. (Formality) *We have an equivalence*[4] *of complexes of sheaves on* $\mathrm{FM}_{H,T}(n)$

$$\mathrm{For} \,:\, \mathcal{H}_\bullet(\Omega^\bullet_{PA,Dol}, \mathrm{d}) \xrightarrow{\sim} (\Omega^\bullet_{PA,Dol}, \mathrm{d})$$

respecting the operad structure on $\mathrm{FM}_{H,T}(n)$.

Conjecture 1.4.2. (Feynman graphs) *There is a complex* $(\mathrm{Graphs}_{H,T}, \mathrm{d}_{\mathrm{Graphs}})$ *of sheaves with*

$$(\mathrm{Graphs}_{H,T}, \mathrm{d}_{\mathrm{Graphs}}) \xrightarrow{\sim} (\Omega^\bullet_{PA,Dol}, \mathrm{d}), \qquad [\Gamma] \mapsto \omega_\Gamma.$$

1.4.3. *Remark.* In physics papers [GKW; Wa], the differential form $\beta = \beta(z_e, \lambda_v)$ is a differential-form-valued *function* on

$$(z_e, \lambda_v) \in \mathrm{Conf}_{\Gamma_1}(\mathbf{R}^T \times \mathbf{C}^H) \times \mathrm{Conf}_{\Gamma_0}(\mathbf{R}^T \times \mathbf{C}^H)$$

the first factor called “Schwinger space" and the second factor called “momentum" in [GKW]. See [Wa, Thm. 3.3.1] and [GKW, (3.24)].

Our **guess** when $T = 0, H = 1$ is that, this comes from the fact that some appropriate generalisation of the moduli stack $\overline{\mathcal{M}}^{\mathrm{fr}}_{0,n+1}$, e.g. with all points framed, has coordinates z_e for each edge e in the dual graph.

1.5. **Questions.**

- Explain how Feynman diagrams show up from the physics perspective, and what this momentum coordinate λ_v and z_e means.
- Understand the “quadratic identities".

[4] Zig-zag of quasiisomorphisms.

References

[CW] Ricardo Campos and Thomas Willwacher. "A model for configuration spaces of points". In: *Algebraic & Geometric Topology* 23.5 (2023), pp. 2029–2106.

[GKW] Davide Gaiotto, Justin Kulp, and Jingxiang Wu. "Higher operations in perturbation theory". In: *arXiv preprint arXiv:2403.13049* (2024).

[Ko] Maxim Kontsevich. "Operads and motives in deformation quantization". In: *Letters in Mathematical Physics* 48 (1999), pp. 35–72.

[LV] Pascal Lambrechts and Ismar Volić. "Formality of the Little N-disks Operad". In: *Memoirs of the American Mathematical Society* 230.1079 (2014), pp. viii+116.

[Wa] Minghao Wang. "Feynman Graph Integrals on $\mathbf{C}^d$". In: *Communications in Mathematical Physics* 406.5 (2025), p. 116.