Spine polytopes and dioperads

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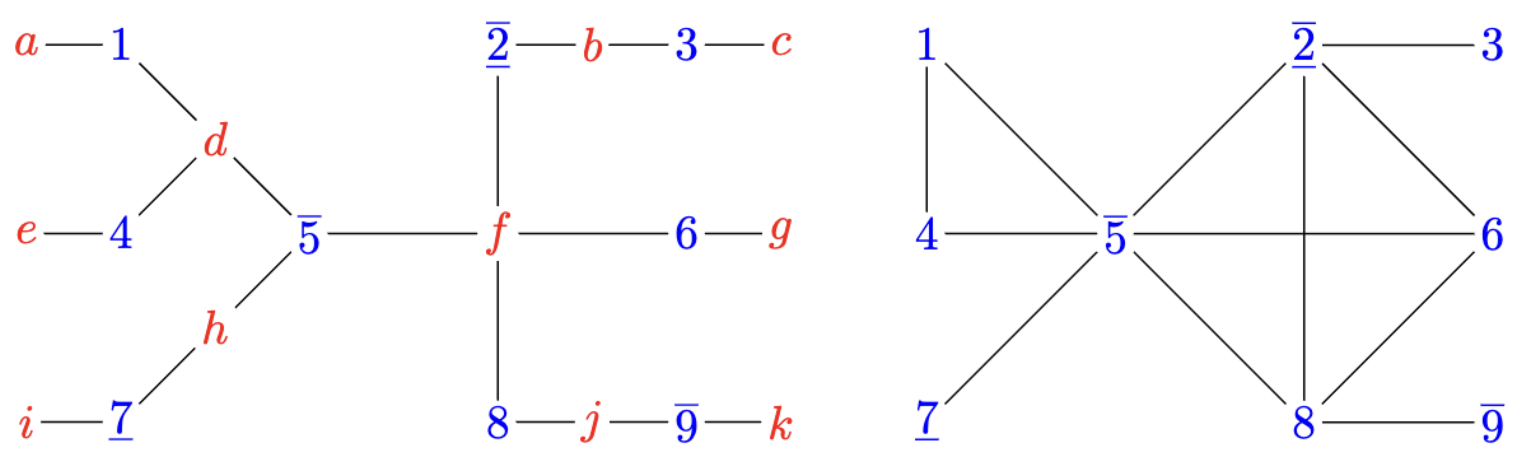
Résumé

We introduce spine polytopes, which are a four-color generalisation of block-graph associahedra, and show that they form a dioperad in polytopes encoding a new type of algebraic structure up to homotopy, which generalises homotopy operads.

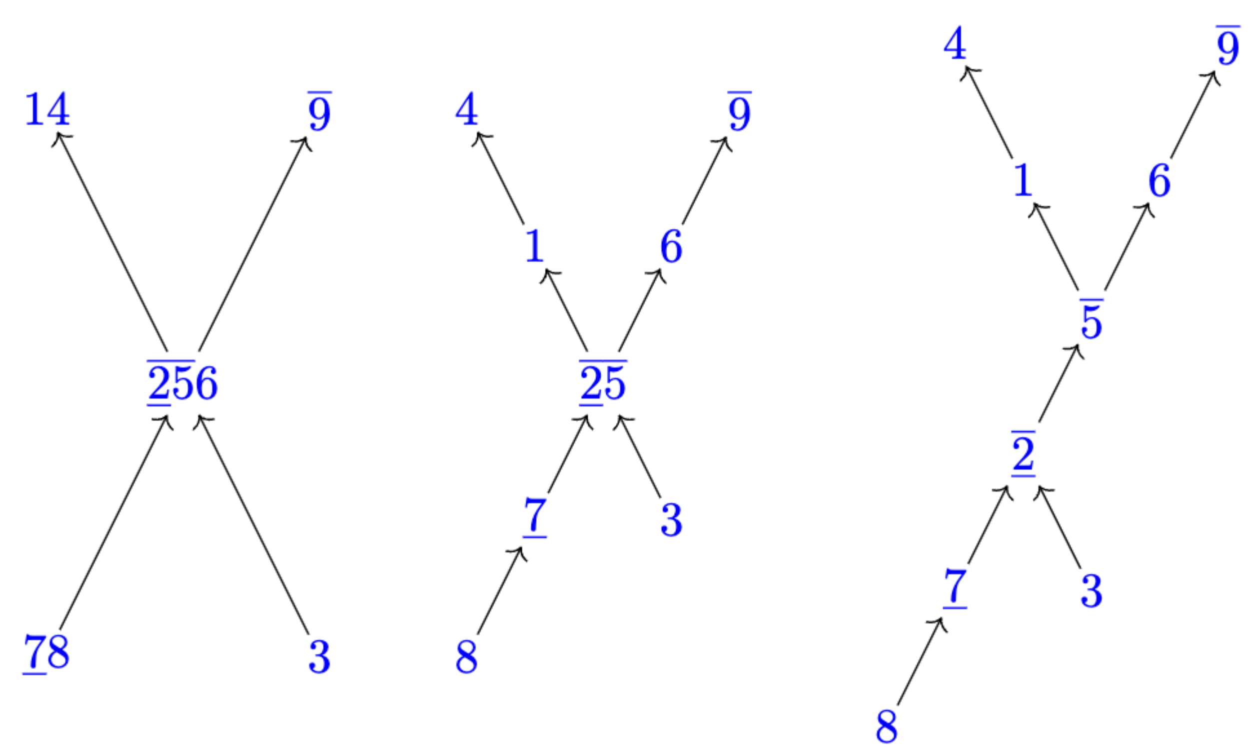
Combinatorics

A *decorated maple tree* is a tree M whose vertices are properly colored in red and blue such that all leaves are red, together with a decoration $\delta : V \to \{\circ, \overline{\circ}, \underline{\circ}, \underline{\overline{\circ}}\}$ of its blue vertices. It has an associated decorated block graph.



A spine on a decorated block graph G is a directed tree S such that

- the nodes of S form a partition of the vertex set V of G, and
- at each node X of S, the source sets of the incoming arcs are contained in distinct connected components of $G \smallsetminus \underline{\delta}(X)$, and the target sets of the outgoing arcs are contained in distinct connected components of $G \smallsetminus \overline{\delta}(X)$.



A node Z of a G-spine is *splittable* if there is an ordered partition $X \sqcup Y = Z$ such that X is contained in a connected component of $G \smallsetminus \underline{\delta}(Y)$ and Y is contained in a connected component of $G \smallsetminus \overline{\delta}(X)$. The set of spines on G forms the spine poset with cover relations given by edge contraction and node splitting.

Geometry

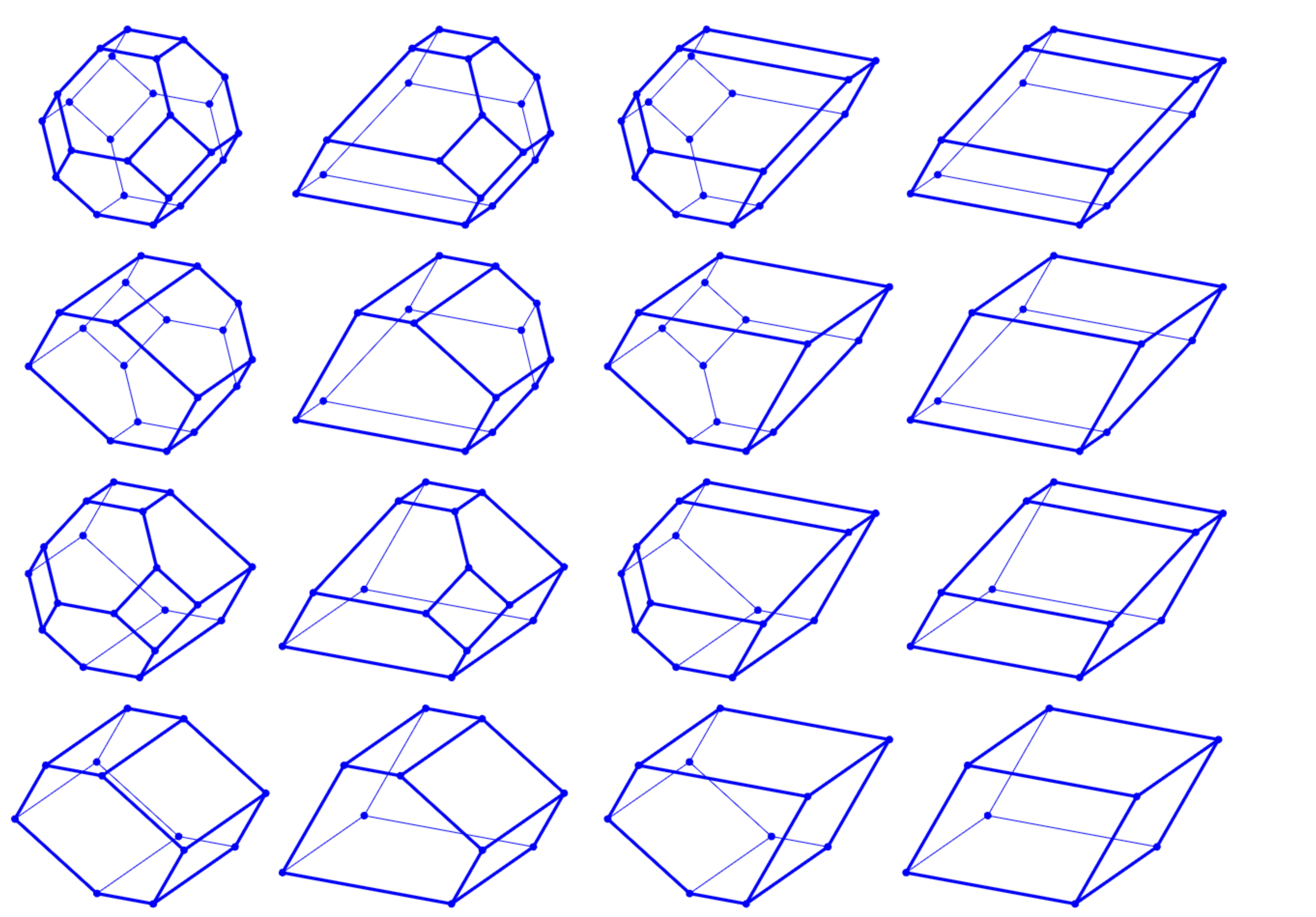
Consider the hyperplane $H_V := \{\boldsymbol{z} \in \mathbb{R}^V \mid \langle \mathbb{1} \mid \boldsymbol{z} \rangle = 0\}$ of $\mathbb{R}^V$. For a spine S, we denote by $\mathsf{C}(S)$ the cone of H_V defined by the inequalities $z_i \leqslant z_j$ whenever there is an oriented path from i to j in S. The spine fan is the collection of cones $\mathsf{C}(S)$ for all spines S on G.

Fix a weight $\omega_p > 0$ for all $p \in \binom{V}{2}$, and define a weight $\omega_X := \sum_{p \in X} \omega_p$ for any subset X of V. For a maximal spine S on G, we denote by $\Pi(S)$ the set of undirected and simple paths joining two nodes of S. For a path $\pi \in \Pi(S)$, we denote by $\partial\pi$ the set of endpoints of π and by $\wedge\pi$ (resp. $\vee\pi$) the set of peaks (resp. valleys) of π.

A *chunk* of G is a subset C of vertices of G such that C and its complement $V \smallsetminus C$ form a splittable partition of the vertex set V.

For any weight matrix ω, the spine fan is the normal fan of the spine polytope defined irredundantly and equivalently as

- the convex hull of the points $\sum_{\pi \in \Pi(S)} \omega_{\partial\pi}\big(\mathbb{1}_{\wedge\pi} - \mathbb{1}_{\vee\pi \smallsetminus \partial\pi}\big)$ for all maximal spines S on G,
- the intersection of the affine hyperplane $\{\boldsymbol{z} \in \mathbb{R}^V \mid \langle \mathbb{1} \mid \boldsymbol{z} \rangle = \omega_V\}$ with the affine halfspaces $\{\boldsymbol{z} \in \mathbb{R}^V \mid \langle \mathbb{1}_C \mid \boldsymbol{z} \rangle \geq \omega_C\}$ for all chunks C of G.



Examples: the *permutahedron* when G is complete or undecorated, the *graph associahedron* when G is down decorated, the *graphical zonotope* when G is fully decorated, the various *permutreehedra* of [2] when G is a path. Note that the construction cannot be extended beyond block graphs!

Algebra

Let X be a set together with a relation $\sim$. An X-*colored* $(\mathbb{S}, \mathbb{S})$-*module* is a family of $(\mathbb{S}_k, \mathbb{S}_l)$-modules $\mathcal{D}(x_1, \dots, x_k; y_1, \dots, y_l)$, indexed by the ordered pairs of ordered sequences $x_1, \dots, x_k$ and $y_1, \dots, y_l$ of elements of X.

An X-colored dioperad is an X-colored $(\mathbb{S}, \mathbb{S})$-module $\mathcal{D}$ together with partial compositions

$$\circ_{ij} : \mathcal{D}(x_1, \dots, x_k; y_1, \dots, y_l) \otimes \mathcal{D}(x'_1, \dots, x'_p; y'_1, \dots, y'_q) \longrightarrow$$
$$\mathcal{D}(x_1, \dots, x_{i-1}, x'_1, \dots, x'_p, x_{i+1}, \dots x_k; y_1, \dots, y_{j-1}, y'_1, \dots, y'_q, y_{j+1}, \dots, y_l)$$

defined whenever $x_i \sim y'_j$, and units $\mathrm{id}_x \in \mathcal{D}(x; x)$ which satisfy appropriate unit, equivariance and associativity axioms.

Any face of a spine polytope is affinely isomorphic to a product of spine polytopes of lower dimensions. Combinatorially, this corresponds to "cutting" a spine along its edges. Using this idea together with the geometric techniques of [1], we endow the family of spine polytopes with a topological cellular dioperad structure. This involves making a compatible choice of cellular approximations of the diagonals $P \to P \times P, x \mapsto (x, x)$.

We show that the image under the cellular chains functor of this dioperad is the minimal homotopy resolution of a Koszul dioperad, encoding a generalization of the notion of homotopy operads (restriction to down decorations) and A-infinity algebras (further restriction to paths). The cellular diagonals then define a tensor product of these new homotopy algebraic structures.

References

[1] G. Laplante-Anfossi. The diagonal of the operahedra. *Advances in Mathematics*, 405:108494, 2022.

[2] V. Pilaud and V. Pons. Permutrees. *Algebraic Combinatorics*, 1(2):173–224, 2018.