Vieille version, voir p17

POISSONS

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Abstract. The purpose of this note is to provide a new combinatorial description of a cellular approximation of the diagonal of the permutahedra.

Introduction

This formula has many possible applications in algebraic topology: 1) iterated cobar construction 2) twisted tensor products 3) Fulton–Sturmfels formula for Losev–Manin spaces (compute explicitly the ring struture on the operational Chow ring -over $\mathbb{Z}$?)

Guillaume: Please make a new line for each sentence as to facilitate comparison between versions in GitHub

1. Preliminaries

1.1. **Original description of the diagonal.** Let us first set up some notations that will be of use throughout the paper. A set $\sigma_I = \bigcup_{i \in I} \sigma_i$ is a *partition* of $[n] := \{1, \ldots, n\}$ if $\bigcup_{i \in I} \sigma_i = [n]$ and $\sigma_i \cap \sigma_j = \varnothing$ for $i \neq j$. We denote by $|\sigma| := |I|$ the size of the partition (its number of blocks). A partition is *ordered* if the indexed set I is equipped with a total order; in what follows we shall use $I = [k]$ for $k \in \mathbb{N}$.

Let us recall the combinatorial formula for the cellular approximation of the diagonal of the permutahedra from [Lap22, Theorem 3.16]. Let $n \geqslant 1$, and let us write

$$D(n) := \{(I, J) \mid I, J \subset \{1, \ldots, n\}, |I| = |J|, I \cap J = \varnothing, \min(I \cup J) \in I\}.$$

Let $\vec{v} \in R^n$ be such that $\forall (I, J) \in D(n)$, we have $\sum_{i \in I} v_i > \sum_{j \in J} v_j$, and let $P \subset \mathbb{R}^n$ denote the standard $(n-1)$-dimensional permutahedron. For any pair (σ^1, σ^2) of ordered partitions of $[n]$, we have

$$(\sigma^1, \sigma^2) \in \operatorname{Im} \triangle_{(P, \vec{v})} \iff \forall (I, J) \in D(n), \exists l, \left| \left(\bigcup_{1 \leqslant k \leqslant l} \sigma^1_k \right) \cap I \right| > \left| \left(\bigcup_{1 \leqslant k \leqslant l} \sigma^1_k \right) \cap J \right| \text{ or}$$
$$\exists l', \left| \left(\bigcup_{1 \leqslant k \leqslant l'} \sigma^2_k \right) \cap I \right| < \left| \left(\bigcup_{1 \leqslant k \leqslant l'} \sigma^2_k \right) \cap J \right| .$$

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We shall denote by $\triangle$ the set of pairs of ordered partitions of $[n]$ which satisfy the above condition.

There is an equivalent description of $\triangle$ which has the following form:

Proposition 1.1 ([Lap22])**.** *For a two ordered partitions $\sigma^1, \sigma^2 \subset [n]$, we have*

$$(\sigma^1,\sigma^2) \in \triangle \iff \forall (I,J) \in D(\sigma^1,\sigma^2), \exists l, \left|\left(\bigcup_{1\leqslant k\leqslant l} \sigma^1_k\right) \cap I\right| > \left|\left(\bigcup_{1\leqslant k\leqslant l} \sigma^1_k\right) \cap J\right| \text{ or}$$

$$\exists l', \left|\left(\bigcup_{1\leqslant k\leqslant l'} \sigma^2_k\right) \cap I\right| < \left|\left(\bigcup_{1\leqslant k\leqslant l'} \sigma^2_k\right) \cap J\right| .$$

Here, $D(\sigma^1,\sigma^2) \subset D(n)$ is a proper subset of $D(n)$ which depends on the choice of (σ^1,σ^2), and comes from the geometry of the situation, see [Lap22, Theorem 1.26] for more details. For our present purposes, it will be enough to restrict our attention to facets of $\triangle$, that is pairs (σ^1,σ^2) which satisfy $|\sigma^1| + |\sigma^2| = n+1$. In this case, $D(\sigma^1,\sigma^2)$ has $n-1$ elements, and admits the following description.

For any subset $\sigma_i \subset [n]$, let $\vec{\sigma}_i \in \mathbb{R}^n$ denote the boolean vector whose coordinates are 1 in position j if $j \in \sigma_i$ and 0 otherwise. Given a facet (σ^1,σ^2) of $\triangle$, one can consider the system of equations $\langle \vec{\sigma}^1_i, x\rangle = 0$, $\langle \vec{\sigma}^2_j, x\rangle = 0$ given by the blocks of both partitions. For geometric reasons (see the proof of [Lap22, Theorem 1.26]), the solution of this system is $x = 0$. Now we will be interested in the solutions of the systems associated to the pairs (ρ^1,σ^2) and (σ^1,ρ^2) where ρ^1 (resp. ρ^2) has been obtained from σ^1 (resp. σ^2) by merging two adjacent blocks.

Proposition 1.2. *There is a bijection between the set $D(\sigma^1,\sigma^2)$ and the solutions to the systems of equations of the form (ρ^1,σ^2) and (σ^1,ρ^2).*

Proof. For any $z \in (\mathring{\sigma}^1 + \mathring{\sigma}^2)/2$, the face $\sigma^2 \cap \rho_z\sigma^1$ of $P \cap \rho_z P$ is a vertex of the polytope $P \cap \rho_z P$. The faces of the form $\sigma^2 \cap \rho_z\rho^1$ and $\rho^2 \cap \rho_z\sigma^1$ are the edges of $P \cap \rho_z P$ which are adjacent to the vertex $\sigma^2 \cap \rho_z\sigma^1$. By definition $D(\sigma^1,\sigma^2)$ describes the directions of these edges, and the translation is made as follows: for a given (I,J), define the corresponding direction $\vec{d}$ by $d_i := 1$ if $i \in I$, $d_j := -1$ if $j \in J$, and $d_k := 0$ otherwise. We refer to [Lap22, Section 1.5] for more details. $\square$

Guillaume: To be rewritten in a self-contained way, and with precise references

We will sometimes refer to the elements of $D(\sigma^1,\sigma^2)$ the minimal (I,J)-pairs.

2. Facets of the diagonal

In this section we establish a bijection between the facets of $\triangle$ and a family of pairs of unordered partitions introduced and enumerated in a series of 3 papers [Che69, CG71, KUC82]. An intermediary bijection to a type of bipartite tree is of particular importance and provides [...]. In particular, we obtain that the number of facets in $\triangle_n$ is $2(n+1)^{n-2}$ (A007334), and more precisely that the pairs of dimension $(k, n-k)$ are counted by the formula $\frac{1}{k+1}\binom{n+1}{k}(k+1)^{n-k}(n+1-k)^k$.

Guillaume: Notations to be uniformized...

2.1. **Essential complementary partitions and bipartite trees.** Let us recall some basic definitions and results from the series of papers [Che69, CG71, KUC82].

Definition 2.1. A set of *distinct representatives* of a partition is a set $R \subset [n]$ such that $\forall i \in I, |P_i \cap R| = 1$.

Definition 2.2. A pair of partitions $P = (P_L, P_R) := (\cup_{l \in L} P_l, \cup_{r \in R} P_r)$ is

- *complementary* if there exists $I \subset [n]$ and $p \in I$ such that I and $(V \backslash I) \cup \{p\}$ are distinct representatives of P_L and P_R, respectively.
- *essential* if there does not exists proper subsets $K \subset [n], L' \subset L$ and $R' \subset R$ such that $P' := (P_{L'}, P_{R'})$ is a complementary partition of K.

We shall denote the set of all essential complementary pairs of partitions by $\mathcal{E}$. Let us emphasize that the pairs of partitions of $\mathcal{E}$ are *unordered*.

Example 2.3. n=2, n=3

Kurt: To do

A *tree* is a simply connected graph with no cycles. A *bipartite graph* is a graph whose vertices are partitioned into two sets such that vertices in one set are only adjacent to vertices in the other, we say it is *ordered* if one of the sets is considered smaller than the other and we denote the partition (V_L, V_R). We say a graph with n edges is *edge labelled* if there exists a bijection between the edges and $\{1, \ldots, n\}$. Let $\mathcal{B}$ denote the set of edge labelled ordered bipartite trees.

Proposition 2.4 ([KUC82, Theorem 3])**.** *Essential complementary partitions and labelled bipartite trees are in bijection through $G : \mathcal{E} \to \mathcal{B}$ and $P : \mathcal{B} \to \mathcal{E}$, where*

- *G takes a pair (P_L, P_R) and constructs partitioned vertices (V_L, V_R). For each $i \in \{1, \ldots, n\}$ an edge is added between v_l and v_r if $i \in P_l$ and $i \in P_r$.*
- *P takes a tree from $\mathcal{B}$ and labels the vertices (V_L, V_R) by the edges which are adjacent to them. The labels of the vertices can then be interpreted as a pair of partitions.*

Guillaume: Consider changing notation for functions vs sets

Corollary 2.5 ([KUC82])**.** *The number of essential complementary partitions is $|\mathcal{E}_n| = 2(n+1)^{n-2}$.*

2.2. **Bijection with the facets of the diagonal.** In this section we denote by $\triangle$ be the set of pairs of ordered partitions of $[n]$ labeling *facets* of the diagonal $\triangle$.

Theorem 2.6. *Facets of the diagonal and essential complementary partitions are in bijection through the inverse functions $u : \triangle \to \mathcal{E}$ and $o : \mathcal{E} \to \triangle$, where*

1. *The function u forgets the order of the ordered partition pair.*
2. *The function o uniquely orders an essential complementary partition pair via the minimal (I, J)-pairs defining the diagonal.*

We shall prove this theorem by establishing the necessary total order, showing that the functions are well defined, and then showing that they are injective.

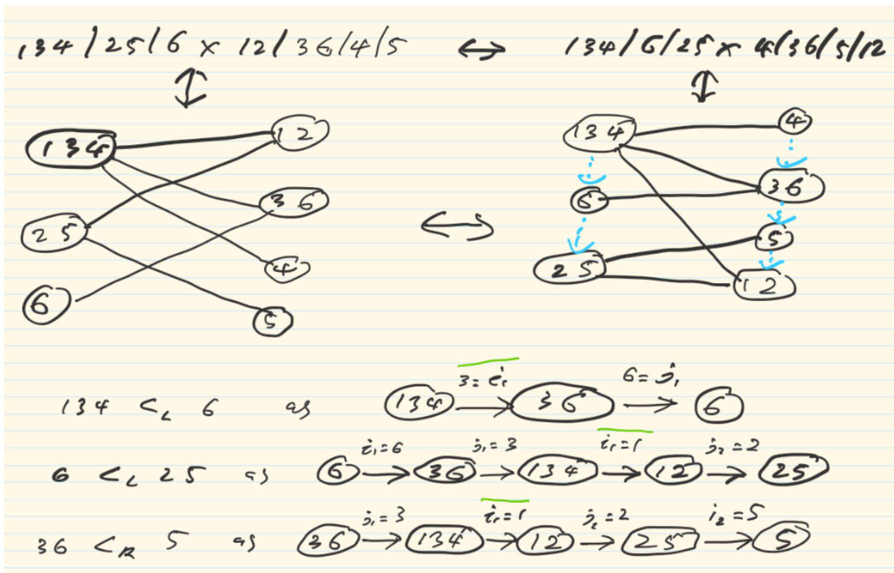


FIGURE 1. The bijection between ordered partitions and bipartite trees.

Lemma 2.7. *The function $u : \triangle \to \mathcal{E}$ that forgets the order in a pair of partitions is well defined.*

Proof. Let $P \in \triangle_n$. Then $G(u(P))$ is a graph with $l + r = n + 1$ vertices, and n edges. Furthermore, as no vertices can be isolated it must be the case that this graph is a tree. It is straightforward to verify that $G(u(P))$ must be labeled bipartite tree, but here is how we may explicitly produce the necessary distinct representatives using an algorithm of [KUC82, Theorem 2].

Let G' be a copy of $G(u(P))$. While there is a vertex of degree 1 in G' delete it and add the sole edge of that vertex as a distinct representative of the corresponding partition of that vertex. As G' is a tree this process can continue until there is a single edge connecting two vertices of degree 1. This edge specifies the element p of the distinct representatives. □

Construction 2.8. *For $P = (P_L, P_R) \in \mathcal{E}$ an essential complementary pair, we construct total orders on P_L and P_R in three steps:*

1. *For $l, l' \in L$ there exists a unique minimal set of edges $p_{l,l'}$ of even cardinality connecting $V(P_l)$ and $V(P_{l'})$ in $G(P)$ (similar for R). We partition this set of edges as $I \cup J$ where I and J are each pairwise non-adjacent, and I contains the minimal edge.*
2. *Orient each path so that I points left to right, and J points right to left (same orientation for P_L and P_R).*
3. *We say $P_l < P_{l'}$ (or $P_r < P_{r'}$) if the constructed path points from $V(P_l) \to V(P_{l'})$ ($V(P_r) \to V(P_{r'})$).*

Proof. We first show our binary relation is well defined before verifying that it defines a total order on $G(P)$ and hence P via the bijection of Proposition 2.4.

As $G(P)$ is a bipartite tree, every vertex is connected, and every path connecting two vertices on the same side must be of even length. As I and J are each pairwise non-adjacent, they must partition the path in an alternating fashion i.e. $p = (I_{i_1}, J_{j_1}, I_{i_2}, J_{j_2}, ...,)$, hence we can orient the path by forcing I to point left and J to point right.

This order is clearly total, reflexive (by convention) and anti-symmetric, what remains to be checked is its transitivity.

Let p_{ab} denote the unique maximal path between two vertices a and b on the left of $G(P)$, that is two blocks of P_L. Let I_{ab} denote the set of left-to-right edges in this path, and let J_{ab} denote its complement. Then, we have

$$a < b \iff \min(I_{ab} \cup J_{ab}) = \min(I_{ab}) \ . \tag{2.1}$$

Suppose now that $a < b$ and $b < c$. Since $p_{ac} = (p_{ab} \cup p_{bc}) \backslash (p_{ab} \cap p_{bc})$, we have

$$I_{ac} = (I_{ab} \cup I_{bc}) \backslash (J_{ab} \cup J_{bc}) \text{ and } J_{ac} = (J_{ab} \cup J_{bc}) \backslash (I_{ab} \cup I_{bc}) \ ,$$

and from the condition (2.1) above it is clear that $\min(I_{ac} \cup J_{ac}) = \min(I_{ac})$, which completes the proof of the transitivity for the total order on P_L. The proof for P_R is similar. □

This order far from being arbitrary provides the unique way to order an essential complementary partition pair into an ordered partition pair of $\triangle$, as we shall demonstrate next.

First we need a geometrical lemma.

Proposition 2.9. *The paths between adjacents vertices of P_L or P_R are in bijection with the minimal (I, J)-pairs.*

Proof. By Proposition 1.2, it suffices to show that the paths between adjacent vertices of P_L are in bijection with the solutions of the system of equations of the form (ρ^1, σ^2). To ease notation let us write ρ for ρ^1 and σ for σ^1. Suppose that ρ is obtained from σ by merging the two blocks σ_a and σ_b. The two equations $\langle \vec{\sigma}_a, x \rangle = 0$ and $\langle \vec{\sigma}_b, x \rangle = 0$ now become $\langle \vec{\sigma}_a + \vec{\sigma}_b, x \rangle = 0$; nothing else changes in the system. Since the solution to the system (σ^1, σ^2) was $x = 0$, now the solution is of dimension 1, and it is given precisely by the path between a and b in $G(P)$. Such a path is given by an alternating sequence of vertices and edges $\sigma_1 := \sigma_a, e_1, \sigma_2, e_2, \ldots, e_{k-1}, \sigma_k := \sigma_b$. Every edge $e_i \in \{1, \ldots, n\}$ is by definition the intersection $\sigma_i \cap \sigma_{i+1}$; thus it is the only common non-zero coordinate between $\vec{\sigma}_i$ and $\vec{\sigma}_{i+1}$. Thus the path encodes the series of equations $x_{e_1} + x_{e_{k-1}} = 0$, $x_{e_1} + x_{e_2} = 0$, $x_{e_2} + x_{e_3} = 0$, $\ldots$, $x_{e_{k-2}} + x_{e_{k-1}} = 0$. Thus, $x_{e_1} = 1$, $x_{e_2} = -1$, $x_{e_3} = 1$, $\ldots$, $x_{e_{k-2}} = 1$, $x_{e_{k-1}} = -1$ is a basis of one-dimensional space of solutions, and it gives the corresponding minimal (I, J)-pair. □

Lemma 2.10. *The function $o : \mathcal{E} \to \triangle$ that orders an essential complementary pair is well defined.*

Proof. Let $P = (P_L, P_R) \in \mathcal{E}$ and consider $o(P)$. We first show that every (I, J)-condition, for $(I, J) \in D(n)$, which corresponds to a path between vertices is satisfied. In particular, this statement will be true for minimal (I, J)-pairs, which will be enough in virtue of Proposition 1.1. Suppose I, J corresponds to a path between two vertices on the left, i.e.

$$V(P_l) = V_{L_1} \xrightarrow{i_1} V_{R_1} \xrightarrow{j_1} V_{L_2} \xrightarrow{i_2} \dots \xrightarrow{i_k} V_{R_{k-1}} \xrightarrow{j_k} V_{L_k} = V(P_{l'})$$

By construction we have that $I = \{i_1, ..., i_k\}, J = \{j_1, ..., j_k\} \in D(n)$ (note we are ordering I and J by the path, so it is not necessarily the case that $\min I = i_1$). Furthermore, each sub partition of P_R either contains a single element of I and a single element of J, or it contains no elements of I and no elements of J. As such for any ordering of the sub-partitions of P_R we have that

$$\forall m, \left| \bigcup_{1 \leqslant k \leqslant m} P_{R,k} \cap I \right| = \left| \bigcup_{1 \leqslant k \leqslant m} P_{R,k} \cap J \right|$$

Hence in order for this $D(n)$ condition to be satisfied it must be the case that for some ordering of the sub-partitions of P_L we have

$$\exists m, \left| \bigcup_{1 \leqslant k \leqslant m} P_{L,k} \cap I \right| > \left| \bigcup_{1 \leqslant k \leqslant m} P_{L,k} \cap J \right|$$

Every sub-partition of P_L excluding the lth and l'th either contains no elements of both I and J, or it contains a single element of I and a single element of J. So the only way for the condition to be satisfied is for P_l to come before $P_{l'}$, which is precisely what is required by the total order.

If I, J correspond to a path between two vertices on the right,

$$V(P_r) = V_{R_1} \xrightarrow{j_1} V_{L_1} \xrightarrow{i_1} V_{L_2} \xrightarrow{j_2} \dots \xrightarrow{j_k} V_{L_{k-1}} \xrightarrow{1_k} V_{R_k} = V(P_{r'})$$

then a similar chain of logic implies we must have an ordering of the sub-partitions of P_R such that

$$\exists m, | \bigcup_{1 \leqslant k \leqslant m} P_{R,k} \cap I | < | \bigcup_{1 \leqslant k \leqslant m} P_{R,k} \cap J |$$

and this can only happen if P_r comes before $P_{r'}$. □

Remark 2.11. It would be interesting to know if there is a geometrical interpretation of the paths that are not between adjacent vertices.

To complete the proof of Theorem 2.6, it remains to show that both $u : \triangle \to \mathcal{E}$ and $o : \mathcal{E} \to \triangle$ are injective, with the other function being their inverse.

Proof of Theorem 2.6. The forgetful function u is clearly the inverse to o as forgetting any assigned order will clearly return the original essential complementary partition pair. The ordering function o is the inverse to u as it returns the sole ordering of the sub-partitions which is compatible with the $D(n)$ conditions. □

2.3. **Combinatorial formula for facets of the diagonal.** From Theorem 2.6, we can deduce a formula for the number of facets of the diagonal:

Proposition 2.12. *The number of pairs of ordered partitions of dimension $(k, n-k)$ which correspond to facets of the diagonal is given by:*

$$\frac{1}{k+1}\binom{n+1}{k}(k+1)^{n-k}(n+1-k)^k. \tag{2.2}$$

Proof. According to Theorem Theorem 2.6, pairs of ordered partitions of dimension $(k, n-k)$ which correspond to facets of the diagonal are in one-to-one correspondence with bipartite trees with $k+1$ black vertices, $n-k+1$ white vertices and $n+1$ edges labeled from 1 to $n+1$.

We do not prove exactly here the proposition but a slightly modified version: Rooted bipartite trees with $k+1$ black vertices and $n-k+1$ white vertices such that:

- a black vertex is distinguished and called *the root*
- the $n+1$ non-root vertices are labeled,
- every label between 1 and $n+1$ is used exactly once.

are counted by:

$$\binom{n+1}{k}(k+1)^{n-k}(n+1-k)^k. \tag{2.3}$$

Let us construct such a bipartite tree.

First, there are $\binom{n+1}{k}$ ways to choose the labels for black vertices (white vertices being labeled by the non-chosen labels). We denote by $\mathcal{B}$ this set of labels.

Moreover, the labeled black vertices are different from the root, hence they should have a white parent : there are $n+1-k$ ways to choose the parent of any labeled black vertex. We thus have $(n+1-k)^k$ ways to build corollas with labeled black leaves and a white root, called bi-colored corollas (or sometimes just corollas) in the sequel.

Finally, we arrange bi-colored corollas in a rooted bipartite tree by adapting the algorithm which convert a Prüfer code to a tree. Here what is called *Prüfer code* is a word of length $n-k$ over the alphabet $\mathcal{B} \cup \{\bullet\}$, where $\bullet$ stands for the non-labeled black vertex. Let us start with a word $c = c_1 \dots c_{n-k}\bullet$ of length $n-k+1$ and the set $\mathcal{T} = \mathcal{S} \cup \{\bullet\}$ of $n-k+2$ bi-colored corollas augmented with the unlabeled black vertex. We apply Algorithm 1. Let us first prove it termination and correctness. The equality $\mathrm{length}(c) = \mathrm{Card}(\mathcal{T}) - 1$ is a loop invariant for the While loop: indeed at each iteration of the loop, the length of c and the number of elements in $\mathcal{T}$ decrease exactly by one. It ensures the termination of the loop and the fact that $\mathcal{T}$ contains a unique element when exiting the loop. Moreover, the set of trees $\mathcal{T}$ contains at each steps exactly one unlabeled black vertex, k labeled black vertices and $n-k+1$ white vertices. Finally, when adding an edge between two trees, one can only get a tree. Moreover, as the edge is added between a white root and the label of a black vertex, the obtained tree is indeed bipartite.

To prove that this algorithm defines a bijection between the pairs of Prüfer code and set of bipartite rooted trees, let us give the algorithm which convert a rooted bipartite tree in such a pair in Algorithm .

Algorithm 1: Prüfer algorithm : from a word to a tree

Input: a word $c = c_1 \dots c_i$ and a set $\mathcal{T}$ of i bi-colored trees with white root and one bi-colored tree with an unlabeled black root
Output: a bipartite rooted tree

while length$(c) > 0$ **do**
  /* loop invariant: length$(c) = \text{Card}(\mathcal{T}) - 1$, at each iteration, the length of c decreases by 1 */
  t $\leftarrow \min\{a \in \mathcal{T} |$ none of the c_i is a label in $a\}$ // Here the order is given by the order on the labels of the root (as the tree with a black root does not satisfy the condition)
  p $\leftarrow$ tree of $\mathcal{T}$ to which belongs the first letter of c // Note that it cannot be t itself
  Remove t and p from $\mathcal{T}$ // Decrease the cardinality of $\mathcal{T}$ by two
  Add an edge between the root of t and the first letter of c and add the obtained tree to $\mathcal{T}$ // Increase the cardinality of $\mathcal{T}$ by one
  Remove the first letter of c // Decrease the length of c by one

Return the unique element of $\mathcal{T}$

Algorithm 2: Prüfer algorithm : from a tree to a word

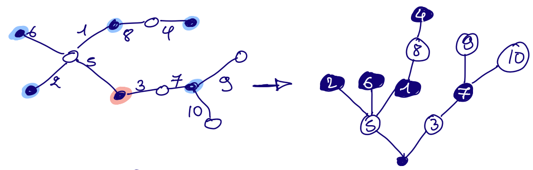
Input: a bipartite rooted tree A
Output: a word $c = c_1 \dots c_i$ and a set $\mathcal{T}$ of i bi-colored trees with white root, except one bi-colored tree with an unlabeled black root

$c \leftarrow$ empty word //
$\mathcal{T} \leftarrow$ empty set // Initialization
while *A has more than one vertex* **do**
  /* At each iteration, the number of white vertices decreases by 1 */
  t $\leftarrow \min\{w \in \mathcal{T} | w$ is a white vertex whose children are leaves $\}$ // Here the order is the one on white labels
  $c \leftarrow c$ concatenated with label of the parent of t // This label is a black vertex
  Remove the edge between t and its parent: the root part goes in A and the corolla in $\mathcal{T}$ // Increase the cardinality of $\mathcal{T}$ by one

Return the pair $(c, \mathcal{T} \cup A)$

This second algorithm terminates as the number of white vertices decreases strictly by one at each iterations. Moreover, every letter added to c is the label of a black vertices, and every tree added to $\mathcal{T}$ is a bi-colored corolla or $\bullet$ (the tree with only the non-labeled black root). Finally, the cardinality of the set of bipartite trees is one more than the length of c. As this algorithm is the classical reverse algorithm of the first one, it ends the proof. Bérénice: Do I need to add more details ? □

Example 2.13. Bérénice: Prüfer algorithm applied on an example : TO DO



$\binom{n+1}{k}$

↳ choix des n^os des sommets noirs
(et des sommets blancs du coup)

Code de Prüfer

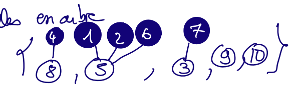
↳
1 → 5
2 → 5
4 → 8
6 → 5
7 → 3

$(n-k+1)$ choix

$\rightsquigarrow (n+1-k)^k$

Reste à assembler ces corolles en arbre

$1 \cdot 77$



Guillaume: New section?

2.4. **The Saneblidze–Umble description.** Here we prove that the diagonal $\triangle$ admits a combinatorial description completely analogous to that of the Saneblidze–Umble diagonal [SU04]. A direct corollary is that the two diagonals are in bijection, and moreover that the SU diagonal can be obtained from a certain choice of chambers in the fundamental hyperplane arrangements of the permutahedra, resolving a conjecture made in [Lap22]. In particular, this provides an alternative proof that all known diagonals on the associahedra agree [SU22].

Given any permutation, one can canonically produce a facet of the diagonal in the following way.

Definition 2.14. A pair of ordered partitions (σ, τ) is called *strong complementary* if σ is obtained from a permutation by merging the adjacent elements which are descreasing, and τ is obtained from the same permutation by merging the adjacent elements which are increasing.

An example is shown in Figure 2. The same argument as in Lemma 2.7 shows that the underlying pair of partitions is an essential complementary pair. Moreover we see directly that any path between adjacent vertices has length 2, and that its minimum is always traversed from left to right, hence all strong complementary pairs are in $\triangle$.

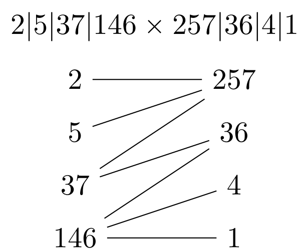


FIGURE 2. The strong complementary pair associated to the partition 2|5|7|3|6|4|1.

There is a natural partial order on the facets of $\triangle$. For two elements $x, y \in [n]$, we say that the set $\{x, y\}$ is an *inversion* of a facet (σ, τ) if we have that $x < y$, the element x appears before y in σ, and the element y appears before x in τ.

Definition 2.15. We say that two facets $(\sigma, \tau) \leqslant (\sigma', \tau')$ are comparable if the set of inversions of (σ, τ) is contained in the set of inversions of (σ', τ').

It is immediate to see that this defines a poset. Guillaume: and even a lattice if a maximal and a minimal element are added?

Proposition 2.16. *The set of inversions of a facet is in bijection with its set of edge crossings. Moreover, the set of facets with no crossings is the set of strong complementary pairs.*

Proof. For the first part of the statement, it is clear that every inversion gives rise to a crossing. For the converse, one needs to check that an anti-inversion, where y appears before x in σ and x appears before y in τ, cannot occur in a facet of $\triangle$; this follows immediately from the (I, J)-conditions for $|I| = |J| = 1$. The second part of the statement follows from the fact that facets of the diagonal with no crossings are in bijection with permutations. As explained above, from a partition one obtains a strong complementary pair, which is in $\triangle$. In the other way around, given a strong complementary pair, one can read-off the partition in the associated tree, which has no crossings, by going along the edges from top to bottom, see Figure 2. $\square$

We aim now at characterizing the cover relations of this poset. Let σ be an ordered partition. We say that a pair of adjacent blocks $\sigma_i|\sigma_{i+1}$ of σ is *admissible* if there is an element $\rho \in (\sigma_{i+1} \backslash \max \sigma_{i+1})$ with $\rho < \min \sigma_i$. Such an element ρ is said to be *critical*. Let us define the *left shift operator* L, which takes an admissible pair $\sigma_i|\sigma_{i+1}$ in σ and creates a new ordered partition $L(\sigma)$ where the critical element ρ is shifted one block to the left: we have $L(\sigma)_i := \sigma_i \cup \rho$, $L(\sigma)_{i+1} := \sigma_{i+1} \backslash \rho$, and $L(\sigma)_j := \sigma_j$ for all $j \neq i, i+1$.

In the same fashion, the *right shift operator* R sends an element $\rho \in (\sigma_i \backslash \max \sigma_i)$ with $\rho < \min \sigma_{i+1}$ one block to the right; the resulting ordered partition $R(\sigma)$ is such that $R(\sigma)_i := \sigma_i \backslash \rho$ and $R(\sigma)_{i+1} := \sigma_{i+1} \cup \rho$.

Definition 2.17. The facets of the *dual SU diagonal* are the pairs or ordered partitions that can be obtained from an strong complementary pair (σ, τ) by iterated applications of the left shift operator L on the first term σ, and iterated applications of the right shift operator R on the second term τ.

As its name suggests, the dual SU diagonal is obtained from the definition of the SU diagonal [SU04] by replacing "max" with "min" and the symbol "$>$" by the symbol "$<$", see the description following Example 1 in [SU22].

Theorem 2.18. *The facets of the diagonal $\triangle$ and the facets of the dual SU diagonal are equal.*

Proof. Since we consider only the action of L on the first term of the pair (σ, τ) and the action of R on the second term, we will prove statements for L, the ones for R are similar.

First we show that every facet of $\triangle$ is a facet of the dual SU diagonal. Let (σ, τ) be a pair in $\triangle$ and suppose that it satisfies the following property: for all pairs of consecutive blocks $\sigma_i|\sigma_{i+1}$ in σ, if $\min \sigma_i < \max \sigma_{i+1}$, then the unique path between σ_i and σ_{i+1} is given by $\{\min \sigma_i, \max \sigma_{i+1}\}$. We claim that (σ, τ) must be a strong complementary pair. To see this, suppose that (σ, τ) is *not* a strong complementary pair; therefore by Proposition 2.16 there exists an edge crossing. This implies that there is a minimal crossing, i.e. a crossing between edges adjacent to two consecutive blocks $\sigma_i|\sigma_{i+1}$. Since edges that are incident to a block are always in increasing order from bottom to top (this is a direct consequence of the (I, J)-conditions for $|I| = |J| = 1$), there is a crossing between $\min \sigma_i$ and $\max \sigma_{i+1}$, which contradicts the property assumed above. So, there is no crossing in (σ, τ) and according to Proposition 2.16, we have that (σ, τ) is a strong complementary pair. The proof of the inclusion is now complete: if we are given a pair of facets (σ, τ) in $\triangle$ which has at least one crossing, then there is at least one possible application of the inverse operators L^{-1} or R^{-1}, and by induction we obtain a facet of the dual SU diagonal.

Second, we show that every facet of the dual SU diagonal is in $\triangle$. We already know that strong complementary pairs are in $\triangle$. Thus, it suffices to prove that if a dual SU pair (σ, τ) is in $\triangle$, then $(L(\sigma), \tau)$ is also in $\triangle$. We first show that for any admissible pair $\sigma_i|\sigma_{i+1}$ and critical element ρ, the unique path γ between σ_i and σ_{i+1} is such that $\rho < \min \gamma$. We use induction on the number of applications of L. The statement is clearly true for strong complementary pairs. Suppose that it holds for (σ, τ). We observe that the paths in $(L(\sigma), \tau)$ are precisely the following: either they do not contain ρ, in which case they are paths of (σ,τ); or they do contain ρ, in which case they are obtained from paths in (σ,τ) by inserting at some place the unique path γ between σ_i and σ_{i+1}, or its inverse. Since $\rho < \min\gamma$, adding the path γ does not change the minima of the paths in $(L(\sigma),\tau)$, with respect to the ones in (σ,τ). Now, the only element of $(L(\sigma),\tau)$ which might be critical and that was not already critical in (σ,τ) is the element ρ that has just been shifted by L. Thus, any critical element of $(L(\sigma),\tau)$ which is different from ρ satisfies the desired property. If ρ itself is critical in $(L(\sigma),\tau)$, we need to consider two cases.

If ρ is in the unique path γ' between $L(\sigma)_{i-1} = \sigma_{i-1}$ and $L(\sigma)_i = \sigma_i \cup \rho$, then

If ρ is not in γ', then we must have $\rho < \min\gamma'$ since $\rho < \min(\sigma_{i-1}\cup\sigma_i)$.

Thus, we must have $\rho < \min\gamma'$.

With this result in hand,

Thus, all minima of paths in $(L(\sigma),\tau)$ are traversed from left to right, and we have $(L(\sigma),\tau) \in \triangle$. □

2.17 ds v1

Proposition 2.19. *The cover relations of the poset of facets are precisely the pairs of the form $(\sigma,\tau) \lessdot (L(\sigma),\tau)$ and $(\sigma,\tau) \lessdot (\sigma, R(\tau))$ for some L and R.*

Proof. From the proof of Theorem 2.18, we know that for any facet $(\sigma,\tau) \in \triangle$, we have that the pairs $(L(\sigma),\tau)$ and $(\sigma,R(\tau))$ are indeed facets of $\triangle$.

We show first that the left shift operator creates inversions, that is, for any $(\sigma,\tau) \in \triangle$, we have that $(\sigma,\tau) \leqslant (L(\sigma),\tau)$. Observe that shifting a critical element to the left in σ cannot delete any inversion: if $x < y$ and x precedes y in σ, then x must precede y also in $L(\sigma)$. Now, we need to show that for $x < y$, if either y comes before x in σ, or both x and y are in the same block of σ, then y must come before x in τ. But this follows immediately from the (I,J)-condition for $I = \{x\}$ and $J = \{y\}$ defining $\triangle$. The result then follows, since ρ and $\max\sigma_{i+1}$ are in the same block of σ; thus $\max\sigma_{i+1}$ comes before ρ in τ and the pair $(\rho, \max\sigma_{i+1})$ is an inversion of $(L(\sigma),\tau)$, which is not an inversion of (σ,τ).

It remains to show that if $(\sigma,\tau) \leqslant (\sigma',\tau) \leqslant (L(\sigma),\tau)$, then we have either $\sigma = \sigma'$ or $\sigma' = L(\sigma)$. Indeed, if there is an inversion (x,y) of $(L(\sigma),\tau)$ which is not an inversion of (σ,τ), then it must be that $x = \rho$. To the contrary, we have $\sigma = \sigma'$, which completes the proof. □

Done in proof if $\rho \in \gamma$ and $\rho \neq \min\gamma$ (or see above)

One can thus interpret the left and right shift operators introduced by Saneblidze–Umble as generators of the poset of facets of the diagonal, with minimal elements the strong complementary pairs.

Now, we end with two important consequences of the preceding results. Consider the symmetry s of the $(n-1)$-dimensional permutahedron, which consists in the reflection with respect to the hyperplane $x_1 + x_n = x_2 + x_{n-1}$. It sends an ordered partition $\sigma := \sigma_1|\ldots|\sigma_k$ to the ordered partition $s\sigma := n - \sigma_k + 1|\ldots|n - \sigma_1 + 1$, where $n - \sigma_i + 1$ is the set $\{n - j + 1 \mid j \in \sigma_i\}$.

Corollary 2.20. *The diagonal action of the symmetry s of the permutahedron, sending a pair (σ,τ) to the pair $(s\sigma, s\tau)$, defines a bijection between the terms of the diagonal $\triangle$ and the terms of the Saneblidze–Umble diagonal.*

Proof. According to Theorem 2.18, it suffices to show that the symmetry s sends the SU diagonal to the dual SU diagonal. □

This allows us to prove a conjecture made in [Lap22, Remark 3.19].

Corollary 2.21. *The Saneblidze–Umble diagonal is given by the following choice of chambers in the fundamental hyperplane arrangement of the permutahedron: any vector $\vec{v}$ with strictly decreasing coordinates and which satisfy $\sum_{i\in I} v_i > \sum_{j\in J} v_j$ for all $I, J \subset \{1,\ldots,n\}$ such that $I \cap J = \varnothing, |I| = |J| \geqslant 2$ and $\max(I \cup J) \in I$ induces the Saneblidze–Umble diagonal on the $(n-1)$-dimensional standard permutahedron.*

Proof. These orientation vectors are obtained precisely from the ones defining $\triangle$ (see Section 1) via the symmetry described in Corollary 2.20 above [...]. □

This gives a geometric, alternative proof of the fact that all known diagonals of the associahedra agree [SU22].

Guillaume: Thus, (I,J)-description for the SU diagonal...

Guillaume: Now we have matrix representation for free!

2.5. **Comultiplicativity.** Guillaume: Conjectural

Here we show that $\triangle$ and the SU diagonal are isomorphic, and that they are the only two comultiplicative diagonals on the permutahedra that respect the weak Bruhat order on the vertices. (??)

The isomorphism of Corollary 2.20 is a comultiplicative isomorphism.

3. Vertices of the diagonal

We are now interested in characterizing the pairs of vertices that occur in the diagonal, that is pairs of permutations $(\sigma_1, \sigma_2) \in \triangle$.

Theorem 3.1. *There exists $(I, J) \in D(n)$ such that $\forall k, |\sigma_1^1 \cdots \sigma_1^k \cap I| \leqslant |\sigma_1^1 \cdots \sigma_1^k \cap J|$ and $\forall l, |\sigma_1^1 \cdots \sigma_1^l \cap I| \geqslant |\sigma_1^1 \cdots \sigma_1^l \cap J|$ (diagonal condition) if and only if $\exists (I', J') = (\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_m\}) \in D(m)$, $m \leqslant n$, such that*

$$\sigma_1 \cap (I' \cup J') = j_1 i_1 j_2 i_2 \cdots j_n i_n$$

and

$$\sigma_2 \cap (I' \cup J') = i_2 j_1 i_3 j_2 \cdots i_n j_{n-1} i_1 j_n \ ,$$

where $i_1 = \min(I' \cup J')$ (fish condition).

Proof.
- If a pair of permutations $(\sigma_1, \sigma_2) \in \mathfrak{S}_N^2$ satisfies the fish condition, then there exist two sets I and J of same cardinality such that $\min(I) < \min(J)$. Denoting σ_1 and σ_2 by two words of size N $\sigma_1^1 \ldots \sigma_1^N$ and $\sigma_2^1 \ldots \sigma_2^N$, then the pair $((\sigma_1, \sigma_2), (I, J))$ satisfies that for any k in $[\![1; N]\!]$, $|\sigma_1^1 \ldots \sigma_1^k \cap J| \geqslant |\sigma_1^1 \ldots \sigma_1^k \cap I|$ and $|\sigma_2^1 \ldots \sigma_2^k \cap I| \geqslant |\sigma_2^1 \ldots \sigma_2^k \cap J|$, hence the diagonal condition.
- We will now prove the converse. Let us presume that (σ_1, σ_2) is a pair of permutations satisfying the diagonal condition for a pair of sets $(I, J) \in D(n)$, minimal for the inclusion of sets.

Case $n = 1$:
If $|I| = |J| = 1$, then it follows directly from the diagonal condition above that $\sigma_{1|I\cup J} = j_1 i_1$ and $\sigma_{1|I\cup J} = i_1 j_1$, hence the fish condition is satisfied.

Case $n > 1$:
In this case, the proof is made by absurdum by considering the number of "well-placed" elements of I and J in σ_1 and σ_2. In what follows, for any set E, σ_i^E will stands for $(\sigma_i)_{|E}$. We write also $n_{i,k}^E$ for the number of elements of E in the k first letters of σ_i. The main argument in each of the small proofs below is the same: if the permutations do not satisfy the pattern described above, then it is possible to find an appropriate pair of elements $(i,j) \in I \times J$ such that $(I-i, J-j)$ satisfies the diagonal condition, hence contradicting the minimality of (I,J).

We first prove that the leftmost element of σ_1^I is i_1. Indeed, if it is not the case, we consider i, the leftmost element in σ_1^I and j the leftmost element in σ_2^J. The pair $(I-i, J-j)$ is in $D(n-1)$, as i is different from i_1. Moreover, it is clear that the diagonal condition still holds for $((\sigma_1, \sigma_2), (I,J))$. As this would contradict the minimality of (I,J), the leftmost element of σ_1^I is i_1.

We then prove that $\sigma_1^{I\cup J}$ starts by $j_1 i_1$ and that this j_1 is exactly the leftmost element in σ_2^J. On that purpose, we suppose that either i_1 is preceeded by several elements of J or that the unique element of J is not the leftmost one in σ_2^J. We then adapt the previous argument by choosing i to be the leftmost element in $\sigma_1^{I-\{i_1\}}$ and j the leftmost element in σ_2^J. The pair $(I-i, J-j)$ is in $D(n-1)$. Let us briefly explain while the diagonal condition would still be fulfilled in this case. If j is after i_1 in σ_1, then the difference $n_{1,k}^{J-j} - n_{1,k}^{I-i}$ is greater than $n_{1,k}^J - n_{1,k}^I$ for any k, hence is non negative. If j is before i_1 in σ_1, then by hypothesis, the difference $n_{1,k}^{J-j} - n_{1,k}^{I-i}$ is:

- strictly positive before i_1 an greater than 1 just before i_1
- non negative after i_1
- increase between i_1 and i
- is equal to $n_{1,k}^J - n_{1,k}^I$ after i,

hence is always non negative. Moreover, if i is after j in σ_2, the diagonal condition is clearly satisfied. If i is before j, then the difference $n_{2,k}^{I-i} - n_{1,k}^{J-j}$ is:

- strictly positive before j an greater than 1 just before j
- is equal to $n_{2,k}^I - n_{1,k}^J$ after j,

hence is always non negative. In short, if i_1 is preceeded by several elements of J or the unique element of J is not the leftmost one in σ_2^J, we obtain a contradiction with the minimality of (I,J).

Let us now consider the biggest $k \geqslant 1$ such that $\sigma_1^{I\cup J}$ begins with $j_1 i_1 j_2 i_2 \ldots j_k i_k$ and $\sigma_2^{I\cup J}$ begins with $i_2 j_1 i_3 j_2 \ldots i_k j_{k-1} w j_k$, where w is a word with letters in I. We want to show that $k = n$. Let us first remark that if $k = n$, $w = i_1$. If $1 \leqslant k < n$, then the sets $\tilde{I} = I - \{i_1, \ldots, i_k\}$ and $\tilde{J} = J - \{j_1, \ldots, j_k\}$ are non empty. Let us choose i_{k+1} to be the leftmost element in $\sigma_1^{\tilde{I}}$ and j_{k+1} the leftmost element in $\sigma_2^{\tilde{J}}$. We thus have $\sigma_1^{I\cup J} = j_1 i_1 j_2 i_2 \ldots j_k i_k w' i_{k+1} \ldots$, where w' is in J and $\sigma_2^{I\cup J} = i_2 j_1 i_3 j_2 \ldots i_k j_{k-1} w j_k w'' j_{k+1} \ldots$, where w and w' are words with letters in I. The pair $(I - i_{k+1}, J - j_{k+1})$ is in $D(n-1)$. Following the study as in the previous case, σ_1 always satisfies the diagonal condition for $(I - i_{k+1}, J - j_{k+1})$ and σ_2 satisfies it if and only if $w \neq i$. By minimality of (I, J), we then have $w = i_{k+1}$. If $k+1 = n$, we are done as the only possible word in J is j_{k+1}, hence $w' = j_{k+1}$. Otherwise, we can choose i_{k+2} to be the leftmost element in $\sigma_1^{\tilde{I} - i_{k+1}}$. Using the same reasoning as above, we show that $((\sigma_1, \sigma_2), (I - i_{k+2}, J - j_{k+1}))$ satisfies the diagonal condition if and only if $w' \neq j_{k+1}$. To sum up, the only possibility for (I, J) to be minimal is to have $k = n$, which implies the fish condition.

□

Corollary 3.2. *For any pair of permutations* $(\sigma_1, \sigma_2$*, there exists* $(I, J) \in D(n)$ *such that* $((\sigma_1, \sigma_2), (I, J))$ *satisfies the diagonal condition if and only if there exists* $(I', J') \in E(m)$*,* $m < n$ *such that* $((\sigma_1, \sigma_2), (I', J'))$ *satisfies the fish condition, with*

$$E(m) = \{(I, J) \in D(m) \,|\, \min(J) < \min(I - \min(I)), |[\![1;k]\!] \cap J| > |[\![1;k]\!] \cap I| \text{ if } |[\![1;k]\!] \cap J| \geqslant 2 \text{ and } I \subsetneq [\![1;k]\!]\} \tag{3.1}$$

Proof. It follows directly from the fish condition: if the fish condition is satisfied, as inversions of σ_1 are included in inversions of σ_2, we get $j_{k-1}, j_k < i_k$ for any $k > 1$. □

4. Tables

In this section we present low dimensional computations of the enumeration results obtained above and we connect them to other known combinatorial objects.

dim	**0**
0	1

dim	**0**	**1**
0	3	1
1	1	

dim	**0**	**1**	**2**
0	17	12	1
1	12	6	
2	1		

dim	**0**	**1**	**2**	**3**
0	149	162	38	1
1	162	150	24	
2	38	24		
3	1			

FIGURE 3. Number of pairs of faces in the cellular image of the diagonal 0, 1, 2 and 3-dimensional permutahedra.

Acknowledgements.

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dim	**0**	**1**	**2**	**3**	**4**
0	1809	2660	1080	110	1
1	2660	3540	1200	80	
2	1080	1200	270		
3	110	80			
4	1				

dim	**0**	**1**	**2**	**3**	**4**	**5**
0	28399	52635	30820	6165	302	1
1	52635	90870	67580	7785	240	
2	30820	47580	20480	2160		
3	6165	7785	2160			
4	302	240				
5	1					

Figure 4. Number of pairs of faces in the cellular image of the diagonal 4 and 5-dimensional permutahedra.

Pairs $(F, G) \in \operatorname{Im} \triangle_{(P,\vec{v})}$	**Polytopes**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	[OEI22]
	Associahedra	1	2	6	22	91	408	1938	A000139
$\dim F + \dim G = \dim P$	Multiplihedra	1	2	8	42	254	1678	11790	to appear
	Permutahedra	1	2	8	50	432	4802	65536	A007334
	Associahedra	1	3	13	68	399	2530	16965	A000260
$\dim F = \dim G = 0$	Multiplihedra	1	3	17	122	992	8721	80920	to appear
	Permutahedra	1	3	17	149	1809	28399	550297	A213507

Figure 5. Number of pairs of faces in the cellular image of the diagonal of the associahedra, multiplihedra and permutahedra of dimension $0 \leqslant \dim P \leqslant 6$, induced by any good orientation vector.

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Wish list

→ enumerate the other faces

→ Guillaume's conj (2.5)
(entre les 2 diags)

→ what about multiplihedra / associahedra?

Last Proof

THE COMBINATORICS OF THE PERMUTAHEDRON DIAGONALS

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GUILLAUME LAPLANTE-ANFOSSI, AND KURT STOECKL

Abstract. The purpose of this note is to provide a new combinatorial description of a cellular approximation of the diagonal of the permutahedra.

Introduction

This formula has many possible applications in algebraic topology: 1) iterated cobar construction 2) twisted tensor products 3) Fulton–Sturmfels formula for Losev–Manin spaces (compute explicitly the ring struture on the operational Chow ring -over $\mathbb{Z}$?)

Guillaume: Please make a new line for each sentence as to facilitate comparison between versions in GitHub

1. Preliminaries

1.1. **Original description of the diagonal.** Let us first set up some notations that will be of use throughout the paper. A set $\sigma_I := \bigcup_{i\in I} \sigma_i$ is a *partition* of $[n] := \{1, \ldots, n\}$ if $\bigcup_{i\in I} \sigma_i = [n]$ and $\sigma_i \cap \sigma_j = \varnothing$ for $i \neq j$. The subsets σ_i are called *blocks*. We denote by $|\sigma| := |I|$ the size of the partition (its number of blocks). A partition is *ordered* if the indexing set I is equipped with a total order; in what follows we shall use $I = [k]$ for $k \in \mathbb{N}$.

Let us recall the combinatorial formula for the cellular approximation of the diagonal of the permutahedra from [Lap22, Theorem 3.16]. Let $n \geqslant 1$, and let us write

$$D(n) := \{(I, J) \mid I, J \subset \{1, \ldots, n\}, |I| = |J|, I \cap J = \varnothing, \min(I \cup J) \in I\}.$$

Let $\vec{v} \in \mathbb{R}^n$ be such that $\forall (I, J) \in D(n)$, we have $\sum_{i\in I} v_i > \sum_{j\in J} v_j$, and let $P \subset \mathbb{R}^n$ denote the standard $(n-1)$-dimensional permutahedron. For any pair (σ, τ) of ordered partitions of $[n]$, we have

$$(\sigma,\tau) \in \operatorname{Im} \triangle_{(P,\vec{v})} \iff \begin{array}{l} \forall (I, J) \in D(n), \exists k \in [n], \left|\sigma_{[k]} \cap I\right| > \left|\sigma_{[k]} \cap J\right| \text{ or} \\ \exists l \in [n], \left|\tau_{[l]} \cap I\right| < \left|\tau_{[l]} \cap J\right| . \end{array}$$

We shall denote by $\triangle$ the set of pairs of ordered partitions of $[n]$ which satisfy the above condition.

There is an equivalent description of $\triangle$ which has the following form:

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2010 *Mathematics Subject Classification.* Primary [...]; Secondary 18M70.
Key words and phrases. Polytopes [...]

Proposition 1.1 ([Lap22])**.** *For a two ordered partitions $\sigma, \tau \subset [n]$, we have*

$$(\sigma,\tau) \in \triangle \iff \forall (I,J) \in D(\sigma,\tau), \exists k \in [n], \left|\sigma_{[k]} \cap I\right| > \left|\sigma_{[k]} \cap J\right| \ \textit{or}$$
$$\exists l \in [n], \left|\tau_{[l]} \cap I\right| < \left|\tau_{[l]} \cap J\right| \ .$$

Here, $D(\sigma,\tau) \subset D(n)$ is a proper subset of $D(n)$ which depends on the choice of (σ,τ), and comes from the geometry of the situation, see [Lap22, Theorem 1.26] for more details. For our present purposes, it will be enough to restrict our attention to facets of $\triangle$, that is pairs (σ,τ) which satisfy $|\sigma| + |\tau| = n+1$. In this case, $D(\sigma,\tau)$ has $n-1$ elements, and admits the following description.

For any subset $\sigma_i \subset [n]$, let $\vec{\sigma}_i \in \mathbb{R}^n$ denote the boolean vector whose coordinates are given by 1 in position j if $j \in \sigma_i$ and 0 otherwise. Given a facet (σ,τ) of $\triangle$, one can consider the system of equations $\langle \vec{\sigma}_i, x\rangle = 0$, $\langle \vec{\tau}_j, x\rangle = 0$ given by the blocks of both partitions. For geometric reasons (see the proof of [Lap22, Theorem 1.26]), the solution of this system is $x = 0$. Now we will be interested in the solutions of the systems associated to the pairs (σ',τ) and (σ,τ') where σ' (resp. τ') has been obtained from σ (resp. τ) by merging two adjacent blocks.

→ sol° of $Mx<0$ with $M := \begin{bmatrix}\sigma \\ \tau\end{bmatrix}$

Proposition 1.2. *There is a bijection between the set $D(\sigma,\tau)$ and the solutions to the systems of equations of the form (σ',τ) and (σ,τ').*

?

Proof. For any $z \in (\mathring{\sigma} + \mathring{\tau})/2$, the face $\tau \cap \rho_z\sigma$ of $P \cap \rho_z P$ is a vertex of the polytope $P \cap \rho_z P$. The faces of the form $\tau \cap \rho_z \sigma'$ and $\tau' \cap \rho_z\sigma$ are the edges of $P \cap \rho_z P$ which are adjacent to the vertex $\tau \cap \rho_z\sigma$. By definition $D(\sigma,\tau)$ describes the directions of these edges, and the translation is made as follows: for a given pair (I,J), define the corresponding direction $\vec{d}$ by its coordinates $d_i := 1$ if $i \in I$, $d_j := -1$ if $j \in J$, and $d_k := 0$ otherwise. We refer to [Lap22, Section 1.5] for more details. □

We will sometimes refer to the elements of $D(\sigma,\tau)$ as the *minimal (I,J)-pairs*.

Guillaume: Need to be rewritten in a more self-contained way, with more precise references?

Oui

2. Facets of the diagonal

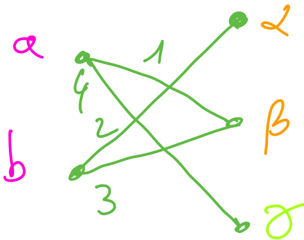
In this section we establish a bijection between the facets of $\triangle$ and a family of pairs of unordered partitions introduced and enumerated in a series of 3 papers [Che69, CG71, KUC82]. An intermediary bijection to a type of bipartite tree is of particular importance and provides [...]. In particular, we obtain that the number of facets in the image of the diagonal $\triangle_n$ of the n-dimensional permutahedron is $2(n+1)^{n-2}$ (A007334), and more precisely that the pairs of dimensions $(k, n-k)$ are counted by the formula $\frac{1}{k+1}\binom{n+1}{k}(k+1)^{n-k}(n+1-k)^k$. Guillaume: OEIS ref?

2.1. **Essential complementary partitions and bipartite trees.** Let us recall some basic definitions and results from the series of papers [Che69, CG71, KUC82].

Definition 2.1. A set of *distinct representatives* of a partition σ_I is a set $R \subset [n]$ such that $\forall i \in I, |\sigma_i \cap R| = 1$.

Exple : 14|23 × 4|13|2 ⚠ dim complementaire

F G



$F_1 = 1234$

$(F_1, G) \to \begin{pmatrix}1&1&1&1\\0&0&0&1\\1&0&1&0\\0&1&0&0\end{pmatrix}$ solut° : $X = \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$

(soit aller de a à b)

$G_1 = 134|2$ $(F, G_1) \to \begin{pmatrix}1&0&0&1\\0&1&1&0\\1&0&1&1\\0&1&0&0\end{pmatrix}$ solut° $\to$?

$G_2 = 4|123$ $(F, G_2) \to \begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&0&1\\1&1&1&0\end{pmatrix}$ solut° : $\to$

Voir cours p. fin janvier

(2|13, 1|23)
(3|12, 1|23)
(3|12, 2|13)

Definition 2.2. A pair of partitions (σ_I, τ_J) is said to be *complementary* if there exists $R \subset [n]$ and $r \in R$ such that R and $([n]\backslash R) \cup \{r\}$ are distinct representatives of σ_I and τ_J, respectively. It is furthermore *essential* if there does not exist proper subsets $I' \subset I$, $J' \subset J$ and $K \subset [n]$ such that $(\sigma_{I'}, \tau_{J'})$ is a complementary partition of K.

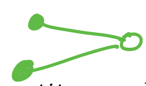
We shall denote the set of all essential complementary pairs of partitions by $\mathcal{E}$. Let us emphasize that the pairs of partitions of $\mathcal{E}$ are *unordered*.

Guillaume: Notations to be uniformized from here

Ça je prendrais en exercice

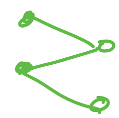
({1}{12}, {12}) + sym

Example 2.3. n=2, n=3

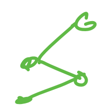
n=2  n=3  ({1}{12}{13}, {123}) ×2

Kurt: To do

A *tree* is a simply connected graph with no cycles. A *bipartite graph* is a graph whose vertices are partitioned into two sets such that vertices in one set are only adjacent to vertices in the other, we say it is *ordered* if one of the sets is considered smaller than the other and we denote the partition (V_L, V_R). We say a graph with n edges is *edge labelled* if there exists a bijection between the edges and $\{1, \dots, n\}$. Let $\mathcal{B}$ denote the set of edge labelled ordered bipartite trees.



({1}{123}, {12}{23})



Proposition 2.4 ([KUC82, Theorem 3])**.** *Essential complementary partitions and labelled bipartite trees are in bijection through $G : \mathcal{E} \to \mathcal{B}$ and $P : \mathcal{B} \to \mathcal{E}$, where*

- *G takes a pair (P_L, P_R) and constructs partitioned vertices (V_L, V_R). For each $i \in \{1, \dots, n\}$ an edge is added between v_l and v_r if $i \in P_l$ and $i \in P_r$.*
- *P takes a tree from $\mathcal{B}$ and labels the vertices (V_L, V_R) by the edges which are adjacent to them. The labels of the vertices can then be interpreted as a pair of partitions.*

({12}{3}, {1}{13})
({2}{13}, {1}{123})

Guillaume: Consider changing notation for functions vs sets

(to check!)

Corollary 2.5 ([KUC82])**.** *The number of essential complementary partitions is $|\mathcal{E}_n| = 2(n+1)^{n-2}$.*

({12}{3}, {13}{12})

2.2. **Bijection with the facets of the diagonal.** In this section we denote by $\triangle$ be the set of pairs of ordered partitions of $[n]$ labeling *facets* of the diagonal $\triangle$.

hum...

Theorem 2.6. *Facets of the diagonal and essential complementary partitions are in bijection through the inverse functions $u : \triangle \to \mathcal{E}$ and $o : \mathcal{E} \to \triangle$, where*

(1) *The function u forgets the order of the ordered partition pair.*
(2) *The function o uniquely orders an essential complementary partition pair via the minimal (I, J)-pairs defining the diagonal.*

We shall prove this theorem by establishing the necessary total order, showing that the functions are well defined, and then showing that they are injective.

Lemma 2.7. *The function $u : \triangle \to \mathcal{E}$ that forgets the order in a pair of partitions is well defined.*

Proof. Let $P \in \triangle_n$. Then $G(u(P))$ is a graph with $l + r = n + 1$ vertices, and n edges. Furthermore, as no vertices can be isolated it must be the case that this graph is a tree. It is straightforward to verify that $G(u(P))$ must be labeled bipartite tree, but here is how we

Je peux le faire aussi

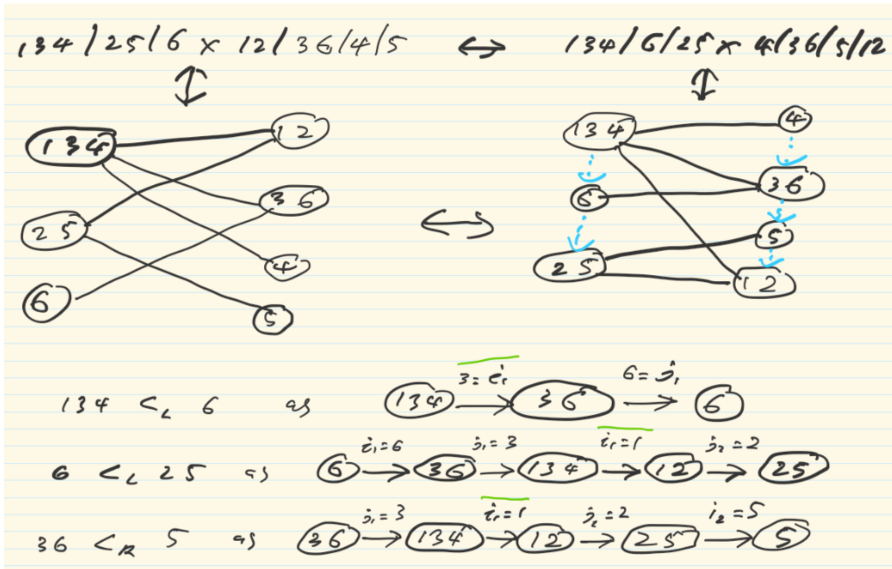


FIGURE 1. The bijection between ordered partitions and bipartite trees.

may explicitly produce the necessary distinct representatives using an algorithm of [KUC82, Theorem 2].

Let G' be a copy of $G(u(P))$. While there is a vertex of degree 1 in G' delete it and add the sole edge of that vertex as a distinct representative of the corresponding partition of that vertex. As G' is a tree this process can continue until there is a single edge connecting two vertices of degree 1. This edge specifies the element p of the distinct representatives. $\square$

Descri pro de l'ordre

Construction 2.8. *For $P = (P_L, P_R) \in \mathcal{E}$ an essential complementary pair, we construct total orders on P_L and P_R in three steps:*

(1) *For $l, l' \in L$ there exists a unique minimal set of edges $p_{l,l'}$ of even cardinality connecting $V(P_l)$ and $V(P_{l'})$ in $G(P)$ (similar for R). We partition this set of edges as $I \cup J$ where I and J are each pairwise non-adjacent, and I contains the minimal edge.*

(2) *Orient each path so that I points left to right, and J points right to left (same orientation for P_L and P_R).*

(3) *We say $P_l < P_{l'}$ (or $P_r < P_{r'}$) if the constructed path points from $V(P_l) \to V(P_{l'})$ $(V(P_r) \to V(P_{r'})$.*

Proof. We first show our binary relation is well defined before verifying that it defines a total order on $G(P)$ and hence P via the bijection of Proposition 2.4.

As $G(P)$ is a bipartite tree, every vertex is connected, and every path connecting two vertices on the same side must be of even length. As I and J are each pairwise non-adjacent, they must partition the path in an alternating fashion i.e. $p = (I_{i_1}, J_{j_1}, I_{i_2}, J_{j_2}, ...,)$, hence we can orient the path by forcing I to point left and J to point right.

This order is clearly total, reflexive (by convention) and anti-symmetric, what remains to be checked is its transitivity.

Let p_{ab} denote the unique maximal path between two vertices a and b on the left of $G(P)$, that is two blocks of P_L. Let I_{ab} denote the set of left-to-right edges in this path, and let J_{ab} denote its complement. Then, we have

$$a < b \iff \min(I_{ab} \cup J_{ab}) = \min(I_{ab}) \ . \tag{2.1}$$

Suppose now that $a < b$ and $b < c$. Since $p_{ac} = (p_{ab} \cup p_{bc}) \backslash (p_{ab} \cap p_{bc})$, we have

$$I_{ac} = (I_{ab} \cup I_{bc}) \backslash (J_{ab} \cup J_{bc}) \text{ and } J_{ac} = (J_{ab} \cup J_{bc}) \backslash (I_{ab} \cup I_{bc}) \ ,$$

and from the condition (2.1) above it is clear that $\min(I_{ac} \cup J_{ac}) = \min(I_{ac})$, which completes the proof of the transitivity for the total order on P_L. The proof for P_R is similar. □

This order far from being arbitrary provides the unique way to order an essential complementary partition pair into an ordered partition pair of $\triangle$, as we shall demonstrate next.

First we need a geometrical lemma.

Proposition 2.9. *The paths between adjacents vertices of P_L or P_R are in bijection with the minimal (I, J)-pairs.*

Proof. By Proposition 1.2, it suffices to show that the paths between adjacent vertices of P_L are in bijection with the solutions of the system of equations of the form (ρ^1, σ^2). To ease notation let us write ρ for ρ^1 and σ for σ^1. Suppose that ρ is obtained from σ by merging the two blocks σ_a and σ_b. The two equations $\langle \vec{\sigma}_a, x \rangle = 0$ and $\langle \vec{\sigma}_b, x \rangle = 0$ now become $\langle \vec{\sigma}_a + \vec{\sigma}_b, x \rangle = 0$; nothing else changes in the system. Since the solution to the system (σ^1, σ^2) was $x = 0$, now the solution is of dimension 1, and it is given precisely by the path between a and b in $G(P)$. Such a path is given by an alternating sequence of vertices and edges $\sigma_1 := \sigma_a, e_1, \sigma_2, e_2, \ldots, e_{k-1}, \sigma_k := \sigma_b$. Every edge $e_i \in \{1, \ldots, n\}$ is by definition the intersection $\sigma_i \cap \sigma_{i+1}$; thus it is the only common non-zero coordinate between $\vec{\sigma}_i$ and $\vec{\sigma}_{i+1}$. Thus the path encodes the series of equations $x_{e_1} + x_{e_{k-1}} = 0$, $x_{e_1} + x_{e_2} = 0$, $x_{e_2} + x_{e_3} = 0$, …, $x_{e_{k-2}} + x_{e_{k-1}} = 0$. Thus, $x_{e_1} = 1$, $x_{e_2} = -1$, $x_{e_3} = 1$, …, $x_{e_{k-2}} = 1$, $x_{e_{k-1}} = -1$ is a basis of one-dimensional space of solutions, and it gives the corresponding minimal (I, J)-pair. □

Pourquoi?

Lemma 2.10. *The function $o : \mathcal{E} \to \triangle$ that orders an essential complementary pair is well defined.*

Proof. Let $P = (P_L, P_R) \in \mathcal{E}$ and consider $o(P)$. We first show that every (I, J)-condition, for $(I, J) \in D(n)$, which corresponds to a path between vertices is satisfied. In particular, this statement will be true for minimal (I, J)-pairs, which will be enough in virtue of Proposition 1.1. Suppose I, J corresponds to a path between two vertices on the left, i.e.

à lire

$$V(P_l) = V_{L_1} \xrightarrow{i_1} V_{R_1} \xrightarrow{j_1} V_{L_2} \xrightarrow{i_2} \ldots \xrightarrow{i_k} V_{R_{k-1}} \xrightarrow{j_k} V_{L_k} = V(P_{l'})$$

By construction we have that $I = \{i_1, ..., i_k\}$, $J = \{j_1, ..., j_k\} \in D(n)$ (note we are ordering I and J by the path, so it is not necessarily the case that $\min I = i_1$). Furthermore, each sub partition of P_R either contains a single element of I and a single element of J, or it contains no elements of I and no elements of J. As such for any ordering of the sub-partitions of P_R we have that

$$\forall m, \left| \bigcup_{1\leqslant k\leqslant m} P_{R,k} \cap I \right| = \left| \bigcup_{1\leqslant k\leqslant m} P_{R,k} \cap J \right|$$

Hence in order for this $D(n)$ condition to be satisfied it must be the case that for some ordering of the sub-partitions of P_L we have

$$\exists m, \left| \bigcup_{1\leqslant k\leqslant m} P_{L,k} \cap I \right| > \left| \bigcup_{1\leqslant k\leqslant m} P_{L,k} \cap J \right|$$

Every sub-partition of P_L excluding the lth and l'th either contains no elements of both I and J, or it contains a single element of I and a single element of J. So the only way for the condition to be satisfied is for P_l to come before $P_{l'}$, which is precisely what is required by the total order.

If I, J correspond to a path between two vertices on the right,

$$V(P_r) = V_{R_1} \xrightarrow{j_1} V_{L_1} \xrightarrow{i_1} V_{L_2} \xrightarrow{j_2} \dots \xrightarrow{j_k} V_{L_{k-1}} \xrightarrow{1_k} V_{R_k} = V(P_{r'})$$

then a similar chain of logic implies we must have an ordering of the sub-partitions of P_R such that

$$\exists m, | \bigcup_{1\leqslant k\leqslant m} P_{R,k} \cap I| < | \bigcup_{1\leqslant k\leqslant m} P_{R,k} \cap J|$$

and this can only happen if P_r comes before $P_{r'}$. □

Remark 2.11. It would be interesting to know if there is a geometrical interpretation of the paths that are not between adjacent vertices.

To complete the proof of Theorem 2.6, it remains to show that both $u : \triangle \to \mathcal{E}$ and $o : \mathcal{E} \to \triangle$ are injective, with the other function being their inverse.

Proof of Theorem 2.6. The forgetful function u is clearly the inverse to o as forgetting any assigned order will clearly return the original essential complementary partition pair. The ordering function o is the inverse to u as it returns the sole ordering of the sub-partitions which is compatible with the $D(n)$ conditions. □

2.3. **Combinatorial formula for facets of the diagonal.** From Theorem 2.6, we can deduce a formula for the number of facets of the diagonal:

Proposition 2.12. *The number of pairs of ordered partitions of dimension $(k, n-k)$ which correspond to facets of the diagonal is given by:*

$$\frac{1}{k+1}\binom{n+1}{k}(k+1)^{n-k}(n+1-k)^k. \tag{2.2}$$

Proof. According to Theorem Theorem 2.6, pairs of ordered partitions of dimension $(k, n-k)$ which correspond to facets of the diagonal are in one-to-one correspondence with bipartite trees with $k+1$ black vertices, $n-k+1$ white vertices and $n+1$ edges labeled from 1 to $n+1$.

We do not prove exactly here the proposition but a slightly modified version: Rooted bipartite trees with $k+1$ black vertices and $n-k+1$ white vertices such that:

- a black vertex is distinguished and called *the root*
- the $n+1$ non-root vertices are labeled,
- every label between 1 and $n+1$ is used exactly once.

are counted by:

$$\binom{n+1}{k}(k+1)^{n-k}(n+1-k)^k. \tag{2.3}$$

Let us construct such a bipartite tree.

First, there are $\binom{n+1}{k}$ ways to choose the labels for black vertices (white vertices being labeled by the non-chosen labels). We denote by $\mathcal{B}$ this set of labels.

Moreover, the labeled black vertices are different from the root, hence they should have a white parent : there are $n+1-k$ ways to choose the parent of any labeled black vertex. We thus have $(n+1-k)^k$ ways to build corollas with labeled black leaves and a white root, called bi-colored corollas (or sometimes just corollas) in the sequel.

Finally, we arrange bi-colored corollas in a rooted bipartite tree by adapting the algorithm which convert a Prüfer code to a tree. Here what is called *Prüfer code* is a word of length $n-k$ over the alphabet $\mathcal{B} \cup \{\bullet\}$, where $\bullet$ stands for the non-labeled black vertex. Let us start with a word $c = c_1 \dots c_{n-k}\bullet$ of length $n-k+1$ and the set $\mathcal{T} = \mathcal{S} \cup \{\bullet\}$ of $n-k+2$ bi-colored corollas augmented with the unlabeled black vertex. We apply Algorithm 1. Let us first prove it termination and correctness. The equality $\text{length}(c) = \text{Card}(\mathcal{T}) - 1$ is a loop invariant for the While loop: indeed at each iteration of the loop, the length of c and the number of elements in $\mathcal{T}$ decrease exactly by one. It ensures the termination of the loop and the fact that $\mathcal{T}$ contains a unique element when exiting the loop. Moreover, the set of trees $\mathcal{T}$ contains at each steps exactly one unlabeled black vertex, k labeled black vertices and $n-k+1$ white vertices. Finally, when adding an edge between two trees, one can only get a tree. Moreover, as the edge is added between a white root and the label of a black vertex, the obtained tree is indeed bipartite.

To prove that this algorithm defines a bijection between the pairs of Prüfer code and set of bipartite rooted trees, let us give the algorithm which convert a rooted bipartite tree in such a pair in Algorithm .

This second algorithm terminates as the number of white vertices decreases strictly by one at each iterations. Moreover, every letter added to c is the label of a black vertices, and every tree added to $\mathcal{T}$ is a bi-colored corolla or $\bullet$ (the tree with only the non-labeled black root). Finally, the cardinality of the set of bipartite trees is one more than the length of c. As this algorithm is the classical reverse algorithm of the first one, it ends the proof. Bérénice: Do I need to add more details ? □

Example 2.13. Bérénice: Prüfer algorithm applied on an example : TO DO

Algorithm 1: Prüfer algorithm : from a word to a tree

Input: a word $c = c_1 \dots c_i$ and a set $\mathcal{T}$ of i bi-colored trees with white root and one bi-colored tree with an unlabeled black root
Output: a bipartite rooted tree

while length$(c) > 0$ **do**
- /* loop invariant: $\mathrm{length}(c) = \mathrm{Card}(\mathcal{T}) - 1$, at each iteration, the length of c decreases by 1 */
- t $\leftarrow \min\{a \in \mathcal{T} |$ none of the c_i is a label in $a\}$ // Here the order is given by the order on the labels of the root (as the tree with a black root does not satisfy the condition)
- p $\leftarrow$ tree of $\mathcal{T}$ to which belongs the first letter of c // Note that it cannot be t itself
- Remove t and p from $\mathcal{T}$ // Decrease the cardinality of $\mathcal{T}$ by two
- Add an edge between the root of t and the first letter of c and add the obtained tree to $\mathcal{T}$ // Increase the cardinality of $\mathcal{T}$ by one
- Remove the first letter of c // Decrease the length of c by one

Return the unique element of $\mathcal{T}$

Algorithm 2: Prüfer algorithm : from a tree to a word

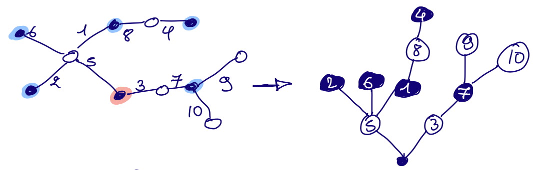
Input: a bipartite rooted tree A
Output: a word $c = c_1 \dots c_i$ and a set $\mathcal{T}$ of i bi-colored trees with white root, except one bi-colored tree with an unlabeled black root

$c \leftarrow$ empty word //
$\mathcal{T} \leftarrow$ empty set // Initialization
while *A has more than one vertex* **do**
- /* At each iteration, the number of white vertices decreases by 1 */
- t $\leftarrow \min\{w \in \mathcal{T} | w$ is a white vertex whose children are leaves $\}$ // Here the order is the one on white labels
- $c \leftarrow c$ concatenated with label of the parent of t // This label is a black vertex
- Remove the edge between t and its parent: the root part goes in A and the corolla in $\mathcal{T}$ // Increase the cardinality of $\mathcal{T}$ by one

Return the pair $(c, \mathcal{T} \cup A)$

Guillaume: New section? Yes

2.4. **The Saneblidze–Umble description.** Here we prove that the diagonal $\triangle$ admits a combinatorial description completely analogous to that of the Saneblidze–Umble diagonal [SU04]. A direct corollary is that the two diagonals are in bijection, and moreover that the SU diagonal can be obtained from a certain choice of chambers in the fundamental hyperplane arrangements of the permutahedra, resolving a conjecture made in [Lap22]. In



$\binom{n+1}{k}$

↳ choix des n^os des sommets noirs

(et des sommets blancs du coup)

Code de Prüfer

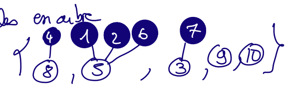
1 → 5
2 → 5
4 → 8
6 → 5
7 → 3

$(n-k+1)$ choix

⇝ $(n+1-k)^k$

Reste à assembler ces corolles en arbre

1 • 77



à écrire au propre

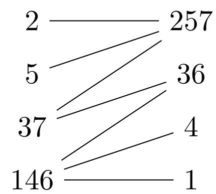
particular, this provides an alternative proof that all known diagonals on the associahedra agree [SU22].

Given any permutation, one can canonically produce a facet of the diagonal in the following way.

Definition 2.14. A pair of ordered partitions (σ, τ) is called *strong complementary* if σ is obtained from a permutation by merging the adjacent elements which are decreasing, and τ is obtained from the same permutation by merging the adjacent elements which are increasing.

An example is shown in Figure 2. The same argument as in Lemma 2.7 shows that the underlying pair of partitions is an essential complementary pair. Moreover we see directly that any path between adjacent vertices has length 2, and that its minimum is always traversed from left to right, hence all strong complementary pairs are in $\triangle$.

$2|5|37|146 \times 257|36|4|1$



257|36|4|1
2|5|37|146
permutation 2573641 ?

FIGURE 2. The strong complementary pair associated to the ~~partition~~ permutation 2|5|7|3|6|4|1.

There is a natural partial order on the facets of $\triangle$. For two elements $x, y \in [n]$, we say that the set $\{x, y\}$ is an *inversion* of a facet (σ, τ) if we have that $x < y$, the element x appears before y in σ, and the element y appears before x in τ.

Definition 2.15. We say that two facets $(\sigma, \tau) \leqslant (\sigma', \tau')$ are comparable if the set of inversions of (σ, τ) is contained in the set of inversions of (σ', τ'). + example

It is immediate to see that this defines a poset. Guillaume: and even a lattice if a maximal and a minimal element are added?

Proposition 2.16. *The set of inversions of a facet is in bijection with its set of edge crossings. Moreover, the set of facets with no crossings is the set of strong complementary pairs.*

Proof. For the first part of the statement, it is clear that every inversion gives rise to a crossing. For the converse, one needs to check that an anti-inversion, where y appears before x in σ and x appears before y in τ, cannot occur in a facet of $\triangle$; this follows immediately from the (I, J)-conditions for $|I| = |J| = 1$. The second part of the statement follows from the fact that facets of the diagonal with no crossings are in bijection with permutations. As explained above, from a partition one obtains a strong complementary pair, which is in $\triangle$. In the other way around, given a strong complementary pair, one can read-off the partition in the associated tree, which has no crossings, by going along the edges from top to bottom, see Figure 2. □

We aim now at characterizing the cover relations of this poset. Let (σ, τ) be a facet of $\triangle$. We say that a pair of adjacent blocks $\sigma_i|\sigma_{i+1}$ of σ is *admissible* if there is an element $\rho \in (\sigma_{i+1} \backslash \max \sigma_{i+1})$ such that $\rho < \min \sigma_i$ and $\rho < \min \gamma$ for γ the unique path between σ_i and σ_{i+1}. Such an element ρ is said to be *critical*. Let us define the *left shift operator* L, which takes an admissible pair $\sigma_i|\sigma_{i+1}$ in σ and creates a new ordered partition $L(\sigma)$ where the critical element ρ is shifted one block to the left: we have $L(\sigma)_i := \sigma_i \cup \rho$, $L(\sigma)_{i+1} := \sigma_{i+1} \backslash \rho$, and $L(\sigma)_j := \sigma_j$ for all $j \neq i, i+1$.

Expl : n=2

$n = 3$

In the same fashion, the *right shift operator* R sends an element $\rho \in (\tau_i \backslash \max \tau_i)$ with $\rho < \min \tau_{i+1}$ and $\rho < \min \gamma$ one block to the right; the resulting ordered partition $R(\tau)$ is such that $R(\tau)_i := \tau_i \backslash \rho$ and $R(\tau)_{i+1} := \tau_{i+1} \cup \ \rho$.

Definition 2.17. The facets of the *dual SU diagonal* are the pairs or ordered partitions that can be obtained from an strong complementary pair (σ, τ) by iterated applications of the left shift operator L on the first term σ, and iterated applications of the right shift operator R on the second term τ.

We shall show shortly that this definition recovers a dual of the SU diagonal on the permutahedra [SU04].

Lemma 2.18. *Applying the left or right shift to a pair* $(\sigma, \tau) \in \triangle$ *does not change the minima of the paths between adjacent vertices.*

Proof. We give the argument for the left shift, the one for the right one is similar. We observe that the paths in $(L(\sigma), \tau)$ are precisely the following: either they do not contain ρ, in which case they are paths of (σ, τ); or they do contain ρ, in which case they are obtained from paths in (σ, τ) by inserting at some place the unique path γ between σ_i and σ_{i+1}, or its inverse. Since $\rho < \min \gamma$, adding the path γ does not change the minima of the paths in $(L(\sigma), \tau)$, with respect to the ones in (σ, τ). □

Theorem 2.19. *The facets of the diagonal* $\triangle$ *and the facets of the dual SU diagonal are equal.*

Proof. Since we consider only the action of L on the first term of the pair (σ, τ) and the action of R on the second term, we will prove statements for L, the ones for R are similar.

First we show that every facet of $\triangle$ is a facet of the dual SU diagonal. Let (σ, τ) be a pair in $\triangle$ and suppose that it satisfies the following property: for all pairs of consecutive blocks $\sigma_i|\sigma_{i+1}$ in σ, if $\min \sigma_i < \max \sigma_{i+1}$, then the unique path γ between σ_i and σ_{i+1} has length 2 and is given by $\{\min \sigma_i, \max \sigma_{i+1}\}$. In particular, we have $\min \sigma_i \geqslant \min \gamma$. We claim that (σ, τ) must be a strong complementary pair. To see this, suppose that (σ, τ) is *not* a strong complementary pair; therefore by Proposition 2.16 there exists an edge crossing. This implies that there is a minimal crossing, i.e. a crossing between edges adjacent to two consecutive blocks $\sigma_i|\sigma_{i+1}$. Since edges that are incident to a block are always in increasing order from bottom to top (this is a direct consequence of the (I, J)-conditions for $|I| = |J| = 1$), there is a crossing between $\min \sigma_i$ and $\max \sigma_{i+1}$, which contradicts the property assumed above. So, there is no crossing in (σ, τ) and according to Proposition 2.16, we have that (σ, τ) is a strong complementary pair. The proof of the inclusion is now complete: if we are given a pair of facets (σ, τ) in $\triangle$ which has at least one crossing, then there is a pair of consecutive blocks $\sigma_i|\sigma_{i+1}$ such that $\min \sigma_i < \max \sigma_{i+1}$ and $\min \sigma_i < \min \gamma$, and one can apply the inverse operator L^{-1} shifting $\min \sigma_i$ to the right; and by induction we obtain a facet of the dual SU diagonal.

Second, we show that every facet of the dual SU diagonal is in $\triangle$. We already know that strong complementary pairs are in $\triangle$. Thus, it suffices to prove that if a dual SU pair (σ, τ) is in $\triangle$, then $(L(\sigma), \tau)$ is also in $\triangle$. Lemma 2.18 shows that the minima of paths between consecutive vertices in $(L(\sigma), \tau)$ are the same as the ones in (σ, τ). Thus, all minima of paths in $(L(\sigma), \tau)$ are traversed from left to right, and we have $(L(\sigma), \tau) \in \triangle$. $\square$

Proposition 2.20. *The cover relations of the poset of facets are precisely the pairs of the form $(\sigma, \tau) \lessdot (L(\sigma), \tau)$ and $(\sigma, \tau) \lessdot (\sigma, R(\tau))$ for some L and R.*

Proof. From the proof of Theorem 2.19, we know that for any facet $(\sigma, \tau) \in \triangle$, we have that the pairs $(L(\sigma), \tau)$ and $(\sigma, R(\tau))$ are indeed facets of $\triangle$.

We show first that the left shift operator creates inversions, that is, for any $(\sigma, \tau) \in \triangle$, we have that $(\sigma, \tau) \leqslant (L(\sigma), \tau)$. Observe that shifting a critical element to the left in σ cannot delete any inversion: if $x < y$ and x precedes y in σ, then x must precede y also in $L(\sigma)$. Now, we need to show that for $x < y$, if either y comes before x in σ, or both x and y are in the same block of σ, then y must come before x in τ. But this follows immediately from the (I, J)-condition for $I = \{x\}$ and $J = \{y\}$ defining $\triangle$. The result then follows, since ρ and $\max \sigma_{i+1}$ are in the same block of σ; thus $\max \sigma_{i+1}$ comes before ρ in τ and the pair $(\rho, \max \sigma_{i+1})$ is an inversion of $(L(\sigma), \tau)$, which is not an inversion of (σ, τ).

It remains to show that if $(\sigma, \tau) \leqslant (\sigma', \tau) \leqslant (L(\sigma), \tau)$, then we have either $\sigma = \sigma'$ or $\sigma' = L(\sigma)$. Indeed, if there is an inversion (x, y) of $(L(\sigma), \tau)$ which is not an inversion of (σ, τ), then it must be that $x = \rho$. To the contrary, we have $\sigma = \sigma'$, which completes the proof. $\square$

Now we give the definition of the Saneblidze–Umble diagonal [SU04], following the description below Example 1 in [SU22], and replace "max" with "min" and the symbol ">" with the symbol "<".

Let (σ, τ) be a pair of ordered partitions. We say that a pair of adjacent blocks $\sigma_i|\sigma_{i+1}$ of σ is *SU admissible* if there is a non-empty subset $\rho \subset (\sigma_{i+1} \backslash \max \sigma_{i+1})$ such that $\max \rho < \min \sigma_i$. Let us define the *subset left shift operator* L^i, which takes an SU admissible pair $\sigma_i|\sigma_{i+1}$ in σ and creates a new ordered partition $L^i(\sigma)$ where the subset ρ is shifted one block to the left: we have $L^i(\sigma)_i := \sigma_i \cup \rho$, $L^i(\sigma)_{i+1} := \sigma_{i+1} \backslash \rho$, and $L^i(\sigma)_j := \sigma_j$ for all $j \neq i, i+1$. In the same fashion, the *subset right shift operator* R^i sends a subset $\rho \subset (\tau_i \backslash \max \tau_i)$ with $\max \rho < \min \tau_{i+1}$ one block to the right; the resulting ordered partition $R^i(\tau)$ is such that $R^i(\tau)_i := \tau_i \backslash \rho$ and $R^i(\tau)_{i+1} := \tau_{i+1} \cup \rho$.

Definition 2.21 (Dual SU diagonal, second definition)**.** The facets of the dual SU diagonal are the pairs of ordered partitions of the form $(L^{i_k} \cdots L^{i_1}(\sigma), R^{j_l} \cdots R^{j_1}(\tau))$ obtained from a strong complementary pair (σ, τ) by iterated application of subset left and right shifts operators, where moreover $i_1 > \cdots > i_k$ is decreasing and $j_1 < \cdots < j_l$ is increasing.

Lemma 2.22. *Applying the subset left or right shift to a pair (σ, τ) in the dual SU diagonal does not change the minima of the paths between adjacent vertices.*

Proof. We analyse the left shift operator, the case of the right shift is similar. First, we observe that any critical subset $\rho \subset L^{i_k} \cdots L^{i_1}(\sigma)_j, j \leqslant i_k$ satisfies $\max \rho < \min \gamma$, where γ is the unique path between $L^{i_k} \cdots L^{i_1}(\sigma)_{j-1} = \sigma_{j-1}$ and $L^{i_k} \cdots L^{i_1}(\sigma)_j$. Indeed, the path γ is the same as the path between σ_{j-1} and σ_j in (σ, τ), and is equal to $\{\min \sigma_{j-1}, \max \sigma_j\}$; but since ρ is critical we must have $\max \rho < \min \sigma_{j-1} = \min \gamma$. The rest of the proof is the same as for Lemma 2.18, with ρ replaced by $\max \rho$. $\square$

Proposition 2.23. *The two definitions of the dual SU diagonal coincide.*

Proof. We analyse the left shift operator, the case of the right shift is similar. First, we observe that any subset left shift $L^i(\sigma)$ can be decomposed in a series of left shifts: since $\max\rho < \min\gamma$ by the proof of Lemma 2.22, we can first shift $\max\rho$ to the left, then $\max(\rho\setminus\max\rho)$, and so on until the entire subset ρ has been shifted to the left. This shows that any facet in the second definition of the dual SU diagonal is also a facet in the first definition.

For the reverse inclusion, we show by induction on the number of left shifts that a pair (σ', τ) where σ' has been obtained by iterated left shifts can also be obtained by iterated subset left shifts, that is, can be written in the form $(L^{i_k}\cdots L^{i_1}(\sigma), \tau)$. The base case of a strong complementary pair is trivial. Suppose that we are given a pair (σ', τ) where σ' has been obtained by a sequence of $l+1$ left shifts. By the induction hypothesis, the first l left shifts can be rewritten as a sequence of subset left shifts, i.e. in the form $L^{i_k}\cdots L^{i_1}(\sigma)$ where $i_1 > \cdots > i_k$ are descreasing. Now suppose that the $(l+1)$-th left shift occurs between $L^{i_k}\cdots L^{i_1}(\sigma)_j$ and $L^{i_k}\cdots L^{i_1}(\sigma)_{j+1}$, with $j > i_k$ (otherwise, we are done!). Let us denote by γ the unique path between $L^{i_k}\cdots L^{i_1}(\sigma)_j$ and $L^{i_k}\cdots L^{i_1}(\sigma)_{j+1}$. By definition, we have that the critical element ρ satisfies $\rho < \min\gamma$. But by Lemma 2.22, this minimum is the same as the minimum of the path between σ_j and σ_{j+1}, which is just $\min\sigma_j$. Thus we have $\rho < \min\sigma_j$ (as well as $\rho < \min L^{i_k}\cdots L^{i_1}(\sigma)_j$, by criticality), and so ρ can be integrated in a new or an existing subset left shift of the family $L^{i_k}\cdots L^{i_1}$, which completes the proof. $\square$

One can thus interpret the left and right shift operators for singletons introduced by Saneblidze–Umble as generators of the poset of facets of the diagonal, with minimal elements the strong complementary pairs.

Now, we end with two important consequences of the preceding results. Consider the symmetry s of the $(n-1)$-dimensional permutahedron, which consists in the reflection with respect to the hyperplane $x_1 + x_n = x_2 + x_{n-1}$. It sends an ordered partition $\sigma := \sigma_1|\ldots|\sigma_k$ to the ordered partition $s\sigma := n - \sigma_k + 1|\ldots|n - \sigma_1 + 1$, where $n - \sigma_i + 1$ is the set $\{n - j + 1 \mid j \in \sigma_i\}$.

Corollary 2.24. *The diagonal action of the symmetry s of the permutahedron, sending a pair (σ, τ) to the pair $(s\sigma, s\tau)$, defines a bijection between the terms of the diagonal $\triangle$ and the terms of the Saneblidze–Umble diagonal.*

Proof. According to Theorem 2.19, it suffices to show that the symmetry s sends the SU diagonal to the dual SU diagonal. $\square$

This allows us to prove a conjecture made in [Lap22, Remark 3.19].

Corollary 2.25. *The Saneblidze–Umble diagonal is given by the following choice of chambers in the fundamental hyperplane arrangement of the permutahedron: any vector $\vec{v}$ with strictly decreasing coordinates and which satisfy $\sum_{i\in I} v_i > \sum_{j\in J} v_j$ for all $I, J \subset \{1,\ldots,n\}$ such that $I \cap J = \emptyset, |I| = |J| \geqslant 2$ and $\max(I \cup J) \in I$ induces the Saneblidze–Umble diagonal on the $(n-1)$-dimensional standard permutahedron.*

Proof. These orientation vectors are obtained precisely from the ones defining $\triangle$ (see Section 1) via the symmetry described in Corollary 2.24 above [...]. $\square$

This gives a geometric, alternative proof of the fact that all known diagonals of the associahedra agree [SU22].

Guillaume: Thus, (I, J)-description for the SU diagonal...

Guillaume: Now we have matrix representation for free!

2.5. Comultiplicativity. Guillaume: Conjectural

Here we show that $\triangle$ and the SU diagonal are isomorphic, and that they are the only two comultiplicative diagonals on the permutahedra that respect the weak Bruhat order on the vertices. (??)

The isomorphism of Corollary 2.24 is a comultiplicative isomorphism.

3. Vertices of the diagonal

We are now interested in characterizing the pairs of vertices that occur in the diagonal, that is pairs of permutations $(\sigma_1, \sigma_2) \in \triangle$.

Theorem 3.1. *There exists $(I, J) \in D(n)$ such that $\forall k, |\sigma_1^1 \cdots \sigma_1^k \cap I| \leqslant |\sigma_1^1 \cdots \sigma_1^k \cap J|$ and $\forall l, |\sigma_1^1 \cdots \sigma_1^l \cap I| \geqslant |\sigma_1^1 \cdots \sigma_1^l \cap J|$ (diagonal condition) if and only if $\exists (I', J') = (\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_m\}) \in D(m)$, $m \leqslant n$, such that*

$$\sigma_1 \cap (I' \cup J') = j_1 i_1 j_2 i_2 \cdots j_n i_n$$

and

$$\sigma_2 \cap (I' \cup J') = i_2 j_1 i_3 j_2 \cdots i_n j_{n-1} i_1 j_n \ ,$$

where $i_1 = \min(I' \cup J')$ (fish condition).

Proof.
- If a pair of permutations $(\sigma_1, \sigma_2) \in \mathfrak{S}_N^2$ satisfies the fish condition, then there exist two sets I and J of same cardinality such that $\min(I) < \min(J)$. Denoting σ_1 and σ_2 by two words of size N $\sigma_1^1 \ldots \sigma_1^N$ and $\sigma_2^1 \ldots \sigma_2^N$, then the pair $((\sigma_1, \sigma_2), (I, J))$ satisfies that for any k in $[\![1; N]\!]$, $|\sigma_1^1 \ldots \sigma_1^k \cap J| \geqslant |\sigma_1^1 \ldots \sigma_1^k \cap I|$ and $|\sigma_2^1 \ldots \sigma_2^k \cap I| \geqslant |\sigma_2^1 \ldots \sigma_2^k \cap J|$, hence the diagonal condition.
- We will now prove the converse. Let us presume that (σ_1, σ_2) is a pair of permutations satisfying the diagonal condition for a pair of sets $(I, J) \in D(n)$, minimal for the inclusion of sets.

  Case $n = 1$:
  If $|I| = |J| = 1$, then it follows directly from the diagonal condition above that $\sigma_{1|I \cup J} = j_1 i_1$ and $\sigma_{1|I \cup J} = i_1 j_1$, hence the fish condition is satisfied.

  Case $n > 1$:
  In this case, the proof is made by absurdum by considering the number of "well-placed" elements of I and J in σ_1 and σ_2. In what follows, for any set E, σ_i^E will stands for $(\sigma_i)_{|E}$. We write also $n_{i,k}^E$ for the number of elements of E in the k first letters of σ_i. The main argument in each of the small proofs below is the same: if the permutations do not satisfy the pattern described above, then it is possible to

find an appropriate pair of elements $(i, j) \in I \times J$ such that $(I - i, J - j)$ satisfies the diagonal condition, hence contradicting the minimality of (I, J).

We first prove that the leftmost element of σ_1^I is i_1. Indeed, if it is not the case, we consider i, the leftmost element in σ_1^I and j the leftmost element in σ_2^J. The pair $(I - i, J - j)$ is in $D(n-1)$, as i is different from i_1. Moreover, it is clear that the diagonal condition still holds for $((\sigma_1, \sigma_2), (I, J))$. As this would contradict the minimality of (I, J), the leftmost element of σ_1^I is i_1.

We then prove that $\sigma_1^{I \cup J}$ starts by $j_1 i_1$ and that this j_1 is exactly the leftmost element in σ_2^J. On that purpose, we suppose that either i_1 is preceeded by several elements of J or that the unique element of J is not the leftmost one in σ_2^J. We then adapt the previous argument by choosing i to be the leftmost element in $\sigma_1^{I - \{i_1\}}$ and j the leftmost element in σ_2^J. The pair $(I - i, J - j)$ is in $D(n-1)$. Let us briefly explain while the diagonal condition would still be fulfilled in this case. If j is after i_1 in σ_1, then the difference $n_{1,k}^{J-j} - n_{1,k}^{I-i}$ is greater than $n_{1,k}^J - n_{1,k}^I$ for any k, hence is non negative. If j is before i_1 in σ_1, then by hypothesis, the difference $n_{1,k}^{J-j} - n_{1,k}^{I-i}$ is:

– strictly positive before i_1 an greater than 1 just before i_1
– non negative after i_1
– increase between i_1 and i
– is equal to $n_{1,k}^J - n_{1,k}^I$ after i,

hence is always non negative. Moreover, if i is after j in σ_2, the diagonal condition is clearly satisfied. If i is before j, then the difference $n_{2,k}^{I-i} - n_{1,k}^{J-j}$ is:

– strictly positive before j an greater than 1 just before j
– is equal to $n_{2,k}^I - n_{1,k}^J$ after j,

hence is always non negative. In short, if i_1 is preceeded by several elements of J or the unique element of J is not the leftmost one in σ_2^J, we obtain a contradiction with the minimality of (I, J).

Let us now consider the biggest $k \geqslant 1$ such that $\sigma_1^{I \cup J}$ begins with $j_1 i_1 j_2 i_2 \ldots j_k i_k$ and $\sigma_2^{I \cup J}$ begins with $i_2 j_1 i_3 j_2 \ldots i_k j_{k-1} w j_k$, where w is a word with letters in I. We want to show that $k = n$. Let us first remark that if $k = n$, $w = i_1$. If $1 \leqslant k < n$, then the sets $\tilde{I} = I - \{i_1, \ldots, i_k\}$ and $\tilde{J} = J - \{j_1, \ldots, j_k\}$ are non empty. Let us choose i_{k+1} to be the leftmost element in $\sigma_1^{\tilde{I}}$ and j_{k+1} the leftmost element in $\sigma_2^{\tilde{J}}$. We thus have $\sigma_1^{I \cup J} = j_1 i_1 j_2 i_2 \ldots j_k i_k w' i_{k+1} \ldots$, where w' is in J and $\sigma_2^{I \cup J} = i_2 j_1 i_3 j_2 \ldots i_k j_{k-1} w j_k w'' j_{k+1} \ldots$, where w and w' are words with letters in I. The pair $(I - i_{k+1}, J - j_{k+1})$ is in $D(n-1)$. Following the study as in the previous case, σ_1 always satisfies the diagonal condition for $(I - i_{k+1}, J - j_{k+1})$ and σ_2 satisfies it if and only if $w \neq i$. By minimality of (I, J), we then have $w = i_{k+1}$. If $k+1 = n$, we are done as the only possible word in J is j_{k+1}, hence $w' = j_{k+1}$. Otherwise, we can choose i_{k+2} to be the leftmost element in $\sigma_1^{\tilde{I} - i_{k+1}}$. Using the same reasoning as above, we show that $((\sigma_1, \sigma_2), (I - i_{k+2}, J - j_{k+1}))$ satisfies the diagonal condition if and only if $w' \neq j_{k+1}$. To sum up, the only possibility for (I, J) to be minimal is to have $k = n$, which implies the fish condition.

$\square$

Corollary 3.2. *For any pair of permutations* $(\sigma_1, \sigma_2$, *there exists* $(I, J) \in D(n)$ *such that* $((\sigma_1, \sigma_2), (I, J))$ *satisfies the diagonal condition if and only if there exists* $(I', J') \in E(m)$, $m < n$ *such that* $((\sigma_1, \sigma_2), (I', J'))$ *satisfies the fish condition, with*

$$E(m) = \{(I, J) \in D(m) | \min(J) < \min(I - \min(I)), |[\![1; k]\!] \cap J| > |[\![1; k]\!] \cap I| \\ \text{if } |[\![1; k]\!] \cap J| \geqslant 2 \text{ and } I \subsetneq [\![1; k]\!]\} \tag{3.1}$$

Proof. It follows directly from the fish condition: if the fish condition is satisfied, as inversions of σ_1 are included in inversions of σ_2, we get $j_{k-1}, j_k < i_k$ for any $k > 1$. $\square$

4. Tables

In this section we present low dimensional computations of the enumeration results obtained above and we connect them to other known combinatorial objects.

dim	**0**
0	1

dim	**0**	**1**
0	3	1
1	1	

dim	**0**	**1**	**2**
0	17	12	1
1	12	6	
2	1		

dim	**0**	**1**	**2**	**3**
0	149	162	38	1
1	162	150	24	
2	38	24		
3	1			

FIGURE 3. Number of pairs of faces in the cellular image of the diagonal 0, 1, 2 and 3-dimensional permutahedra.

dim	**0**	**1**	**2**	**3**	**4**
0	1809	2660	1080	110	1
1	2660	3540	1200	80	
2	1080	1200	270		
3	110	80			
4	1				

dim	**0**	**1**	**2**	**3**	**4**	**5**
0	28399	52635	30820	6165	302	1
1	52635	90870	67580	7785	240	
2	30820	47580	20480	2160		
3	6165	7785	2160			
4	302	240				
5	1					

FIGURE 4. Number of pairs of faces in the cellular image of the diagonal 4 and 5-dimensional permutahedra.

Acknowledgements.

Pairs $(F,G) \in \mathrm{Im}\,\triangle_{(P,\vec{v})}$	**Polytopes**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	[OEI22]
$\dim F + \dim G = \dim P$	Associahedra	1	2	6	22	91	408	1938	A000139
	Multiplihedra	1	2	8	42	254	1678	11790	to appear
	Permutahedra	1	2	8	50	432	4802	65536	A007334
$\dim F = \dim G = 0$	Associahedra	1	3	13	68	399	2530	16965	A000260
	Multiplihedra	1	3	17	122	992	8721	80920	to appear
	Permutahedra	1	3	17	149	1809	28399	550297	A213507

Figure 5. Number of pairs of faces in the cellular image of the diagonal of the associahedra, multiplihedra and permutahedra of dimension $0 \leqslant \dim P \leqslant 6$, induced by any good orientation vector.

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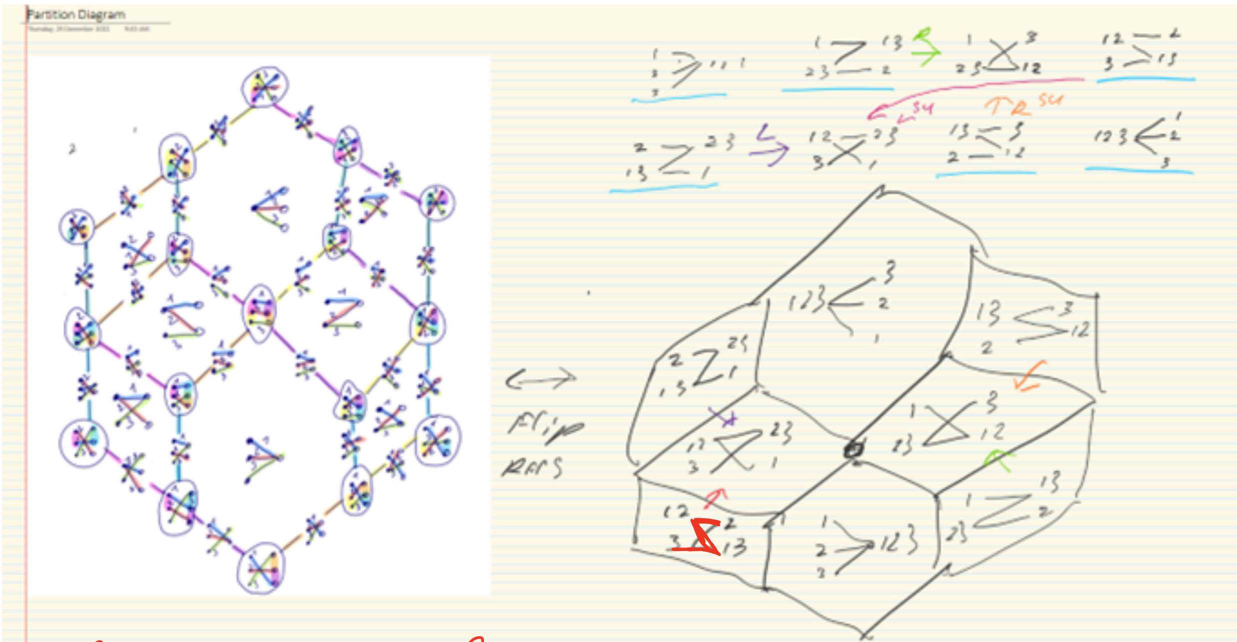
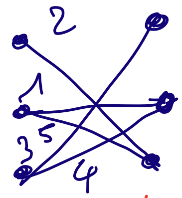


What is the shift operator?

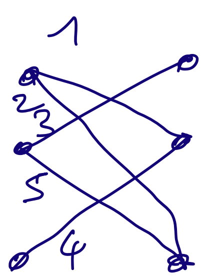
2|15|34 $\times$ 25|14|3



Cannot left shift 1 then 3 → 12|35|4 $\times$ 25|14|3

not in cliq

$\ell = 3 >$ min $\sigma = 1$



$\sigma_i | \sigma_{i+1}$ s.t. $\exists\, p$ ...

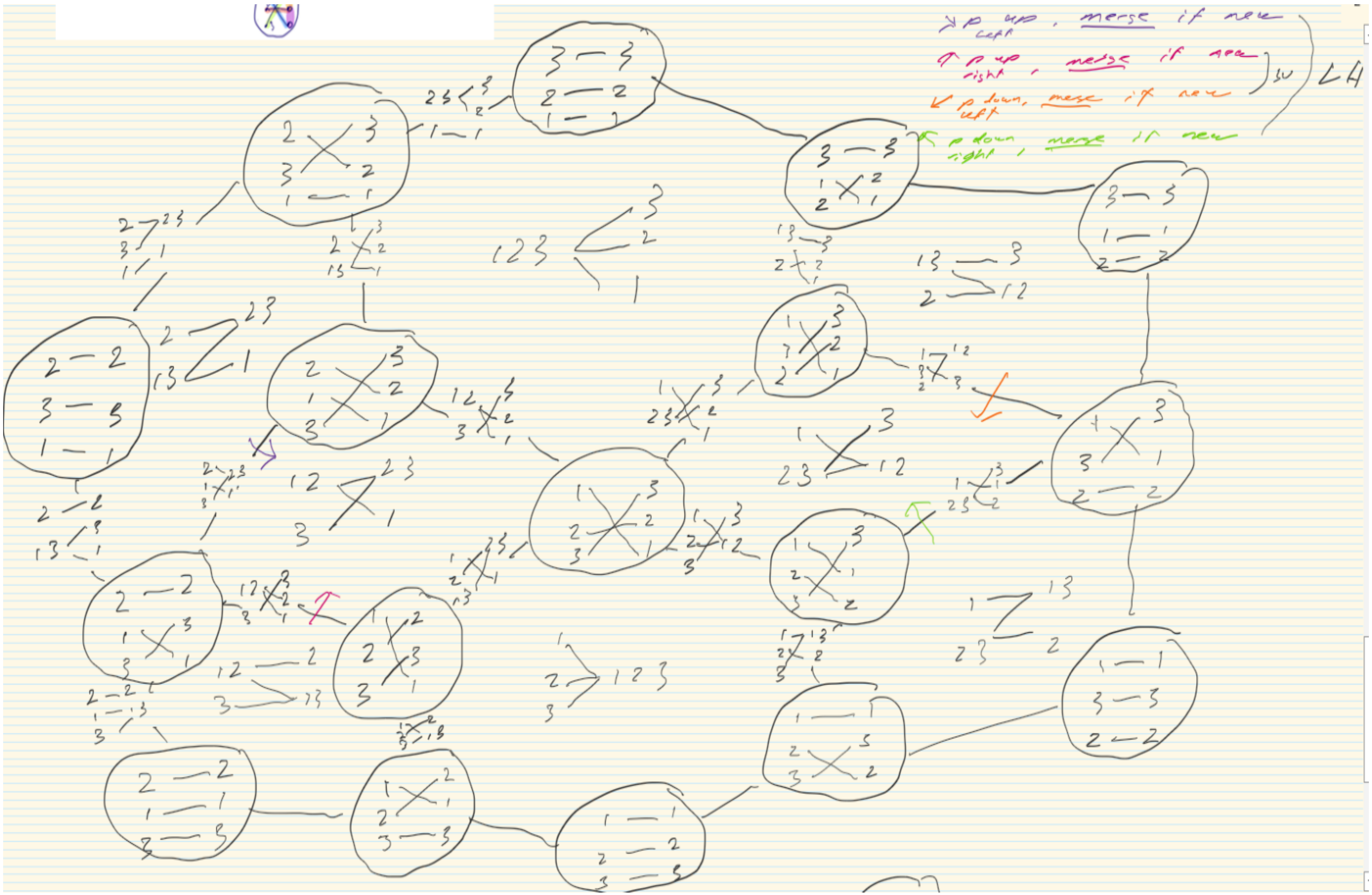
and for $j \leq i$ no inv. using elem of σ_i on RHS

+ Idea proof in mail 29/12 Th 26

SU → apply the operators in lex order only

shift ⟺ $e < \min \sigma$?

Expl: 5|17|4|236, 57|146|3|2


↗ p up left, merge if new
↑ p up right, merge if new
↙ p down left, merge if new
↖ p down right, merge if new
3U
LH

RHS inverted

Ext edges → no crossing

Int → crossings

same dir edges corresponds

to shift

↳ generalised shift with all internal edges

bij facets of Δ(associahedron)
(pairs of dual faces ...) ⟷ Synchronized intervals
[D, E] avec un type (type = T → N
| → E)

GEOMETRIC REALIZATIONS OF TAMARI INTERVAL LATTICES VIA CUBIC COORDINATES

CAMILLE COMBE

Abstract. We introduce cubic coordinates, which are integer words encoding intervals in the Tamari lattices. Cubic coordinates are in bijection with interval-posets, themselves known to be in bijection with Tamari intervals. We show that in each degree the set of cubic coordinates forms a lattice, isomorphic to the lattice of Tamari intervals. Geometric realizations are naturally obtained by placing cubic coordinates in space, highlighting some of their properties. We consider the cellular structure of these realizations. Finally, we show that the poset of cubic coordinates is shellable.

+ What about permutohedron? Ref: A007334: 1, 2, 8, 50, ...
PB (= niveaux avec un carpe?

Contents

Date: January 2, 2023.
Key words and phrases. Tamari lattices; Tamari intervals; interval-posets; posets; geometric realizations; cubical complexes.

Introduction

The Tamari lattices are partial orders having extremely rich combinatorial and algebraic properties. These partial orders are defined on the set of binary trees and rely on the rotation operation [Tam62]. We are interested in the intervals of these lattices, meaning the pairs of comparable binary trees. Tamari intervals of size n also form a lattice. The number of these objects is given by a formula that was proved by Chapoton [Cha06]:

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}. \tag{0.0.1}$$

Strongly linked with associahedra, Tamari lattices have been recently generalized in many ways [BPR12, PRV17]. In this process, the number of intervals of these generalized lattices have also been enumerated through beautiful formulas [BMFPR12, FPR17]. Many bijections between Tamari intervals of size n and other combinatorial objects are known. For instance, a bijection with 3-connected planar triangulations is presented by Bernardi and Bonichon in [BB09] (see also [Fan18]). It has been proved by Châtel and Pons that Tamari intervals are in bijection with interval-posets of the same size [CP15].

We provide in this paper a new bijection with Tamari intervals, which is inspired by interval-posets. More precisely, we first build two words of size n from the Tamari diagrams [Pal86] of a binary tree. If they satisfy a certain property of compatibility, we build a Tamari interval diagram from these two words. We show that Tamari interval diagrams and interval-posets are in bijection. Then we propose a new encoding of Tamari intervals, by building $(n-1)$-tuples of numbers from Tamari interval diagrams. We call these tuples cubic coordinates. This new encoding has two obvious virtues: it is very compact and it gives a way of comparing in a simple manner two Tamari intervals, through a fast algorithm. On the other hand, some properties of Tamari intervals translate nicely in the setting of cubic coordinates. For instance, synchronized Tamari intervals [FPR17] become cubic coordinates with no zero entry. Besides, cubic coordinates provide naturally a geometric realization of the lattice of Tamari intervals, by seeing them as space coordinates. Indeed, all cubic coordinates of size n can be placed in the space $\mathbb{R}^{n-1}$. By drawing their cover relations, we obtain a directed graph. This gives us a realization of cubic coordinate lattices, which we call cubic realization. This realization leads us to many questions, in particular about the cells it contains. We characterize these cells in a combinatorial way, and we deduce a formula to compute the volume of the cubic realization in the geometrical sense. Another direction, more topological, involves the shellability of partial order. We show, drawing inspiration from the work of Björner and Wachs [BW96, BW97], that the cubic coordinates poset is EL-shellable, and as a consequence its associated complex is shellable.

This article is organized in three sections.

The first section is dedicated to reminders about some definitions, such as binary trees, Tamari intervals and interval-posets, and sets out the conventions used. Because of its key role in this work, the bijection between Tamari intervals and interval-posets is also recalled in this section.

In the second section, we define Tamari interval diagrams and show that they are in bijection, size by size, with interval-posets. We then define cubic coordinates and show that they are in bijection, size by size, with Tamari interval diagrams. Using this two bijections, and after having endowed the set of cubic coordinates with a partial order, we show that there is a poset isomorphism between the poset of cubic coordinates and the poset of Tamari intervals.

As pointed out above, the poset of cubic coordinates can then be realized geometrically. This cubic realization and the cells that compose it are the object of the third section. For each cell, we then associate a synchronized cubic coordinate, which is a cubic coordinate without letter 0. By relying upon this particular cubic coordinate, we give a formula to compute the volume of the cubic realization. Finally, we extend the result of Björner and Wachs on the Tamari posets to the Tamari interval posets, by showing that the cubic coordinate posets are EL-shellable.

This article is a complete version of [Com19]. All the proofs are given and several new results are presented, such as the EL-shellability of cubic coordinate posets.

General notations and conventions. Throughout this article, for all words u, we denote by u_i the i-th letter of u. For any integers i and j, $[i, j]$ denotes the set $\{i, i+1, \dots, j\}$. For any integer i, $[i]$ denotes the set $[1, i]$. All posets considered in this article are finite.

1. Preliminaries

In this first section we provide some basic notions of combinatorics and the conventions used afterwards. For this, we recall the definitions of lattices, binary trees, Tamari intervals, and interval-posets. Also, we recall the bijection given in [CP15].

1.1. Posets and lattices.

A *partially ordered set*, commonly called *poset*, is a pair $(\mathcal{P}, \preccurlyeq_{\mathcal{P}})$. When the context is clear, we simply denote this pair by $\mathcal{P}$.

When two elements x and y of $\mathcal{P}$ satisfy $x \preccurlyeq_{\mathcal{P}} y$, then we say that x and y are *comparable*. Otherwise, they are *incomparable*.

Let $x, y \in \mathcal{P}$ such that $x \preccurlyeq_{\mathcal{P}} y$ and $x \neq y$. The element y *covers* x, denoted by $x \lessdot_{\mathcal{P}} y$, for the partial order $\preccurlyeq_{\mathcal{P}}$ if, for all $z \in \mathcal{P}$ such that $x \preccurlyeq_{\mathcal{P}} z \preccurlyeq_{\mathcal{P}} y$, either $z = x$ or $z = y$. The binary relation $\lessdot_{\mathcal{P}}$ is called the *covering relation* of the poset $\mathcal{P}$. By a slight abuse of notation, the set of elements (x, y) such that $x \lessdot_{\mathcal{P}} y$ is also denoted by $\lessdot_{\mathcal{P}}$.

A *maximal element* of $\mathcal{P}$ is an element x such that if there is $y \in \mathcal{P}$ such that $x \preccurlyeq_{\mathcal{P}} y$, then $y = x$. Likewise, a *minimal element* of $\mathcal{P}$ is an element y such that if there is $x \in \mathcal{P}$ such that $x \preccurlyeq_{\mathcal{P}} y$, then $x = y$. A poset $\mathcal{P}$ is *bounded* if it has a unique maximal element and a unique minimal element for $\preccurlyeq_{\mathcal{P}}$.

Since a partial order is transitive, one can realize posets or lattices by knowing only covering relations. The natural way to realize posets is to draw their *Hasse diagrams*, by drawing a edge between all x and y in $\mathcal{P}$ such that $(x, y) \in \lessdot_{\mathcal{P}}$. For any $(x, y) \in \lessdot_{\mathcal{P}}$, we choose the convention to represent x at the top and y at the bottom in the Hasse diagrams. We will keep this convention for all realizations.

Let $x, y \in \mathcal{P}$, the *join* between x and y, denoted by $\vee_{\mathcal{P}}(x, y)$ (or $x \vee_{\mathcal{P}} y$), is defined by

$$\vee_{\mathcal{P}}(x, y) := \min_{\preccurlyeq_{\mathcal{P}}}\{z \in \mathcal{P} \ : \ x \preccurlyeq_{\mathcal{P}} z \text{ and } y \preccurlyeq_{\mathcal{P}} z\}. \tag{1.1.1}$$

The *meet* between x and y, denoted by $\wedge_{\mathcal{P}}(x, y)$ (or $x \wedge_{\mathcal{P}} y$), is defined by

$$\wedge_{\mathcal{P}}(x, y) := \max_{\preccurlyeq_{\mathcal{P}}}\{z \in \mathcal{P} \ : \ z \preccurlyeq_{\mathcal{P}} x \text{ and } z \preccurlyeq_{\mathcal{P}} y\}. \tag{1.1.2}$$

A poset $\mathcal{P}$ is a *join-semilattice* if for all $x, y \in \mathcal{P}$, $\vee_{\mathcal{P}}(x, y)$ exists. Likewise, a poset $\mathcal{P}$ is a *meet-semilattice* if for all $x, y \in \mathcal{P}$, $\wedge_{\mathcal{P}}(x, y)$ exists. A poset $(\mathcal{L}, \preccurlyeq_{\mathcal{L}})$ is a *lattice* if $\mathcal{L}$ is a join-semilattice and a meet-semilattice.

Let $\mathcal{P}$ be a poset and $u^{(1)}, u^{(2)} \in \mathcal{P}$ such that $u^{(1)} \preccurlyeq_{\mathcal{P}} u^{(2)}$. An *interval* $(u^{(1)}, u^{(2)})$ is the set of all elements between $u^{(1)}$ and $u^{(2)}$. The set of intervals of $\mathcal{P}$ is denoted by $\mathrm{int}(\mathcal{P})$. The *poset of intervals* of a poset $\mathcal{P}$ is the poset on the set $\mathrm{int}(\mathcal{P})$ endowed with the partial order $\preccurlyeq_{\mathrm{int}(\mathcal{P})}$ defined, for all $(u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)}) \in \mathrm{int}(\mathcal{P})$, by

$$(u^{(1)}, u^{(2)}) \preccurlyeq_{\mathrm{int}(\mathcal{P})} (v^{(1)}, v^{(2)}) \text{ if } u^{(1)} \preccurlyeq_{\mathcal{P}} v^{(1)} \text{ and } u^{(2)} \preccurlyeq_{\mathcal{P}} v^{(2)}. \tag{1.1.3}$$

In the same way, for $(u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)}) \in \mathrm{int}(\mathcal{L})$ such that $(u^{(1)}, u^{(2)}) \preccurlyeq_{\mathrm{int}(\mathcal{L})} (v^{(1)}, v^{(2)})$, a covering relation for the partial order $\preccurlyeq_{\mathrm{int}(\mathcal{L})}$ is defined.

The property of being a lattice is preserved under this construction.

Proposition 1.1.1. *If $(\mathcal{L}, \preccurlyeq_{\mathcal{L}})$ is a lattice, then $(\mathrm{int}(\mathcal{L}), \preccurlyeq_{\mathrm{int}(\mathcal{L})})$ is a lattice.*

Proof. Let $(u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)}) \in \mathrm{int}(\mathcal{L})$. First, we have to show that $\vee_{\mathcal{L}}(u^{(1)}, v^{(1)}) \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$. By the definition of the join, one has $u^{(2)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$ and $v^{(2)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$. Furthermore, since $u^{(1)} \preccurlyeq_{\mathcal{L}} u^{(2)}$ and $v^{(1)} \preccurlyeq_{\mathcal{L}} v^{(2)}$, one has $u^{(1)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$ and $v^{(1)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$. In addition, $\vee_{\mathcal{L}}(u^{(1)}, v^{(1)})$ is the minimal element of $\mathcal{L}$ satisfying $u^{(1)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(1)}, v^{(1)})$ and $v^{(1)} \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(1)}, v^{(1)})$. Thus, $\vee_{\mathcal{L}}(u^{(1)}, v^{(1)}) \preccurlyeq_{\mathcal{L}} \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})$.

From (1.1.3), one has

$$\begin{aligned}
&\vee_{\mathrm{int}(\mathcal{L})}\left((u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)})\right) \\
&= \min_{\preccurlyeq_{\mathrm{int}(\mathcal{L})}}\{(w^{(1)}, w^{(2)}) \in \mathrm{int}(\mathcal{L}) \ : \ (u^{(1)}, u^{(2)}) \preccurlyeq_{\mathrm{int}(\mathcal{L})} (w^{(1)}, w^{(2)}), (v^{(1)}, v^{(2)}) \preccurlyeq_{\mathrm{int}(\mathcal{L})} (w^{(1)}, w^{(2)})\} \\
&= \min_{\preccurlyeq_{\mathrm{int}(\mathcal{L})}}\{(w^{(1)}, w^{(2)}) \in \mathrm{int}(\mathcal{L}) \ : \ u^{(1)} \preccurlyeq_{\mathcal{L}} w^{(1)}, u^{(2)} \preccurlyeq_{\mathcal{L}} w^{(2)}, v^{(1)} \preccurlyeq_{\mathcal{L}} w^{(1)}, v^{(2)} \preccurlyeq_{\mathcal{L}} w^{(2)}\} \\
&= \left(\vee_{\mathcal{L}}(u^{(1)}, v^{(1)}), \vee_{\mathcal{L}}(u^{(2)}, v^{(2)})\right).
\end{aligned} \tag{1.1.4}$$

The case of the meet $\wedge_{\mathrm{int}(\mathcal{L})}\left((u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)})\right) = \left(\wedge_{\mathcal{L}}(u^{(1)}, v^{(1)}), \wedge_{\mathcal{L}}(u^{(2)}, v^{(2)})\right)$ is symmetrical. □

1.2. **Rooted trees and binary trees.** A *rooted tree*, or simply a *tree* in our context, is defined recursively as a node together with a (possibly empty) sequence of rooted trees. We shall use the standard terminology about trees like *root*, *edge*, *child*, *descendant*, *subtree*, *etc.* The *size* of a tree is its number of nodes. The nodes of the trees considered in this work are labeled by positive integers. We draw trees with the root at the top, where a node is depicted by ○ with its label inside the circle. A *forest* is a sequence of trees. From a forest $\mathfrak{f}$ of n trees, it is always possible to build a tree $\mathfrak{t}$ by taking the root of each element of $\mathfrak{f}$ and by linking all these roots to an artificial node, such that this artificial node become the root of $\mathfrak{t}$. The size of the obtained tree is one plus the sum of all sizes of trees in $\mathfrak{f}$.

A *binary tree* (or *2-tree*) $\mathfrak{t}$ is either a leaf or a node attached through two edges to two binary trees, which are called respectively the *left subtree* and the *right subtree* of $\mathfrak{t}$. Recall that the *size* of a binary tree is its number of nodes. We denote by $\mathrm{T}_2(n)$ the set of binary trees of size n. The set of binary trees is enumerated by Catalan numbers. We draw binary trees with the root at the top and the leaves at the bottom, where a node is depicted by ○ and a leaf is depicted by ▫ (see for instance Figure 1).

Let $\mathfrak{t} \in \mathrm{T}_2(n)$. Each node of $\mathfrak{t}$ is numbered recursively, starting with the left subtree, then the root, and ending with the right subtree. An example is given in Figure 1. This numbering then establishes a total order on the nodes of a binary tree called the *infix order*. Afterwards, this numbering is used to refer to the nodes. The sequence of nodes numbered from 1 to n forms the *infix traversal*.

When the size n of $\mathfrak{t}$ satisfies $n \geqslant 1$, the *canopy* of $\mathfrak{t}$ is the word of size $n - 1$ on the alphabet $\{0, 1\}$ built by assigning to each leaf of $\mathfrak{t}$ a letter as follows. Any leaf oriented to the left (resp. right) is labeled by 0 (resp. 1). The canopy of $\mathfrak{t}$ is the word obtained by reading from left to right the labels thus established, forgetting the first and the last one (since there are always respectively 0 and 1). For instance, the binary tree in Figure 1 has for canopy the word 0110100. There is a link between infix order of a binary tree and its canopy. For a node of index i for the infix order in a tree $\mathfrak{t}$, the right subtree of i is a leaf oriented to the right if and only if the i-th letter of the canopy of $\mathfrak{t}$ is 1. The left subtree of i is a leaf oriented to the left if and only if the $(i - 1)$-th letter of the canopy of $\mathfrak{t}$ is 0. The two direct implications can be proved by induction on the set of binary trees, for instance, see Lemma 4.3. of [Gir12]. The converses are simply given by the definition of the canopy.

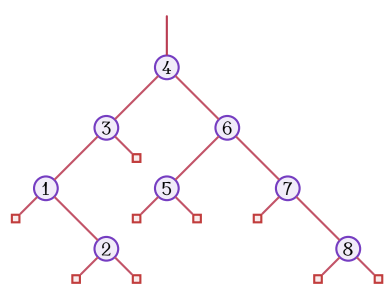


FIGURE 1. A binary tree of size 8 and the numbering of its nodes in the infix order.

A fundamental operation in binary trees is the *right rotation* [Tam62]. Let k and l be the indices for the infix order of two nodes of a binary tree $\mathfrak{t}$, such that the node k is the left child of the node l. Right rotation locally changes the tree $\mathfrak{t}$ so that l becomes the right child of k (see Figure 2). Equivalently, this means that the local configuration $((\mathfrak{a}, \mathfrak{b}), \mathfrak{c})$ becomes $(\mathfrak{a}, (\mathfrak{b}, \mathfrak{c}))$, where $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ are the subtrees shown in Figure 2.

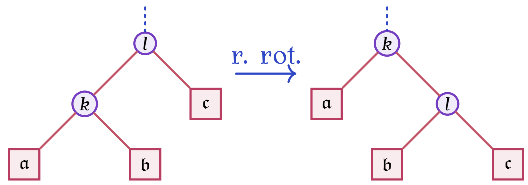


FIGURE 2. Right rotation of edge (k, l) in $\mathfrak{t}$ (on the left), where $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ are any subtrees.

1.3. **Tamari intervals and interval-posets.** For any $n \geqslant 0$, let $\mathfrak{s}, \mathfrak{t} \in \mathrm{T}_2(n)$. We set $\mathfrak{s} \preccurlyeq_{\mathrm{ta}} \mathfrak{t}$ if either $\mathfrak{t} = \mathfrak{s}$ or $\mathfrak{t}$ is obtained by successively applying one or more right rotations in $\mathfrak{s}$. The set $\mathrm{T}_2(n)$ endows with $\preccurlyeq_{\mathrm{ta}}$ is the *Tamari lattice* of order n [HT72]. Moreover, $\mathfrak{s}$ is covered by $\mathfrak{t}$, denoted by $\mathfrak{s} \lessdot_{\mathrm{ta}} \mathfrak{t}$, if $\mathfrak{t}$ is obtained from $\mathfrak{s}$ by performing one right rotation.

In the literature, the Tamari lattice is closely related to the *associahedron*, or the *Stasheff polytope* after the work of Stasheff. More precisely, the Hasse diagram of the Tamari lattice is the 1-skeleton of the associahedron.

Let $\mathfrak{s}, \mathfrak{t} \in \mathrm{T}_2(n)$. A *Tamari interval* of *size* n is an interval $(\mathfrak{s}, \mathfrak{t})$ for the Tamari order $\preccurlyeq_{\mathrm{ta}}$. The set of Tamari intervals of size n is denoted by $\mathrm{int}(\mathrm{T}_2(\mathrm{n}))$.

The Tamari interval lattice is the set $\mathrm{int}(\mathrm{T}_2(\mathrm{n}))$ endowed with the partial order $\preccurlyeq_{\mathrm{int(ta)}}$. Let $n \geqslant 0$ and $(\mathfrak{s}, \mathfrak{t}), (\mathfrak{s}', \mathfrak{t}') \in \mathrm{int}(\mathrm{T}_2(\mathrm{n}))$, following (1.1.3), we have that $(\mathfrak{s}, \mathfrak{t}) \preccurlyeq_{\mathrm{int(ta)}} (\mathfrak{s}', \mathfrak{t}')$ if $\mathfrak{s} \preccurlyeq_{\mathrm{ta}} \mathfrak{s}'$ and $\mathfrak{t} \preccurlyeq_{\mathrm{ta}} \mathfrak{t}'$. According to Proposition 1.1.1, the poset so defined is a lattice. Moreover, it follows from the definition of $\preccurlyeq_{\mathrm{int(ta)}}$ that $(\mathfrak{s}', \mathfrak{t}')$ covers $(\mathfrak{s}, \mathfrak{t})$ if

- ⋆ either $\mathfrak{s}'$ is obtained by a single right rotation of an edge in $\mathfrak{s}$ and $\mathfrak{t}' = \mathfrak{t}$,
- ⋆ or $\mathfrak{t}'$ is obtained by a single right rotation of an edge in $\mathfrak{t}$ and $\mathfrak{s}' = \mathfrak{s}$.

It is known from [Cha06] that Tamari intervals of size n are enumerated by

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}. \tag{1.3.1}$$

The first numbers are

$$1, 1, 3, 13, 68, 399, 2530, 16965. \tag{1.3.2}$$

This sequence is Sequence **A000260** of [Slo].

Interval-posets are posets introduced by Châtel and Pons in [CP15] in order to study the Tamari lattice. Indeed, there is a poset isomorphism between the Tamari interval lattices and the set of interval-posets endowed with a certain partial order.

Let $n \geqslant 0$ and $\{\pi_1, \dots, \pi_n\}$ be a set of n symbols numbered from 1 to n. An *interval-poset* π is a partial order $\lhd$ on the set $\{\pi_1, \dots, \pi_n\}$ such that

(i) if $i < k$ and $\pi_k \lhd \pi_i$, then for all π_j such that $i < j < k$, one has $\pi_j \lhd \pi_i$,
(ii) if $i < k$ and $\pi_i \lhd \pi_k$, then for all π_j such that $i < j < k$, one has $\pi_j \lhd \pi_k$.

The *size* of an interval-poset is the cardinality of its underlying set. The set of interval-posets of size n is denoted by $\mathrm{IP}(n)$, and the elements of interval-poset are called *vertices*.

The two conditions (i) and (ii) of interval-posets are referred to as *interval-poset properties*. For any $i < j$, the relations $\pi_j \triangleleft \pi_i$ are known as *decreasing relations* and the relations $\pi_i \triangleleft \pi_j$ are known as *increasing relations*.

As it is shown in Figure 4b, the Hasse diagram of interval-posets can be drawn as directed graph where two vertices π_i and π_j are related by an arrow from π_i to π_j (resp. π_j to π_i) if $\pi_i \triangleleft \pi_j$ (resp. $\pi_j \triangleleft \pi_i$) where $i < j$.

Let $n \geqslant 0$ and $(\mathfrak{s}, \mathfrak{t}) \in \mathrm{int}(\mathrm{T}_2(\mathrm{n}))$ and $\pi \in \mathrm{IP}(n)$. We will recall a bijection ρ relating on the one hand the restriction of π to its decreasing relations with the binary tree $\mathfrak{s}$, and on the other hand the restriction of π to its increasing relations with the binary tree $\mathfrak{t}$.

Thus, from the restriction of π to its decreasing (resp. increasing) relations we build a forest referred to as the *decreasing* (resp. *increasing*) *forest*, such that if $\pi_j \triangleleft \pi_i$ with $i < j$ (resp. $j < i$), then the node j is a descendant of the node i. Otherwise, if $\pi_j \ntriangleleft \pi_i$ with $i < j$ (resp. $j < i$) the node j is placed to the right (resp. left) of the node i.

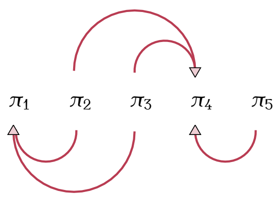
Note that we obtain a decreasing (resp. increasing) forest formed by trees labelled from the roots to the leaves in increasing (resp. decreasing) order. Moreover, the prefix (resp. suffix) traversal of the decreasing (resp. increasing) forest gives the sequence of labels $1, \ldots, n$. Let us add a virtual root node (without label) on the top of both decreasing and increasing forests to form two trees. We denote by $\mathfrak{s}'$ and $\mathfrak{t}'$ the trees respectively obtained from the decreasing and the increasing forests.

Let ρ be the map sending π to the pair $(\mathfrak{s}, \mathfrak{t})$ of binary trees defined such that the tree $\mathfrak{s}$ (resp. $\mathfrak{t}$) is the unique binary tree obtained by reading $\mathfrak{s}'$ (resp. $\mathfrak{t}'$) in the following way. For all label i, j in $\mathfrak{s}'$ (resp. $\mathfrak{t}'$), if a node j is a descendant of a node i in $\mathfrak{s}'$ (resp. $\mathfrak{t}'$), then j becomes a right (resp. left) descendant of the node i in $\mathfrak{s}$ (resp. $\mathfrak{t}$). If a node i is a left (resp. right) brother of a node j in $\mathfrak{s}'$ (resp. $\mathfrak{t}'$), then i becomes a left (resp. right) descendant of the node j in $\mathfrak{s}$ (resp. $\mathfrak{t}$).

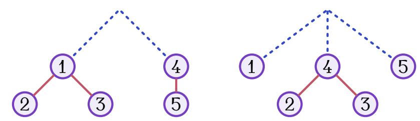
Figure 3 gives an example of construction by the bijection ρ of a Tamari interval from an interval-poset of size 5.

In this section, we shall draw interval-posets as follows. For any $i < j$, if $\pi_j \triangleleft \pi_i$ and there is no vertex π_k such that $\pi_k \triangleleft \pi_i$ and $j < k$, then we draw an arrow with source π_j and target π_i from below as shown in the example in Figure 4. Symmetrically, if $\pi_j \triangleleft \pi_k$ and $j < k$ and if there is no π_i such that $\pi_i \triangleleft \pi_k$ and $i < j$, then we draw an arrow with source π_j and target π_k from above. We refer to this directed graph with two types of arrows as the *minimalist representation* of π.

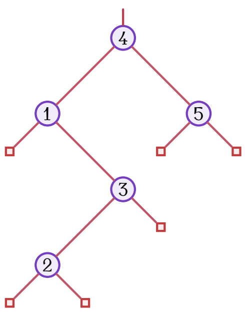
The closure for the interval-poset properties is given by adding the decreasing relations $\pi_j \triangleleft \pi_i$ for any relation $\pi_k \triangleleft \pi_i$ and by adding the increasing relations $\pi_j \triangleleft \pi_k$ for any relation $\pi_i \triangleleft \pi_k$, for any $i < j < k$. By taking the reflexive closure and the closure for the interval-poset properties, an interval-poset is obtained from the minimalist representation. The interest of the minimalist representation is justified later, in particular with Theorem 2.2.3. It is important to represent the decreasing relations and the increasing relations independently.



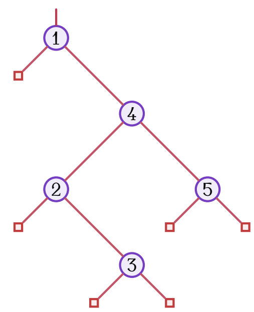
(A) Interval-poset of size 5.



(B) Decreasing (on the left) and increasing (on the right) forests

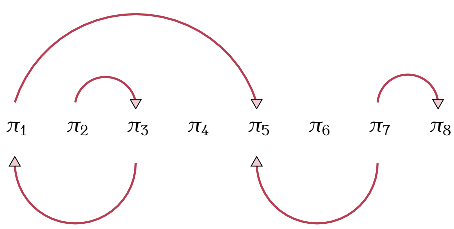


(C) Left binary tree.

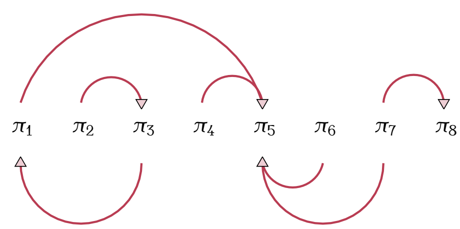


(D) Right binary tree.

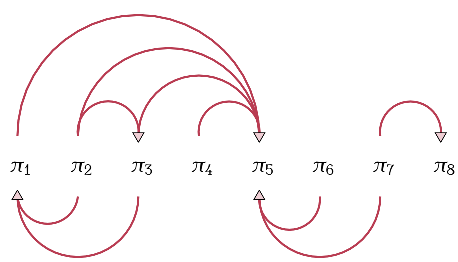
FIGURE 3. Construction of a Tamari interval from an interval-poset by ρ.



(A) Minimalist representation.



(B) Hasse diagram.



(C) Diagram with all apparent (except reflexive) relations.

FIGURE 4. Different representations of an interval-poset of size 8.

Let $n \geqslant 0$ and $\pi, \pi' \in \mathrm{IP}(n)$ and $(\mathfrak{s}, \mathfrak{t}) := \rho(\pi)$, $(\mathfrak{s}', \mathfrak{t}') := \rho(\pi')$. Let $(\star)$ (resp. $(\diamond)$) the following condition: π' is obtained from π by adding (resp. removing) only decreasing (resp. increasing) relations of target a vertex π_k, such that if only one of these decreasing (resp. increasing) relations is removed (resp. added), then either π is obtained or the object obtained is not an interval-poset.

For the sequel, we need to recall that $(\mathfrak{s}', \mathfrak{t}')$ covers $(\mathfrak{s}, \mathfrak{t})$ if and only if π and π' satisfy either $(\star)$ or $(\diamond)$.

Lemma 1.3.1. *The interval-posets π and π' satisfy $(\star)$ (resp. $(\diamond)$) for the vertex π_i (resp. π_j) if and only if $\mathfrak{s}'$ (resp. $\mathfrak{t}'$) is obtained by a unique right rotation of the edge (i, j) in $\mathfrak{s}$ (resp. $\mathfrak{t}$) and $\mathfrak{t}' = \mathfrak{t}$ (resp. $\mathfrak{s}' = \mathfrak{s}$).*

Proof. Suppose π and π' satisfy $(\star)$ for the vertex π_i. Therefore, π' has more decreasing relations of target π'_i than the vertex π_i in π. Suppose that the vertices π_j and π_i are not related in π, and that π'_j and π'_i are related in π', with $i < j$. Then, by the interval-poset property (i), for any π'_k such that $i < k < j$, $\pi'_k \triangleleft \pi'_i$. Moreover, if we remove only one of these decreasing relations, we obtain either π or an object that is no longer an interval-poset. This means that the number of descending relations added in π' is minimal, or equivalently, that the vertex π_j is closest to the vertex π_i such that π_j and π_i are not related in π and $i < j$. This case is depicted in Figure 5. By the bijection ρ, add

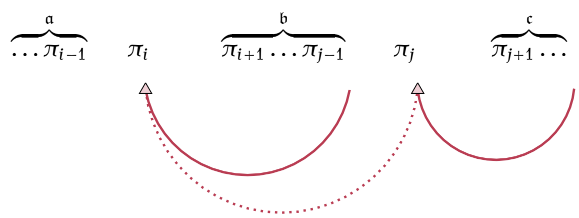
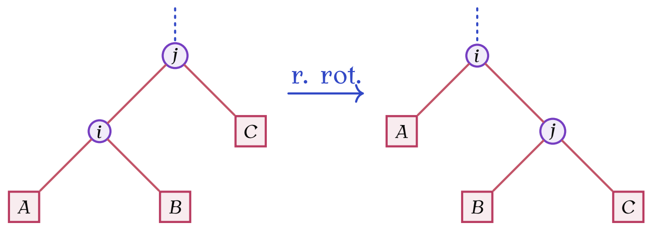


FIGURE 5. Interval-poset of the decreasing forest before (without dotted line) and after (with dotted line) the right rotation of the edge (i, j), where $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ may be empty.

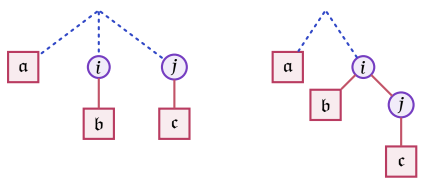
these decreasing relations of target π_i in π leads to the decreasing forest induced by $\mathfrak{s}'$ represented by Figure 6b. A unique right rotation is then made between the trees $\mathfrak{s}$ and $\mathfrak{s}'$ (see Figure 6a). Furthermore, since the increasing relations are unchanged between π and π', the increasing forests induced by $\mathfrak{t}$ and $\mathfrak{t}'$ are the same, and thus $\mathfrak{t}' = \mathfrak{t}$.

Reciprocally, suppose that $\mathfrak{s}'$ is obtained by a unique right rotation of the edge (i, j) in $\mathfrak{s}$ and that $\mathfrak{t}' = \mathfrak{t}$. The case is depicted by Figure 6a, and the two decreasing forests induced by $\mathfrak{s}$ and $\mathfrak{s}'$ are depicted by Figure 6b. By the bijection ρ, we then obtain the interval-poset whose restriction to decreasing relations is shown by Figure 5. Since $\mathfrak{t}' = \mathfrak{t}$, the increasing relations of the interval-posets associated with $(\mathfrak{s}, \mathfrak{t})$ and $(\mathfrak{s}', \mathfrak{t}')$ are the same. Finally, π' is obtained by adding only decreasing relations of target π_i in π. Furthermore, if only one of these relations is removed, then either π is obtained, or the object obtained is not an interval-poset. This means that π and π' satisfy $(\star)$.

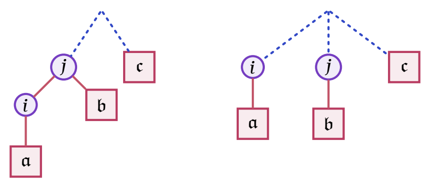
Symmetrically, we show that π and π' satisfy $(\diamond)$ for π_j if and only if $\mathfrak{t}'$ is obtained by a unique right rotation of the edge (i, j) in $\mathfrak{t}$ and $\mathfrak{s}' = \mathfrak{s}$. Figure 6c and Figure 7 depicts this case. $\square$



(A) Binary trees $\mathfrak{s}$ and $\mathfrak{s}'$ (resp. $\mathfrak{t}$ and $\mathfrak{t}'$).



(B) Decreasing forests induced by $\mathfrak{s}$ and $\mathfrak{s}'$.



(C) Increasing forests induced by $\mathfrak{t}$ and $\mathfrak{t}'$.

FIGURE 6. Right rotation of the edge (i, j) in the binary tree $\mathfrak{s}$ (resp. $\mathfrak{t}$), where $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are subtrees.

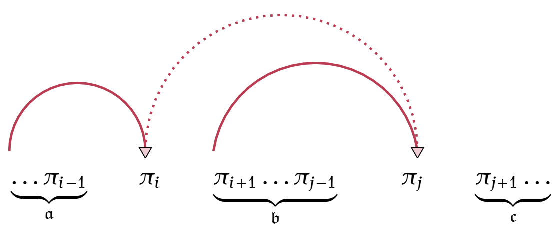


FIGURE 7. Interval-poset of the increasing forest before (with dotted lines) and after (without dotted lines) the right rotation of the edge (i, j), where $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ may be empty.

2. Cubic coordinates and Tamari intervals

The aim of this section is to build the poset of the cubic coordinates, then to establish the poset isomorphism between this poset and the poset of the Tamari intervals. To achieve this goal, we first define the Tamari interval diagrams based on the interval-posets. The cubic coordinates are then obtained from the Tamari interval diagrams.

2.1. **Tamari interval diagrams.** Let us give the definition of a Tamari diagram, as formulated in [BW97]. For any $n \geqslant 0$, a *Tamari diagram* is a word u of length n on the alphabet $\mathbb{N}$ which satisfies the two following conditions:

(i) $0 \leqslant u_i \leqslant n - i$ for all $i \in [n]$,
(ii) $u_{i+j} \leqslant u_i - j$ for all $i \in [n]$ and $j \in [0, u_i]$.

The *size* of a Tamari diagram is its number of letters. For instance, the sets of Tamari diagrams of size 2, 3 and 4 are

$$\begin{gathered}\{00, 10\}, \qquad \{000, 100, 010, 200, 210\}, \\ \{0000, 0010, 0100, 0200, 0210, 1000, 1010, 2000, 2100, 3000, 3010, 3100, 3200, 3210\}.\end{gathered} \tag{2.1.1}$$

In the literature, Tamari diagrams are also known as bracket vectors, objects inspired by the right bracketing introduced in [HT72] by Huang and Tamari. Furthermore, Tamari diagrams are known to be enumerated by Catalan numbers

$$\mathrm{cat}(n) := \frac{1}{n+1}\binom{2n}{n}. \tag{2.1.2}$$

A dual version of Tamari diagrams can be defined by considering the opposite of Conditions (i) and (ii). For any $n \geqslant 0$, a *dual Tamari diagram* is a word v of length n on the alphabet $\mathbb{N}$ which satisfies the two following conditions:

(i) $0 \leqslant v_i \leqslant i - 1$ for all $i \in [n]$,
(ii) $v_{i-j} \leqslant v_i - j$ for all $i \in [n]$ and $j \in [0, v_i]$.

The *size* of a dual Tamari diagram is its number of letters. In other words, $v = v_1 \dots v_n$ is a dual Tamari diagram if and only if $v_n \dots v_1$ is a Tamari diagram.

Note that the first condition of a Tamari diagram u and of a dual Tamari diagram v of size n implies that $u_n = 0$ and $v_1 = 0$.

A graphical representation of a Tamari diagram u of size n by needles and diagonals provides a simple way to check Condition (ii) of a Tamari diagram. For each position $i \in [n]$, we draw a needle from the point $(i-1, 0)$ to the point $(i-1, u_i)$ in the Cartesian plane. Condition (ii) says that one can draw lines of slope -1 passing through the x-axis and the top of each needle without crossing any other needle. For instance, the Tamari diagram 9021043100 is drawn by Figure 8. One can observe that none of its diagonals, drawn as dotted lines, crosses a needle.

Likewise, a graphical representation can be given for the dual Tamari diagram v of size n. One draws v in the same way as Tamari diagram, and Condition (ii) says that one can draw lines of slope 1 passing through the x-axis and the top of each needle without crossing any other needle. Figure 8 also depicts the dual Tamari diagram 0010040002.

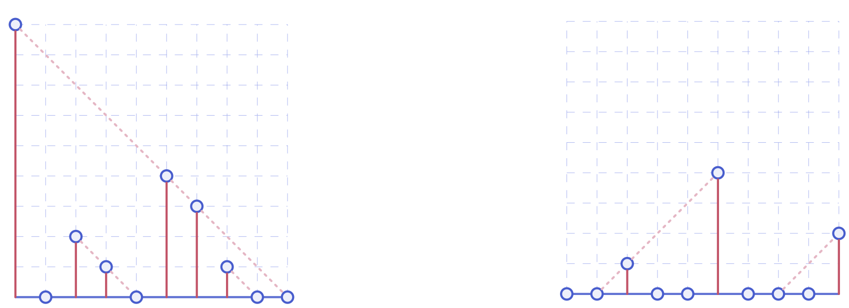


FIGURE 8. A Tamari diagram 9021043100 (on the left) and a dual Tamari diagram 0010040002 (on the right) of size 10.

Figure 9. A binary tree and the associated Tamari diagram of the same size.

For any $n \geqslant 0$, the set of Tamari diagrams of size n is in bijection with $\mathrm{T}_2(n)$. Indeed, one builds from a Tamari diagram u of size n a binary tree $\mathfrak{s}$ recursively as follows. If $n = 0$, $\mathfrak{s}$ is defined as the leaf. Otherwise, let i be the smallest position in u such that u_i is the maximum allowed value, namely $n - i$. Then $\mathfrak{s}_1 := u_1 \dots u_{i-1}$ and $\mathfrak{s}_2 := u_{i+1} \dots u_n$ are also Tamari diagrams. One forms $\mathfrak{s}$ by grafting the binary trees obtained recursively by this process applied on $\mathfrak{s}_1$ and on $\mathfrak{s}_2$ to a new node. Reciprocally, for each node of index i of the tree $\mathfrak{s}$, labeled with an infix traversal, the value of the i-th letter of the corresponding Tamari diagram is given by the number of nodes in the right subtree of the node i. The complete demonstration is given in [Pal86].

In the case of dual Tamari diagrams, the construction of the binary tree $\mathfrak{t}$ is also recursive, except that it is the maximum position i in the dual Tamari diagram whose value is the highest allowed on that section of the word that should be chosen first. Similarly for the reciprocal, the procedure is identical, except that the value of the i-th letter in the dual Tamari diagram is given by the number of nodes in the left subtree of the node i in the tree $\mathfrak{t}$.

For instance, in Figure 1, the Tamari diagram is 10040210 and the dual Tamari diagram is 00230100. Figure 9 depicts the corresponding binary tree of the Tamari diagram 1003010.

Let $n \geqslant 0$ and u be a Tamari diagram, and v be a dual Tamari diagram, both of size n. The diagrams u and v are *compatible* if there are no i, j with $1 \leqslant i < j \leqslant n$ such that $u_i \geqslant j - i$ and $v_j \geqslant j - i$. If u and v are compatible, then the pair (u, v) is called *Tamari interval diagram*. The set of Tamari interval diagrams of size n is denoted by $\mathrm{TID}(n)$.

In other words, a Tamari diagram u of size n and a dual Tamari diagram v of size n are compatible if for any needle of position i and height $v_i \neq 0$ in v (resp. $u_i \neq 0$ in u), there is no needle of position j and height greater than or equal to $i - j$ in u (resp. $j - i$ in v) with $i - v_i \leqslant j \leqslant i - 1$ (resp. $i + 1 \leqslant j \leqslant i + u_i$) and $i \in [n]$.

For example, the two diagrams in Figure 8 are compatible. Figure 10 gives two other examples of two incompatible diagrams 00400000 and 00003000, and two compatible diagrams 04000000 and 00000030. Hereinafter, if u and v are compatible, we can also say that u and v satisfy the compatibility condition.

As for Tamari diagrams and dual Tamari diagrams, a graphical representation of the Tamari interval diagram is also possible, as shown in Figure 10. Figure 11 gives the

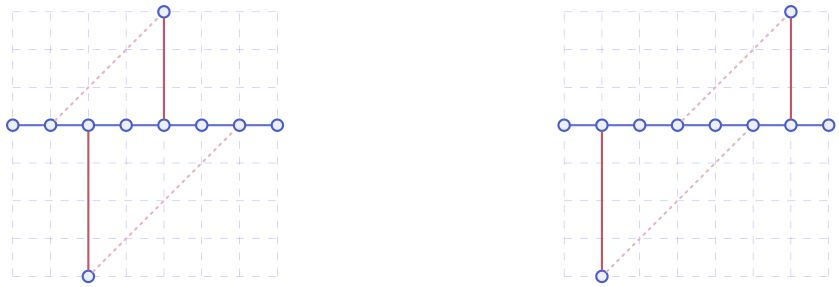


FIGURE 10. Two incompatible diagrams (on the left) and two compatible diagrams (on the right).

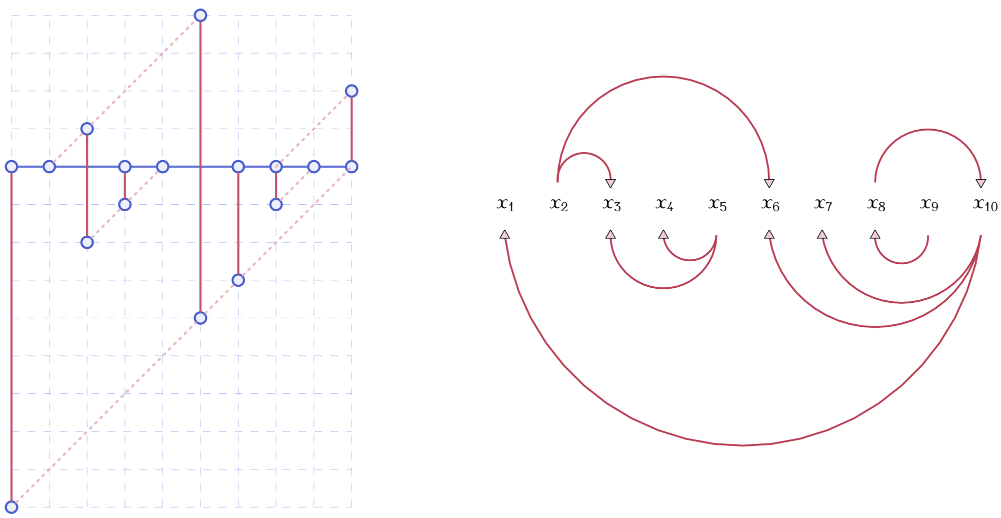


FIGURE 11. A Tamari interval diagram of size 10 (on the left) and its associated interval-poset (on the right).

representation of the Tamari interval diagram $(9021043100, 0010040002)$ formed by the two diagrams seen in Figure 8 which are compatible, where we have simply considered the symmetry relative to the abscissa axis of the Tamari diagram, and placed it under the dual Tamari diagram. Thus, the Tamari diagram u is drawn below and the dual Tamari diagram v is drawn above. With such a representation, it is then easy to verify that u and v are compatible. Indeed, any needle of u that is below the diagonal linking the top of the needle in position j in v to the abscissa point $j - v_j$, has a diagonal that intersects the x-axis strictly before the position j. Symmetrically, any needles of v that is above a diagonal linking the top of the needle in position i in u to the abscissa point $i + u_i$, has a diagonal that intersects the x-axis strictly after the position i.

One consequence of the compatibility condition is that each needle of non-zero height in the dual Tamari diagram v is always preceded by a needle of u of zero height. Symmetrically, each non-zero height needle in the Tamari diagram u is always followed by a needle of v of zero height. In other words, for any $i \in [n]$, u_i and v_{i+1} can both be zero, but cannot both be non-zero.

2.2. **Link with interval-posets.** Let us show that there is a bijection between the set of Tamari interval diagrams and the set of interval-posets of the same size.

Let $n \geqslant 0$ and χ be the map sending a Tamari interval diagram (u, v) of size n to the relation

$$(\{\pi_1, \dots, \pi_n\}, \lhd) \tag{2.2.1}$$

where $\pi_{i+l} \lhd \pi_i$ for all $i \in [n]$ and $0 \leqslant l \leqslant u_i$, and $\pi_{i-k} \lhd \pi_i$ for all $i \in [n]$ and $0 \leqslant k \leqslant v_i$.

Proposition 2.2.1. *For any $n \geqslant 0$, the map χ has values in* $\mathrm{IP}(n)$*.*

Proof. Let $(u, v) \in \mathrm{TID}(n)$ and $\pi := \chi(u, v)$. First, we show that $\lhd$ is a partial order, then that interval-poset properties are satisfied.

(1) By the definition of χ one has $\pi_{i+l} \lhd \pi_i$ and $\pi_{i-k} \lhd \pi_i$ with $0 \leqslant l \leqslant u_i$ and $0 \leqslant k \leqslant v_i$ for all $\pi_i \in \pi$. Specifically, $\pi_i \lhd \pi_i$. This shows that π is reflexive.
(2) Let π_i, π_j and π_k be vertices of π with $i < j < k$.
   (a) Suppose that $\pi_j \lhd \pi_i$ and that $\pi_k \lhd \pi_j$. Then $\pi_j \lhd \pi_i$ implies that there is an integer $0 \leqslant i' \leqslant u_i$ such that $j = i + i'$. Therefore, by Condition (ii) of a Tamari diagram, $u_j = u_{i+i'} \leqslant u_i - i'$. Likewise, $\pi_k \lhd \pi_j$ implies that there is an integer $0 \leqslant j' \leqslant v_j$ such that $k = j + j'$. Still by the same condition, one has $u_k = u_{j+j'} \leqslant u_j - j'$. By using these two inequalities, we obtain that $u_i \geqslant u_k + i' + j'$. Since $i' + j' = k - i$, then we have $u_i \geqslant k - i$, which implies by the definition of χ that $\pi_k \lhd \pi_i$ in π.
   (b) Suppose that $\pi_j \lhd \pi_i$ and that $\pi_i \lhd \pi_k$. Therefore, $\pi_j \lhd \pi_k$ because $\pi_i \lhd \pi_k$ implies that each vertex between π_i and π_k is in relation with π_k.
   (c) Suppose that $\pi_i \lhd \pi_j$ and that $\pi_j \lhd \pi_k$. Then $\pi_i \lhd \pi_j$ implies that there is an integer $0 \leqslant i' \leqslant v_i$ such that $i = j - i'$. By Condition (ii) of a dual Tamari diagram, $v_i = v_{j-i'} \leqslant v_j - i'$. Likewise, $\pi_j \lhd \pi_k$ implies that there is an integer $0 \leqslant j' \leqslant v_j$ such that $j = k - j'$. By the same condition (ii), $v_j = v_{k-j'} \leqslant v_k - j'$. By these two inequalities, one has $v_k \geqslant v_i + i' + j'$. Since $i' + j' = k - i$, one has $v_k \geqslant k - i$, which implies by the definition of χ that $\pi_i \lhd \pi_k$ in π.
   (d) Suppose that $\pi_j \lhd \pi_k$ and that $\pi_k \lhd \pi_i$. Then $\pi_j \lhd \pi_i$ because $\pi_k \lhd \pi_i$ implies that all vertex between π_i and π_k is in relation with π_i.

   This shows that π is transitive. Notice that it is impossible to have the case $\pi_i \lhd \pi_k$ and $\pi_k \lhd \pi_j$ since π is the image of a Tamari interval diagram. Getting this case would contradict the fact that u and v are compatible. Similarly, the case $\pi_i \lhd \pi_j$ and $\pi_k \lhd \pi_i$ is impossible.
(3) Let $i < j$ and π_i, π_j be vertices of π. Suppose that $\pi_j \lhd \pi_i$ and that $\pi_i \lhd \pi_j$. By the definition of χ, $\pi_j \lhd \pi_i$ if and only if $u_i \geqslant j - i$. Likewise, $\pi_i \lhd \pi_j$ if and only if $v_j \geqslant j - i$. However, since u and v are compatible, this case is impossible. This shows that π is antisymmetric.

(4) The definition of χ implies directly that π satisfies the interval-poset properties, namely that for all π_i, π_j and π_k vertices of π with $i < j < k$, if $\pi_k \triangleleft \pi_i$, then $\pi_j \triangleleft \pi_i$, and if $\pi_i \triangleleft \pi_k$, then $\pi_j \triangleleft \pi_k$.

□

Let $n \geqslant 0$ and χ' be the map sending an interval-poset π of size n on a pair of words $(u, v) \in \mathbb{N}^n \times \mathbb{N}^n$, such that for all $i \in [n]$,

$$u_i := \#\{\pi_j \in \pi : \pi_j \triangleleft \pi_i \text{ and } i < j\}; \tag{2.2.2}$$

$$v_j := \#\{\pi_i \in \pi : \pi_i \triangleleft \pi_j \text{ and } i < j\}. \tag{2.2.3}$$

Lemma 2.2.2. *Let $n \geqslant 0$, $\pi \in \mathrm{IP}(n)$ and $(u, v) := \chi'(\pi)$. If $u_i \geqslant j - i$ (resp. $v_j \geqslant j - i$), then $\pi_j \triangleleft \pi_i$ (resp. $\pi_i \triangleleft \pi_j$), with $0 \leqslant i \leqslant j \leqslant n$.*

Proof. According to (2.2.2), the fact that $u_i \geqslant j - i$ means that there are at least $j - i$ vertices in decreasing relation to the vertex π_i. By the point (i) of interval-poset properties, this implies in particular that $\pi_j \triangleleft \pi_i$. Respectively, we show with the point (ii) of interval-poset properties that $v_j \geqslant j - i$ implies that $\pi_i \triangleleft \pi_j$. □

Theorem 2.2.3. *For any $n \geqslant 0$, the map $\chi : \mathrm{TID}(n) \to \mathrm{IP}(n)$ is bijective.*

Proof. Let us show that χ' is the inverse map of χ. Let $n \geqslant 0$, $\pi \in \mathrm{IP}(n)$ and $(u, v) := \chi'(\pi)$.

(1) Since π is an interval-poset, there are at most $n - i$ vertices of π in decreasing relation to π_i and at most $i - 1$ vertices of π in increasing relation to π_i for all $i \in [n]$. Therefore, the word u satisfies Condition (i) of a Tamari diagram and the word v satisfies Condition (i) of a dual Tamari diagram.

(2) Let π_i and π_{i+j} be vertices of π such that $i \in [n]$ and $j \in [0, u_i]$. By Lemma 2.2.2, the fact that $u_i \geqslant j$ means that $\pi_{i+j} \triangleleft \pi_i$. Thus, by transitivity of interval-posets, one has that for any $i + j \leqslant k \leqslant n$, if $\pi_k \triangleleft \pi_{i+j}$, then $\pi_k \triangleleft \pi_i$. Thus, $u_{i+j} + j \leqslant u_i$, which implies Condition (ii) of a Tamari diagram.
Symmetrically, Condition (ii) of a dual Tamari diagram is checked by considering π_i and π_{i-j} vertices of π such that $i \in [n]$ and $j \in [0, v_i]$.

(3) For all i, j such that $1 \leqslant i < j \leqslant n$ and $u_i \geqslant j - i$, suppose that $v_j \geqslant j - i$. By Lemma 2.2.2, the relation $u_i \geqslant j - i$ implies that $\pi_j \triangleleft \pi_i$. Likewise, the relation $v_j \geqslant j - i$ means that $\pi_i \triangleleft \pi_j$. Both of these implications lead to a contradiction with the antisymmetric nature of interval-posets. Necessarily, we have $v_j < j - i$, which implies that u and v are compatible.

The pair (u, v) is a Tamari interval diagram of size n. Finally, it is clear that $\chi(u, v) = \pi$ by construction. Therefore, the map χ is surjective.

Let (u, v) and (u', v') be two Tamari interval diagrams of size n, such that $(u, v) \neq (u', v')$ and such that $\chi(u, v) := \pi$ and $\chi(u', v') := \pi'$. So there is at least one letter of (u, v) and (u', v') such that $u_i \neq u'_i$ or $v_i \neq v'_i$, for $i \in [n]$. Therefore, the number of vertices of π in relation to the vertex π_i associated with the component u_i and v_i by χ is different from the number of vertices of π' in relation to the vertex π'_i associated with the component u'_i and v'_i by χ, we thus have $\pi \neq \pi'$. This shows that the map χ is injective. □

The minimalist representation of the interval-posets defined in Section 1 allows us to describe a direct construction of the corresponding Tamari interval diagram. Indeed, let us consider the minimalist representation of an interval-poset π of size n. For any relation $\pi_j \lhd \pi_i$ (resp. $\pi_i \lhd \pi_j$) drawn, with $1 \leqslant i < j \leqslant n$, we set $u_i := j - i$ (resp. $v_j := j - i$) and all other elements not involved in any relation to 0. This forms a pair of words (u, v) which is the inverse image of π by χ.

An example is given by Figure 11, where a Tamari interval diagram and its interval-poset which is its image by χ are shown.

2.3. **Cubic coordinates.** We describe in this part the set of cubic coordinates, and we show that there is a bijection between this set and the set of Tamari interval diagrams.

An $(n-1)$-tuple c on $\mathbb{Z}$ is a *cubic coordinate* if there is a Tamari interval diagram (u, v) of size n such that

$$c = (u_1 - v_2, u_2 - v_3, \dots, u_{n-1} - v_n). \tag{2.3.1}$$

The size of a cubic coordinate is its number of components plus one. The set of cubic coordinates of size n is denoted by $\mathrm{CC}(n)$. For instance, $(9, -1, 2, 1, -4, 4, 3, 1, -2)$ is a cubic coordinate of size 10 since there is the Tamari interval diagram $(9021043100, 0010040002)$ satisfying the conditions of the definition.

Besides, for any $n \geqslant 1$, let ϕ be the map sending an $(n-1)$-tuple c on $\mathbb{Z}$ to a pair (u, v) of words on $\mathbb{N}$, both of length n, such that u satisfies $u_n = 0$ and for any $i \in [n-1]$,

$$u_i = \max(c_i,\ 0), \tag{2.3.2}$$

and v satisfies $v_1 = 0$ and for any $2 \leqslant i \leqslant n$,

$$v_i = |\min(c_{i-1},\ 0)|. \tag{2.3.3}$$

Theorem 2.3.1. *For any $n \geqslant 0$, the map $\phi : \mathrm{CC}(n) \to \mathrm{TID}(n)$ is bijective.*

Proof. Let c and c' be two cubic coordinates of size n such that $c \neq c'$. Then there is a component c_i such that $c_i \neq c'_i$, with $i \in [n-1]$. By the map ϕ, one has then $u_i \neq u'_i$ or $v_{i+1} \neq v'_{i+1}$, namely $(u, v) \neq (u', v')$. Which shows that the map ϕ is injective.

Let $(u, v) \in \mathrm{TID}(n)$. Let $c := (u_1 - v_2, u_2 - v_3, \dots, u_{n-1} - v_n)$, the $(n-1)$-tuple whose components are given by the difference between u_i and v_{i+1} for any $i \in [n-1]$. Now if $u_i \neq 0$, then $v_{i+1} = 0$ for any $i \in [n-1]$. Therefore, $\phi(c) = (u, v)$, where (u, v) is indeed a Tamari interval diagram by hypothesis. By the definition of a cubic coordinate, one can conclude that $c \in \mathrm{CC}(n)$. This shows that the map ϕ is surjective. □

Therefore, by the map ϕ it is possible to build a cubic coordinate from a Tamari interval diagram and reciprocally. Graphically, we have to shift the upper part of a Tamari interval diagram (corresponding to the dual Tamari diagram) to the left by one position and collect the height of the needles from left to right. Then, we put a positive sign for the needles of the lower part of the Tamari interval diagram (corresponding to the Tamari diagram) and a negative sign for the upper part, and we forget the last needle of zero height. To reconstruct a Tamari interval diagram from a cubic coordinate, we reconstruct the needles of the Tamari diagram and the dual Tamari diagram from the components of the cubic coordinate in the same way, and then we shift the dual Tamari diagram to the right by one position.

Using the map χ we can then directly give the cubic coordinate of an interval-poset π. In the same way that we shift the dual Tamari diagram one position to the left, we shift all the increasing relations of the interval-poset to the left by one vertex. Then, for each vertex π_i, we count the number of elements in increasing or decreasing relation of target π_i, out of reflexive relation, for all $i \in [n-1]$. These numbers become the components of positive sign if it is a decreasing relation, negative otherwise, of the cubic coordinate. As the increasing relations have been shifted, the number associated with the vertex π_n is always zero. Therefore, this vertex is forgotten for the cubic coordinate. In the same way, to construct an interval-poset from a cubic coordinate with each component of a cubic coordinate, we rebuild the increasing and decreasing relations on $n-1$ vertices, we add the vertex π_n, then we shift the increasing relations to the right.

Lemma 2.3.2. *Let $n \geqslant 0$ and $c \in \mathrm{CC}(n)$ such that there is a component $c_i \neq 0$, for $i \in [n-1]$. Let c' be the $(n-1)$-tuple such that $c'_i = 0$ and $c'_j = c_j$ for any $j \neq i$, with $j \in [n-1]$. Then c' is a cubic coordinate.*

Proof. Let $(u', v') := \phi(c')$ and (u'_j, v'_{j+1}) be the pair of letters corresponding to c'_j by the map ϕ, with $j \in [n-1]$. Since $c'_i = 0$, then $(u'_i, v'_{i+1}) = (0, 0)$. By hypothesis, all other pairs of letters are the same as those of $(u, v) := \phi(c)$. In order to show that c' is a cubic coordinate, we have to show that (u', v') is a Tamari interval diagram, namely that (u', v') satisfies the conditions of a Tamari diagram, of a dual Tamari diagram, and of compatibility. Clearly, with $(u'_i, v'_{i+1}) = (0, 0)$, all these conditions are satisfied for (u', v'). □

Depending on the case, either the definition of cubic coordinates or the definition of Tamari interval diagrams is used, as it is done for the proof of Lemma 2.3.2. For example, the following results are stated for Tamari interval diagrams.

Let $n \geqslant 0$. A Tamari interval diagram (u, v) of size n is *synchronized* if either $u_i \neq 0$ or $v_{i+1} \neq 0$ for any $i \in [n-1]$.

Likewise, a cubic coordinate c of size n is synchronized if $c_i \neq 0$ for any $i \in [n-1]$. The set of synchronized cubic coordinates of size n is denoted by $\mathrm{SCC}(n)$.

A Tamari interval $(\mathfrak{s}, \mathfrak{t})$ is synchronized if and only if the binary trees $\mathfrak{s}$ and $\mathfrak{t}$ have the same canopy [FPR17, PRV17]. The definition of the canopy is recalled in Section 1.

Proposition 2.3.3. *Let $n \geqslant 0$ and $(u, v) \in \mathrm{TID}(n)$. The Tamari interval diagram (u, v) is synchronized if and only if $\rho(\chi(u, v))$ is a synchronized Tamari interval.*

Proof. If (u, v) is not synchronized, then there is an index $i \in [n-1]$ such that $u_i = 0$ and $v_{i+1} = 0$. Let $\pi := \chi(u, v)$ be the interval-poset associated to (u, v), and $(\mathfrak{s}, \mathfrak{t}) := \rho(\chi(u, v))$. The two binary trees $\mathfrak{s}$ and $\mathfrak{t}$ are not synchronized if there is at least one letter of some index j in the canopy of the tree $\mathfrak{s}$ that is different from the letter of the same index j in the canopy of $\mathfrak{t}$. Let us show that (u, v) is not synchronized if and only if the binary trees $\mathfrak{s}$ and $\mathfrak{t}$ are not synchronized.

The letter u_i is equal to 0 if and only if there is no descending relation of target π_i in π, namely, if and only if the node i has no right child in the tree $\mathfrak{s}$ (see Section 1.3). To summarize, $u_i = 0$ if and only if the right subtree of the node i is a leaf oriented to the right. Now, as recall in Section 1.2, a leaf linked to the node i is oriented to the right if and only if the i-th letter in the canopy corresponding to $\mathfrak{s}$ is 1.

Symmetrically, $v_{i+1} = 0$ if and only if there is no increasing relation of target π_{i+1} in π, namely, if and only if the node $i+1$ has no left child in the tree $\mathfrak{t}$. Then, $v_{i+1} = 0$ if and only if the left subtree of the node $i+1$ is a leaf oriented to the left. As seen in Section 1, a leaf linked to the node $i+1$ is oriented to the left if and only if the i-th letter in the canopy corresponding to $\mathfrak{t}$ is 0.

To conclude, $u_i = 0$ and $v_{i+1} = 0$ if and only if the letter of index i in the canopy of the tree $\mathfrak{s}$ is different from the letter of index i in the canopy of the tree $\mathfrak{t}$. Therefore, (u, v) is not synchronized if and only if the binary trees $\mathfrak{s}$ and $\mathfrak{t}$ are not synchronized. □

An interval-poset π of size $n \geqslant 3$ is *new* if

(1) there is no decreasing relation of source π_n,
(2) there is no increasing relation of source π_1,
(3) there is no relation $\pi_{i+1} \lhd \pi_{j+1}$ and $\pi_j \lhd \pi_i$ with $i < j$.

The definition of a new interval-poset is given in [Rog20].

For any $n \geqslant 3$, a Tamari interval diagram (u, v) of size n is *new* if the following conditions are satisfied

(i) $0 \leqslant u_i \leqslant n - i - 1$ for all $i \in [n-1]$,
(ii) $0 \leqslant v_j \leqslant j - 2$ for all $j \in [2, n]$,
(iii) $u_k < l - k - 1$ or $v_l < l - k - 1$ for all $k, l \in [n]$ such that $k + 1 < l$.

Proposition 2.3.4. *Let $n \geqslant 3$ and $(u, v) \in \mathrm{TID}(n)$. The Tamari interval diagram (u, v) is new if and only if $\chi(u, v)$ is a new interval-poset.*

Proof. Let us show that $\pi := \chi(u, v)$ is not new if and only if (u, v) is not new. Theorem 2.2.3 leads to three cases.

- ⋆ Let us consider the negation of (i) of a new Tamari interval diagram by assuming that $u_i = n-i$. By Lemma 2.2.2, this implies that $\pi_n \lhd \pi_i$ with $i \in [n-1]$. Reciprocally, if $\pi_n \lhd \pi_i$ with $i \in [n-1]$, then by the point (i) of interval-poset properties, all vertices between π_i and π_n are in decreasing relation to π_i. Since $u_i := \#\{\pi_j \in \pi : \pi_j \lhd \pi_i \text{ and } i < j\}$, it implies that $u_i = n - i$.
- ⋆ Likewise, by Lemma 2.2.2, if $v_j = j - 1$, then $\pi_1 \lhd \pi_j$ with $j \in [2, n]$. By the point (ii) of interval-poset properties, we get the converse property.
- ⋆ According to Lemma 2.2.2, if $u_i \geqslant j - i$, then $\pi_j \lhd \pi_i$, and if $v_{j+1} \geqslant j - i$, then $\pi_{i+1} \lhd \pi_{j+1}$ with $i < j$. We obtain the two converse properties with respectively the point (i) and the point (ii) of interval-poset properties. Specifically, by setting $l := j+1$ and $k := i$, we find the formulation of the negation of (iii) of a new Tamari interval diagram, with $k + 1 < l$.

□

In [Rog20] it is shown that a Tamari interval is new if and only if the associated interval-poset is new. With Proposition 2.3.4 we get the following result.

Proposition 2.3.5. *Let $n \geqslant 3$ and $(u, v) \in \mathrm{TID}(n)$. The Tamari interval diagram (u, v) is new if and only if $\rho(\chi(u, v))$ is a new Tamari interval.*

Proposition 2.3.6. *Let $n \geqslant 3$ and $(u, v) \in \mathrm{TID}(n)$. If (u, v) is synchronized, then (u, v) is not new.*

Proof. Assume by contradiction that (u, v) is synchronized and new. Since (u, v) is new, one has $u_i < n - i$ for $i \in [n - 1]$, and $v_j < j - 1$ for $j \in [2, n]$. In particular, $u_{n-1} = 0$ and $v_2 = 0$. This implies, since (u, v) is synchronized, that $u_1 \neq 0$ and $v_n \neq 0$. Furthermore, since (u, v) is new, Condition (iii) of a Tamari interval diagram is satisfied. Specifically, for any $k \in [n - 2]$, either $u_k < 1$ or $v_{k+2} < 1$. Let us denote by $(*)$ this condition. Assuming that $u_1 \neq 0$, since (u, v) is synchronized, one has either $u_2 \neq 0$ or $v_3 \neq 0$. By $(*)$, the second choice is impossible, thus $u_2 \neq 0$. By the same reasoning, for every $k \in [n - 2]$, $u_k \neq 0$. However, also by assumption one has $v_n \neq 0$. Therefore, $u_{n-2} \neq 0$ and $v_n \neq 0$ which is a contradiction with $(*)$. □

2.4. **Order structure.** Firstly, we endow the set of cubic coordinates with an order relation. Then we show that there is an isomorphism between this poset and the poset of Tamari intervals. The two bijections constructed in the first two parts of Section 2 allow us to establish this poset isomorphism.

Let $n \geqslant 0$ and $c, c' \in \mathrm{CC}(n)$. We set that $c \preccurlyeq c'$ if and only if $c_i \leqslant c'_i$ for all $i \in [n - 1]$. Endowed with $\preccurlyeq$, the set $\mathrm{CC}(n)$ is a poset called the *cubic coordinate poset*.

Recall that the map ϕ is defined at the beginning of Section 2.3 and the map χ is defined at the beginning of Section 2.2. Let $(\mathfrak{s}, \mathfrak{t}), (\mathfrak{s}', \mathfrak{t}') \in \mathrm{int}(\mathrm{T}_2(\mathrm{n}))$ and let $\psi := \phi^{-1} \circ \chi^{-1} \circ \rho^{-1}$ be the map from the Tamari interval poset to the cubic coordinate poset $\mathrm{CC}(n)$.

For the next results in all this section, let us denote by $c := \psi(\mathfrak{s}, \mathfrak{t})$, $c' := \psi(\mathfrak{s}', \mathfrak{t}')$ and $(u, v) := \phi(c)$, $(u', v') := \phi(c')$, and $\pi := \chi(u, v)$, $\pi' := \chi(u', v')$.

Lemma 2.4.1. *If $(\mathfrak{s}', \mathfrak{t}')$ covers $(\mathfrak{s}, \mathfrak{t})$, then there is a unique different component c_i between c and c' such that $c_i < c'_i$ and there is no cubic coordinate c'' different from c and c' such that $c \preccurlyeq c'' \preccurlyeq c'$.*

Proof. By Lemma 1.3.1 we know that $(\mathfrak{s}', \mathfrak{t}')$ covers $(\mathfrak{s}, \mathfrak{t})$ if and only if π and π' satisfy either $(\star)$ or $(\diamond)$. Let us assume that π and π' satisfy either $(\star)$ or $(\diamond)$ for the vertex π_i. By using (2.2.2) and (2.2.3), two cases are possible.

- $\star$ Suppose that π and π' satisfy $(\star)$, then since only decreasing relations are added in π' relative to π, only u' is modified in (u', v') relative to (u, v). Furthermore, since π' is obtained by adding decreasing relations of target π_i in π, only the letter u'_i in u' is increased relative to u. Moreover, since the number of descending relations added in π is minimal, there cannot be any Tamari interval diagram between (u, v) and (u', v'), and thus no cubic coordinate between c and c'. In the end, the image by ϕ^{-1} of (u', v') is the cubic coordinate c' with $c'_i = u'_i$ and $c'_j = c_j$ for any $j \neq i$.

- $\star$ Suppose that π and π' satisfy ($\diamond$), the arguments are roughly the same, with the difference that this time, only increasing relations are removed in π' relative to π. We obtain that only the component $c'_{i-1} = -v'_i$ of c' has increased relative to c.

In both cases, the implication is true. $\square$

Note that if there is a unique different component c_i between c and c' such that $c_i < c'_i$ and there is no cubic coordinate c'' different from c and c' such that $c \preccurlyeq c'' \preccurlyeq c'$, then in particular c' covers c. Thus, Lemma 2.4.1 has the consequence that if $(\mathfrak{s}', \mathfrak{t}')$ covers $(\mathfrak{s}, \mathfrak{t})$, then c' covers c.

Lemma 2.4.2. *Let $n \geqslant 0$ and $c, c' \in \mathrm{CC}(n)$. If $c \preccurlyeq c'$, then there is a cubic coordinate c'' such that $u'' = u$ and $v'' = v'$, where $(u'', v'') := \phi(c'')$.*

Proof. The composition of bijections $\phi^{-1} \circ \chi^{-1}$ associates a pair of words (u, v) to a pair of comparable binary trees $(\mathfrak{s}, \mathfrak{t})$ such that u encodes the binary tree $\mathfrak{s}$ and v encodes the binary tree $\mathfrak{t}$. By this composition, u (resp. v) is obtained by counting in $\mathfrak{s}$ (resp. $\mathfrak{t}$) the number of left (resp. right) descendants of each node for the infix order. Additionally, we know that if $(\mathfrak{s}, \mathfrak{t}) \preccurlyeq_{\mathrm{int(ta)}} (\mathfrak{s}', \mathfrak{t}')$, then the interval $(\mathfrak{s}, \mathfrak{t}')$ is a Tamari interval because we always have $\mathfrak{s} \preccurlyeq_{\mathrm{ta}} \mathfrak{s}' \preccurlyeq_{\mathrm{ta}} \mathfrak{t}'$. The construction of $\phi^{-1} \circ \chi^{-1}$ and the fact that $(\mathfrak{s}, \mathfrak{t}')$ is a Tamari interval imply that the pair (u, v') is always a Tamari interval diagram. Therefore, c'' is a cubic coordinate. $\square$

For any $c, c' \in \mathrm{CC}(n)$, let

$$\mathrm{D}^{-}\left(c, c'\right) := \left\{d \ : \ c_d \neq c'_d \text{ and } c'_d \leqslant 0\right\}, \tag{2.4.1}$$

and

$$\mathrm{D}^{+}\left(c, c'\right) := \left\{d \ : \ c_d \neq c'_d \text{ and } c_d \geqslant 0\right\}. \tag{2.4.2}$$

Now consider the case where c and c' share either their Tamari diagrams or their associated dual Tamari diagrams, then we have the two following lemmas.

Lemma 2.4.3. *Let $n \geqslant 0$ and $c, c' \in \mathrm{CC}(n)$. If $c \preccurlyeq c'$ such that $u = u'$ and $\#\mathrm{D}^{-}(c, c') = r$, then there is a chain*

$$\left(c = c^{(0)}, c^{(1)}, \ldots, c^{(r-1)}, c^{(r)} = c'\right), \tag{2.4.3}$$

such that $\#\mathrm{D}^{-}\left(c^{(i-1)}, c^{(i)}\right) = 1$ for all $i \in [r]$.

Proof. Let

$$\mathrm{D}^{-}\left(c, c'\right) = \{d_1, d_2, \ldots, d_r\} \tag{2.4.4}$$

with $d_{k-1} < d_k$ for all $k \in [2, r]$. For any $k \in [r]$, let $c^{(k)}$ be a tuple obtained by replacing in c all the components c_{d_i} by the components c'_{d_i} for $i \in [k]$. The tuple $c^{(k)}$ is a cubic coordinate. Indeed, by denoting $\phi(c^{(k)})$ by $(u^{(k)}, v^{(k)})$, one has that $u^{(k)} = u = u'$, so the compatibility with $v^{(k)}$ is always satisfied. Therefore, the only thing to check is that $v^{(k)}$ is a dual Tamari diagram. Condition (i) is naturally satisfied. Since $c \leqslant c'$, one has $v_i \geqslant v'_i$ for all $i \in [n]$. Therefore, Condition (ii) is satisfied because for $i \in [d_k]$ and $j \in [i+1, n]$,

$v_i^{(k)} = v'_i$ and $v_j^{(k)} = v_j$, and so $v_j^{(k)} - v_i^{(k)} = v_j - v'_i \geqslant v_j - v_i \geqslant j - i$. The word $v^{(k)}$ is then a dual Tamari diagram. Consider the chain

$$\left(c = c^{(0)}, c^{(1)}, \dots, c^{(r-1)}, c^{(r)} = c'\right). \tag{2.4.5}$$

For all $i \in [r]$, since we change only one component between $c^{(i-1)}$ and $c^{(i)}$, one has $\#\mathrm{D}^-\left(c^{(i-1)}, c^{(i)}\right) = 1$. □

Lemma 2.4.4. *Let $n \geqslant 0$ and $c, c' \in \mathrm{CC}(n)$. If $c \preccurlyeq c'$ such that $v = v'$ and $\mathrm{D}^+(c, c') = s$, then there is a chain*

$$\left(c = c^{(0)}, c^{(1)}, \dots, c^{(s-1)}, c^{(s)} = c'\right), \tag{2.4.6}$$

such that $\#\mathrm{D}^+\left(c^{(i-1)}, c^{(i)}\right) = 1$ for all $i \in [s]$.

Proof. The proof is similar to the demonstration of Lemma 2.4.3. Let

$$\mathrm{D}^+\left(c, c'\right) = \{d_1, d_2, \dots, d_s\} \tag{2.4.7}$$

with $d_{k-1} < d_k$ for all $k \in [2, s]$. For any $k \in [s]$, let $c^{(k)}$ be a tuple obtained by replacing in c all the components c_{d_i} by the components c'_{d_i} for $i \in [k]$. As we did in the proof of Lemma 2.4.3, we can check that, for any $k \in [s]$, the tuple $c^{(k)}$ is a cubic coordinate. Then, by consider the chain

$$\left(c = c^{(0)}, c^{(1)}, \dots, c^{(s-1)}, c^{(s)} = c'\right), \tag{2.4.8}$$

one has that $\#\mathrm{D}^+\left(c^{(i-1)}, c^{(i)}\right) = 1$ for all $i \in [s]$.

□

Theorem 2.4.5. *For any $n \geqslant 0$, the map ψ is a poset isomorphism.*

Proof. The map ψ is an isomorphism of posets if ψ and its inverse preserves the partial order. As these relations are transitive, Lemma 2.4.1 gives the direct implication. Suppose that $c \preccurlyeq c'$. According to Lemma 2.4.2, Lemma 2.4.3 and Lemma 2.4.4 there is always a chain between c and c' such that the components are independently increasing one by one. So we can see what happens when we change only one component c_i by c'_i at any step between c and c'.

Obviously, if $c_i = c'_i$, then $u_i = u'_i$ and $v_{i+1} = v'_{i+1}$ and no changes are made between the corresponding binary tree pairs. Suppose that $c_i < c'_i$, then three cases are possible.

- ⋆ Suppose that c'_i is positive and c_i is positive or null. The image by ϕ of c and c' differ for the letter u_i, namely $c'_i = u'_i$ and $c_i = u_i$, and $v_{i+1} = v'_{i+1} = 0$. The difference of a letter u_i between (u, v) and (u', v') is directly translated by the map χ: the interval-poset π' has more decreasing relations of target π_i than the vertex π_i in π. By the map ρ, it means that to go from the tree $\mathfrak{s}$ to the tree $\mathfrak{s}'$ at least one right rotation of the edge (i, j) is made, where j is the father of the node i in $\mathfrak{s}$.
- ⋆ Symmetrically, assume that c'_i is negative or null, then $c'_i = -v'_{i+1}$, $c_i = -v_{i+1}$ and $u_i = u'_i = 0$. By the map χ, the interval-poset π' has less decreasing relations of target π_{i+1} than the vertex π_{i+1} in π. This implies by ρ that to pass from the tree $\mathfrak{t}$ to the tree $\mathfrak{t}'$ at least one right rotation of the edge $(k, i+1)$ is made, where k is the right child of the node $i+1$ in $\mathfrak{t}$.

- ⋆ Finally, with Lemma 2.4.2, the case where c_i is negative and c'_i is positive falls into the conjunction of the two previous cases.

Therefore, $c \preccurlyeq c'$ implies that $(\mathfrak{s}, \mathfrak{t}) \preccurlyeq_{\mathrm{int(ta)}} (\mathfrak{s}', \mathfrak{t}')$. Hence, the map ψ is an isomorphism of posets. □

Let us denote by $\lessdot$ the covering relation of the poset $\mathrm{CC}(n)$.

Proposition 2.4.6. *Let $n \geqslant 0$ and $c, c' \in \mathrm{CC}(n)$ such that $c \lessdot c'$. Then, there is a unique different component between c and c'.*

Proof. It is a consequence of Theorem 2.4.5 and Lemma 2.4.1. □

The following diagram provides a summary of the applications used in Section 2. Recall that $\psi = \phi^{-1} \circ \chi^{-1} \circ \rho^{-1}$, therefore this diagram of poset isomorphisms is commutative.

$$\begin{tikzcd} \mathrm{TID}(n) \arrow[r, "\chi"] & \mathrm{IP}(n) \arrow[d, "\rho"] \\ \mathrm{CC}(n) \arrow[u, "\phi"] & \mathrm{int}(\mathrm{T}_2(\mathrm{n})) \arrow[l, "\psi"'] \end{tikzcd} \tag{2.4.9}$$

A consequence of the poset isomorphism ψ is that the order dimension [MP90, Tro02] of the poset of Tamari intervals is at most $n - 1$.

3. Geometric properties

In this section, we give a very natural geometrical realization for the lattices of cubic coordinates. After defining the cells of this realization, we give some properties related to them. Finally, we show that the lattice of the cubic coordinates is EL-shellable.

3.1. Cubic realizations.

Theorem 2.4.5 provides a simpler translation of the order relation between two Tamari intervals. We provide the geometrical realization induced by this order relation, which is natural for cubic coordinates. In a combinatorial way we study the cells formed by this realization.

For any $n \geqslant 0$, the *cubic realization* of $\mathrm{CC}(n)$ is the geometric object $\mathfrak{C}(\mathrm{CC}(n))$ defined in the space $\mathbb{R}^{n-1}$ and obtained by placing for each $c \in \mathrm{CC}(n)$ a vertex of coordinates $(c_1, \ldots, c_{n-1})$, and by forming for each $c, c' \in \mathrm{CC}(n)$ such that $c \lessdot c'$ an edge between c and c'. Every edge of $\mathfrak{C}(\mathrm{CC}(n))$ is parallel to some vector in the canonical basis of $\mathbb{R}^{n-1}$.

Figure 12 shows the cubic realization of CC(3), where the elements are the vertices and the edges are the covering relations. Figure 13 shows the cubic realization of CC(4). In these drawings the negative sign components are denoted with a bar.

In algebraic topology, to define the tensor products of A_∞-algebras, one can use a cell complex called the *diagonal of the associahedron*. This complex has notably been studied by Loday [Lod11], by Saneblidze and Umble [SU04], and by Markl and Shnider [MS06]. More recently, there is a description of this object in [MTTV21]. The realization of this complex seems to be identical to the cubic realization, up to continuous deformation.

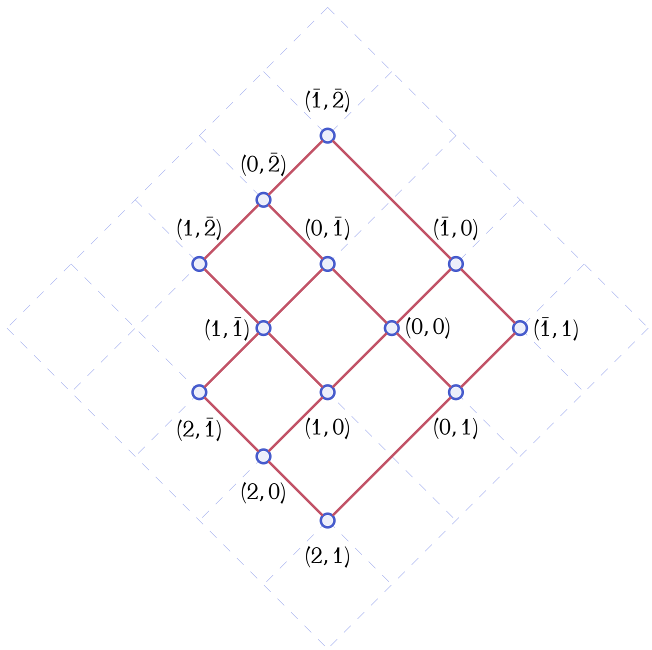


FIGURE 12. $\mathfrak{C}(\mathrm{CC}(3))$.

3.2. **Covering map.** Let $n \geqslant 0$. We define the set of

- ⋆ *input-wings* as the set $\mathcal{I}(\mathrm{CC}(n))$ containing any $c \in \mathrm{CC}(n)$ which covers exactly $n-1$ elements,
- ⋆ *output-wings* as the set $\mathcal{O}(\mathrm{CC}(n))$ containing any $c \in \mathrm{CC}(n)$ which is covered by exactly $n-1$ elements.

Let $n \geqslant 0$ and $c \in \mathrm{CC}(n)$. For $i \in [n-1]$, the *covering map* $\uparrow_i$ sends c to its covering differing only at index i, when such covering exists. We denote by $\uparrow c_i$ the letter which differs in $\uparrow_i(c)$.

In particular, for $n \geqslant 0$, a cubic coordinate c of size n is an output-wing if for any $i \in [n-1]$, $\uparrow_i(c)$ is well-defined.

Let $n \geqslant 0$ and $c \in \mathrm{CC}(n)$, and $(u, v) := \phi(c)$. If $\uparrow c_i$ is positive, then the letter u_i increases and becomes equal to $\uparrow c_i$ and v_{i+1} is equal to 0. Then, we define $\uparrow u_i := \uparrow c_i$. If $\uparrow c_i$ is negative or null, then v_{i+1} decreases and becomes equal to $|\uparrow c_i|$ and u_i is equal to 0. Then, we set $\downarrow v_{i+1} := -\uparrow c_i$.

Lemma 3.2.1. *Let $n \geqslant 0$ and $c \in \mathrm{CC}(n)$, and $i \in [n-1]$ such that $\uparrow_i(c)$ is well-defined. Then,*

(i) if $c_i < 0$, then $\uparrow c_i \leqslant 0$,
(ii) if $c_i \geqslant 0$, then $\uparrow c_i > 0$.

Proof. Let us show the first implication, the second being obvious because the covering map always strictly increases a component. Let $c_i < 0$, and let c' be the $(n-1)$-tuple such that $c'_i = 0$ and $c'_j = c_j$ for any $j \neq i$, with $j \in [n-1]$. By Lemma 2.3.2, c' is a cubic coordinate. As $c \leqslant c'$ and they differ only at the i-th component, by the definition of $\uparrow_i(c)$, we have $c \leqslant \uparrow_i(c) \leqslant c'$, thus $\uparrow c_i \leqslant c'_i = 0$. □

Let $c \in \mathrm{CC}(n)$. For all $i \in [n]$, let

$$\Uparrow_i(c) := \uparrow_i(\uparrow_{i+1} \dots (\uparrow_{n-1}(\uparrow_n(c)))), \tag{3.2.1}$$

with the convention that $\uparrow_n(c) := c$. For instance, for $c \in \mathrm{CC}(5)$, $\Uparrow_2(c) = \uparrow_2(\uparrow_3(\uparrow_4(\uparrow_5(c))))$.

Lemma 3.2.2. *Let $n \geqslant 0$ and $c \in \mathcal{O}(\mathrm{CC}(n))$. For all $i \in [n]$, $\Uparrow_i(c)$ is a cubic coordinate.*

Proof. For $i = n$, one has by convention that $\Uparrow_n(c)$ is a cubic coordinate. Let us suppose that for $i \in [n-1]$, $\Uparrow_{i+1}(c)$ is a cubic coordinate, and let us show that $\Uparrow_i(c)$ is also a cubic coordinate. Depending on the sign of $\Uparrow_{i+1}(c)_i$, two cases are possible.

Suppose that $\Uparrow_{i+1}(c)_i < 0$. In this case, consider c' the $(n-1)$-tuple obtained from $\Uparrow_{i+1}(c)$ by replacing the component $\Uparrow_{i+1}(c)_i$ by 0. By Lemma 2.3.2, c' is a cubic coordinate. Since $\Uparrow_{i+1}(c)_i < 0$ one has $\Uparrow_{i+1}(c) \preccurlyeq c'$. If c' covers $\Uparrow_{i+1}(c)$, then $c' = \Uparrow_i(c)$. Otherwise, it is always possible to find another cubic coordinate c'' between $\Uparrow_{i+1}(c)$ and c' such that $c'' = \Uparrow_i(c)$. In both cases, $\Uparrow_i(c)$ is a cubic coordinate.

Suppose that $\Uparrow_{i+1}(c)_i \geqslant 0$. Let us set $(u, v) := \phi(c)$, and $(x, y) := \phi(\Uparrow_{i+1}(c))$. Since u_i is not changed yet in x, one has $x_i = u_i$. Due to Condition (ii) of a Tamari diagram and the compatibility condition, there are two configurations, involving indices, which can make contradiction with the fact that (x, y) is still a Tamari interval diagram when x_i becomes $\uparrow x_i$.

(1) If there is an index j such that $1 \leqslant i < j \leqslant n$ and $y_j \geqslant j - i$ in y, then, since $y_j < v_j$, one has $v_j \geqslant j - i$ in v. By the compatibility condition, that implies $u_i < j - i$ in u. Moreover, since c is assumed to be an output-wing, $u_i < j - i - 1$ in u, so that u_i can be increased. This inequality remains true in x.
(2) If there is an index h such that $1 \leqslant i - h \leqslant u_h$, by Condition (ii) of a Tamari diagram, $u_i \leqslant u_h - i + h$ in u. This remains true in x because components with index smaller than i remain unchanged between c and $\Uparrow_{i+1}(c)$. Furthermore, since c is an output-wing, then $u_i < u_h - i + h$. This inequality remains true for $\Uparrow_{i+1}(c)$.

With these two configurations, let us build a cubic coordinate c' different from $\Uparrow_{i+1}(c)$ only for $\Uparrow_{i+1}(c)_i$, depending on which choices are available to increase u_i. Let us set $(u', v') := \phi(c')$.

(a) Suppose there is a j satisfying (1), and there is no h satisfying (2) in $\Uparrow_{i+1}(c)$. In this case, by choosing the minimal index j such that (1) holds, we set $u'_i := j - i - 1$ in c'. Thus, u'_i is also minimized, and since $u'_i < j - i$, the compatibility condition is satisfied in c'. Furthermore, since $\Uparrow_{i+1}(c)$ is assumed to be a cubic coordinate, all conditions in a Tamari diagram and a dual Tamari diagram are satisfied for c'. Therefore, our candidate c' is a cubic coordinate. Note that in the construction of c', other possible not minimal j satisfying (1) will not cause any problem.
(b) Suppose there is an h satisfying (2), and there is no j satisfying (1) in $\Uparrow_{i+1}(c)$. Then, by choosing the minimal index h such that (2) holds, we set $u'_i := u'_h -$

$i+h$. Therefore, Condition (ii) of a Tamari diagram is satisfied for u'. Also, by Condition (i) of a Tamari diagram, $u'_h \leqslant n-h$ which implies $u'_i \leqslant n-i$. Finally, the compatibility condition is also satisfied because it was assumed that there was no j satisfying (1). The tuple c' is thus a cubic coordinate. As for the previous case, other possible not minimal h satisfying (2) will not cause any problem.

(c) Suppose there is a j and an h satisfying (1) and (2) in $\Uparrow_{i+1}(c)$. In this case, we set $u'_i := \min\{u'_h - i + h,\ j - i - 1\}$. By the two previous cases, the tuple c' is a cubic coordinate.

(d) Otherwise, we set $u'_i := n - i$. The tuple c' is a cubic coordinate.

In any case, for u'_i fixed in c', either c' covers $\Uparrow_{i+1}(c)$, and so $c' = \Uparrow_i(c)$, or there is a cubic coordinate c'' between $\Uparrow_{i+1}(c)$ and c' such that $c'' = \Uparrow_i(c)$. In both cases, $\Uparrow_i(c)$ is a cubic coordinate, and differs by only one component from c'. □

Let $n \geqslant 0$ and $c \in \mathcal{O}(\mathrm{CC}(n))$. The cubic coordinate $\Uparrow_1(c)$ is the *corresponding input-wing* of c (the name comes from a corollary of Theorem 3.3.1). For instance $c = (0,-1,1,-1,-5,0,1,-1,-3)$ is an output-wing, and its corresponding input-wing is $\Uparrow_1(c) = (1,0,2,0,-4,3,2,0,-2)$. By Lemma 3.2.2 such an element does exist. Note that performing the covering map on c in a different order than the one prescribed by (3.2.1) does not always result in the corresponding input-wing. This observation can already be made on the two pentagons of Figure 12.

3.3. Cells and synchronized cubic coordinates.

In Figure 12 and Figure 13, we notice that a "cellular" organization appears. Thanks to the cubic coordinates, a combinatorial definition of these cells is provided. The aim is to have a better understanding of the realization of the cubic coordinate posets as a geometrical object.

For any $n \geqslant 0$, let $c, c' \in \mathrm{CC}(n)$ such that $c \preccurlyeq c'$. A *cell* is the set of points

$$\langle c, c' \rangle := \left\{ x \in \mathbb{R}^{n-1} : c_i \leqslant x_i \leqslant c'_i \text{ for all } i \in [n-1] \right\}. \tag{3.3.1}$$

By the definition, a cell is an orthotope, that is, a parallelotope whose edges are all mutually orthogonal or parallel. The *dimension* $\dim\langle c, c'\rangle$ of a cell $\langle c, c'\rangle$ is its dimension as an orthotope and it satisfies $\dim\langle c,c'\rangle = \#\mathrm{D}(c,c')$, where $\mathrm{D}(c,c') := \{d : c_d \neq c'_d\}$.

From now on, we denote by c^{out} any output-wing and by c^{in} its corresponding input-wing. Any particular cell $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$ formed by an output-wing and by its corresponding input-wing is called a *cell-wing*.

A consequence of Lemma 3.2.1 is that for any cell-wing $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$ of dimension $n-1$, for all $i \in [n-1]$,

(i) if $c^{\mathrm{out}}_i < 0$, then $c^{\mathrm{in}}_i \leqslant 0$,
(ii) if $c^{\mathrm{out}}_i \geqslant 0$, then $c^{\mathrm{in}}_i > 0$.

Theorem 3.3.1. *Let $n \geqslant 1$ and $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$ be a cell-wing of dimension $n-1$, and c be a $(n-1)$-tuple such that for all $i \in [n-1]$, the component c_i is equal either to c^{out}_i or to c^{in}_i. Then c is a cubic coordinate.*

Proof. If all the components of c are equal to those of c^{out} (resp. to those of c^{in}), then c is a cubic coordinate. Suppose this is not the case, meaning that c has components of c^{out} and c^{in}.

Let us denote $(u_i^{\text{out}}, v_{i+1}^{\text{out}})$ (resp. $(u_i^{\text{in}}, v_{i+1}^{\text{in}})$) the pair of letters corresponding to c_i^{out} (resp. c_i^{in}) and (u_i, v_{i+1}) the one corresponding to c_i for any $i \in [n-1]$. By hypothesis on c^{out} and c^{in} the letter u_i which is equal to u_i^{out} or u_i^{in} satisfies $0 \leqslant u_i \leqslant n - i$ for any $i \in [n]$. Similarly, the letter v_i which is equal to v_i^{out} or v_i^{in} satisfies $0 \leqslant v_i \leqslant i - 1$ for any $i \in [n]$. In order to show that c is a cubic coordinate, let us prove that u satisfies Condition (ii) of a Tamari diagram, v satisfies Condition (ii) of a dual Tamari diagram and (u, v) satisfies the compatibility condition.

(i) Let us show that for any choice of letters u_i and u_{i+j} with $i \in [n]$ and $j \in [0, u_i]$ one has $u_{i+j} \leqslant u_i - j$.
   - ⋆ If u_i and u_{i+j} are equal respectively to u_i^{out} and to u_{i+j}^{out} (resp. to u_i^{in} and to u_{i+j}^{in}), then Condition (ii) of a Tamari diagram is satisfies because c^{out} (resp. c^{in}) is a cubic coordinate.
   - ⋆ Suppose that $u_i = u_i^{\text{in}}$ and $u_{i+j} = u_{i+j}^{\text{out}}$. By the definition of c^{in} one has $u_{i+j}^{\text{out}} < u_{i+j}^{\text{in}}$. However $u_{i+j}^{\text{in}} \leqslant u_i^{\text{in}} - j$ because c^{in} is a cubic coordinate. Therefore, Condition (ii) of a Tamari diagram is satisfied.
   - ⋆ Suppose that $u_i = u_i^{\text{out}}$ and $u_{i+j} = u_{i+j}^{\text{in}}$. Let $c' :=\Uparrow_{i+j} (c^{\text{out}})$. According to Lemma 3.2.2 c' is a cubic coordinate such that $c'_i = u_i^{\text{out}}$ and $c'_{i+j} = u_{i+j}^{\text{in}}$. Since Condition (ii) of a Tamari diagram is satisfied for c', it must also be satisfied for c.

(ii) Condition (ii) of a dual Tamari diagram is satisfied with similar arguments given for the previous case, applied to the dual Tamari diagram v.

(iii) Rather than showing the compatibility condition as it is stated, let us show the contrapositive. That is, for every $1 \leqslant i < j \leqslant n$ such that $v_j \geqslant j - i$, let us show that $u_i < j - i$.
   - ⋆ Clearly, if u_i and v_j are equal to u_i^{out} and v_j^{out} (resp. to u_i^{in} and v_j^{in}), then the compatibility condition is satisfied.
   - ⋆ Suppose that $u_i = u_i^{\text{out}}$ and $v_j = v_j^{\text{in}}$. If $v_j^{\text{in}} \geqslant j - i$, then for c^{out} one has $v_j^{\text{out}} \geqslant j - i$ because $v_j^{\text{in}} < v_j^{\text{out}}$. Since c^{out} is a cubic coordinate, this implies that $u_i^{\text{out}} < j - i$.
   - ⋆ Suppose that $u_i = u_i^{\text{in}}$ and $v_j = v_j^{\text{out}}$. If $v_j^{\text{out}} \geqslant j - i$, then for all $k \in [i, j-1]$, $u_k^{\text{out}} < j - k$ because c^{out} is a cubic coordinate and then satisfies the compatibility condition. Moreover, since $c^{\text{out}} \in \mathcal{O}(\mathrm{CC}(n))$ each component can be minimally increased independently of the others, thus $u_k^{\text{out}} < j - k - 1$ for all $k \in [i, j-1]$. For the same reason $u_{i+h} < u_i - h$ for all $h \in [0, u_i]$. These two reasons imply that if one builds the cubic coordinate $c' =\Uparrow_i (c^{\text{out}})$, then by the definition of the covering map one has $c'_i = u'_i < j - i$, because at worst, the covering map sends u_i^{out} to $j - i - 1$ (we have already seen this in the proof of Lemma 3.2.2). However, by the definition of c^{in} one has $u_i^{\text{in}} = u'_i$, that is $u_i^{\text{in}} < j - i$. Therefore, the compatibility condition between u^{in} and v_j^{out} is satisfied for c.

Thus, for all choices of letters of u and v one has that c is a cubic coordinate. □

One of the direct consequences of Theorem 3.3.1 is that for every cell-wing $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$, at least 2^{n-1} cubic coordinates belong to this cell.

This theorem also implies that a corresponding input-wing covers $n-1$ cubic coordinates, and so is in particular an input-wing.

Moreover, due to the fact the Tamari interval lattice is self-dual, the number of output-wings is equal to the number of input-wings. Therefore, by Theorem 2.4.5, an input-wing is always a corresponding input-wing of some output-wing.

Let $n \geqslant 0$, and $\epsilon \in \{-1, 1\}^{n-1}$, and $c \in \mathrm{CC}(n)$. The *ϵ-region* of c is the set

$$\mathcal{R}_\epsilon(c) := \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \ : \ x_i < c_i \text{ if } \epsilon_i = -1, \ x_i > c_i \text{ otherwise}\}. \tag{3.3.2}$$

The cubic coordinate c is *external* if there is $\epsilon \in \{-1, 1\}^{n-1}$ such that $\mathrm{CC}(n) \cap \mathcal{R}_\epsilon(c) = \emptyset$. The ϵ-region $\mathcal{R}_\epsilon(c)$ is then *empty*. Otherwise, c is *internal*.

Proposition 3.3.2. *Let $n \geqslant 0$ and $c \in \mathrm{CC}(n)$. If c is internal, then $\phi(c)$ is a new Tamari interval diagram.*

Proof. Instead, let us show that if $\phi(c)$ is not new, then c is external. Let us denote (u_i, v_{i+1}) the pair of letters corresponding to c_i by the map ϕ for $i \in [n-1]$.

Tamari interval diagram $\phi(c)$ is not new if there is

(1) either $i \in [n-1]$ such that $u_i = n - i$,
(2) or $j \in [2, n]$ such that $v_j = j - 1$,
(3) or $k, l \in [n]$ such that $u_k = l - k - 1$ and $v_l = l - k - 1$ with $k + 1 < l$.

Suppose there is some i satisfying (1), then there cannot be a cubic coordinate c' such that $c'_i > c_i$ because, by the definition of a Tamari diagram, $c'_i \leqslant n - i$. Similarly, if we assume that there is j satisfying (2), then there cannot be a cubic coordinate c' such that $c'_{j-1} < c_{j-1}$ because by the definition of a dual Tamari diagram, $c'_{j-1} \geqslant 1 - j$. If (3) is satisfied, then there cannot be a cubic coordinate c' such that $c'_k > c_k$ and $c'_{l-1} < c_{l-1}$. Indeed, if the letters u_k and v_l are increased in c, then the compatibility condition is contradicted, so the result cannot be a cubic coordinate. Since in each case at least one ϵ-region is empty, c is external. □

Proposition 3.3.3. *Let $n \geqslant 0$ and $c \in \mathrm{SCC}(n)$. Then c is external.*

Proof. By Proposition 2.3.6 we know that if c is synchronized, then $\phi(c)$ is not new. Now, we just saw from Proposition 3.3.2 that if $\phi(c)$ is not new, then c is external. □

We know that each cell-wing contains at least 2^{n-1} cubic coordinates on the edges. Now, let us show that it is possible to associate bijectively each cell-wing to a synchronized cubic coordinate.

Let $n \geqslant 1$ and $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$ be a cell-wing of dimension $n-1$ and γ be the map defined by

$$\gamma(c_i^{\mathrm{out}}, c_i^{\mathrm{in}}) := \begin{cases} c_i^{\mathrm{out}} & \text{if } c_i^{\mathrm{out}} < 0, \\ c_i^{\mathrm{in}} & \text{if } c_i^{\mathrm{out}} \geqslant 0, \end{cases} \tag{3.3.3}$$

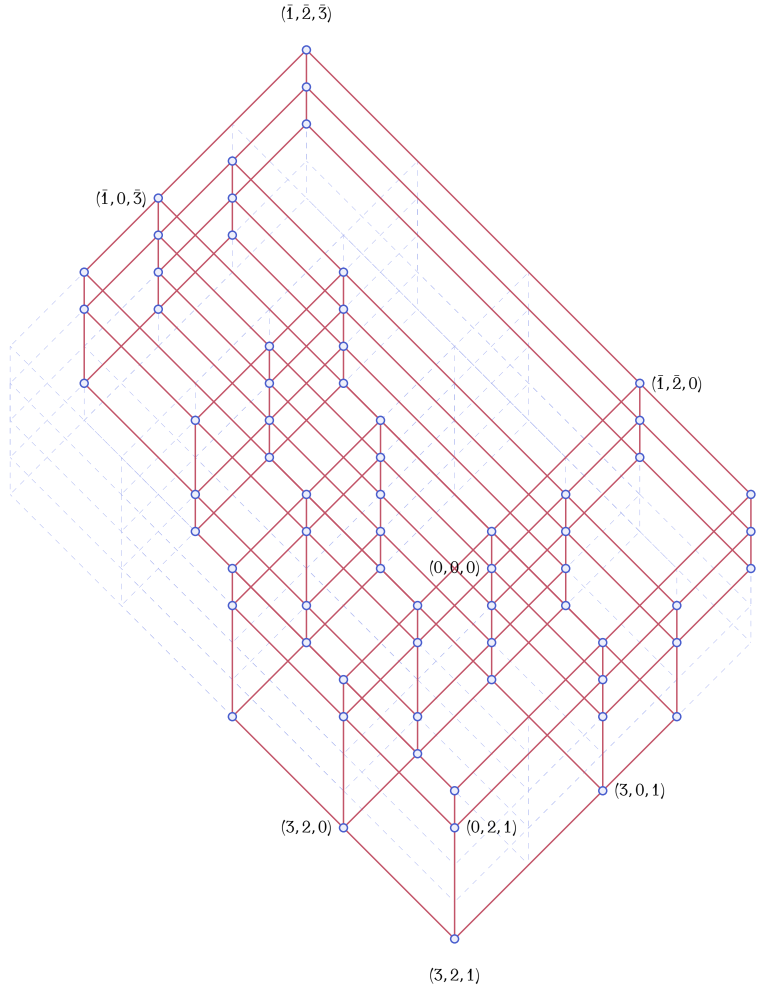


FIGURE 13. $\mathfrak{C}(\mathrm{CC}(4))$.

for all $i \in [n-1]$. Note that the components returned by the map γ are never zero. Let denote by $(u_i^{\mathrm{out}}, v_{i+1}^{\mathrm{out}})$ (resp. $(u_i^{\mathrm{in}}, v_{i+1}^{\mathrm{in}})$) the pair of letters corresponding to c_i^{out} (resp. c_i^{in}) by the map ϕ, for any $i \in [n-1]$. Thus, the map γ becomes

$$\gamma(c_i^{\mathrm{out}}, c_i^{\mathrm{in}}) := \begin{cases} -v_{i+1}^{\mathrm{out}} & \text{if } c_i^{\mathrm{out}} < 0, \\ u_i^{\mathrm{in}} & \text{if } c_i^{\mathrm{out}} \geqslant 0. \end{cases} \tag{3.3.4}$$

Let Γ be the map defined by

$$\Gamma\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle := (\gamma(c_1^{\mathrm{out}}, c_1^{\mathrm{in}}), \gamma(c_2^{\mathrm{out}}, c_2^{\mathrm{in}}), \dots, \gamma(c_{n-1}^{\mathrm{out}}, c_{n-1}^{\mathrm{in}})). \tag{3.3.5}$$

For instance, the cell-wing $\langle(0,-1,1,-1,-5,0,1,-1,-3),(1,0,2,0,-4,3,2,0,-2)\rangle$ is sent by Γ to $(1,-1,2,-1,-5,3,2,-1,-3)$.

Theorem 3.3.4. *For any $n \geqslant 1$, the map Γ is a bijection from the set of cell-wings of dimension $n-1$ to $\mathrm{SCC}(n)$.*

Proof. The components of $\Gamma\langle c^{\text{out}}, c^{\text{in}}\rangle$ belong to either c^{out} or c^{in}. In both cases, it is a non-zero component. According to Theorem 3.3.1, $\Gamma\langle c^{\text{out}}, c^{\text{in}}\rangle$ is therefore a cubic coordinate of size n. Moreover, this cubic coordinate is synchronized because none of its components is null.

Let $\langle c^{\text{out}}, c^{\text{in}}\rangle$ and $\langle e^{\text{out}}, e^{\text{in}}\rangle$ be two cell-wings of dimension $n-1$ such that $\Gamma\langle c^{\text{out}}, c^{\text{in}}\rangle = \Gamma\langle e^{\text{out}}, e^{\text{in}}\rangle$. Let us denote $(u_i^{\text{out}}, v_{i+1}^{\text{out}})$ (resp. $(u_i^{\text{in}}, v_{i+1}^{\text{in}})$) the pair of letters corresponding to c_i^{out} (resp. c_i^{in}) and $(x_i^{\text{out}}, y_{i+1}^{\text{out}})$ (resp. $(x_i^{\text{in}}, y_{i+1}^{\text{in}})$) the pair of letters corresponding to e_i^{out} (resp. e_i^{in}) by the map ϕ, for all $i \in [n-1]$.

To suppose that $\Gamma\langle c^{\text{out}}, c^{\text{in}}\rangle = \Gamma\langle e^{\text{out}}, e^{\text{in}}\rangle$ is equivalent to suppose that for all $i \in [n-1]$, $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = \gamma(e_i^{\text{out}}, e_i^{\text{in}})$. The map Γ is injective if, for every $i \in [n-1]$, $c_i^{\text{out}} = e_i^{\text{out}}$ and $c_i^{\text{in}} = e_i^{\text{in}}$. Suppose that there is some index i such that $c_i^{\text{out}} \neq e_i^{\text{out}}$ or $c_i^{\text{in}} \neq e_i^{\text{in}}$, and we take the smallest such index. Then, two cases have to be considered: either $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = u_i^{\text{in}}$ or $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = -v_{i+1}^{\text{out}}$.

(1) Suppose that $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = u_i^{\text{in}}$.
- ⋆ In this case, $\gamma(e_i^{\text{out}}, e_i^{\text{in}}) = x_i^{\text{in}}$ and $u_i^{\text{in}} = x_i^{\text{in}}$. Moreover, since $u_i^{\text{in}} \neq 0$ (resp. $x_i^{\text{in}} \neq 0$), then necessarily $v_{i+1}^{\text{in}} = 0$ (resp. $y_{i+1}^{\text{in}} = 0$). Therefore, $c_i^{\text{in}} = e_i^{\text{in}}$.
- ⋆ On the other hand , the fact that $u_i^{\text{in}} > 0$ (resp. $x_i^{\text{in}} > 0$) implies by Lemma 2.3.2 that $0 \leqslant u_i^{\text{out}} < u_i^{\text{in}}$ and $v_{i+1}^{\text{out}} = 0$ (resp. $0 \leqslant x_i^{\text{out}} < x_i^{\text{in}}$ and $y_{i+1}^{\text{out}} = 0$). Thus, one has $v_{i+1}^{\text{out}} = y_{i+1}^{\text{out}}$. Therefore, the only way for the hypothesis to be true is that $u_i^{\text{out}} \neq x_i^{\text{out}}$.

  Without loss of generality, suppose that $u_i^{\text{out}} < x_i^{\text{out}}$. By the definition of the covering map, one has $x_i^{\text{out}} < x_i^{\text{in}}$. This implies, in addition to the hypothesis that $x_i^{\text{in}} = u_i^{\text{in}}$, that $u_i^{\text{out}} < x_i^{\text{out}} < u_i^{\text{in}}$.

  Let $c := \Uparrow_{i+1}(c^{\text{out}})$ and $e := \Uparrow_{i+1}(e^{\text{out}})$, both cubic coordinates by Lemma 3.2.2. By construction, $c_j = c_j^{\text{out}}$ (resp. $e_j = e_j^{\text{out}}$) for all $j \in [i]$ and $c_k = c_k^{\text{in}}$ (resp. $e_k = e_k^{\text{in}}$) for all $k \in [i+1, n-1]$.

  By minimality of i, we have that $c_j = e_j$ for all $j \in [i]$. Moreover, by the hypothesis that $\Gamma\langle c^{\text{out}}, c^{\text{in}}\rangle = \Gamma\langle e^{\text{out}}, e^{\text{in}}\rangle$, we have that $u_k^{\text{in}} = x_k^{\text{in}}$ for $k \in [i+1, n-1]$. Indeed, if $u_k^{\text{in}} > 0$ (resp. $x_k^{\text{in}} > 0$) then necessarily $\gamma(c_k^{\text{out}}, c_k^{\text{in}}) = u_k^{\text{in}}$ (resp. $\gamma(e_k^{\text{out}}, e_k^{\text{in}}) = x_k^{\text{in}}$) and so $u_k^{\text{in}} = x_k^{\text{in}}$. Otherwise, $u_k^{\text{in}} = x_k^{\text{in}} = 0$. Note that because we know nothing about v_k^{in} and y_k^{in} for $k \in [i+2, n]$, we cannot say that $\uparrow_i(c)$ and $\uparrow_i(e)$ are equal.

  Now, let c' be a tuple such that $c'_i = x_i^{\text{out}}$ and $c'_j = c_j$ for all $j \neq i$ and let (u', v') the pair of words corresponding to c' by the map ϕ. Let us show that c' is a cubic coordinate.

  By construction, since the word v' is the dual Tamari diagram of c, v' is a dual Tamari diagram. Likewise, since the word u' is the Tamari diagram of $\uparrow_i(e)$, u' is a Tamari diagram.

  Moreover, we know that between c, c' and $\uparrow_i(c)$, only one positive letter changes, with $c_i = u_i^{\text{out}}$, $c'_i = x_i^{\text{out}}$ and $\uparrow c_i = u_i^{\text{in}}$, and we have established that $u_i^{\text{out}} < x_i^{\text{out}} < u_i^{\text{in}}$. Since the letter u_i^{in} satisfies the compatibility condition with the letters of v^{in} in $\uparrow_i(c)$, then all letter lower in position i satisfies this condition

as well. Therefore, u' and v' are compatible and c' is a cubic coordinate distinct from c and $\uparrow_i (c)$ such that $c \preccurlyeq c' \preccurlyeq \uparrow_i (c)$.
However, if c' is a cubic coordinate, then by the definition of the covering map $\uparrow c_i := u_i^{\text{in}} = x_i^{\text{out}}$, and so $\uparrow_i (c) := \Uparrow_i (c^{\text{out}}) = c'$. This is not possible with the assumption that $u_i^{\text{in}} = x_i^{\text{in}}$, and so that $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = \gamma(e_i^{\text{out}}, e_i^{\text{in}})$.

(2) Suppose that $\gamma(c_i^{\text{out}}, c_i^{\text{in}}) = -v_{i+1}^{\text{out}}$. In this case $\gamma(e_i^{\text{out}}, e_i^{\text{in}}) = -y_{i+1}^{\text{out}}$ and $v_{i+1}^{\text{out}} = y_{i+1}^{\text{out}}$. By rephrasing the arguments of the case (1) for the dual, we show that $c_i^{\text{out}} = e_i^{\text{out}}$ and $c_i^{\text{in}} = e_i^{\text{in}}$.

This shows that the map Γ is injective.

Now let us show that the cardinal of the set of cell-wings of dimension $n-1$ is equal to the cardinal of $\mathrm{SCC}(n)$. Recall that the set of cells of size n is exactly $\mathcal{O}(\mathrm{CC}(n))$. Furthermore, by the poset isomorphism ψ we know that these elements are the Tamari intervals having $n-1$ elements covering in the Tamari interval lattices. In [Cha18] Chapoton shows that the set of these Tamari intervals has the same cardinal as the set of synchronized Tamari intervals (see Theorem 2.1 and Theorem 2.3 from [Cha18]). Finally, Proposition 2.3.3 allows us to conclude that the cardinal of $\mathrm{SCC}(n)$ and the cardinal of the set of cell-wings of dimension $n-1$ are equal. Thus, the map Γ is bijective. □

Let us also defined the map $\bar{\gamma}$ by

$$\bar{\gamma}(c_i^{\text{out}}, c_i^{\text{in}}) := \begin{cases} c_i^{\text{in}} & \text{if } c_i^{\text{out}} < 0, \\ c_i^{\text{out}} & \text{if } c_i^{\text{out}} \geqslant 0, \end{cases} \tag{3.3.6}$$

for all $i \in [n-1]$. Then $\bar{\Gamma}$ is defined by

$$\bar{\Gamma}\langle c^{\text{out}}, c^{\text{in}}\rangle := (\bar{\gamma}(c_1^{\text{out}}, c_1^{\text{in}}), \bar{\gamma}(c_2^{\text{out}}, c_2^{\text{in}}), \dots, \bar{\gamma}(c_{n-1}^{\text{out}}, c_{n-1}^{\text{in}})). \tag{3.3.7}$$

By Theorem 3.3.1, $\bar{\Gamma}\langle c^{\text{out}}, c^{\text{in}}\rangle$ is a cubic coordinate belonging to $\langle c^{\text{out}}, c^{\text{in}}\rangle$, called *opposite cubic coordinate*. For the synchronized cubic coordinate c associated with $\langle c^{\text{out}}, c^{\text{in}}\rangle$ by Γ, denote c^{op} the opposite cubic coordinate. All the components of c^{op} are different from those of c, and these differences are the greatest possible. For any synchronized cubic coordinate c, such a cubic coordinate c^{op} always exists and is unique.

Note that the map Γ only returns the positive components of c^{in} and the negative components of c^{out}. Conversely, the map $\bar{\Gamma}$ returns the positive components of c^{out} and the negative components of c^{in}. We already know that the latter combination is always possible for any comparable cubic coordinates according to Lemma 2.4.2. On the other hand, this is not the case for the first mentioned combination.

3.4. **Volume of $\mathfrak{C}(\mathrm{CC})$.** Now let us take a closer look at the geometry of the cubic realization. We already know that there are at least 2^{n-1} cubic coordinates forming an outline of each cell-wing. The following notions will allow us to say more.

A point x of $\mathbb{R}^{n-1}$ is *inside* a cell $\langle c, c'\rangle$ if, for any $i \in [n-1]$, $c_i \neq c'_i$ implies $c_i < x_i < c'_i$. A cell $\langle c, c'\rangle$ is *pure* if there is no cubic coordinate inside $\langle c, c'\rangle$. The *volume* $\operatorname{vol}\langle c, c'\rangle$ of

$\langle c, c' \rangle$ is its volume as an orthotope and it satisfies

$$\operatorname{vol}\langle c, c' \rangle = \prod_{i \in \mathrm{D}(c,c')} (c'_i - c_i). \tag{3.4.1}$$

Lemma 3.4.1. *Let $n \geqslant 1$ and $\langle c^{\text{out}}, c^{\text{in}} \rangle$ be a cell-wing of dimension $n-1$. The cell $\langle c^{\text{out}}, c^{\text{in}} \rangle$ is pure.*

Proof. Suppose there is a cubic coordinate c such that $c_i^{\text{out}} < c_i < c_i^{\text{in}}$ for all $i \in [n-1]$. By Lemma 3.2.1 we know that if $c_i^{\text{out}} < 0$, then $c_i^{\text{in}} \leqslant 0$ and if $c_i^{\text{out}} \geqslant 0$, then $c_i^{\text{in}} > 0$. However, since $c_i^{\text{out}} < c_i < c_i^{\text{in}}$, then c_i is different from 0. In the end, if such a cubic coordinate c exists, it would be synchronized. But then, there would be a cubic coordinate both synchronized and internal by hypothesis. This is impossible according to Proposition 3.3.3. □

We showed with Theorem 3.3.1 that each cell-wing contains at least 2^{n-1} cubic coordinates. By Lemma 3.4.1, we know that each cell-wing $\langle c^{\text{out}}, c^{\text{in}} \rangle$ is pure, and then has only cubic coordinates on its border.

Let $n \geqslant 1$ and $\langle c^{\text{out}}, c^{\text{in}} \rangle$ be a cell-wing of dimension $n-1$. Since between c^{out} and c^{in} all components are different, one has $\mathrm{D}(c^{\text{out}}, c^{\text{in}}) = n-1$, and so the volume of $\langle c^{\text{out}}, c^{\text{in}} \rangle$ satisfies

$$\operatorname{vol}\langle c^{\text{out}}, c^{\text{in}} \rangle = \prod_{i=1}^{n-1} (c_i^{\text{in}} - c_i^{\text{out}}). \tag{3.4.2}$$

Let us denote by c^0 the cubic coordinate such that $c_i^0 = 0$ for any $i \in [n-1]$. To compute $\operatorname{vol}\langle c^{\text{out}}, c^{\text{in}} \rangle$ from the synchronized cubic coordinate c associated by Γ, we must first compute the volume of the cell formed by c^0 and c.

By Lemma 3.2.1, any cell-wing is included in an ϵ-region of the c^0 cubic coordinate. This means that no cell-wing can be cut by a line passing by the origin c^0 and a cubic coordinate of the form $(0, \dots, 0, 1, 0, \dots, 0)$ or $(0, \dots, 0, -1, 0, \dots, 0)$.

According to Lemma 2.3.2, for any cubic coordinate, replacing any component by 0 gives a cubic coordinate. In other words, for any cubic coordinate c, there are $n-1$ cubic coordinates related to c which are its projections on the lines passing by c^0 and a cubic coordinate of the form $(0, \dots, 0, 1, 0, \dots, 0)$ or $(0, \dots, 0, -1, 0, \dots, 0)$. Therefore, even if c^0 and c are not comparable, we consider the cell, denoted by $\langle c \rangle$, between c^0 and c, such that the volume of this cell satisfies

$$\operatorname{vol}\langle c \rangle = \prod_{i \in \mathrm{D}(c,c^0)} |c_i|. \tag{3.4.3}$$

Note that the dimension of a cell is less than or equal to $n-1$. Moreover, $\langle c \rangle$ can be no-pure, and may even contain other cells of the same dimension.

By the map Γ, the components of the synchronized cubic coordinate c of the cell-wing $\langle c^{\text{out}}, c^{\text{in}} \rangle$ are the greatest in absolute value between c^{out} and c^{in}. Therefore, in the cell-wing $\langle c^{\text{out}}, c^{\text{in}} \rangle$, c is the furthest cubic coordinate from c^0. In particular, $\langle c \rangle$ contains the cell-wing $\langle c^{\text{out}}, c^{\text{in}} \rangle$ and the dimension of $\langle c \rangle$ is $n-1$.

Let $n \geqslant 0$ and $c \in \mathrm{SCC}(n)$. Since by the definition, all components of c are different from 0, one has $\mathrm{D}(c, c^0) = n - 1$. Therefore,

$$\operatorname{vol}\langle c\rangle = \prod_{i=1}^{n-1} |c_i|. \tag{3.4.4}$$

Let us endow the set $\mathrm{SCC}(n)$ with the partial order $\preccurlyeq_{\mathrm{s}}$ such that for $c, c' \in \mathrm{SCC}(n)$ one has $c' \preccurlyeq_{\mathrm{s}} c$ if c'_i and c_i have the same sign and $|c'_i| \leqslant |c_i|$ for any $i \in [n-1]$.

Lemma 3.4.2. *For any $n \geqslant 1$, let $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$ be a cell-wing of dimension $n-1$, and $c := \Gamma\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$. For any $x \in \mathbb{R}^{n-1}$ such that $x \in \langle c\rangle$, if $x \notin \langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$, then there is $c' \in \mathrm{SCC}(n)$ different from c such that $c' \preccurlyeq_{\mathrm{s}} c$ and $x \in \langle c'\rangle$.*

Proof. Let c^{op} be the opposite cubic coordinate of c. Since $x \notin \langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$ and $x \in \langle c\rangle$, then necessarily $c^{op} \neq c^0$. For the same reasons, there is an index i such that $|x_i| < |c_i^{op}|$ where $c_i^{op} \neq 0$. Let us build from such index i the $(n-1)$-tuple ∇c such that $\nabla c_i = c_i^{op}$ and $\nabla c_j = c_j$ for all $j \neq i$. According to Theorem 3.3.1, ∇c is a cubic coordinate and belongs to the cell-wing $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$. Also, ∇c is a synchronized cubic coordinate which satisfies $\nabla c \preccurlyeq_{\mathrm{s}} c$ and which is different from c. We can then associate to ∇c a cell, which is strictly included in $\langle c\rangle$. Then $x \in \langle \nabla c\rangle$. □

Since by Lemma 3.4.1 all cell-wings are pure, Lemma 3.4.2 implies that $\langle c\rangle \subseteq \coprod_{c' \preccurlyeq_{\mathrm{s}} c} \Gamma^{-1}(c')$, and since the reciprocal inclusion is obvious, one has the following result.

Lemma 3.4.3. *Let $n \geqslant 0$ and $c \in \mathrm{SCC}(n)$. Then*

$$\langle c\rangle = \coprod_{c' \preccurlyeq_{\mathrm{s}} c} \Gamma^{-1}(c'). \tag{3.4.5}$$

Let $n \geqslant 0$ and $c \in \mathrm{SCC}(n)$. The *synchronized volume* of c is defined by

$$\operatorname{sv}(c) := \operatorname{vol}\langle c\rangle - \sum_{\substack{c' \preccurlyeq_{\mathrm{s}} c \\ c' \neq c}} \operatorname{sv}(c'). \tag{3.4.6}$$

Note that (3.4.6) is a Möbius inversion [Sta12].

Proposition 3.4.4. *Let $n \geqslant 1$ and $\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$ be a cell-wing of dimension $n-1$. By setting $c := \Gamma\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle$, we have*

$$\operatorname{vol}\langle c^{\mathrm{out}}, c^{\mathrm{in}}\rangle = \operatorname{sv}(c). \tag{3.4.7}$$

Proof. This is a consequence of Lemma 3.4.3 and of (3.4.6). □

With Proposition 3.4.4 we are able to compute, for any $n \geqslant 0$, the volume of $\mathfrak{C}(\mathrm{CC}(n))$ depending on synchronized cubic coordinates,

$$\operatorname{vol}(\mathfrak{C}(\mathrm{CC}(n))) = \sum_{c \in \mathrm{SCC}(n)} \operatorname{sv}(c). \tag{3.4.8}$$

3.5. **EL-shellability.** In [BW96] and [BW97], Björner and Wachs generalized the method of labellings of the cover relations of graded posets to the case of non-graded posets. In particular, they showed the EL-shellability of the Tamari poset [BW97].

Let $\mathcal{P}$ be a bounded poset and Λ be a poset, and $\lambda : \lessdot_{\mathcal{P}} \to \Lambda$ be a map. For any saturated chain $\left(x^{(1)}, \dots, x^{(k)}\right)$ of $\mathcal{P}$, we set

$$\lambda\left(x^{(1)}, \dots, x^{(k)}\right) := \left(\lambda\left(x^{(1)}, x^{(2)}\right), \dots, \lambda\left(x^{(k-1)}, x^{(k)}\right)\right). \tag{3.5.1}$$

We say that a saturated chain of $\mathcal{P}$ is *λ-increasing* (resp. *λ-decreasing*) if its image by λ is an increasing (resp. decreasing) word for the order relation $\preccurlyeq_\Lambda$. We say also that a saturated chain $\left(x^{(1)}, \dots, x^{(k)}\right)$ of $\mathcal{P}$ is *λ-smaller* than a saturated chain $\left(y^{(1)}, \dots, y^{(k)}\right)$ of $\mathcal{P}$ if $\lambda\left(x^{(1)}, \dots, x^{(k)}\right)$ is smaller than $\lambda\left(y^{(1)}, \dots, y^{(k)}\right)$ for the lexicographic order induced by $\preccurlyeq_\Lambda$. The map λ is called *EL-labeling* (edge lexicographic labeling) of $\mathcal{P}$ if for any $x, y \in \mathcal{P}$ satisfying $x \preccurlyeq_{\mathcal{P}} y$, there is exactly one λ-increasing saturated chain from x to y, and this chain is λ-minimal among all saturated chains from x to y. Any bounded poset that admits an EL-labeling is *EL-shellable* [BW96, BW97].

The EL-shellability of a poset $\mathcal{P}$ implies several topological and order theoretical properties of the associated order complex $\Delta(\mathcal{P})$ built from $\mathcal{P}$. Recall that the faces of this simplicial complex are all the chains of $\mathcal{P}$. Moreover, if $\mathcal{P}$ has at most one λ-decreasing chain between any pair of elements, then the Möbius function of $\mathcal{P}$ takes values in $\{-1, 0, 1\}$. In this case, the simplicial complex associated with each open interval of $\mathcal{P}$ is either contractible or has the homotopy type of a sphere [BW97].

For the sequel, we set Λ as the poset $\mathbb{Z}^3$ wherein elements are ordered lexicographically. Let $(c, c') \in \lessdot$ such that, for $i \in [n-1]$, $c_i < c'_i$, and let $\lambda : \lessdot \to \mathbb{Z}^3$ be the map defined by

$$\lambda(c, c') := (\varepsilon, i, c_i), \tag{3.5.2}$$

where $\varepsilon := \begin{cases} -1 & \text{if } c_i < 0, \\ 1 & \text{else.} \end{cases}$

Note that by Proposition 2.4.6, the index i such that $c_i < c'_i$ is unique.

Theorem 3.5.1. *For any $n \geqslant 0$, the map λ is an EL-labeling of $\mathrm{CC}(n)$. Moreover, there is at most one λ-decreasing chain between any pair of elements of $\mathrm{CC}(n)$.*

Proof. Let $c, c' \in \mathrm{CC}(n)$ such that $c \preccurlyeq c'$. By Lemma 2.4.2, there is a cubic coordinate c'' such that $u'' = u$ and $v'' = v'$ with $(u'', v'') := \phi(c'')$. Let

$$\mathrm{D}^{-}\left(c, c''\right) = \{d_1, d_2, \dots, d_r\} \tag{3.5.3}$$

with $d_{k-1} < d_k$ for all $k \in [2, r]$, and

$$\mathrm{D}^{+}\left(c'', c'\right) = \{d'_1, d'_2, \dots, d'_s\}, \tag{3.5.4}$$

with $d'_{k-1} < d'_k$ for all $k \in [2, s]$.

By Lemma 2.4.3, there is a chain between c and c''

$$\left(c, c^{(1)}, \dots, c^{(r-1)}, c^{(r)} = c''\right), \tag{3.5.5}$$

where, for $k \in [r]$, $c^{(k)}$ be a cubic coordinate obtained by replacing in c all the components c_{d_i} by the components c''_{d_i} for $i \in [k]$.

By Lemma 2.4.4, there is a chain between c'' and c'

$$\left(c'', c'^{(1)}, \dots, c'^{(s-1)}, c'^{(s)} = c'\right), \tag{3.5.6}$$

where, for $k \in [s]$, $c'^{(k)}$ be a cubic coordinate obtained by replacing in c'' all the components c''_{d_i} by the components c'_{d_i} for $i \in [k]$.

Let us consider the chain obtained by concatenating the two chains (3.5.5) and (3.5.6). Since in this chain only one component differs between two consecutive cubic coordinates, a saturated chain μ can be constructed by considering all the cubic coordinates between them. For both chains (3.5.5) and (3.5.6), the components are independently increasing one by one from the left to the right. By construction, it implies that μ is λ-increasing for the lexicographic order induced by (3.5.2).

Moreover, any other choice of saturated chain between c and c' implies choosing, at a certain step k, a greater label for the lexicographical order than the label (ε, k, c_k) of μ, and then having to choose the label (ε, k, c_k) afterwards. Thus, in addition to being λ-increasing, the saturated chain μ is unique and is λ-minimal among all saturated chains from c to c'.

If a saturated chain λ-decreasing exists between c and c', it is built by first changing the different and negative components between c and c'' from right to left, and then changing the different and positive components between c'' and c' from right to left. For the same reason that any saturated λ-increasing chain is unique for any interval, if it exists, the λ-decreasing chain is also unique. ☐

For instance, in Figure 12, the λ-increasing saturated chain between $(-1,-2)$ and $(2,1)$ is the chain

$$((-1,-2), (0,-2), (0,-1), (0,0), (1,0), (2,0), (2,1)), \tag{3.5.7}$$

and

$$\lambda\left((-1,-2), \dots, (2,1)\right) = ((-1,1,-1), (-1,2,-2), (-1,2,-1), (1,1,0), (1,1,1), (1,2,0)). \tag{3.5.8}$$

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My manuscript has no associated data.

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Bref.

explicative

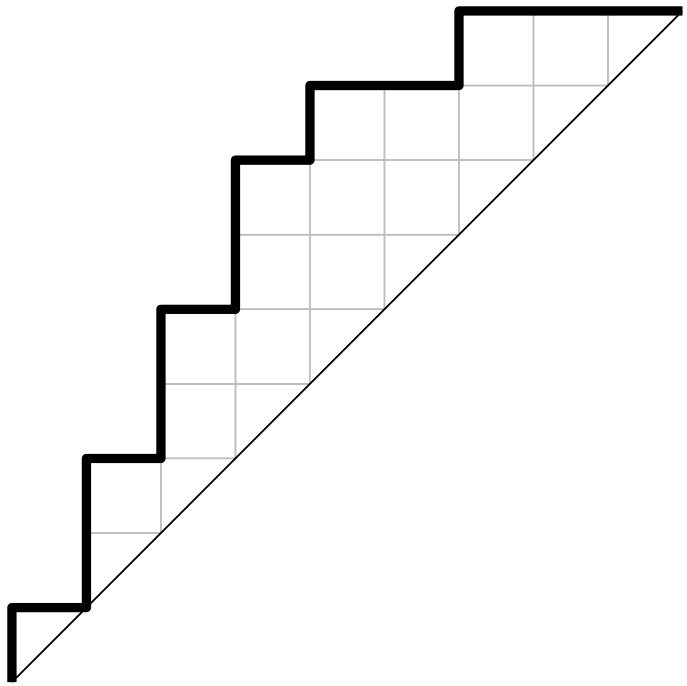
synchronized

Tamari-like intervals and planar maps

Wenjie Fang
TU Graz

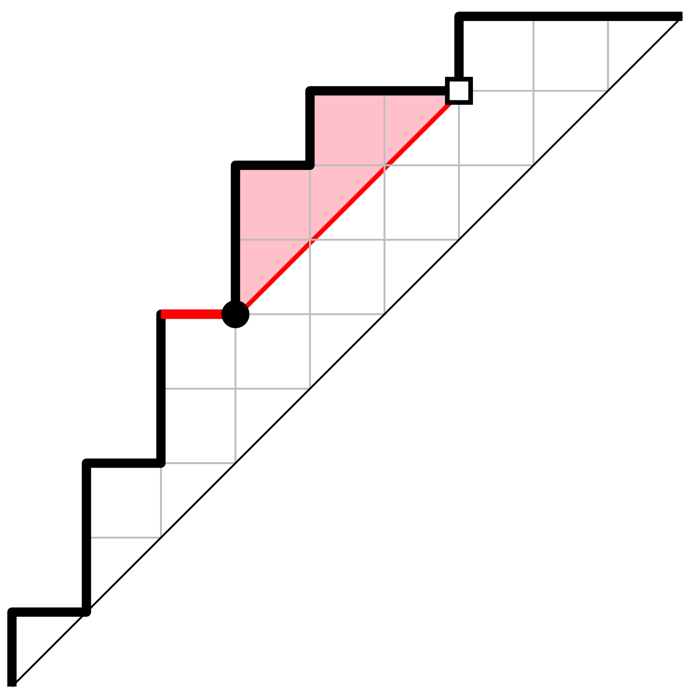
Workshop on Enumerative Combinatorics, 19 October 2017
Erwin Schrödinger Institute

Dyck paths and Tamari lattice, ...



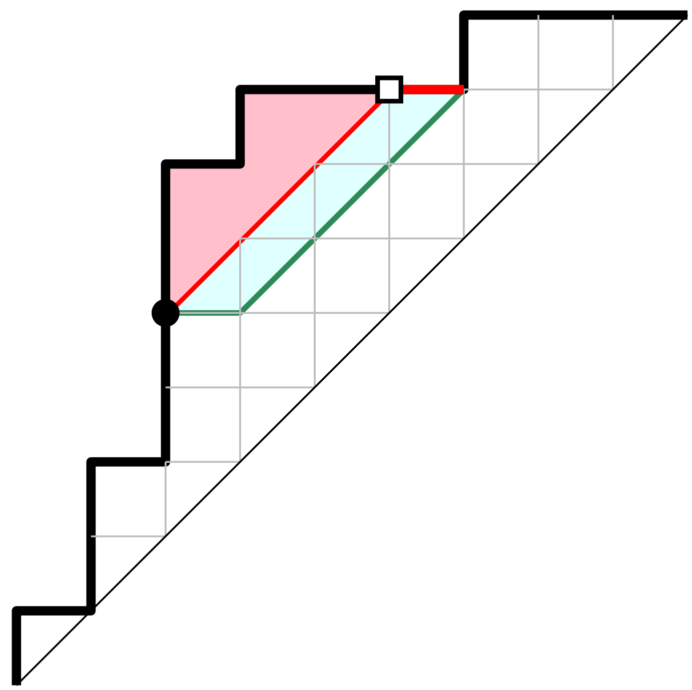
Dyck path: n north(N) and n east(E) steps, always above the diagonal
Counted by the n-th Catalan numbers $\mathrm{Cat}(n) = \frac{1}{2n+1}\binom{2n+1}{n}$

Dyck paths and Tamari lattice, ...



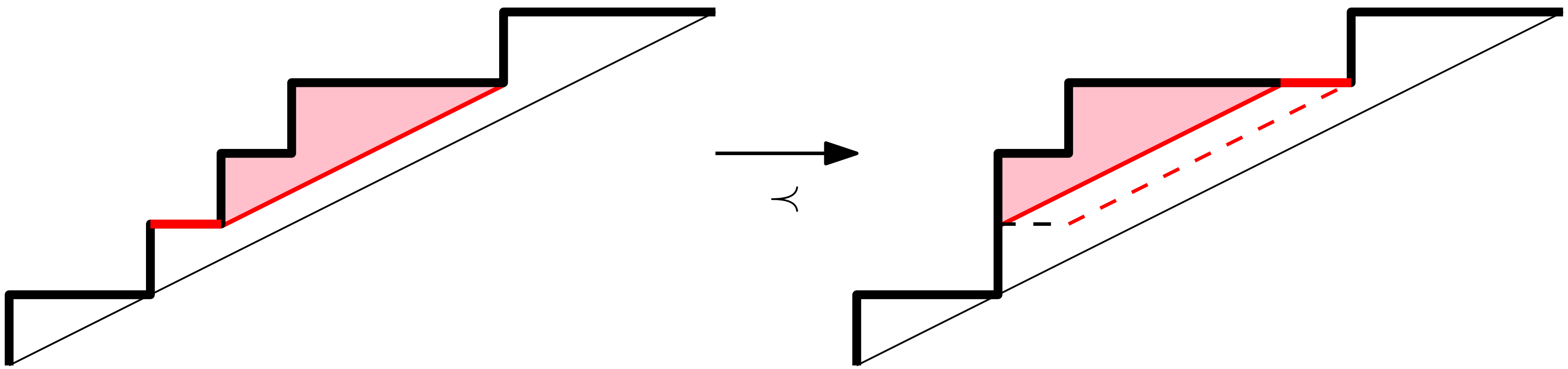
Covering relation: take a valley point •, find the next point □ with the same distance to the diagonal ...

Dyck paths and Tamari lattice, ...



... and push the segment to the left. This gives the **Tamari lattice** (Huang-Tamari 1972).

..., m-Tamari lattice, ...



m-ballot paths: n north steps, mn east steps, above the "m-diagonal".

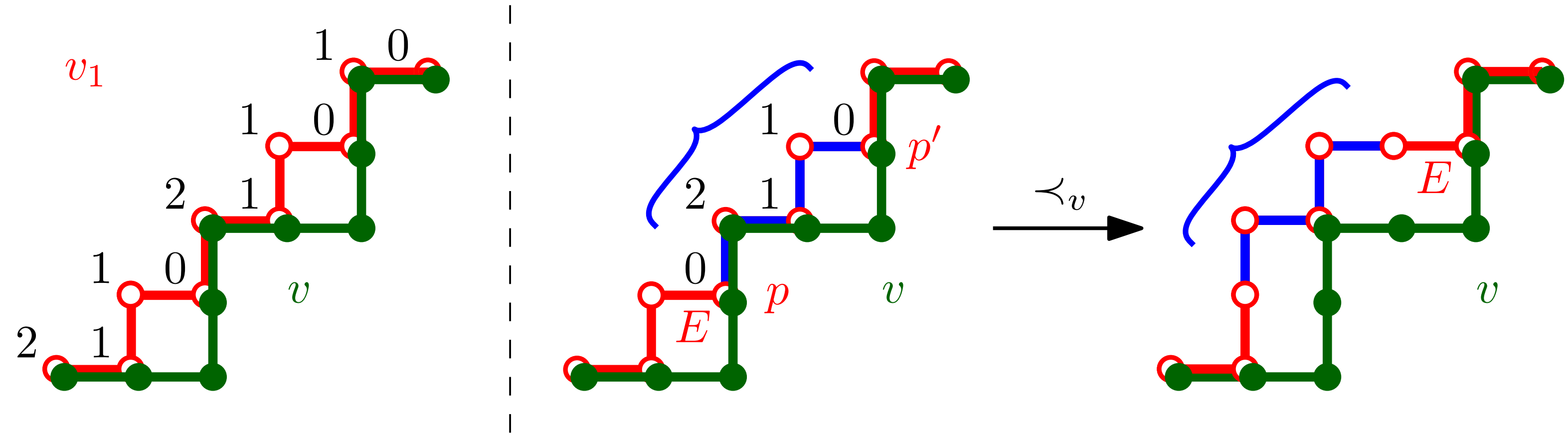
Counted by Fuss-Catalan numbers $\mathrm{Cat}_m(n) = \frac{1}{mn+1}\binom{mn+1}{n}$.

A similar covering relation gives the **m-Tamari lattice** (Bergeron 2010).

… and beyond.

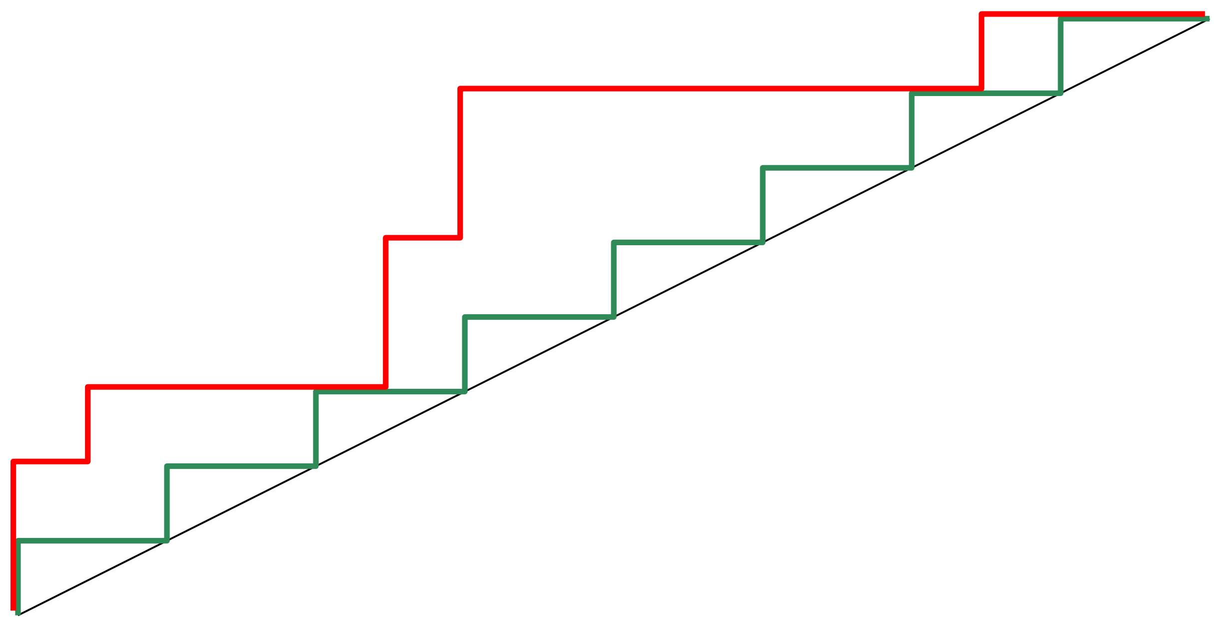
But we can use an arbitrary path v as "diagonal"!

Horizontal distance = # steps one can go without crossing v



Generalized Tamari lattice (Préville-Ratelle and Viennot 2014): $\mathrm{TAM}(v)$ over arbitrary v (called the **canopy**) with N, E steps.

… and beyond.

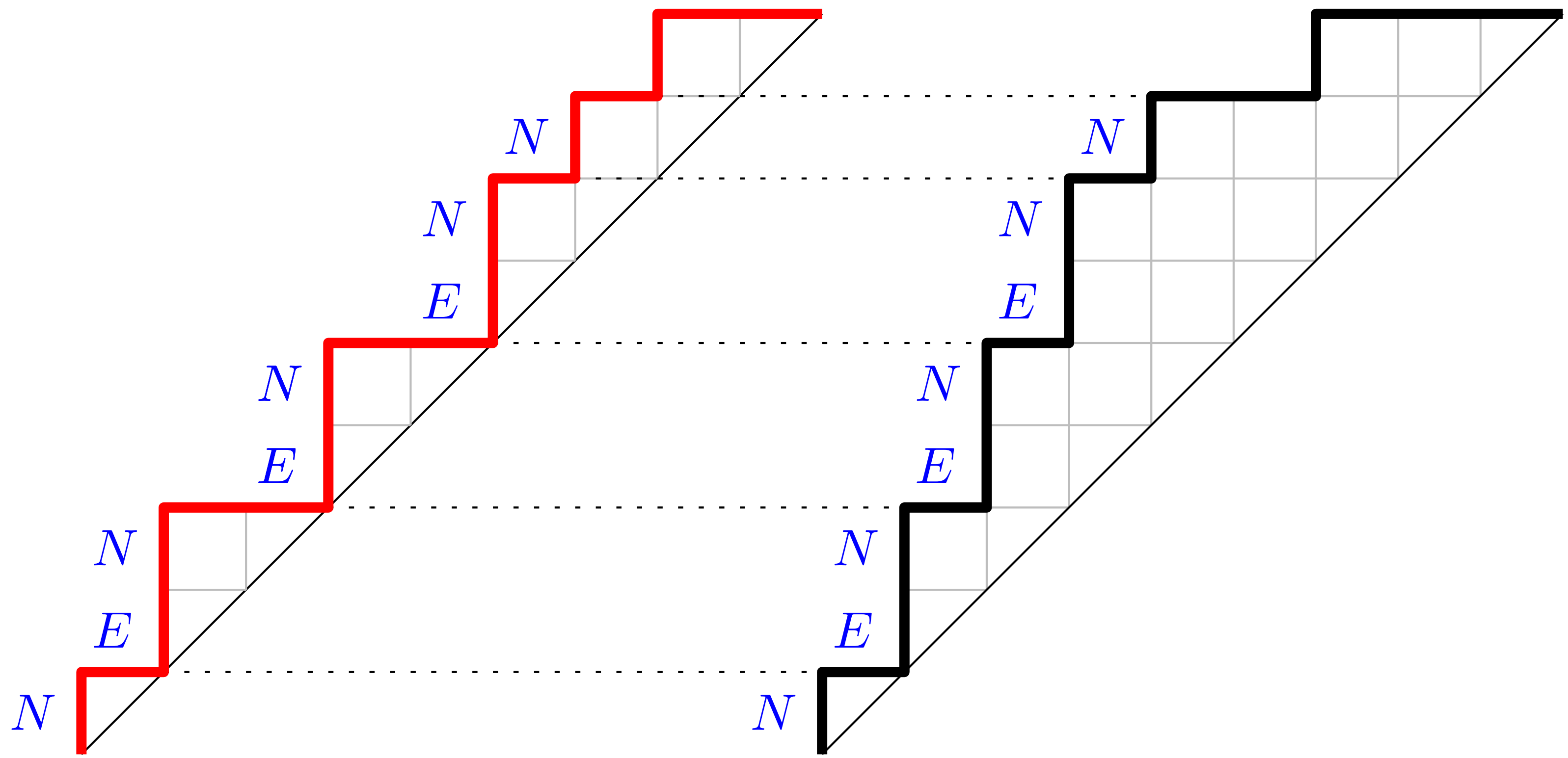


$\textsc{Tam}((NE^m)^n) \simeq m$-Tamari lattice

Type of a Dyck path

North step: followed by an east step $\to N$, by a north step $\to E$.

Mind the change!



Type: $NENENENN$

The two paths have the same type, therefore **synchronized**.

The next level: intervals

Interval in a lattice: $[a, b]$ with comparable $a \leq b$

Motivation: conjecturally related to the dimension of diagonal coinvariant spaces

For generalized Tamari intervals:

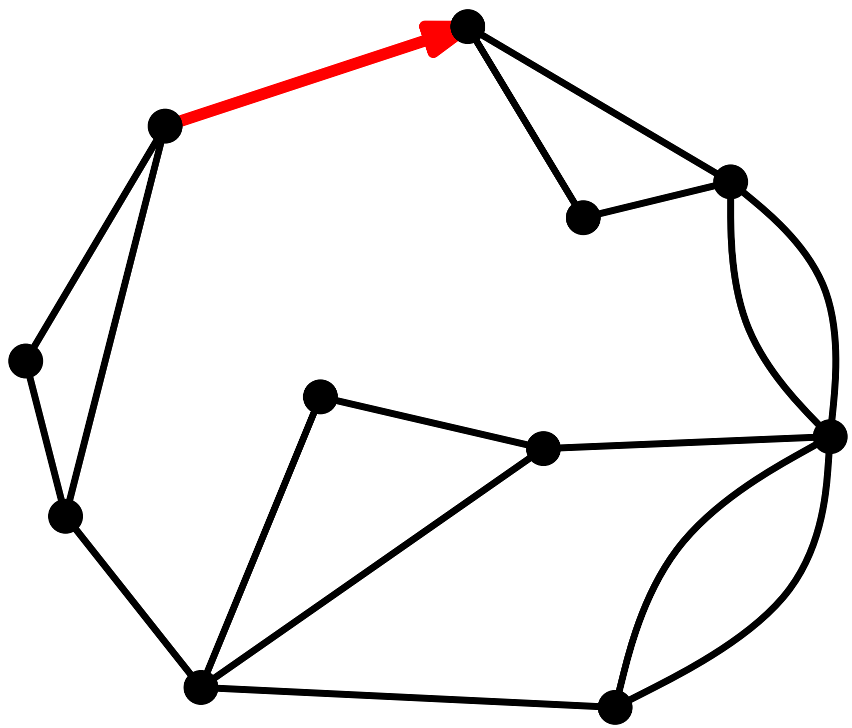
Interval in $\text{Tam}(v)$ with v of length $n-1$ $\Leftrightarrow$ **synchronized interval** of length $2n$, *i.e.*, Tamari interval $[D, E]$ with D and E of the same type.

How exactly?

For Tamari and m-Tamari intervals:

- Counting: Bousquet-Mélou, Chapoton, Chapuy, Fusy, Préville-Ratelle, Viennot, ...
- Interval poset: Chapoton, Châtel, Pons, ...
- λ-terms: N. Zeilberger, ...
- Planar maps

What is a planar map?



Planar map: embedding of a connected multigraph on the plane (loops and multiple edges allowed), defined up to homeomorphism, cutting the plane into **faces**

Planar maps are **rooted** at an edge on the infinite outer face.

Intervals that count like planar maps

Chapoton 2006: # intervals in Tamari lattice of size $n =$

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

$=$ # 3-connected planar triangulations with $n+3$ vertices (Tutte 1963)
$=$ # bridgeless planar maps with n edges (Walsh and Lehman 1975)

Bousquet-Mélou, Fusy and Préville-Ratelle 2011:
intervals in m-Tamari lattice of size $n =$

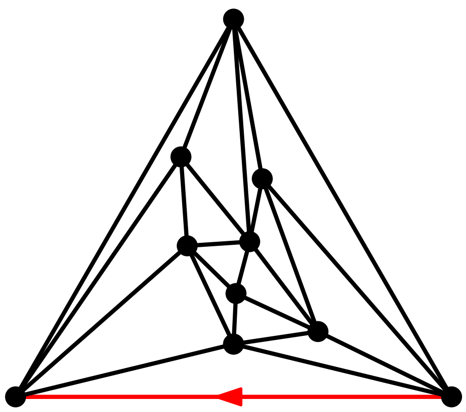
$$\frac{m+1}{n(mn+1)}\binom{n(m+1)^2+m}{n-1},$$

and it also looks like an enumeration of planar maps!

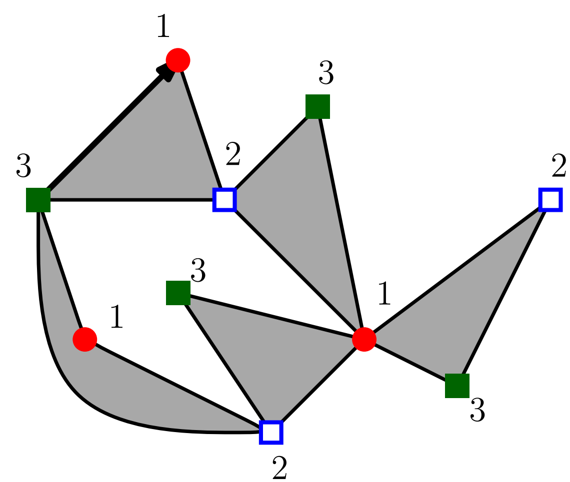
Labeled version: Bousquet-Mélou, Chapuy and Préville-Ratelle 2013

Deeper connections

For Tamari intervals and 3-connected planar triangulations: bijective proof using orientations (Bernardi and Bonichon 2009)

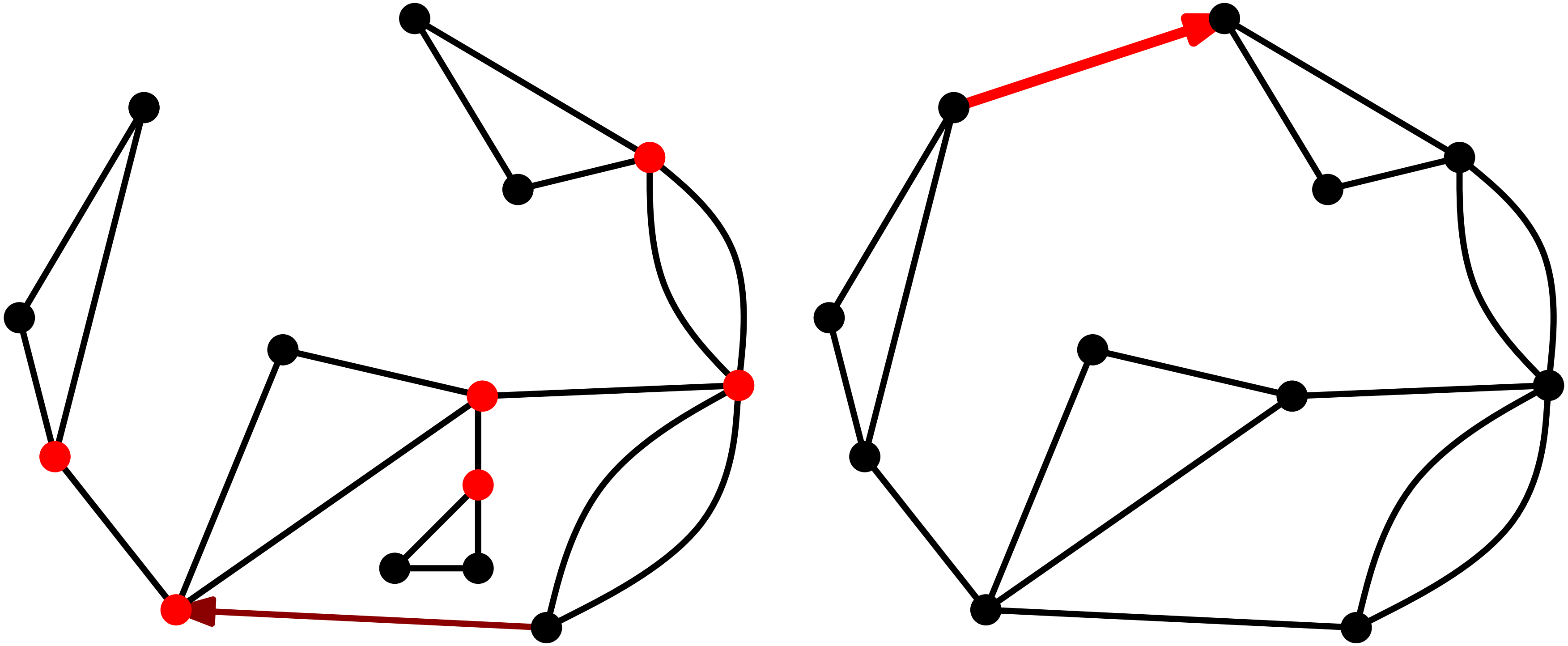


For m-Tamari intervals, the formal method used to solve for its generating function (the "differential-catalytic" method) can also be used on planar m-constellations.



Any other links? Especially for generalized Tamari intervals...

Non-separable planar maps



A **cut vertex** cuts the map into two sets of edges.
A **non-separable planar map** is a planar map without cut vertex.

Another type of intervals that counts like map

Theorem (W.F. and Louis-François Préville-Ratelle 2016)

There is a natural bijection between intervals in $\textsc{Tam}(v)$ *for all possible* v *of length* n *and non-separable planar maps with* $n+2$ *edges.*

Intermediate object: **decorated trees**

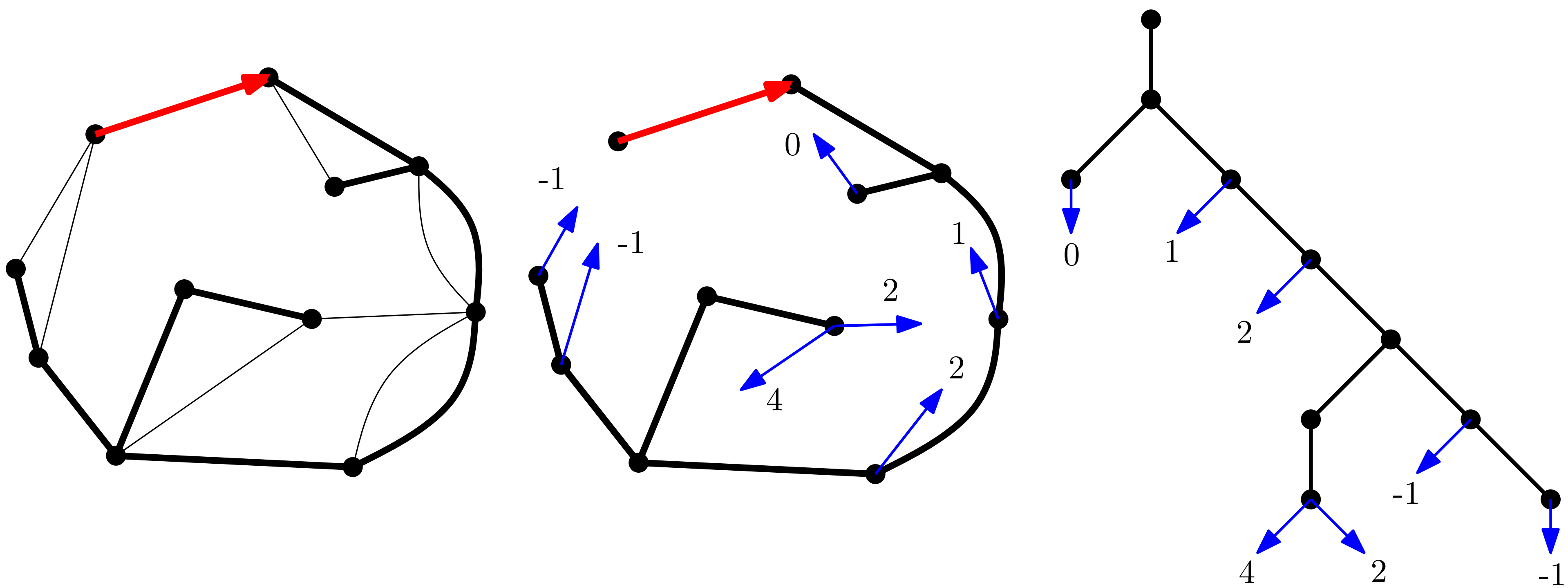
Corollary

The total number of intervals in $\textsc{Tam}(v)$ *for all possible* v *of length* n *is*

$$\sum_{v \in (N,E)^n} \mathrm{Int}(\textsc{Tam}(v)) = \frac{2}{(n+1)(n+2)} \binom{3n+3}{n}.$$

This formula was first obtained in (Tutte 1963).

What are decorated trees?



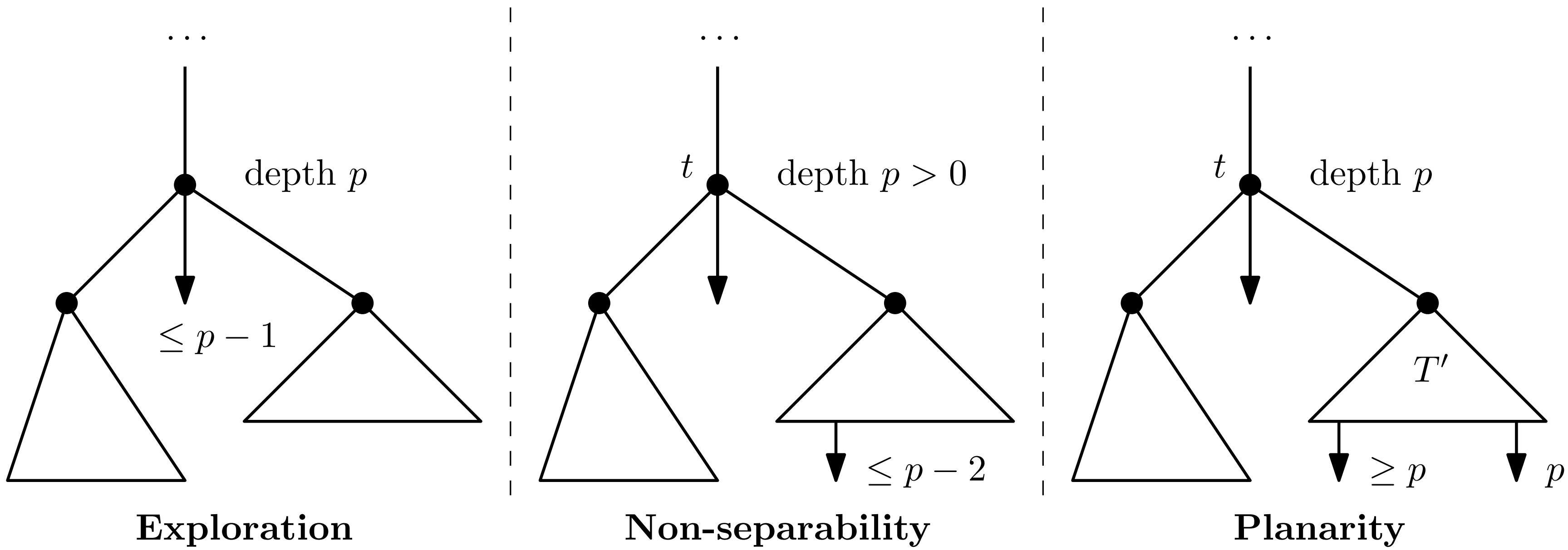
Property

If the exploration of an edge e adjacent to a vertex u reaches an already visited vertex w, then w is an ancestor of u.

Characterizing decorated trees

A **decorated tree** is a rooted plane tree with labels ≥ -1 on leaves such that (depth of the root is 0):

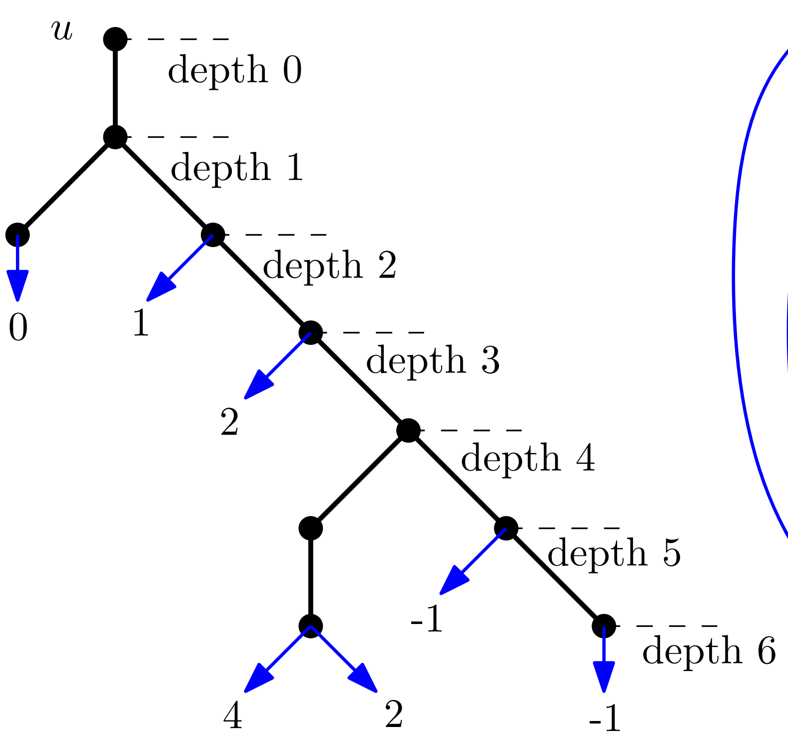
1. (Exploration) For a leaf ℓ of a node of depth p, the label of ℓ is $< p$;
2. (Non-separability) For a non-root node u of depth p, there is at least one descendant leaf with label $\leq p-2$ (the first such leaf is the **certificate** of u);
3. (Planarity) For t a node of depth p and T' a direct subtree of t, if a leaf ℓ in T' is labeled p, every leaf in T' before ℓ has a label $\geq p$.

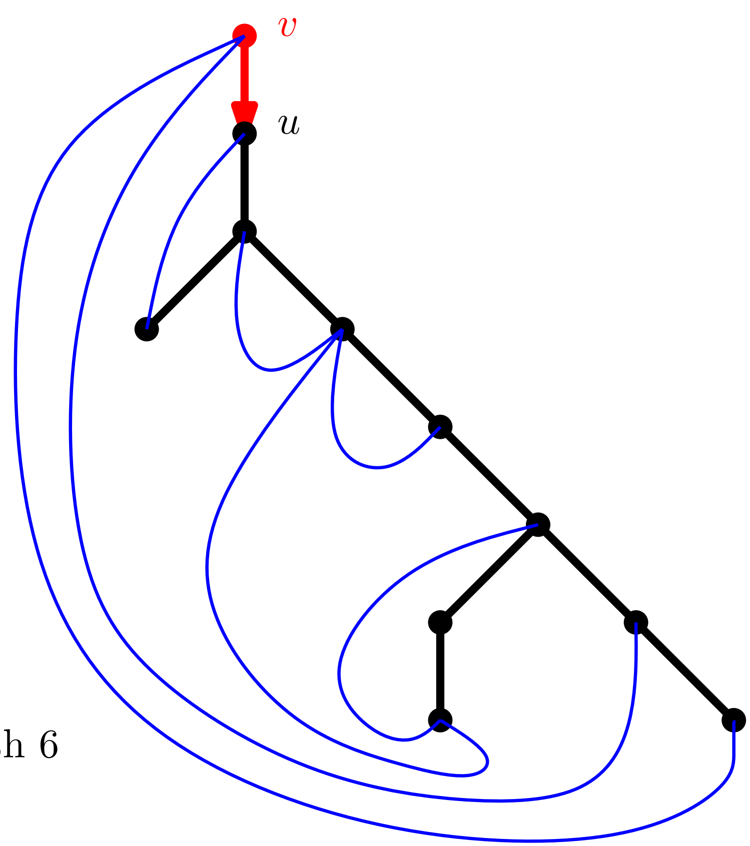


From maps to trees

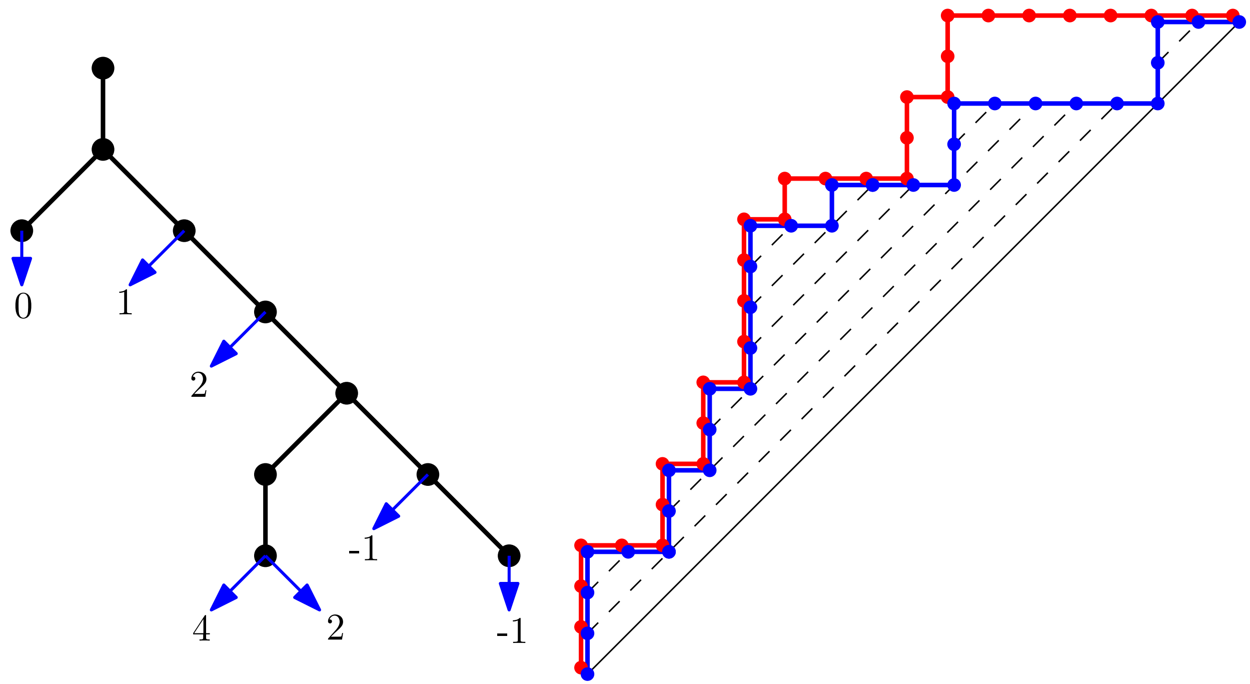
Just glue leaves with label d to their ancestor of depth d.

Only one way to glue back to a planar map.



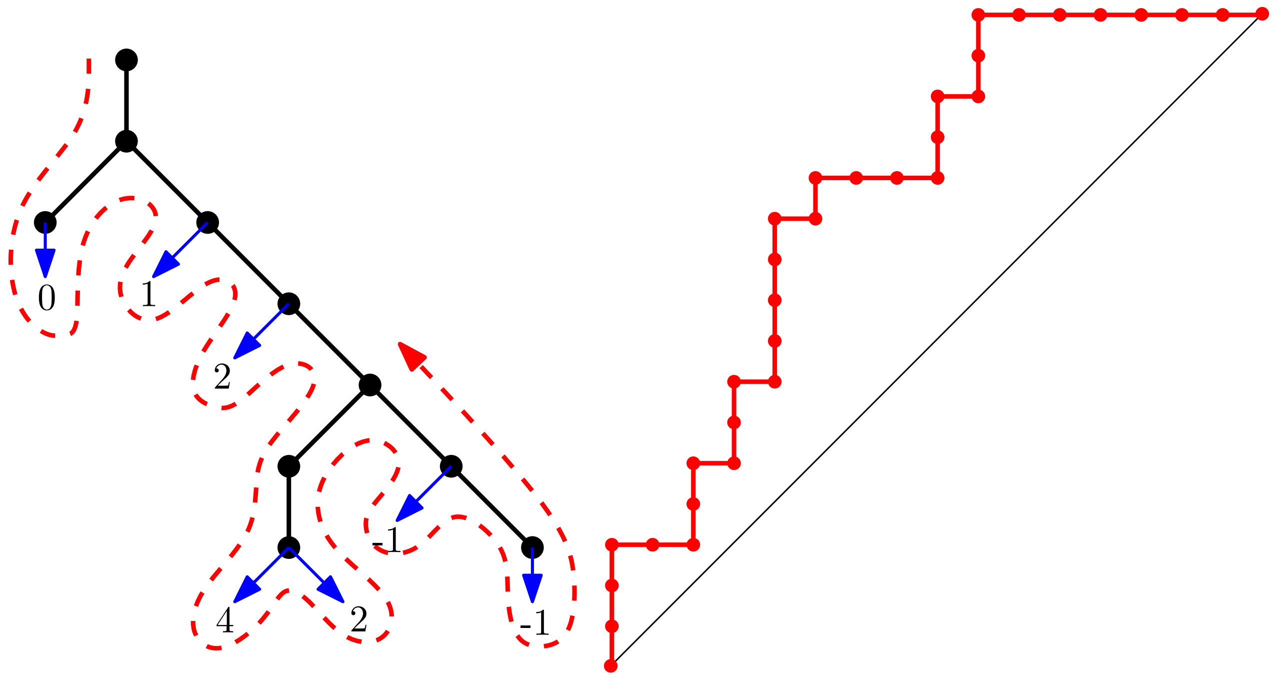


From trees to intervals



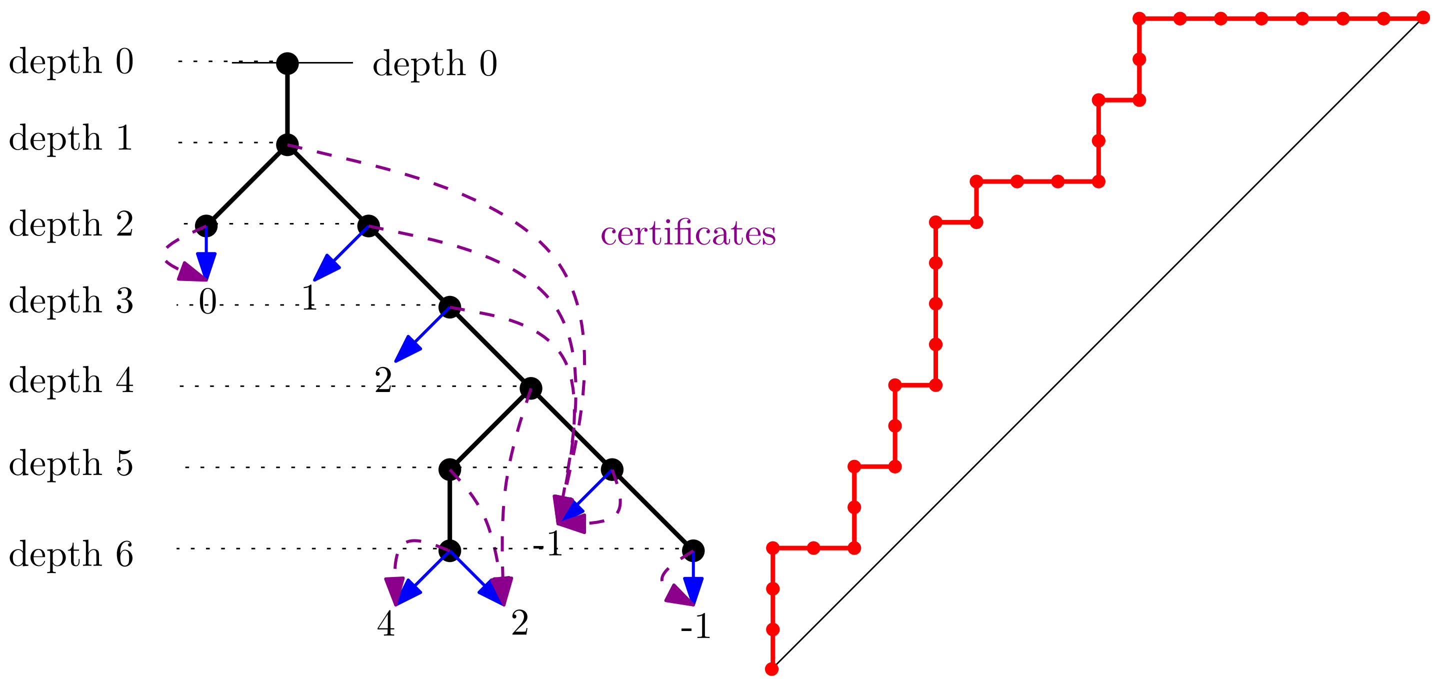
From a decorated tree T to a synchronized interval $[\mathrm{P}(T), \mathrm{Q}(T)]$

From trees to intervals



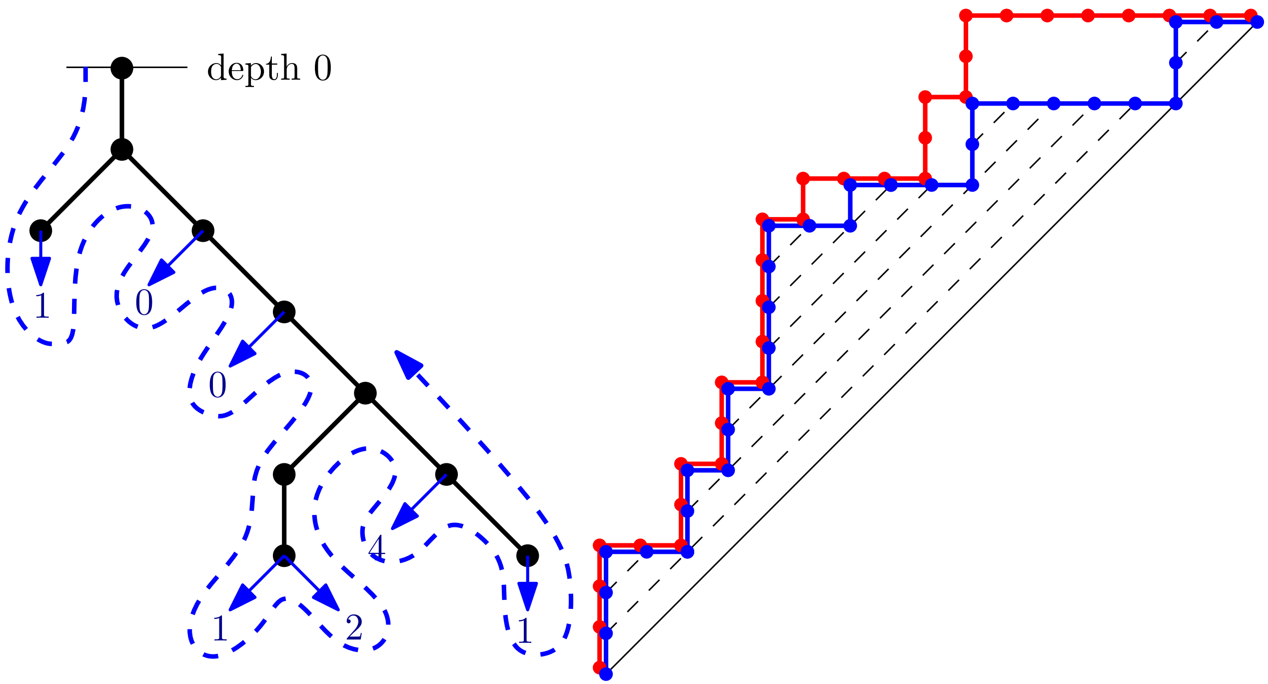
Path Q: a traversal

From trees to intervals



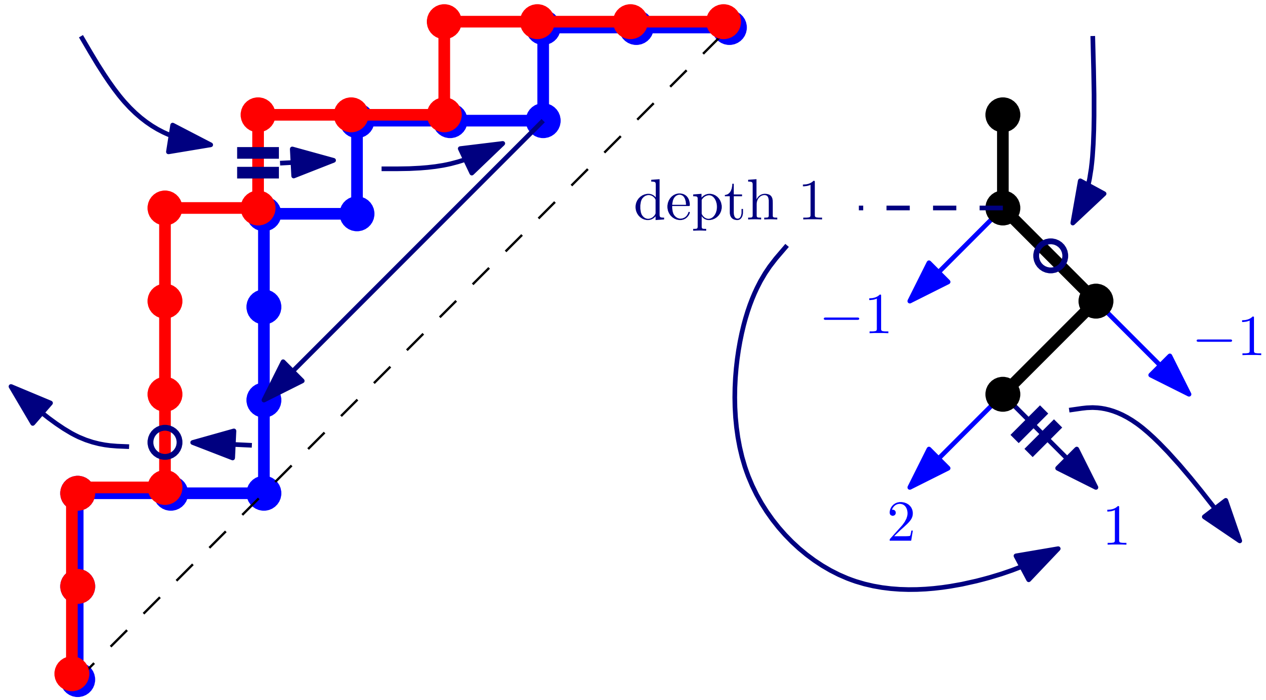
Function c: for a leaf ℓ, $c(\ell)$ =#nodes with ℓ as certificate

From trees to intervals

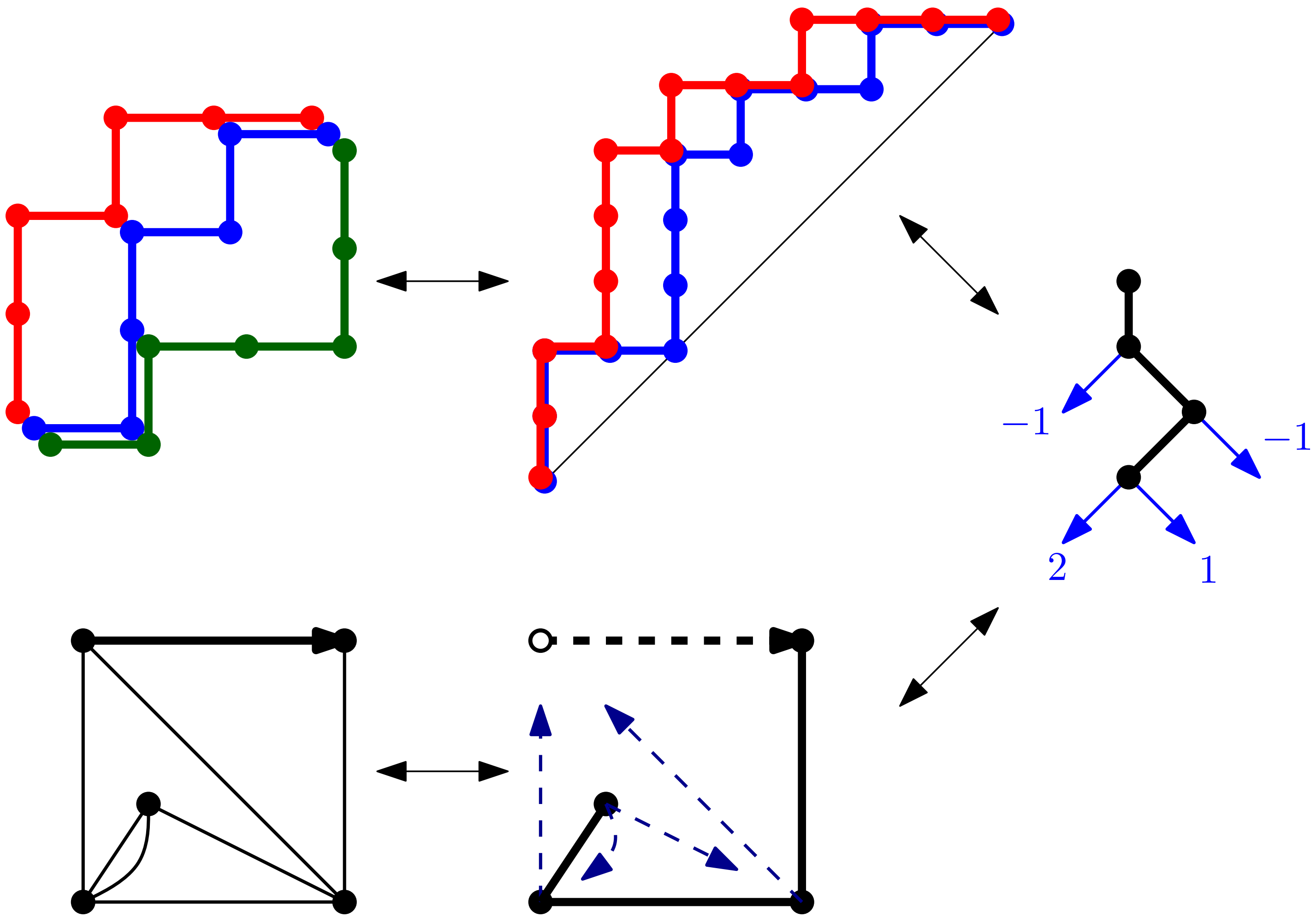


Path P: an altered traversal where descents are $c(\ell)+1$

The other direction



The whole bijection



Structural result

Our bijections are canonical w.r.t. appropriate recursive decompositions of related objects.

Theorem (W.F. 2017)

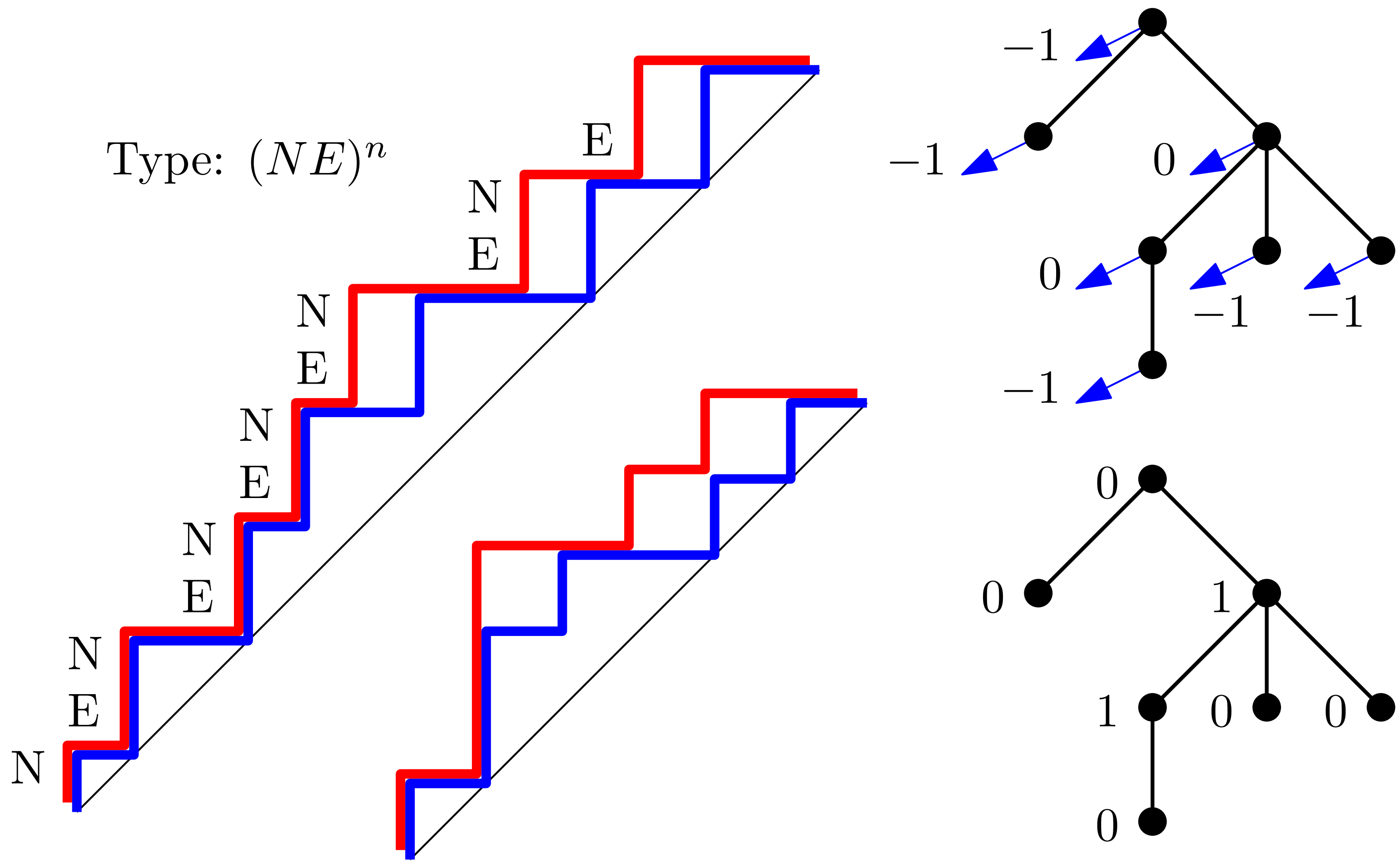
Under our bijections, the involution from intervals in $\textsc{Tam}(v)$ *to those in* $\textsc{Tam}(\overleftarrow{v})$ *is equivalent to map duality.*

Also connection with β-(1,0) trees (Cori, Schaeffer, Jacquard, Kitaev, de Mier, Steingrímsson, ...), leading to a bijective proof of a result in Kitaev–de Mier(2013).

Also equi-distribution results on various statistics

Restriction to the original Tamari intervals...

Tamari lattice = $\textsc{Tam}((NE)^n)$

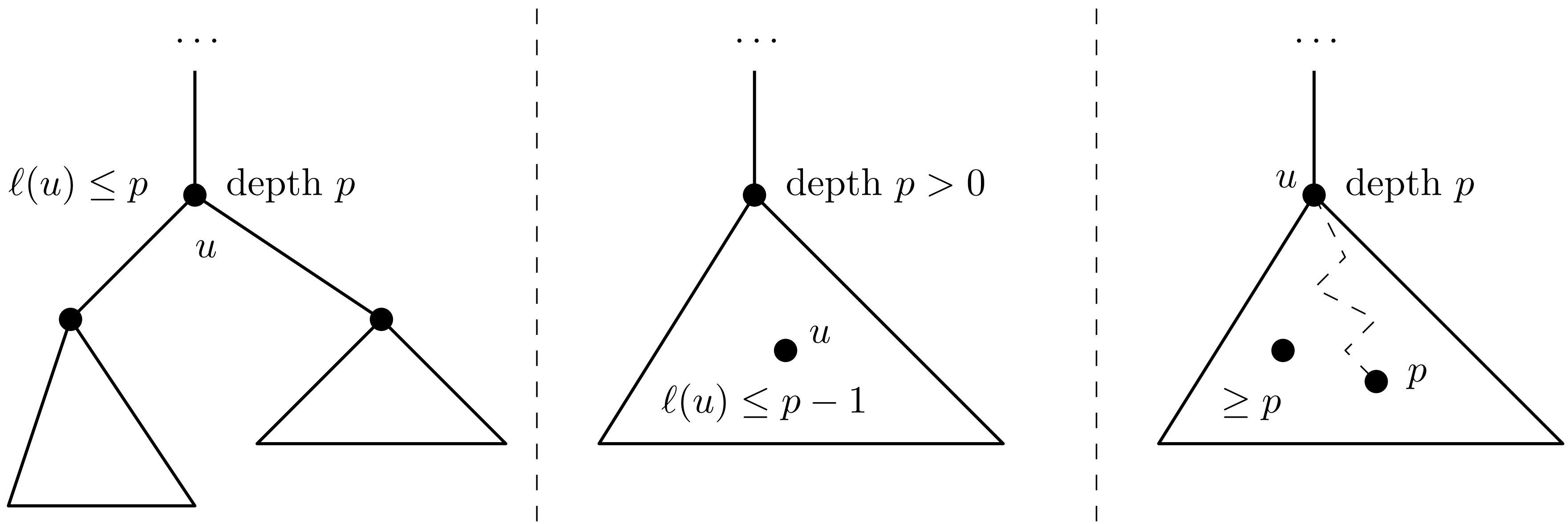


Restriction to type $(NE)^n$: decorated trees where each leaf is the first child of each internal node.

Sticky tree

Decorated trees restricted in $\text{TAM}((NE)^n) \rightsquigarrow$ **sticky trees**

A **sticky tree** is a plane tree with a label $\ell(u) \geq 0$ on each node u such that:



Exploration **Absence of bridges** **Planarity**

Essentially adapted from the condition of decorated trees! Now every non-root node has a certificate, which is a node (and can be itself).

Bijections to classical objects

Theorem (W.F. 2017+)

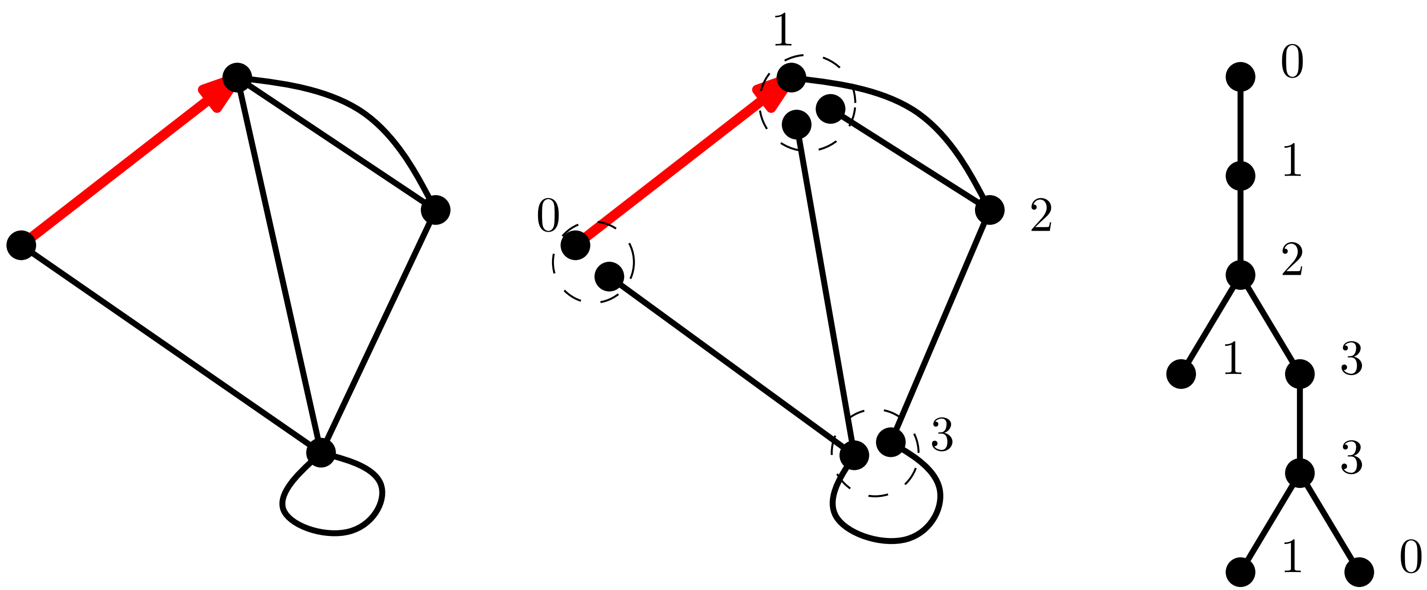
Sticky trees with n edges are in natural bijection with

1. *Tamari intervals with n up steps;*
2. *bridgeless planar maps with n edges;*
3. *3-connected triangulations with $n+3$ vertices.*

A new bijective proof of (1) $=$ (3), different from (Bernardi–Bonichon 2009).

Also a new bijective (and direct!) proof of (2) $=$ (3), different from the recursive ones in (Wormald 1980) and (Fusy 2010).

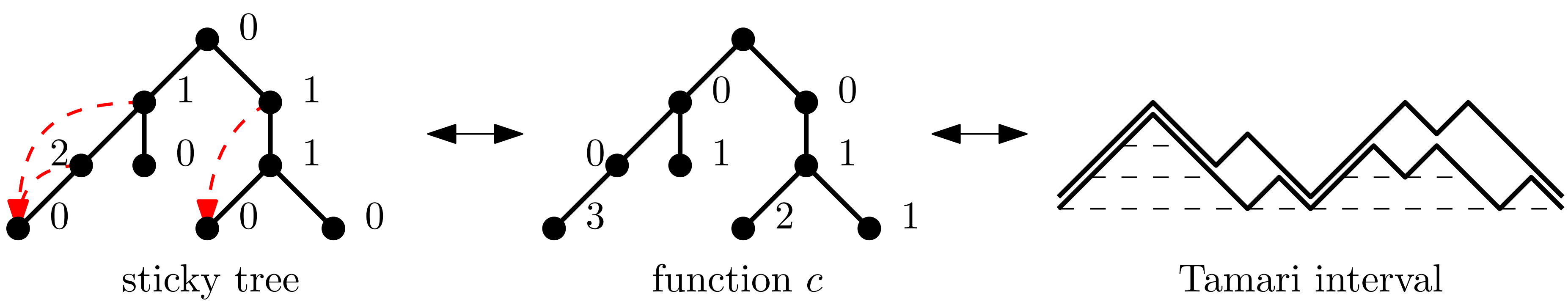
Bijection with bridgeless planar maps



An exploration on edges

There is also a bijection between sticky trees and 3-connected planar triangulations (with a different exploration process)

Bijection with Tamari intervals



Also with closed flows of plane forests (Chapoton–Châtel–Pons 2014), recovering a result therein.

General discussion

- Other related lattices (Stanley, Kreweras, ...) and planar maps (bipartite, constellations)?
- Other structures (*e.g.* 2-stack-sortable permutations)?
- Asymptotic aspects of these objects (statistics, limit shape, ...)?
- Restricted bijections on m-Tamari lattice?

Some interesting sequences...

Number of intervals in $\textsc{Tam}(w^n)$ with w a word in $\{N, E\}$?

Observation

For $w = N^a E N^b$, the number of intervals in $\textsc{Tam}(w^n)$ is of the form

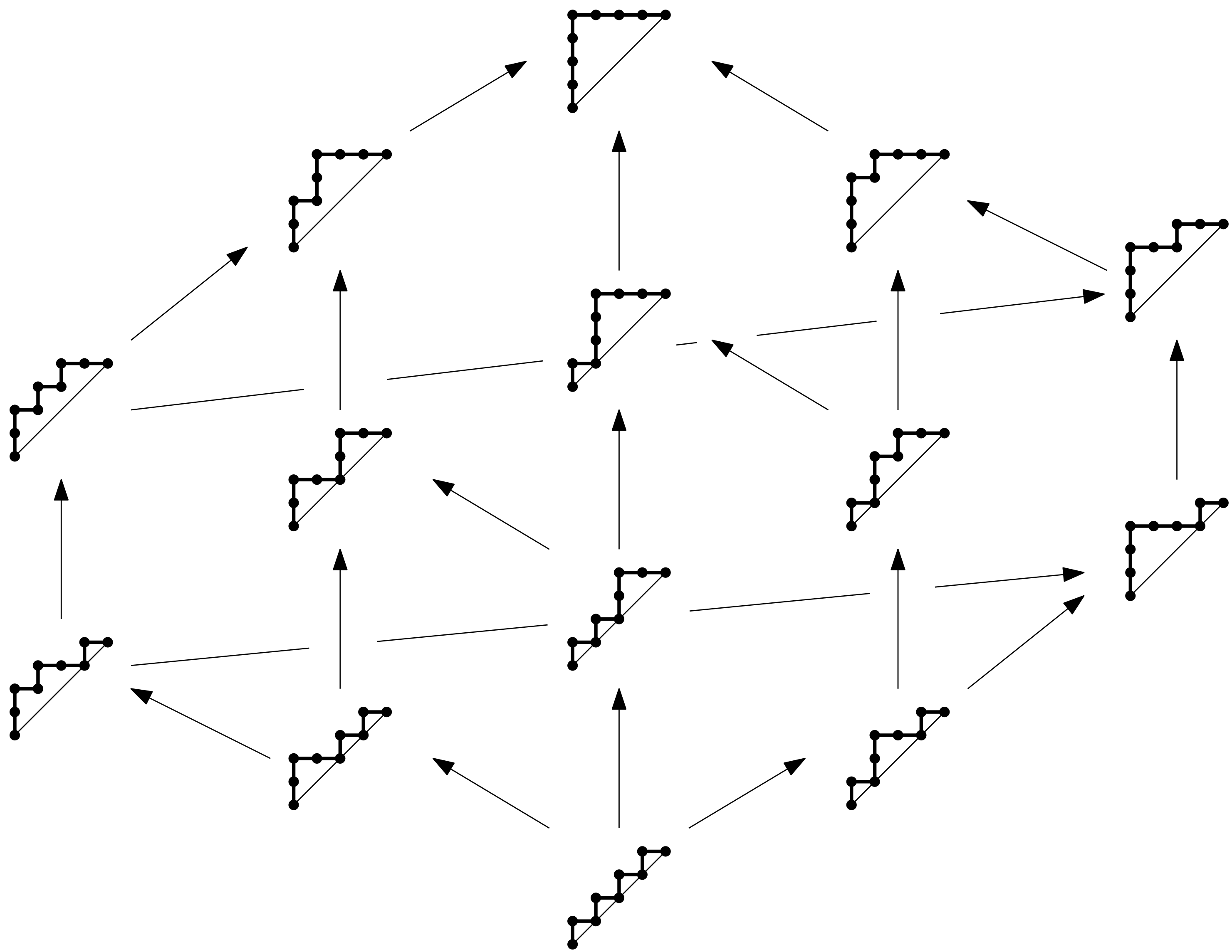
$$\frac{k_{a,b}+1}{n(\ell_{a,b}n+1)}\binom{(a+b+1)^2 n + k_{a,b}}{n-1},$$

where $k_{a,b}$ and $\ell_{a,b}$ are integers. What are these constants?

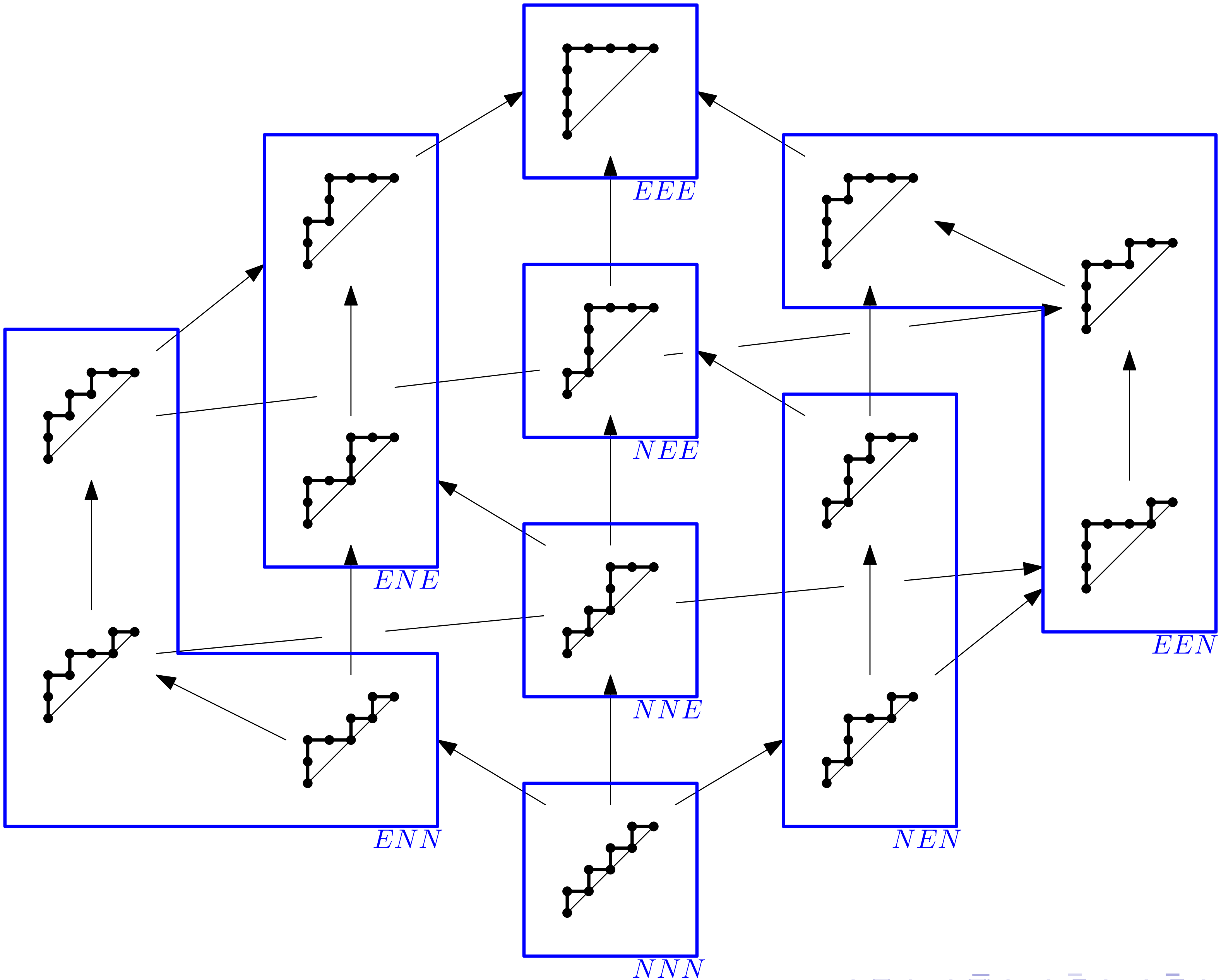
For $w = NNEE$: $1, 20, 755, 37541, 2177653, \ldots$
For $w = NEEN$: $6, 164, 7019, 373358, 22587911, \ldots$

What are these sequences?

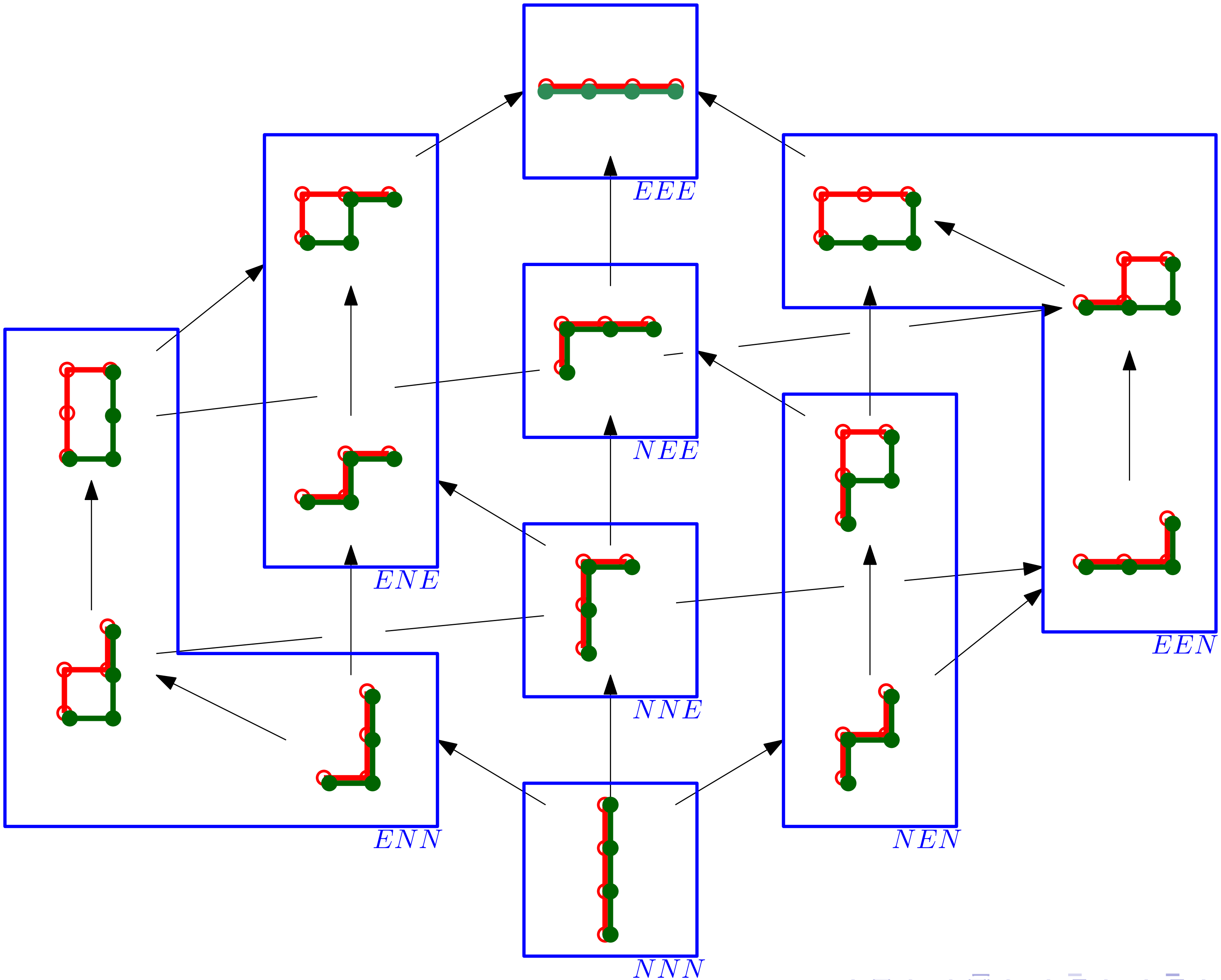
Partitionning the Tamari lattice by type



Partitionning the Tamari lattice by type



Partitionning the Tamari lattice by type



Partitioning the Tamari lattice by type

Delest and Viennot (1984): There is a bijection between Dyck path of length $2n$ and an element in $\textsc{Tam}(v)$ for some v of length $n-1$.

Theorem (Préville-Ratelle and Viennot (2014))

The Tamari lattice of order n is partitioned by path types into 2^{n-1} sublattices, each isomorphic to the generalized Tamari lattice $\textsc{Tam}(v)$ with v the type (a word in N, E of length $n-1$).

Theorem (Préville-Ratelle and Viennot (2014))

The lattice $\textsc{Tam}(v)$ is isomorphic to the order dual of $\textsc{Tam}(\overleftarrow{v})$, where $\overleftarrow{v}$ is the word v read from right to left, with the substitution $N \leftrightarrow E$.

And back...

Article de base
Synchronized

FPSAC 2015, Daejeon, South Korea DMTCS proc. **FPSAC'15**, 2015, 133–144

An extension of Tamari lattices

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Abstract. For any finite path v on the square lattice consisting of north and east unit steps, we construct a poset $\mathrm{Tam}(v)$ that consists of all the paths lying weakly above v with the same endpoints as v. For particular choices of v, we recover the traditional Tamari lattice and the m-Tamari lattice. In particular this solves the problem of extending the m-Tamari lattice to any pair (a, b) of relatively prime numbers in the context of the so-called rational Catalan combinatorics.

For that purpose we introduce the notion of canopy of a binary tree and explicit a bijection between pairs (u, v) of paths in $\mathrm{Tam}(v)$ and binary trees with canopy v. Let $\overleftarrow{v}$ be the path obtained from v by reading the unit steps of v in reverse order and exchanging east and north steps. We show that the poset $\mathrm{Tam}(v)$ is isomorphic to the dual of the poset $\mathrm{Tam}(\overleftarrow{v})$ and that Tam(v) is isomorphic to the set of binary trees having the canopy v, which is an interval of the ordinary Tamari lattice. Thus the usual Tamari lattice is partitioned into (smaller) lattices $\mathrm{Tam}(v)$, where the v's are all the paths of length $n - 1$ on the square lattice.

We explain possible connections between the poset $\mathrm{Tam}(v)$ and (the combinatorics of) the generalized diagonal coinvariant spaces of the symmetric group.

Résumé. Pour tout chemin v sur le réseau carré formé de pas Nord et Est, nous construisons un ensemble partiellement ordonné $\mathrm{Tam}(v)$ dont les éléments sont les chemins au dessus de v et ayant les mêmes extrémités. Pour certains choix de v nous retrouvons le classique treillis de Tamari ainsi que son extension m-Tamari. En particulier nous résolvons le problème d'étendre le treillis m-Tamari à toute paire (a, b) d'entiers premiers entre eux dans le contexte de la combinatoire rationnelle de Catalan.

Pour ceci nous introduisons la notion de canopée d'un arbre binaire et explicitons une bijection entre les paires (u, v) de chemins dans $\mathrm{Tam}(v)$ et les arbres binaires ayant la canopée v. Soit $\overleftarrow{v}$ le chemin obtenu en lisant les pas en ordre inverse et en échangeant les pas Est et Nord. Nous montrons que $\mathrm{Tam}(v)$ est isomorphe au dual de $\mathrm{Tam}(\overleftarrow{v})$ et que $\mathrm{Tam}(v)$ est isomorphe à l'ensemble des arbres binaires ayant la canopée v, qui est un intervalle du treillis de Tamari ordinaire. Ainsi le traditionnel treillis de Tamari admet une partition en plus petits treillis $\mathrm{Tam}(v)$, où les v sont tous les chemins de longueur $n - 1$ sur le réseau carré. Enfin nous explicitons les liens possibles entre l'ensemble ordonné $\mathrm{Tam}(v)$ et (la combinatoire des) espaces diagonaux coinvariants généralisés du groupe symétrique.

Keywords: Tamari lattice, m-Tamari lattice, rational Catalan combinatorics, diagonal coinvariant spaces.

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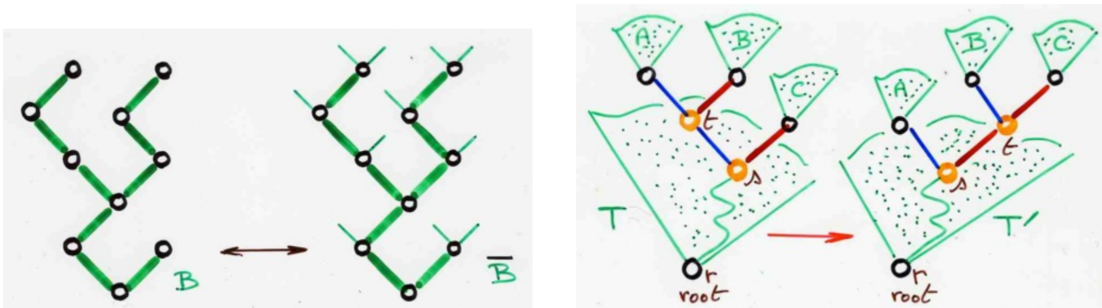

Fig. 1: A binary tree B and its associated complete binary tree $\overline{B}$ (left). Right rotation on a complete binary tree: the covering relation in the Tamari lattice (right).

1 Introduction

In this paper, we generalize the m-Tamari lattice to posets of arbitrary paths, as it is explained in section 2. We prove that these posets are actually lattices, that they satisfy a duality property, and that they partition the ordinary Tamari lattice into intervals. We first introduce some basic definitions in section 1.1 and some motivations in section 1.2.

1.1 Basic definitions

A binary tree is either empty or a triple (L, r, R) where L and R are binary trees (left and right subtrees) and r is the root of the binary tree. If L (resp. R) is not empty, its root is called the left (resp. right) child of r. A binary tree is complete when all vertices have either two children (internal vertices) or no child (external vertices). Figure 1 displayed the classical bijection between binary trees and complete binary trees. We denote by $\overline{B}$ the complete binary tree associated to B.

These two families of trees are enumerated by the well studied Catalan numbers $C_n = \frac{1}{2n+1}\binom{2n+1}{n}$.

We now define the Tamari lattice. The complete binary trees with n interior vertices can be equipped with a rotation. As in Figure 1, consider a complete binary tree $\overline{T}$ with an internal vertex s such that the left child of s, denoted by t, is also an internal vertex. Let A be the left subtree of t, B the right subtree of t and C the right subtree of s. Let $\overline{T'}$ be the complete binary tree constructed from $\overline{T}$ such that t becomes the right child of s, A the left subtree of s, B the left subtree of t and C the right subtree of t. This operation from $\overline{T}$ to $\overline{T'}$ is called a right rotation, and the operation from $\overline{T'}$ to $\overline{T}$ is called a left rotation. The right rotation define the covering relation of the well known Tamari lattice (see [27, 13]), denoted by $\overline{T} \prec \overline{T'}$.

In this article, we consider a path to be a (finite) walk on the square lattice, starting at (0,0), consisting of north and east unit steps denoted by N and E respectively. The set of ballot paths of height n is the set of paths that consist of n north steps, n east steps and lie weakly above the diagonal, that is, weakly above the path $(NE)^n$. They are also counted by the Catalan numbers. By applying a clockwise rotation of 45 degrees on ballot paths so that the diagonal becomes horizontal, these ballot paths become the well known Dyck paths (see Figure 2). The ballot paths can be generalized with a parameter m that is a positive integer. The m-ballot paths are the paths that consist of n north steps, mn east steps and lie weakly above the line $y = \frac{x}{m}$, that is, weakly above the path $(NE^m)^n$.

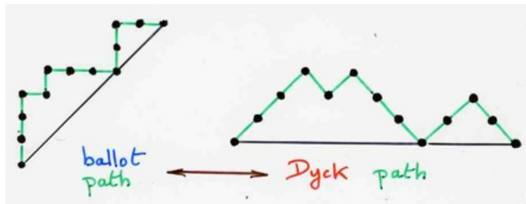

Fig. 2: Ballot path and Dyck path.

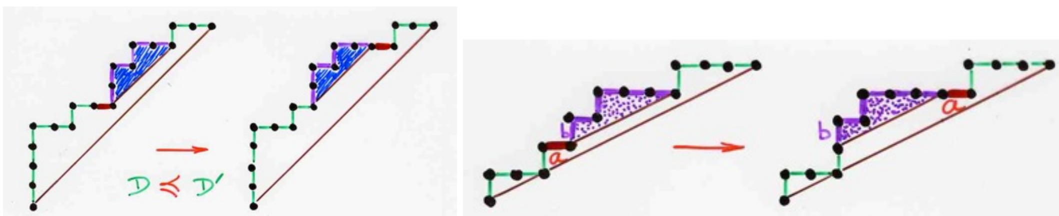

Fig. 3: The Tamari covering relation for ballot (Dyck) path (left). The covering relation in the m-Tamari lattice (m=2) (right).

Using the bijection between complete binary trees with n internal vertices and ballot paths of height n, which is called the postorder traversal on edges, the covering relations for the Tamari lattice can be translated into the following procedure on ballot paths. Let D be a ballot path of height n. Let E be an east step that precedes a north step in D. Draw a diagonal of slope 1 starting at the right extremity of E until it touches D again. Construct D' from D by switching E and the portion of the path above this diagonal. Then the covering relation in the Tamari lattice based on ballot paths becomes $D \prec D'$ (see Figure 3 for such a covering relation). Motivated by the higher diagonal coinvariant spaces of the symmetric group, the covering relation on ballot paths is generalized in [5] to m-ballot paths by mimicking the above procedure as follows. Let D be an m-ballot path. Let E be an east step that precedes a north step in D. Draw a diagonal of slope $\frac{1}{m}$ starting at the right endpoint of E until it touches D again. Construct D' from D by switching E and the portion of the path above this diagonal. Then the covering relation in the m-Tamari lattice is given by $D \prec D'$ (see Figure 3 for an example). For more on these lattices and for enumerations of their intervals, we refer the reader to section 5.

1.2 Rational Catalan combinatorics (a, b)

Let a and b be two relatively prime integers. We consider paths starting at $(0, 0)$ on the square lattice with north and east steps and strictly above the line $y = \frac{a}{b}x$, excluding the start and end points (see [7]). They are called (a, b)-ballot paths (or (a, b)-Dyck paths), and their study is the subject of very recent work under the term "rational Catalan combinatorics" (see [2, 3, 14, 15, 20] for more on this subject). The classical ballot paths and their extensions with any integer m are particular cases of such (a, b)-ballot paths. The simple Catalan ballot paths are obtained by putting $(a, b) = (n, n+1)$, and their m-extensions are obtained by putting $(a, b) = (n, mn + 1)$.

An open question is to give an extension of the Tamari lattice, and more generally of the m-Tamari lattice to any pair (a, b) of relatively prime integers. We propose an answer to this question, by giving a far more general extension of these Tamari lattices and in particular give a construction of a rational (a, b)-Tamari lattice.

2 Extension: The Tamari lattice Tam(v), where v is an arbitrary path

Let v be an arbitrary path, starting at (0,0). Consider all the lattice paths lying weakly above v that start at (0,0) and finish at the end of v. We define the poset Tam(v) on this set of paths with a covering relation. Let u be such a path above v. Let p be a lattice point on u. We define the horizontal distance $\text{horiz}_v(p)$ to be the maximum number of east steps that can be added to the right of p without crossing v. An example of these horizontal distances is given in Figure 4 (left). Suppose that p is preceded by an east step E and

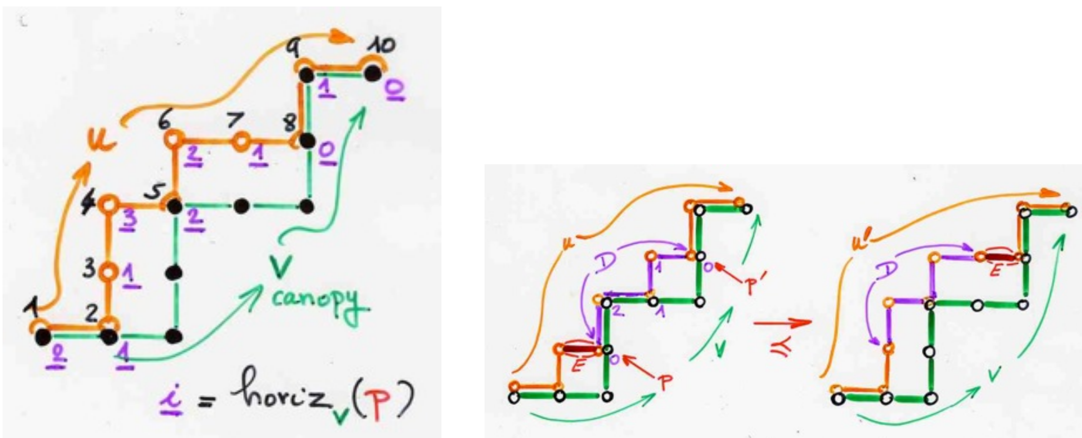

Fig. 4: A pair (u,v) of paths with the horizontal distance $horiz_v(p)$ (left). The covering relation defining the poset Tam(v)(right).

followed by a north step in u. Let p' be the first lattice point in u that is after p and such that $\text{horiz}_v(p') = \text{horiz}_v(p)$. As in Figure 4 (right), let $D_{[p,p']}$ be the subpath of u that starts at p and finishes at p'. Let u' be the path obtained from u by switching E and $D_{[p,p']}$. We define the covering relation to be $u \prec_v u'$ (see Figure 4 (right) for an example). Then the poset Tam(v) is the transitive closure $\leq_v$ of this relation. It is easy to see that Tam$((NE^m)^n)$ is the m-Tamari lattice.

For v an arbitrary path, let $\overleftarrow{v}$ be the path obtained by reading v backward and replacing the east steps by north steps and vice versa. We can now state our main results:

Theorem 1 *For any path v, Tam(v) is a lattice.*

Recall from the first section that the usual Tamari lattice on complete binary trees with n interior vertices is isomorphic to the lattice Tam$((NE)^n)$.

Theorem 2 *The lattice Tam(v) is isomorphic to the dual of Tam$(\overleftarrow{v})$.*

For any pair (a, b) of relatively prime integers, we define the lattice Tam(a, b) as to be the lattice Tam(v) where v is the minimum path above the segment passing through the origin $(0, 0)$ and the point (a, b). The duality between Tam(v) and Tam$(\overleftarrow{v})$ becomes the duality between Tam(a, b) and Tam(b, a) (see Figure 5). In this Figure we have drawn with brown dotted arrows the covering relation for the Young lattices $Y(v)$ of Ferrers diagrams included in the Ferrers diagram defined by the path v. The lattice $Y(v)$ can be seen as a refinement of the lattice Tam(v), and in that case, by the simple symmetry exchanging rows and columns, the lattice $Y(v)$ is isomorphic to $Y(\overleftarrow{v})$.

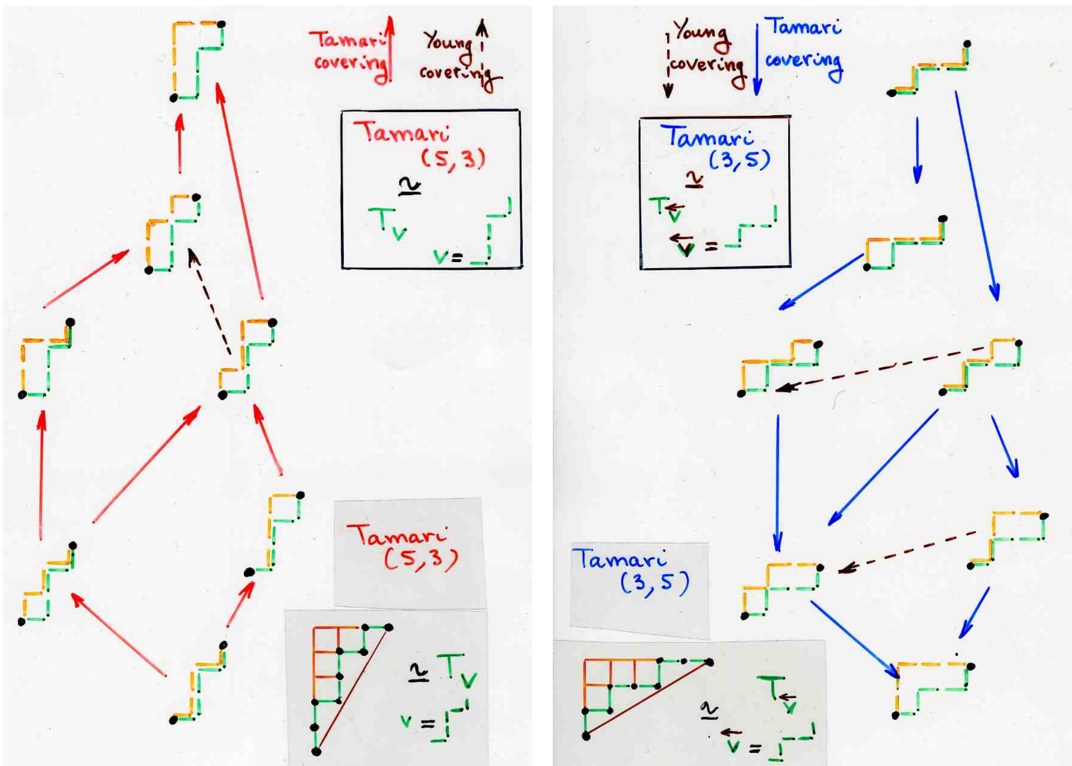

Fig. 5: Duality and rational Catalan combinatorics: the lattices Tam(5,3) and Tam(3,5) .

Theorem 3 *The usual Tamari lattice Tam$((NE)^n)$ can be partitioned into disjoint intervals $I(v)$ indexed by the unitary paths v consisting of a total of $n-1$ east and north steps, i.e.*

$$Tam((NE)^n) = \bigcup_{|v|=n-1} I(v),$$

where each $I(v) \cong Tam(v)$.

An example of Theorem 3 is given in Figure 6.

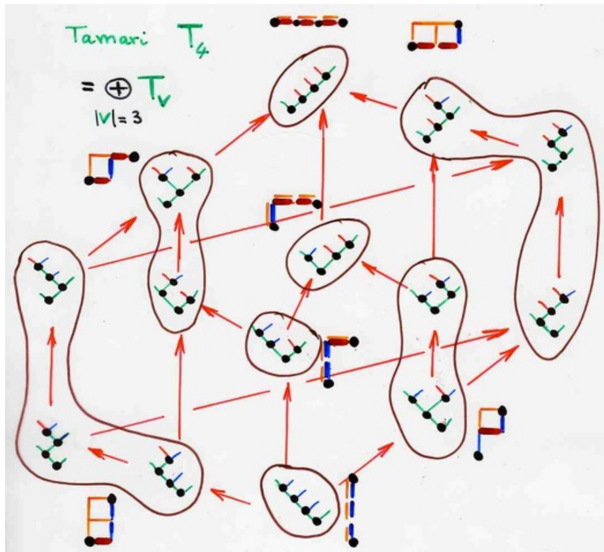

Fig. 6: The decomposition of the Tamari lattice on complete binary trees with 4 interior vertices into the union of 8 disjoint intervals $\mathrm{Tam}(v)$ (Theorem 3).

3 Canopy of a binary tree

For any binary tree B, we construct a word $w(B)$ on the alphabet $\{a, \bar{a}, b, \bar{b}\}$. Walking clockwise around B and starting at the root, we write the letter a when we walk on a left edge for the first time and $\bar{a}$ when we walk on a left edge for the second time. Similarly we repeat with the letters b and $\bar{b}$ for the right edges (see Figure 7 (left) for an example).

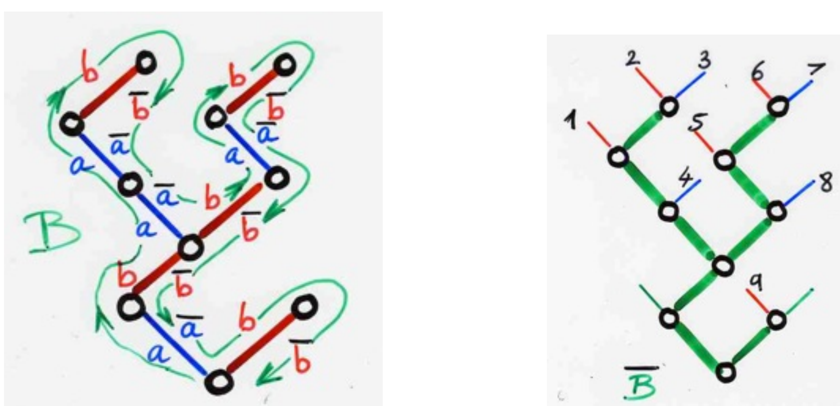

Fig. 7: Walk around a binary tree B: the word w(B) (left). Second definition of the canopy (right).

From $w(B)$, we construct two subwords. The first subword $u(B)$ is obtained by keeping track only of

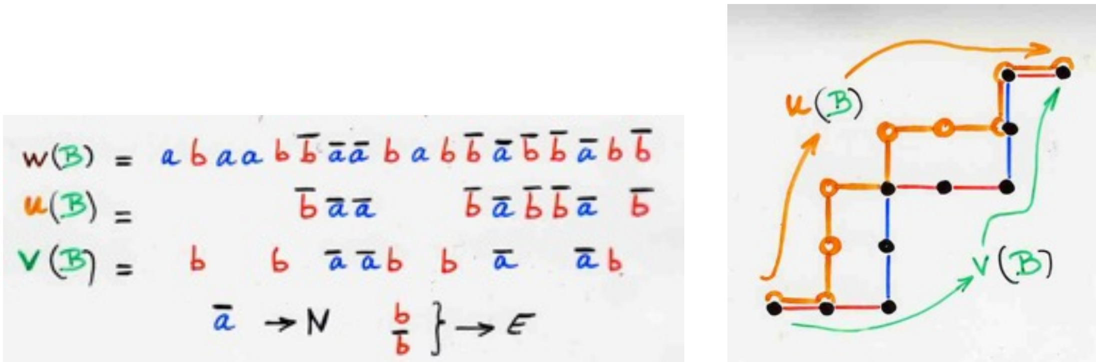

Fig. 8: The words $w(B), u(B)$ and $v(B)$ associated to the binary tree in Figure 7 (left). The pair (u,v) of paths associated to a binary tree in Figure 7(right).

the two letters $\{\bar{a}, \bar{b}\}$ in $w(B)$. We identify a path with $u(B)$ by replacing in this sequence the letter $\bar{a}$ by a north step and $\bar{b}$ by an east step. The canopy $v(B)$ of B, which is also a subword of $w(B)$, is obtained similarly by keeping track only of the letters $\{\bar{a}, b\}$. We identify a path with $v(B)$ by replacing in this sequence the letter $\bar{a}$ by a north step and b by an east step[(i)]. The concept of the canopy was introduced using a different terminology in [21]. For the binary tree in Figure 7 (left), we show an example of all these words in Figure 8 (left) and draw the paths $u(B)$ and $v(B)$ in Figure 8 (right).

It is no accident that we use the same letter v for the canopy as the letter that defines the poset in the previous section. Before explaining this, we can mention an easy property:

Lemma 4 *For any given binary tree B, the path $u(B)$ is weakly above the canopy (also a path) $v(B)$.*

Let $\overline{B}$ be a complete binary tree with n vertices. It is not difficult to prove that the canopy can also be defined using the following equivalent definition.

The second definition of the canopy, which is defined in [21], can be described as follows. Walking around $\overline{B}$ clockwise starting at the root, record the sequence of left and right external edges, except the first and last external edges. From this sequence, construct a path by changing the right external edges into north steps and the left external edges into east steps. The path obtained is also the canopy (see Figure 7 (right) for an example). Because of this definition, we define the interior canopy of the complete binary tree $\overline{B}$ to be the canopy of B.

In the proofs of the main theorems, we need a third definition of the canopy. We do not explicit its definition here. It is based on ordering the vertices of the binary tree B in the so-called symmetric order (also called infix or in-order), and considering a certain order on the left edges of B. The sequence of "right heights" of these left edges is related to the sequence of distances between the pairing vertical edges of the paths u and v (the labels of the path v in Figure 9).

4 Proofs of the main theorems

The proofs of the main theorems rely on the following 3 propositions.

(i) So by abuse of notation, we will refer to the paths $u(B)$ and $v(B)$.

Proposition 5 *The map defined in Section 3 associating the pair (u, v) to a binary tree is a bijection from the set of binary trees with n vertices to the set of unordered pairs of non-crossing paths, consisting each of a total of $n - 1$ north and east steps, with the same endpoints.*

The bijection between binary trees and pairs of paths (u, v) was introduced in a different form ([12], [28] slides 42-53). We have described here a new version of the bijection which fits our purpose. We describe the reverse bijection in terms of a "push-gliding algorithm", which is a kind of "jeu de taquin" on a binary tree attached to a path. The algorithm starts with an empty binary tree and the path v where the vertical edges are labeled by the horizontal distance to the path u. Reading the path v, from bottom to top, sliding is performed according to each labeled vertical edge and pushing according to each horizontal edge. For each pushing, a new edge is introduced in the binary tree attached to the path. At the end the path is empty and we get the binary tree related to pair (u, v), (see Figure 9).

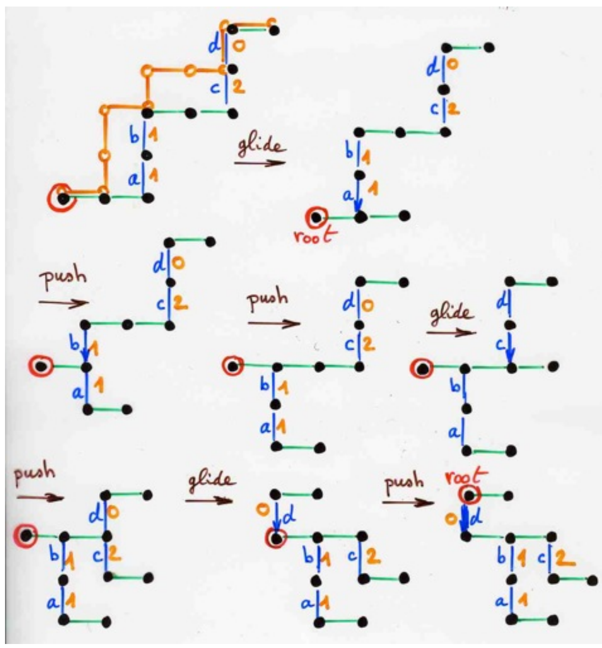

Fig. 9: The "push-gliding" algorithm

Proposition 6 *The set $I(v)$ of complete binary trees having interior canopy v is an interval of the ordinary Tamari lattice on complete binary trees with $|v| + 1$ interior vertices.*

This proposition is well known, see for example Proposition 3.5 in [22] where the Boolean lattice appears as a quotient of the Tamari lattice. The intervals $\mathrm{I}(v)$ are the fibres over points of the usual map from the Tamari lattice to the Boolean lattice. These intervals can also be viewed as the images under the map from the symmetric group to the Tamari lattice of the set of permutations with a fixed descent set. Here we give a direct combinatorial proof. The max and min element of the interval can be defined easily from the "push-gliding" algorithm mentioned above.

Proposition 7 *For any path v, the poset $I(v)$ is isomorphic to Tam(v).*

The proof relies on a lemma relating the sequence of horizontal distances $\mathrm{horiz}_v(p)$ of the vertices p of the path u to the sequence of right heights of the vertices of the corresponding binary tree B when traversed in postorder. Then the proof continues by studying the relation between the canopy and the rotation in binary tree. For binary trees with a given canopy, the rotations are restricted, and such rotations can be shown to be equivalent to the covering relation for Tam(v) using the lemma mentioned previously.

We can now prove our main theorems of section 2.

Proof of Theorem 1: An interval of a lattice is always a lattice, therefore I(v) is a lattice by Proposition 6, and so is Tam(v) by Proposition 7. □

Proof of Theorem 2: After applying a reflection to a binary tree with canopy v, it is easy to see using the second definition of canopy, that the canopy of this tree (obtained by reflection) is precisely $\overleftarrow{v}$. The left and right rotation on a complete binary tree are exchanged and we deduce from the constructions of Tam(v) and Tam$(\overleftarrow{v})$ that these lattices are isomorphic up to duality (also called anti-isomorphic). □

Proof of Theorem 3: We partition the complete binary trees with n interior vertices into sets of trees with the same interior canopy. We then apply Proposition 7 to each set of trees. □

5 Connections with the diagonal coinvariant spaces and perspectives

Our work has been influenced by the combinatorics of the "generalized" diagonal coinvariant spaces of the symmetric group. We give a brief description of the subject here, and refer the reader to [4, 5, 8, 9, 16] for more details.

Let $X = (x_{i,j})_{\substack{1\le i\le k\\ 1\le j\le n}}$ be a matrix of variables. A permutation σ of the symmetric group $\mathfrak{S}_n$ permutes the variables columnwise by $\sigma(X) = (x_{i,\sigma(j)})_{\substack{1\le i\le k\\ 1\le j\le n}}$, i.e. $\sigma(x_{i,j}) = x_{i,\sigma(j)}$. This action can be directly extended to all the polynomials in $\mathbb{C}[X]$. All the variables in a same row of X are said to be contained in the same set of variables. Since X contains k rows, there are k sets of variables. Let $\mathcal{J}$ be the ideal generated by constant free invariant polynomials under this action. The diagonal coinvariant spaces of $\mathfrak{S}_n$ are defined as $\mathcal{DR}_{k,n} := \mathbb{C}[X]\big/\mathcal{J}$. They can be generalized using an additional parameter m that is a positive integer. The higher diagonal coinvariant spaces of the symmetric group are defined as $\mathcal{DR}^m_{k,n} := \varepsilon^{m-1} \otimes \mathcal{A}^{m-1}\big/\mathcal{J}\mathcal{A}^{m-1}$, where ϵ is the sign representation and $\mathcal{A}$ is the ideal generated by alternants, i.e. polynomials $f(X)$ such that $\sigma f(X) = \varepsilon(\sigma)f(X)$, $\forall\sigma\in\mathfrak{S}_n$. Note that $\mathcal{DR}_{k,n} = \mathcal{DR}^1_{k,n}$. The $\mathcal{DR}^m_{k,n}$ are representations of $\mathfrak{S}_n$ because the action given above can be applied to the quotient space $\mathcal{DR}^m_{k,n}$. They are graded with respect to the degree of each set of variables. We denote the subspace of alternants of $\mathcal{DR}^m_{k,n}$ by $\mathcal{DR}^{m\ \varepsilon}_{k,n}$.

In the case $k = 1$, they are classical [26] and the dimensions of $\mathcal{DR}^{m\ \varepsilon}_{1,n}$ and $\mathcal{DR}^m_{1,n}$ are given by 1 and $n!$, respectively.

In the case $k = 2$, they were first defined and studied by Garsia and Haiman because of their connections with the Macdonald polynomials. It was proven by Haiman [18] that the dimensions of $\mathcal{DR}^{m\ \varepsilon}_{2,n}$ and $\mathcal{DR}^m_{2,n}$ are given by $\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{mn}$ and $(mn+1)^{n-1}$, respectively. The first number corresponds to the number of m-ballot paths of height n and the second one to the number of m-parking functions of height n. The m-parking functions of height n are simply the m-ballot paths labelled on the north steps, with the labels in the set $\{1, 2, ..., n\}$ such that consecutive north steps are labelled increasingly. The spaces $\mathcal{DR}_{2,n}^m$ have been studied by many researchers for more than 20 years. Despite that, there are still some important unresolved conjectures left in the field. We mention only one here. The m-shuffle conjecture [17] states that the graded Frobenius series of $\mathcal{DR}_{2,n}^m$ is equal to a q,t-weighted sum on m-parking functions, which involves the combinatorial statistics $area$ and $dinv$, and some quasi-symmetric functions associated to these m-parking functions.

For the case $k = 3$, Haiman [19] conjectured in the 1990's that the dimensions of ${\mathcal{DR}_{3,n}}^{\varepsilon}$ and $\mathcal{DR}_{3,n}$ are equal to $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ and $2^n(n+1)^{n-2}$ respectively.

Independently of all this story, Chapoton [10] proved in 2006 that the number of intervals in the Tamari lattice based on complete binary trees with n interval vertices is given by $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$. In 2008, the m-Tamari lattice was introduced in [5] and it was conjectured that the number of intervals and labelled intervals in the m-Tamari lattice are given by $\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}$ and $(m+1)^n(mn+1)^{n-2}$, respectively. A labelled interval in the m-Tamari lattice is simply an interval where the top path is decorated as a m-parking function. Refinements of these two results were proven in [8, 9].

The duality that is proved in this article shows that the number of intervals in Tam$((N^mE)^n)$ is the same as in the m-Tamari lattice Tam$((NE^m)^n)$. Using refinements and calculations, it seems that the number of labelled intervals in Tam$((NE^m)^n)$ is equal to the number of labelled intervals on east steps in Tam$((N^mE)^n)$, where the labelled intervals on east steps are defined by assigning the labels in the set $\{1, 2, ..., n\}$ on east steps of the upper path, and such that the labels on consecutive east steps are increasing. Note that for $m = 1$, this is easy to prove since you can obtain without difficulty the same functional equations for both cases from recurrences. But we have not been able to do so in the case $m > 1$. It would be interesting to see if the ideas presented in [11] could help prove this equality.

More recently, some researchers (see [1, 2, 6, 20]) have extended the combinatorics of the $\mathcal{DR}_{2,n}^m$ by considering paths and parking functions above the line with endpoints (0,0) and (b, a), where a, b are arbitrary positive integers[(ii)]. They defined the combinatorial statistics area and dinv on these objects. So this rational Catalan combinatorics can be seen as the combinatorics of some possible generalizations of the spaces $\mathcal{DR}_{2,n}^m$. Even though these spaces have not yet been shown to exist, some preliminary calculations [6] suggest that they do. One might try now to define a dinv statistic on paths and parking functions above an arbitrary path consisting of east and north steps, even if it is not known to be possible.

It remains to be seen if our lattices Tam(v), for arbitrary paths v, will give a combinatorial setup for the not yet defined generalizations[(iii)] of the spaces $\mathcal{DR}_{3,n}^m$. It will be interesting to verify this as the theory of the "generalized" diagonal coinvariant spaces develops.

We finish this article by mentioning that in a forthcoming paper [23], it will be shown that the total number of intervals in the lattices Tam(v), for all the paths v of length n, is given by $\frac{2(3n+3)!}{(n+2)!(2n+3)!}$, which is the same as the number of rooted non-separable planar maps with $n + 2$ edges (sequence A000139 of OEIS).

(ii) Note that the paths above the line with endpoints (0,0) and (mn, n) are the same as the paths above the line with endpoints (0,0) and $(mn + 1, n)$, this is why we use the term extension.

(iii) As it is explained in the previous paragraph, at the moment it is not known if these generalizations exist.

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Mail Kurt

0.4 Enumerating vertices

By admissible labelled constructions (or equivalent) (see for instance pg 100-104 of analytic combinatorics) we can show that the vertices are enumerated by A213507 through

$$V = Set(C') \implies V(z) = Exp(C'(z))$$

where $C'(z) = \sum_{n=1}^{\infty} \frac{(n-1)!C_n}{n!} z^n$ i.e. this is the egf of some labelled Catalan family i.e. $C'_n = (n-1)!C_n$ (see A065866 and A000108).

So need to identify a labelled set structure on the vertices, then identify the residual labelled Catalan structure. The labelled set structure tells that

- the 3 vertices for $n = 2$ correspond to
  - 2 from C'_2
  - 1 from $C'_1 * C'_1$ (labelled product) ✓
- the 17 vertices for $n = 3$ correspond to
  - 10 from C'_3 ✓
  - 6 from $C'_2 * C'_1$ (labelled product) ✓
  - 1 from $C'_1 * C'_1 * C'_1$ ✓

$C_1' = C_1 = 1$

$C_2' = (n-1)! \times \frac{1}{n+1}\binom{2n}{n} = \frac{1}{3} \times \frac{4\times 3}{2} = 2$

$C_3' = \frac{2}{4} \times \binom{6}{3} = \frac{2\times \cancel{6} \times 5 \times \cancel{4}}{\cancel{4}\times\cancel{3\times 2}} = 10$

$C_4' = \frac{6}{5} \times \binom{8}{4} = \frac{\cancel{6}\times\cancel{8}\times 7\times 6\times \cancel{5}}{\cancel{5}\times\cancel{4}\times\cancel{3\times 2}} = 84$

One labelled set structure (could be other more natural ways to decompose) seems to be

Definition 1. *Let (σ, τ) be a vertex. We define $B(\sigma, \tau)$ as follows*

- *Form the bipartite graph for σ, τ.*
- *This bipartite graph has $D_1, ..., D_k$ disconnected components (with same number of vertices on each side) if crossings are viewed as establishing connections. The components are ordered top to bottom.*
- *We form $B_1, ..., B_i$ where $i \leq k$ by merging adj Ds if the first has a larger minimum than the second. We repeat this merging until no further are possible.*
- *As such final Bs satisfy* $\min B_1 < \min B_2 < ... < \min B_i$

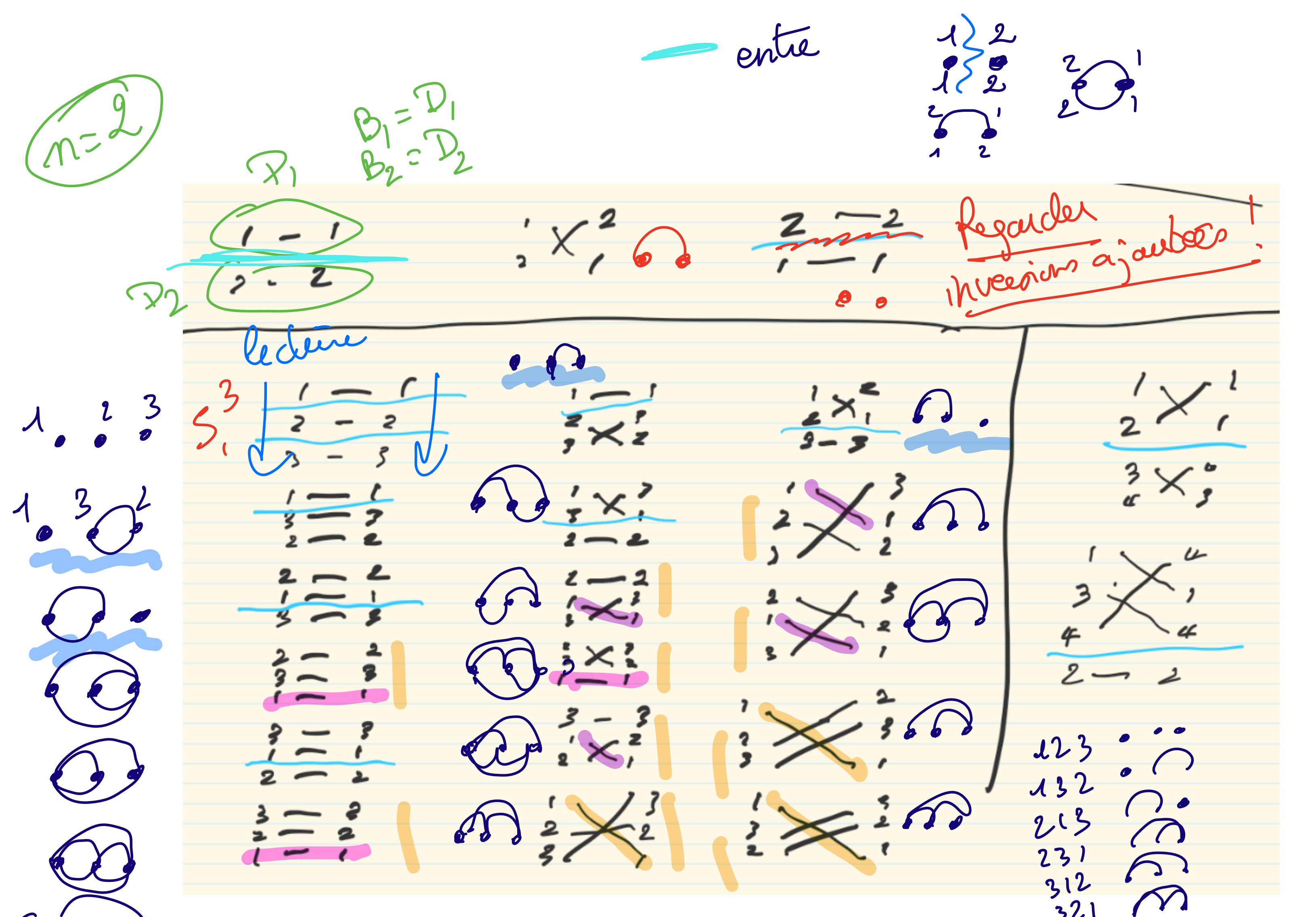
e.g. for $n = 2, 3$ (and 2 examples for $n = 4$)

$$\exp\left(z + 2\frac{z^2}{2} + 10\frac{z^3}{6} + 84\frac{z^4}{4!}\right)$$

$$= z + 2\frac{z^2}{2} + 10\frac{z^3}{6} + 84\frac{z^4}{4!} + \frac{\left(z + 2\frac{z^2}{2}\right)^2}{2} + \frac{z^3}{6} + \frac{z^4}{4!}$$

$$= z + 2\frac{z^2}{2} + 10\frac{z^3}{6} + 84\frac{z^4}{4!} + \left(z^2 + 4\frac{z^3}{2} + 4\frac{z^4}{4}\right)\frac{1}{2}$$

$$\frac{4}{4}z^3 = 6\frac{z^3}{6}$$

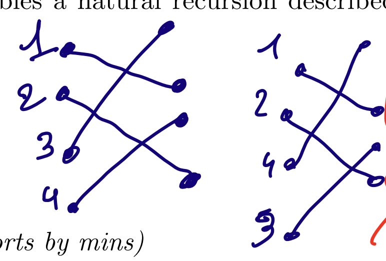


This identified set structure seems somewhat natural as it enables a natural recursion described by following lemma.

1 à la fin

a croise

Lemma 1. *We have the following equivalent characterisations*



1. *A vertex (σ, τ) is in the diagonal*
2. $\forall j, B_j(\sigma, \tau)$ *is in the diagonal*
3. $\prod_j B_j(\sigma, \tau)/R$ *in diagonal (labelled product, R in affect sorts by mins)*

ne marche pas

conditions sur Block Catalan peut être bordel : l croise qqn qui croise qqn qui ... qui croise le dernier en bas à gauche

Proof. 1 implies 2 and 3 implies 1 are both immediate. For 2 implies 3 we first show that $B_1; B_2 + |B_1|; ...; B_i + |B_1| + ... + |B_{i-1}| =: (\sigma', \tau')$ is a vertex.

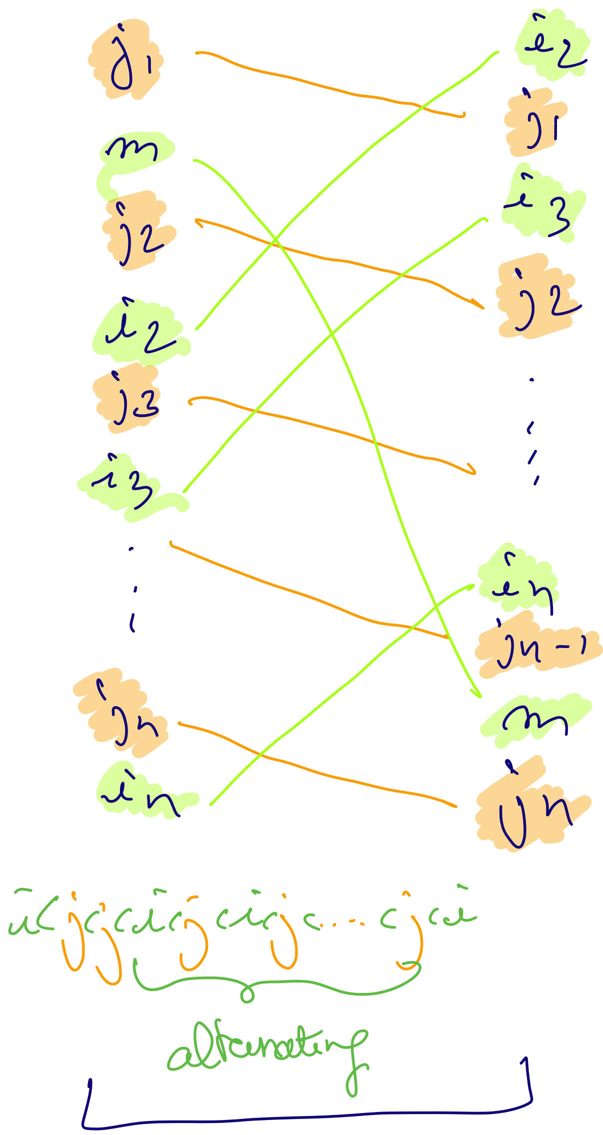
- If $(I, J \in D(n) \cap range(1, |B_1|)$ then B_1 forces I, J cond to be satisfied.
- Similarly if $(I, J \in D(n) \cap range(B_1 + ...B_k, B_1 + ... + B_k + B_{k+1})$ then B_{k+1} forces it to be satisfied.
- If I, J across multiple sets, then observe that

$$\exists l : |\cup_{1 \le k \le l} B_k \cap I| > |\cup_{1 \le k \le l} B_k \cap J| \vee \exists l : |\cup_{1 \le k \le l} B_k \cap I| < |\cup_{1 \le k \le l} B_k \cap J|$$

one side of the above must be true and

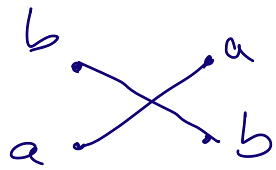
$$\exists l : |\cup_{1 \le k \le l} B_k \cap I| > |\cup_{1 \le k \le l} B_k \cap J| \implies \exists l' : |\cup_{1 \le k \le l} \sigma' \cap I| > |\cup_{1 \le k \le l} \sigma' \cap J|$$
$$\exists l : |\cup_{1 \le k \le l} B_k \cap I| < |\cup_{1 \le k \le l} B_k \cap J| \implies \exists l' : |\cup_{1 \le k \le l} \tau' \cap I| < |\cup_{1 \le k \le l} \tau' \cap J|$$

Motif Poisson :



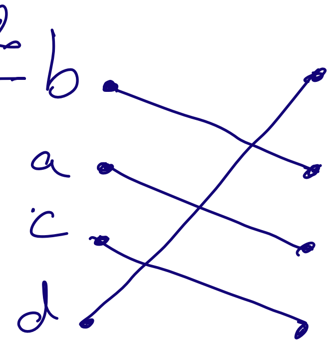
condition sur blocs primaires

$\underline{n=1}$



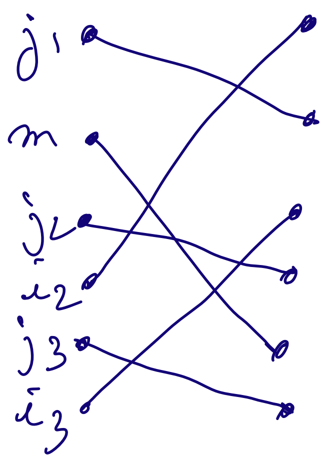
$<$

$\underline{n=2}$



with $a<b,\ c<d$

$\underline{n=3}$



with $m < j_{\sigma(1)} < j_{\sigma(2)} < i_{\tau(2)} < j_{\sigma(3)} < i_{\tau(3)}$

for a $\sigma \in \mathfrak{S}_3$

$\tau \in \mathfrak{S}_{\{2,3\}}$

? de la diagonale ?

Now we need to show that a shuffle permutation σ_s acting across the blocks produces a vertex. i.e. σ_s is a permutation such that $r(\sigma_s B_j) = B_j$ and $\min \sigma_s B_1 < \min \sigma_s B_2 < ... < \min \sigma_s B_k$.

- Can't see how a shuffle perm would alter the prior argument, so is this not immediate...

$\square$

From here need to

- better characterise the 'blocks' ~~no min en bas à droite~~ Non car croisements!
- show that these blocks are enumerated by $(n-1)!C_n$. Some potential approaches
  - do it 'directly'
  - identify an action of S_{n-1} then enumerate the equivalence classes by C_n. (actions appear non-trivial, maybe another set description produces better actions?)
  - biject to nestings of complete binary trees ($n-1$ internal edges)
  - other tree links... maybe some tie in to generalisation of canopy idea?
  - Maybe lattice structure given by generalised rotations on sets of labelled trees?
  - Maybe the blocks interact nicely with theorem 3.1?

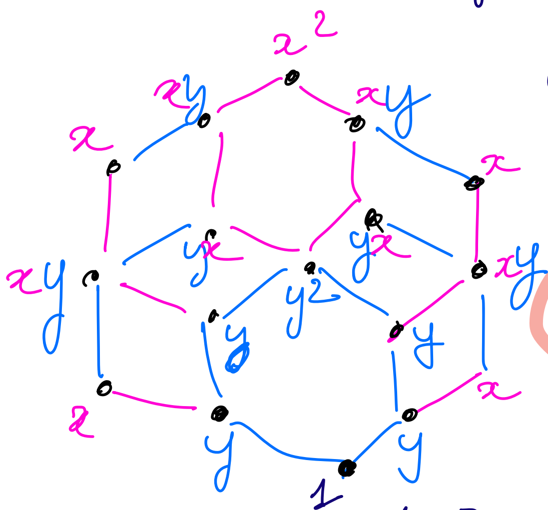
Rque de Guillaume

↳ Trouver combinatoire derrière le h-vecteur

(Paramètre pour avoir

- 1
- $x+y+1$
- $x^2+6xy+y^2+4x+4y+1$ ← facettes
- $x^3+24x^2y+24xy^2+y^3+11x^2+54xy$
  $+11y^2+11x+11y+1$

$n=2$

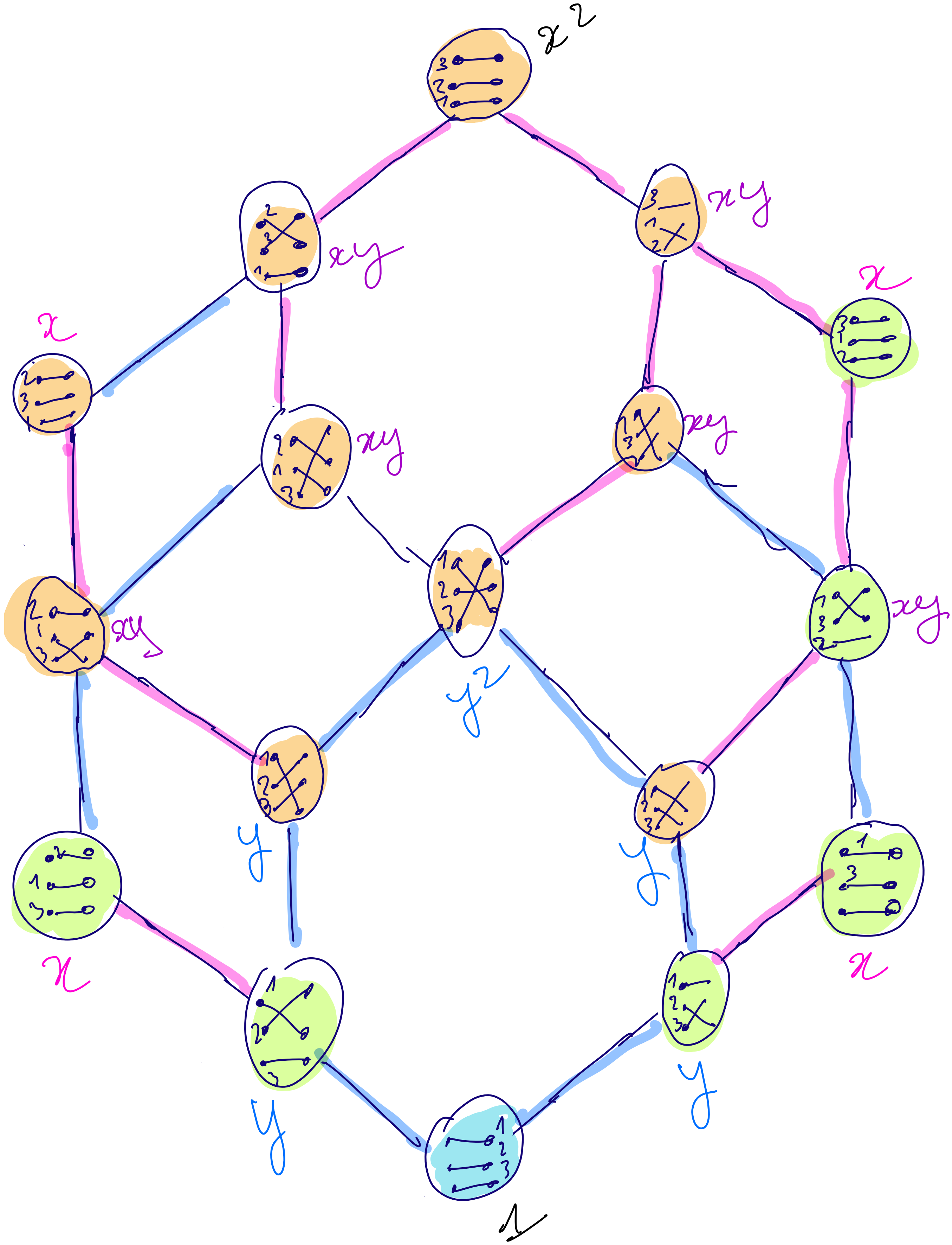


↳ stats sur paires de permutations qui évitent ...

Regarder les arêtes entrantes

Conj fausse : $y^{des(\sigma_1^{-1})}\, x^{des(\sigma_1^{-1}\sigma_2)}$

Ref : mail Zimbra



x^2
xy
xy
x
xy
xy
x
xy
y^2
xy
y
y
x
x
y
y
1

Autre descript° d'ensemble d'objets de Catalan que celle de Knuth?
Compatible avec les polynômes?