Réunion de travail

Synchrone = int canopée

	Sommets	Faces	Facettes
Associaèdre	Intervalles de Tamari $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ $k=0$	f-vecteur $f_k = \sum_{x \le y} \binom{in(x)+out(y)}{k}$ $= h_k$ (canonical complex des intervalles) CONJECTURES Pilaud $\frac{2}{(3n+1)(3n+2)}\binom{n-1}{k}\binom{4n+1-k}{n+1}$ Chapoton $\sum_{x\le y}\binom{\text{ceux où canopée } \neq \text{ de } k}{k}$ à l'arête donnée par involution Tamari (celle de Viviane ?)	En bij avec • intervalles synchrones (un sommet distingué par chaque facette) • Arbres ternaires gauches [Fonlupt / Schaeffer-Jacquard] $k=n-1$ $\frac{2}{(3n+1)(3n+2)}\binom{3n+2}{n+1}$ $= \frac{2}{(2n+1)(2n)}\binom{3n}{n+1}$
Permutoèdre	?	?	En bijection avec les arbres planaires bipartis décorés / essential complementary partitions

• Série génératrice avec symétries ?

Rec Matthieu: (σ_1, σ_2) qui stabilisent $(1 \dots i)$ $(i+1 \dots n)$
si pas de ∞ alors pas de ∞ ?

Intervalles synchrones [Chapoton / Combe]

Formule associaèdre : (F,G) tq $\max(F) \leq \min(G)$

Tonks → étiquetage des arbres : 1 aux feuilles

Schaeffer / Jacquard

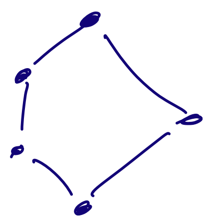
↳ arbres binaires gauches  v ← $\in [\![1, v]\!]$

• Vincent utilise canonical join pour encoder (diagramme d'arcs)

Tamari ⟶ représentants canoniques → arcs

f-vecteur associaèdre

$n=2$ 13, 18, 6



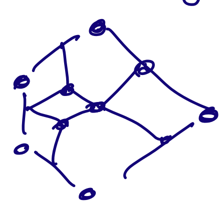
$f_k = \binom{0}{k} + 6\binom{1}{k} + 6\binom{2}{k}$

$k=0 \to 13$

$k=1 \to 18$

$k=2 \to 6$

$n=3$ (68, 144, 99, 22)



(13 sommets
18 arêtes
6 2-faces)

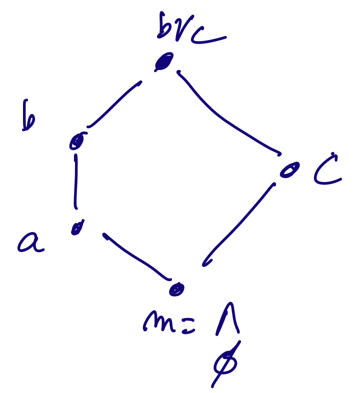
$n=4$ (395, 1140, 1197, 546, 91)

C'est le f-vecteur de la deleg et h-vecteur du canonical complex

Canonical join complex : chaque élément comme un join

??

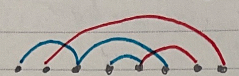
nested set complex



∨ = join

pensé à partir de ce présentant de Boulet

Tamari ~~lattice~~ interval : semi-crossings diagrams

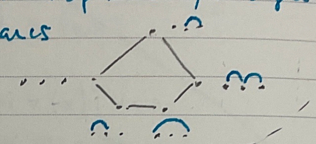


$\boxed{1}$ non-crossing

non-crossing

forbidden

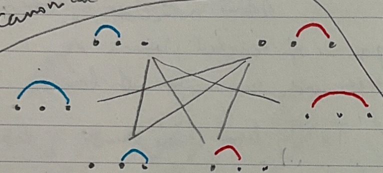
element of Tamari ; in-degree in # arcs



$\boxed{2}$ semi-crossings

forbidden =

canonical complex



Vincent a un papier sur ce complexe !

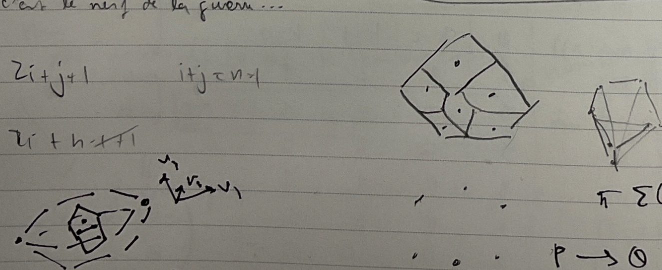
$$\sum_{x<y} \binom{in(x)+out(y)}{k} = \sum_j \binom{j}{k} \#\{x<y \mid in\,x + out(y) = k\} = \sum_j \binom{j}{k} f_k = h_k(\text{canonical complex})$$

calcul de ce h-vecteur est extrêmement rapide !!

qu'est-ce qui m'empêche de faire ces révisions ce soir ?? publis acceptés, c'est le nerf de la guerre...

$2i+j+1$ $\quad i+j=n-1$

$2i+h-t+1$



$\pi\ \Sigma(P,R) = \Sigma(Q,R)$

$P \to Q$

R

$$\Sigma(P,(v_1,v_2)) \overset{?}{=} \Sigma(\Sigma(P,v_1),v_2)$$

$P \xrightarrow[\pi]{\Sigma(P,v_1,v_2)} R^2$

$\Sigma(P,v_1) \quad R^1$

$\pi(\Sigma(P,v_1)), \Sigma(R^2,R^1))$

thm (Vincent) :

inter...

in(...

What I wan...

Intervals giv...

all faces in

faces = (13, ...

(68

(39...

petits f...

donc sûrem...

(dans Chapter

19.01.2023

la prop de KV

une formul...

qu'il y a de...

qui nous intére...

$\Sigma(P, v_1 v_2) \prec$

x une ...

ε_x une (...

// x d...

$\langle x \rangle = \langle 0, x \rangle$

Lire papier de Vincent

À lire pour canonical complex

THE CANONICAL COMPLEX OF THE WEAK ORDER

DORIANN ALBERTIN AND VINCENT PILAUD

Abstract. We define and study the canonical complex of a finite semidistributive lattice L. It is the simplicial complex on the join or meet irreducible elements of L which encodes each interval of L by recording the canonical join representation of its bottom element and the canonical meet representation of its top element. This complex behaves properly with respect to lattice quotients of L, in the sense that the canonical complex of a quotient of L is the subcomplex of the canonical complex of L induced by the join or meet irreducibles of L uncontracted in the quotient. We then describe combinatorially the canonical complex of the weak order on permutations in terms of semi-crossing arc bidiagrams, formed by the superimposition of two non-crossing arc diagrams of N. Reading. We provide explicit direct bijections between the semi-crossing arc bidiagrams and the weak order interval posets of G. Châtel, V. Pilaud and V. Pons. Finally, we provide an algorithm to describe the Kreweras maps in any lattice quotient of the weak order in terms of semi-crossing arc bidiagrams.

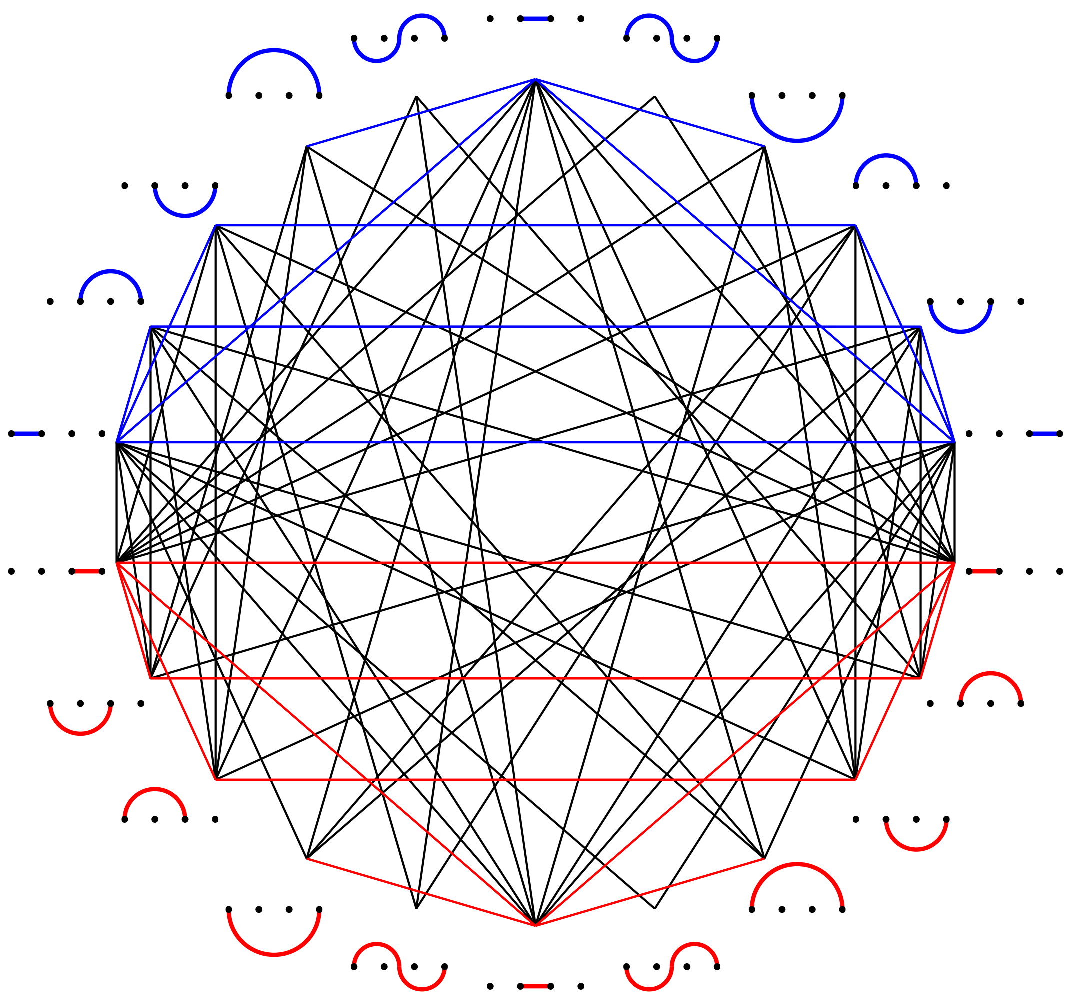


Figure 1. The canonical complex of the weak order on $\mathfrak{S}_4$ labeled by arcs.

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Contents

1. Introduction

A finite lattice L is join semidistributive when any element admits a canonical join representation (see *e.g.* [FN95] for a classical reference on lattices). This enables us to define the canonical join complex of L [Rea15, Bar19], whose vertices are the join irreducible elements of L and whose simplices are the canonical join representations in L. When L is both join and meet semidistributive, it thus admits both a canonical join complex and a canonical meet complex which are actually isomorphic flag simplicial complexes [Bar19].

In the first part of this paper, we define the canonical complex of a finite semidistributive lattice L, a larger flag simplicial complex where the canonical join complex and the canonical meet complex naturally live and interact. More precisely, its vertex set is the disjoint union of the set of join irreducible elements of L with the set of meet irreducible elements of L, and its simplices are the disjoint unions $J \sqcup M$ of a canonical join representation J in L with a canonical meet representation M in L such that $\bigvee J \le \bigwedge M$. In other words, each interval $[x, y]$ in L contributes to a simplex of the canonical complex given by the disjoint union of the canonical join representation of x with the canonical meet representation of y. This provides a model for the intervals of L which is compatible with lattice quotients. Namely, the canonical complex of a quotient $L/{\equiv}$ is the subcomplex of the canonical complex of L induced by the join and meet irreducibles of L uncontracted by the congruence $\equiv$.

In the second part of this paper, we study the combinatorics of the canonical complex of the weak order. N. Reading showed in [Rea15] that join irreducible permutations correspond to certain arcs wiggling around the horizontal axis, and that canonical join representations of permutations correspond to non-crossing arc diagrams. We show that the elements of the canonical complex can be interpreted as semi-crossing arc bidiagrams, defined as pairs $\delta_\vee \sqcup \delta_\wedge$ of non-crossing arc diagrams where only certain types of crossings are allowed between an arc of $\delta_\vee$ and an arc of $\delta_\wedge$. It thus follows that the canonical complex of any quotient of the weak order is isomorphic to a subcomplex of the semi-crossing complex induced by arcs contained in an upper ideal of the subarc order. We then provide explicit direct bijections between the semi-crossing arc bidiagrams and the weak order interval posets of G. Châtel, V. Pilaud and V. Pons [CPP19], which are both in bijection with the intervals of the weak order. Finally, we provide an algorithm to describe the Kreweras maps in any lattice quotient of the weak order in terms of semi-crossing arc bidiagrams, generalizing the classical Kreweras complement on non-crossing partitions.

2. The canonical complex of a finite semidistributive lattice

This section deals with canonical meet and join representations in a finite semidistributive lattice and its quotients. We start with a recollection on join semidistributive lattices, their canonical join representations, their canonical join complexes, their Kreweras maps, and their lattice congruences (Section 2.1). We then define the canonical complex of a semidistributive lattice L which encodes the intervals of L and contains both the canonical join complex and the canonical meet complex of L (Section 2.2).

2.1. **Recollection on lattices.**

We start by a quick recollection on semidistributive lattices, canonical representations, canonical complexes, Kreweras maps and lattice congruences. All the material covered here is classical, we refer for instance to [FN95, Rea16, Rea15, Bar19]. Following [Bar19, Exm. 10], we illustrate this section with the case of distributive lattices.

2.1.1. *Join representations and semidistributive lattices.* Consider a finite lattice $(L, \leq, \vee, \wedge)$ where $\vee$ is the join and $\wedge$ is the meet. We see $\vee$ and $\wedge$ as internal binary operators on L and try to factorize the elements of L in some canonical way. It is first important to understand the irreducible elements for $\vee$ and $\wedge$.

Definition 1. An element $x \in L$ is called *join* (resp. *meet*) *irreducible* if it covers (resp. is covered by) a unique element denoted $x_\star$ (resp. $x^\star$). We denote by $\mathcal{JI}(L)$ (resp. $\mathcal{MI}(L)$) the subposet of L induced by the set of join (resp. meet) irreducible elements of L.

Definition 2. A *join representation* of $x \in L$ is a subset $J \subseteq L$ such that $x = \bigvee J$. Such a representation is *irredundant* if $x \neq \bigvee J'$ for any strict subset $J' \subsetneq J$. The irredundant join representations in L are antichains of L, and are ordered by containement of the lower sets of their elements (*i.e.* $J \leq J'$ if and only if for any $y \in J$ there exists $y' \in J'$ such that $y \leq y'$ in L). The *canonical join representation* of x, denoted $\mathbf{cjr}(x)$, is the minimal irredundant join representation of x for this order, when it exists.

Note that when it exists, $\mathbf{cjr}(x)$ is an antichain of $\mathcal{JI}(L)$. The following statement characterizes the lattices where canonical join representations exist.

Proposition 3 ([FN95, Thm. 2.24 & Thm. 2.56])**.** *A finite lattice L is join semidistributive when the following equivalent conditions hold:*

(i) *$x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$,*
(ii) *for any cover relation $x \lessdot y$ in L, the set*

$$K_\vee(x, y) := \{z \in L \mid z \not\leq x \text{ but } z \leq y\} = \{z \in L \mid x \vee z = y\}$$

has a unique minimal element $k_\vee(x, y)$ (which is then automatically join irreducible),
(iii) *any element of L admits a canonical join representation.*

Moreover, the canonical join representation of $y \in L$ is $\mathbf{cjr}(y) = \{k_\vee(x, y) \mid x \lessdot y\}$.

Note that in a finite join semidistributive lattice L, we can associate to any meet irreducible element m of L a join irreducible element $\kappa_\vee(m) := k_\vee(m, m^\star)$ of L. Moreover, the existence of canonical join representations enable us to consider the following complex, illustrated in Figure 2. It was initially defined in [Rea15], where a combinatorial model was provided for the weak order (see Section 3.1), and studied in [Bar19] for arbitrary finite semidistributive lattices.

Definition 4. The *canonical join complex* $\mathcal{CJC}(L)$ of a finite join semidistributive lattice L is the simplicial complex on $\mathcal{JI}(L)$ whose faces are the canonical join representations of the elements of L.

The *meet semidistributivity*, the maps $K_\wedge$, $k_\wedge$ and $\kappa_\wedge$, the *canonical meet representation* $\mathbf{cmr}(x)$ and the *canonical meet complex* $\mathcal{CMC}(L)$ are all defined dually. A lattice L is *semidistributive* if it is both meet and join semidistributive. In this case, the maps $\kappa_\vee$ and $\kappa_\wedge$ define inverse bijections between $\mathcal{MI}(L)$ and $\mathcal{JI}(L)$, and the complexes $\mathcal{CJC}(L)$ and $\mathcal{CMC}(L)$ behave particularly nicely.

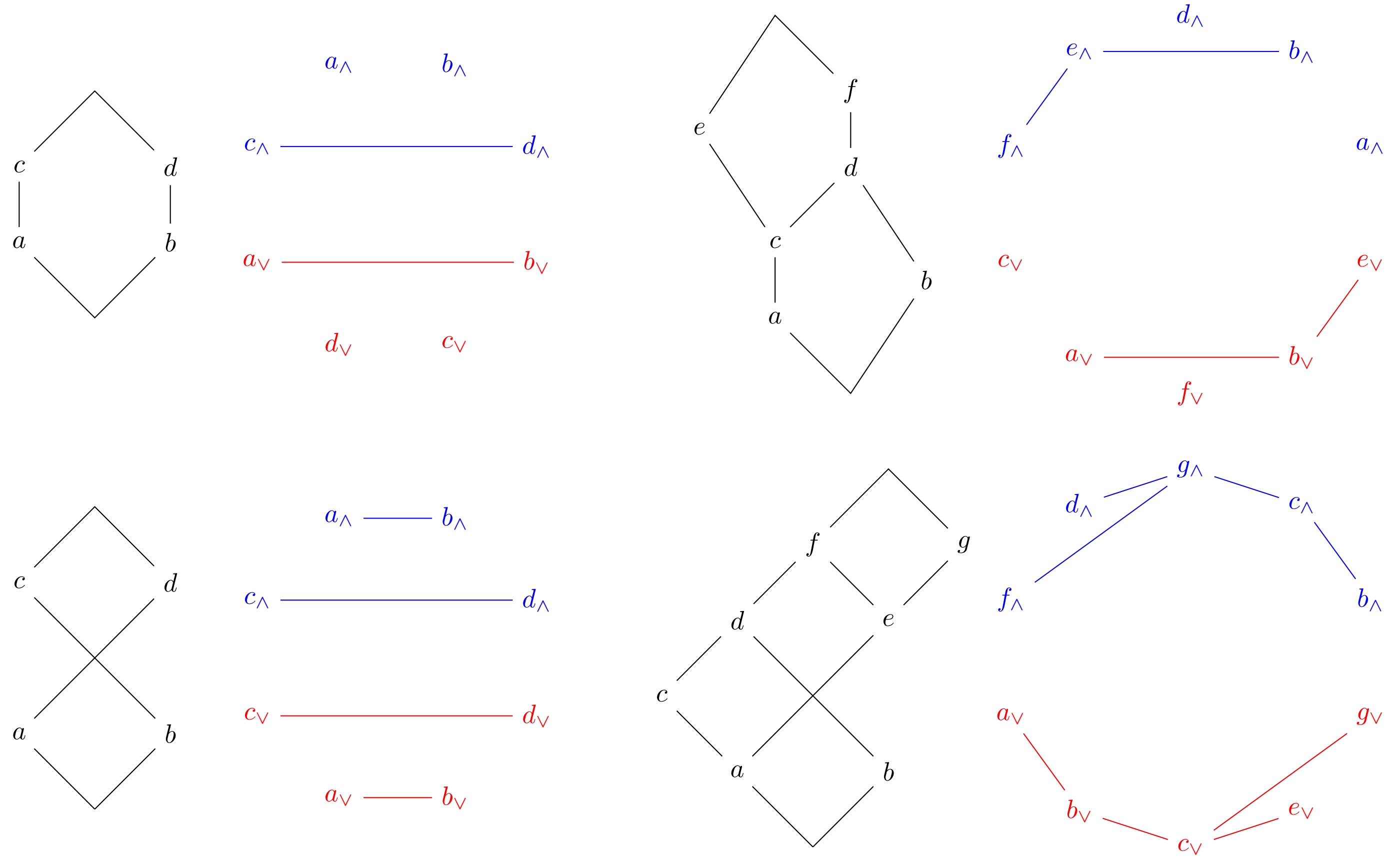

FIGURE 2. Some semidistributive lattices and their canonical join (red) and meet (blue) complexes. The letters label all join or meet irreducible elements, and we denote by $x_\vee$ (resp. $x_\wedge$) the element x when it is considered as a join (resp. meet) irreducible. Note that we always consistently color joinands in red and meetands in blue. The bottom two lattices are distributive while the top two are only semidistributive.

Proposition 5 ([Bar19, Thm. 2 & Coro. 5])**.** *If L is a finite semidistributive lattice, then*

(i) $\mathcal{CJC}(L)$ and $\mathcal{CMC}(L)$ are flag simplicial complexes (i.e. their minimal non-faces are edges, or equivalently they are the clique complexes of their graphs),

(ii) the maps $\kappa_\vee$ and $\kappa_\wedge$ induce inverse isomorphisms between $\mathcal{CMC}(L)$ and $\mathcal{CJC}(L)$.

In fact, it was proved in [Bar19, Thm. 2] that $\mathcal{CJC}(L)$ is flag if and only if L is semidistributive. We will not use the "only if" direction in this paper.

Example 6 (Distributive lattices)**.** The name semidistributivity actually comes from the well understood class of distributive lattices. A lattice L is distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$. Note that the dual condition $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in L$ is actually equivalent to the primal one. The fundamental theorem for distributive lattices affirms that L is distributive if and only if it is isomorphic to the lattice of lower sets of its join irreducible poset P. In other words, any antichain of join irreducible elements in P forms a canonical join representation in L. To be more precise, consider, for an antichain A of P, the two lower sets

$$j_A := \{x \in P \mid x \le y \text{ for some } y \in A\} \qquad \text{and} \qquad m^A := \{x \in P \mid x \not\ge y \text{ for all } y \in A\}.$$

Said differently, A is the set of maximal elements of j_A and the set of minimal elements of $P \smallsetminus m^A$. For $y \in P$, we abbreviate $j_{\{y\}}$ into j_y and $m^{\{y\}}$ into m^y. Then

- the join (resp. meet) irreducibles of L are precisely the lower sets j_y (resp. m^y) for $y \in P$,
- the map $\kappa_\vee$ (resp. $\kappa_\wedge$) is given by $\kappa_\vee(m^y) = j_y$ (resp. $\kappa_\wedge(j_y) = m^y$),
- the canonical join representation of j_A and the canonical meet representation of m^A are
$$\mathbf{cjr}(j_A) = \{j_y \mid y \in A\} \qquad \text{and} \qquad \mathbf{cmr}(m^A) = \{m^y \mid y \in A\}.$$
- the canonical join and meet complexes $\mathcal{CJC}(L)$ and $\mathcal{CMC}(L)$ are both (isomorphic to) the clique complex on the incomparability graph of P.

See [Bar19, Exm. 10].

2.1.2. *Kreweras maps.* In a semidistributive lattice L, each element has both a canonical join representation and a canonical meet representation. It is natural to consider the maps that exchange the canonical join representation with the canonical meet representation of the same element.

Definition 7. The *Kreweras maps* $\eta_\vee : \mathcal{CMC}(L) \to \mathcal{CJC}(L)$ and $\eta_\wedge : \mathcal{CJC}(L) \to \mathcal{CMC}(L)$ are defined by
$$\eta_\vee(M) := \mathbf{cjr}\big(\bigwedge M\big) \qquad \text{and} \qquad \eta_\wedge(J) := \mathbf{cmr}\big(\bigvee J\big).$$

Note that some authors call Kreweras maps the compositions $\eta_\vee \circ \kappa_\wedge : \mathcal{CJC}(L) \to \mathcal{CJC}(L)$ and $\eta_\wedge \circ \kappa_\vee : \mathcal{CMC}(L) \to \mathcal{CMC}(L)$, see for instance [Bar19]. As will be discussed in Example 52, the Kreweras maps for the Tamari lattice are closely related to the classical Kreweras complement on non-crossing partitions. For the moment, we recall that the Kreweras maps for the distributive lattices are related to rowmotion.

Example 8 (Distributive lattices)**.** With the notations of Example 6, for an antichain A in P, we denote by $\mathrm{row}_\vee(A)$ the set of maximal elements of m^A and by $\mathrm{row}_\wedge(A)$ the set of minimal elements of $P \smallsetminus j_A$. In other words, we have $m^A = j_{\mathrm{row}_\vee(A)}$ and $j_A = m_{\mathrm{row}_\wedge(A)}$. Hence, by Example 6, the Kreweras maps $\eta_\vee$ and $\eta_\wedge$ are given by
$$\eta_\vee(\{m^y \mid y \in A\}) = \{j_y \mid y \in \mathrm{row}_\vee(A)\} \qquad \text{and} \qquad \eta_\wedge(\{j_y \mid y \in A\}) = \{m^y \mid y \in \mathrm{row}_\wedge(A)\}.$$
See [Bar19, Rem. 32].

2.1.3. *Lattice congruences.* We now discuss quotients of the lattice L, considered as an algebraic structure with two internal binary operators $\vee$ and $\wedge$. We thus need equivalence relations on L that respects $\vee$ and $\wedge$.

Definition 9. A *congruence* $\equiv$ on L is an equivalence relation on L such that $x \equiv x'$ and $y \equiv y'$ implies $x \vee y \equiv x' \vee y'$ and $x \wedge y \equiv x' \wedge y'$. Equivalently, the equivalence classes are intervals, and the maps $\pi_\downarrow^\equiv$ and $\pi^\uparrow_\equiv$ sending an element to the minimum and maximum elements in its congruence class are order preserving.

Definition 10. The *lattice quotient* $L/{\equiv}$ is the lattice structure on the congruence classes, where for any two congruence classes X and Y,

- the order is given by $X \le Y$ if and only if $x \le y$ for some representatives $x \in X$ and $y \in Y$,
- the join $X \vee Y$ (resp. meet $X \wedge Y$) is the congruence class of $x \vee y$ (resp. $x \wedge y$) for any representatives $x \in X$ and $y \in Y$.

Note that the lattice quotient $L/{\equiv}$ is isomorphic to the subposet of L induced by the minimal (or maximal) elements in their congruence classes. This subposet is a join (resp. meet) subsemilattice of L but may fail to be a sublattice of L. We now consider all congruences of L.

Definition 11. The *congruence lattice* $\mathrm{con}(L)$ is the set of all congruences of L ordered by refinement.

The congruence lattice $\mathrm{con}(L)$ is a distributive lattice where the meet is the intersection of relations and the join is the transitive closure of union of relations. For any join irreducible element $j \in \mathcal{JI}(L)$, we denote by $\mathrm{con}(j)$ the unique minimal congruence of L that *contracts* j, that is with $j_\star \equiv j$. It turns out that $\mathrm{con}(j)$ is join irreducible in $\mathrm{con}(L)$ and that all join irreducible congruences in $\mathrm{con}(L)$ are of this form. Hence, any congruence of L is completely determined by the set of join irreducible elements of L that it contracts. We denote by $\mathcal{UJI}(\equiv)$ the set of join irreducible elements of L uncontracted by $\equiv$. Not all subsets of join irreducible elements of L are of the form $\mathcal{UJI}(\equiv)$ for some congruence $\equiv$ of L. The possible subsets are governed by the following relation.

Definition 12. For $j, j' \in \mathcal{JI}(L)$, we say that j *forces* j', and write $j \succcurlyeq j'$, if $\operatorname{con}(j) \ge \operatorname{con}(j')$, that is if any congruence contracting j also contracts j'.

The forcing relation is a preorder $\preccurlyeq$ (*i.e.* a transitive and reflexive, but not necessarily antisymmetric, relation) on $\mathcal{JI}(L)$, whose upper sets correspond to the congruences of L.

Proposition 13 ([Rea16, Prop. 9-5.16])**.** *The following conditions are equivalent for $J \subseteq \mathcal{JI}(L)$:*

- *J is an upper set of the forcing preorder (i.e. $j \succcurlyeq j'$ and $j \in J$ implies $j' \in J$).*
- *$J = \mathcal{UJI}(\equiv)$ for some congruence $\equiv$ of L.*

As already mentioned, the set $\mathcal{UJI}(\equiv)$ characterizes $\equiv$. It moreover enables to understand the elements of L which are minimal in their congruence classes and their canonical join and meet representations as follows.

Proposition 14 ([Rea16, Prop. 9-5.29])**.** *Let $\equiv$ be a congruence of a finite join semidistributive lattice L. Then*

- *an element $x \in L$ is minimal in its congruence class if and only if $\mathbf{cjr}(x) \subseteq \mathcal{UJI}(\equiv)$,*
- *the quotient $L/\equiv$ is join semidistributive and the canonical joinands of a congruence class X in $L/\equiv$ are the congruence classes of the canonical joinands of the minimal element in X.*

Proposition 14 translates as follows to the canonical join complex $\mathcal{CJC}(L)$.

Proposition 15. *Let $\equiv$ be a congruence on a finite join semidistributive lattice L. Then the canonical join complex $\mathcal{CJC}(L/\equiv)$ of the quotient $L/\equiv$ is isomorphic to the subcomplex $\mathcal{CJC}(\equiv)$ of the canonical join complex $\mathcal{CJC}(L)$ of L induced by $\mathcal{UJI}(\equiv)$.*

We will need the following statement relating the canonical join representation of an element x of L with the canonical join representation of the minimal element $\pi_\downarrow^\equiv(x)$ in its equivalence class. We provide a proof here as we have not found this statement explicitly in the literature.

Proposition 16. *Let $\equiv$ be a congruence of a finite join semidistributive lattice L. For any $x \in L$, the lower ideal of L generated by $\mathbf{cjr}(x)$ contains $\mathbf{cjr}\big(\pi_\downarrow^\equiv(x)\big)$.*

Proof. The proof works by induction on the size of the interval $[\pi_\downarrow^\equiv(x), x]$. The statement is immediate if $\pi_\downarrow^\equiv(x) = x$. Otherwise, there exists $j \in \mathbf{cjr}(x) \smallsetminus \mathcal{UJI}(\equiv)$. Let $y := \bigvee \big(\mathbf{cjr}(x) \bigtriangleup \{j, j_\star\}\big)$, where $\bigtriangleup$ denotes the symmetric difference. Since $j \equiv j_\star$, we have $x \equiv y$ and thus $\pi_\downarrow^\equiv(x) = \pi_\downarrow^\equiv(y)$. Since y has a join representation strictly contained in the lower ideal of L generated by $\mathbf{cjr}(x)$, we have $y < x$ so that $[\pi_\downarrow^\equiv(y), y]$ is strictly contained in $[\pi_\downarrow^\equiv(x), x]$. By induction hypothesis, we thus obtain that $\mathbf{cjr}\big(\pi_\downarrow^\equiv(y)\big)$ is contained in the lower ideal of L generated by $\mathbf{cjr}(y)$. Moreover, observe that $J := \big(\mathbf{cjr}(x) \smallsetminus \{j\}\big) \cup \mathbf{cjr}(j_\star)$ is a join representation for y, so that $\mathbf{cjr}(y)$ is contained in the lower ideal of L generated by J, which is itself contained in the lower ideal of L generated by $\mathbf{cjr}(x)$ (since any element of $\mathbf{cjr}(j_\star)$ is lower than $j_\star$ and thus than j). We conclude that $\mathbf{cjr}\big(\pi_\downarrow^\equiv(x)\big)$ is indeed in the lower ideal of L generated by $\mathbf{cjr}(x)$. $\square$

Note that this property is quite specific to $\pi_\downarrow^\equiv(x)$. Namely, a relation $x \le y$ does not imply any inclusion between the lower ideals of L generated by $\mathbf{cjr}(x)$ and $\mathbf{cjr}(y)$ in general (see *e.g.* Figure 2). Proposition 16 ensures that we can look for $\mathbf{cjr}\big(\pi_\downarrow^\equiv(x)\big)$ among the antichains of join irreducible elements of L uncontracted by $\equiv$ and below a join irreducible element of $\mathbf{cjr}(x)$. Unfortunately, it remains difficult in general to describe $\mathbf{cjr}\big(\pi_\downarrow^\equiv(x)\big)$ because not all such antichains define a canonical join representation of L.

Dual statements hold using meets instead of joins, and we denote by $\mathcal{UMI}(\equiv)$ the meet irreducible elements of L uncontracted by $\equiv$, and by $\mathcal{CMC}(\equiv)$ the subcomplex of $\mathcal{CMC}(L)$ induced by $\mathcal{UMI}(\equiv)$ for a congruence $\equiv$ on L. Due to Proposition 15, we will always work with the subcomplexes $\mathcal{CJC}(\equiv)$ and $\mathcal{CMC}(\equiv)$ rather than with the complexes $\mathcal{CJC}(L/\equiv)$ and $\mathcal{CMC}(L/\equiv)$.

When the lattice L is semidistributive, the two sets $\mathcal{UJI}(\equiv)$ and $\mathcal{UMI}(\equiv)$ and the two subcomplexes $\mathcal{CJC}(\equiv)$ and $\mathcal{CMC}(\equiv)$ are connected by the maps $\kappa_\vee$ and $\kappa_\wedge$. We provide a proof here as we have not found this statement explicitly in the literature.

Proposition 17. *Let $\equiv$ be a congruence on a finite semidistributive lattice L. Then we have $\mathcal{UJI}(\equiv) = \kappa_\vee\big(\mathcal{UMI}(\equiv)\big)$ and $\mathcal{UMI}(\equiv) = \kappa_\wedge\big(\mathcal{UJI}(\equiv)\big)$. Hence, the maps $\kappa_\vee$ and $\kappa_\wedge$ induce inverse isomorphisms between the subcomplexes $\mathcal{CMC}(\equiv)$ and $\mathcal{CJC}(\equiv)$.*

Proof. Let $m \in \mathcal{MI}(L)$ and $j = \kappa_\vee(m)$. By definition, we have $m \vee j = m^\star$ and $m \vee j_\star = m$. Hence, $j \equiv j_\star$ implies that $m = m \vee j_\star \equiv m \vee j = m^\star$. In other words, $\kappa_\vee\big(\mathcal{UMI}(\equiv)\big) \subseteq \mathcal{UJI}(\equiv)$. By symmetry, we have $\kappa_\wedge\big(\mathcal{UJI}(\equiv)\big) \subseteq \mathcal{UMI}(\equiv)$. Since $\kappa_\vee$ and $\kappa_\wedge$ are reversed bijections, this yields equalities. The last sentence of the statement thus follows from Proposition 5 (ii). $\square$

Example 18 (Distributive lattices)**.** In a distributive lattice L, there is no forcing at all. Hence, any subset of join irreducible elements of L defines a congruence of L. In other words, with the notations of Example 6, any subset Y of P defines a congruence $\equiv_Y$ with

$$\mathcal{UJI}(\equiv_Y) = \{j_y \mid y \in Y\} \quad \text{and} \quad \mathcal{UMI}(\equiv_Y) = \{m^y \mid y \in Y\}.$$

The lattice quotient $L/{\equiv}$ is again distributive and isomorphic to the lattice of lower ideals of the restriction of the poset P to Y.

2.2. **The canonical complex.** We now define another complex that connects the canonical join complex $\mathcal{CJC}(L)$ to the canonical meet complex $\mathcal{CMC}(L)$ using intervals of L. This complex is illustrated in Figure 3.

Definition 19. The *canonical complex* $\mathcal{CC}(L)$ of a finite semidistributive lattice L is the simplicial complex whose

- ground set is the **disjoint** union $\mathcal{JI}(L) \sqcup \mathcal{MI}(L)$ of the sets of join irreducible and of meet irreducible elements of L, and
- faces are the **disjoint** unions $J \sqcup M$ where $J \in \mathcal{CJC}(L)$ is a canonical join representation, $M \in \mathcal{CMC}(L)$ is a canonical meet representation, and $\bigvee J \le \bigwedge M$.

To avoid any confusion, let us insist here that when an element $x \in L$ is both join irreducible and meet irreducible, then it appears twice in $\mathcal{CC}(L)$ and may appear twice in some faces of $\mathcal{CC}(L)$, once as a join irreducible element and once as a meet irreducible element. We will thus always write the faces of $\mathcal{CC}(L)$ explicitly as disjoint unions. In our pictures, we denote by $x_\vee$ (resp. $x_\wedge$) and color red (resp. blue) the element $x \in L$ considered as a join (resp. meet) irreducible.

Our first observation is that, while the canonical join and meet complexes $\mathcal{CJC}(L)$ and $\mathcal{CMC}(L)$ encode the individual elements of L, the canonical complex $\mathcal{CC}(L)$ encodes the intervals of L.

Definition 20. The *canonical representation* of an interval $[x, y]$ of a semidistributive lattice L is the **disjoint** union $\mathbf{cjr}(x) \sqcup \mathbf{cmr}(y)$.

Proposition 21. *For a finite semidistributive lattice L, the faces of the canonical complex $\mathcal{CC}(L)$ are precisely the canonical representations of the intervals of L.*

Proof. An interval $[x, y]$ of L corresponds to a face $\mathbf{cjr}(x) \sqcup \mathbf{cmr}(y)$ of $\mathcal{CC}(L)$ since $\mathbf{cjr}(x) \in \mathcal{CJC}(L)$, $\mathbf{cmr}(y) \in \mathcal{CMC}(L)$ and $\bigvee \mathbf{cjr}(x) = x \le y = \bigwedge \mathbf{cmr}(y)$. Conversely, a face $J \sqcup M$ of $\mathcal{CC}(L)$ corresponds to the interval $[\bigvee J, \bigwedge M]$ of L. $\square$

We next observe that $\mathcal{CC}(L)$ contains both $\mathcal{CJC}(L)$ and $\mathcal{CMC}(L)$ as induced subcomplexes.

Proposition 22. *For a finite semidistributive lattice L, the canonical join (resp. meet) complex $\mathcal{CJC}(L)$ (resp. $\mathcal{CMC}(L)$) is the subcomplex of the canonical complex $\mathcal{CC}(L)$ induced by the join (resp. meet) irreducible elements $\mathcal{JI}(L)$ (resp. $\mathcal{MI}(L)$).*

Proof. We have $J \in \mathcal{CJC}(L) \iff J \sqcup \varnothing \in \mathcal{CC}(L)$ and $M \in \mathcal{CMC}(L) \iff \varnothing \sqcup M \in \mathcal{CC}(L)$. $\square$

Our next statement is the analogue of Proposition 5 (i).

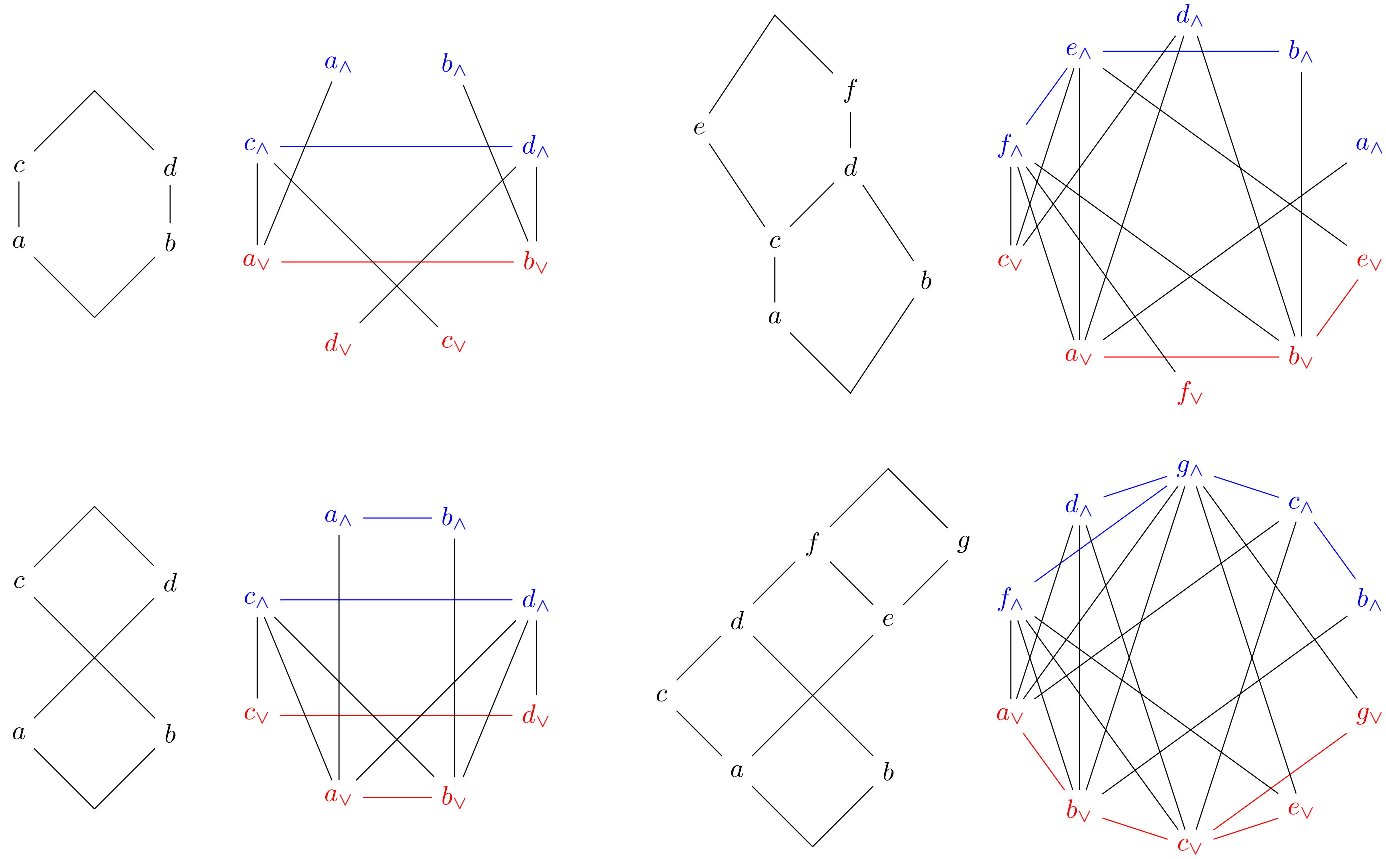

FIGURE 3. The canonical complexes of the semidistributive lattices of Figure 2. The corresponding join (resp. meet) canonical complexes of Figure 2 are highlighted in red (resp. blue). Since the canonical complexes are flag by Propositions 5 and 23, it is sufficient to represent their graphs. The letters label all join or meet irreducible elements, and we denote by $x_\vee$ (resp. $x_\wedge$) the element x when it is considered as a join (resp. meet) irreducible. The vertices of the canonical complex are positioned so that the map κ of Remark 25 acts by central symmetry.

Proposition 23. *For a finite semidistributive lattice L, the canonical complex $\mathcal{CC}(L)$ is a flag simplicial complex.*

Proof. Consider $J \subseteq J'$ and $M \subseteq M'$ such that $J' \sqcup M' \in \mathcal{CC}(L)$. Then $J \in \mathcal{CJC}(L)$ since $J' \in \mathcal{CJC}(L)$ and $M \in \mathcal{CMC}(L)$ since $M' \in \mathcal{CMC}(L)$ (since $\mathcal{CJC}(L)$ and $\mathcal{CMC}(L)$ are simplicial complexes), and $\bigvee J \le \bigvee J' \le \bigwedge M' \le \bigwedge M$. Hence $J \sqcup M \in \mathcal{CC}(L)$ so that $\mathcal{CC}(L)$ is indeed a simplicial complex.

Consider $J \subseteq \mathcal{JI}(L)$ and $M \subseteq \mathcal{MI}(L)$ such that any two elements of $J \sqcup M$ form a face of $\mathcal{CC}(L)$. Then $J \in \mathcal{CJC}(L)$ and $M \in \mathcal{CMC}(L)$ (since $\mathcal{CJC}(L)$ and $\mathcal{CMC}(L)$ are flag by Proposition 5 (i)), and $\bigvee J \le \bigwedge M$ since $j \le m$ for any $j \in J$ and $m \in M$. $\square$

Our next statement says that $\mathcal{CC}(L)$ is not only a simplicial complex, it is actually naturally embedded on the boundary of a cross-polytope.

Proposition 24. *For any $j \in \mathcal{JI}(L)$, the pair $\{j, \kappa_\wedge(j)\}$ is not in $\mathcal{CC}(L)$. Hence, for any labeling $\lambda : \mathcal{JI}(L) \to [|\mathcal{JI}(L)|]$, the map sending j to $\boldsymbol{e}_{\lambda(j)}$ and $\kappa_\wedge(j)$ to $-\boldsymbol{e}_{\lambda(j)}$ defines an embedding of $\mathcal{CC}(L)$ to the boundary of the $|\mathcal{JI}(L)|$-dimensional cross-polytope.*

Proof. By definition, $j \not\le \kappa_\wedge(j)$ so that $\{j, \kappa_\wedge(j)\}$ is not in $\mathcal{CC}(L)$. The second sentence thus follows from Proposition 23. $\square$

Remark 25. Denote by κ the map on $\mathcal{JI}(L) \sqcup \mathcal{MI}(L)$ defined by $\kappa(m) := \kappa_\vee(m)$ for $m \in \mathcal{MI}(L)$ and $\kappa(j) := \kappa_\wedge(j)$ for $j \in \mathcal{JI}(L)$. It corresponds to the central symmetry on the corresponding cross-polytope. By Proposition 5 (ii), the two subcomplexes $\mathcal{CJC}(L)$ and $\mathcal{CMC}(L)$ are symmetric under the action of κ. However, the full canonical complex $\mathcal{CC}(L)$ is not invariant under the action of κ. See Figure 3 for examples.

Example 26. The canonical complex of the boolean lattice on $[n]$ is isomorphic to the boundary of the n-dimensional cross-polytope.

Finally, analogously to Proposition 15, the canonical complex is compatible with lattice congruences of L.

Proposition 27. *For any congruence $\equiv$ of a finite semidistributive lattice L, the canonical complex $\mathcal{CC}(L/{\equiv})$ of the quotient $L/{\equiv}$ is isomorphic to the subcomplex $\mathcal{CC}(\equiv)$ of the canonical complex $\mathcal{CC}(L)$ of L induced by the disjoint union $\mathcal{UJI}(\equiv) \sqcup \mathcal{UMI}(\equiv)$ of the join and meet irreducible elements of L uncontracted by $\equiv$.*

Proof. This immediately follows from Proposition 15 and its dual version. ☐

Example 28 (Distributive lattices)**.** With the notations of Example 6, we have $j_y \subseteq m^z \iff y \not\geq z$. Hence, the canonical complex $\mathcal{CC}(L)$ is the clique complex of the graph whose vertex set is made of two copies $P_\vee$ and $P_\wedge$ of P and whose edge set is the union of two copies $I_\vee$ and $I_\wedge$ of the incomparability graph of P with the edges $\{y_\vee, z_\wedge\}$ for $y \not\geq z$ in P.

3. Semi-crossing arc bidiagrams

In this section, we apply the lattice theoretic results presented in Section 2 to the classical weak order on permutations. We first recall that the canonical join and meet representations of the weak order can be encoded as non-crossing arc diagrams as defined by N. Reading in [Rea15] (Section 3.1). We then briefly study the restriction of the weak order on join or meet irreducibles in terms of arcs (Section 3.2). We then describe the intervals and the canonical complex of the weak order in terms of semi-crossing arc diagrams (Section 3.3). We then provide direct bijections between the semi-crossing arc bidiagrams and the weak order interval posets of [CPP19] (Section 3.4). Finally, we provide an algorithm to compute the Kreweras maps in any lattice quotient of the weak order, generalizing the classical Kreweras complement on non-crossing partitions (Section 3.5).

3.1. Non-crossing arc diagrams.

We consider the following classical order on the set $\mathfrak{S}_n$ of permutations of $[n] := \{1, \dots, n\}$, illustrated in Figure 4.

Definition 29. An *inversion* of a permutation $\sigma \in \mathfrak{S}_n$ is a pair (u, v) with $1 \leq u < v \leq n$ and $\sigma^{-1}(u) > \sigma^{-1}(v)$ (in other words, u is smaller than v but u appears after v in σ). The (right) *weak order* of size n is the order on permutations of $\mathfrak{S}_n$ defined by inclusion of their inversion sets.

Note that a cover relation in the weak order corresponds to the swap of two values σ_i and σ_{i+1} at consecutive positions. The swap is increasing in the weak order if i is an *ascent* *i.e.* $\sigma_i < \sigma_{i+1}$, and decreasing if i is a *descent* *i.e.* $\sigma_i > \sigma_{i+1}$.

It is classical that the weak order is a semidistributive lattice (the lattice property was proved in [GR63, Bjö84], the semidistributivity in [LCdPB94]). We now describe its join (resp. meet) irreducible elements and its canonical join (resp. meet) representations in terms of the arcs and non-crossing arc diagrams introduced by N. Reading in [Rea15].

Definition 30 ([Rea15])**.** An *arc* is a quadruple (a, b, A, B) where $1 \leq a < b \leq n$ and $A \sqcup B = \,]a, b[$ forms a partition of $]a, b[\, := \{a{+}1, \dots, b{-}1\}$. Two arcs $\alpha := (a, b, A, B)$ and $\alpha' := (a', b', A', B')$ *cross* if there exist $u \neq v$ such that $u \in (A' \cup \{a', b'\}) \cap (B \cup \{a, b\})$ and $v \in (A \cup \{a, b\}) \cap (B' \cup \{a', b'\})$. A *non-crossing arc diagram* (or *NCAD* for short) is a collection of pairwise non-crossing arcs. The *non-crossing complex* is the clique complex of the non-crossing relation on all arcs.

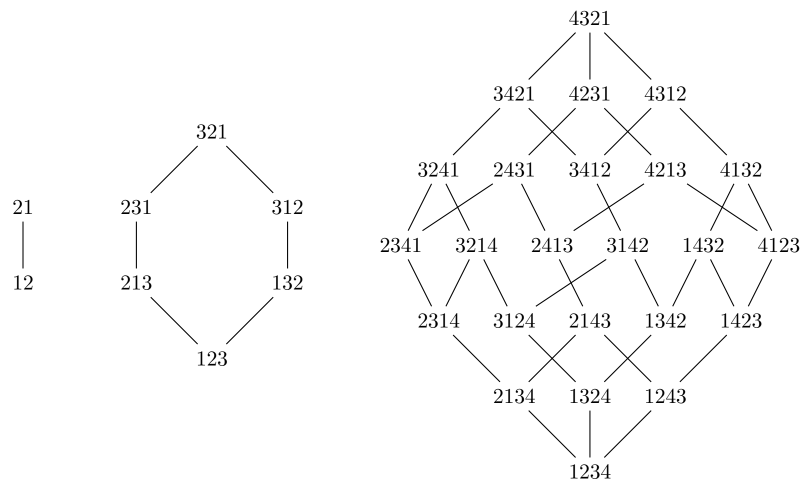

FIGURE 4. Hasse diagrams of the right weak orders of size 2, 3, and 4.

Remark 31. Visually, an arc (a, b, A, B) is represented by an x-monotone curve wiggling around the horizontal axis, starting at a and ending at b, and passing above points of A and below points of B. Two arcs cross if they cross in their interiors or start at the same point or end at the same point (but they do not cross if one ends where the other starts). See Figure 5 for illustrations of arcs and of non-crossing arc diagrams.

Observe that the join (resp. meet) irreducible elements of the weak order are precisely the permutations with exactly one descent (resp. ascent). Hence, we associate to an arc $\alpha := (a, b, A, B)$ with $A := \{a_1 < \cdots < a_k\}$ and $B := \{b_1 < \cdots < b_\ell\}$

- a join irreducible permutation $\boldsymbol{\sigma}_\vee(\alpha) := [1, \dots, (a{-}1), a_1, \dots, a_k, b, a, b_1, \dots, b_\ell, (b{+}1), \dots, n]$,
- a meet irreducible permutation $\boldsymbol{\sigma}_\wedge(\alpha) := [n, \dots, (b{+}1), a_k, \dots, a_1, a, b, b_\ell, \dots, b_1, (a{-}1), \dots, 1]$,

where we use the one-line notation of permutations $\sigma = [\sigma_1, \dots, \sigma_n]$.

Consider now a permutation $\sigma \in \mathfrak{S}_n$ represented by its permutation table formed by dots at coordinates (σ_i, i) for $i \in [n]$. Draw segments between consecutive dots (σ_i, i) and $(\sigma_{i+1}, i + 1)$, colored red for a descent $\sigma_i > \sigma_{i+1}$ and blue for an ascent $\sigma_i < \sigma_{i+1}$. Finally, flatten the picture vertically to the horizontal line, allowing segments to bend but not to pass points. The resulting picture is the superimposition of a set $\boldsymbol{\delta}_\vee(\sigma)$ of red arcs and a set $\boldsymbol{\delta}_\wedge(\sigma)$ of blue arcs. See Figure 5. More formally, $\boldsymbol{\delta}_\vee(\sigma) := \{\boldsymbol{\alpha}_\vee(\sigma, i) \mid \sigma_i > \sigma_{i+1}\}$ and $\boldsymbol{\delta}_\wedge(\sigma) := \{\boldsymbol{\alpha}_\wedge(\sigma, i) \mid \sigma_i < \sigma_{i+1}\}$ where

$$\boldsymbol{\alpha}_\vee(\sigma, i) := (\sigma_{i+1}, \sigma_i, \{\sigma_j \mid j < i \text{ and } \sigma_i > \sigma_j > \sigma_{i+1}\}, \{\sigma_j \mid j > i+1 \text{ and } \sigma_i > \sigma_j > \sigma_{i+1}\}),$$
$$\text{and} \quad \boldsymbol{\alpha}_\wedge(\sigma, i) := (\sigma_i, \sigma_{i+1}, \{\sigma_j \mid j < i \text{ and } \sigma_i < \sigma_j < \sigma_{i+1}\}, \{\sigma_j \mid j > i+1 \text{ and } \sigma_i < \sigma_j < \sigma_{i+1}\}).$$

Proposition 32 ([Rea15]). *The map $\boldsymbol{\delta}_\vee$ (resp. $\boldsymbol{\delta}_\wedge$) is a bijection between the set of permutations of $\mathfrak{S}_n$ and the set of non-crossing arc diagrams of size n. Moreover, the canonical join (resp. meet) representation of a permutation $\sigma \in \mathfrak{S}_n$ is given by $\mathbf{cjr}(\sigma) = \{\boldsymbol{\sigma}_\vee(\alpha_\vee) \mid \alpha_\vee \in \boldsymbol{\delta}_\vee(\sigma)\}$ (resp. $\mathbf{cmr}(\sigma) = \{\boldsymbol{\sigma}_\wedge(\alpha_\wedge) \mid \alpha_\wedge \in \boldsymbol{\delta}_\wedge(\sigma)\}$). Hence, the canonical join (resp. meet) complex of the weak order is isomorphic to the non-crossing complex.*

By construction, the non-crossing arc diagrams are adapted to the maps $\kappa_\vee$ and $\kappa_\wedge$ and to quotients of the weak order. First, our next statement says that the maps $\kappa_\vee$ and $\kappa_\vee$ only change the colors of the arcs. We provide a proof as we have not found it explicitly in [Rea15].

Proposition 33. *$\kappa_\vee(\boldsymbol{\sigma}_\wedge(\alpha)) = \boldsymbol{\sigma}_\vee(\alpha)$ and $\kappa_\wedge(\boldsymbol{\sigma}_\vee(\alpha)) = \boldsymbol{\sigma}_\wedge(\alpha)$ for any arc α.*

Proof. Let $\alpha := (a, b, A, B)$, let $\sigma := \boldsymbol{\sigma}_\wedge(\alpha) = [n, \dots, (b{+}1), a_k, \dots, a_1, a, b, b_\ell, \dots, b_1, (a{-}1), \dots, 1]$ and let $\sigma^\star = [n, \dots, (b + 1), a_k, \dots, a_1, b, a, b_\ell, \dots, b_1, (a - 1), \dots, 1]$ denote the only element

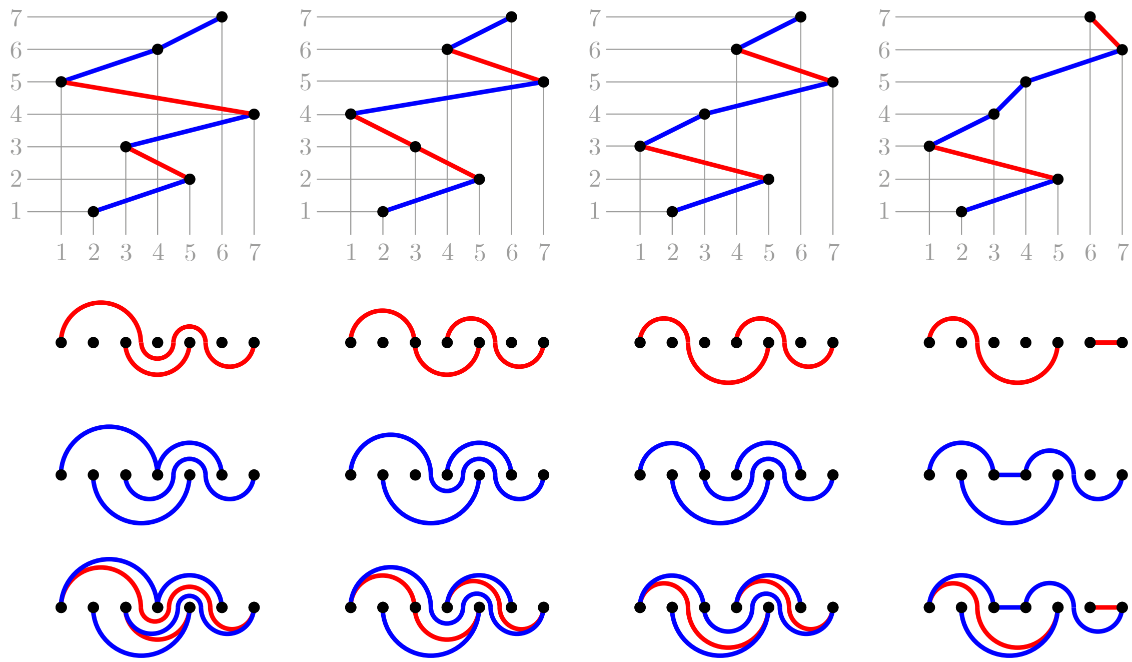

FIGURE 5. NCADs and SCABs of the permutations 2537146, 2531746, 2513746, and 2513476. The first line represents the table $(\sigma(i), i)$ of a permutation σ with ascents in blue and descents in red, the second line is the join diagram $\boldsymbol{\delta}_\vee(\sigma)$, the third line is the meet diagram $\boldsymbol{\delta}_\wedge(\sigma)$, and the fourth line is the superimposition $\boldsymbol{\delta}_\vee(\sigma) \sqcup \boldsymbol{\delta}_\wedge(\sigma)$.

covering σ. Any permutation τ with $\tau \not\leq \sigma$ but $\tau \leq \sigma^\star$ must have all elements of A before b before a before all elements of B. Therefore, the minimal such permutation is clearly $\boldsymbol{\sigma}_\vee(\alpha) = [1, \dots, (a-1), a_1, \dots, a_k, b, a, b_1, \dots, b_\ell, (b+1), \dots, n]$. $\square$

Proposition 34 ([Rea15])**.** *For any arcs $\alpha := (a, b, A, B)$ and $\alpha' := (a', b', A', B')$, the join irreducible $\boldsymbol{\sigma}_\vee(\alpha)$ forces the join irreducible $\boldsymbol{\sigma}_\vee(\alpha')$ if and only if α is a subarc of α', meaning that $a' \leq a < b \leq b'$ and $A \subseteq A'$ while $B \subseteq B'$. Hence, to each upper ideal I of the subarc order corresponds a lattice congruence $\equiv_I$ of the weak order, and the canonical join (resp. meet) complex of the quotient of the weak order by $\equiv_I$ is isomorphic to the non-crossing complex on I.*

Remark 35. Visually, α is a subarc of α' if the endpoints of α are weakly in between the endpoints of α', and α follows α' between its endpoints. The subarc order on arcs of size 3 to 5 is represented in Figure 6.

Example 36. The prototypical congruence of the weak order is the *sylvester congruence* $\equiv_{\mathrm{sylv}}$ [LR98, HNT05], which can be defined equivalently as

- the fiber of the binary search tree insertion (inserting a permutation from right to left),
- the congruence where each class is the set of linear extensions of a binary tree (labeled in inorder and oriented toward its root),
- the transitive closure of the rewriting rule $UacVbW \equiv UcaVbW$ for $1 \leq a < b < c \leq n$,
- the congruence corresponding to the upper ideal of the subarc order given by all up arcs $(a, b,]a, b[, \varnothing)$ (or equivalently, generated by the long up arc $(1, n,]1, n[, \varnothing)$.

Hence, a permutation is minimal (resp. maximal) in its class if and only if it avoids the pattern 312 (resp. 132). The quotient of the weak order by the sylvester congruence is (isomorphic to) the classical *Tamari lattice* [Tam51, HT72], whose elements are the binary trees on n nodes and whose cover relations are rotations in binary trees. The canonical join representations in the Tamari lattice correspond to non-crossing sets of up arcs, also known as *non-crossing partitions*.

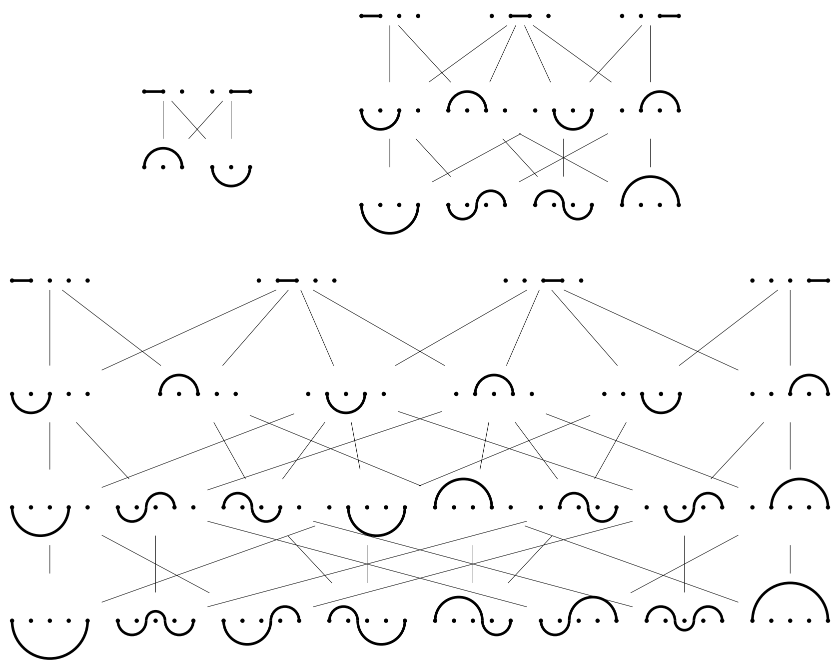

FIGURE 6. The subarc order on arcs of sizes 3 (top left), 4 (top right), and 5 (bottom).

The sylvester congruence was extended in [Rea06] to Cambrian congruences and in [PP18] to permutree congruences.

3.2. **Weak order on arcs.** We now briefly compare join or meet irreducible elements in the weak order in terms of arcs. For this, we first observe that the inversions of $\boldsymbol{\sigma}_\vee(\alpha)$ and $\boldsymbol{\sigma}_\wedge(\alpha)$ are easily read on the arc α.

Lemma 37. *For any arc $\alpha := (a, b, A, B)$ and any $u < v$, the pair (u, v) is an inversion of $\boldsymbol{\sigma}_\vee(\alpha)$ (resp. of $\boldsymbol{\sigma}_\wedge(\alpha)$) if and only if $u \in B \cup \{a\}$ and $v \in A \cup \{b\}$ (resp. if $u \notin A \cup \{a\}$ or $v \notin B \cup \{b\}$).*

Proof. Immediate from the definition $\boldsymbol{\sigma}_\vee(\alpha) := [1, \dots, (a-1), a_1, \dots, a_k, b, a, b_1, \dots, b_\ell, (b+1), \dots, n]$ and $\boldsymbol{\sigma}_\wedge(\alpha) := [n, \dots, (b+1), a_k, \dots, a_1, a, b, b_\ell, \dots, b_1, (a-1), \dots, 1]$. $\square$

Corollary 38. *For any two arcs $\alpha := (a, b, A, B)$ and $\alpha' := (a', b', A', B')$, we have*

(i) $\boldsymbol{\sigma}_\vee(\alpha) \le \boldsymbol{\sigma}_\vee(\alpha')$ if and only if $a \in B' \cup \{a'\}$ and $b \in A' \cup \{b'\}$, and $A \subseteq A'$ and $B \subseteq B'$,
(ii) $\boldsymbol{\sigma}_\wedge(\alpha) \le \boldsymbol{\sigma}_\wedge(\alpha')$ if and only if $a' \in B \cup \{a\}$ and $b' \in A \cup \{b\}$, and $A' \subseteq A$ and $B' \subseteq B$,
(iii) $\boldsymbol{\sigma}_\vee(\alpha) \le \boldsymbol{\sigma}_\wedge(\alpha')$ if and only if there is no $u < v$ such that $u \in (A' \cup \{a'\}) \cap (B \cup \{a\})$ and $v \in (A \cup \{b\}) \cap (B' \cup \{b'\})$.

Remark 39. Figure 7 shows the weak order on arcs defined by $\alpha \le \alpha'$ if $\boldsymbol{\sigma}_\vee(\alpha) \le \boldsymbol{\sigma}_\vee(\alpha')$. Visually, $\alpha \le \alpha'$ if α is a subarc of α' which starts weakly below α' and ends weakly above α'. Note that $\alpha := (a, b, A, B)$ covers at most two arcs, namely $(\min B, b, A \cap]\min b, b[, B \smallsetminus \min B)$ and $(a, \max A, A \smallsetminus \max A, B \cap]a, \max A[)$ when they are defined. Similar remarks hold for the order defined by $\boldsymbol{\sigma}_\wedge(\alpha)$ instead of $\boldsymbol{\sigma}_\vee(\alpha)$.

Remark 40. As illustrated in Figure 7, the weak order on join irreducible of $\mathfrak{S}_n$ has interesting enumerative properties. Let us just mention here that it has

- $2^n - n - 1$ elements (permutations with a single descent, or arcs) [OEI10, A000295],
- $2^{n+1} - n^2 - n - 2$ cover relations (in bijection with arcs of size $n+1$ crossing the horizontal axis, or with subsets of $[n+1]$ crossing their complement) [OEI10, A324172],
- $n(n+1)2^{n-2}$ intervals (including the singletons) [OEI10, A001788].

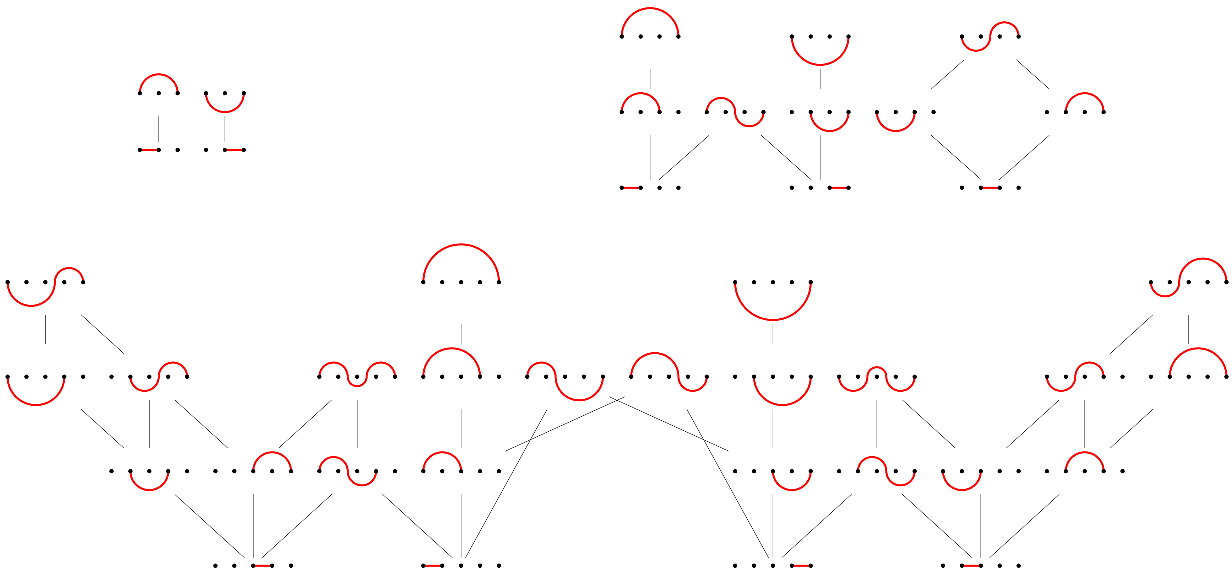

FIGURE 7. The weak orders of size 3 (top left), 4 (top right), and 5 (bottom) restricted to their join irreducibles represented by the corresponding arcs.

3.3. **Semi-crossing arc bidiagrams.** We now describe the canonical complex of the weak order as defined in Section 2.2 in terms of the following combinatorial objects, illustrated in Figure 8.

Definition 41. A *semi-crossing arc bidiagram* (or *SCAB* for short) is a disjoint union $\delta_\vee \sqcup \delta_\wedge$ of non-crossing arc diagrams such that for any $\alpha_\vee := (a_\vee, b_\vee, A_\vee, B_\vee) \in \delta_\vee$ and $\alpha_\wedge := (a_\wedge, b_\wedge, A_\wedge, B_\wedge) \in \delta_\wedge$, there is no $u < v$ with $u \in (A_\wedge \cup \{a_\wedge\}) \cap (B_\vee \cup \{a_\vee\})$ and $v \in (A_\vee \cup \{b_\vee\}) \cap (B_\wedge \cup \{b_\wedge\})$. The *semi-crossing complex* is the simplicial complex whose ground set contains two copies $\alpha_\vee$ and $\alpha_\wedge$ of each arc α and whose simplices are all semi-crossing arc bidiagrams.

Remark 42. Visually, a semi-crossing arc bidiagram $\delta_\vee \sqcup \delta_\wedge$ is a collection of arcs such that

- no two arcs of $\delta_\vee$ (resp. of $\delta_\wedge$) cross in their interiors, or start or end at the same points,
- no two arcs $\alpha_\vee \in \delta_\vee$ and $\alpha_\wedge \in \delta_\wedge$ cross in their interiors with $\alpha_\vee$ going up and $\alpha_\wedge$ going down at the crossing, or start at the same point with $\alpha_\vee$ leaving above $\alpha_\wedge$, or end at the same point with $\alpha_\vee$ arriving below $\alpha_\wedge$ at this point.

Remark 43. Before going further, we report in Table 1 on the number of semi-crossing arc bidiagrams $\delta_\vee \sqcup \delta_\wedge$ according to the cardinalities $|\delta_\vee|$ and $|\delta_\wedge|$ for $n = 2$ to 6.

In these tables, observe that

- the first row (resp. column) corresponds to the intervals $[\sigma, w_\circ]$ (resp. $[e, \sigma]$) for the permutations σ of $[n]$ with k ascents (resp. descents) and are thus counted by the Eulerian numbers [OEI10, A008292],
- the last row (resp. column) corresponds to the single interval $[w_\circ, w_\circ]$ (resp. $[e, e]$).

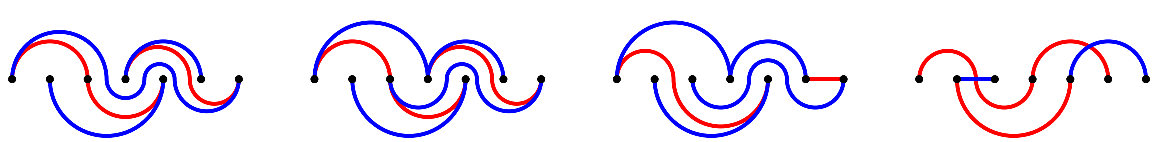

FIGURE 8. SCABs of the intervals [2531746, 2531746], [2531746, 2537146], [2513476, 2537146] and [5264137, 6574231].

	0	1
0	1	1
1	1	0

	0	1	2
0	1	4	1
1	4	6	0
2	1	0	0

	0	1	2	3
0	1	11	11	1
1	11	54	24	0
2	11	24	2	0
3	1	0	0	0

	0	1	2	3	4
0	1	26	66	26	1
1	26	300	420	80	0
2	66	420	320	20	0
3	26	80	20	0	0
4	1	0	0	0	0

	0	1	2	3	4	5
0	1	57	302	302	57	1
1	57	1340	4145	2505	240	0
2	302	4145	8270	3035	120	0
3	302	2505	3035	562	5	0
4	57	240	120	5	0	0
5	1	0	0	0	0	0

Table 1. The number of semi-crossing arc bidiagrams $\delta_\vee \sqcup \delta_\wedge$ according to the cardinalities $|\delta_\vee|$ and $|\delta_\wedge|$ for $n = 2$ to 6.

We now connect the semi-crossing arc bidiagrams with the canonical complex of the weak order using Corollary 38.

Proposition 44. *The map $[\sigma, \tau] \mapsto \boldsymbol{\delta}_\vee(\sigma) \sqcup \boldsymbol{\delta}_\wedge(\tau)$ is a bijection between the intervals of the weak order on $\mathfrak{S}_n$ and the semi-crossing arc bidiagrams. Hence, the canonical complex of the weak order is isomorphic to the semi-crossing complex.*

Proof. By Proposition 32, the maps $\sigma \mapsto \boldsymbol{\delta}_\vee(\sigma)$ and $\tau \mapsto \boldsymbol{\delta}_\wedge(\tau)$ are both bijections from permutations to non-crossing arc diagrams. Moreover, $\sigma \le \tau$ if and only if each canonical joinand of σ is smaller than each canonical meetand of τ, which is equivalent to each arc of $\boldsymbol{\delta}_\vee(\sigma)$ being semi-crossing each arc of $\boldsymbol{\delta}_\wedge(\tau)$ by Corollary 38 (iii). Hence, $[\sigma, \tau] \mapsto \boldsymbol{\delta}_\vee(\sigma) \sqcup \boldsymbol{\delta}_\wedge(\tau)$ is a bijection from intervals to semi-crossing arc bidiagrams. Finally, $\boldsymbol{\delta}_\vee(\sigma) \sqcup \boldsymbol{\delta}_\wedge(\tau)$ corresponds to the canonical representation of $[\sigma, \tau]$ since $\boldsymbol{\delta}_\vee(\sigma)$ corresponds to the canonical join representation of σ and $\boldsymbol{\delta}_\wedge(\tau)$ corresponds to the canonical meet representation of τ. $\square$

For instance, the canonical complexes of the weak orders on $\mathfrak{S}_3$ and $\mathfrak{S}_4$ are illustrated in Figures 1 and 9. As usual, since the canonical complex is flag by Proposition 23, we only represent its graph. The central symmetry corresponds to the map κ of Remark 25, which just corresponds to the exchange of color of the arcs by Proposition 33.

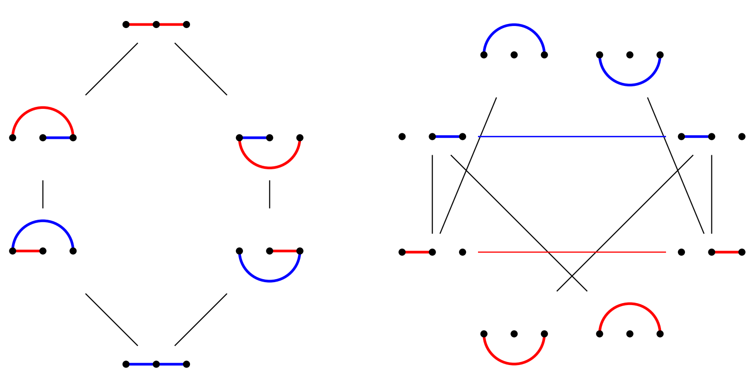

Figure 9. The weak order on $\mathfrak{S}_3$ with permutations labeled by semi-crossing arc bidiagrams, and the canonical complex of $\mathfrak{S}_3$ with join and meet irreducible permutations labeled by arcs.

We now characterize the semi-crossing arc bidiagrams corresponding to singleton intervals. As illustrated in Figure 5, these semi-crossing arc bidiagrams are certain paths. We thus define the *source* $s(\alpha)$ and the *target* $t(\alpha)$ of an arc $\alpha := (a, b, A, B)$ in a semi-crossing arc bidiagram $\delta_\vee \sqcup \delta_\wedge$ as $s(\alpha) = b$ and $t(\alpha) = a$ if $\alpha \in \delta_\vee$ and $s(\alpha) = a$, and $t(\alpha) = b$ if $\alpha \in \delta_\wedge$.

Proposition 45. *The following conditions are equivalent for a semi-crossing arc bidiagram $\delta_\vee \sqcup \delta_\wedge$:*

(i) $\delta_\vee \sqcup \delta_\wedge = \boldsymbol{\delta}_\vee(\sigma) \sqcup \boldsymbol{\delta}_\wedge(\sigma)$ for a permutation $\sigma \in \mathfrak{S}_n$,

(ii) there is an labeling $\alpha_1 = (a_1, b_1, A_1, B_1), \dots, \alpha_{n-1} = (a_{n-1}, b_{n-1}, A_{n-1}, B_{n-1})$ of $\delta_\vee \sqcup \delta_\wedge$ such that $t(\alpha_i) = s(\alpha_{i+1})$ and $a_i \notin B_j$ for $1 \le i < j \le n-1$.

If these conditions hold, then the arcs of $\delta_\vee \sqcup \delta_\wedge$ do not contain crossings in their interiors.

Proof. For (i) $\Rightarrow$ (ii), the arc α_i is the arc $\boldsymbol{\alpha}_\vee(\sigma, i)$ if $\sigma_i > \sigma_{i+1}$ and $\boldsymbol{\alpha}_\wedge(\sigma, i)$ if $\sigma_i < \sigma_{i+1}$ described before Proposition 32. For (ii) $\Rightarrow$ (i), the permutation σ is given by $[s(\alpha_1), t(\alpha_1), \dots, t(\alpha_{n-1})]$. $\square$

Finally, let us insist again here that this combinatorial model for the intervals of the weak order is adapted to the study of its quotients. The next statement follows from Propositions 33 and 34.

Proposition 46. *For any lower ideal I of the subarc order, the canonical complex of the quotient of the weak order by $\equiv_I$ is isomorphic to the subcomplex of the semi-crossing complex induced by $\{\alpha_\vee \mid \alpha \in I\} \sqcup \{\alpha_\wedge \mid \alpha \in I\}$.*

We conclude with a conjecture motivated by Proposition 45 and checked by computer experiments for all lattice quotients of the weak order on $\mathfrak{S}_n$ for $n \le 5$.

Conjecture 47. *The semi-crossing arc bidiagram corresponding to an inclusion minimal interval in a lattice quotient of the weak order does not contain any crossing in the interior of its arcs.*

3.4. **Weak order interval posets.** Following a classical result of A. Björner and M. Wachs [BW91, Thm. 6.8], G. Châtel, V. Pilaud and V. Pons already studied in [CPP19] a family of posets in bijection with the intervals of the weak order. For a poset $\triangleleft$ on $[n]$ and $1 \le u < v \le n$, we say that the relation $u \triangleleft v$ is *increasing* and that the relation $v \triangleleft u$ is *decreasing* (and we write $u \triangleright v$ for decreasing relations).

Proposition 48 ([BW91, Thm. 6.8] & [CPP19, Prop. 26])**.** *The following conditions are equivalent for a poset $\triangleleft$ on $[n]$:*

- *the linear extensions of $\triangleleft$ form an interval $[\sigma, \tau]$ of the weak order,*
- *there are $\sigma \le \tau$ in the weak order such that the increasing relations of $\triangleleft$ are the non-inversions of τ and the decreasing relations are the inversions of σ,*
- *$a \triangleleft c$ implies $a \triangleleft b$ or $b \triangleleft c$, and $a \triangleright c$ implies $a \triangleright b$ or $b \triangleright c$ for all $1 \le a < b < c \le n$.*

Such a poset is called a weak order interval poset (or WOIP for short).

Although they are specific to the weak order and do not behave well with respect to its quotients, the WOIPs are combinatorial objects in bijection with intervals of the weak order, and thus with SCABs. It is thus relevant to provide direct explicit bijections between SCABs and WOIPs, which are illustrated in Figure 10. We need the following definitions, illustrated in Figure 10.

From SCABs to WOIPs. For any arc $\alpha := (a, b, A, B)$, we denote by $\triangleleft_\alpha$ the set of increasing relations $u \triangleleft_\alpha v$ (by $\triangleright_\alpha$ the set of decreasing relations $u \triangleright_\alpha v$) where $u < v$ with $u \in A \cup \{a\}$ and $v \in B \cup \{b\}$. For a NCAD δ, we denote by $\triangleleft_\delta$ (resp. $\triangleright_\delta$) the transitive closure of the union of the increasing relations $\{\triangleleft_{\alpha_\vee} \mid \alpha_\vee \in \delta_\vee\}$ (resp. of the decreasing relations $\{\triangleright_{\alpha_\wedge} \mid \alpha_\wedge \in \delta_\wedge\}$). Finally, to a SCAB $\delta_\vee \sqcup \delta_\wedge$ we associate the WOIP $\triangleright_{\delta_\vee} \sqcup \triangleleft_{\delta_\wedge}$ (see Proposition 49).

From WOIPs to SCABs. Fix a WOIP $\triangleright \sqcup \triangleleft$ where $\triangleright$ denote the decreasing relations and $\triangleleft$ denote the increasing relations. We say that an increasing (resp. decreasing) cover relation $a \lessdot b$ (resp. $a \gtrdot b$) is *maximal* if there is no cover relation $a' \lessdot b$ (resp. $a' \gtrdot b$) with $a' < a$ or $a \lessdot b'$ (resp. $a \gtrdot b'$) with $b < b'$. To a maximal decreasing (resp. increasing) cover relation $a \gtrdot b$ (resp. $a \lessdot b$), we associate the arc $\boldsymbol{\alpha}(a \gtrdot b) := (a, b,]a, b[\cap \langle a \rangle_\downarrow,]a, b[\cap \langle b \rangle^\uparrow)$ (resp. $\boldsymbol{\alpha}(a \lessdot b) := (a, b,]a, b[\cap \langle b \rangle_\downarrow,]a, b[\cap \langle a \rangle^\uparrow)$), where $\langle x \rangle_\downarrow$ and $\langle x \rangle^\uparrow$ denote the lower and upper ideals of $\triangleleft$ generated by x. We denote by $\boldsymbol{\delta}(\triangleright)$

(resp. $\boldsymbol{\delta}(\lhd)$) the set of arcs $\boldsymbol{\alpha}(a \rhd b)$ (resp. $\boldsymbol{\alpha}(a \lhd b)$) for all maximal decreasing (resp. increasing) cover relations of $\lhd$. Finally, to the WOIP $\rhd \sqcup \lhd$, we associate the SCAB $\boldsymbol{\delta}(\rhd) \sqcup \boldsymbol{\delta}(\lhd)$ (see Proposition 49).

Proposition 49. *The maps $\delta_\vee \sqcup \delta_\wedge \mapsto \rhd_{\delta_\vee} \sqcup \lhd_{\delta_\wedge}$ and $\rhd \sqcup \lhd \mapsto \boldsymbol{\delta}(\rhd) \sqcup \boldsymbol{\delta}(\lhd)$ are inverse bijections between SCABs and WOIPs.*

Proof. Consider an interval $[\sigma, \tau]$ of the weak order corresponding to a SCAB $\delta_\vee \sqcup \delta_\wedge$. For any arc $\alpha_\vee$, the decreasing relations of $\rhd_{\alpha_\vee}$ are precisely the inversions of $\boldsymbol{\sigma}_\vee(\alpha_\vee)$ by Lemma 37. Hence, the decreasing relations of $\rhd_{\delta_\vee}$ are precisely the inversions of $\sigma = \bigvee \{\boldsymbol{\sigma}_\vee(\alpha_\vee) \mid \alpha_\vee \in \delta_\vee\}$ (since the inversion set of a join is the transitive closure of the union of the inversion sets of the joinands). Similarly, the increasing relations of $\lhd_{\delta_\wedge}$ are precisely the non-inversions of $\tau = \bigwedge \{\boldsymbol{\sigma}_\wedge(\alpha_\wedge) \mid \alpha_\wedge \in \delta_\wedge\}$. Hence $\rhd_{\delta_\vee} \sqcup \lhd_{\delta_\wedge}$ is indeed the WOIP of the interval $[\sigma, \tau]$ by Proposition 48 (ii). Finally, to see that the two maps are inverse to each other, we just need to observe that the relations $a \lhd_{\delta_\wedge} b$ created out of the extremities of the arcs (a, b, A, B) of $\delta_\wedge$ are precisely the maximal cover relations of $\lhd_{\delta_\wedge}$ (and similarly for $\rhd_{\delta_\vee}$). $\square$

3.5. **Kreweras maps in quotients of the weak order.** We finally describe the Kreweras maps defined in Section 2.1.2 in all quotients of the weak order in terms of semi-crossing arc bidiagrams. For this, we first connect the canonical join representation of a permutation to the canonical join representation of the minimal element in its class for a given congruence, as illustrated in Figure 12. In the following proposition, we call weak order on arcs the order $\alpha \le \alpha'$ if $\boldsymbol{\sigma}_\vee(\alpha) \le \boldsymbol{\sigma}_\vee(\alpha')$ (see Section 3.2 and Figure 7).

Proposition 50. *Consider an upper ideal I of the subarc order and a permutation σ. Let X be the intersection of I with the lower ideal generated by the non-crossing arc diagram $\boldsymbol{\delta}_\vee(\sigma)$ in the weak order on arcs. Let Y be the set of arcs (a, b, A, B) of X such that there is $a < p < b$ such that both arcs $(a, p, A \cap \,]a, p[, B \cap \,]a, p[)$ and $(p, b, A \cap \,]p, b[, B \cap \,]p, b[)$ belong to X. Then the non-crossing arc diagram $\boldsymbol{\delta}_\vee\big(\pi_\downarrow^{\equiv_I}(\sigma)\big)$ is the set of maximal elements of $X \smallsetminus Y$ in the weak order on arcs.*

Proof. By Propositions 14 and 16, $\boldsymbol{\delta}_\vee\big(\pi_\downarrow^{\equiv_I}(\sigma)\big)$ is a non-crossing arc diagram contained in X. Among all options, we need to choose the non-crossing arc diagram with maximal join. This implies that the arcs of Y cannot appear in $\boldsymbol{\delta}_\vee\big(\pi_\downarrow^{\equiv_I}(\sigma)\big)$ since

$$\boldsymbol{\sigma}_\vee(a, b, A, B) \le \boldsymbol{\sigma}_\vee(a, p, A \cap \,]a, p[, B \cap \,]a, p[) \vee \boldsymbol{\sigma}_\vee(p, b, A \cap \,]p, b[, B \cap \,]p, b[)$$

for any arc (a, b, A, B) and any $a < p < b$ by Lemma 37. We finally claim that any two incomparable elements in $X \smallsetminus Y$ cannot cross. Hence, the maximal elements of $X \smallsetminus Y$ form a non-crossing arc diagram, which must therefore be $\boldsymbol{\delta}_\vee\big(\pi_\downarrow^{\equiv_I}(\sigma)\big)$. To prove the claim, consider two arcs $\alpha := (a, b, A, B)$ and $\alpha' := (a', b', A', B')$ in X which are incomparable and crossing. Assume for instance that there are $u < v$ such that $u \in (A \cup \{a\}) \cap (B' \cup \{a'\})$ and $v \in (A' \cup \{b'\}) \cap (B \cup \{b\})$. We can moreover assume that α and α' agree on $]u, v[$, meaning that $A \cap \,]u, v[\; = A' \cap \,]u, v[$ and $B \cap \,]u, v[\; = B' \cap \,]u, v[$. Since α and α' are incomparable, this implies that $u \neq a$ or $v \neq b$.

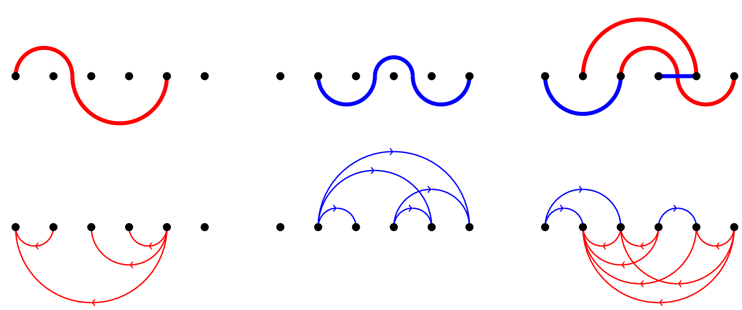

Figure 10. The bijection between SCABs (top) and WOIPs (bottom). We represent a WOIP $\lhd$ with arcs joining $a < b$ from above (resp. below) in blue (resp. red) when $a \lhd b$ (resp. $a \rhd b$).

Moreover, if $u \neq a$ and $v \neq b$, then there is a crossing between any arc larger than α and any arc larger than α' in the weak order on arcs, so that α and α' cannot both belong to X. We conclude that either $u \neq a$ or $v \neq b$, so that precisely one of the arcs

$$(a, u, A \cap]a, u[, B \cap]a, u[) \quad \text{and} \quad (v, b, A \cap]v, b[, B \cap]v, b[)$$

is non-trivial (not reduced to a single point). Moreover, this arc belongs to X since α does, and the arc $(u, v, A \cap]u, v[, B \cap]u, v[) = (u, v, A' \cap]u, v[, B' \cap]u, v[)$ belongs to X since α' does. This implies that α is in Y as it can be decomposed into exactly two subarcs that belong to X. $\square$

This enables us to compute the Kreweras maps in quotients of the weak order directly on non-crossing arc diagrams. For this, let us extend the notations of Section 2.1.2 to quotients and transport them to non-crossing arc diagrams. For an upper ideal I of the subarc order, each equivalence class of $\equiv_I$ is an interval $[x, y]$ of the weak order and thus corresponds to two non-crossing arc diagrams $\delta_\vee := \boldsymbol{\delta}_\vee(x)$ and $\delta_\wedge := \boldsymbol{\delta}_\wedge(y)$. We denote by $\eta^I_\vee$ and $\eta^I_\wedge$ the two opposite maps defined by $\eta^I_\vee(\delta_\wedge) = \delta_\vee$ and $\eta^I_\wedge(\delta_\vee) = \delta_\wedge$. We just write $\eta_\vee$ and $\eta_\wedge$ when I is the set of all arcs. Note that $\eta_\vee = \boldsymbol{\delta}_\vee \circ \boldsymbol{\delta}_\wedge^{-1}$ and $\eta_\wedge = \boldsymbol{\delta}_\wedge \circ \boldsymbol{\delta}_\vee^{-1}$ are easily computed from the descriptions of the maps $\boldsymbol{\delta}_\vee$ and $\boldsymbol{\delta}_\wedge$ (see Section 3.1) and of their inverses (see the explicit description in [Rea15]). Proposition 50 enables to compute $\eta^I_\vee$ and $\eta^I_\wedge$ in general.

Corollary 51. *Consider an upper ideal I of the subarc order and a non-crossing arc diagram $\delta_\wedge$ with all arcs in I. Then the non-crossing arc diagram $\eta^I_\vee(\delta_\wedge)$ is obtained from $\eta_\vee(\delta_\wedge)$ by applying the algorithm of Proposition 50.*

Example 52. When I is the upper ideal of up arcs corresponding to the sylvester congruence, the description of Corollary 51 can be translated to the classical description of the Kreweras complement of a non-crossing partition. Namely, the Kreweras complement of a non-crossing partition is obtained by shifting the points and connecting the points in the same connected component. See Figure 11.

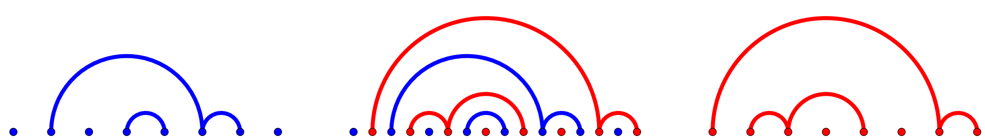

FIGURE 11. Classical Kreweras complement on non-crossing partitions.

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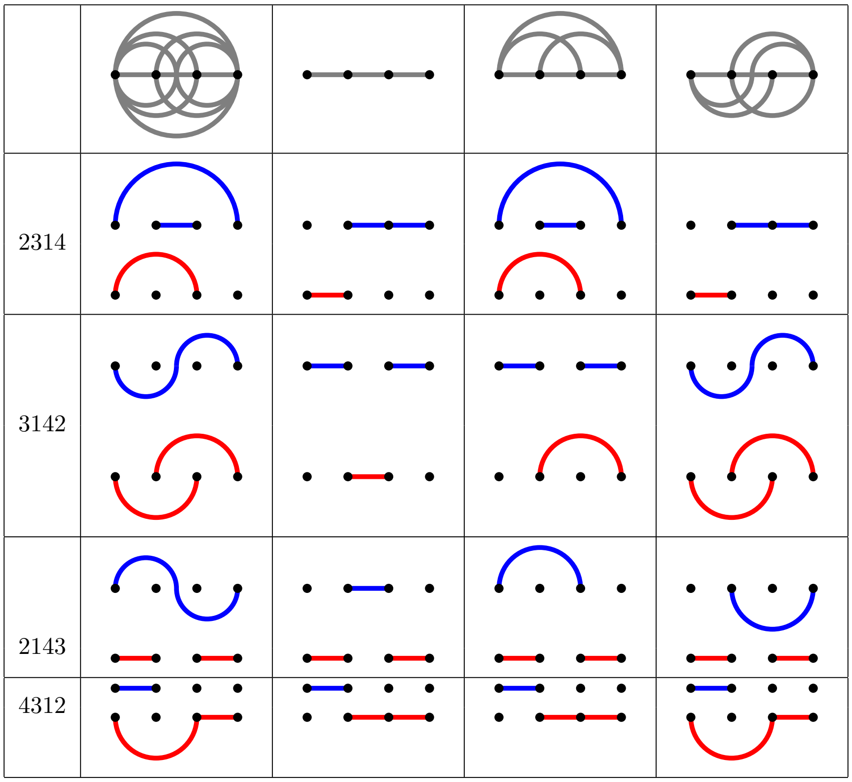

FIGURE 12. Illustrations of computations of $\boldsymbol{\delta}_{\vee}\big(\pi_{\downarrow}^{\equiv_{I}}(\sigma)\big)$ and $\eta_{\vee}^{I}(\delta_{\wedge})$ in the trivial congruence, the descent congruence, the sylvester congruence, and a generic congruence.

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$$\frac{2}{(3n+1)(3n+2)}\binom{4n+1}{n+1} = \frac{2\,(4n+1)!}{(3n+1)(3n+2)\,(n+1)!\,(3n)!}$$

$$= \frac{2\,(4n+1)!}{(3n+2)!\,(n-1)!} \times \frac{1}{n(n+1)}$$

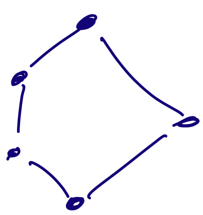
$$= \frac{2}{n(n+1)}\binom{4n+1}{n-1} \quad ✓$$

- Que donne comptage des intervalles synchrones pour les arbres de parking ?

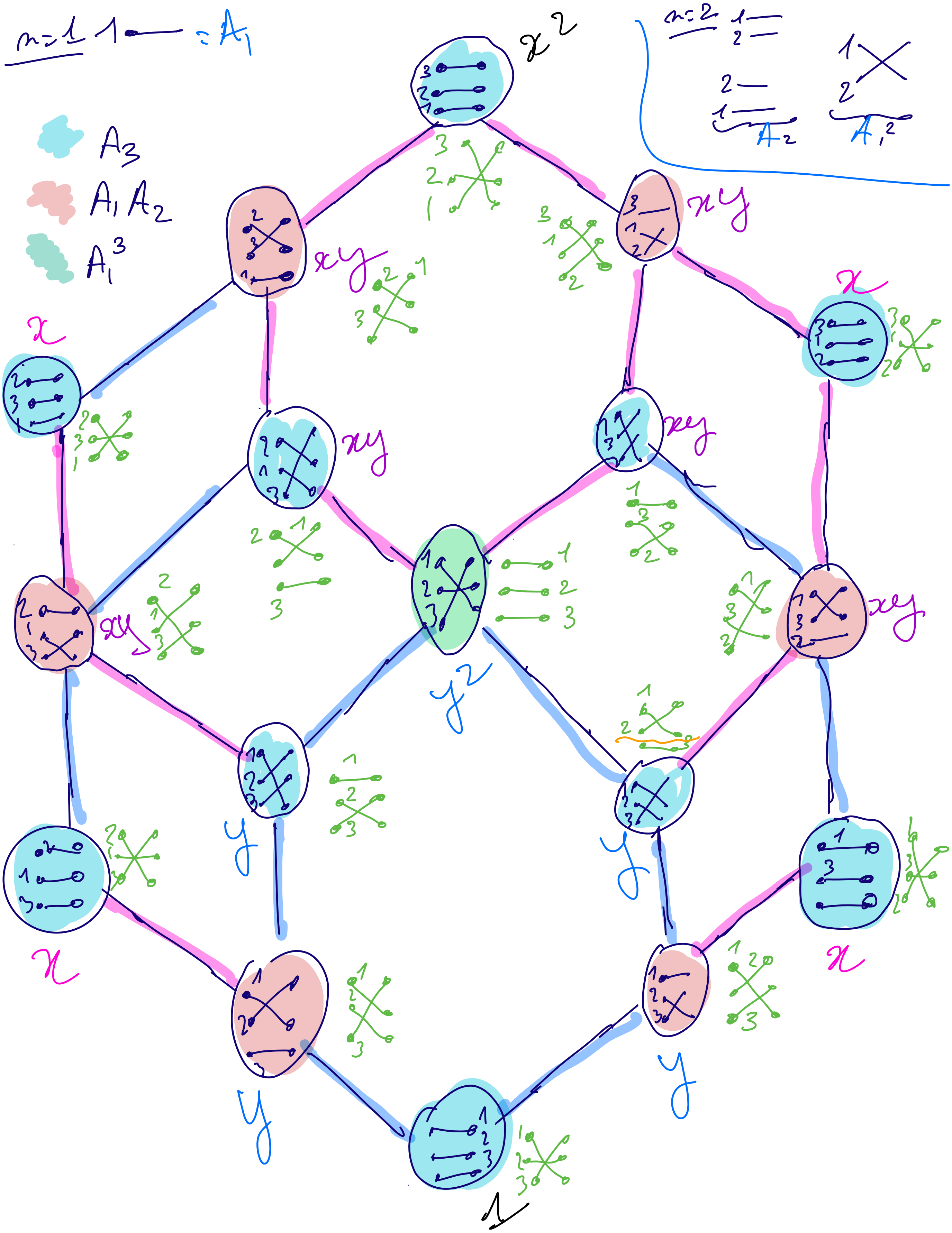
Retour f-vecteur

f-vecteur associaèdre

$n=2$ 13,18,6



$$f_k = \binom{0}{k} + 6\binom{1}{k} + 6\binom{2}{k}$$


n=1 1 = A1
n=2
A2
A1 2
A3
A1 A2
A1 3
x 2
xy
x
y 2
y
1

$n=4$ $\quad 1, 12, \underset{12+40}{52}, 84$

celui au milieu $\left(A_1^4 \rightarrow 1 \right.$ ① ② ③ ④ (min, max)

$A_4 \rightarrow (n-1)! \times C_n = 84$

$A_1 A_3 \rightarrow$ ① $\quad 4\times 10 = 40$

$A_2^2 \rightarrow = 12$

$3\times 2\times 2$

$A_1^2 A_2 \rightarrow 12$

sommets du permutoèdre : $\boxed{24}$

où $A_n = (n-1)! C_n$

14 facettes

8 hexagones et 6 carrés à l'ext.

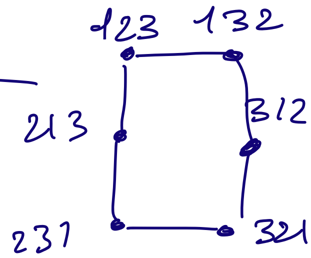
Saneblidze – Umble → Subdivision des cubes qui assure la partie en traitement récursif

avec left shift et right shift

n=1 1 •

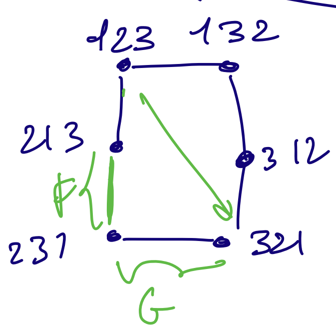
n=2 12 — 21

n=3



(descript° récursive du permutoèdre)

(F, G) tq $\max(F) \leq \min(G)$ ← faces de la subdivision cubique



12|3 × 23|1

2|13 × 23|1

12|3 × 2|13

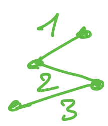
13|2 × 3|12

1|23 × 13|2

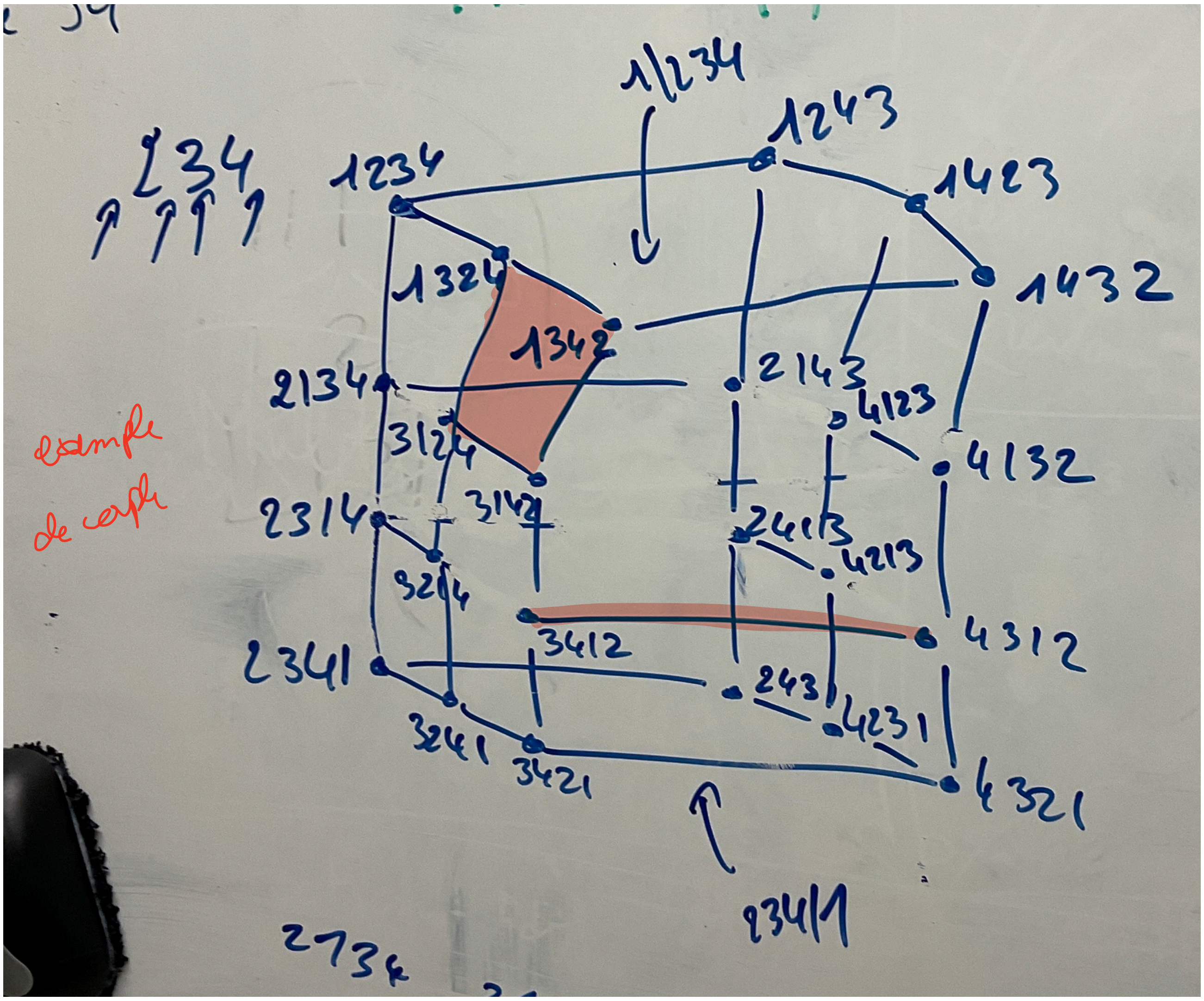
1|23 × 3|12

1|2|3 × 123

123 × 3|2|1

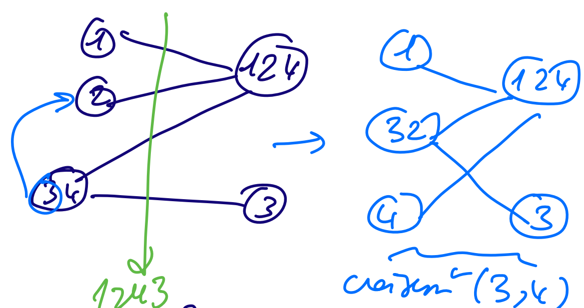


8 éléments



déf 2 " Strong complementary pairs " (correspond aux arbres sans croisements)

Remarque $1243 \xrightarrow{(\sigma,\tau)} 1|2|34 \times 124|3$



$(L(\sigma),\tau)$ encore ds diag

$(\sigma, R(\tau))$ aussi ds la diag $\rightarrow$ $n=2$: 6 paires complémentaires

↳ Poset sur facettes de la diag par inclusion de croisements

→ Voir ce poset des croisements

n=2

$12|3 \times 23|1$

$2|13 \times 23|1$

$12|3 \times 2|13$

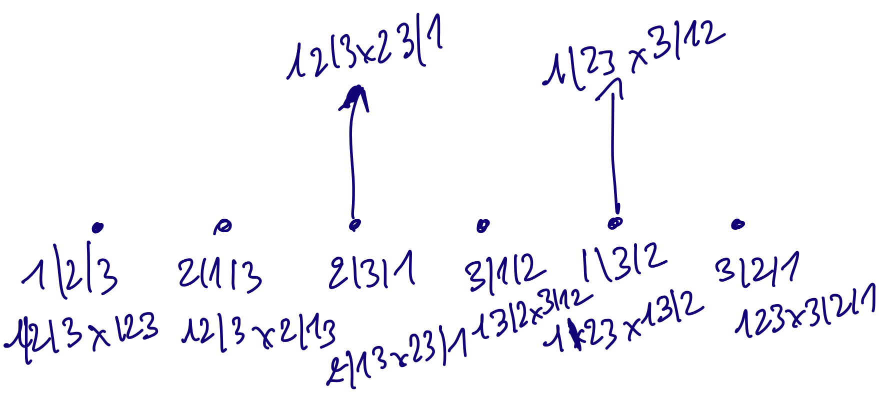
$13|2 \times 3|12$

$1|23 \times 13|2$

$1|23 \times 3|12$

$1|2|3 \times 123$

$123 \times 3|2|1$



Voir Dessin de Kurt → sur le document

→ Explicat° de Saneblidze ?

↳ Applicat° aux faces ?

Obj 1 : i) Énumérer les faces (sommets puis général)

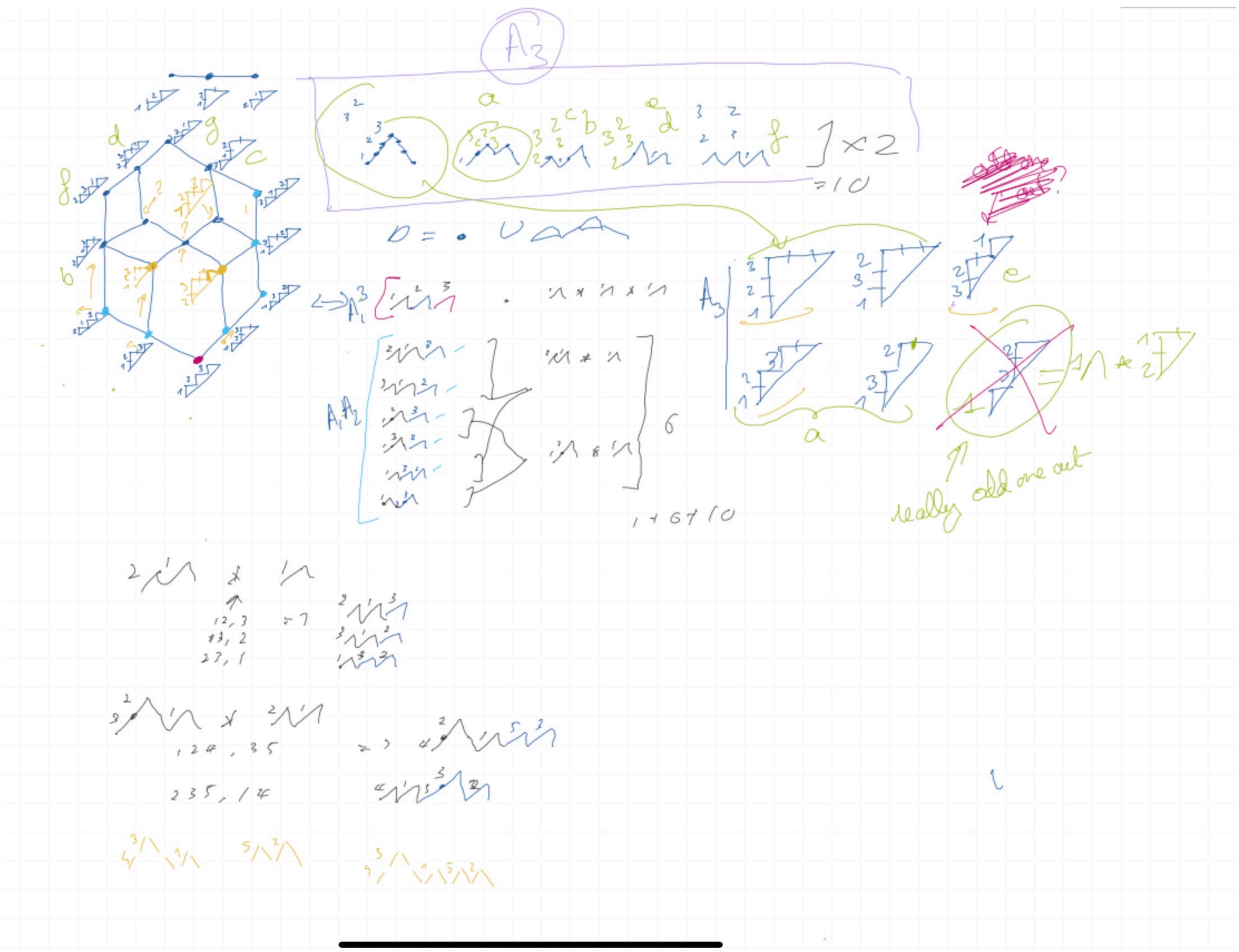
2) unicité ?

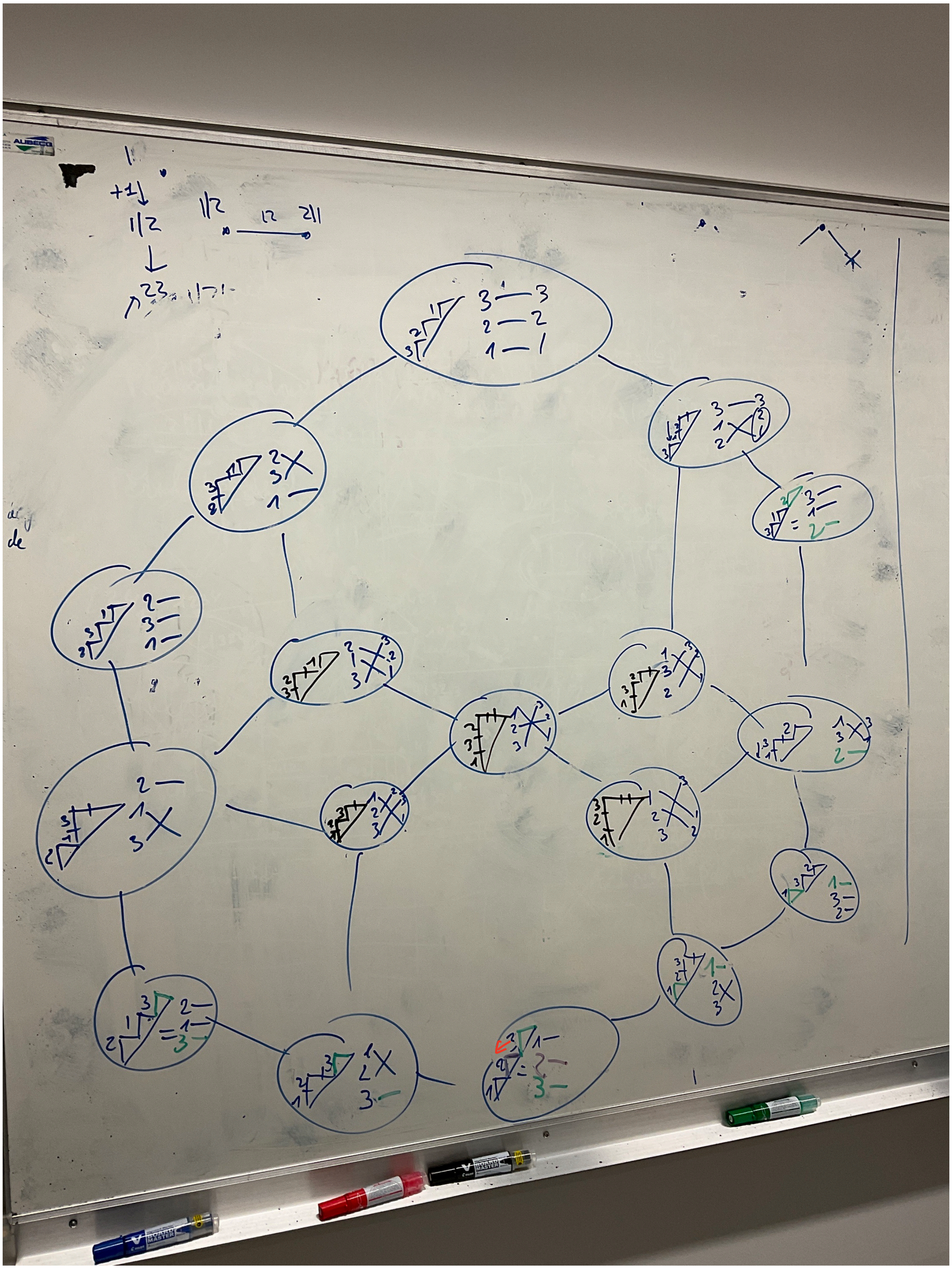
10h → RDV Ruth

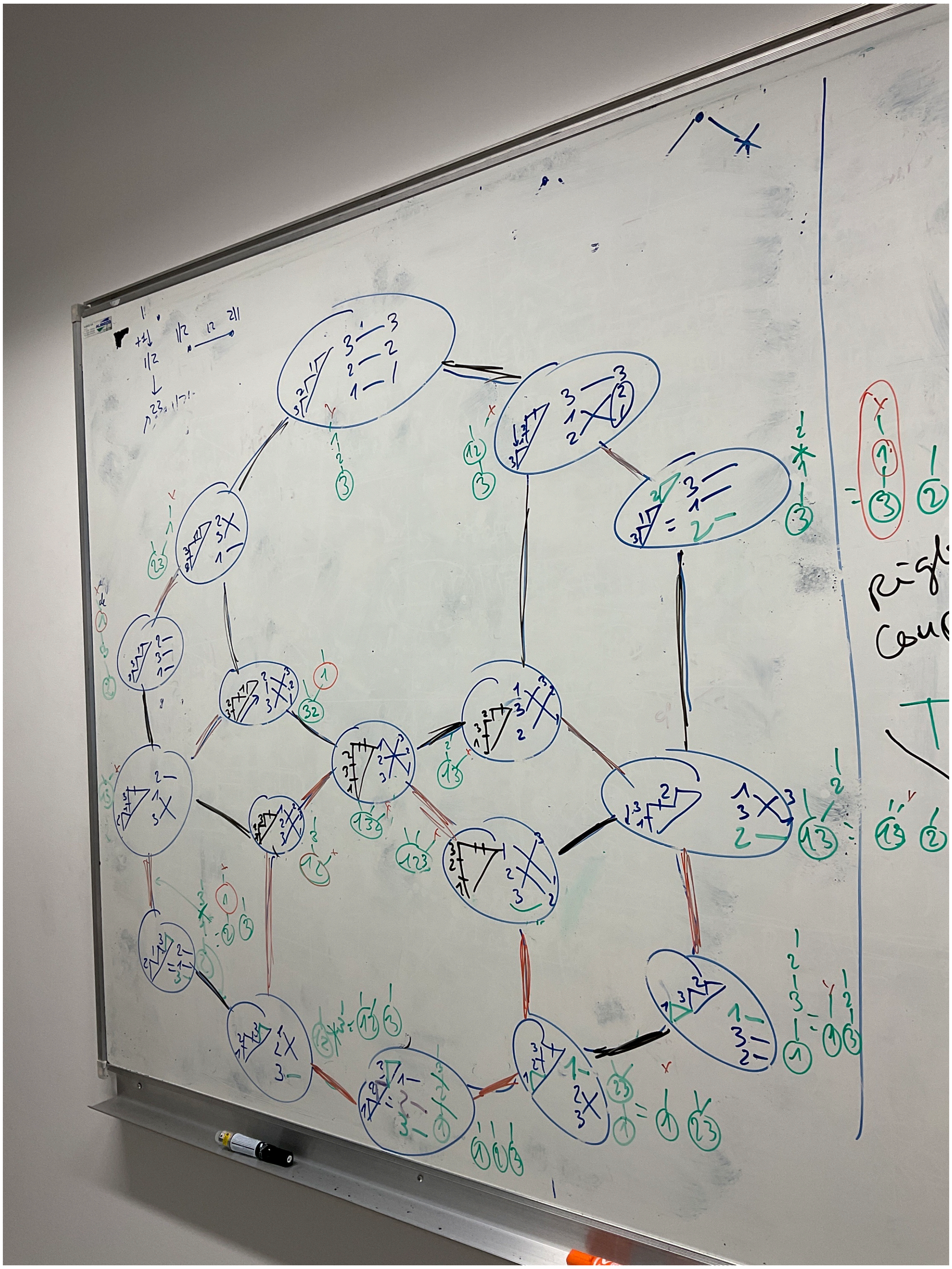
| 3) projections?

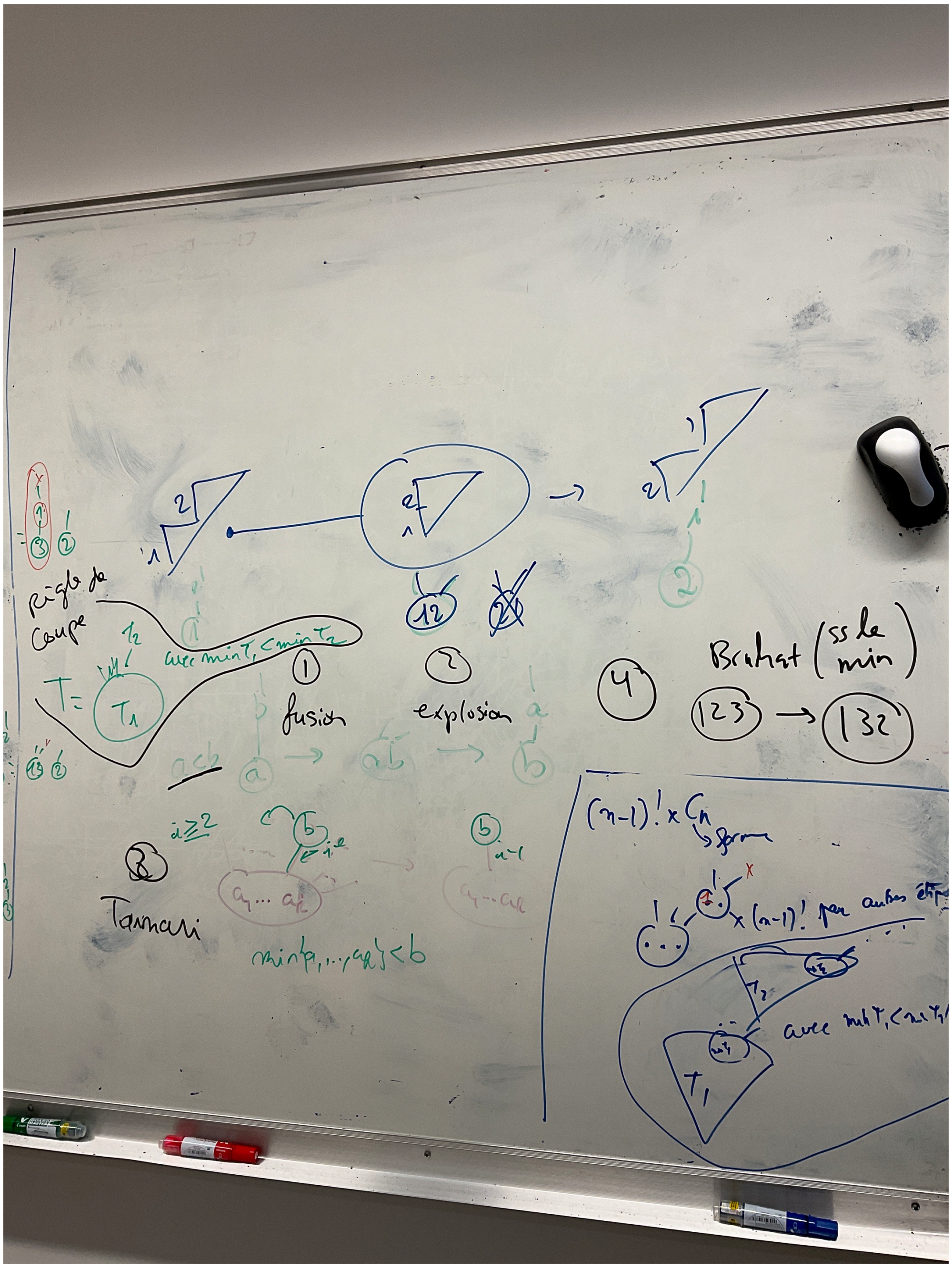
Reprise sur l'idée de Matthieu → formule récursive?

bloc =



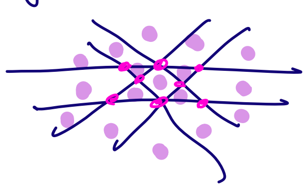


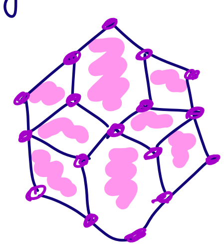




Rque Sylvie : Lien avec article Stanley-Postnikov

~> Polynôme de Poincaré → f-vecteur

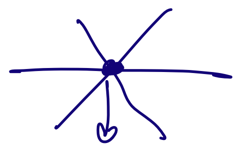




On perturbe l'arrangement de Tresse dans la direction

v tq $\forall (I, J) \in \mathcal{D}_n$)

$$\sum_{i \in I} v_i > \sum_{j \in J} v_j$$



$x_i - x_j = 0$

$x_i - x_j = v_i - v_j$ avec $\vec{v}$ vecteur par lequel on perturbe

$\longrightarrow$ dit dans papier de Bernardi mais peut-on l'adapter ?

arXiv:1604.06554v4 [math.CO] 12 Jun 2021

DEFORMATIONS OF THE BRAID ARRANGEMENT AND TREES

DEDICATED TO IRA GESSEL FOR HIS RETIREMENT

OLIVIER BERNARDI

Abstract. We establish general counting formulas and bijections for deformations of the braid arrangement. Precisely, we consider real hyperplane arrangements such that all the hyperplanes are of the form $x_i - x_j = s$ for some integer s. Classical examples include the braid, Catalan, Shi, semiorder and Linial arrangements, as well as graphical arrangements. We express the number of regions of any such arrangement as a signed count of decorated plane trees. The characteristic and coboundary polynomials of these arrangements also have simple expressions in terms of these trees.

We then focus on certain "well-behaved" deformations of the braid arrangement that we call *transitive*. This includes the Catalan, Shi, semiorder and Linial arrangements, as well as many other arrangements appearing in the literature. For any transitive deformation of the braid arrangement we establish a simple bijection between regions of the arrangement and a set of labeled plane trees defined by local conditions. This answers a question of Gessel.

1. Introduction

In this article we establish enumerative and bijective results about classical families of hyperplane arrangements. Specifically, we consider real hyperplane arrangements made of a finite number of hyperplanes of the form

$$H_{i,j,s} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = s\},$$

with $i, j \in \{1, \ldots, n\}$ and $s \in \mathbb{Z}$. We shall call them *deformations of the braid arrangement*. In particular, given an integer n and a finite set of integers S, the *S-braid arrangement in dimension n*, denoted $\mathcal{A}_S(n)$, is the arrangement made of the hyperplanes $H_{i,j,s}$ for all $1 \le i < j \le n$ and all $s \in S$. Classical examples include the *braid*, *Catalan*, *Shi*, *semiorder*, and *Linial* arrangements, which correspond to $S = \{0\}$, $\{-1,0,1\}$, $\{0,1\}$, $\{-1,1\}$, and $\{1\}$ respectively. These arrangements are represented[1] in Figure 1. We refer the reader to [34] or [41] for an introduction to the general theory of hyperplane arrangements.

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This work was partially supported by NSF grant DMS-1400859. Part of it was completed while visiting the MIT in the Spring 2016. Many thanks to Ira Gessel for suggesting this problem and providing many valuable inputs, and to Sam Hopkins and Alexander Postnikov for interesting discussions. We are also grateful to an anonymous referee for suggesting additional references and making many other valuable comments.

[1]Here and later, the arrangements are represented by drawing their intersection with the hyperplane $H_0 = \{(x_1, \ldots, x_n) \mid x_1 + \ldots + x_n = 0\}$, which is orthogonal to all the hyperplanes of any deformation of the braid arrangement.

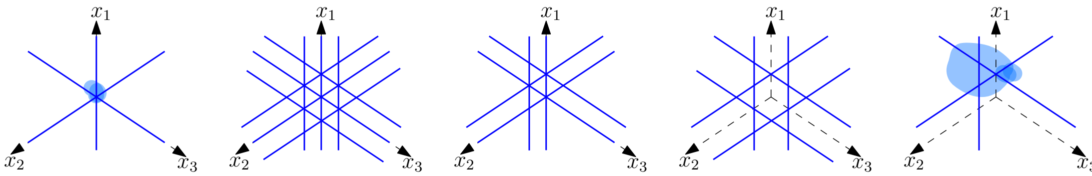


FIGURE 1. The braid, Catalan, Shi, semiorder, and Linial arrangements in dimension $n = 3$ (seen from the direction $(1, 1, 1)$).

There is an extensive literature on counting regions of deformations of the braid arrangement, starting with the work of Shi [38, 39]. Important seminal results on this enumerative question were established by Stanley [43, 40], Postnikov and Stanley [37], and Athanasiadis [6, 5]. Since then, the subject has become quite popular among combinatorialists, and many beautiful counting formulas and bijections were discovered for various families of arrangements; see in particular [7, 10, 7, 11, 1, 3, 15, 23, 19, 25, 27, 26, 31] (see also Section 2 for additional references).

About a decade ago, an interesting pattern was observed by Ira Gessel. In an unpublished manuscript, Gessel obtained an equation for the generating function of labeled binary trees counted according to ascents and descents along left or right edges[2]. By specializing his equation, Gessel observed (based on known results for arrangements) that each of the five classical arrangements families $\mathcal{A}_S$ defined above (braid, Catalan, Shi, semi-order, Linial) can be associated to a simple family $\mathcal{B}_S$ of binary trees (characterized by some ascent and descent conditions), in such a way that the regions of $\mathcal{A}_S(n)$ are equinumerous to trees with n nodes in $\mathcal{B}_S$ (see Section 2.3). This opened the question of explaining these mysterious enumerative identities between regions of arrangements and binary trees; and attempts at answering this question were made for instance in [15, 19].

It is the goal of this paper to explain that Gessel's observation is not a mere coincidence, but rather the manifestation of a more general theory which unifies and extends many previous results. This paper has two parts. The first part gives enumerative results which apply to every deformation of the braid arrangement. The second part establishes bijections for every deformations of the braid arrangement satisfying a certain *transitivity condition*. In the rest of this introduction, we give a detailed outline of the paper and a preview of our results.

In the first part of the paper (Section 2 to 7), we deal with arbitrary deformations of the braid arrangement. For any such arrangement in dimension n, we express the number of regions as a *signed* count of some (decorated, labeled, k-ary) trees with n nodes, that we call *boxed trees*. These results are given in Section 3 for the case of S-braid arrangements (Theorem 3.4), and in Section 4 for the general case (Theorem 4.2). When the arrangement satisfy the *transitivity condition*, our counting result simplifies greatly and the regions are shown to be equinumerous to a family of (labeled, plane) trees satisfying certain ascent and descent conditions (Theorems 3.8 and 4.6).

[2]Gessel's result was then rederived in two different ways by Kalikow [28] and Drake [17].

In Section 5, we generalize the previous counting results by expressing the characteristic polynomial and coboundary polynomial (equivalently, Tutte polynomial) of any deformation of the braid arrangement in terms of boxed trees (Theorem 5.2). In Section 6, we use the preceding expressions in terms of boxed trees in order to establish equations for the generating function of the number of regions, and more generally for the generating function of coboundary polynomials. We thereby recover and extend many known results.

All the proofs for our counting results are gathered in Section 7. The proof is inspired by statistical mechanics considerations (although some of the arguments can alternately be interpreted in light of the finite field method), and has three steps which could informally be described as follows:

- at the first step (Lemma 7.1) we express the coboundary polynomials of the arrangements as a signed count of some decorated graphs (directly encoding the central subarrangements),
- at the second step (Lemma 7.3) we express the generating function of these decorated graphs in terms of a 1-dimensional gas model (by using a version of Mayers' theory of cluster integrals),
- at the third step (Lemma 7.5) we rearrange the information about the gas model configurations in order to encode them in terms of boxed trees.

In the second part of the paper (Section 8), we establish bijections for regions of *transitive* deformations of the braid arrangement. These are arrangements made up of hyperplanes $H_{i,j,s}$, where the triples (i,j,s) satisfy certain conditions (see Definition 4.3). Examples of transitive arrangements include the S-braid arrangements for $S \subseteq \{-1,0,1\}$, or S an interval containing 1, and the G-Shi arrangement for any graph G. As mentioned earlier, for transitive arrangements our enumerative result simplifies and the regions are found to be equinumerous to some simple families of (labeled, plane) trees. In Section 8 we establish, for every transitive arrangement, a direct bijection between the regions of the arrangement and the corresponding family of trees (Theorem 8.8). Our bijection is surprising explicit: given a tree it is very simple to determine all the linear inequalities that define the corresponding region of the arrangement. In order to illustrate this fact, we now present the bijection in the case of the Linial arrangement, which is illustrated in Figure 2.

Example 1.1. The regions of the Linial arrangement $\mathcal{A}_{\{1\}}(n)$ are in bijection with the set $\mathcal{T}_{\{1\}}(n)$ of binary trees with n labeled node satisfying the following condition: *for all node $u \in [n]$ having at least one child which is a node, the rightmost such child v is such that $v < u$.*

The bijection Ψ associates to any tree T in $\mathcal{T}_{\{1\}}(n)$, the region $\rho(T)$ of $\mathcal{A}_{\{1\}}(n)$ made of the points $(x_1,\ldots,x_n)$ satisfying the following inequalities for all $1 \le i < j \le n$: $x_i - x_j < 1$ *if and only if either* $\mathrm{drift}(i) \le \mathrm{drift}(j)$ *or* $\mathrm{drift}(i) = \mathrm{drift}(j)+1$ *and i appears before j in the postfix order of T*, where $\mathrm{drift}(v)$ is the number of ancestors of v (including v) which are right-children. See Figure 2 for the case $n=3$.

In the case of the Catalan arrangement $\mathcal{A}_{\{-1,0,1\}}(n)$ (and the generalization $\mathcal{A}_{[-m..m]}(n)$) our bijection builds on a classical construction. In the case of the Shi arrangement $\mathcal{A}_{\{0,1\}}(n)$ (and the generalization $\mathcal{A}_{[-m+1..m]}(n)$) our bijection is a close relative to a bijection of Athanasiadis and Linusson [11]; see Section 9.1. But already in the case

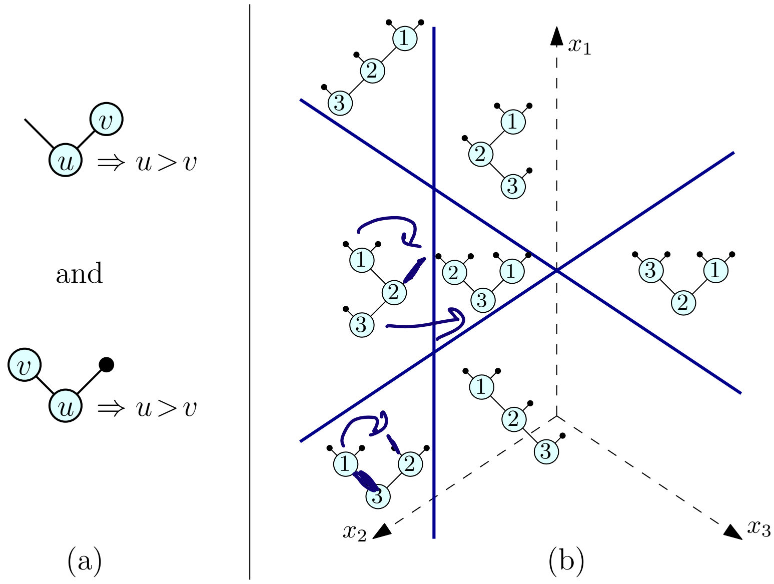


FIGURE 2. (a) The condition defining the trees in $\mathcal{T}_{\{1\}}(n)$. (b) The bijection Ψ between the regions of the Linial arrangement $\mathcal{A}_{\{1\}}(3)$ and the trees in $\mathcal{T}_{\{1\}}(3)$.

of the Linial arrangement $\mathcal{A}_{\{1\}}(n)$, no direct bijection was known between the regions of $\mathcal{A}_{\{1\}}(n)$ and other combinatorial objects[3] (although several families of trees where known to be equinumerous to the regions of $\mathcal{A}_{\{1\}}(n)$ [37, 36, 35]). We give short direct proofs of our bijective results in Section 8.2 in the case of the Catalan, Shi, semiorder and Linial arrangements. However, in the general case, we only prove surjectivity of our mapping and conclude to bijectivity by invoking the counting results established in the first part of the paper.

In Section 9, we explain the relation between the trees described in Example 1.1 and the *local binary search trees* which were known to be equinumerous to the regions of $\mathcal{A}_{\{1\}}(n)$, and we conclude with some remarks and open questions.

2. Definitions and known results

In this section we set our notation about arrangements and trees, and we recall some known counting results for the regions of the deformations of the braid arrangement.

[3]The bijections in [15, 19] are not defined on the regions themselves, but rather on some combinatorial objects, called *gain graphs without broken circuit*, which are known to be equinumerous to the regions by a non-bijective argument (Zaslavsky formula [46] allows to express the number of regions as a signed count of gain graphs, and after a suitable sign-reversing involution, one is left with the gain graphs without broken circuit).

2.1. **Basic definitions.**

A *real hyperplane arrangement* in dimension n is a set $\mathcal{A}$ of affine hyperplanes of $\mathbb{R}^n$. For instance, the braid arrangement $\mathcal{A}_{\{0\}}(n)$ is the set $\{H_{i,j,0}\}_{1\le i<j\le n}$ of $\binom{n}{2}$ hyperplanes. The *regions of* $\mathcal{A}$ are connected components of $\mathbb{R}^n \setminus \bigcup_{H\in\mathcal{A}} H$. We denote by $r_{\mathcal{A}}$ the number of regions. For instance, it is easy to see that $r_{\mathcal{A}_{\{0\}}(n)} = n!$.

We denote $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $a, b \in \mathbb{Z}$, we denote $[a..b] = \{i \in \mathbb{Z} \mid a \le i \le b\}$, and $[b] = [1..b]$. For a set S, we denote by $|S|$ the cardinality. For a ring R, we denote by $R[t]$ and $R[[t]]$ respectively the set of polynomials and formal power series in t with coefficients in R. We extend the notation to several variables so that $R[y][[t_1, t_2]]$ is the set of formal power series in t_1, t_2 with coefficients in $R[y]$. For $G(t) \in R[[t]]$, we denote by $[t^k]G(t)$ the coefficient of t^k in G.

2.2. **Labeled plane trees.**

A *tree* is a finite connected acyclic graph. A *rooted plane tree* is a tree with a vertex distinguished as the *root*, together with an ordering of the children of each vertex. A vertex in a rooted plane tree is a *leaf* if it has no children, and a *node* otherwise. We think of the children of a node u of a rooted plane tree as being "ordered from left to right", and we adopt this convention in all our figures. For a child v of u we call *left siblings of* v the children of u (including leaves) which are on the left of v, that is, smaller than v in the ordering of the children of u.

We denote by $\mathcal{T}$ the set of rooted plane trees with labeled nodes (if the tree has n nodes, then the nodes have distinct labels in $[n]$, while the leaves are not labeled). We denote by $\mathcal{T}^{(m)}$ the set of $(m+1)$*-ary* trees in $\mathcal{T}$ (i.e. the trees such that every node has $m+1$ children). We also denote by $\mathcal{T}^{(m)}(n)$ the set of trees with n nodes in $\mathcal{T}^{(m)}$. A tree in $\mathcal{T}^{(2)}(13)$ is represented in Figure 3(a). For a non-root node v of $T \in \mathcal{T}$, we denote by parent(v) the parent of v, and lsib(v) the number of left-siblings of v. We compare nodes of T according to their labels, so that $u < v$ means that the label of u is less than the label of v.

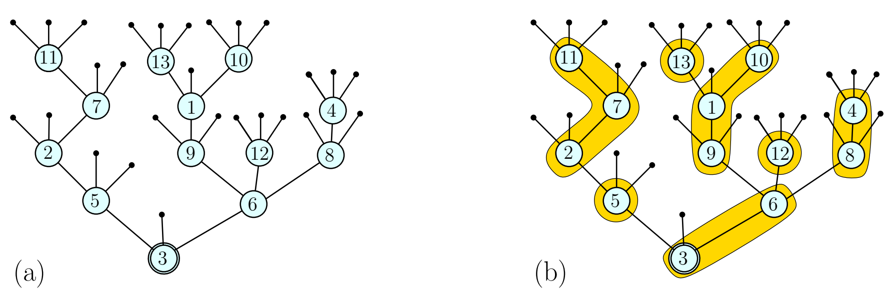


FIGURE 3. (a) A (rooted plane node-labeled) tree in $\mathcal{T}^{(2)}(13)$. (b) A **S**-boxed tree for $S = \{-1, 2\}$ (note for instance that $-1 \in S$ imposes that if a box contains both a node u and its middle child v, then $u > v$).

2.3. **Known counting results about deformed braid arrangements.** As mentioned in the introduction, it has been observed by Gessel, that for every set $S \subseteq \{-1,0,1\}$, the regions of the S-braid arrangement $\mathcal{A}_S(n)$ are equinumerous to a certain family of trees in $\mathcal{T}^{(1)}(n)$. Up to symmetry, we only need to consider the braid, Catalan, Shi, semiorder, and Linial arrangements, which are represented in Figure 1. We now describe the corresponding families of trees (see Figure 4). A non-root node v of a tree $T \in \mathcal{T}^{(1)}$ is called *left node* (resp. *right node*) if $\mathrm{lsib}(v) = 0$ (resp. $\mathrm{lsib}(v) = 1$). Here are the identities that Gessel observed (based on known counting results about hyperplane arrangements, and a new formula he established for trees in $\mathcal{T}^{(1)}(n)$ counted according to the number of left and right ascents and descents):

- The regions of the *Catalan arrangement* $\mathcal{A}_{\{-1,0,1\}}(n)$ are equinumerous to the trees in $\mathcal{T}^{(1)}(n)$.
- The regions of the *Shi arrangement* $\mathcal{A}_{\{0,1\}}(n)$ are equinumerous to the trees in $\mathcal{T}^{(1)}(n)$ such that
  (i) for every right node v, $\mathrm{parent}(v) > v$.
- The regions of the *semiorder arrangement* $\mathcal{A}_{\{-1,1\}}$ are equinumerous to the trees in $\mathcal{T}^{(1)}(n)$ such that
  (ii) for every left node v, if the right-sibling of v is a leaf then $\mathrm{parent}(v) > v$.
- The regions of the *Linial arrangement* $\mathcal{A}_{\{1\}}(n)$ are equinumerous to the trees in $\mathcal{T}^{(1)}(n)$ such that
  (iii) for every left node v, $\mathrm{parent}(v) > v$, and for every right node v, $\mathrm{parent}(v) < v$.

Based on these observations, Gessel raised the question of finding a uniform, possibly bijective, explanation of these five correspondences between arrangements and binary trees (see [20] or [23, Section 1]). It is the goal of this paper to provide such an explanation (and more). Let us mention however that the family of trees that will appear in the framework of the current paper for the Linial arrangement is not given by the Condition (iii), but instead by the Condition (iii') represented in Figure 4. The bijective link between Conditions (iii) and (iii') is explained in Section 9.2. In fact, Condition (iii') is simply the combination of Conditions (i) and (ii). The fact that the family of trees associated to the intersection $\mathcal{A}_{\{1\}}(n) = \mathcal{A}_{\{0,1\}}(n) \cap \mathcal{A}_{\{-1,1\}}(n)$ is the intersection of the family of trees associated to $\mathcal{A}_{\{0,1\}}(n)$ and $\mathcal{A}_{\{-1,1\}}(n)$ is an instance of a general feature of the theory developed in the present paper (see Remark 3.10).

Let us now recall the relevant references for the identities observed above, and some natural generalizations. First, observe that $\mathcal{T}^{(1)}(n)$ has cardinality $n!\mathrm{Cat}(n)! = \frac{(2n)!}{(n+1)!}$, because there are $\mathrm{Cat}(n)$ binary trees with n nodes and $n!$ ways of labeling their nodes. More generally, $\mathcal{T}^{(m)}(n)$ has cardinality $\frac{((m+1)n)!}{(mn+1)!}$. On the other hand, it is classical that the m*-Catalan arrangement* $\mathcal{A}_{[-m..m]}(n)$ has $\frac{((m+1)n)!}{(mn+1)!}$ regions (see e.g. [43, Section 4]). We will recall a bijective proof of this fact in Section 8.1.

Next, observe that the number of trees in $\mathcal{T}^{(1)}(n)$ satisfying Condition (i) is $(n+1)^{n-1}$ because these trees are easily seen to be in bijection with Cayley trees with $n+1$ vertices. The fact that the number of regions of the Shi arrangement $\mathcal{A}_{\{0,1\}}(n)$ is also $(n+1)^{n-1}$ was first established by Shi [38]. In fact, Shi further showed that for all m the number

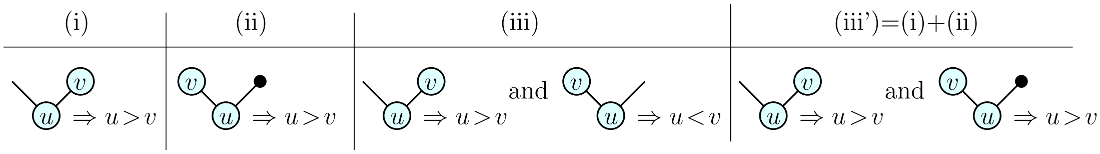


FIGURE 4. The conditions (i), (ii), (iii) appearing in the literature for the classes of trees equinumerous to the regions of the Shi, semiorder and Linial arrangements. The characterization (iii') proved in this paper for the Linial arrangement appears to be new (see Section 9.2 for a bijection between (iii) and (iii')). Nodes are represented by labeled discs, while leaves are represented by black dots (here, the nature of some vertices is left unspecified).

of regions of the m-Shi arrangement $\mathcal{A}_{[-m+1..m]}(n)$ is $(mn+1)^{n-1}$, which is the number of m-parking functions of size n. Since then, at least two distinct bijective proofs of this fact have been given [11, 40], besides non-bijective proofs [37, 15, 9]. We discuss these bijections further in Section 9.1. As we will see, $(mn+1)^{n-1}$ is also the number of trees in $\mathcal{T}^{(m)}(n)$ such that if a node v is the right-most child of a node u, then $u > v$ (this generalizes (i)).

The identity between the regions of the semiorder arrangement and the trees in $\mathcal{T}^{(1)}(n)$ satisfying Condition (ii) is equivalent to a result of Chandon [14]. More generally, the *m-semiorder arrangement* $\mathcal{A}_{[-m..m]\setminus\{0\}}(n)$ has regions equinumerous to trees in $\mathcal{T}^{(m)}(n)$ such that if a node v is the leftmost child of a node u, and all its siblings are leaves, then $u > v$ (this generalizes (ii)). This fact is easily deduced from the generating function equation given in [37, Theorem 7.1].

Lastly, the identity between the regions of the Linial arrangement and the counting sequence of the trees in $\mathcal{T}^{(1)}(n)$ satisfying (iii) was conjectured by Linial and Ravid, and proved in [37] and independently in [6] by equating two generating functions. However, as mentioned above, the family of trees which appear naturally in our framework are characterized by Condition (iii') instead of Condition (iii). More generally, the *m-Linial arrangement* $\mathcal{A}_{[-m+1..m]\setminus\{0\}}(n)$ has regions equinumerous to the subset of trees in $\mathcal{T}^{(m)}(n)$ such that if a node v is the right-most child of a node u, then $u > v$, and moreover, if a node v is the leftmost child of a node u, and all its siblings are leaves, then $u > v$ (this generalizes (iii')).

Although we cannot give an exhaustive bibliography about the enumerative study of deformations of the braid arrangement, we should mention a few additional references which are relevant to the present article. Formulas for the number of regions of several additional deformations of the braid arrangements (for instance $\mathcal{A}_{[-\ell..m]}(n)$ for $\ell \geq -1$) are given in [37]. The characteristic and coboundary polynomials of some of the arrangements above have been computed in [9, 1]. In a different direction, several deformations of the braid arrangements associated to a graph $G = ([n], E)$ have been considered in the literature. The most classical is the *G-graphical arrangement* made of the hyperplanes $H_{i,j,0}$, for all $\{i, j\} \in E$. Another important example is the *G-Shi arrangement* considered for instance in [3, 11]. This arrangement is made of the hyperplanes $H_{i,j,0}$ for all $i,j \in [n]$, and $H_{i,j,1}$ for all $\{i,j\} \in E$ with $i<j$ (so that it is the braid arrangement if G has no edge, and the Shi arrangement if $G = K_n$). In yet another direction, several authors have considered deformed braid arrangements with hyperplanes $H_{i,j,s}$ for generic, non-integer values of s (see e.g. [37, 40, 41, 27]), but we will not consider such situations here.

3. Counting regions of S-braid arrangements

In this section we present our counting results for the regions of S-braid arrangements. Throughout this section, S is a finite set of integers, $m = \max(|s|,\ s \in S)$, and n is a non-negative integer.

We start with the definition of S-*boxed trees* (Definition 3.3), and then express the number of regions of $\mathcal{A}_S(n)$ as a signed count of S-boxed trees (Theorem 3.4). Then we restrict our attention to *transitive sets* S (Definition 3.5) and for them we express the number of regions of $\mathcal{A}_S(n)$ as an unsigned count of trees (Theorem 3.8).

In order to define S-boxed trees, we first need to define S-cadet sequences.

Definition 3.1.
- Let T be a tree in $\mathcal{T}$, and let u be a node. If one of the children of u is a node, we call the rightmost such child the *cadet-node* of u, and denote it by $\text{cadet}(u)$.[4]
- A *cadet sequence* is a non-empty sequence $(v_1, \ldots, v_k)$ of nodes such that for all i in $[k-1]$, $v_{i+1} = \text{cadet}(v_i)$.
- A S-*cadet sequence* is a cadet sequence $(v_1, \ldots, v_k)$ such that for all $1 \leq i < j \leq k$, if $\sum_{p=i+1}^{j} \text{lsib}(v_p) \in S \cup \{0\}$ then $v_i < v_j$, and if $-\sum_{p=i+1}^{j} \text{lsib}(v_p) \in S$ then $v_i > v_j$.

Note that an S-cadet sequence $(v_1, \ldots, v_k)$ of $T \in \mathcal{T}^{(m)}$ satisfies in particular $\text{lsib}(v_j) \in [0..m] \setminus \{s \in S \mid -s \in S\}$ for all $j \in [2..k]$.

Example 3.2. Let $T \in \mathcal{T}^{(m)}$.
- For $S = [-m..m]$, the S-cadet sequences of T contain a single vertex.
- For $S = [-\ell..m]$ with $0 \leq \ell \leq m$, the S-cadet sequences of T are the cadet sequences $(v_1, \ldots, v_k)$ satisfying $v_1 < v_2 < \cdots < v_k$ and $\text{lsib}(v_p) \in [\ell+1..m]$ for all $p \in [2..k]$.
- For $S = [-\ell..m] \setminus \{0\}$ with $0 \leq \ell \leq m$, the S-cadet sequences of T are the cadet sequences $(v_1, \ldots, v_k)$ satisfying $v_1 < v_2 < \cdots < v_k$ and $\text{lsib}(v_p) \in \{0\} \cup [\ell+1..m]$ for all $p \in [2..k]$.
- For $S = \{-2, 0, 1, 2\}$, the S-cadet sequences of T have size at most 2, and the S-cadet sequences of size 2 are of the form (v_1, v_2) with $\text{lsib}(v_2) = 1$ and $v_1 < v_2$.

Definition 3.3. A *boxed tree* is a pair (T, B), where T is in $\mathcal{T}$, and B is a set of cadet sequences partitioning the set of nodes of T (that is, every node of T is contained in exactly one cadet sequence in B). The pair (T, B) is an S-*boxed tree* if $T \in \mathcal{T}^{(m)}$, and

[4] The term *cadet* is used here in its genealogical meaning of *youngest heir*.

B contains only S-cadet sequences. We denote by $\mathcal{U}_S(n)$ the set of S-boxed trees with n nodes.

We represent boxed trees as trees decorated with boxes partitioning the nodes into cadet sequences, as in Figure 3(b). We can now state the main result of this section.

Theorem 3.4. *Let S be a finite set of integers and n be a positive integer. The number of regions of the hyperplane arrangement $\mathcal{A}_S(n)$ is*

$$r_S(n) = \sum_{(T,B)\in\mathcal{U}_S(n)} (-1)^{n-|B|}. \tag{3.1}$$

The proof of Theorem 3.4 is delayed to Section 7. We will now give a simpler expression for $r_S(n)$ in the cases where the set S is "well-behaved". More precisely, we now introduce the notion of *transitivity* for a set S, which implies a drastic simplification of the definition of S-cadet-sequences (Lemma 3.11), and allows one to define a simple sign reversing involution on S-boxed trees.

Definition 3.5. A set S of integers is called *transitive* if it satisfies the following conditions for all integers $s, t \notin S$:

- if $st > 0$, then $s + t \notin S$,
- if $s > 0$ and $t \leq 0$, then $s - t \notin S$ and $t - s \notin S$.

Example 3.6.

- All the subsets of $\{-1, 0, 1\}$ are transitive.
- All the intervals of integers containing 1 are transitive.
- Sets of the form $S = I \setminus k\mathbb{Z}$, where I is an interval containing 1 are transitive.
- Sets S such that $[-\lfloor m/2\rfloor..\lfloor m/2\rfloor] \subseteq S \subseteq [-m..m]$ for some m are transitive.
- A set S such that $\{-s,\ s \in S\} = S$ is transitive if and only if the set of positive integers not in S is closed under addition (equivalently, 0 together with the positive integer not in S form what is called a *numerical semigroup*; see [4] for references on numerical semigroups).
- A set S such that $[m] \subseteq S \subseteq [-m..m]$ is transitive if and only if the set of negative integers not in S is closed under addition.

Definition 3.7. We denote by $\mathcal{T}_S(n)$ the set of trees T in $\mathcal{T}^{(m)}(n)$ such that all nodes u, v satisfying $\mathrm{cadet}(u) = v$ further satisfies the following:
Condition(S): if $\mathrm{lsib}(v) \notin S \cup \{0\}$ then $u < v$, and if $-\mathrm{lsib}(v) \notin S$ then $u > v$.

Theorem 3.8. *If S is transitive, then regions of the hyperplane arrangement $\mathcal{A}_S(n)$ are equinumerous to the trees in $\mathcal{T}_S(n)$.*

Example 3.9.

- $\mathcal{T}_{[-m..m]}(n) = \mathcal{T}^{(m)}(n)$.
- $\mathcal{T}_{[-m+1..m]}(n)$ is the set of trees in $\mathcal{T}^{(m)}(n)$, such that any non-root node having no right-sibling (not even leaves) is less than its parent.
- $\mathcal{T}_{[m]}(n)$ is the set of trees in $\mathcal{T}^{(m)}(n)$, such that such that any cadet-node v is less than its parent.
- More generally, for $0 \leq \ell \leq m$, $\mathcal{T}_{[-\ell..m]}(n)$ is the set of trees in $\mathcal{T}^{(m)}(n)$, such that any cadet-node v having more than ℓ left-sibling is less than its parent. And $\mathcal{T}_{[-\ell..m]\setminus\{0\}}(n)$ is the set of trees in $\mathcal{T}^{(m)}(n)$, such that any cadet-node v having either no left sibling or more than ℓ left-siblings is less than its parent.

Remark 3.10. For any sets $S, S' \subset \mathbb{Z}$, $\mathcal{T}_S(n) \cap \mathcal{T}_{S'}(n) = \mathcal{T}_{S\cap S'}(n)$. For instance the set $\mathcal{T}_{\{1\}}(n)$ of trees associated to Linial arrangement, is the intersection of the set of trees $\mathcal{T}_{\{0,1\}}(n)$ associated to the Shi arrangement, and the set of trees $\mathcal{T}_{\{-1,1\}}(n)$ associated to the semiorder arrangement.

Moreover, each element $s \in [-m..m] \setminus S$ gives a simple condition for trees in $\mathcal{T}_S(n)$: for $s > 0$ the condition is that a cadet node with s left siblings is greater than its parent, while for $s \leq 0$ the condition is that a cadet node with $-s$ left siblings is less than its parent. This is represented in Figure 5.

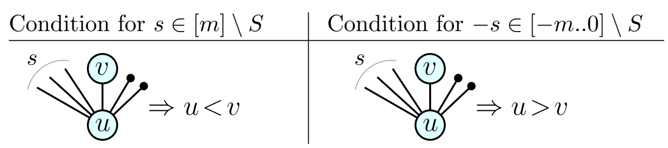


FIGURE 5. Conditions for trees to be in $\mathcal{T}_S(n)$. Each element $s \in [-m..m] \setminus S$ imposes one condition.

Theorem 3.8 is an easy consequence of Theorem 3.4 and the following lemma.

Lemma 3.11. *Suppose that the set S is transitive. In this case, a cadet sequence $(v_1, \ldots, v_k)$ is an S-cadet sequence if and only if for all $i \in [k-1]$,*
(*) if $\mathrm{lsib}(v_{i+1}) \in S \cup \{0\}$ then $v_i < v_{i+1}$, and if $-\mathrm{lsib}(v_{i+1}) \in S$ then $v_i > v_{i+1}$.

Proof. It is clear that the condition (*) is necessary. We now prove that it is sufficient, by induction on k. The case $k = 1$ is trivial. Now suppose that $k > 1$ and $\gamma = (v_1, \ldots, v_k)$ satisfies (*). Since $\gamma' = (v_1, \ldots, v_{k-1})$ satisfies (*), it is an S-cadet sequence. Hence we only need to check that for all $i \in [k-1]$,

$$(**)\ \text{if} \sum_{p=i+1}^{k} \mathrm{lsib}(v_p) \in S \cup \{0\} \text{ then } v_i < v_k, \text{ and if } -\sum_{p=i+1}^{k} \mathrm{lsib}(v_p) \in S \text{ then } v_i > v_k.$$

The case $i = k-1$ of (**) is directly given by (*). We now consider $i \in [k-2]$, and consider several cases. Suppose first that $v_i < v_{k-1} < v_k$. In this case, $-\sum_{p=i+1}^{k-1} \mathrm{lsib}(v_p) \notin S$ (since γ' is an S-cadet sequence), $-\mathrm{lsib}(v_k) \notin S$ (since γ satisfies (*)), hence $-\sum_{p=i+1}^{k} \mathrm{lsib}(v_p) \notin S$ (since S is transitive), hence Condition (**) holds for i. The case $v_i > v_{k-1} > v_k$ is treated similarly. Suppose next that $v_i > v_{k-1} < v_k$. In this case, $\sum_{p=i+1}^{k-1} \mathrm{lsib}(v_p) \notin S \cup \{0\}$ (since γ' is an S-cadet sequence), $-\mathrm{lsib}(k) \notin S$ (since γ satisfies (*)), hence $\sum_{p=i+1}^{k} \mathrm{lsib}(v_p) \notin S \cup \{0\}$ and $-\sum_{p=i+1}^{k} \mathrm{lsib}(v_p) \notin S$ (since S is transitive), hence Condition (**) holds for i. The case $v_i < v_{k-1} > v_k$ is treated similarly. Thus Condition (**) holds for all i, and γ is an S-cadet sequence. □

Proof of Theorem 3.8. Let $T \in \mathcal{T}^{(m)}(n)$, and let $v = \mathrm{cadet}(u)$. We claim that that u and v can be in the same box of an S-boxed tree (T, B) if and only if u and v do not satisfy Condition(S) of Definition 3.7. Indeed, by Lemma 3.11, in the case $u < v$ (resp. $u > v$) the vertices u and v can be in the same box if and only if $-\mathrm{lsib}(v) \notin S$ (resp. $\mathrm{lsib}(v) \notin S \cup \{0\}$), and this holds if and only if Condition(S) does not hold.

For a tree T in $\mathcal{T}^{(m)}(n)$, we denote $\mathcal{B}_T = \{B \mid (T,B) \in \mathcal{U}_S(n)\}$. By Theorem 3.4,

$$r_S(n) = \sum_{T\in\mathcal{T}_S(n)} \sum_{B\in\mathcal{B}_T} (-1)^{n-|B|} + \sum_{T\in\mathcal{T}^{(m)}(n)\setminus\mathcal{T}_S(n)} \sum_{B\in\mathcal{B}_T} (-1)^{n-|B|}. \tag{3.2}$$

By the above claim, for all T in $\mathcal{T}_S$, $\mathcal{B}_T$ contain a single element because every node of T must be in a different box. Thus the first sum of (3.2) contributes $|\mathcal{T}_S(n)|$. We now prove that the second sum is 0 using a sign reversing involution. For a tree $T \in \mathcal{T}^{(m)}(n) \setminus \mathcal{T}_S(n)$, we pick the smallest vertex $v = \text{cadet}(u)$ such that Condition(S) does not hold, and define an involution φ on $\mathcal{B}_T$ as follows:

- if u and v are in the same box of B, then $\varphi(B)$ is obtained by splitting the box containing them between u and v,
- if u and v are in different boxes of B, then $\varphi(B)$ is obtained by merging these boxes.

Lemma 3.11 ensures that $\varphi(B) \in \mathcal{B}_T$ in the second situation. Since φ is an involution on $\mathcal{B}_T$ changing the number of boxes by ± 1, we get $\sum_{B\in\mathcal{B}_T}(-1)^{n-|B|} = 0$. Hence the second sum in (3.2) contributes 0. $\square$

4. General deformations of the braid arrangement

In this section we extend the results of Section 3 to general deformations of the braid arrangement. We fix a positive integer N and an $\binom{N}{2}$-tuple of finite sets of integers $\mathbf{S} = (S_{a,b})_{1\le a<b\le N}$. The arrangement in $\mathbb{R}^N$ made of the hyperplanes

note cadre

$$H_{a,b,s} = \{(x_1,\dots,x_N) \in \mathbb{R}^N \mid x_a - x_b = s\},$$

for all $1 \le a < b \le N$ and all $s \in S_{a,b}$ is called ***S**-braid arrangement*, and is denoted $\mathcal{A}_{\mathbf{S}}$. Note that if $S_{a,b} = S$ for all a, b, then $\mathcal{A}_{\mathbf{S}} = \mathcal{A}_S(N)$.

We will now extend Theorem 3.4 to $\mathbf{S}$-braid arrangements. Let $m = \max(|s|,\ s \in \cup S_{a,b})$. For $1 \le a < b \le N$, we denote $S_{b,a} = S_{a,b}$, $S^-_{a,b} = \{s \ge 0 \mid -s \in S_{a,b}\}$, and $S^-_{b,a} = \{s > 0 \mid s \in S_{a,b}\} \cup \{0\}$.

$S_{a,b} = \{v_a - v_b\}$ $a<b$ pour ceux $S^-_{a,b} = \{0\}$ $S^-_{b,a} = S_{a,b} \cup \{0\}$

Definition 4.1. A cadet sequence $(v_1,\dots,v_k)$ of $T \in \mathcal{T}^{(m)}(N)$ is an ***S**-cadet sequence* if for all $1 \le i < j \le k$, $\sum_{p=i+1}^{j} \text{lsib}(v_p) \notin S^-_{v_i,v_j}$. An ***S**-boxed tree* is a boxed tree (T,B) with $T \in \mathcal{T}^{(m)}(N)$, and B containing only $\mathbf{S}$-cadet sequences. We denote by $\mathcal{U}_{\mathbf{S}}$ the set of $\mathbf{S}$-boxed trees.

"left sibling"

Theorem 4.2. *The number of regions of the hyperplane arrangement $\mathcal{A}_{\mathbf{S}}$ is*

$$r_{\mathbf{S}} = \sum_{(T,B)\in\mathcal{U}_{\mathbf{S}}} (-1)^{n-|B|}. \tag{4.1}$$

The condition $\sum_{p=i+1}^{j} \text{lsib}(v_p) \notin S^-_{v_i,v_j}$ is equivalent to: if $\sum_{p=i+1}^{j} \text{lsib}(v_p) \in S_{v_i,v_j} \cup \{0\}$ then $v_i < v_j$, and if $-\sum_{p=i+1}^{j} \text{lsib}(v_p) \in S_{v_i,v_j}$ then $v_i > v_j$. In particular, if $S_{a,b} = S$ for all a, b, then $\mathcal{U}_{\mathbf{S}} = \mathcal{U}_S(N)$. Hence Theorem 4.2 generalizes Theorem 3.4.

Theorem 4.2 will be extended in the next section and its proof is delayed to Section 7. We now generalize Theorem 3.8 to $\mathbf{S}$-braid arrangements.

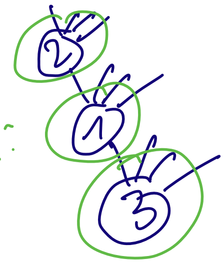
Exple:

$\vec{v} = (2^{n-i})$ $S_{a,b} = (2^{n-a} - 2^{n-b}) = 2^{n-b}(2^{b-a} - 1)$

$a<b$

Qd arbre est peigne gauche, S-cadet sequence tout
séparée

<u>Exple</u> :



$2 \notin S_{3,2}$ et $1 \notin S_{2,1}$

$1+2 \in S^-_{3,1}$

Pas transitif

Definition 4.3. The tuple $\mathbf{S}$ is said *transitive* if for all distinct integers $a, b, c \in [N]$ the following condition holds: if $s \notin S^-_{a,b}$ and $t \notin S^-_{b,c}$, then $s + t \notin S^-_{a,c}$.

It is easy to see that if $S_{a,b} = S$ for all $a, b \in [N]$, then $\mathbf{S}$ is transitive if and only if S is transitive.

Example 4.4. If for all $a, b \in [N]$, $[-\lfloor m/2 \rfloor .. \lfloor m/2 \rfloor] \subseteq S_{a,b} \subseteq [-m..m]$, then $\mathbf{S}$ is transitive.

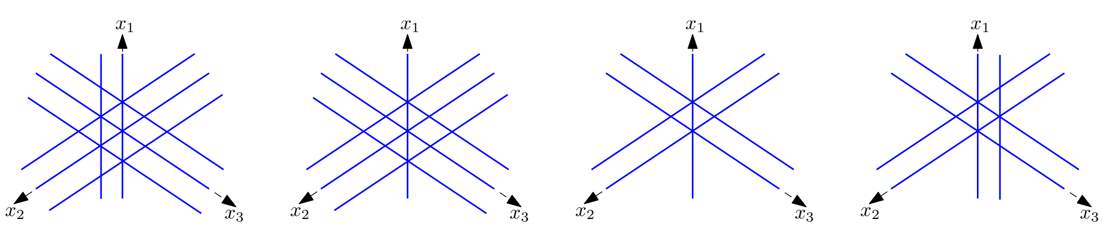
Definition 4.5. We denote by $\mathcal{T}_\mathbf{S}$ the set of trees T in $\mathcal{T}^{(m)}(N)$ such that any pair of nodes u, v such that $\text{cadet}(u) = v$ satisfies $\text{lsib}(v) \in S^-_{u,v}$.

Theorem 4.6. *If $\mathbf{S} = (S_{a,b})_{1 \leq a < b \leq N}$ is transitive, then the regions of $\mathcal{A}_\mathbf{S}$ are equinumerous to the trees in $\mathcal{T}_\mathbf{S}$.*

Remark 4.7. The condition $\text{lsib}(v) \notin S^-_{u,v}$ is equivalent to: if $\text{lsib}(v) \notin S_{u,v} \cup \{0\}$ then $u < v$, and if $-\text{lsib}(v) \in S_{u,v}$ then $u > v$. In particular, if $S_{a,b} = S$ for all $a, b \in [N]$, then $\mathcal{T}_\mathbf{S} = \mathcal{T}_S(N)$. Hence Theorem 4.6 generalizes Theorem 3.8.

Example 4.8. Let $G = ([N], E)$ be a graph and let S, S' be two finite sets of integers. Let $G(S, S')$ be the tuple $\mathbf{S} = (S_{a,b})_{1 \leq a < b \leq N}$ defined by $S_{a,b} = S$ if $\{a, b\} \in E$ and $S_{a,b} = S'$ otherwise. Several cases are represented in Figure 6.

(1) For $S = \{-1, 0, 1\}$ and $S' = \{0, 1\}$, the tuple $\mathbf{S} = G(S, S')$ is transitive for any graph G, and $\mathcal{T}_\mathbf{S}$ is the set of trees in $\mathcal{T}^{(1)}(N)$ such that if a node v is the right child of u, then either $\{u, v\} \in E$ or $u > v$ (or both).
(2) For $S = \{-1, 0, 1\}$ and $S' = \{0\}$, the tuple $\mathbf{S} = G(S, S')$ is transitive for any graph G, and $\mathcal{T}_\mathbf{S}$ is the set of trees in $\mathcal{T}^{(1)}(N)$ such that if a node v is the right child of u, then $\{u, v\} \in E$.
(3) For $S = \{0, 1\}$ and $S' = \{0\}$, the tuple $\mathbf{S} = G(S, S')$ is transitive for any graph G, and $\mathcal{T}_\mathbf{S}$ is the set of trees in $\mathcal{T}^{(1)}(N)$ such that if a node v is the right child of u, then $\{u, v\} \in E$ and $u > v$.
(4) For $S = \{0, 1\}$ and $S' = \{-1, 0\}$, the tuple $\mathbf{S} = G(S, S')$ is transitive for any graph G, and $\mathcal{T}_\mathbf{S}$ is the set of trees in $\mathcal{T}^{(1)}(N)$ such that if a node v is the right child of u, then either ($\{u, v\} \in E$ and $u > v$) or ($\{u, v\} \notin E$ and $u < v$).



$S = \{-1,0,1\}$, $S' = \{0,1\}$ $\quad$ $S = \{-1,0,1\}$, $S' = \{0\}$ $\quad$ $S = \{0,1\}$, $S' = \{0\}$, $\quad$ $S = \{0,1\}$, $S' = \{-1,0\}$

FIGURE 6. Some transitive deformations of the braid arrangement. These arrangements have the form $A_{G(S,S')}$, where G is the graph having vertex set [3] and edges $\{1, 2\}$ and $\{1, 3\}$.

Proof of Theorem 4.6. Given Theorem 4.2, we only need to prove that if $\mathbf{S} = (S_{a,b})_{1\leq a<b\leq n}$ is transitive, then

$$\sum_{(T,B)\in\mathcal{U}_{\mathbf{S}}} (-1)^{|B|} = |\mathcal{T}_{\mathbf{S}}|. \tag{4.2}$$

The proof of (4.2) is almost identical to that of Theorem 3.8, except that Lemma 3.11 is replaced by the following claim: *if $\boldsymbol{S}$ is transitive, a cadet sequence $(v_1,\ldots,v_k)$ of $T\in\mathcal{T}^{(m)}(N)$ is a $\boldsymbol{S}$-cadet sequence if and only if for all $i\in[k-1]$,* $\mathrm{lsib}(v_{i+1})\notin S^-_{v_i,v_{i+1}}$.
Proof of the claim: It is clear that $\mathrm{lsib}(v_{i+1})\notin S^-_{v_i,v_{i+1}}$ is necessary. We now prove that it is sufficient, by induction on k. The case $k=1$ is trivial. Now suppose that $k>1$, and $\gamma=(v_1,\ldots,v_k)$ is cadet sequence such that for all $i\in[k-1]$, $\mathrm{lsib}(v_{i+1})\notin S^-_{v_i,v_{i+1}}$. We want to prove that γ is a $\mathbf{S}$-cadet sequence. By the induction hypothesis, $\gamma'=(v_1,\ldots,v_{k-1})$ is a $\mathbf{S}$-cadet sequence, so we only need to prove that for all $i\in[k-1]$, $\sum_{p=i+1}^{k}\mathrm{lsib}(v_p)\notin S^-_{v_i,v_k}$. This is true by hypothesis for $i=k-1$. Moreover, for $i\in[k-1]$, $\sum_{p=i+1}^{k-1}\mathrm{lsib}(v_p)\notin S^-_{i,k-1}$ (since γ' is a $\mathbf{S}$-cadet sequence), and $\mathrm{lsib}(v_k)\notin S^-_{k-1,k}$ (by hypothesis), hence $\sum_{p=i+1}^{k}\mathrm{lsib}(v_p)\notin S^-_{i,k}$ (since $\mathbf{S}$ is transitive). Hence γ is a $\mathbf{S}$-cadet sequence. This proves the claim.
One can then define a sign reversing involution on $\mathbf{S}$-boxed trees showing (4.2), exactly as in the proof of Theorem 3.8. □

5. Characteristic and coboundary polynomials of the deformations of the braid arrangement

In this section we refine our preceding counting results by expressing the characteristic and coboundary polynomials of deformed braid arrangements in terms of boxed trees.

5.1. Characteristic and coboundary polynomials.

For a hyperplane arrangement $\mathcal{A}\subset\mathbb{R}^n$, we denote by $\chi_{\mathcal{A}}(q)$ its *characteristic polynomial* of $\mathcal{A}$, and by $P_{\mathcal{A}}(q,y)$ its *coboundary polynomial*. Recall from [16] that the coboundary polynomial is defined by

$$P_{\mathcal{A}}(q,y) = \sum_{\mathcal{B}\subseteq\mathcal{A},\ \cap_{H\in\mathcal{B}}H\neq\emptyset} q^{\dim(\cap_{H\in\mathcal{B}}H)}(y-1)^{|\mathcal{B}|},$$

and that $\chi_{\mathcal{A}}(q)=P_{\mathcal{A}}(q,0)$.

The characteristic polynomial $\chi_{\mathcal{A}}$ contains a lot of information about the arrangement $\mathcal{A}$. In particular, by a result of Zaslavsky [46], the number of regions $r_{\mathcal{A}}$ and the number of *relatively bounded*[5] regions $b_{\mathcal{A}}$ are evaluations of $\chi_{\mathcal{A}}(q)$:

$$\begin{aligned} r_{\mathcal{A}} &= (-1)^n\chi_{\mathcal{A}}(-1),\\ b_{\mathcal{A}} &= (-1)^{\mathrm{rank}(\mathcal{A})}\chi_{\mathcal{A}}(1), \end{aligned} \tag{5.1}$$

where $\mathrm{rank}(\mathcal{A})$ is the dimension of the vector space generated by the vectors normal to the hyperplanes of $\mathcal{A}$. The characteristic polynomial is also equivalent to the Poincaré

[5]A region of $\mathcal{A}$ is *relatively bounded* if its intersection with the subspace generated by the vectors normal to the hyperplanes of $\mathcal{A}$ is bounded.

polynomial of the cohomology ring of the complexification of $\mathcal{A}$; see [33]. The coboundary polynomial is equivalent to the Tutte polynomial $T_{\mathcal{A}}(x,y)$ of $\mathcal{A}$ (that is, the Tutte polynomial of the semi-matroid associated with $\mathcal{A}$, in the sense of Ardila [1, 2]):

$$T_{\mathcal{A}}(x,y) = (y-1)^{-\operatorname{rank}(\mathcal{A})} P_{\mathcal{A}}((x-1)(y-1), y).$$

5.2. **Expressing the coboundary polynomial in terms of boxed trees.** In order to express the coboundary polynomials of deformed braid arrangements in terms of boxed trees, we will consider arrangements of all dimensions. Let $\widehat{\mathbf{S}} = (S_{a,b})_{1\leq a\leq b\leq N}$ be an $\binom{N+1}{2}$-tuple of finite sets, let $m = \max(|s|,\ s \in \cup S_{a,b})$, and let $\mathbf{n} = (n_1,\ldots,n_N) \in \mathbb{N}^N$. We denote $|\mathbf{n}| = n_1 + \ldots + n_N$, and

$$V(\mathbf{n}) = \{(a,i) \mid a \in [N], i \in [n_a]\}.$$

We endow $V(\mathbf{n})$ with the *lexicographical order*, that is, we denote $(a,i) < (b,j)$ if either $a < b$, or $a = b$ and $i < j$. For $u = (a,i)$ and $v = (b,j) \in V(\mathbf{n})$, we denote $S_{u,v} = S_{v,u} = S_{a,b}$, and if $u < v$ we denote $S^-_{u,v} = \{s \geq 0 \mid -s \in S_{a,b}\}$ and $S^-_{v,u} = \{s > 0 \mid s \in S_{a,b}\} \cup \{0\}$. Lastly, we define $\mathcal{A}_{\widehat{\mathbf{S}}}(\mathbf{n})$ as the arrangement in $\mathbb{R}^{|\mathbf{n}|}$ with hyperplanes

$$H_{u,v,s} = \{(x_w)_{w\in V(\mathbf{n})} \mid x_u - x_v = s\},$$

for all $u < v$ in $V(\mathbf{n})$ and all $s \in S_{u,v}$.

Note that $\mathcal{A}_{\widehat{\mathbf{S}}}(\mathbf{n})$ identifies with the arrangement $\mathcal{A}_{\widehat{\mathbf{S}}(\mathbf{n})}$, where

$$\widehat{\mathbf{S}}(\mathbf{n}) = (S'_{u,v})_{1\leq u<v\leq|\mathbf{n}|} \tag{5.2}$$

with $S'_{u,v} = S_{a,b}$ for all $u \in \left[1+\sum_{i=1}^{a-1} n_i \,..\, \sum_{i=1}^{a} n_i\right]$ and $v \in \left[1+\sum_{i=1}^{b-1} n_i \,..\, \sum_{i=1}^{b} n_i\right]$. For instance, $\mathcal{A}_{\widehat{\mathbf{S}}}(1,1,\ldots,1) = \mathcal{A}_{\widehat{\mathbf{S}}}$, and $\mathcal{A}_{\widehat{\mathbf{S}}}(n_1,0,\ldots,0) = \mathcal{A}_{S_{1,1}}(n_1)$. We now describe boxed trees related to the arrangement $\mathcal{A}_{\widehat{\mathbf{S}}}(\mathbf{n})$.

- We denote by $\mathcal{T}^{(m)}(\mathbf{n})$ the set of rooted plane $(m+1)$-ary trees with $|\mathbf{n}|$ nodes labeled with distinct labels in $V(\mathbf{n})$.
- A cadet sequence $(v_1,\ldots,v_k)$ of $T \in \mathcal{T}^{(m)}(\mathbf{n})$ is *admissible* if $v_i < v_{i+1}$ for all $i \in [k-1]$ such that $\operatorname{lsib}(v_{i+1}) = 0$. A boxed tree (T,B) is *admissible* if all the sequences in B are admissible. We denote by $\mathcal{U}^{(m)}(\mathbf{n})$ the set of admissible boxed trees (T,B) with $T \in \mathcal{T}^{(m)}(\mathbf{n})$.
- The $\widehat{\boldsymbol{S}}$*-energy* of a cadet sequence $(v_1,\ldots,v_k)$ of T is the number of pairs $\{i,j\}$ with $1 \leq i < j \leq k$, such that $\sum_{p=i+1}^{j} \operatorname{lsib}(v_p) \in S^-_{v_i,v_j}$.[6] The $\widehat{\boldsymbol{S}}$*-energy* of a boxed tree $(T,B) \in \mathcal{U}^{(m)}(\mathbf{n})$, denoted $\operatorname{energy}_{\widehat{\mathbf{S}}}(T,B)$, is the sum of the energies of the cadet sequences in B.
- We denote by $\mathcal{U}_{\widehat{\mathbf{S}}}(\mathbf{n})$ the set of boxed trees $(T,B) \in \mathcal{U}^{(m)}(\mathbf{n})$ such that $\operatorname{energy}_{\widehat{\mathbf{S}}}(T,B) = 0$.

[6]The terminology *energy* used here is related to the interpretation of the current counting problem (about the coboundary polynomial of arrangements) in terms of a gas model which will be introduced in Section 7.

Note that any boxed tree in $\mathcal{U}_{\widehat{\mathbf{S}}}(\mathbf{n})$ is admissible. Moreover, $\mathcal{U}_{\widehat{\mathbf{S}}}(n_1,0,\ldots,0) = \mathcal{U}_{S_{1,1}}(n_1)$, and $\mathcal{U}_{\widehat{\mathbf{S}}}(1,1,\ldots,1) = \mathcal{U}_{\mathbf{S}}$, where $\mathbf{S} = (S_{a,b})_{1\leq a<b\leq N}$.

We denote by $\mathcal{T}_{\widehat{\mathbf{S}}}(\mathbf{n})$ the set of trees T in $\mathcal{T}^{(m)}(\mathbf{n})$ such that any pair of nodes u, v such that $\mathrm{cadet}(u) = v$ satisfies $\mathrm{lsib}(v) \in S^-_{u,v}$. Note that $\mathcal{T}_{\widehat{\mathbf{S}}}(n_1,0,\ldots,0) = \mathcal{T}_{S_{1,1}}(n_1)$, and $\mathcal{T}_{\widehat{\mathbf{S}}}(1,1,\ldots,1) = \mathcal{T}_{\mathbf{S}}$, where $\mathbf{S} = (S_{a,b})_{1\leq a<b\leq N}$. We say that $\widehat{\mathbf{S}}$ is *multi-transitive* if $\widehat{\mathbf{S}}(\mathbf{n})$ is transitive (in the sense of Definition 4.3) for all $\mathbf{n} \in \mathbb{N}^N$. Note that for $N = 1$ $\widehat{\mathbf{S}}$ is multi-transitive if and only if $S_{1,1}$ is transitive.

Example 5.1. If for all $a, b \in [N]$, $[-\lfloor m/2\rfloor \,..\, \lfloor m/2\rfloor] \subseteq S_{a,b} \subseteq [-m..m]$, then $\widehat{\mathbf{S}}$ is multi-transitive. Also, if $S_{a,a}$ is transitive for all $a \in [N]$, and $S_{a,b} = [-m..m]$ for all $a < b$, then $\widehat{\mathbf{S}}$ is multi-transitive.

We can now express the coboundary polynomial of the arrangements. Given indeterminates $t_1,\ldots,t_N$, we denote $\mathbf{t} = (t_1,\ldots,t_N)$, $\mathbf{t}^{\mathbf{n}} = \prod_{a=1}^N t_a^{n_a}$, and $\mathbf{n}! = \prod_{a=1}^N n_a!$. We denote

$$P_{\widehat{\mathbf{S}}}(q,y,\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{N}^N} P_{\mathcal{A}_{\widehat{\mathbf{S}}}(\mathbf{n})}(q,y)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}, \tag{5.3}$$

$$\chi_{\widehat{\mathbf{S}}}(q,\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{N}^N} \chi_{\mathcal{A}_{\widehat{\mathbf{S}}}(\mathbf{n})}(q)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}, \tag{5.4}$$

$$R_{\widehat{\mathbf{S}}}(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{N}^N} r_{\mathcal{A}_{\widehat{\mathbf{S}}}(\mathbf{n})}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}, \tag{5.5}$$

In the above definition, we adopt the convention $P_{\mathcal{A}_{\widehat{\mathbf{S}}}(0,\ldots,0)}(q,y) = 1$ (coboundary polynomial of the empty semi-matroid). By (5.1), $\chi_{\widehat{\mathbf{S}}}(q,\mathbf{t}) = P_{\widehat{\mathbf{S}}}(q,0,\mathbf{t})$ and $R_{\widehat{\mathbf{S}}}(\mathbf{t}) = \chi_{\widehat{\mathbf{S}}}(-1,-\mathbf{t})$, where $-\mathbf{t} = (-t_1,\ldots,-t_N)$.

Theorem 5.2. *Let $\widehat{\boldsymbol{S}} = (S_{a,b})_{1\leq a\leq b\leq N}$ be an $\binom{N+1}{2}$-tuple of finite sets of integers, and let $m = \max(|s|,\ s \in \cup S_{a,b})$. Then $P_{\widehat{\boldsymbol{S}}}(q,y,\boldsymbol{t})$ is related to boxed trees by*

$$P_{\widehat{\boldsymbol{S}}}(q,y,\boldsymbol{t}) = \left(\sum_{\boldsymbol{n}\in\mathbb{N}^N} \frac{\boldsymbol{t}^{\boldsymbol{n}}}{\boldsymbol{n}!} \sum_{(T,B)\in\mathcal{U}^{(m)}(\boldsymbol{n})} (-1)^{|B|} y^{\mathrm{energy}_{\widehat{\boldsymbol{S}}}(T,B)}\right)^{-q}. \tag{5.6}$$

In particular,

$$\chi_{\widehat{\boldsymbol{S}}}(q,\boldsymbol{t}) = R_{\widehat{\boldsymbol{S}}}(-\boldsymbol{t})^{-q}, \tag{5.7}$$

and

$$R_{\widehat{\boldsymbol{S}}}(\boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{N}^N} \frac{\boldsymbol{t}^{\boldsymbol{n}}}{\boldsymbol{n}!} \sum_{(T,B)\in\mathcal{U}_{\widehat{\boldsymbol{S}}}(\boldsymbol{n})} (-1)^{|\boldsymbol{n}|-|B|}. \tag{5.8}$$

Moreover, if $\widehat{\boldsymbol{S}}$ is multi-transitive, then

$$R_{\widehat{\boldsymbol{S}}}(\boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{N}^N} \frac{\boldsymbol{t}^{\boldsymbol{n}}}{\boldsymbol{n}!}|\mathcal{T}_{\widehat{\boldsymbol{S}}}(\boldsymbol{n})|. \tag{5.9}$$

Note that Equation (5.8) implies Theorem 3.4 (for $S = S_{1,1}$) by extracting the coefficient of $t_1^n t_2^0 \dots t_N^0$, and Theorem 4.2 by extracting the coefficient of $t_1 t_2 \cdots t_N$. Several applications of Theorem 5.2 are given in Section 6.

5.3. **Remarks about the case of graphical arrangements (case $m = 0$).** In this subsection we consider the special case $m = 0$ of Theorem 5.2 which corresponds to graphical arrangements, in order to highlight how our results are related to some known results about graph colorings and acyclic orientations.

Remember that the *G-graphical arrangement* associated to a (simple, undirected) graph $G = ([n], E)$ is the n-dimensional arrangement $\mathcal{A}(G)$ made of the hyperplanes $H_{i,j,0}$ for all $\{i, j\} \in E$. It follows easily from the definitions that the number of regions of $\mathcal{A}(G)$ is the number of acyclic orientations of G[7]. Also, as we now recall, the characteristic and coboundary polynomials of $\mathcal{A}(G)$ are related to the colorings of G. Recall that the *partition function of the Potts model* on G, is the unique bivariate polynomial $P_G(q, y)$ such that for all positive integer q,

$$P_G(q, y) = \sum_{f:V\to[q]} y^{\mathrm{mono}(f)},$$

where the sum is over all possible colorings of the vertices in q colors, and $\mathrm{mono}(f)$ is the number of edges of G with both endpoints of the same color. The specialization $\chi_G(q) = P_G(q, 0)$ counting the *proper colorings* of G in q colors is called *chromatic polynomial* of G. It is known [16] that for any graph G, the coboundary polynomial $P_{\mathcal{A}(G)}(q, y)$ is equal to $P_G(q, y)$, and in particular the characteristic polynomial $\chi_{\mathcal{A}(G)}(q)$ is equal to the chromatic polynomial $\chi_G(q)$.

We can now interpret the case $m = 0$ of Theorem 5.2 in terms of graphs. Let $\widehat{\mathbf{S}} = (S_{a,b})_{1\leq a\leq b\leq N}$ be a tuple such that each set $S_{a,b}$ is either empty or equal to $\{0\}$. Note that $\mathcal{A}_{\widehat{\mathbf{S}}(\mathbf{n})}$ is the G-graphical arrangement for the graph $G = G_{\widehat{\mathbf{S}}}(\mathbf{n})$ with vertex set $V(\mathbf{n})$ and edges $\{u, v\}$ for all $u = (a, i)$ and $v = (b, j)$ such that $S_{a,b} = \{0\}$. Moreover, the boxed trees in $\mathcal{U}^{(0)}(\mathbf{n})$ are simply paths with boxes partitioning the vertices, such that in each box the vertices are in increasing order. Hence $\mathcal{U}^{(0)}(\mathbf{n})$ can simply be interpreted as the set of *ordered set partitions* of $V(\mathbf{n})$. Lastly, the $\widehat{\mathbf{S}}$-energy of a subset U of $V(\mathbf{n})$ is the number of edges of $G_{\widehat{\mathbf{S}}}(\mathbf{n})$ *induced* by U (that is, edges of $G_{\widehat{\mathbf{S}}}(\mathbf{n})$ with both endpoints in U). So the right-hand side of (5.6) counts ordered partitions of $V(\mathbf{n})$ according to the number of edges it induces.

Example 5.3. Consider the case $N = 2$, $S_{1,1} = S_{22} = \emptyset$ and $S_{1,2} = \{0\}$. In this case, $G = G_{\widehat{\mathbf{S}}}(n_1, n_2)$ is the *complete bipartite graph* K_{n_1,n_2}. Hence, for a subset U of $V(n_1, n_2)$ containing k_1 vertices of the form $(1, i)$ and k_2 vertices of the form $(2, i)$, the

[7]In this natural correspondence, the direction of the edge (i, j) indicates which side of the hyperplanes $H_{i,j,0}$ the region is, or equivalently the inequality between the coordinates x_i and x_j.

number of edges of G induced by U is k_1k_2. Thus (5.6) gives

$$\begin{aligned}\sum_{(n_1,n_2)\in\mathbb{N}^2} P_{K_{n_1,n_2}}(q,y)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!} &= \left(\sum_{(n_1,n_2)\in\mathbb{N}^2}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}\sum_{\substack{(U_1,\ldots,U_r)\\ \text{ordered partition of } V(n_1,n_2)}}\prod_{i=1}^{r} y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(U_i)}\right)^{-q}\\ &= \left(\frac{1}{1+\sum_{(k_1,k_2)\in\mathbb{N}^2\setminus\{(0,0)\}}\frac{y^{k_1k_2}t_1^{k_1}t_2^{k_2}}{k_1!k_2!}}\right)^{-q}\\ &= \left(\sum_{(k_1,k_2)\in\mathbb{N}^2}\frac{y^{k_1k_2}t_1^{k_1}t_2^{k_2}}{k_1!k_2!}\right)^{q},\end{aligned}$$

where the second equality directly follows basic generating function principles upon interpreting ordered set partitions as labeled sequences of sets (see [18, Chapter 2]).

Now, for an arbitrary graph $G=([N],E)$, we consider $\widehat{\mathbf{S}}=(S_{a,b})_{1\le a\le b\le N}$ with $S_{a,a}=\{0\}$ for all $a\in[N]$, and for $a<b$, $S_{a,b}=\{0\}$ if $\{a,b\}\in E$ and $S_{a,b}=\emptyset$ otherwise. Then, (5.7) gives

$$\chi_G(q)=[t_1t_2\cdots t_N]R_{\widehat{\mathbf{S}}}(-\mathbf{t})^{-q}=(-1)^N[t_1t_2\cdots t_N]R_{\widehat{\mathbf{S}}}(\mathbf{t})^{-q}. \tag{5.10}$$

This equation relates the proper colorings of G (left-hand side) to the acyclic orientations of induced subgraphs of G (right-hand side). For instance, it gives $(-1)^N\chi_G(-1)=[t_1t_2\cdots t_N]R_{\widehat{\mathbf{S}}}(\mathbf{t})$ which is the number of regions of $\mathcal{A}_{\widehat{\mathbf{S}}(1,1,\ldots,1)}=\mathcal{A}(G)$, or equivalently the number of acyclic orientations of G. This is precisely the interpretation of $\chi_G(-1)$ given by Stanley in [42]. More generally, for $U=\{u_1,\ldots,u_k\}\subseteq[N]$, the coefficient $[t_{u_1}t_{u_2}\cdots t_{u_k}]R_{\widehat{\mathbf{S}}}(\mathbf{t})$ is by definition the number of regions of the $G[U]$-graphical arrangement, where $G[U]$ is the subgraph of G induced by U. Thus $[t_{u_1}t_{u_2}\cdots t_{u_k}]R_{\widehat{\mathbf{S}}}(\mathbf{t})$ is the number of acyclic orientations of $G[U]$. Using this interpretation, one can recover from (5.10) the interpretation of $(-1)^N\chi(-q)$ given in [42] (it counts the total number of acyclic orientations induced on the blocks of an ordered set partition of length q of the vertices of G).

For the readers with some familiarity with the theory of heaps (see for instance [45, 29]), we mention an interpretation of (5.10) in this context. Consider the heap structure associated to the graph G: the pieces are the vertices of G, and two pieces overlap if they correspond to adjacent vertices. In this context, $R_{\widehat{\mathbf{S}}}(\mathbf{t})$ can be interpreted as the generating function of the heaps of pieces associated to G (where the variable t_i counts the number of pieces associated to the vertex i of G). Indeed, the regions of $\mathcal{A}_{\widehat{\mathbf{S}}(\mathbf{n})}$ are in one-to-one correspondence with the acyclic orientations of $G_{\widehat{\mathbf{S}}(\mathbf{n})}$, which are in one-to-one correspondence with the heaps having n_i pieces associated to the vertex i of G. Hence, by [45, Proposition 5.3], $I(\mathbf{t}):=R_{\widehat{\mathbf{S}}}(-\mathbf{t})^{-1}$ is the generating function of *trivial heaps*, or equivalently, *independent sets* of G (that is, sets of non-adjacent vertices). Thus, through the theory of heaps (5.10) becomes transparent: it simply expresses the fact that a proper q-coloring of G is a q-tuple of independent sets partitioning the vertices. More generally, we get a simple expression for the generating function of chromatic polynomials $\chi_{\widehat{\mathbf{S}}}(q,\mathbf{t})$:

$$\chi_{\widehat{\mathbf{S}}}(q,\mathbf{t}) = I(\mathbf{t})^q = \left(\sum_{U\subseteq[N],\text{ independent set of }G} \prod_{i\in U} t_i \right)^q.$$

6. Generating functions

In this section, we use Theorem 5.2 in order to give equations for the generating functions $P_{\widehat{\mathbf{S}}}(q,y,\mathbf{t})$, $\chi_{\widehat{\mathbf{S}}}(q,\mathbf{t})$, and $R_{\widehat{\mathbf{S}}}(\mathbf{t})$. These equations simply translate the decomposition of boxed trees obtained by deleting the box containing the root.

6.1. A universal generating function equation.

Let $\widehat{\mathbf{S}} = (S_{a,b})_{1\le a\le b\le N}$ be an $\binom{N+1}{2}$-tuple of finite sets, and let $m = \max(|s|,\ s\in \cup S_{a,b})$. In order to characterize the generating functions associated to the arrangements $\mathcal{A}_{\widehat{\mathbf{S}}}(\mathbf{n})$, we need to define some combinatorial structures encoding admissible cadet sequences for trees in $\mathcal{T}^{(m)}(\mathbf{n})$.

Definition 6.1. A (m,N)-*configuration* of *size* $\mathbf{k}\in\mathbb{N}^N$, is a pair $\gamma = ((d_1,\dots,d_{|\mathbf{k}|-1}),(u_1,\dots,u_{|\mathbf{k}|}))$ such that $\{u_1,\dots,u_{|\mathbf{k}|}\} = V(\mathbf{k})$, $d_1,\dots,d_{|\mathbf{k}|-1}\in[0..m]$, and if $d_i=0$ then $u_i<u_{i+1}$.

We denote by $|\gamma| = \mathbf{k}$ the size of γ. The *width* of γ is $\mathrm{wid}(\gamma) = d_1+\dots+d_{|\mathbf{k}|-1}+m+1$, and the $\widehat{\boldsymbol{S}}$-*energy* of γ, denoted $\mathrm{energy}_{\widehat{\mathbf{S}}}(\gamma)$, is the number of pairs $\{i,j\}$ with $1\le i< j\le|\mathbf{k}|$ such that $\sum_{p=i}^{j-1} d_p \in S^-_{u_i,u_j}$.

Remark 6.2. To an admissible cadet sequence $(v_1,\dots,v_k)$ of a tree $T\in\mathcal{T}^{(m)}(\mathbf{n})$, we associate an (m,N)-configuration $\gamma = ((d_1,\dots,d_{k-1}),(u_1,\dots,u_k))$ defined as follows. Denoting $\mathbf{k}=(k_1,\dots,k_N)$ with $k_a = |\{i\in[n_a]\mid (a,i)\in\{v_1,\dots,v_k\}\}|$, we set

- $d_i = \mathrm{lsib}(v_{i+1})$ for all $i\in[k-1]$,
- $(u_1,\dots,u_k)$ is the unique order-preserving relabeling of $(v_1,\dots,v_k)$ in $V(\mathbf{k})$.

It is clear that γ is an (m,N)-configuration of size $\mathbf{k}$, and that $\mathrm{energy}_{\widehat{\mathbf{S}}}(\gamma)$ is the $\widehat{\mathbf{S}}$-energy of the cadet sequence $(v_1,\dots,v_k)$. Moreover, $\mathrm{wid}(\gamma)$ is the number of children of $v_1,\dots,v_k$ which are neither in $\{v_2,\dots,v_k\}$ nor right-siblings of $v_2,\dots,v_k$.

For $\mathbf{k}\in\mathbb{N}^N$, we denote by $\mathcal{C}^{(m)}(\mathbf{k})$ the set of (m,N)-configurations of size $\mathbf{k}$, and we denote by $\mathcal{C}_{\widehat{\mathbf{S}}}(\mathbf{k})$ the subset of configurations having $\widehat{\mathbf{S}}$-energy 0. We also denote $\mathcal{C}^{(m,N)} = \bigcup_{\mathbf{k}\neq(0,\dots,0)} \mathcal{C}^{(m)}(\mathbf{k})$ and $\mathcal{C}_{\widehat{\mathbf{S}}} = \bigcup_{\mathbf{k}\neq(0,\dots,0)} \mathcal{C}_{\widehat{\mathbf{S}}}(\mathbf{k})$, where the unions are over non-zero tuples in $\mathbb{N}^N$. Finally, we denote

$$\Gamma_{\widehat{\mathbf{S}}}(x,y,\mathbf{t}) = \sum_{\gamma\in\mathcal{C}^{(m,N)}} x^{\mathrm{wid}(\gamma)} y^{\mathrm{energy}_S(\gamma)} \frac{\mathbf{t}^{|\gamma|}}{|\gamma|!},$$

and

$$\Gamma_{\widehat{\mathbf{S}}}(x,\mathbf{t}) = \Gamma_{\widehat{\mathbf{S}}}(x,0,\mathbf{t}) = \sum_{\gamma\in\mathcal{C}_{\widehat{\mathbf{S}}}} x^{\mathrm{wid}(\gamma)} \frac{\mathbf{t}^{|\gamma|}}{|\gamma|!}.$$

We now state the general form of the generating function equation.

Theorem 6.3. *The generating function of coboundary polynomials $P_{\widehat{\mathbf{S}}}(q,y,\boldsymbol{t})$ (defined by (5.3)) is equal to $\widetilde{P}_{\widehat{\mathbf{S}}}(y,\boldsymbol{t})^{-q}$, where $\widetilde{P}_{\widehat{\mathbf{S}}}(y,\boldsymbol{t})$ is the unique series in $\mathbb{Q}[y][[t_1\dots,t_N]]$ satisfying*

$$\widetilde{P}_{\widehat{\mathbf{S}}}(y,\boldsymbol{t}) = 1-\Gamma_{\widehat{\mathbf{S}}}(\widetilde{P}_{\widehat{\mathbf{S}}}(y,\boldsymbol{t}),y,\boldsymbol{t}). \tag{6.1}$$

In particular, the generating function of regions $R_{\widehat{\mathbf{S}}}(\boldsymbol{t})$ (defined by (5.5)) is the unique series in $\mathbb{Q}[[t_1,\dots,t_N]]$ satisfying

$$R_{\widehat{\mathbf{S}}}(\boldsymbol{t}) = 1-\Gamma_{\widehat{\mathbf{S}}}(R_{\widehat{\mathbf{S}}}(\boldsymbol{t}),-\boldsymbol{t}). \tag{6.2}$$

Example 6.4. Let $N=2$ and $S_{1,1}=[-2..2]$, $S_{1,2}=[-1..2]$, and $S_{2,2}=\{-2,0,1,2\}$. Then we have $m=2$ and $\mathcal{C}_S=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5\}$, where $\gamma_1=((),(v_1))$, $\gamma_2=((),(v_2))$ $\gamma_3=((1),(v_2,v_3))$, $\gamma_4=((2),(v_1,v_2))$, and $\gamma_5=((2,1),(v_1,v_2,v_3))$ with $v_1=(1,1)$, $v_2=(2,1)$, $v_3=(2,2)$. Thus $\Gamma_{\widehat{\mathbf{S}}}(x,\mathbf{t})=(t_1+t_2)x^3+t_2^2x^4/2+t_1t_2x^5+t_1t_2^2x^6/2$ and

$$R_{\widehat{\mathbf{S}}}(\mathbf{t})=1+(t_1+t_2)R_{\widehat{\mathbf{S}}}(\mathbf{t})^3-t_2^2R_{\widehat{\mathbf{S}}}(\mathbf{t})^4/2-t_1t_2R_{\widehat{\mathbf{S}}}(\mathbf{t})^5+t_1t_2^2R_{\widehat{\mathbf{S}}}(\mathbf{t})^6/2.$$

This gives

$$R_{\widehat{\mathbf{S}}}(\mathbf{t})=1+t_1+t_2+3\,{t_1}^2+5\,t_1t_2+5/2\,{t_2}^2+12\,{t_1}^3+28\,{t_1}^2t_2+25\,t_1{t_2}^2+17/2\,{t_2}^3+\dots.$$

Proof. By Theorem 5.2, $P_{\widehat{\mathbf{S}}}(q,y,\mathbf{t})=\widetilde{P}_{\widehat{\mathbf{S}}}(y,\mathbf{t})^{-q}$ for

$$\widetilde{P}_{\widehat{\mathbf{S}}}(y,\mathbf{t}):=\sum_{\mathbf{n}\in\mathbb{N}^N}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}\sum_{(T,B)\in\mathcal{U}^{(m)}(\mathbf{n})}(-1)^{|B|}y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(T,B)}.$$

We now consider the decomposition boxed trees $(T,B)\in\mathcal{U}^{(m)}(\mathbf{n})$ for $|\mathbf{n}|>0$. Consider the cadet sequence $\beta=(v_1,\dots,v_k)\in B$ containing the root v_1 of T. By Remark 6.2, we can associate to β an (m,N)-configuration γ. Deleting the vertices $v_1,\dots,v_k$ of T and the right-siblings of $v_2,\dots,v_{k-1}$ (which, by definition, are leaves), gives a sequence of $\mathrm{wid}(\gamma)$ subtrees. Hence, the class $\mathcal{U}^{(m,N)}=\bigcup_{\mathbf{n}\in\mathbb{N}^N}\mathcal{U}^{(m)}(\mathbf{n})$ admits the following recursive equation

$$\mathcal{U}^{(m,N)}=1+\sum_{\gamma\in\mathcal{C}^{(m,N)}}\{\gamma\}\star\mathrm{Seq}_{\mathrm{wid}(\gamma)}(\mathcal{U}^{(m,N)}),$$

where $\star$ denotes the *product*, and Seq_ℓ denotes the ℓ*-sequences* construction for *labeled combinatorial classes* (see e.g. [18, Chapter 2]). This gives

$$\widetilde{P}_{\widehat{\mathbf{S}}}(y,\mathbf{t})=1+\sum_{\gamma\in\mathcal{C}^{(m,N)}}\left(-y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(\gamma)}\frac{\mathbf{t}^{|\gamma|}}{|\gamma|!}\right)\times\left(\widetilde{P}_{\widehat{\mathbf{S}}}(y,\mathbf{t})\right)^{\mathrm{wid}(\gamma)},$$

which is (6.1). Moreover, (6.1) implies (6.2), because (5.1) gives $R_{\widehat{\mathbf{S}}}(\mathbf{t})=\widetilde{P}_{\widehat{\mathbf{S}}}(0,-\mathbf{t})$. $\square$

6.2. **Generating functions for S-braid arrangements (case N=1).**
In this subsection, we explore in more details the case of S-braid arrangements (case $N=1$ of Theorem 6.3). When S is transitive we obtain simpler equations for the generating function of regions. We then recover and extend several classical results.

For a set of integers S, we denote $\mathcal{C}_S = \mathcal{C}_{(S)}$. Hence, $\mathcal{C}_S$ is the set of pairs $\gamma = ((d_1,\ldots,d_{k-1}),(v_1,\ldots,v_k))$ such that

- $\{v_1,\ldots,v_k\} = [k]$,
- $d_1,\ldots,d_{k-1} \in [0..m]$, where $m = \max(|s|,\ s \in S)$
- for all $0 \leq i < j \leq k$ either $v_i < v_j$ and $-\sum_{p=i}^{j-1} d_p \notin S$, or $v_i > v_j$ and $\sum_{p=i}^{j-1} d_p \notin S \cup \{0\}$.

Equation (6.2) then gives the following characterization of $R_S(t) = \sum_{n\geq 0} r_{\mathcal{A}_S(n)} \frac{t^n}{n!}$:

$$R_S(t) = 1 - \Gamma_S(R_S(t), -t), \tag{6.3}$$

where $\Gamma_S(x,t) = \sum_{\gamma\in\mathcal{C}_S} x^{\mathrm{wid}(\gamma)} \frac{t^{|\gamma|}}{|\gamma|!}$.

Example 6.5. For $S = [-3..3] \setminus \{-2,1\}$, we have $m=3$ and $\mathcal{C}_S = \{\gamma_k,\ k \geq 1\} \cup \{\gamma'\}$, where $\gamma_k = ((2,2,\ldots,2),(1,2,\ldots,k))$ and $\gamma' = ((1),(2,1))$. Thus $\Gamma_S(x,t) = \sum_{k\geq 1} t^k \frac{x^{2k+2}}{k!} + t^2x^5/2 = x^2(e^{tx^2}-1) + t^2x^5/2$ and

$$R_S(t) = 1 + R_S(t)^2(1 - e^{-tR_S(t)^2}) - t^2R_S(t)^5/2.$$

Next, we give an expression for $\Gamma_S(x,\mathbf{t})$ when S is transitive. For $k>1$ we denote by $\mathfrak{S}_k$ the set of permutations of $[k]$. For $\pi \in \mathfrak{S}_k$, we let $\mathrm{asc}(\pi) = \{i \in [k-1] \mid \pi(i) < \pi(i+1)\}$ and $\mathrm{des}(\pi) = \{i \in [k-1] \mid \pi(i) > \pi(i+1)\}$ be the number of ascents and descents of π respectively. We denote

$$\Lambda(u,v,t) = \sum_{k=1}^{\infty} \sum_{\pi\in\mathfrak{S}_k} u^{\mathrm{asc}(\pi)} v^{\mathrm{des}(\pi)} \frac{t^k}{k!}.$$

This is the generating function of the *homogeneous Eulerian polynomials.* Clearly, $\Lambda(u,u,t) = \frac{t}{1-tu}$ and $\Lambda(u,0,t) = \frac{e^{tu}-1}{u}$. More generally, it is known [13] that

$$\Lambda(u,v,t) = \frac{e^{tu} - e^{tv}}{u\,e^{tv} - v\,e^{tu}}. \tag{6.4}$$

Proposition 6.6. *If $S \subseteq \mathbb{Z}$ is transitive, and $m = \max(|s|,\ s \in S)$ then*

$$\Gamma_S(x,t) = x^{m+1}\Lambda(\mu(x),\nu(x),t), \tag{6.5}$$

where $\mu(x) = \sum_{-d\in[-m..0]\setminus S} x^d$, and $\nu(x) = \sum_{d\in[m]\setminus S} x^d$. Thus, $R_S(t)$ is the unique solution of

$$R_S(t) = 1 - R_S(t)^{m+1}\Lambda(\mu(R_S(t)),\nu(R_S(t)),-t). \tag{6.6}$$

Example 6.7. For $S = [-m..m]$, we have $\mu(x) = \nu(x) = 0$. Hence $\Gamma_S(x,t) = tx^{m+1}$, and $R_S(t) = 1 + t\,R_S(t)^{m+1}$.

Proof. Lemma 3.11 gives a simple characterization of $\mathcal{C}_S$. Namely $\gamma = ((d_1,\dots,d_{k-1}),(u_1,\dots,u_k))$ is in $\mathcal{C}_S$ if and only if $u_1,\dots,u_k$ is a permutation of $[k]$, and for all $i \in [k-1]$, $d_i \in [0..m]$ and either ($v_i < v_{i+1}$ and $-d_i \notin S$) or ($v_i > v_{i+1}$ and $d_i \notin S \cup \{0\}$). Thus, for each ascent i of the permutation $u_1,\dots,u_k$, $-d_i$ is in $[-m..0] \setminus S$, and for each descent i, d_i is in $[m] \setminus S$. This gives (6.5). ☐

Let us first consider the special case $[m] \subseteq S$. Equation (6.8) below is [37, Theorem 9.1] of Postnikov and Stanley.

Corollary 6.8. *If $[m] \subseteq S \subset [-m..m]$ and $\{s < 0, s \notin S\}$ is closed under addition, then*

$$R_S(t) = 1 + R_S(t)^{m+1}\frac{1-e^{-t\,\mu(R_S(t))}}{\mu(R_S(t))}, \tag{6.7}$$

where $\mu(x) = \sum_{-d\in[-m..0]\setminus S} x^d$. In particular, if $S = [-\ell..m]$ with $\ell \in [-1..m-1]$, then

$$R_S(t)^{m-\ell} = \exp\left(t\frac{R_S(t)^{m+1} - R_S(t)^{\ell+1}}{R_S(t)-1}\right). \tag{6.8}$$

Proof. A set S satisfying the assumptions is transitive. Moreover $\nu(x) = 0$ so that $\Lambda(\mu(x),\nu(x),t) = \frac{e^{t\mu(x)}-1}{\mu(x)}$. Thus (6.6) gives (6.7). In the particular case $S = [-\ell..m]$ we also have $\mu(x) = \frac{x^{m+1}-x^{\ell+1}}{x-1}$, and (6.7) readily gives (6.8). ☐

In the special case $S = -S$, we recover [43, Theorem 2.3] of Stanley and [43, Theorem 2.4] (written in a slightly different form) which is credited to Athanasiadis.

Corollary 6.9 ([43])**.** *If $S \subseteq \mathbb{Z}$ satisfies $0 \in S$, $\{-s,\ s \in S\} = S$, and $\mathbb{N} \setminus S$ is closed under addition, then for all $n > 0$,*

$$r_{A_S(n)} = (n-1)![x^{n-1}]\left(1 + x\sum_{d\in[0..m]\cap S}(x+1)^d\right)^n. \tag{6.9}$$

where $m = \max(|s|,\ s \in S)$. Moreover,

$$R_{S\setminus\{0\}}(t) = R_S(1-e^{-t}). \tag{6.10}$$

Proof. A set S satisfying the assumptions is transitive. Moreover, $\mu(x) = \nu(x)$, so that (6.6) becomes

$$R_S(t) = 1 + \frac{t\,R_S(t)^{m+1}}{1+t\,\nu(R_S(t))}, \tag{6.11}$$

where $\nu(x) = \sum_{d\in[m]\setminus S} x^d$. This gives $\tilde{R}(t) = t\Theta(\tilde{R}(t))$, where $\tilde{R}(t) = R_S(t) - 1$ and $\Theta(x) = (x+1)^{m+1} - x\,\nu(x+1) = 1 + x\sum_{d\in[0..m]\cap S}(x+1)^d$. Hence, Lagrange inversion formula gives (6.9). Moreover, $S \setminus \{0\}$ is also transitive, and

$$\Gamma_{S\setminus\{0\}}(x,t) = x^{m+1}\Lambda(\nu(x)+1,\nu(x),t) = x^{m+1}\frac{e^t-1}{1-(e^t-1)\nu(x)} = \Gamma_S(x,e^t-1).$$

Thus (6.6) becomes

$$R_{S\setminus\{0\}}(t) = 1 + \frac{(1-e^{-t})\, R_{S\setminus\{0\}}(t)^{m+1}}{1+(1-e^{-t})\,\nu(R_{S\setminus\{0\}}(t))}.$$

Comparing this equation with (6.11) gives (6.10). $\square$

6.3. Generating functions for transitive arrangements in the case $N > 1$.

In this subsection, we return to the general case $N \geq 1$ and consider several transitive arrangements.

Theorem 6.10. *Suppose $\widehat{\boldsymbol{S}} = (S_{a,b})_{1\leq a\leq b\leq N}$ is multi-transitive. Let $\Gamma_1(x, \boldsymbol{t}), \ldots, \Gamma_N(x, \boldsymbol{t})$ be the series defined by the system of linear equations*

(6.12)

$$\Gamma_a(x, \boldsymbol{t}) = \Lambda(\mu_{a,a}(x), \nu_{a,a}(x), t_a) \cdot \left(1 + \sum_{b=1}^{a-1} \nu_{b,a}(x)\Gamma_b(x, \boldsymbol{t}) + \sum_{b=a+1}^{N} \mu_{a,b}(x)\Gamma_b(x, \boldsymbol{t})\right),$$

where Λ is defined by (6.4), and for all $1 \leq a \leq b \leq N$, $\mu_{a,b}(x) = \sum_{-d\in[-m..0]\setminus S_{a,b}} x^d$, and $\nu_{a,b}(x) = \sum_{d\in[m]\setminus S_{a,b}} x^d$. Then $\Gamma_{\widehat{\boldsymbol{S}}}(x, \boldsymbol{t}) = x^{m+1} \sum_{a=1}^{N} \Gamma_a(x, \boldsymbol{t})$ so that

$$R_{\widehat{\boldsymbol{S}}}(\boldsymbol{t}) = 1 - R_{\widehat{\boldsymbol{S}}}(\boldsymbol{t})^{m+1} \sum_{a=1}^{N} \Gamma_a(R_{\widehat{\boldsymbol{S}}}(\boldsymbol{t}), -\boldsymbol{t}).$$

Proof. Let $\mathcal{C}_{\widehat{\mathbf{S}},a}$ be the set of configurations $\gamma = ((d_1, \ldots, d_{k-1}), (v_1, \ldots, v_k)) \in \mathcal{C}_{\widehat{\mathbf{S}}}$ such that v_1 has the form (a, i) for some i. We claim that $\Gamma_a(x, \mathbf{t})$ is the generating function of configurations in $\mathcal{C}_{\widehat{\mathbf{S}},a}$. More precisely,

$$\Gamma_a(x, \mathbf{t}) = \sum_{\gamma\in\mathcal{C}_{\widehat{\mathbf{S}}}} x^{\mathrm{wid}(\gamma)-m-1} \frac{\mathbf{t}^{|\gamma|}}{|\gamma|!}.$$

Indeed, $\mathcal{C}_{\widehat{\mathbf{S}},a}$ has a simple description (see proof of Theorem 4.6), and (6.12) simply translates the decomposition of configurations in $\mathcal{C}_{\widehat{\mathbf{S}},a}$, at the first $p \in [k]$ such that v_p has the form (b, j) with $b \neq a$ (the term 1 in (6.12) corresponds to the case where there is no such p). $\square$

As an illustration of Theorem 6.10, we treat two examples inspired by [22].

Example 6.11. Suppose first that $S_{a,a} = \{-1, 0, 1\}$ for all $a \in [N]$ and $S_{a,b} = \{-1, 0\}$ for all $1 \leq a < b \leq N$. Then (6.12) reads $\Gamma_a(x, \mathbf{t}) = t_a(1 + \sum_{b=1}^{a-1} x\Gamma_b(x, \mathbf{t}))$. This gives

$\Gamma_a(x, \mathbf{t}) = \sum_{k>0} x^{k-1} \sum_{1\leq i_1<i_2<\cdots<i_k=a} \prod_{j=1}^{k} t_{i_j}$, so that

$$\Gamma_{\widehat{\mathbf{S}}}(x, \mathbf{t}) = x \sum_{k>0} x^k \sum_{1\leq i_1<i_2<\cdots<i_k\leq N} \prod_{j=1}^{n} t_{i_j} = x\left(\prod_{a=1}^{N} 1 + t_a x\right) - x.$$

Thus, (6.6) gives

$$R_{\widehat{\mathbf{S}}}(\mathbf{t}) = \prod_{a=1}^{N} \frac{1}{1 - t_a R_{\widehat{\mathbf{S}}}(\mathbf{t})}. \tag{6.13}$$

Now consider $\widehat{\mathbf{S}}' = (S'_{a,b})_{1 \le a \le b \le N}$ with $S'_{a,a} = \{0\}$ for all $a \in [N]$ and $S'_{a,b} = \{0,1\}$ for all $1 \le a < b \le N$. Equation (6.12) reads $\Gamma_a(x,\mathbf{t}) = \dfrac{t_a}{1 - t_a x}(1 + \sum_{b=a+1}^{N} x\Gamma_b(x,\mathbf{t}))$.

hence $\Gamma_a(x,\mathbf{t}) = \sum_{k>0} x^{k-1} \sum_{a=i_1 \le i_2 \le \cdots \le i_k \le N} \prod_{j=1}^{k} t_{i_j}$, and

$$\Gamma_{\widehat{\mathbf{S}}'}(x,\mathbf{t}) = x \sum_{k>0} x^k \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le N} \prod_{j=1}^{n} t_{i_j} = x \left(\prod_{a=1}^{N} \frac{1}{1 - t_a x} \right) - x.$$

Thus, (6.6) gives

$$R_{\widehat{\mathbf{S}}'}(\mathbf{t}) = \prod_{a=1}^{N} 1 + t_a R_{\widehat{\mathbf{S}}'}(\mathbf{t}). \tag{6.14}$$

Note that (6.13) and (6.14) imply that $R_{\widehat{\mathbf{S}}}(\mathbf{t})$ and $R_{\widehat{\mathbf{S}}'}(\mathbf{t})$ are symmetric functions in $(t_1, \ldots, t_N)$, which is not obvious from the definition. This is a special case of a result proved in [22]. In fact, it follows from (5.9) that $R_{\widehat{\mathbf{S}}}(\mathbf{t}) = 1 + B(1,1,0,1,\mathbf{t})$ and $R_{\widehat{\mathbf{S}}'}(\mathbf{t}) = 1 + B(1,1,1,0,\mathbf{t})$ for the series $B(u_1,u_2,v_1,v_2,\mathbf{t})$ considered in [22] which counts trees in $\bigcup_{\mathbf{n} \in \mathbb{N}^N} \mathcal{T}^{(1)}(\mathbf{n})$ according to certain ascent and descent statistics. Accordingly, (6.13) and (6.14) are special cases of the equation given for $B(u_1,u_2,v_1,v_2,\mathbf{t})$ in [22].

We now state the extensions of Corollary 6.9 to $N > 1$.

Corollary 6.12. *Suppose that $\widehat{\boldsymbol{S}} = (S_{a,b})_{1 \le a \le b \le N}$ is multi-transitive, and that for all $1 \le a \le b \le N$, the set $S_{a,b}$ contains 0 and satisfies $\{-s,\ s \in S_{a,b}\} = S_{a,b}$. Then for all $\boldsymbol{n} \neq (0,\ldots,0)$,*

$$r_{\mathcal{A}_{\widehat{\mathbf{S}}}(\boldsymbol{n})} = [\boldsymbol{t}^{\boldsymbol{n}}] \left(\left(\sum_{a=1}^{N} t_a \right) \left(\prod_{b=1}^{N} g_b(\boldsymbol{t})^{n_b - 1} \right) \det \left(\delta_{i,j}\, g_i(\boldsymbol{t}) - t_j \frac{\partial g_i(\boldsymbol{t})}{\partial t_j} \right)_{i,j \in [N]} \right),$$

where $g_a(\boldsymbol{t}) = 1 + \sum_{b=1}^{N} t_b \sum_{d \in [0..m] \cap S_{a,b}} \left(1 + \sum_{c=1}^{N} t_c \right)^d$, and $\delta_{i,j}$ is the Kronecker delta.

Example 6.13. Let $N = 2$ and let $\widehat{\mathbf{S}} = (S_{a,b})_{1 \le a \le b \le 2}$ with $S_{1,1} = S_{2,2} = \{-1,0,1\}$ and $S_{1,2} = \{0\}$. Then $g_1(t_1,t_2) := 1 + t_1(2 + t_1 + t_2) + t_2$, $g_2(t_1,t_2) := 1 + t_2(2 + t_1 + t_2) + t_1$, the determinant is $(1 + t_1 + t_2)(1 - t_1^2 - t_2^2)$ and Corollary 6.12 gives

$$r_{\mathcal{A}_{\widehat{\mathbf{S}}}(n_1,n_2)} = [t_1^{n_1} t_2^{n_2}] \left((t_1 + t_2)(1 + t_1 + t_2)(1 - t_1^2 - t_2^2)\, g_1(t_1,t_2)^{n_1 - 1}\, g_2(t_1,t_2)^{n_2 - 1} \right).$$

Corollary 6.12 is an application of the multivariate Lagrange inversion formula [21, 24] that we now recall for the readers' convenience.

Lemma 6.14 (Lagrange inversion formula)**.** *Let $\boldsymbol{t} = (t_1, \ldots, t_N)$ be indeterminates. Let $g_1(\boldsymbol{t}), \ldots, g_N(\boldsymbol{t})$ be series in $\mathbb{C}[[\boldsymbol{t}]]$ with non-zero constant terms. Let $f_1(\boldsymbol{t}), \ldots, f_N(\boldsymbol{t})$ be the unique series in $\mathbb{C}[[\boldsymbol{t}]]$ such that for all $i \in [N]$ $f_i(\boldsymbol{t}) = t_i g_i(f_1(\boldsymbol{t}), \ldots, f_N(\boldsymbol{t}))$. Then for all $a \in [N]$ and for all tuples $\boldsymbol{n} = (n_1, \ldots, n_N) \in \mathbb{N}^N$,*

$$[\boldsymbol{t}^{\boldsymbol{n}}] f_a(\boldsymbol{t}) = [\boldsymbol{t}^{\boldsymbol{n}}] \left(t_a \cdot \left(\prod_{b=1}^{N} g_b(\boldsymbol{t})^{n_b - 1} \right) \cdot \det \left(\delta_{i,j}\, g_i(\boldsymbol{t}) - t_j \frac{\partial g_i(\boldsymbol{t})}{\partial t_j} \right)_{i,j \in [N]} \right).$$

Proof of Corollary 6.12. Since for all $a \in [N]$, $\Lambda(\mu_{a,a}(x), \nu_{a,a}(x), t_a) = \frac{t_a}{1 - t_a \nu_{a,a}(x)}$, Equation (6.12) can be rewritten as $\Gamma_a(x, \mathbf{t}) = t_a(1 + \sum_{b=1}^{N} \nu_{a,b}(x) \Gamma_b(x, \mathbf{t}))$. Hence, denoting $R(\mathbf{t}) = R_{\widehat{\mathbf{S}}}(\mathbf{t}) - 1$ and $R_a(\mathbf{t}) = -R_{\widehat{\mathbf{S}}}(\mathbf{t})^{m+1} \Gamma_a(R_{\widehat{\mathbf{S}}}(\mathbf{t}), -\mathbf{t})$, we get

$$R(\mathbf{t}) = \sum_{a=1}^{N} R_a(\mathbf{t}), \text{ and for all } a \in [N],\ R_a(\mathbf{t}) = t_a\, g_a(R_1(\mathbf{t}), \ldots, R_N(\mathbf{t})),$$

where

$$g_a(r_1, \ldots, r_N) = \left(1 + \sum_{c=1}^{N} r_c\right)^{m+1} - \sum_{b=1}^{N} r_b\, \nu_{a,b} \left(1 + \sum_{c=1}^{N} r_c\right) = 1 + \sum_{b=1}^{N} r_b \sum_{d \in [0..m] \cap S_{a,b}} \left(1 + \sum_{c=1}^{N} r_c\right)^{d}.$$

Applying Lemma 6.14 gives the result. □

Corollary 6.15. *Suppose that $\widehat{\boldsymbol{S}} = (S_{a,b})_{1 \le a \le b \le N}$ is multi-transitive, and that for some $a \in [N]$, $S_{a,a}$ contains 0 and satisfies $\{-s,\ s \in S_{a,a}\} = S_{a,a}$. Let $\widehat{\boldsymbol{S}}'$ be the same tuple as $\widehat{\boldsymbol{S}}$ except $S_{a,a}$ is replaced by $S_{a,a} \setminus \{0\}$. Then,*

$$R_{\widehat{\boldsymbol{S}}'}(\boldsymbol{t}) = R_{\widehat{\boldsymbol{S}}}(t_1, \ldots, t_{a-1}, 1 - e^{-t_a}, t_{a+1}, \ldots, t_N).$$

Proof. Let $\Gamma_1(x, \mathbf{t}), \ldots, \Gamma_N(x, \mathbf{t})$ be the series satisfying (6.12) for the tuple $\widehat{\mathbf{S}}$ and $\Gamma'_1(x, \mathbf{t}), \ldots, \Gamma'_N(x, \mathbf{t})$ their analogues for $\widehat{\mathbf{S}}'$. As in the proof of Corollary 6.9, we get for all $i \in [N]$ $\Gamma'_i(x, \mathbf{t}) = \Gamma'_i(x, \mathbf{t}')$, where $\mathbf{t}' = (t_1, \ldots, t_{a-1}, e^{t_a} - 1, t_{a+1}, \ldots, t_N)$. Thus $\Gamma_{\widehat{\mathbf{S}}'}(x, \mathbf{t}) = \Gamma_{\widehat{\mathbf{S}}}(x, \mathbf{t}')$, and $\Gamma_{\widehat{\mathbf{S}}'}(x, -\mathbf{t}) = \Gamma_{\widehat{\mathbf{S}}}(x, -\mathbf{t}'')$, for $\mathbf{t}'' = (t_1, \ldots, t_{a-1}, 1 - e^{-t_a}, t_{a+1}, \ldots, t_N)$. Hence (6.2) implies $R_{\widehat{\mathbf{S}}'}(\mathbf{t}) = R_{\widehat{\mathbf{S}}}(\mathbf{t}'')$. □

Corollary 6.16. *Suppose that $\widehat{\boldsymbol{S}} = (S_{a,b})_{1 \le a \le b \le N}$ is multi-transitive, and that for all $a < b$, $S_{a,b} = S$ for some set S containing 0 and such that $\{-s,\ s \in S\} = S$. Then*

$$\Gamma_{\widehat{\boldsymbol{S}}}(x, \boldsymbol{t}) = x^{m+1} \frac{\Delta(x, \boldsymbol{t})}{1 - \nu(x) \Delta(x, \boldsymbol{t})},$$

where $\Delta(x, \boldsymbol{t}) = \displaystyle\sum_{a=1}^{N} \frac{\Lambda(\mu_a(x), \nu_a(x), t_a)}{1 + \nu(x) \Lambda(\mu_a(x), \nu_a(x), t_a)}$, $\mu_a(x) = \displaystyle\sum_{-d \in [-m..0] \setminus S_{a,a}} x^d$, $\nu_a(x) = \displaystyle\sum_{d \in [m] \setminus S_{a,a}} x^d$, *and* $\nu(x) = \displaystyle\sum_{d \in [m] \setminus S} x^d$.

Example 6.17. Let $N = 2$ and $\widehat{\mathbf{S}} = (S_{a,b})_{1 \le a \le b \le 2}$ with $S_{1,1} = \{-1, 0, 1\}$, $S_{2,2} = \{0, 1\}$ and $S_{1,2} = \{0\}$. Then with the notation of Corollary 6.16, we have $\nu(x) = x$, $\Delta(x, \mathbf{t}) =$

$\dfrac{t_1}{1+t_1x} + \dfrac{e^{t_2x}-1}{x+(e^{t_2x}-1)x}$ and

$$\Gamma_{\widehat{\mathbf{S}}}(x,\mathbf{t}) = -\frac{x\left(2\,t_1x\mathrm{e}^{t_2x} - t_1x + \mathrm{e}^{t_2x} - 1\right)}{t_1x\mathrm{e}^{t_2x} - t_1x - 1}.$$

Proof. Equation (6.12) can be rewritten as

$$\Gamma_a(x,\mathbf{t}) = \Lambda(\mu_a(x),\nu_a(x),t_a)(1+\nu(x)(\Gamma_{\widehat{\mathbf{S}}}(x,\mathbf{t})/x^{m+1} - \Gamma_a(x,\mathbf{t})).$$

Thus $\Gamma_a(x,\mathbf{t}) = \dfrac{\Lambda(\mu_a(x),\nu_a(x),t_a)}{1+\nu(x)\Lambda(\mu_a(x),\nu_a(x),t_a)}(1+\nu(x)\Gamma_{\widehat{\mathbf{S}}}(x,\mathbf{t})/x^{m+1})$, and

$$\Gamma_{\widehat{\mathbf{S}}}(x,\mathbf{t}) = x^{m+1}\sum_{a=1}^{N}\Gamma_a(x,\mathbf{t}) = \Delta(x,\mathbf{t})\,(x^{m+1}+\nu(x)\Gamma_{\widehat{\mathbf{S}}}(x,\mathbf{t})).$$

□

Remark 6.18. Our results for the number of regions of a multi-transitive arrangement $\mathcal{A}_{\widehat{\mathbf{S}}}(\mathbf{n})$ have been derived from (5.8) using the decomposition of boxed trees in $\mathcal{U}_{\widehat{\mathbf{S}}}(\mathbf{n})$. They could alternately be obtained from (5.9) using the decomposition of trees in $\mathcal{T}_{\widehat{\mathbf{S}}}(\mathbf{n})$. When $S_{a,b} = -S_{a,b}$ for all $a,b \in [N]$ one can simply use the decomposition obtained by deleting the root. In the general case, the decomposition would correspond to deleting all the vertices in the longest cadet sequence starting at the root.

7. Proofs

As explained above, Theorem 5.2 implies Theorems 3.4 and Theorems 4.2. It remains to prove Theorem 5.2. The proof breaks into three steps corresponding to Lemmas 7.1, 7.3, and 7.5 below.

We first express the coboundary polynomial of any deformation of the braid arrangement as a weighted count of graphs. We denote by $\mathcal{G}_n$ the set of graphs (without loops nor multiple edges) with vertex set $[n]$.

Lemma 7.1. *Let $n \in \mathbb{N}$, and let $\boldsymbol{S} = (S_{u,v})_{1\le u<v\le n}$ be an $\binom{n}{2}$-tuple of finite sets of integers, and let $m = \max(|s|,\ s \in \cup S_{u,v})$. The coboundary polynomial of $\mathcal{A}_{\boldsymbol{S}}$ is*

$$P_{\mathcal{A}_{\boldsymbol{S}}}(q,y) = \sum_{G\in\mathcal{G}_n}(y-1)^{e(G)}q^{c(G)}|\mathcal{W}_{\boldsymbol{S}}(G)|,$$

where $e(G)$ and $c(G)$ are the number of edges and connected components of G respectively, and $\mathcal{W}_{\boldsymbol{S}}(G)$ is the set of tuples $(x_1,\ldots,x_n) \in \mathbb{Z}^n$ such that

- *for all edge $\{u,v\}$ of G with $u<v$, $x_u - x_v$ is in $S_{u,v}$,*
- *for all vertex v of G such that v is smallest in its connected component, $x_v = 0$.*

Example 7.2. Let $n=3$ and $S_{1,2} = S_{1,3} = \{-2,1\}$ and $S_{2,3} = \{-2,-1,1\}$. Let G be the graph with vertex set $[3]$ and edges $\{1,2\}$ and $\{2,3\}$. Then
$\mathcal{W}_{\mathbf{S}}(G) = \{(0,-1,-2),(0,-1,0),(0,-1,1),(0,2,1),(0,2,3),(0,2,4)\}$.

Proof. To a subarrangement $\mathcal{B} \subseteq \mathcal{A}_{\mathbf{S}}$, we associate the graph $G_{\mathcal{B}} \in \mathcal{G}_n$ with arcs $\{u, v\}$ for all $u < v$ such that there exists $s \in S_{u,v}$ such that $H_{a,b,s} \in \mathcal{B}$. We say that $\mathcal{B}$ is *central* if $\cap_{H \in \mathcal{B}} H \neq \emptyset$. If $\mathcal{B}$ is central, for each edge $\{u, v\}$ of $G_{\mathcal{B}}$, with $u < v$, there is a unique value $s \in S_{u,v}$ such that $H_{u,v,s} \in \mathcal{B}$, and we denote this value $\mathcal{B}(u, v)$. We also denote $\mathcal{B}(v, u) = -\mathcal{B}(u, v)$. Clearly, a point $(x_1, \ldots, x_n)$ is in $\bigcap_{H \in \mathcal{B}} H$ if and only if for any path $v_1, v_2, \ldots, v_k$ in $G_{\mathcal{B}}$,

$$x_{v_1} - x_{v_k} = \sum_{i=1}^{k-1} \mathcal{B}(v_i, v_{i+1}).$$

Hence, there is a unique point $\mathbf{x}_{\mathcal{B}} = (x_1, \ldots, x_n)$ in $\bigcap_{H \in \mathcal{B}} H$ such that $x_v = 0$ for all $v \in [n]$ such that v is the smallest vertex in its connected component of $G_{\mathcal{B}}$. Moreover, $\dim(\bigcap_{H \in \mathcal{B}} H) = \mathrm{c}(G_{\mathcal{B}})$. Note also that $\mathbf{x}_{\mathcal{B}}$ is in $\mathcal{W}_{\mathbf{S}}(G_{\mathcal{B}})$, and that $\mathcal{B}$ is uniquely determined by the pair $(G_{\mathcal{B}}, \mathbf{x}_{\mathcal{B}})$. Lastly, any pair $(G, \mathbf{x})$ where $G \in \mathcal{G}_n$ and $\mathbf{x} \in \mathcal{W}_{\mathbf{S}}(G)$ comes from a central subarrangement $\mathcal{B}$. Thus,

$$\begin{aligned} P_{\mathcal{A}_{\mathbf{S}}}(q, y) &= \sum_{\mathcal{B} \subseteq \mathcal{A}_{\mathbf{S}} \text{ central}} (y-1)^{|\mathcal{B}|} q^{\dim\left(\bigcap_{H \in \mathcal{B}} H\right)} \\ &= \sum_{(G, \mathbf{x}),\ G \in \mathcal{G}_n,\ \mathbf{x} \in \mathcal{W}_{\mathbf{S}}(G)} (y-1)^{\mathrm{e}(G)} q^{\mathrm{c}(G)} \\ &= \sum_{G \in \mathcal{G}_n} (y-1)^{\mathrm{e}(G)} q^{\mathrm{c}(G)} |\mathcal{W}_{\mathbf{S}}(G)|. \end{aligned}$$

$\square$

Our second step relates the generating function $P_{\widehat{\mathbf{S}}}(q, y, \mathbf{t})$ of coboundary polynomials, to a generating function $Z_{\widehat{\mathbf{S}}}(\delta, y, \mathbf{t})$ of tuples of integers. We fix $N > 0$ and an $\binom{N+1}{2}$-tuple $\widehat{\mathbf{S}} = (S_{a,b})_{1 \leq a < b \leq N}$ of finite sets of integers. As before, $m = \max(|s|,\ s \in \cup S_{a,b})$ and for $\mathbf{n} \in \mathbb{N}^N$ and $u, v \in V(\mathbf{n})$ with $u = (a, i)$, $v = (b, j)$ and $u < v$, we denote $S_{u,v} = S_{a,b}$, $S_{v,u} = S_{a,b}$, $S^-_{u,v} = \{s \geq 0 \mid -s \in S_{a,b}\}$, and $S^-_{v,u} = \{s > 0 \mid s \in S_{a,b}\} \cup \{0\}$.

For a positive integer δ, and $\mathbf{n} \in \mathbb{N}^N$, we denote

$$Z_{\widehat{\mathbf{S}}, \mathbf{n}}(\delta, y) = \sum_{\mathbf{x} = (x_v)_{v \in V(\mathbf{n})} \in [\delta]^{|\mathbf{n}|}} y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(\mathbf{x})},$$

where $\mathrm{energy}_{\widehat{\mathbf{S}}}(\mathbf{x})$ is number of pairs $(u, v) \in V(\mathbf{n})^2$, with $u < v$, such that $x_u - x_v \in S_{u,v}$. For instance, the $\widehat{\mathbf{S}}$-energy of the tuple $\mathbf{x}$ in Figure 7 is 1. By convention, we set $Z_{\widehat{\mathbf{S}}, (0, \ldots, 0)}(\delta, y) = 1$.

Lemma 7.3. *The generating functions $P_{\widehat{\mathbf{S}}}(q, y, \boldsymbol{t})$ and*

$$Z_{\widehat{\mathbf{S}}}(\delta, y, \boldsymbol{t}) = \sum_{\boldsymbol{n} \in \mathbb{N}^N} Z_{\widehat{\mathbf{S}}, \boldsymbol{n}}(\delta, y) \frac{\boldsymbol{t}^{\boldsymbol{n}}}{\boldsymbol{n}!},$$

are related by

$$\frac{1}{q} \log(P_{\widehat{\mathbf{S}}}(q, y, \boldsymbol{t})) = \lim_{\delta \to \infty} \frac{1}{\delta} \log(Z_{\widehat{\mathbf{S}}}(\delta, y, \boldsymbol{t})). \tag{7.1}$$

Equation (7.1) is to be understood as an identity for formal power series in $t_1, \ldots, t_N$:

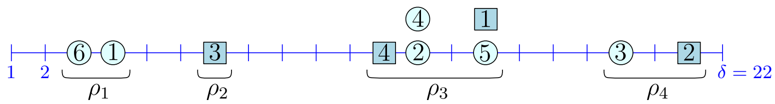


FIGURE 7. Let $\delta = 22$, $N = 2$, and $\mathbf{n} = (6,4)$. In the figure, the tuple $\mathbf{x} = (x_{1,1}, \ldots, x_{1,6}, x_{2,1}, \ldots, x_{2,4}) = (4, 13, 19, 13, 15, 3, 15, 21, 7, 12)$ is represented by indicating a "round-particle" labeled i in position $x_{1,i}$ for all $i \in [6]$, and a "square-particle" labeled i in position $x_{2,i}$ for all $i \in [4]$. For $\widehat{\mathbf{S}} = (S_{a,b})_{1 \le a \le b \le N}$ with $S_{1,1} = S_{2,2} = \{-1, 2\}$ and $S_{1,2} = \{-1, 0, 2\}$, the $\widehat{\mathbf{S}}$-energy is $\mathrm{energy}_{\widehat{\mathbf{S}}}(\mathbf{x}) = 1$, given by the pair $\{(1,5),(2,1)\}$ (because $x_{1,5} - x_{2,1} = 0 \in S_{(1,5),(2,1)} = \{-1, 0, 2\}$). The tuple $\mathbf{x}$ has four runs $\rho_1, \ldots, \rho_4$.

- *the limit is taken coefficient by coefficient in $t_1, \ldots, t_N$,*
- *for a series in formal power series $A(t_1, \ldots, t_N)$ such that $A(0, \ldots, 0) = 1$, we denote by $\log(A(\boldsymbol{t}))$ the formal power series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(A(\boldsymbol{t})-1)^n}{n}$.*

Proof. Let $\mathcal{G}_{\mathbf{n}}$ be the set of graphs with vertex set $V(\mathbf{n})$, and let $\mathcal{G}^{(N)} = \bigcup_{\mathbf{n} \in \mathbb{N}^N} \mathcal{G}_{\mathbf{n}}$. For $G \in \mathcal{G}_{\mathbf{n}}$, we denote by $\mathcal{W}_{\widehat{\mathbf{S}}}(G)$ the set of tuples $(x_v)_{v \in V(\mathbf{n})} \in \mathbb{Z}^{|\mathbf{n}|}$ such that

- for all edge $\{u, v\}$ of G with $u < v$, $x_u - x_v \in S_{u,v}$,
- for all vertex v of G which is smallest in its connected component, $x_v = 0$.

Recall that the arrangement $\mathcal{A}_{\widehat{\mathbf{S}}}(\mathbf{n})$ identifies with the arrangement $\mathcal{A}_{\widehat{\mathbf{S}}(\mathbf{n})}$, where the tuple $\widehat{\mathbf{S}}(\mathbf{n})$ is given by (5.2). Up to this identification, Lemma 7.1 gives

$$P_{\mathcal{A}_{\widehat{\mathbf{S}}}(\mathbf{n})}(q, y) = \sum_{G \in \mathcal{G}_{\mathbf{n}}} (y-1)^{\mathrm{e}(G)} q^{\mathrm{c}(G)} |\mathcal{W}_{\widehat{\mathbf{S}}}(G)|,$$

hence

$$P_{\widehat{\mathbf{S}}}(q, y, \mathbf{t}) = \sum_{\mathbf{n} \in \mathbb{N}^N} \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} \sum_{G \in \mathcal{G}_{\mathbf{n}}} (y-1)^{\mathrm{e}(G)} q^{\mathrm{c}(G)} |\mathcal{W}_{\widehat{\mathbf{S}}}(G)|. \tag{7.2}$$

We now apply the multivariate exponential formula. We think of $\mathcal{G}^{(N)}$ as the combinatorial class of graphs with N types of vertices, with the vertices of each type being *well-labeled* (that is, the n_a vertices of type a have distinct labels in $[n_a]$). The *size* of $G \in \mathcal{G}_{\mathbf{n}}$ is $\mathbf{n} = (n_1, \ldots, n_N)$ and the *weight* of G is $(y-1)^{\mathrm{e}(G)} q^{\mathrm{c}(G)} |\mathcal{W}_{\widehat{\mathbf{S}}}(G)|$. The weight is multiplicative over connected components (and unchanged by order preserving relabeling of the vertices of each type). Hence the multivariate exponential formula applies (see e.g. [44]), and taking the logarithm of $P_{\widehat{\mathbf{S}}}(q, y, \mathbf{t})$ amounts to selecting the connected graphs in $\mathcal{G}^{(N)}$. This gives,

$$\frac{1}{q} \log(P_{\widehat{\mathbf{S}}}(q, y, \mathbf{t})) = \sum_{\mathbf{n} \in \mathbb{N}^N} \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} \sum_{G \in \mathcal{G}_{\mathbf{n}} \text{ connected}} (y-1)^{\mathrm{e}(G)} |\mathcal{W}_{\widehat{\mathbf{S}}}(G)|. \tag{7.3}$$

Next, observe that

$$\begin{aligned}
|\mathcal{Z}_{\widehat{\mathbf{S}},\mathbf{n}}(\delta,y)| &= \sum_{(x_v)_{v\in V(\mathbf{n})}\in[\delta]^{|\mathbf{n}|}} \prod_{u,v\in V(\mathbf{n}),\ u<v} \left(1+(y-1)\cdot \mathbf{1}_{x_u-x_v\in S_{u,v}}\right),\\
&= \sum_{(x_v)_{v\in V(\mathbf{n})}\in[\delta]^{|\mathbf{n}|}} \sum_{G\in\mathcal{G}_\mathbf{n}} (y-1)^{\mathrm{e}(G)} \left(\prod_{\{u,v\} \text{ edge of } G,\ u<v} \mathbf{1}_{x_u-x_v\in S_{u,v}}\right),\\
&= \sum_{G\in\mathcal{G}_\mathbf{n}} (y-1)^{\mathrm{e}(G)} |\mathcal{W}_{\widehat{\mathbf{S}},\delta}(G)|,
\end{aligned}$$

where $\mathbf{1}$ is the indicator function, and $\mathcal{W}_{\widehat{\mathbf{S}},\delta}(G)$ is the set of tuples $(x_v)_{v\in V(\mathbf{n})}\in[\delta]^{|\mathbf{n}|}$ such that for all edge $\{u,v\}$ of G with $u<v$, $x_u-x_v\in S_{u,v}$. The graph weight $(y-1)^{\mathrm{e}(G)}|\mathcal{W}_{\widehat{\mathbf{S}},\delta}(G)|$ is multiplicative over connected components, hence by the multivariate exponential formula,

$$\begin{aligned}
\log(Z_{\widehat{\mathbf{S}},\delta}(\mathbf{t})) &= \log\left(\sum_{\mathbf{n}\in\mathbb{N}^N} \frac{\mathbf{t}^\mathbf{n}}{\mathbf{n}!} \sum_{G\in\mathcal{G}_\mathbf{n}} (y-1)^{\mathrm{e}(G)}|\mathcal{W}_{\widehat{\mathbf{S}},\delta}(G)|\right),\\
&= \sum_{\mathbf{n}\in\mathbb{N}^N} \frac{\mathbf{t}^\mathbf{n}}{\mathbf{n}!} \sum_{G\in\mathcal{G}_\mathbf{n},\ \text{connected}} (y-1)^{\mathrm{e}(G)}|\mathcal{W}_{\widehat{\mathbf{S}},\delta}(G)|. \qquad (7.4)
\end{aligned}$$

It only remains to prove that for any connected graph $G\in\mathcal{G}_\mathbf{n}$,

$$\lim_{\delta\to\infty}\frac{1}{\delta}|\mathcal{W}_{\widehat{\mathbf{S}},\delta}(G)| = |\mathcal{W}_{\widehat{\mathbf{S}}}(G)|. \qquad (7.5)$$

It is easy to see that, the tuples in $\mathcal{W}_{\widehat{\mathbf{S}},\delta}(G)$ are translations of tuples in $\mathcal{W}_{\widehat{\mathbf{S}}}(G)$, and the number of translations is of order δ. More precisely,

$$\mathcal{W}_{\widehat{\mathbf{S}},\delta}(G)=\{(x_v+\theta)_{v\in V(\mathbf{n})}\mid (x_v)_{v\in V(\mathbf{n})}\in W_{\widehat{\mathbf{S}}}(G),\ \text{and } 1-\min(x_v)_{v\in V(\mathbf{n})}\leq\theta\leq\delta-\max(x_v)_{v\in V(\mathbf{n})}\}.$$

The tuples above are all distinct, and for any $(x_v)_{v\in V(\mathbf{n})}\in W_{\widehat{\mathbf{S}}}(G)$, $\max(x_v)_{v\in V(\mathbf{n})}-\min(x_v)_{v\in V(\mathbf{n})}\leq m\cdot|\mathbf{n}|$. Thus

$$(\delta-m\cdot|\mathbf{n}|)\,|\mathcal{W}_{\widehat{\mathbf{S}}}(G)|\leq|\mathcal{W}_{\widehat{\mathbf{S}},\delta}(G)|\leq\delta\,|\mathcal{W}_{\widehat{\mathbf{S}}}(G)|.$$

This shows (7.5), and completes the proof. □

Remark 7.4. The proof of Lemma 7.3 is reminiscent of Mayers' theory of cluster integrals (see e.g. [12, 30, 32]). In this perspective, the right-hand side of (7.1) corresponds to the pressure of the infinite volume limit of a discrete gas model (where particles of type a and b interacts according to a soft-core potential of shape $S_{a,b}$, and energy of interaction y). Alexander Postnikov also pointed out to the author that Lemma 7.3 could alternatively be obtained by using the finite field method pioneered by Athanasiadis [6] and adapted to the calculation of coboundary polynomials in [1]. However, the situation of deformed braid arrangement is distinguished by the fact that the parameter q appears merely as an exponent of the generating function $P_{\widehat{\mathbf{S}}}(q,y,\mathbf{t})$; see (5.7). This fact, which is a direct consequence of Lemma 7.1 and already appears in [43, Theorem 1.2], allows one to focus the remaining analysis on a single value of q, namely $+\infty$.

Our last step, relates the generating function $Z_{\widehat{\mathbf{S}}}(\delta, y, \mathbf{t})$ of point configurations to the generating function of boxed trees.

Lemma 7.5. *Let*

$$U_{\widehat{\boldsymbol{S}}}(y, \boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{N}^N} \frac{\boldsymbol{t}^{\boldsymbol{n}}}{\boldsymbol{n}!} \sum_{(T,B)\in\mathcal{U}^{(m)}(\boldsymbol{n})} (-1)^{|B|} y^{\mathrm{energy}_{\widehat{\boldsymbol{S}}}(T,B)},$$

and

$$U^{\bullet}_{\widehat{\boldsymbol{S}}}(y, \boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{N}^N} \frac{\boldsymbol{t}^{\boldsymbol{n}}}{\boldsymbol{n}!} \sum_{(T,B)\in\mathcal{U}^{(m)}_{\boldsymbol{S}}(\boldsymbol{n})} (|B| + \mathrm{leaf}(T))(-1)^{|B|} y^{\mathrm{energy}_{\widehat{\boldsymbol{S}}}(T,B)},$$

where $\mathrm{leaf}(T)$ *is the number leaves of* T. *For all* $\boldsymbol{n} \in \mathbb{N}^N$, *and for all* $\delta > m \cdot |\boldsymbol{n}|$,

$$Z_{\widehat{\boldsymbol{S}},\boldsymbol{n}}(\delta, y) = [\boldsymbol{t}^{\boldsymbol{n}}] U_{\widehat{\boldsymbol{S}}}(y, \boldsymbol{t})^{-\delta-m-2} U^{\bullet}_{\widehat{\boldsymbol{S}}}(y, \boldsymbol{t}). \tag{7.6}$$

We will use a counting result about tuples of plane trees. We recall that the *prefix order* of the vertices of a rooted plane tree is the total order for which any vertex v is less than its children and all the descendants of v are less than the right siblings of v.

Claim 7.6. *Let* $\alpha, w_1 \dots, w_r$ *be positive integers. Let* $\tau(\alpha; w_1, \dots, w_r)$ *be the set of tuples* $(T_1, \dots, T_\alpha)$, *where* $T_1, \dots, T_\alpha$ *are rooted plane trees, and* T_α *has a marked vertex, such that, denoting* $c_{i,1}, \dots, c_{i,k_i}$ *the number of children of the nodes of* T_i *in prefix order, one has*

$$(c_{1,1}, \dots, c_{1,k_1}, c_{2,1}, \dots, c_{2,k_2}, \dots, c_{\alpha,1}, \dots, c_{\alpha,k_\alpha}) = (w_1, \dots, w_r).$$

Then,

$$|\tau(\alpha; w_1, \dots, w_r)| = \binom{\alpha + w_1 + \cdots + w_r}{r}.$$

Proof. The proof is represented in Figure 8. Let $\mathcal{P}$ be the set of lattice paths on $\mathbb{Z}$ starting at 0, and having every step greater or equal to -1. Let $\mathcal{P}_{-1}$ be the set of paths in $\mathcal{P}$ ending at -1, and let $\mathcal{P}^+_{-1} \subset \mathcal{P}_{-1}$ be the subset of paths remaining non-negative until the last step. Recall that the map ϕ which associates to a rooted plane tree T the path P with steps $c_1 - 1, c_2 - 1, \dots, c_n - 1$ where $c_1, \dots, c_n$ are the number of children of the vertices of T taken in prefix order is a bijection between rooted plane trees and $\mathcal{P}^+_{-1}$ (see e.g. [44, Chapter 5.3]). Moreover by the cycle lemma, there is a n-to-1 correspondence between the paths with n steps in $\mathcal{P}_{-1}$ and the paths with n steps in $\mathcal{P}^+_{-1}$. Thus the map ϕ induces a bijection between $\mathcal{P}_{-1}$ and rooted plane trees with a marked vertex.

Now let $\mathcal{P}(\alpha; w_1, \dots, w_r)$ be the set of path in $\mathcal{P}$ having $\alpha + w_1 + \ldots + w_r - r$ steps -1, and r non-negative steps $w_1 - 1, \dots, w_r - 1$ in this order. Clearly, $|\mathcal{P}(\alpha; w_1, \dots, w_r)| = \binom{\alpha + w_1 + \cdots + w_r}{r}$, and paths in $\mathcal{P}(\alpha; w_1, \dots, w_r)$ ends at $-\alpha$. We consider the decomposition of paths in $\mathcal{P}(\alpha; w_1, \dots, w_r)$ at the first time they reach $-1, -2, \dots, -\alpha + 1$, as represented in Figure 8. This gives a bijection between $\mathcal{P}(\alpha; w_1, \dots, w_r)$ and the set of tuples $(P_1, \dots, P_\alpha)$ such that $P_1, \dots, P_{\alpha-1} \in \mathcal{P}^+_{-1}$, $P_\alpha \in \mathcal{P}_{-1}$ and there is a total of $\alpha + w_1 + \ldots + w_r - r$ steps -1, and r non-negative steps $w_1, \dots, w_r$ in this order. Combining this decomposition with ϕ gives a bijection between $\mathcal{P}(\alpha; w_1, \dots, w_r)$ and $\tau(\alpha; w_1, \dots, w_r)$, thereby proving the claim. $\square$

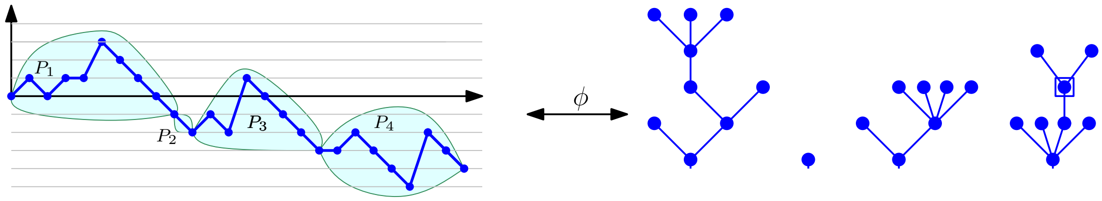


FIGURE 8. A path P in $\mathcal{P}(4;2,2,1,3,2,4,1,2,4)$, and its decomposition into four paths $P_1,\ldots,P_4$ with $P_1,P_2,P_3 \in \mathcal{P}_{-1}^{+}$, and P_4 in $\mathcal{P}_{-1}$. Here the vertical direction corresponds to the value of the path, while the horizontal direction represents time.

Proof of Lemma 7.5. We will relate both sides of (7.6) to the generating function of (m,N)-configurations (see Definition 6.1). Let $(T,B) \in \mathcal{U}_{\mathbf{S}}^{(m)}(\mathbf{n})$. We associate to (T,B) a rooted plane tree R obtained by contracting each box into a node. More precisely, if $\beta = (v_1,\ldots,v_k)$ is a cadet sequence in B, then we delete all the right siblings of the nodes $v_2,\ldots,v_k$ (these are leaves of T) and contract the edges (v_i,v_{i+1}) of all $i \in [k-1]$. If β corresponds to the pth node of R in prefix order of R, we denote $L_p = \{v_1,\ldots,v_k\}$ and we denote by γ_p the (m,N)-configuration corresponding to β in the sense of Remark 6.2. Clearly, the boxed tree (T,B) is uniquely determined by the triple $(R,(\gamma_1,\ldots,\gamma_{|B|}),(L_1,\ldots,L_{|B|}))$. Hence, the boxed trees $(T,B) \in \mathcal{U}_{\mathbf{S}}^{(m)}$ are in bijection with triples $(R,(\gamma_1,\ldots,\gamma_n),(L_1,\ldots,L_n))$, where

- R is a rooted plane tree, and n is the number of nodes of R,
- $\gamma_1,\ldots,\gamma_n$ are (m,N)-configurations of width $\mathrm{child}(v_1),\ldots,\mathrm{child}(v_n)$ respectively, where $v_1,\ldots,v_n$ are the nodes of R in prefix order, and $\mathrm{child}(v)$ is the number of children of the node v (including leaves),
- and $(L_1,\ldots,L_n)$ is a set partition of $V(|\gamma_1|+\cdots+|\gamma_n|)$ such that for all $p \in [n]$, $|\gamma_p| = (k_{p,1},\ldots,k_{p,N})$, where $k_{p,a} = |\{i \mid (a,i) \in L_p\}|$.

Lastly, there are

$$\frac{(|\gamma_1|+\cdots+|\gamma_n|)!}{\prod_{p=1}^{n}|\gamma_p|!}$$

ways to choose the set partition $(L_1,\ldots,L_n)$. Hence

$$U_{\mathbf{S}}(y,\mathbf{t}) = \sum_{R\in\mathcal{R}} \prod_{v \text{ node of } R} \left(- \sum_{\gamma\in\mathcal{C}^{(m,N)} \ | \ \mathrm{wid}(\gamma)=\mathrm{child}(v)} y^{\mathrm{energy}_{\mathbf{S}}(\gamma)} \frac{\mathbf{t}^{|\gamma|}}{|\gamma|!} \right),$$

where $\mathcal{R}$ is the set of rooted plane trees and $\mathcal{C}^{(m,N)}$ is the set of (m,M)-configurations. Similarly,

$$U_{\mathbf{S}}^{\bullet}(y,\mathbf{t}) = \sum_{R\in\mathcal{R}} \mathrm{v}(R) \prod_{v \text{ node of } R} \left(- \sum_{\gamma\in\mathcal{C}^{(m,N)} \ | \ \mathrm{wid}(\gamma)=\mathrm{child}(v)} y^{\mathrm{energy}_{\mathbf{S}}(\gamma)} \frac{\mathbf{t}^{|\gamma|}}{|\gamma|!} \right),$$

where $\mathrm{v}(R)$ is the number of vertices of R.

Next, we express $Z_{\widehat{\mathbf{S}},\mathbf{n}}(\delta, y)$ in terms of (m, N)-configurations. Let $\mathbf{n} \in \mathbb{N}^N$, and let $\mathbf{x} = (x_v)_{v \in V(\mathbf{n})} \in [\delta]^{|\mathbf{n}|}$. Intuitively, we think of each coordinate x_v as the position of a particle in the space $[\delta]$, and we will distinguish *runs* which are groups of particles that are close to one another. This is represented in Figure 7. Let $v'_1, \ldots, v'_{|\mathbf{n}|} \in V(\mathbf{n})$ be defined by $\{v'_1, \ldots, v'_{|\mathbf{n}|}\} = V(\mathbf{n})$, and the conditions $x_{v'_i} \leq x_{v'_{i+1}}$, and if $x_{v'_i} = x_{v'_{i+1}}$ then $v'_i < v'_{i+1}$. We denote $d_i = x_{v'_{i+1}} - x_{v'_i}$ for all $i \in [|\mathbf{n}| - 1]$, and adopt the convention $d_0 = d_{|\mathbf{n}|} = \infty$. A *run* of $\mathbf{x}$ is a subsequence $\rho = (v'_i, v'_{i+1}, \ldots, v'_j)$, with $1 \leq i \leq j \leq |\mathbf{n}|$, such that $d_{i-1} > m$, $d_j > m$, and for all $i \leq p < j$, $d_p \leq m$. We define the *position* of ρ as $\mathrm{pos}(\rho) = x_{v'_i}$, the *width* of ρ as $\mathrm{wid}(\rho) = x_{v'_j} - x_{v'_i} + m + 1$, the *labels* of ρ as $\mathrm{lab}(\rho) = \{v'_i, \ldots, v'_j\}$, and the *size* of ρ as $|\rho| = \mathbf{k} = (k_1, \ldots, k_N)$ where $k_a = |\{i \mid (a, i) \in \mathrm{lab}(\rho)\}|$. For instance, the tuple $\mathbf{x}$ represented in Figure 7 has four runs having position 3, 7, 12, and 19 respectively, width 4, 3, 6, and 5 respectively, and size $(2, 0)$, $(0, 1)$, $(3, 2)$, and $(1, 1)$ respectively. Lastly, the *configuration* of ρ is $\mathrm{config}(\rho) = ((d_i, d_{i+1}, \ldots, d_{j-1}), (u_i, \ldots, u_j))$, where $(u_i, \ldots, u_j)$ is the unique order preserving relabeling of $(v'_i, \ldots, v'_j)$ in $V(\mathbf{k})$. For instance, in Figure 7, $\mathrm{config}(\rho_3) = ((1, 0, 2, 0), ((2, 2), (1, 1), (1, 2), (1, 3), (2, 1)))$. Note that $\gamma = \mathrm{config}(\rho)$ is in $\mathcal{C}^{(m)}(\mathbf{k})$, and $\mathrm{wid}(\rho) = \mathrm{wid}(\gamma)$. Moreover it is easy to see that

$$\mathrm{energy}_{\widehat{\mathbf{S}}}(\mathbf{x}) = \sum_{i=1}^{r} \mathrm{energy}_{\widehat{\mathbf{S}}}(\mathrm{config}(\rho_i)),$$

where $\rho_1, \ldots, \rho_r$ are the runs of $\mathbf{x}$ (because pairs of particles at distance greater than m do not contribute to the $\widehat{\mathbf{S}}$-energy). Moreover, the tuple $\mathbf{x}$ is completely determined by the positions, labels, and configurations of its runs $\rho_1, \ldots, \rho_r$. The configurations of the runs are arbitrary, and given the configurations $\gamma_1, \ldots, \gamma_r$ of the runs there are

$$\binom{\delta + m + r - \mathrm{wid}(\gamma_1) - \ldots - \mathrm{wid}(\gamma_r)}{r}$$

ways to choose the positions $(p_1, \ldots, p_r) \in [\delta]^r$ (since the only constraints are $p_i + \mathrm{wid}(\gamma_i) \leq p_{i+1}$ for all $i \in [r-1]$, and $p_r + \mathrm{wid}(\gamma_r) \leq \delta + m + 1$). Also, there are $\dfrac{\mathbf{n}!}{\prod_{i=1}^{r} |\gamma_i|!}$ ways to choose the labels. Thus,

(7.7)
$$Z_{\widehat{\mathbf{S}},\mathbf{n}}(\delta, y) = \mathbf{n}! \sum_{r=0}^{\infty} \sum_{\substack{\gamma_1, \ldots, \gamma_r \in \mathcal{C}^{(m,N)} \\ |\gamma_1| + \cdots + |\gamma_r| = \mathbf{n}}} \binom{\delta + m + r - \mathrm{wid}(\gamma_1) - \cdots - \mathrm{wid}(\gamma_r)}{r} \prod_{i=1}^{r} \frac{y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(\gamma_i)}}{|\gamma_i|!}.$$

In order to prove (7.6), we will now consider negative values of δ. Let us denote $\mathrm{Pol}_r(x) = \frac{x(x-1)\cdots(x-r+1)}{r!}$. This is a polynomial in x, such that for all $x \in \mathbb{N}$, $\binom{x}{r} = \mathrm{Pol}_r(x)$. Let

(7.8)
$$\widetilde{Z}_{\widehat{\mathbf{S}},\mathbf{n}}(\delta, y) = \mathbf{n}! \sum_{r=0}^{\infty} \sum_{\substack{\gamma_1, \ldots, \gamma_r \in \mathcal{C}^{(m,N)} \\ |\gamma_1| + \cdots + |\gamma_r| = \mathbf{n}}} \mathrm{Pol}_r\left(\delta + m + r - \mathrm{wid}(\gamma_1) - \cdots - \mathrm{wid}(\gamma_r)\right) \prod_{i=1}^{r} \frac{y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(\gamma_i)}}{|\gamma_i|!}.$$

This is a polynomial in δ and y which coincides with $Z_{\widehat{\mathbf{S}},\mathbf{n}}(\delta, y)$ for all integer $\delta > m\cdot|\mathbf{n}|$, because any $\gamma \in \mathcal{C}^{(m,N)}$ satisfies $\mathrm{wid}(\gamma) - 1 \leq m \cdot |\gamma|$. It remains to prove that for all δ,

$$\widetilde{Z}_{\widehat{\mathbf{S}},\mathbf{n}}(\delta, y) = [\mathbf{t}^{\mathbf{n}}]U_{\widehat{\mathbf{S}}}(y, \mathbf{t})^{-\delta-m-2}U^{\bullet}_{\widehat{\mathbf{S}}}(y, \mathbf{t}). \tag{7.9}$$

We observe that both sides of (7.9) are polynomials in δ. Indeed, $U_{\widehat{\mathbf{S}}}(y, (0, \dots, 0)) = 1$, hence the series

$$U_{\widehat{\mathbf{S}}}(y, \mathbf{t})^{-\delta} = \exp(-\delta \log(U_{\mathbf{S}}(y, \mathbf{t})) = \exp\left(\delta \sum_{k\geq 1} \frac{(1 - U_S(\mathbf{t}))^k}{k}\right)$$

has coefficients which are polynomial in δ. Thus, in order to prove (7.9), it suffices to prove it for infinitely many values of $\delta \in \mathbb{C}$. Let α be a positive integer, let $\delta = -m - 1 - \alpha$, and let

$$\widetilde{Z}_{\widehat{\mathbf{S}}}(\delta, y, \mathbf{t}) \quad = \quad \sum_{\mathbf{n}\in\mathbb{N}^N} \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} \widetilde{Z}_{\widehat{\mathbf{S}},\mathbf{n}}(\delta, y).$$

We have

$$\begin{aligned}
\widetilde{Z}_{\widehat{\mathbf{S}}}(\delta, y, \mathbf{t}) &= \sum_{r=0}^{\infty} \sum_{\gamma_1,\dots,\gamma_r\in\mathcal{C}^{(m,N)}} \mathrm{Pol}_r\left(r - 1 - \alpha - \mathrm{wid}(\gamma_1) - \cdots - \mathrm{wid}(\gamma_r)\right) \prod_{i=1}^{r} \frac{y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(\gamma_i)} t^{|\gamma_i|}}{|\gamma_i|!} \\
&= \sum_{r=0}^{\infty} \sum_{\gamma_1,\dots,\gamma_r\in\mathcal{C}^{(m,N)}} (-1)^r \binom{\alpha + \mathrm{wid}(\gamma_1) + \cdots + \mathrm{wid}(\gamma_r)}{r} \prod_{i=1}^{r} \frac{y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(\gamma_i)} t^{|\gamma_i|}}{|\gamma_i|!}.
\end{aligned}$$

Using Claim 7.6 gives

$$\begin{aligned}
\widetilde{Z}_{\widehat{\mathbf{S}}}(\delta, y, \mathbf{t}) &= \sum_{r=0}^{\infty} (-1)^r \sum_{\gamma_1,\dots,\gamma_r\in\mathcal{C}^{(m,N)}} \tau(\alpha; \mathrm{wid}(\gamma_1), \dots, \mathrm{wid}(\gamma_r)) \prod_{i=1}^{r} \frac{y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(\gamma_i)} t^{|\gamma_i|}}{|\gamma_i|!} \\
&= \left(\sum_{R\in\mathcal{R}} \prod_{v \text{ node of } R} - \sum_{\gamma\in\mathcal{C}^{(m,N)},\ \mathrm{wid}(\gamma)=\mathrm{child}(v)} \frac{y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(\gamma)} t^{|\gamma|}}{|\gamma|!}\right)^{\alpha-1} \\
&\quad \times \left(\sum_{R\in\mathcal{R}} \mathrm{v}(R) \prod_{v \text{ node of } R} - \sum_{\gamma\in\mathcal{C}^{(m,N)},\ \mathrm{wid}(\gamma)=\mathrm{child}(v)} \frac{y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(\gamma)} t^{|\gamma|}}{|\gamma|!}\right) \\
&= U_{\widehat{\mathbf{S}}}(y, t)^{-\delta-m-2} U^{\bullet}_{\widehat{\mathbf{S}}}(y, t).
\end{aligned}$$

for all $\delta \leq -m - 2$. Thus, $\widetilde{Z}_{\widehat{\mathbf{S}}}(\delta, y, \mathbf{t}) = U_{\widehat{\mathbf{S}}}(y, t)^{-\delta-m-2} U^{\bullet}_{\widehat{\mathbf{S}}}(y, t)$ for all $\delta \in \mathbb{C}$, and extracting the coefficient of $\mathbf{t}^{\mathbf{n}}$ gives (7.9). $\square$

We can now complete the proof of Theorem 5.2. By Lemma 7.5,

$$\lim_{\delta\to\infty} \frac{Z_{\widehat{\mathbf{S}},\delta}(\delta, y, \mathbf{t})}{U_{\widehat{\mathbf{S}}}(y, \mathbf{t})^{-\delta-m-2} U^{\bullet}_{\widehat{\mathbf{S}}}(y, \mathbf{t})} = 1,$$

where the limit is taken coefficient by coefficient in $\mathbf{t}$. Thus, by Lemma 7.3,

$$\begin{aligned}
\frac{1}{q}\log(P_{\widehat{\mathbf{S}}}(q,y,\mathbf{t})) &= \lim_{\delta\to\infty}\frac{1}{\delta}\log(Z_{\widehat{\mathbf{S}},\delta}(\delta,y,\mathbf{t}))\\
&= \lim_{\delta\to\infty}\frac{-\delta-m-2}{\delta}\log\left(U_{\widehat{\mathbf{S}}}(y,\mathbf{t})\right)+\lim_{\delta\to\infty}\frac{1}{\delta}\log\left(U^{\bullet}_{\widehat{\mathbf{S}}}(y,\mathbf{t})\right)\\
&\quad+\lim_{\delta\to\infty}\log\left(\frac{Z_{\widehat{\mathbf{S}},\delta}(\delta,y,\mathbf{t})}{U_{\widehat{\mathbf{S}}}(y,\mathbf{t})^{-\delta-m-2}U^{\bullet}_{\widehat{\mathbf{S}}}(y,\mathbf{t})}\right)\\
&= -\log(U_{\widehat{\mathbf{S}}}(y,\mathbf{t})).
\end{aligned}$$

Hence,

$$P_{\widehat{\mathbf{S}}}(q,y,\mathbf{t}) = (U_{\widehat{\mathbf{S}}}(y,\mathbf{t}))^{-q} = \left(\sum_{\mathbf{n}\in\mathbb{N}^N}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}\sum_{(T,B)\in\mathcal{U}^{(m)}(\mathbf{n})}(-1)^{|B|}y^{\mathrm{energy}_{\widehat{\mathbf{S}}}(T,B)}\right)^{-q}.$$

This gives (5.6), and setting $y=0$ gives (5.8). Lastly, in the case where $\widehat{\mathbf{S}}$ is multi-transitive, (5.9) follows from (5.8) via (4.2).

8. A simple bijection for regions of transitive deformations of the braid arrangement

In this section we present a simple bijection between regions of the arrangement $\mathcal{A}_{\mathbf{S}}$ and the trees in $\mathcal{T}_{\mathbf{S}}(n)$ for any transitive tuple $\mathbf{S}$. We first recall a bijection between the regions of $\mathcal{A}^{(m)}(n)$ and labeled parenthesis systems in Subsection 8.1, and combine it with a convenient bijection between parenthesis systems and trees. Then, we treat the cases of the Shi, semiorder and Linial arrangements in Subsection 8.2, before treating the general case in Subsection 8.3.

8.1. Preliminary: bijection between regions of $\mathcal{A}_{[-m..m]}(n)$ and $\mathcal{T}^{(m)}(n)$.

We first recall a classical encoding of the regions of $\mathcal{A}_{[-m..m]}(n)$ by labeled, non-nesting, parenthesis systems. An example is represented in Figure 9.

A *m-parenthesis system* of *size* n is a word w on the alphabet $\{\alpha,\beta\}$ with n letters α and mn letters β, such that no prefix of w contains more β's than m times the number of α's. It is well known that there are $\mathrm{Cat}^{(m)}(n)=\frac{((m+1)n)!}{n!(mn+1)!}$ m-parenthesis systems of size n, and that such words bijectively encode rooted plane $(m+1)$-ary trees with n nodes (see e.g. [44, Chapter 5.3]). A *m-sketch* of size n is a word $\tilde{w}$ obtained from a parenthesis system w by replacing the ith letter α by the letter $\alpha_{\pi(i)}$ for some permutation π of $[n]$. We denote by $\tilde{\mathcal{D}}^{(m)}(n)$ the set of m-sketches of size n. Clearly, $|\tilde{\mathcal{D}}^{(m)}(n)| = n!\mathrm{Cat}^{(m)}(n)=\frac{((m+1)n)!}{(mn+1)!}=|\mathcal{T}^{(m)}(n)|$. We now describe bijections between the regions of $\mathcal{A}_{[-m..m]}(n)$, and the sets $\tilde{\mathcal{D}}^{(m)}(n)$ and $\mathcal{T}^{(m)}(n)$. The case $m=1$, $n=3$ is represented in Figures 10 and 13.

We first need to *annotate* our sketches. Let $A^{(m)}(n)$ be the alphabet made of the $(m+1)n$ letters $\{\alpha_i^{(s)}\mid i\in[n], s\in[0..m]\}$. We call *$\alpha$-letters* the letters $\alpha_i^{(0)}$ for $i\in[n]$, and *β-letters* the letters $\alpha_i^{(s)}$ for $i\in[n], s\in[m]$. For a word $\hat{w}$ on the alphabet $A^{(m)}(n)$, we say that the letter $\alpha_i^{(s)}$ is *active* in $\hat{w}$ if $s<m$, and $\alpha_i^{(s)}$ appears in $\hat{w}$ but $\alpha_i^{(s+1)}$ does

not. The *annotation* of a sketch $\tilde{w} = \tilde{w}_1 \cdots \tilde{w}_{(m+1)n}$ is the word $\hat{w} = \hat{w}_1 \cdots \hat{w}_{(m+1)n}$ obtained by applying the following rule for $p = 1, 2, \ldots, (m+1)n$: if $\tilde{w}_p = \alpha_i$ then set $\hat{w}_p = \alpha_i^{(0)}$, while if $\tilde{w}_p = \beta$ then set $\hat{w}_p = \alpha_i^{(s+1)}$, where $\alpha_i^{(s)}$ is the first active letter in $\hat{w}_1 \cdots \hat{w}_{p-1}$ (it is easy to see that there is always such a letter). We denote by $\mathcal{D}^{(m)}(n)$ the set of annotated m-sketches of size n. It is easy to see that a word $\hat{w} = \hat{w}_1 \cdots \hat{w}_{(m+1)n}$ is in $\mathcal{D}^{(m)}(n)$ if and only if

(a) $\{\hat{w}_1, \ldots, \hat{w}_{(m+1)n}\} = A^{(m)}(n)$,
(b) for all $i \in [n]$ and all $s \in [m]$, the letter $\alpha_i^{(s-1)}$ appears before $\alpha_i^{(s)}$,
(c) for all $i, j \in [n]$ and all $s, t \in [m]$, if $\alpha_i^{(s-1)}$ appears before $\alpha_j^{(t-1)}$, then $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$.

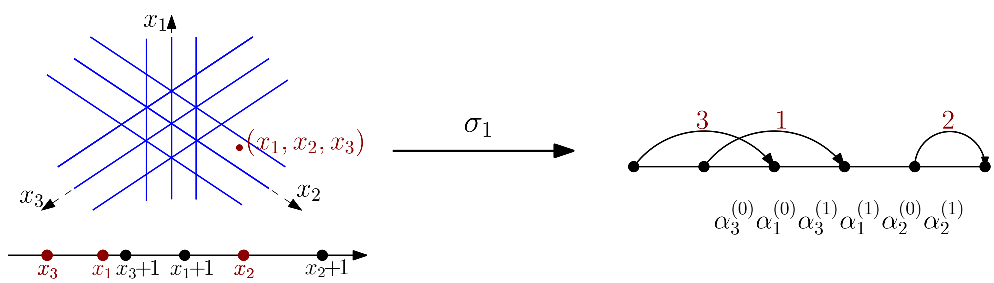


FIGURE 9. The mapping σ_1 associating an annotated 1-sketch to any point $(x_1, \ldots, x_n)$ in $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}_{[-1..1]}(n)} H$. Graphically, the annotated 1-sketch $\hat{w} = \hat{w}_1 \cdots \hat{w}_{2n}$ is represented by a set of non-nesting parentheses on $2n$-points corresponding to the letters. More precisely, if the letters $\alpha_i^{(0)}$ to $\alpha_i^{(1)}$ are in position p and q of $\hat{w}$, then parenthesis labeled i goes from the pth point to the qth point.

We now associate an annotated m-sketch of size n to each region of $\mathcal{A}_{[-m..m]}(n)$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a point in $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}_{[-m..m]}(n)} H$. Observe that the condition $\mathbf{x} \notin \bigcup_{H \in \mathcal{A}_{[-m..m]}(n)} H$ is equivalent to the fact that the numbers $\{x_i + s \mid i \in [n], s \in [0..m]\}$ are all distinct. We define $z_1, \ldots, z_{(m+1)n}$ by the conditions $z_1 < \ldots < z_{(m+1)n}$ and $\{z_1, \ldots, z_{(m+1)n}\} = \{x_i + s \mid i \in [n], s \in [0..m]\}$. Then, we define $\sigma_m(\mathbf{x}) = \hat{w}_1 \hat{w}_2 \cdots \hat{w}_{(m+1)n}$, where $\hat{w}_p = \alpha_i^{(s)}$ if $z_p = x_i + s$. Here are basic properties of the mapping σ_m.

(i) For any $\mathbf{x} \notin \bigcup_{H \in \mathcal{A}_{[-m..m]}(n)} H$, the word $\sigma_m(\mathbf{x})$ is an annotated m-sketch. Indeed, it clearly satisfies the conditions (a), (b), (c).
(ii) The mapping σ_m is constant over each region of $\mathcal{A}_{[-m..m]}(n)$. Indeed, the order of the numbers $\{x_i + s \mid i \in [n], s \in [0..m]\}$ cannot change when $\mathbf{x}$ moves continuously inside $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}_{[-m..m]}(n)} H$.
(iii) The sketch $\sigma_m(\mathbf{x})$ identifies the region containing $\mathbf{x}$. Indeed, for all $i, j \in [n]$, and all $s \in [0..m]$, $x_i - x_j < s$ if $\alpha_i^{(0)}$ appears before $\alpha_j^{(s)}$ in $\sigma_m(\mathbf{x})$ and $x_i - x_j > s$ otherwise.

(iv) For any annotated m-sketch $\hat{w} = \hat{w}_1 \cdots \hat{w}_{(m+1)n}$, there exists $\mathbf{x} \in \mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}_{[-m..m]}(n)} H$ such that $\sigma(\mathbf{x}) = \hat{w}$. Indeed, one can simultaneously define $\mathbf{x} \in \sigma^{-1}(\hat{w})$ and $z_1, \ldots, z_{(m+1)n}$ by applying the following rules for $p = 1, 2, \ldots, (m+1)n$: if $\hat{w}_p = \alpha_i^{(0)}$, then set $z_p = z_{p-1} + 1/2^p$ (with $z_0 = 0$) and set $x_i = z_p$, while if $\hat{w}_p = \alpha_i^{(s)}$ with $s \neq 0$, then set $z_p = x_i + s$.

Properties (i) and (ii) show that σ_m is a mapping from the regions of $\mathcal{A}_{[-m..m]}(n)$ to $\mathcal{D}^{(m)}(n)$. Properties (iii) and (iv) imply that σ_m is a bijection. In particular, this shows that $\mathcal{A}_{[-m..m]}(n)$ has $\frac{((m+1)n)!}{(mn+1)!}$ regions. The bijection σ_1 is represented in Figure 10. Note that (iii) gives the inverse bijection σ_m^{-1} in terms of inequality, while (iv) gives an explicit point in $\sigma_m^{-1}(\hat{w})$.

Next, we describe a bijection ϕ_m between $\mathcal{D}^{(m)}(n)$ and the set $\mathcal{T}^{(m)}(n)$ of $(m+1)$-ary trees with labeled nodes[8]. Let T be a rooted plane tree. We define the *drift* of a vertex v of T as $\mathrm{drift}(v) = \mathrm{lsib}(u_1) + \cdots + \mathrm{lsib}(u_k)$, where $u_0, u_1, \ldots, u_k = v$ are the vertices on the path from the root u_0 to v. We define the total order $\prec_T$ on vertices of T by setting $u \prec_T v$ if either $\mathrm{drift}(u) < \mathrm{drift}(v)$, or $\mathrm{drift}(u) = \mathrm{drift}(v)$ and u appears before v in the postfix order of T (recall that the *postfix order* is the order of appearance of the vertices when turning counterclockwise around the tree starting at the root). The order $\prec_T$ is represented in Figure 11(a).

Let $\tilde{\mathcal{T}}(n)$ be the set of rooted plane trees, with (at most n) nodes labeled with distinct numbers in $[n]$, and some special leaves called *buds* (see Figure 12). If T has some buds, we call *first bud* the least bud of T for the order $\prec_T$. Let $\hat{w} = \hat{w}_1 \ldots \hat{w}_{(m+1)n} \in \mathcal{D}^{(m)}(n)$, and let $\hat{w}^p = \hat{w}_1 \ldots \hat{w}_p$ be its prefix of length p. We define the trees $\tilde{\phi}_m(\hat{w}^0), \ldots, \tilde{\phi}_m(\hat{w}^{(m+1)n}) \in \tilde{\mathcal{T}}(n)$ as follows:

- $\tilde{\phi}_m(\hat{w}^0)$ is the tree with one bud and no other vertex,
- if $p > 0$ and $\hat{w}_p = \alpha_i^{(s)}$ with $s > 0$, then $\tilde{\phi}_m(\hat{w}^p)$ is obtained from $\tilde{\phi}(\hat{w}^{p-1})$ by replacing its first bud by a (non-bud) leaf,
- if $p > 0$ and $\hat{w}_p = \alpha_i^{(0)}$, then $\tilde{\phi}_m(\hat{w}^p)$ is obtained from $\tilde{\phi}(\hat{w}^{p-1})$ by replacing its first bud by a node labeled i with $m+1$ children, all of them buds.

The trees $\tilde{\phi}(\hat{w}^p)$ are represented in Figure 12. It is clear, by induction on p, that $\tilde{\phi}_m(\hat{w}^p)$ has $1 + m\,n_\alpha - n_\beta$ buds, where n_α and n_β are respectively the number of α-letters and β-letters in $\hat{w}^p$. In particular $\tilde{\phi}_m(\hat{w}^p)$ has at least one bud, so that $\tilde{\phi}_m$ is well defined, and $\tilde{\phi}_m(\hat{w})$ has exactly one bud. We denote by $\phi_m(\hat{w})$ the tree in $\mathcal{T}^{(m)}(n)$ obtained from $\tilde{\phi}_m(\hat{w})$ by replacing its bud by a leaf; see Figure 11(b). Before showing that ϕ_m is a bijection, we describe the inverse mapping ψ_m. Let $T \in \mathcal{T}^{(m)}(n)$ and let $u_0 \prec_T u_1 \prec_T \cdots \prec_T u_{(m+1)n}$ be the vertices of T (T has n nodes and $mn+1$ leaves). Let $\psi_m(T)$ be the word $\hat{w} = \hat{w}_1 \ldots \hat{w}_{(m+1)n}$ defined as follows: for all $p \in [(m+1)n]$, if u_p is the $(s+1)$st child of the node i, then $\hat{w}_p = \alpha_i^{(s)}$.

[8]Of course, any classical bijection between m-parenthesis systems and $(m+1)$-ary trees induces a bijection between $\mathcal{D}^{(m)}(n)$ and $\mathcal{T}^{(m)}(n)$ (by sending labels from the parentheses to the nodes). The non-classical bijection ϕ_m is chosen because it is well adapted to the "non-nesting" nature of annotated sketches.

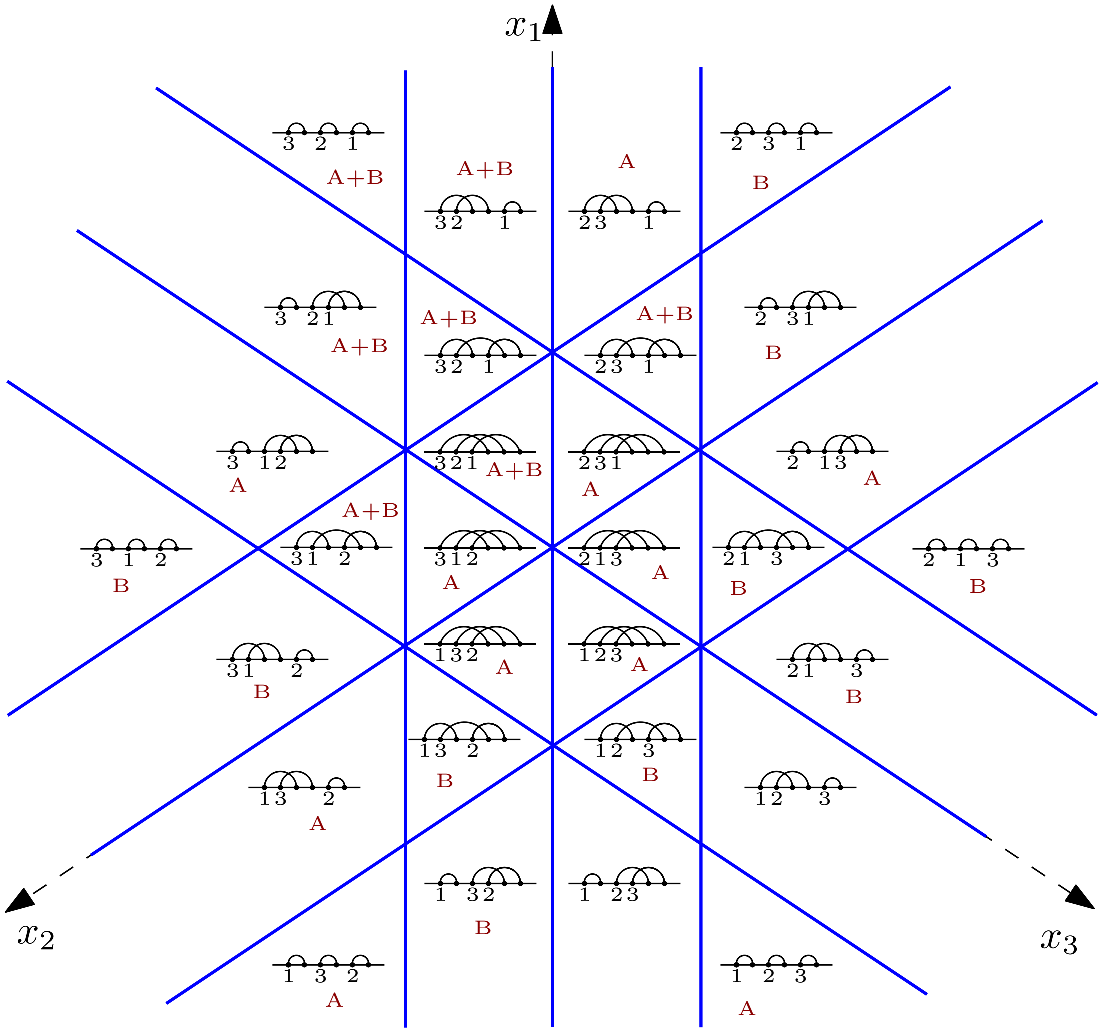


FIGURE 10. The Catalan arrangement $\mathcal{A}_{\{-1,0,1\}}(3)$, and the annotated 1-sketches corresponding to each region. The sketches marked A are Shi maximal, the sketches marked B are semiorder maximal, and those marked both A and B are Linial maximal.

Proposition 8.1. *The mapping ϕ_m is a bijection between $\mathcal{D}^{(m)}(n)$ and $\mathcal{T}^{(m)}(n)$. The inverse mapping is ψ_m. Moreover, for $\hat{w} \in \mathcal{D}^{(m)}(n)$, if the letter following $\alpha_i^{(s)}$ in $\hat{w}$ is $\alpha_j^{(t)}$, then the $(s+1)$st child of the node i in $T = \phi_m(\hat{w})$ is the node j if $t = 0$, and a leaf otherwise.*

Proof. First note that for all $T \in \mathcal{T}^{(m)}(n)$, the word $\psi_m(T)$ clearly satisfies the properties (a), (b), (c) of annotated m-sketches. Hence ψ_m is a mapping from $\mathcal{T}^{(m)}(n)$ to $\mathcal{D}^{(m)}(n)$.

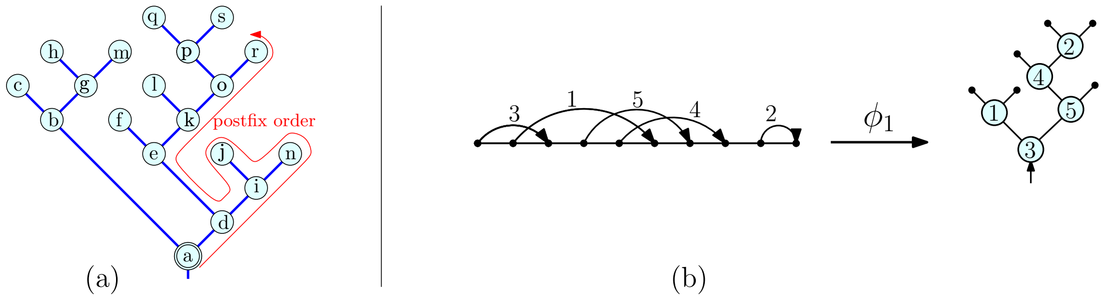


FIGURE 11. (a) The order $\prec_T$ for this binary tree T is $a \prec_T b \prec_T c \prec_T d \prec_T \cdots \prec_T s$. (b) The annotated 1-sketch $\hat{w} = \alpha_3^{(0)}\alpha_1^{(0)}\alpha_3^{(1)}\alpha_5^{(0)}\alpha_4^{(0)}\alpha_1^{(1)}\alpha_5^{(1)}\alpha_4^{(1)}\alpha_2^{(0)}\alpha_2^{(1)}$, and the associated tree $\phi_1(\hat{w}) \in \mathcal{T}^{(1)}(5)$.

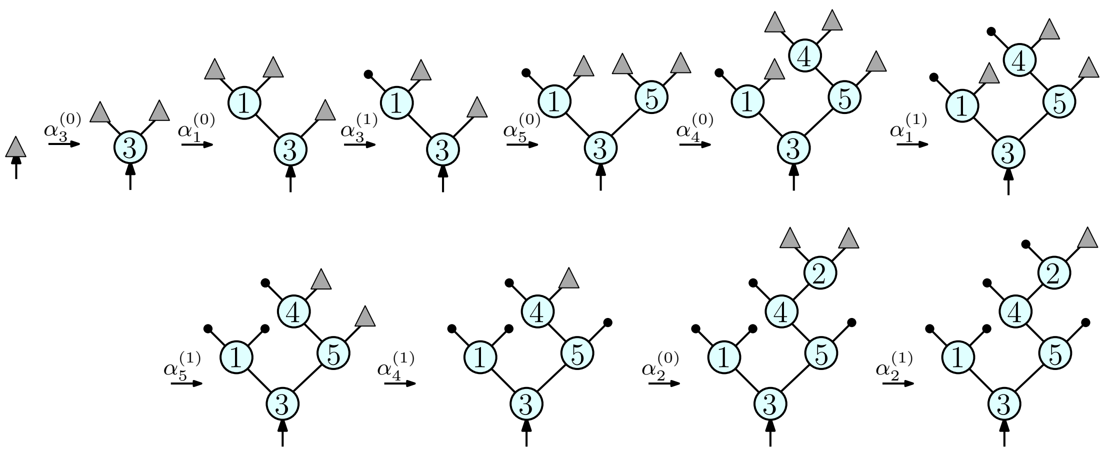


FIGURE 12. The trees $\tilde{\phi}_m(\hat{w}^0), \ldots, \tilde{\phi}_m(\hat{w}^{(m+1)n})$ for $\hat{w} = \alpha_3^{(0)}\alpha_1^{(0)}\alpha_3^{(1)}\alpha_5^{(0)}\alpha_4^{(0)}\alpha_1^{(1)}\alpha_5^{(1)}\alpha_4^{(1)}\alpha_2^{(0)}\alpha_2^{(1)}$. Buds are represented by triangles, and leaves by dots.

Next we give an alternative description of ψ_m. Let $T \in \mathcal{T}^{(m)}(n)$, and let $u_0 \prec_T u_1 \prec_T \cdots \prec_T u_{(m+1)n}$ be its vertices. Let $\tilde{\psi}_m(T)$ be the word $\tilde{w} = \tilde{w}_1 \ldots \tilde{w}_{(m+1)n}$, where $\tilde{w}_p = \alpha_i$ if u_{p-1} is the node labeled i, and $\tilde{w}_p = \beta$ if u_{p-1} is a leaf. We now show that $\hat{w} = \psi_m(T)$ is the annotation of $\tilde{w} = \tilde{\psi}_m(T)$. It is easy to see that for all $q \in [0..(m+1)n]$, if the vertex u_q is a node, then u_{q+1} is its first child. Now let $p \in [(m+1)n]$ such that $\hat{w}_p = \alpha_i^{(0)}$ for some $i \in [n]$. In this case u_p is the first child of the node i, hence u_{p-1} is the node i and $\tilde{w}_p = \alpha_i$. Suppose now that $p \in [(m+1)n]$ is such that $\hat{w}_p = \alpha_i^{(s)}$ for some $s \in [m], i \in [n]$. In this case u_p is not a first child, thus u_{p-1} is a leaf and $\tilde{w}_p = \beta$. This proves that $\hat{w} = \psi_m(T)$ is indeed the annotation of $\tilde{w} = \tilde{\psi}_m(T)$.

Next we prove that $\psi_m \circ \phi_m = \mathrm{Id}$ and $\phi_m \circ \psi_m = \mathrm{Id}$. Let $\hat{w} \in \mathcal{T}^{(m)}(n)$, and let $\hat{w}^p$ be the prefix of length p. Let $T = \phi(\hat{w})$, and let $T^p = \tilde{\phi}_m(\hat{w}^p)$. We claim that the order in which the non-bud vertices are created in the sequence $\tilde{\phi}_m(\hat{w}^0), \tilde{\phi}_m(\hat{w}^1) \ldots, \tilde{\phi}_m(\hat{w}^{(m+1)n}), \phi_m(\hat{w})$ is the same as the order $\prec_T$. Indeed, for all $p \in [(m+1)n]$ the order $\prec_{T^p}$ on the vertices of T^p coincide with the order $\prec_T$. So the first bud of T^p is less than any bud of T^{p+1} for the order $\prec_T$. This establish the claim. From the claim, and the alternative description of ψ_m it follows directly that $\psi_m \circ \phi_m(\hat{w}) = \hat{w}$. And since $|\mathcal{D}^{(m)}(n)| = |\mathcal{T}^{(m)}(n)|$, ϕ_m and ψ_m are inverse bijections.

Lastly, suppose that for $\hat{w} \in \mathcal{D}^{(m)}(n)$ we have $\hat{w}_p = \alpha_i^{(s)}$ and $\hat{w}_{p+1} = \alpha_j^{(t)}$. In this case, u_p is the $(s+1)$st child of the node i (by definition of ψ_m) and u_p is the node j if $s = 0$ and a leaf otherwise (by definition of $\tilde{\psi}_m$). $\square$

Let $\Phi_m = \phi_m \circ \sigma_m$ be the bijection from the the regions of $\mathcal{A}_{[-m..m]}(n)$ to $\mathcal{T}^{(m)}(n)$, and let $\Psi_m = \sigma_m^{-1} \circ \psi_m$ its inverse.

Lemma 8.2. *For $T \in \mathcal{T}^{(m)}(n)$, $\Psi_m(T)$ is the region of $\mathcal{A}_{[-m..m]}(n)$ made of the points $\boldsymbol{x} = (x_1, \ldots, x_n)$ satisfying the following inequalities for all distinct integers $i, j \in [n]$, and all $s \in [0..m]$: $x_i - x_j < s$ if $i \prec_T v$ where v is the $(s+1)$st child of the node j, and $x_i - x_j > s$ otherwise.*

Proof. Let $\mathbf{x} \in \Psi_m(T) = \psi_m(\sigma_m^{-1}(T)$. By Property (iii) of σ_m, $x_i - x_j < s$ if and only if $\alpha_i^{(0)}$ appears before $\alpha_j^{(s)}$ in $\hat{w} = \psi_m(T)$. By definition of ψ_m, this happens if and only if $u \prec_T v$, where u is the first child of the node i. Moreover, since $u \neq v$ and u is the successor of i in the $\prec_T$ order, this happens if and only if $i \prec_T v$. $\square$

8.2. **Bijections for the Shi, semiorder, and Linial arrangements.**

In this section we give a bijection between the regions of $\mathcal{A}_S(n)$, and the trees in $\mathcal{T}_S$ for all $S \subseteq \{-1, 0, 1\}$. Up to symmetry, the interesting cases are $S = \{-1, 0, 1\}$ (Catalan arrangement), $S = \{0, 1\}$ (Shi arrangement), $S = \{-1, 1\}$ (semiorder arrangement), $S = \{1\}$ (Linial arrangement), and $S = \{0\}$ (braid arrangement). We already treated the case $S = \{-1, 0, 1\}$ in the previous Section. For the Shi arrangements our bijection can be seen as a relative to [11] as discussed in Section 9.1. The bijection for the semiorder and Linial arrangements seem to be new.

The basic idea of our bijection for the Shi, semiorder, and Linial arrangements is to think of regions in these arrangements as union of regions of the Catalan arrangement. Then we will choose a canonical representative among these regions, so as to identify regions of the Shi, semiorder and Linial arrangements with certain canonical 1-sketches. We show that the bijection ϕ_1 between $\mathcal{D}^{(1)}(n)$ and $\mathcal{T}^{(1)}(n)$ induces bijections between the canonical 1-sketches for $\mathcal{A}_{\{0,1\}}(n)$, $\mathcal{A}_{\{-1,1\}}(n)$, and $\mathcal{A}_{\{1\}}(n)$ and the trees in $\mathcal{T}_{\{0,1\}}(n)$, $\mathcal{T}_{\{-1,1\}}(n)$, and $\mathcal{T}_{\{1\}}(n)$ respectively. This is represented in Figure 13. Moreover, the bijections induced by $\Phi_1 = \phi_1 \circ \sigma_1$ between regions and trees have simple inverses.

Definition 8.3. Let $\hat{w}$ and $\hat{w}'$ be annotated 1-sketches of size n. We say that $\hat{w}$ and $\hat{w}'$ are related by a *Shi move* if $\hat{w}'$ is obtained from $\hat{w}$ by swapping two consecutive letters $\alpha_i^{(1)}$ and $\alpha_j^{(0)}$ with $i < j$. We say that $\hat{w}$ and $\hat{w}'$ are related by a *semiorder move* if $\hat{w}'$ is obtained from $\hat{w}$ by swapping two consecutive letters $\alpha_i^{(0)}$ and $\alpha_j^{(0)}$ and also two consecutive letters $\alpha_i^{(1)}$ and $\alpha_j^{(1)}$ (for the same pair $\{i, j\}$). We say that $\hat{w}$ and $\hat{w}'$ are related by a *Linial move* if they are related by either a Shi or semiorder move. Lastly, we say that $\hat{w}$ and $\hat{w}'$ are *Shi equivalent* (resp. *semiorder equivalent*, *Linial equivalent*) if one can be obtained from the other by performing a series of Shi (resp. semiorder, Linial) moves.

Let $\hat{w}, \hat{w}' \in \mathcal{D}^{(1)}(n)$ be annotated 1-sketches. Let $\rho = \sigma^{-1}(\hat{w})$ and $\rho' = \sigma^{-1}(\hat{w}')$ be the regions of $\mathcal{A}_{\{-1,0,1\}}(n)$ corresponding to $\hat{w}$ and $\hat{w}'$. Observe that $\hat{w}$ and $\hat{w}'$ are related by the Shi move swapping $\alpha_i^{(0)}$ and $\alpha_j^{(1)}$ if and only if the regions ρ and ρ' are separated only by the hyperplane $H_{i,j,-1} = \{x_i - x_j = -1\}$. Thus, $\hat{w}$ and $\hat{w}'$ are Shi equivalent if and only if one can go from ρ to ρ' only crossing hyperplanes of the forms $H_{i,j,-1}$ for $i < j$. In other words, $\hat{w}$ and $\hat{w}'$ are Shi equivalent if and only if ρ and ρ' are contained in the same region of the Shi arrangement $\mathcal{A}_{\{0,1\}}(n)$. Similarly, $\hat{w}$ and $\hat{w}'$ are related by the semiorder move swapping $\alpha_i^{(0)}$ and $\alpha_j^{(0)}$ (and $\alpha_i^{(1)}$ and $\alpha_j^{(1)}$) if and only if the regions ρ to ρ' are separated only by the hyperplane $H_{i,j,0} = \{x_i - x_j = 0\}$. Thus, $\hat{w}$ and $\hat{w}'$ are semiorder equivalent if and only if ρ and ρ' are contained in the same region of the semiorder arrangement $\mathcal{A}_{\{-1,1\}}(n)$. Also, $\hat{w}$ and $\hat{w}'$ are Linial equivalent if and only if ρ and ρ' are contained in the same region of the Linial arrangement $\mathcal{A}_{\{1\}}(n)$. To summarize:

Lemma 8.4. *Let $\hat{w}$ and $\hat{w}'$ be annotated 1-sketches of size n, and let $\rho = \sigma^{-1}(\hat{w})$ and $\rho' = \sigma^{-1}(\hat{w}')$ be the regions of $\mathcal{A}_{\{-1,0,1\}}(n)$ corresponding to $\hat{w}$ and $\hat{w}'$. The annotated 1-sketches $\hat{w}$ and $\hat{w}'$ are Shi (resp. semiorder, Linial) equivalent if and only if ρ and ρ' are contained in the same region of $\mathcal{A}_{\{0,1\}}(n)$ (resp. $\mathcal{A}_{\{-1,1\}}(n)$, $\mathcal{A}_{\{1\}}(n)$).*

We consider the lexicographic order $\prec$ on $\mathcal{D}^{(1)}(n)$ given by the following order on the alphabet: $\alpha_1^{(1)} \prec \alpha_2^{(1)} \prec \cdots \prec \alpha_n^{(1)} \prec \alpha_1^{(0)} \prec \alpha_2^{(0)} \prec \cdots \prec \alpha_n^{(0)}$. We say that an annotated 1-sketch $\hat{w}$ is *Shi locally-maximal* (resp. *semiorder locally-maximal*, *Linial locally-maximal*) if it is larger than any 1-sketch obtained from $\hat{w}$ by a single Shi (resp. semiorder, Linial) move. We say that an annotated 1-sketch $\hat{w}$ is *Shi maximal* (resp. *semiorder maximal*, *Linial maximal*) if it is larger than any Shi (resp. semiorder, Linial) equivalent 1-sketch. The maximal 1-sketches are indicated in Figure 10.

On the one hand, Lemma 8.4 implies that regions of $\mathcal{A}_{\{0,1\}}(n)$ (resp. $\mathcal{A}_{\{-1,1\}}(n)$, $\mathcal{A}_{\{1\}}(n)$) are in bijection with Shi (resp. semiorder, Linial) maximal 1-sketches in $\mathcal{D}^{(1)}(n)$. On the other hand, locally-maximal 1-sketches are easy to characterize. The following result shows that the two notions actually coincide.

Lemma 8.5. *An annotated 1-sketch $\hat{w} \in \mathcal{D}^{(1)}(n)$ is Shi (resp. semiorder, Linial) maximal if and only if it is Shi (resp. semiorder, Linial) locally-maximal.*

Before proving Lemma 8.5 we explore its consequences.

Corollary 8.6. *The mapping $\Psi_1 = \Phi_1^{-1}$ between $\mathcal{T}^{(1)}(n)$ and the regions of $\mathcal{A}_{\{-1,0,1\}}(n)$ induces a bijection $\Psi_{\{0,1\}}$ (resp. $\Psi_{\{-1,1\}}$, $\Psi_{\{1\}}$) between the trees in $\mathcal{T}_{\{0,1\}}(n)$ (resp. $\mathcal{T}_{\{-1,1\}}(n)$, $\mathcal{T}_{\{1\}}(n)$) and the regions of $\mathcal{A}_{\{0,1\}}(n)$ (resp. $\mathcal{A}_{\{-1,1\}}(n)$, $\mathcal{A}_{\{1\}}(n)$).*

(1) *For* $T \in \mathcal{T}_{\{0,1\}}(n)$ *the region* $\Psi_{\{0,1\}}(T)$ *is defined by the following inequalities for all* $1 \leq i < j \leq n$: $x_i - x_j < 0$ *iff* $i \prec_T j$ *(that is to say, node* i *is less than node* j *for the* $\prec_T$ *order), and* $x_i - x_j < 1$ *iff* $i \prec_T v$, *where* v *is the right child of* j.

(2) *For* $T \in \mathcal{T}_{\{-1,1\}}(n)$ *the region* $\Psi_{\{-1,1\}}(T)$ *is defined by the following inequalities for all* $1 \leq i < j \leq n$: $x_i - x_j > -1$ *iff* $j \prec_T u$, *and* $x_i - x_j < 1$ *iff* $i \prec_T v$, *where* u *is the right child of* i *and* v *is the right child of* j.

(3) *For* $T \in \mathcal{T}_{\{1\}}(n)$ *the region* $\Psi_{\{1\}}(T)$ *is defined by the following inequalities for all* $1 \leq i < j \leq n$: $x_i - x_j < 1$ *iff* $i \prec_T v$, *where* v *is the right child of* j.

Corollary 8.6 is illustrated in Figure 13.

Remark 8.7. The method used above works just as well for the braid arrangement: the induced bijection is between the regions of $\mathcal{A}_{\{0\}}(n)$ and the binary trees with no right child. These trees (which look like paths) are the ones near the origin in Figure 13 (note that they are a subsets of the Shi trees). Of course, such a machinery is unnecessary for this simple case, which could be treated with $m = 0$, but it illustrates the fact that arrangements corresponding to small values of m are embedded in the bijective framework corresponding to larger values of m.

Proof of Corollary 8.6. It is clear that an annotated 1-sketch $\hat{w}$ is Shi locally-maximal if and only if for all $i \in [n]$ the letter $\alpha_i^{(1)}$ is not followed by a letter $\alpha_j^{(0)}$ with $j > i$. By Proposition 8.1, this means that $\hat{w}$ is Shi locally-maximal if and only if for all $i \in [n]$ the right child of the node i in $T = \phi_1(\hat{w})$ is not a node j with $j > i$. In other words, $\hat{w}$ is Shi locally-maximal if and only if $\phi_1(\hat{w})$ is in $\mathcal{T}_{\{0,1\}}$.

Similarly, an annotated 1-sketch $\hat{w}$ is semiorder locally-maximal if and only if for all $i \in [n]$ the letters $\alpha_i^{(0)}$ and $\alpha_i^{(1)}$ are not followed by the letters $\alpha_j^{(0)}$ and $\alpha_j^{(1)}$ respectively, with $j > i$. Note that if $\alpha_i^{(0)}$ is followed by $\alpha_j^{(0)}$ and $\alpha_i^{(1)}$ is followed by a β-letter, then this letter is necessarily $\alpha_j^{(1)}$. Thus, by Proposition 8.1, $\hat{w}$ is semiorder locally-maximal if and only if no node $i \in [n]$ of T has both a left child which is a node $j > i$ and a right child which is a leaf. Thus, $\hat{w}$ is semiorder locally-maximal if and only if $\phi_1(\hat{w})$ is in $\mathcal{T}_{\{-1,1\}}$.

Lastly, an annotated 1-sketch $\hat{w}$ is Linial locally-maximal if and only if it is both Shi locally-maximal and semiorder locally-maximal. Thus $\hat{w}$ is Linial locally-maximal if and only if $\phi_1(\hat{w})$ is in $\mathcal{T}_{\{0,1\}} \cap \mathcal{T}_{\{-1,1\}} = \mathcal{T}_{\{1\}}$.

Moreover, the description of the bijection Ψ_S is immediate from Lemma 8.2 as the inequalities defining the region $\Psi_S(T)$ are a subset of the inequalities defining the region $\Psi_1(T)$ (the inequalities of the form $x_i - x_j < s$ or $x_i - x_j > s$ for $i < j$ and $s \in S$). $\square$

Proof of Lemma 8.5. We first treat the case of the Shi arrangement. Let $\hat{w} = \hat{w}_1 \cdots \hat{w}_{2n}$ and $\hat{w}' = \hat{w}'_1 \cdots \hat{w}'_{2n}$ be two 1-sketches. Suppose that $\hat{w}$ and $\hat{w}'$ are Shi equivalent. It is easy to see (by induction on the number of Shi moves), that

(a) for all $i, j \in [n]$, $\alpha_i^{(0)}$ appears before $\alpha_j^{(0)}$ in $\hat{w}$ if and only if $\alpha_i^{(0)}$ appears before $\alpha_j^{(0)}$ in $\hat{w}'$,

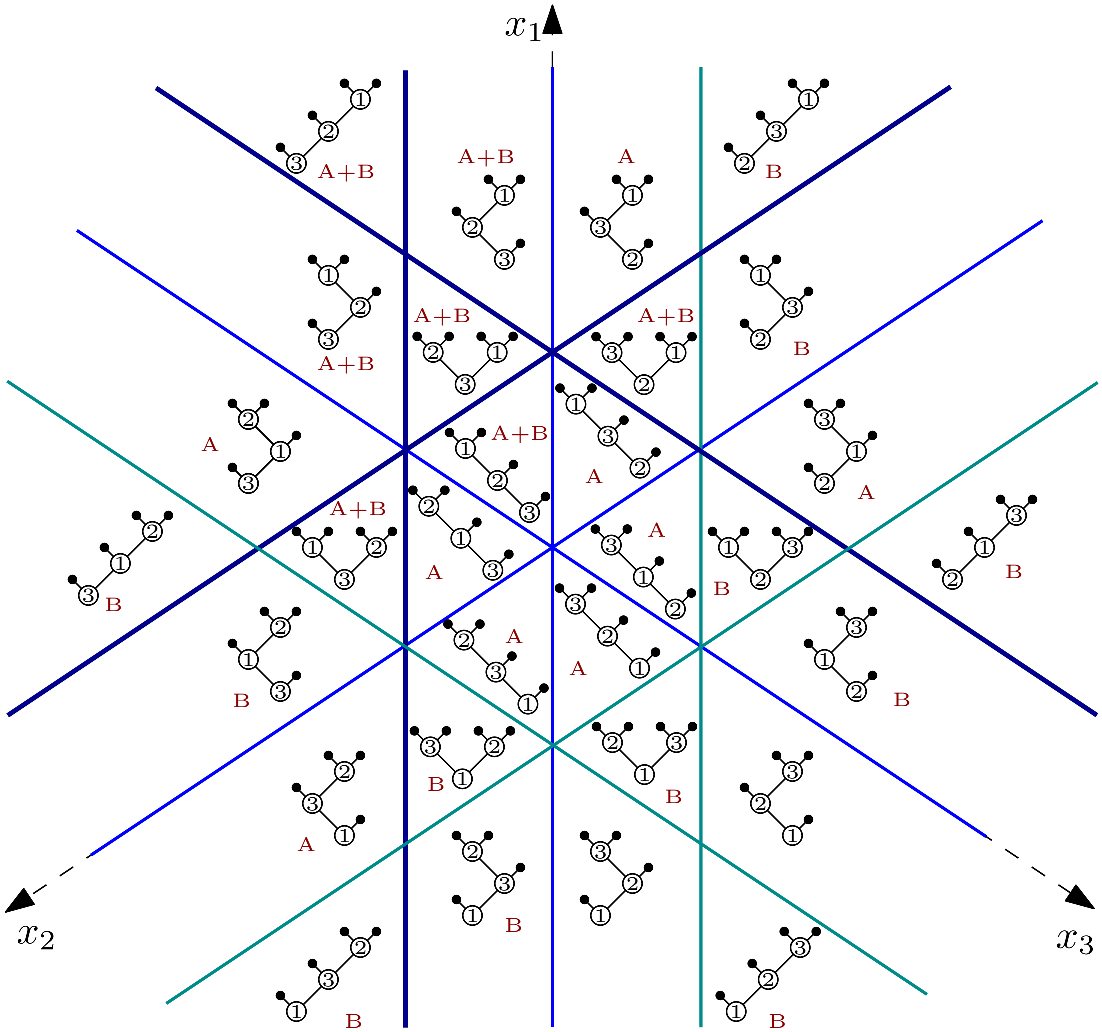


FIGURE 13. The Catalan arrangement $\mathcal{A}_{\{-1,0,1\}}(3)$, and the labeled binary trees corresponding to each region. The trees marked A (including those denoted $A{+}B$) are in bijection with the regions of the Shi arrangement $\mathcal{A}_{\{0,1\}}(3)$. The trees marked B (including those denoted $A{+}B$) are in bijection with the regions of the semiorder arrangement $\mathcal{A}_{\{-1,1\}}(3)$. The trees marked $A+B$ are in bijection with the regions of the Linial arrangement $\mathcal{A}_{\{1\}}(3)$.

(b) for all $i > j \in [n]$, $\alpha_i^{(1)}$ appears before $\alpha_j^{(0)}$ in $\hat{w}$ if and only if $\alpha_i^{(1)}$ appears before $\alpha_j^{(0)}$ in $\hat{w}'$.

Now suppose that $\hat{w}$ is Shi locally-maximal and $\hat{w}'$ is Shi maximal. We want to show $\hat{w} = \hat{w}'$. Suppose by contradiction that they are different, and let $p \in [2n]$ be such that $\hat{w}_p \neq \hat{w}'_p$ and $\hat{w}_k = \hat{w}'_k$ for all $k \in [p-1]$. Since $\hat{w} \prec \hat{w}'$, either

(i) $\hat{w}_p = \alpha_i^{(1)}$ and $\hat{w}'_p = \alpha_j^{(1)}$ with $i < j$,
(ii) $\hat{w}_p = \alpha_i^{(0)}$ and $\hat{w}'_p = \alpha_j^{(0)}$ with $i < j$,
(iii) $\hat{w}_p = \alpha_i^{(1)}$ and $\hat{w}'_p = \alpha_j^{(0)}$ for some i, j.

However case (i) is impossible for 1-sketches: if $\hat{w}_k = \hat{w}'_k$ for all $k \in [p-1]$, $\hat{w}_p = \alpha_i^{(1)}$ and $\hat{w}'_p = \alpha_j^{(1)}$ then $i = j$. Moreover case (ii) is impossible by (a). Hence (iii) holds. Let $q > p$ be such that $\hat{w}_q = \alpha_j^{(0)}$. By (a) and (b), we must have $\hat{w}_{q-1} = \alpha_k^{(1)}$ with $k < j$. But this contradicts the fact that $\hat{w}$ is Shi locally-maximal. Hence $\hat{w} = \hat{w}'$ as wanted.

Next, we treat the case of the semiorder arrangement. Given $\hat{w} \in \mathcal{D}^1(n)$, we say that $i, j \in [n]$ are $\hat{w}$-*exchangeable* if in $\hat{w}$ the letters $\alpha_i^{(0)}$ and $\alpha_j^{(0)}$ are separated only by α-letters, and $\alpha_i^{(1)}$ and $\alpha_j^{(1)}$ are separated only by β-letters. Let $\hat{w}, \hat{w}' \in \mathcal{D}^{(1)}(n)$ be two semiorder equivalent 1-sketches. It is easy to see that $\hat{w}'$ is obtained from $\hat{w}$ by replacing the letters $\alpha_i^{(0)}$ and $\alpha_i^{(1)}$ by $\alpha_{\pi(i)}^{(0)}$ and $\alpha_{\pi(i)}^{(1)}$ for a permutation π of $[n]$ such that for all $i \in [n]$, i and $\pi(i)$ are $\hat{w}$-exchangeable. Now suppose that $\hat{w}$ is semiorder locally-maximal and $\hat{w}'$ is semiorder maximal. We want to show $\hat{w} = \hat{w}'$. Suppose by contradiction that they are different, and let $p \in [2n]$ be such that $\hat{w}_p \neq \hat{w}'_p$ and $\hat{w}_k = \hat{w}'_k$ for all $k \in [p-1]$. Since $\hat{w} \prec \hat{w}'$, either (i), (ii) or (iii) holds. However (i) is impossible as before, and (iii) is impossible because the parenthesis systems underlying $\hat{w}$ and $\hat{w}'$ are equal. Hence, (ii) holds. By the remark above, i, j are $\hat{w}$-equivalent. Hence denoting $\hat{w}_{p+d} = \alpha_j^{(0)}$, we get that for all $c \in [d]$ the letter $\hat{w}_{p+c}$ has the form $\alpha_{i_c}^{(0)}$ for some i_c which is $\hat{w}$-exchangeable with i. Since $\hat{w}$ is locally maximal, we have $i > i_1 > \cdots > i_d = j$. This contradicts $i < j$, hence $\hat{w} = \hat{w}'$ as wanted.

Lastly, we treat the case of the Linial arrangement. Let $\hat{w}, \hat{w}' \in \mathcal{D}^{(1)}(n)$ be two Linial equivalent 1-sketches. It is easy to see that (b) holds. Suppose now that $\hat{w}$ is Linial locally-maximal and $\hat{w}'$ is Linial maximal. We want to show $\hat{w} = \hat{w}'$. Suppose by contradiction that they are different, and let $p \in [2n]$ be such that $\hat{w}_p \neq \hat{w}'_p$ and $\hat{w}_k = \hat{w}'_k$ for all $k \in [p-1]$. Since $\hat{w} \prec \hat{w}'$, either (i), (ii) or (iii) holds. However (i) is impossible as before, so that $\hat{w}'_p = \alpha_j^{(0)}$ for some $j \in [n]$. Let $d > 0$ such that $\hat{w}_{p+d} = \alpha_j^{(0)}$. Suppose first that $\hat{w}_p, \hat{w}_{p+1}, \ldots, \hat{w}_{p+d}$ are all α-letters. We denote $\hat{w}_{p+c} = \alpha_{i_c}^{(0)}$ for all $c \in [0..d]$. In this case, $i_0 < i_d = j$ (since $\hat{w} \prec \hat{w}'$), hence taking the least index i_c we have $i_c < i_{c+1}$ and $i_c < j$. Since $\hat{w}$ is Linial locally-maximal, the letter following $\alpha_{i_c}^{(1)}$ has the form $\alpha_k^{(0)}$ with $k < i_c$ (otherwise it would be $\alpha_{i_{c+1}}^{(1)}$ or $\alpha_k^{(0)}$ with $k > i_c$ and one could do an increasing Linial move) Lastly, since $k < i_c, j$ and $\alpha_k^{(0)}$ is between $\alpha_{i_c}^{(1)}$ and $\alpha_j^{(1)}$ in $\hat{w}$, property (b) implies that $\alpha_k^{(0)}$ is between $\alpha_{i_c}^{(1)}$ and $\alpha_j^{(1)}$ in $\hat{w}'$. Hence $\alpha_{i_c}^{(0)}$ appears before $\alpha_j^{(0)}$ in $\hat{w}'$. We reach a contradiction. It remains to treat the case where $\{\hat{w}_p, \hat{w}_{p+1}, \ldots, \hat{w}_{p+d-1}\}$ contains a β-letter. Let $\alpha_i^{(1)}$ be the last β-letter before $\alpha_j^{(0)}$ in $\hat{w}$, and let $\alpha_{i_0}^{(0)}$ be the letter following $\alpha_i^{(1)}$. By (b), we have $i < j$. Moreover, since $\hat{w}$ is Linial locally-maximal, $i_0 < i$. Since $i_0 < j$ and all the letters between $\alpha_{i_0}^{(0)}$ and $\alpha_j^{(0)}$

are α-letters, the same reasoning as before leads to a contradiction. Hence $\hat{w} = \hat{w}'$ as wanted. $\square$

8.3. **General bijection for transitive deformation of the braid arrangement.** In this section we generalize the strategy adopted in Section 8.3 in order to establish bijections between regions of arbitrary transitive deformations of the braid arrangement, and trees.

We fix a positive integer N and an $\binom{N}{2}$-tuple of finite sets of integers $\mathbf{S} = (S_{i,j})_{1\leq i<j\leq N}$. Recall that $\mathcal{A}_{\mathbf{S}}$ is the arrangement in $\mathbb{R}^N$ made of the hyperplanes $H_{i,j,s}$ for all $1 \leq i < j \leq N$ and all $s \in S_{i,j}$. Recall also that when $\mathbf{S}$ is transitive, the regions of $\mathcal{A}_{\mathbf{S}}$ are equinumerous to the trees $\mathcal{T}_{\mathbf{S}}$ defined in Definition 4.5. For $T \in \mathcal{T}_{\mathbf{S}}$, we denote by $\Psi_{\mathbf{S}}(T)$ the set of points $(x_1, \ldots, x_N)$ satisfying the following inequalities for all $1 \leq i < j \leq N$ and $s \in S_{i,j}$:

- for $s \geq 0$, $x_i - x_j < s$ if the node i is less than the $(s+1)$st child of the node j in the $\prec_T$ order, and $x_i - x_j > s$ otherwise,
- for $s < 0$, $x_i - x_j > s$ if the node j is less than the $(-s+1)$st child of the node i in the $\prec_T$ order, and $x_i - x_j < s$ otherwise.

Our goal is to establish the following result.

Theorem 8.8. *If $\boldsymbol{S} = (S_{i,j})_{1\leq i<j\leq N}$ is transitive (see Definition 4.3), then $\Psi_{\boldsymbol{S}}$ is a bijection between the set $\mathcal{T}_{\boldsymbol{S}}$ of trees and the regions of $\mathcal{A}_{\boldsymbol{S}}$.*

Remark 8.9. The bijections $\Psi_{\mathbf{S}}$ are compatible with refinements of arrangements. Indeed $\mathcal{A}_{\mathbf{S}'}$ is a refinement of $\mathcal{A}_{\mathbf{S}}$ if and only if $\mathbf{S}' = (S'_{i,j})_{1\leq i<j\leq N}$, with $S_{i,j} \subseteq S'_{i,j}$ for all i,j. In this case, $\mathcal{T}_{\mathbf{S}} \subseteq \mathcal{T}_{\mathbf{S}'}$, and for all $T \in \mathcal{T}_{\mathbf{S}}$, $\Psi_{\mathbf{S}'}(T) \subseteq \Psi_{\mathbf{S}}(T)$.

Our strategy to prove Theorem 8.8 is the same as in Section 8.2. Let $m = \max(|s|,\ s \in \cup S_{a,b})$, so that $\mathcal{T}_{\mathbf{S}}$ is a subarrangement of $\mathcal{A}_{[-m..m]}(N)$. We will think of regions of $\mathcal{A}_{\mathbf{S}}$ as equivalence class of regions of $\mathcal{A}_{[-m..m]}(N)$, and the bijection Φ_m defined in Section 8.1 will induce a bijection $\Phi_{\mathbf{S}}$ between regions of $\mathcal{A}_{\mathbf{S}}$ and $\mathcal{T}_{\mathbf{S}}$.

Definition 8.10. Let $\hat{w}, \hat{w}'$ be annotated m-sketches of size N. Let $i,j \in [N]$ with $i < j$, and let $s \in [-m..m]$. We say that $\hat{w}$ and $\hat{w}'$ are related by a (i,j,s)*-move* if for all $k \in [0..m] \cap [-s..m-s]$ the pair of letters $\{\alpha_i^{(k)}, \alpha_j^{(s+k)}\}$ are consecutive in $\hat{w}$, and $\hat{w}'$ is obtained from $\hat{w}$ by swapping each of these pairs of letters. A $\boldsymbol{S}$*-move* is any (i,j,s)-move with $1 \leq i < j \leq N$, and $s \notin S_{i,j}$. We say that $\hat{w}$ and $\hat{w}'$ are $\boldsymbol{S}$*-equivalent* if one can be obtained from the other by performing a series of $\mathbf{S}$-moves.

Example 8.11. Let $m = 1$. The Shi moves defined in Section 8.2 are all the $(i,j,-1)$-moves, and the semiorder moves are all the $(i,j,0)$-moves. Hence the Shi (resp. semiorder, Linial) moves are the $\mathbf{S}$-moves for the tuple $\mathbf{S} = (S_{i,j})_{1\leq i<j\leq N}$ with $S_{i,j} = \{0,1\}$ (resp. $S_{i,j} = \{-1,1\}$, $S_{i,j} = \{1\}$) for all $1 \leq i < j \leq N$.

We consider the following order $\prec$ on the alphabet $\mathcal{A}^{(m)}(N)$: $\alpha_i^{(s)} \prec \alpha_j^{(t)}$ if either $s > t$, or $s = t$ and $i < j$. We now consider the lexicographic order $\prec$ on $\mathcal{D}^{(m)}(n)$ corresponding the order $\prec$ on the letters. An annotated m-sketch $\hat{w} \in \mathcal{D}^{(m)}(n)$ is $\boldsymbol{S}$-*locally-maximal* if it is greater than any m-sketch obtained from $\hat{w}$ by a single $\mathbf{S}$-move.

It is ***S**-maximal* if it is greater than any $\mathbf{S}$-equivalent m-sketch. Lastly, a region ρ of $\mathcal{A}_{[-m..m]}(N)$ is said ***S**-maximal* if the annotated m-sketch $\sigma_m(\rho)$ is $\mathbf{S}$-maximal. We now establish two easy lemmas.

Lemma 8.12. *Each region of $\mathcal{A}_{\mathbf{S}}$ contains a unique $\boldsymbol{S}$-maximal region of $\mathcal{A}_{[-m..m]}(N)$.*

Proof. Let $\hat{w}, \hat{w}' \in \mathcal{D}^{(m)}(N)$, and let $\rho = \sigma_m^{-1}(\hat{w})$ and $\rho' = \sigma_m^{-1}(\hat{w}')$ be the associated regions of $\mathcal{A}_{[-m..m]}(n)$. It is clear that $\hat{w}$ are $\hat{w}'$ are related by a (i,j,s)-move (for $1 \leq i < j \leq N$, and $s \in [-m..m]$) if and only if the regions ρ and ρ' are separated only by the hyperplane $H_{i,j,s}$. Thus $\hat{w}$ and $\hat{w}'$ are $\mathbf{S}$-equivalent if and only if ρ and ρ' are in the same regions of $\mathcal{A}_{\mathbf{S}}$. Thus, for in any region R of $\mathcal{A}_{\mathbf{S}}$, exactly one of the regions ρ of $\mathcal{A}_{[-m..m]}(N)$ contained in R is $\mathbf{S}$-maximal. □

Lemma 8.13. *Let $\hat{w} \in \mathcal{D}^{(m)}(n)$. The sketch $\hat{w}$ is $\boldsymbol{S}$-locally-maximal if and only if the tree $\phi_m(\hat{w})$ is in $\mathcal{T}_{\boldsymbol{S}}$. In other words, ϕ_m induces a bijection between $\boldsymbol{S}$-locally-maximal regions of $\mathcal{A}_{[-m..m]}(N)$ and $\mathcal{T}_{\boldsymbol{S}}$.*

Proof. Let $\hat{w} \in \mathcal{D}^{(m)}(n)$ and let $T = \phi_m(\hat{w})$. For $0 \leq i < j \leq n$, and $s \in [m]$, a (i,j,s)-moves on $\hat{w}$ is possible and gives an annotated m-sketch $\hat{w}' \succ \hat{w}$ if and only if in $\hat{w}$ the letter $\alpha_j^{(s)}$ is immediately followed by $\alpha_i^{(0)}$, and for all $t \in [s+1..m]$, the letters $\alpha_j^{(t)}$ and $\alpha_i^{(t-s)}$ are consecutive. By definition of annotations, this holds if and only if the letter $\alpha_j^{(s)}$ is immediately followed by $\alpha_i^{(0)}$, and for all $t \in [s+1..m]$, the letters $\alpha_j^{(t)}$ is immediately followed by a β-letter. By Proposition 8.1, this holds if and only if in the tree T the node i is the $(s+1)$st child of the node j, and the right siblings of i are leaves (so that $i = \mathrm{cadet}(j)$ and $\mathrm{lsib}(i) = s$).

Similarly, for $0 \leq i < j \leq n$, and $s \in [-m..0]$, a (i,j,s)-moves on $\hat{w}$ is possible and gives an annotated m-sketch $\hat{w}' \succ \hat{w}$ if and only if in $\hat{w}$ the letter $\alpha_i^{(-s)}$ is immediately followed by $\alpha_j^{(0)}$ and for all $t \in [-s+1..m]$, the letters $\alpha_i^{(t)}$ is immediately followed by a β-letter. By Proposition 8.1, this holds if and only if in the tree T the node j is the $(s+1)$st child of the node i, and the right siblings of j are leaves (so that $j = \mathrm{cadet}(i)$ and $\mathrm{lsib}(j) = -s$).

Thus $\hat{w}$ is $\mathbf{S}$-locally-maximal if and only if the tree T satisfies the following property for all $0 \leq i < j \leq n$: if $i = \mathrm{cadet}(j)$ then $\mathrm{lsib}(i) \in S_{i,j} \cup \{0\}$, and if $j = \mathrm{cadet}(i)$ then $-\mathrm{lsib}(j) \in S_{i,j}$. This holds if and only if T is in $\mathcal{T}_{\mathbf{S}}$. □

We now complete the proof of Theorem 8.8. From Lemma 8.2, it is clear that for any tree $T \in \mathcal{T}^{(m)}(N)$, $\Psi_{\mathbf{S}}(T)$ is the region of $\mathcal{A}_{\mathbf{S}}$ containing the region $\Psi_m(T)$ of $\mathcal{A}_{[-m..m]}(N)$. Hence, by Lemma 8.13, the mapping $\Psi_{\mathbf{S}}$ is a surjection between the trees in $\mathcal{T}_{\mathbf{S}}$ and the regions of $\mathcal{A}_{\mathbf{S}}$ containing at least one $\mathbf{S}$-locally-maximal region of $\mathcal{A}_{[-m..m]}(N)$. And since any $\mathbf{S}$-maximal region is $\mathbf{S}$-locally-maximal, Lemma 8.13 ensures that $\Psi_{\mathbf{S}}$ is a surjection between the trees in $\mathcal{T}_{\mathbf{S}}$ and the regions of $\mathcal{A}_{\mathbf{S}}$ (all this holds even if $\mathbf{S}$ is not transitive). Now assuming that $\mathbf{S}$ is transitive, Theorem 4.6, ensures that the regions of $\mathcal{A}_{\mathbf{S}}$ are equinumerous to the trees in $\mathcal{T}_{\mathbf{S}}$, so $\Psi_{\mathbf{S}}$ is actually a bijection.

9. Concluding remarks

We conclude with some additional links to the literature and some open questions.

9.1. Bijections for the Shi arrangement.

We now explain how our bijection $\Psi_{\{0,1\}}$ for the Shi arrangement $\mathcal{A}_{\{0,1\}}(n)$ relates to the existing bijections described in [11] and [40] between regions of $\mathcal{A}_{\{0,1\}}(n)$ and parking functions of size n. The correspondence is represented in Figure 14. Recall that a *parking function of size* n is a n-tuple $(p_1,\ldots,p_n)$ of integers in $[0..n-1]$ such that for all $k\in[n]$, $k\le|\{i\in[n]\mid p_i<k\}|$.

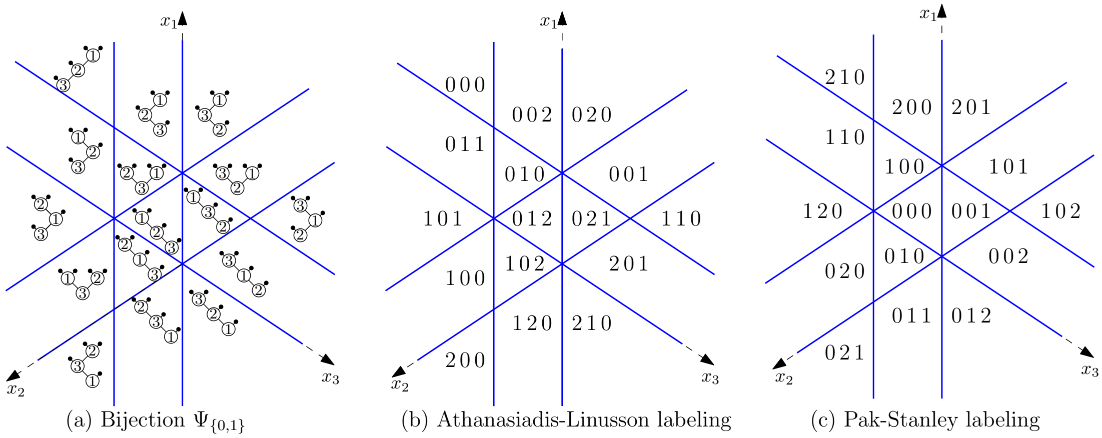


Figure 14. The bijection $\Psi_{\{0,1\}}$, the *Athanasiadis-Linusson labeling* and the *Pak-Stanley labeling*.

The first bijection discovered for the Shi arrangement is the so-called *Pak-Stanley labeling* of the regions described in [40] (and earlier in [43, Section 5] where Igor Pak is credited for suggesting the labeling in the case $m=1$, without proof). This bijection associates to a region ρ of $\mathcal{A}_{\{0,1\}}(n)$ the parking function $(p_1,\ldots,p_n)$, where for all $i\in[n]$,

$$p_i=|\{k\in[i-1]\mid x_k<x_i\}|+|\{k\in[i+1..n]\mid x_k+1<x_i\}|,$$

where $(x_1,\ldots,x_n)$ is any point in the region ρ. This is represented in Figure 14(b). It follows directly from the definition of $\Psi_{\{0,1\}}$ that for any tree $T\in\mathcal{T}_{\{0,1\}}(n)$, the Pak-Stanley labeling of the region $\rho=\Psi_{\{0,1\}}(T)$ is the parking function $\lambda_1(T)=(p_1,\ldots,p_n)$ given by

$$p_i=|\{k\in[i-1]\mid \text{node } k\prec_T \text{node } i\}|+|\{k\in[i+1..n]\mid \text{right child of node } k\preceq_T \text{node } i\}|.$$

Another bijection for the Shi arrangement was established by Athanasiadis and Linusson in [11]. This bijection has two steps. The first step associates to each region ρ of $\mathcal{A}_{\{0,1\}}(n)$ a *diagram* $\delta(\rho)$. The second step associates to the diagram $\delta(\rho)$ a partition function that we call *Athanasiadis-Linusson labeling* of ρ. A reader familiar with [11] will have no difficulty seeing that the diagram $\delta(\rho)$ is closely related to the Shi-maximal 1-sketch that we associated to ρ in Section 8.2. This induces a correspondence between our bijection and the Athanasiadis-Linusson labeling that we now state (the easy proof is omitted). For $T \in \mathcal{T}_{\{0,1\}}(n)$, the Athanasiadis-Linusson labeling of the region $\rho = \Psi_{\{0,1\}}(T)$ is the parking function $\lambda_2(T) = (p_1, \ldots, p_n)$ obtained as follows. For all $i \in [n]$, we consider the path of vertices $v_1, v_2, \ldots, v_\ell$, where v_1 is the node i, v_ℓ is a leaf, and v_{k+1} is the right child of v_k for all $k \in [\ell - 1]$. Then, p_i is the number of leaves greater than v_ℓ for the $\prec_T$ order. This is represented in Figure 14(c).

It is not very hard to see that the correspondence λ_2 is a bijection, and this is why the bijection in [11] can be considered a close relative of $\Psi_{\{0,1\}}$. However, it is less clear why the correspondence λ_1 is a bijection.

9.2. **Regions of the Linial arrangements and binary search trees.**

We now discuss the Linial arrangement $\mathcal{A}_{\{1\}}(n)$. Stanley had conjectured that the regions of $\mathcal{A}_{\{1\}}(n)$ were equinumerous to *binary search trees with* n *nodes*, that is, trees in $\mathcal{T}^{(1)}(n)$ satisfying the Condition (iii) of Figure 4. This fact was proved independently in [37] and [6]. In [37, 35] Postnikov and Stanley listed several combinatorial classes equinumerous to the regions of $\mathcal{A}_{\{1\}}(n)$, and some bijections between them. But, up to now, no bijection was known between these classes and the regions of $\mathcal{A}_{\{1\}}(n)$. We remedy to this situation by giving bijections between regions of $\mathcal{A}_{\{1\}}(n)$, the set $\mathcal{T}_{\{1\}}(n)$ (which was not in the list), and the set $\mathcal{B}(n)$ of binary search trees with n nodes (which was in the list). The bijection between the regions of $\mathcal{A}_{\{1\}}(n)$ and $\mathcal{T}_{\{1\}}(n)$ was established in Section 8.2 (see also Figure 2). We now describe a recursive bijection θ, represented in Figure 15, between $\mathcal{T}_{\{1\}}(n)$ and $\mathcal{B}(n)$.

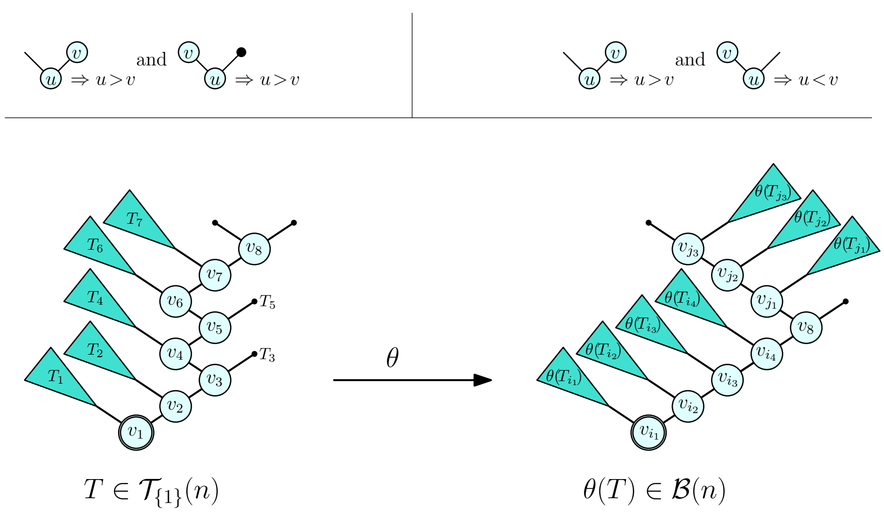


FIGURE 15. The recursive bijection θ from $\mathcal{T}_{\{1\}}(n)$ to $\mathcal{B}(n)$. In this example, exactly two of the trees $\theta(T_{j_1}), \theta(T_{j_2}), \theta(T_{j_3})$ have no node, while the third tree is $\theta(T_p)$ for the only integer $p \in \{1, 2, 4, 6, 7\}$ such that T_p has at least one node, and the root of $\theta(T_p)$ is less than v_p.

For the tree $\tau_0 \in \mathcal{T}_{\{1\}}(0)$ made of one leaf, we define $\theta(\tau_0) = \tau_0 \in \mathcal{B}(0)$. We now consider $n > 0$ and suppose that θ is a well defined bijection from $\mathcal{T}_{\{1\}}(k)$ to $\mathcal{B}(k)$ for all $k < n$. By extension, we may assume that θ is defined on all order-preserving relabeling of trees in $\mathcal{T}_{\{1\}}(k)$ for all $k < n$ (with θ preserving the set of labels). Let T be a tree in $\mathcal{T}_{\{1\}}(n)$, and let v_1 be its root. Let $v_2, v_3, \ldots, v_{k+1}$ be defined by $v_{i+1} = \mathrm{cadet}(v_i)$ for all $i \in [k]$, and the fact that both children of v_{k+1} are leaves. For $i \in [k]$, let T_i be the subtree of T rooted at the child of v_i which is not v_{i+1}; see Figure 15. We denote by I the subset of $[k]$ such that either T_i is a reduced to a leaf which is the left child of v_i, or the root of $\theta(T_i)$ is a node which is greater than v_i. Let $i_1 < \cdots < i_a = k+1$ be the elements of $I \cup \{k+1\}$ and let $k+1 = j_1 > \cdots > j_b$ be the elements of $[k+1] \setminus I$. We then define $\theta(T)$ as follows:

- v_{i_1} is the root,
- for all $p \in [a-1]$, the node v_{i_p} has right child $v_{i_{p+1}}$ and left child the root of the subtree $\theta(T_{i_p})$,
- for all $p \in [b]$, the node v_{j_p} has left child $v_{j_{p+1}}$ (or a leaf for $p = b$), and left child the root of the subtree $\theta(T_{j_p})$.

It is easy to see that $\theta(T)$ is in $\mathcal{B}(n)$ (since $v_1 > v_2 > \cdots > v_k$). It is also easy to see, by induction on n, that θ is a bijection (one of the useful observations to invert θ is that θ transforms subtrees T_j which are right leaves into subtrees which are right leaves). The bijection θ is applied to a tree in $\mathcal{T}_{\{1\}}(10)$ in Figure 16.

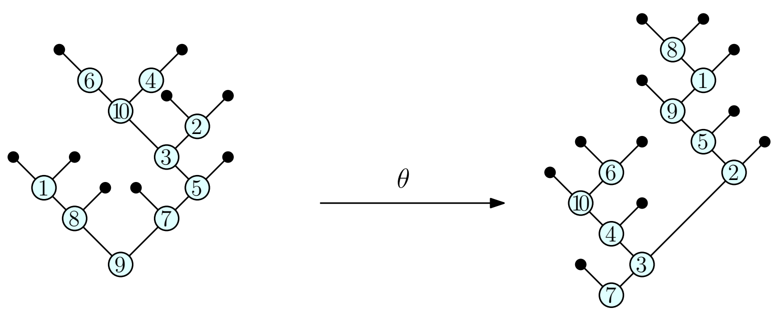


FIGURE 16. The bijection θ from $\mathcal{T}_{\{1\}}(n)$ to $\mathcal{B}(n)$.

9.3. **Open questions.**

The braid arrangement is associated to the root system A_{n-1}, in the sense that the hyperplane have the form $< \alpha, \mathbf{x} >= 0$ for the positive roots α of A_{n-1}. The (deformations of) arrangements corresponding to other root systems are known to share some of the properties of (deformations of) the braid arrangement (see e.g. [8, 37, 39]). Thus, a natural question is whether the results of the present paper can be extended to this more general setting. Another direction for future research is to use the bijections presented here in order to obtain more refined counting formulas for the regions of deformed braid arrangements, by taking into account additional parameters of these regions (in the spirit of e.g. [40, 3]). We now state two open questions.

It was shown in Section 8.3, that when the tuple $\mathbf{S}$ is not transitive, the mapping $\Psi_{\mathbf{S}}$ still gives a surjection between the trees in $\mathcal{T}_{\mathbf{S}}$ and the regions of $\mathcal{A}_{\mathbf{S}}$. Indeed, $\Psi_{\mathbf{S}}$ gives a bijection between the subset of trees corresponding to $\mathbf{S}$-maximal regions and the regions of $\mathcal{A}_{\mathbf{S}}$.

Question 9.1. For a non-transitive tuple $\mathbf{S}$, is it possible to characterize a subset $\widetilde{\mathcal{T}}_{\mathbf{S}}$ of $\mathcal{T}_{\mathbf{S}}$ in bijection with the regions of $\mathcal{A}_{\mathbf{S}}$ via $\Psi_{\mathbf{S}}$?

Let us consider for example the non-transitive set $S = \{-2, 0, 2\}$. The set $\mathcal{T}_{\{-2,0,2\}}(n)$ contains all the trees in $\mathcal{T}^{(2)}(n)$ such that if the rightmost child of any node v is a leaf, then the middle child is also a leaf. However, because $\mathcal{A}_{\{-2,0,2\}}(n)$ is just a dilation of the Catalan arrangement $\mathcal{A}_{\{1,0,1\}}(n)$, we know that the regions of $\mathcal{A}_{\{-2,0,2\}}(n)$ are in bijection with the set $\mathcal{T}^{(1)}(n)$, or equivalently, the set $\widetilde{\mathcal{T}}_{\{-2,0,2\}}(n)$ of trees in $\mathcal{T}_{\{-2,0,2\}}(n)$ such that the middle child of any node is a leaf. In general, one could hope to find the desired subset $\widetilde{\mathcal{T}}_{\mathbf{S}}$ of $\mathcal{T}_{\mathbf{S}}$ either starting from the counting formulas in terms of boxed-trees (Theorem 4.2) and applying some sign-reversing involutions, or by using more direct bijective considerations.

A related problem is to find a more illuminating proof of our bijective results (Theorem 8.8). In the case of the Shi, semiorder, and Linial arrangement we gave a direct proof involving Lemma 8.5 showing that locally-maximal regions are maximal. The argument given there can actually be extended to the m-Shi, m-semiorder and m-Linial arrangements discussed in Section 2.3. However it is unclear whether such an approach would work in the general case (hence removing the need of using Theorem 4.6).

Question 9.2. Is there a direct, preferably geometric, proof that $\mathbf{S}$-locally-maximal regions are $\mathbf{S}$-maximal whenever $\mathbf{S}$ is transitive?

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Hyperplane Arrangements, Interval Orders and Trees

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1 Hyperplane arrangements

The main object of this paper is to survey some recently discovered connections between hyperplane arrangements, interval orders, and trees. We will only indicate the highlights of this development; further details and proofs will appear elsewhere. First we review some basic facts about hyperplane arrangements. A *hyperplane arrangement* is a finite collection $\mathcal{A}$ of affine hyperplanes in a (finite-dimensional) affine space A. We will consider here only the case $A = \mathbb{R}^n$ (regarded as an affine space). The theory of hyperplane arrangements has been extensively developed and has deep connections with many other areas of mathematics, such as algebraic geometry, algebraic topology, and the theory of hypergeometric functions; see for example [16][17]. We will be primarily concerned with the number $r(\mathcal{A})$ of regions of $\mathcal{A}$, i.e., the number of connected components of the space $\mathbb{R}^n - \bigcup_{H\in\mathcal{A}} H$. Closely related to this number is the number $b(\mathcal{A})$ of *bounded* regions of $\mathcal{A}$.

A fundamental object associated with the arrangement $\mathcal{A}$ is its *intersection poset* $L_{\mathcal{A}}$ (actually a meet semilattice), defined as follows. The elements of $L_{\mathcal{A}}$ are the *nonempty* intersections of subsets of the hyperplanes in $\mathcal{A}$, including the empty intersection A. The elements of $L_{\mathcal{A}}$ are ordered by *reverse* inclusion, so in particular $L_{\mathcal{A}}$ has a unique minimal element $\hat{0} = A$. $L_{\mathcal{A}}$ will have a unique maximal element (and thus be a lattice) if and only if the intersection of all the hyperplanes in $\mathcal{A}$ is nonempty. For the basic facts about posets and lattices we are using here, see [27, Ch. 3]. The *characteristic*

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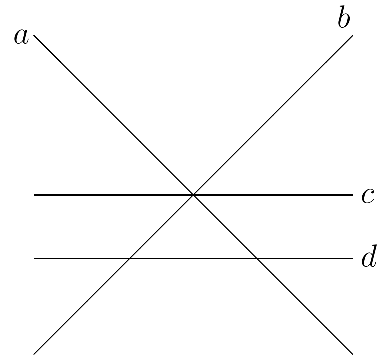

Figure 1: A hyperplane arrangement.

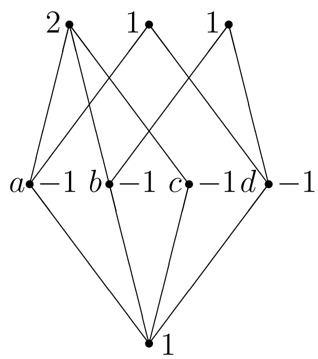

Figure 2: An intersection poset.

polynomial $\chi_{\mathcal{A}}(q)$ of $\mathcal{A}$ is defined by

$$\chi_{\mathcal{A}}(q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) q^{\dim x},$$

where μ denotes the Möbius function of $L_{\mathcal{A}}$ [27, Ch. 3]. Figure 1 illustrates a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^2$, while Figure 2 shows the intersection poset $L_{\mathcal{A}}$, with vertex x labelled with the number $\mu(\hat{0}, x)$, and vertices corresponding to hyperplanes also labelled by the same letter as in Figure 1. From Figure 2 we see that $\chi_{\mathcal{A}}(q) = q^2 - 4q + 4$. The connection between the characteristic polynomial and the number of regions was discovered by Zaslavsky [32, §2].

1.1 Theorem. *With notation as above, we have*

$$\begin{aligned} r(\mathcal{A}) &= (-1)^n \chi_{\mathcal{A}}(-1) &= \sum_{x \in L_{\mathcal{A}}} |\mu(\hat{0}, x)| \\ b(\mathcal{A}) &= (-1)^{\rho(L_{\mathcal{A}})} \chi_{\mathcal{A}}(1) &= \left| \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) \right|, \end{aligned}$$

where $\rho(L_{\mathcal{A}})$ denote the rank (one less than the number of levels) of the intersection poset $L_{\mathcal{A}}$.

An important arrangement, known as the *braid arrangement* and denoted $\mathcal{B}_n$, consists of all hyperplanes $x_i = x_j$, where $1 \leq i < j \leq n$. (See [16, Example 1.10][17, Example 1.9].) It is easy to see that for the braid arrangement we have $r(\mathcal{B}_n) = n!$, since a region of the arrangement is specified by a linear ordering of the n coordinates. (Moreover, $b(\mathcal{B}_n) = 0$ since the origin belongs to all the hyperplanes in $\mathcal{B}_n$.) With a little more work one can if fact show (see e.g. [16, Prop. 2.26][17, Prop. 2.54]) that

$$\chi_{\mathcal{B}_n}(q) = q(q-1)\cdots(q-n+1).$$

The hyperplane arrangements discussed in this paper are closely related to the braid arrangement and could be called modifications or *deformations* of the braid arrangement. Much of this work was done in collaboration with Christos Athanasiadis, Nati Linial, Igor Pak, Alexander Postnikov, and Shmulik Ravid, whose contributions will be noted in the appropriate places. I am also grateful to Persi Diaconis for helpful comments regarding exposition. Our primary concern will be with the following deformation of $\mathcal{B}_n$. Let $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}^n$, with $\ell_i > 0$, and define $\mathcal{A}_\ell$ to be the arrangement in $\mathbb{R}^n$ whose hyperplanes are given by

$$x_i - x_j = \ell_i, \quad i \neq j. \tag{1}$$

A classical theorem of Whitney [30] gives a formula for the characteristic polynomial of any subarrangement $\mathcal{G}$ of the braid arrangement $\mathcal{B}_n$. Such an arrangement is called a *graphical arrangement*, because its set of hyperplanes $x_i = x_j$ may be identified with the edges ij of a graph G with vertices $1, 2, \ldots, n$. Whitney's theorem for the arrangement $\mathcal{G}$ asserts that

$$\chi_{\mathcal{G}}(q) = \sum_{S \subseteq E(G)} (-1)^{\#S} q^{c(S)},$$

where $E(G)$ denotes the set of edges of G, and $c(S)$ is the number of connected components of the spanning subgraph G_S of G with edge set S. Postnikov [20] has generalized Whitney's theorem to subarrangements of arbitrary deformations of the braid arrangement. Rather than state Postnikov's theorem in its full generality here, we will just cite special cases as needed, calling the resulting formula the "Whitney formula" for that arrangement.

For many of the arrangements we will be considering, the characteristic polynomial is actually determined by the number of regions. More precisely, suppose that $\mathcal{A} = (\mathcal{A}_1, \ \mathcal{A}_2, \ldots)$ is a sequence of arrangements such that $\mathcal{A}_n$ is an arrangement in $\mathbb{R}^n$, and every hyperplane in $\mathcal{A}_n$ is parallel to some hyperplane of the braid arrangement $\mathcal{B}_n$. Let S be a k-element subset of $\{1, 2, \ldots, n\}$. Let $\mathcal{A}_n^S$ denote the subarrangement of $\mathcal{A}_n$ consisting of all hyperplanes parallel to $x_i - x_j = 0$ for $i, j \in S$. We call the sequence $\mathcal{A}$ an *exponential sequence* of arrangements if $r(\mathcal{A}_n^S) = r(\mathcal{A}_j)$ for all k-element subsets S of $\{1, 2, \ldots, n\}$, where $1 < k < n$. The following result is a simple consequence of Theorem 1.1 and the exponential formula of enumerative combinatorics (e.g., [25, Cor. 6.2]).

1.2 Theorem. *Let $\mathcal{A}$ be an exponential sequence of arrangements, and write $r_n = r(\mathcal{A}_n)$, $\chi_n(q) = \chi_{\mathcal{A}_n}(q)$. Then*

$$\sum_{n \geq 0} \chi_n(q) \frac{x^n}{n!} = \left(\sum_{n \geq 0} (-1)^n r_n \frac{x^n}{n!} \right)^{-q}.$$

(Equivalently, the sequence $\chi_0(q), \ \chi_1(q), \ \ldots$ is a sequence of polynomials of binomial type in the sense of [21][22].) In particular, if $b_n = b(\mathcal{A}_n)$, then

$$\sum_{n \geq 1} b_n \frac{x^n}{n!} = 1 - \left(\sum_{n \geq 0} r_n \frac{x^n}{n!} \right)^{-1}.$$

2 Interval orders

Let $P = \{I_1, \ldots, I_n\}$ be a collection of closed intervals of positive length on the real line. Partially order the set P by defining $I_i < I_j$ if I_i lies entirely to the left of I_j, i.e., if $I_i = [a, b]$ and $I_j = [c, d]$ then $b < c$. Any partially ordered set isomorphic to P is known as an *interval order*. A basic reference for the theory of interval orders is [5], which gives references to the origins of this subject within economics and psychology. We will be considering *labelled* interval orders whose intervals have specified lengths. Thus given a sequence $\ell = (\ell_1, \dots, \ell_n)$ of positive real numbers, let $\mathcal{I}_\ell$ be the set of all partial orderings of $1, 2, \dots, n$ for which there exists a set of intervals $I_1, \dots, I_n$ satisfying: (a) I_i has length ℓ_i, and (b) $i < j$ in P if and only if I_i lies entirely to the left of I_j. If each $\ell_i = 1$, then the corresponding interval orders are known as *unit interval orders* or *semiorders*, and have been subjected to considerable scrutiny. For other values of ℓ_i there has been considerably less work. The following result shows the main connection between interval orders and deformations of the braid arrangement.

2.1 Theorem. *Let $\ell = (\ell_1, \dots, \ell_n)$ with $\ell_i > 0$. Then*

$$r(\mathcal{A}_\ell) = \#\mathcal{I}_\ell,$$

the number of elements of $\mathcal{I}_\ell$.

The proof of Theorem 2.1 is a straightforward consequence of the relevant definitions. Theorem 2.1 suggests several generalizations of the concept of interval order which may be worth further investigation. (Some work in this direction appears in [3] and [4], but the enumerative aspects are not considered.) Perhaps the most straightforward of these generalizations corresponds to the arrangement

$$x_i - x_j = \ell_i^{(1)}, \dots, \ell_i^{(m_i)},\ i \neq j, \tag{2}$$

for positive integers m_i and real numbers $0 < \ell_i^{(1)} < \ell_i^{(2)} < \cdots < \ell_i^{(m_i)}$. This arrangement corresponds to a collection of *marked* intervals I_i of length $\ell_i^{(m_i)}$. The interval I_i is marked with "dots" at distances $\ell_i^{(1)}, \dots, \ell_i^{(m_i)}$ from the left endpoint (so in particular the right endpoint is marked). We want to count the number of different ways of placing these intervals on the real axis, where two placements P_1 and P_2 are considered the same if for every i and j, the number of marked points of I_i to the left of the left endpoint of I_j is the same for P_1 as for P_2. The number of inequivalent placements ("generalized interval orders") is the number of regions of the arrangement (2). We could even allow $\ell_i^{(1)} = 0$, in which case we must require in the definition of placement that the left endpoint of I_i not coincide with any marked point of another interval. (Thus the order type does not change under small perturbations of the interval placements.)

There is a special case of these generalized interval orders with a further connection with arrangements. It is clear what we mean for two placements P_1 and P_2 of marked intervals to be *isomorphic*, namely, there is a bijection φ between the intervals of P_1 and those of P_2 such that for all intervals I of P_1, $\varphi(I)$ has the same number of marks as I, and for all intervals $I,\ J$ of P_1, the number of marks of I to the left of the left endpoint of J is equal to the number of marks of $\varphi(I)$ to the left of the left endpoint of $\varphi(J)$.

2.2 Theorem. *Let $\ell_1, \ldots, \ell_m > 0$, and let $\mathcal{A}_n$ denote the arrangement in $\mathbb{R}^n$ given by*

$$x_i - x_j = \ell_1, \ldots, \ell_m,\ i \neq j. \tag{3}$$

(Note that this is the special case of (2) when all the marked intervals are identical.) Let $\mathcal{A}_n^0$ denote the arrangement obtained from $\mathcal{A}_n$ by adjoining the hyperplanes $x_i = x_j$, i.e.,

$$\mathcal{A}_n^0 = \mathcal{A}_n \cup \mathcal{B}_n.$$

Then $r(\mathcal{A}_n^0) = n!\nu(\mathcal{A}_n)$, where $\nu(\mathcal{A}_n)$ is the number of nonisomorphic *generalized interval orders corresponding to $\mathcal{A}_n$.*

There is a direct connection between the number of regions of the arrangements $\mathcal{A}_n$ and $\mathcal{A}_n^0$ of the previous theorem, obtained in collaboration with A. Postnikov. Regard $\ell = (\ell_1, \ldots, \ell_m)$ as fixed, and define the generating functions

$$
\begin{aligned}
F_\ell(x) &= \sum_{n\geq 0} r(\mathcal{A}_n)\frac{x^n}{n!} \\
F^0_\ell(x) &= \sum_{n\geq 0} r(\mathcal{A}^0_n)\frac{x^n}{n!} \\
&= \sum_{n\geq 0} \nu(\mathcal{A}_n)x^n.
\end{aligned}
$$

2.3 Theorem. *We have*

$$F_\ell(x) = F^0_\ell(1-e^{-x}).$$

In the special case $\ell = (1)$ (i.e., $m = 1$ and $\ell_1 = 1$), we have that $r(\mathcal{A}_n)$ is the number of labelled semiorders on n points, while $\nu(\mathcal{A}_n)$ is the number of unlabelled (i.e., nonisomorphic) semiorders on n points. It is a well-known result of Wine and Freund [31][5, p. 98][29, p. 195] that this latter number is just the Catalan number $\frac{1}{n+1}\binom{2n}{n}$, so Theorem 2.3 in the case of semiorders may be regarded as determining the number of labelled semiorders. This result is equivalent to a result of Chandon, Lemaire, and Pouget [2]. It is not difficult to show that if $\ell = (1, 2, \ldots, k)$, then

$$\nu(\mathcal{A}_n) = \frac{1}{kn+1}\binom{(k+1)n}{n},$$

generalizing the result of Wine and Freund. For instance, if $n = 3$ and $k = 2$, then we get twelve nonisomorphic placements of three marked intervals, each of length two, with a mark in the center and at the right endpoint. These twelve placements are shown in Figure 3, where each of the three symbols $\bullet$, $\circ$, and $\star$ indicates the left endpoint, center, and right endpoint of an interval. More generally, we have the following result of Athanasiadis [1].

2.4 Theorem. *Let $0 < \ell_1 < \cdots < \ell_m$ be integers such that the set $\{1, 2, 3, \ldots, \} - \{\ell_1, \ldots, \ell_m\}$ is closed under addition (so in particular $\ell_1 = 1$). Let $P(x) = \sum_{j=1}^m x^{\ell_j - 1}$. For $n > 0$ let $R_n(x)$ be the remainder upon dividing $(1 + (x-1)P(x))^n$ by $(1-x)^n$. Let $\mathcal{A}^0_n$ be as in Theorem 2.2. Then $r(\mathcal{A}^0_n)$*

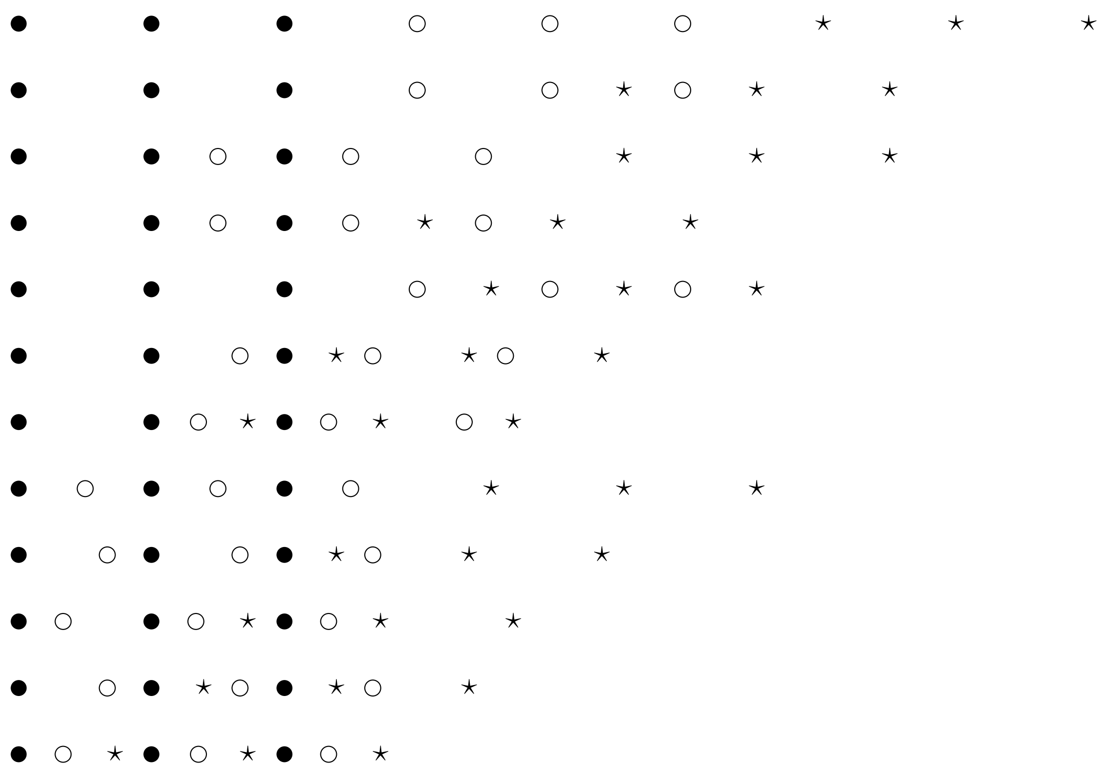

Figure 3: Nonisomorphic marked interval placements.

is equal to the coefficient of x^n *in the Taylor series expansion about* $x = 0$ *of the rational function* $(n-1)!x^{n-1}R_n(1/x)(1-x)^{-n}$.

For any $\ell_1, \ldots, \ell_m > 0$ the symmetric group $\mathfrak{S}_n$ acts on the arrangements $\mathcal{A}_n$ and $\mathcal{A}_n^0$ (by permutation of coordinates), and therefore also acts on the intersection posets of these arrangements. Thus one can apply the representation-theoretic machinery of [26], as has been done by Robert Gill [9] in the case $\ell = (1, 2, \ldots, k)$. For structural properties of generalized interval orders corresponding to the arrangement $x_i - x_j = \ell_1,\ \ell_2$ for $i \neq j$, see [3] and [4]. These generalized interval orders are there called *double semiorders*.

3 Generic interval lengths

An interesting special case of the arrangement $\mathcal{A}_\ell$ occurs when the ℓ_i are *generic*. Intuitively this means that the hyperplanes (1) have as few intersections as possible. More precisely, we mean that the intersection poset of the arrangement $\mathcal{A}_\ell$ is the same as the case when $\ell_1, \ldots, \ell_n$ are linearly independent over the rationals. It is not difficult to determine the exact criterion on $\ell_1, \ldots, \ell_n$ necessary for this condition to hold, though we do not state this result here. We have in particular that $(\ell_1, \ldots, \ell_n)$ is generic if $\ell_1, \ldots, \ell_n$ are linearly independent over the rationals. Hence the set of generic interval lengths $(\ell_1, \ldots, \ell_n)$ is dense in the positive orthant of $\mathbb{R}^n$. (In fact, the set of nongeneric interval lengths has measure 0.) Moreover $(\ell_1, \ldots, \ell_n)$ is generic if $\ell_1, \ldots, \ell_n$ are *superincreasing*, i.e., ℓ_{i+1} is much larger than ℓ_i. It might be interesting to find a characterization of interval orders whose interval lengths are superincreasing in terms of forbidden subposets, similar to the well-known characterizations of interval orders and semiorders (e.g., [5, pp. 28 and 30][29, pp. 86 and 193]).

Define a power series

$$y = 1 + x + 5\frac{x^2}{2!} + 46\frac{x^3}{3!} + 631\frac{x^4}{4!} + 9655\frac{x^5}{5!} + 267369\frac{x^6}{6!} + 7442758\frac{x^7}{7!} + \cdots$$

by the equation

$$1 = y(2 - e^{xy}).$$

Let

$$\begin{aligned} z &= \sum_{n\geq 0} c_n \frac{x^n}{n!} \\ &= 1 + x + 3\frac{x^2}{2!} + 19\frac{x^3}{3!} + 195\frac{x^4}{4!} + 2831\frac{x^5}{5!} + 53703\frac{x^6}{6!} + 1264467\frac{x^7}{7!} \\ &\quad +35661979\frac{x^8}{8!} + 1173865927\frac{x^9}{9!} + 44218244942\frac{x^{10}}{10!} + \cdots \end{aligned}$$

be the unique power series satisfying

$$\frac{z'}{z} = y^2, \quad z(0) = 1.$$

3.1 Theorem. *Let c_n be as above. Then c_n is equal to the number of regions of the arrangement (1), where $\ell_1, \ldots, \ell_n$ are generic.*

Theorem 3.1 shows that the number of labelled interval orders with n generic interval lengths does not depend on the actual lengths (provided they are generic). On the other hand, the posets themselves (or even their isomorphism types) do depend on the choice of lengths.

The basic tool used to prove Theorem 3.1 is Whitney's formula (as discussed in Section 1) for the arrangement $\mathcal{A}_\ell$. The next theorem states this result in a somewhat simplified form. (The case $q = -1$ is all that is needed to prove Theorem 3.1.)

3.2 Theorem. *Let $\mathcal{A}_n$ be the arrangement (1), where $\ell_1, \ldots, \ell_n$ are generic. Then*

$$\chi_{\mathcal{A}_n}(q) = \sum_G (-1)^{e(G)} 2^{b(G)} q^{c(G)},$$

where G ranges over all bipartite graphs on the vertex set $1, 2 \ldots, n$, and where $e(G)$ denotes the number of edges of G, $b(G)$ the number of blocks (maximal doubly connected subgraphs), and $c(G)$ the number of connected components.

Theorem 1.2 applies to the generic arrangement $\mathcal{A}_\ell$, so we obtain the following corollary.

3.3 Corollary. *If $\mathcal{A}_n$ is the arrangement of Theorem 3.2 then we have*

$$\sum_{n\geq 0} \chi_{\mathcal{A}_n}(q)\frac{x^n}{n!} = \left(\sum_{n\geq 0}(-1)^n r(\mathcal{A}_n)\frac{x^n}{n!}\right)^{-q}.$$

In particular,

$$\sum_{n\geq 1} b(\mathcal{A}_n)\frac{x^n}{n!} = 1 - \left(\sum_{n\geq 0} r(\mathcal{A}_n)\frac{x^n}{n!}\right)^{-1}.$$

The first few polynomials $\chi_{\mathcal{A}_n}(q)$ are given by

$$\begin{aligned}
\chi_{\mathcal{A}_1}(q) &= q \\
\chi_{\mathcal{A}_2}(q) &= q^2 - 2q \\
\chi_{\mathcal{A}_3}(q) &= q^3 - 6q^2 + 12q \\
\chi_{\mathcal{A}_4}(q) &= q^4 - 12q^3 + 60q^2 - 122q \\
\chi_{\mathcal{A}_5}(q) &= q^5 - 20q^4 + 180q^3 - 850q^2 + 1780q \\
\chi_{\mathcal{A}_6}(q) &= q^6 - 30q^5 + 420q^4 - 3390q^3 + 15780q^2 - 34082q.
\end{aligned}$$

4 Alternating trees and local search trees

In this section we will be concerned with the arrangement in $\mathbb{R}^n$ given by

$$x_i - x_j = 1,\ 1 \leq i < j \leq n.$$

Denote this arrangement by $\mathcal{L}_n$, and set $r(\mathcal{L}_n) = g_n$. N. Linial conceived the idea of looking at this arrangement, and he and S. Ravid made some computations from which a fascinating conjecture about the value of g_n was obtained. This conjecture was recently proved by Postnikov. We first make a number of relevant combinatorial definitions.

- An *alternating tree* or *intransitive tree* is a labelled tree, say with the $n+1$ vertices $0, 1, \ldots, n$, such that if $a_1, \ldots, a_k$ are the vertices of a

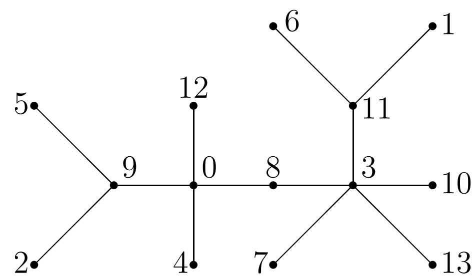

Figure 4: An alternating tree.

path in the tree (in the given order), then either $a_1 < a_2 > a_3 < a_4 > \cdots a_k$ or $a_1 > a_2 < a_3 > a_4 < \cdots a_k$. See Figure 4 for an example. Alternating trees first arose in the work of Gelfand, Graev, and Postnikov [6, §5] (where they are called *admissible* trees), and were further investigated by Postnikov [19]. He showed that if f_n denotes the number of alternating trees on $n + 1$ vertices and if

$$\begin{aligned} y &= \sum_{n\geq 0} f_n \frac{x^n}{n!} \\ &= 1 + x + 2\frac{x^2}{2!} + 7\frac{x^3}{3!} + 36\frac{x^4}{4!} + 246\frac{x^5}{5!} + 2104\frac{x^6}{6!} + \cdots, \end{aligned}$$

then

$$\begin{aligned} y &= e^{\frac{x}{2}(y+1)} \\ f_{n-1} &= \frac{1}{n2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} k^{n-1}. \end{aligned}$$

- A *local binary search tree* (LBST) is a labelled (plane) binary tree, such that every left child has a smaller label than its parent, and every right child has a larger label than its parent. (Compare with the notion of a binary search tree, in which *all* the nonroot vertices of the left subtree of a vertex v have lower labels than v, and similarly for right subtrees.) See Figure 5 for an example. LBST's were first considered by Gessel [7], though not with that terminology. Postnikov [20] found a bijection between alternating trees with $n+1$ vertices and LBST's with n vertices labelled $1, 2, \ldots, n$.

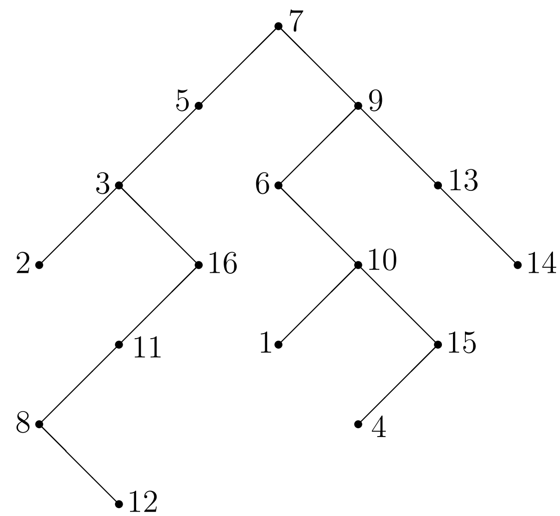

Figure 5: A local binary search tree.

- An easy bijection, obtained independently by A. Postnikov and S. Ravid, shows that g_n is equal to the number of tournaments T on the vertex set $\{1, 2, \ldots, n\}$ such that in every directed cycle of T, there are more edges (i, j) with $i < j$ than with $i > j$.

- It is also easy to see that g_n is equal to the number of partially ordered sets on the vertex set $\{1, 2, \ldots, n\}$ that are the intersection of a semiorder (as defined in Section 2) with the chain $1 < 2 < \cdots < n$. (More generally, if $\ell = (\ell_1, \ldots, \ell_n)$ with $\ell_i > 0$, then the number of regions of the arrangement $x_i - x_j = \ell_i,\ 1 \leq i < j \leq n$, is equal to the number of posets obtained by intersecting an element of $\mathcal{I}_\ell$ with the chain $1 < 2 < \cdots < n$.) Let us call the intersection of a semiorder on the vertex set $\{1, 2, \ldots, n\}$ with the chain $1 < 2 < \cdots < n$ a *sleek poset*. For instance, the poset with cover relations $1 < 2,\ 3 < 2,\ 3 < 4$ is a semiorder. When we intersect it with the chain $1 < 2 < 3 < 4$ we obtain the poset $1 < 2,\ 3 < 4$, which is not a semiorder (or even an interval order) but is sleek. It was shown in collaboration with A. Postnikov that a poset on the vertex set $\{1, 2, \ldots, n\}$ is sleek if and only it contains no induced subposet of the four types shown in Figure 6, where $a < b < c < d$. This is the analogue for sleek posets of the characterization of Scott and Suppes [5, p. 30][28][29, p. 193] of

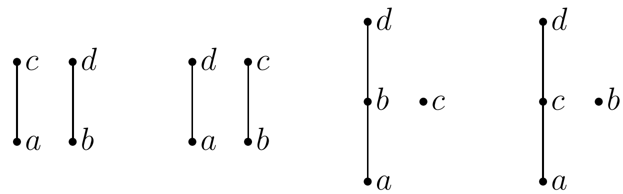

Figure 6: Obstructions to sleekness.

semiorders in terms of two forbidden induced subposets.

- Whitney's formula for the arrangement $\mathcal{L}_n$ yields that

$$g_n = \sum_G (-1)^{\kappa(G)},$$

where G ranges over all bipartite graphs on the vertex set $1, 2, \ldots, n$ such that if $i_1, i_2, \ldots, i_{2k}$ are the vertices of a cycle (in that order), then exactly k indices $1 \leq j \leq 2k$ satisfy $i_j > i_{j+1}$ (where we take subscripts modulo $2k$), and where $\kappa(G)$ denotes the cyclomatic number (number of linearly independent cycles in the mod 2 cycle space) of G.

- Athanasiadis [1] has shown, based on a combinatorial interpretation [17, Thm. 2.3.22] of the characteristic polynomial of an arrangement defined over a finite field, that the characteristic polynomial $\chi_n(q)$ of $\mathcal{L}_n$ is given by

$$\chi_n(q) = q \sum_{k=1}^{n} (k-1)! S(n,k) \sum_{i=0}^{n-k} \binom{n-k}{i} \binom{q-k-i-1}{k-1},$$

where $S(n,k)$ denotes a Stirling number of the second kind.

The primary result on the Linial arrangement $\mathcal{L}_n$ is the following. It was conjectured by this writer on the basis of data supplied by Linial and Ravid, and recently proved by Postnikov.

4.1 Theorem. *For all $n \geq 0$ we have $f_n = g_n$.*

Theorem 1.2 applies to the arrangement $\mathcal{L}_n$ (see Corollary 4.2(a) below), so Theorem 4.1 in fact determines the characteristic polynomial of $\mathcal{L}_n$.

4.2 Corollary. *The characteristic polynomial $\chi_n(q)$ of $\mathcal{L}_n$ is given by*

$$\sum_{n\geq 0} \chi_n(q)\frac{x^n}{n!} = \left(\sum_{n\geq 0}(-1)^n g_n \frac{x^n}{n!}\right)^{-q}.$$

In particular

$$\sum_{n\geq 1} b(\mathcal{L}_n)\frac{x^n}{n!} = 1 - \left(\sum_{n\geq 0} g_n \frac{x^n}{n!}\right)^{-1}.$$

Moreover,

$$\chi_n(q) = \sum_{i=1}^{n}(-1)^{n-i} f_{i,n} q^i,$$

where $f_{i,n}$ is the number of alternating trees on the vertices $0, 1, \ldots n$ such that vertex 0 has degree i.

5 The Shi arrangement and parking functions.

An arrangement closely related to those discussed above is given by $x_i - x_j = 0, 1$ for $1 \leq i < j \leq n$ and will be called the *Shi arrangement* (called by Headley [12, Ch. VI] the *sandwich arrangement* associated with the symmetric group $\mathfrak{S}_n$), denoted $\mathcal{S}_n$. It was first considered by J.-Y. Shi [23] in his investigation of the affine Weyl group $\tilde{A}_n$, so we will call it the *Shi arrangement*. Shi showed the surprising result [23, Cor. 7.3.10]

$$r(\mathcal{S}_n) = (n+1)^{n-1} \tag{4}$$

using group-theoretic techniques. Later Shi [24] generalized his result to other Weyl groups. Headley [11][12, Ch. VI] gave a proof of equation (4) based on Zaslavsky's theorem (Theorem 1.1), and in fact computed the characteristic polynomial $\chi_{\mathcal{S}_n}(q)$, viz.,

$$\chi_{\mathcal{S}_n}(q) = q(q-n)^{n-1}. \tag{5}$$

One can also deduce (5) from (4) using Theorem 1.2; and an elegant "coloring" proof has been given by Athanasiadis [1]. We will give a bijective proof of a refinement of equation (4) related to inversions of trees. This work was done in collaboration with Igor Pak.

In order to motivate our result, first consider the case of the braid arrangement $\mathcal{B}_n$. If w is a permutation of $1, 2 \ldots, n$, then define its *code* or *inversion table* to be the sequence $C(w) = (a_1, \ldots, a_n)$ where a_i is the number of elements j for which $j > i$ and $w^{-1}(j) < w^{-1}(i)$. (The definition of $C(w)$ is sometimes given as a minor variation of our definition here.) In particular, $\sum_i a_i = \ell(w)$, the number of *inversions* of w (or the *length* of w in the sense of Coxeter groups). It is clear that $0 \le a_i \le n - i$, and it is easy to see that C is a bijection from the symmetric group $\mathfrak{S}_n$ to the set of sequences $(a_1, \ldots, a_n)$ with $0 \le a_i \le n - i$. There follows the well-known result [27, Cor. 1.3.10]

$$\sum_{w \in \mathfrak{S}_n} q^{\ell(w)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}). \qquad (6)$$

Now let R_0 be the region of $\mathcal{B}_n$ defined by $x_1 > x_2 > \cdots > x_n$, which we call the *base region*. We will assign an n-tuple $\kappa(R)$ of nonnegative integers to every region R of $\mathcal{B}_n$ as follows. First define $\kappa(R_0) = (0, 0, \ldots, 0)$. Suppose that $\kappa(R)$ has been defined, and that R' is a region such that (a) $\kappa(R')$ has not yet been defined, (b) some hyperplane $x_i = x_j$ (with $i < j$) is a boundary facet of both R and R', and (c) R_0 and R lie on the same side of the hyperplane $x_i = x_j$. Define $\kappa(R') = \kappa(R) + \varepsilon_i$, where ε_i is the ith unit coordinate vector. It is then easy to see that $\kappa(R)$ is well-defined, and that $\kappa(R)$ is the code of some permutation $w \in \mathfrak{S}_n$. Moreover, for any code $C(w)$ with $w \in \mathfrak{S}_n$ there is a unique region R of $\mathcal{B}_n$ with $\kappa(R) = C(w)$. Thus the map $C^{-1}\kappa$ defines a bijection between the regions of $\mathcal{B}_n$ and the symmetric group $\mathfrak{S}_n$, with the property that if $C^{-1}\kappa(R) = w$, then the number of hyperplanes of $\mathcal{B}_n$ separating R from R_0 is $\ell(w)$, the number of inversions of w. We now describe a completely analogous construction for the arrangement $\mathcal{S}_n$.

Let T be a tree with vertices $0, 1, \ldots, n$. An *inversion* of T is a pair $1 \le i < j$ such that vertex j lies on the unique path in T from 0 to i. Write $\ell(T)$ for the number of inversions of T. The polynomial

$$I_{n+1}(q) = \sum_T q^{\ell(T)},$$

summed over all trees with vertices $0, 1, \ldots, n$, is known as the *inversion enumerator* for trees. Analogously to equation (6) (but more difficult to prove [8][15]) we have

$$\sum_{n \geq 0} (1+q)^{\binom{n}{2}} \frac{x^n}{n!} = \exp \sum_{n \geq 1} q^{n-1} I_n(1+q) \frac{x^n}{n!}.$$

Consider now n cars $C_1, \ldots, C_n$ that want to park on a one-way street with parking places $0, 1, \ldots, n-1$ in that order. Each car C_i has a preferred space a_i. The cars enter the street one at a time in the order $C_1, \ldots, C_n$. A car tries to park in its preferred space. If that space is occupied, then it parks in the next available space. If there is no space then the car leaves the street. The sequence $(a_1, \ldots, a_n)$ is called a *parking function* if all the cars can park, i.e., no car leaves the street. (Our definition is a slight variant of the usual definition.) It is not difficult to see that the sequence $(a_1, \ldots, a_n)$ is a parking function if and only if it has at most i terms greater than or equal to $n-i$, for $1 \leq i \leq n$. Equivalently, a parking function is a permutation of the code of a permutation. For further information on parking functions, see [10, §2.6][13]. The number of parking functions of length n is $(n+1)^{n-1}$, and Kreweras [14] in fact gave a bijection C between trees with vertices $0, 1, \ldots, n$ and parking functions such that if $C(T) = (a_1, \ldots, a_n)$ then $a_1 + \cdots + a_n = \binom{n}{2} - \ell(T)$. Thus the number of parking functions $(a_1, \ldots, a_n)$ of length n such that $\sum a_i = k$ is equal to the number of labelled trees with vertices $0, 1, \ldots, n$ and with $\binom{n}{2} - k$ inversions. It follows that the parking function $C(T)$ is a good analogue of the code of a permutation. Theorem 5.1 below makes this analogy even stronger.

Let R_0 be the region of $\mathcal{S}_n$ defined by $x_1 > x_2 > \cdots > x_n$ and $x_1 - x_n < 1$, which we call the *base region*. Equivalently, R_0 is the unique region contained between all pairs of parallel hyperplanes of $\mathcal{S}_n$. We will assign an n-tuple $\lambda(R)$ of nonnegative integers to every region R of $\mathcal{S}_n$ as follows. First define $\lambda(R_0) = (0, 0, \ldots, 0)$. Suppose that $\lambda(R)$ has been defined, and that R' is a region such that (a) $\lambda(R')$ has not yet been defined, (b) some hyperplane H of $\mathcal{S}_n$ is a boundary facet of both R and R', and (c) R_0 and R lie on the same side of the hyperplane H. Define

$$\lambda(R') = \left\{ \begin{array}{ll} \lambda(R) + \varepsilon_i, & \text{if } H \text{ is given by } x_i - x_j = 0 \text{ with } i < j \\ \lambda(R) + \varepsilon_j, & \text{if } H \text{ is given by } x_i - x_j = 1 \text{ with } i < j. \end{array} \right\}$$

It is then easy to see that $\lambda(R)$ is well-defined, and that $\lambda(R)$ is a parking function. Moreover, if $\lambda(R) = (a_1, \ldots, a_n)$, then $a_1 + \cdots + a_n$ is equal to the number of hyperplanes in $\mathcal{S}_n$ separating R from R_0. Not so evident is the following result obtained in collaboration with Igor Pak [18].

5.1 Theorem. *The map λ defined above is a bijection from the regions of $\mathcal{S}_n$ to the set of all parking functions of length n. Consequently, the number of regions R for which i hyperplanes separate R from R_0 is equal to the number of trees on the vertices $0, 1, \ldots, n$ with $\binom{n}{2} - i$ inversions.*

Theorem 5.1 can be reformulated in terms of posets. Given a permutation $w \in \mathfrak{S}_n$, let $P_w = \{(i,j) : 1 \leq i < j \leq n,\ w(i) < w(j)\}$. Partially order P_w by the rule $(i,j) \leq (k,l)$ if $k \leq i < j \leq l$. Let $F(J(P_w), q)$ denote the rank-generating function of the lattice $J(P_w)$ of order ideals of P_w, as defined in [27, pp. 99 and 106]. Then it is not difficult to show that Theorem 5.1 is equivalent to the formula

$$\sum_{w \in \mathfrak{S}_n} F(J(P_w), q) = I_{n+1}(q).$$

Theorem 5.1 suggests a host of additional problems dealing with the Shi arrangement and related arrangements. Many of these problems are currently under investigation.

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