Il semble que les facettes (= paires de dimensions complémentaires) de la diagonale soient en bijection avec les "partitions complémentaires essentielles" (que l'on sait compter !!)

Def : Une paire de partitions (σ, δ) de $\{1,\dots,n\}$ est dite complémentaire s'il existe $I \subset \{1,\dots,n\}$, $i \in I$ tels que

$|I \cap \sigma_k| = 1 \quad \forall$ bloc σ_k de σ

$|((\{1,\dots,n\} \setminus I) \cup \{i\}) \cap \delta_k| = 1 \quad \forall$ bloc δ_k de δ

$I = \{1,2\}$ $J = \{1,3\}$
$i = 1$
$(\{1,3\}\{2\}, \{1,2\}\{3\})$
complémentaires

Si l'on note $J := (\{1,\dots,n\} \setminus I) \cup \{i\}$, on a $I \cup J = \{1,\dots,n\}$
et $I \cap J = \{i\}$

On dit I et J une bonne sélection de représentants (1 pour chaque bloc de chaque partition).

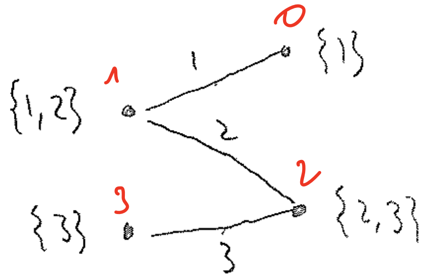
Def : Une paire de partitions complémentaires (σ, δ) est dite essentielle // pce, s'il n'existe aucun sous-ensemble $V \subsetneq \{1,\dots,n\}$ et des unions de blocs $\sigma' \subset \sigma$, $\delta' \subset \delta$ tels que (σ', δ') soit une paire de partitions complémentaires de V.

$(\{\{1,2\},\{3\}\}, \{\{1\},\{2,3\}\})$ est une paire de partitions complémentaires essentielles de $\{1,2,3\}$

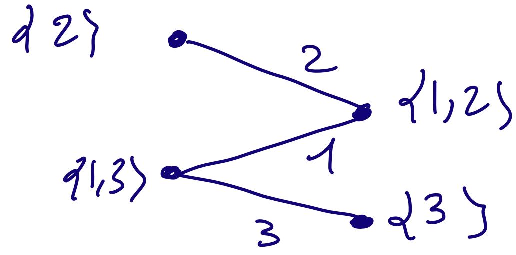
$I = \{1,3\}$, $J = \{1,2\}$ ou $I = \{2,3\}$, $J = \{1,3\}$ ...

$(\{\{1,2\},\{3\} | \{4,6\},\{5\}\}, \{\{1\},\{2,3\} | \{4,5,6\}\})$ est une paire de partitions complémentaires non-essentielle de $\{1,\dots,6\}$.

Les paires de partitions complémentaires enentielles sont en bijection avec les arbres bipartitionnés : dans un tel arbre, à n arêtes, il suffit de se placer à chaque sommet et de lire les arêtes incidentes



$(\{\{1,3\}\{2\}\}, \{\{1,2\}\{3\}\})$



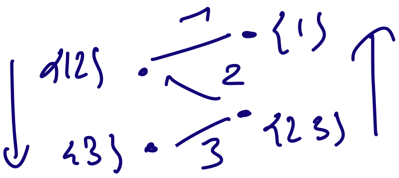
Ces arbres sont faciles à compter : il y en a $2(n+1)^{n-2}$ ← ceux tq plus ℓ[?] des l'ordre lex à G

Échange G↔D

nb arbre de Cayley sur $(n+1)$ sommets

Conjecture : L'application Φ : Diag ⟶ Compl qui à une paire de partitions ordonnées de la diagonale associe sa paire de partition sous-jacente est une bijection entre les facettes de la diagonale et les paires de partitions complémentaires enentielles.

$(1|2|3, 23|1) \longmapsto (\{\{1,2\},\{3\}\}, \{\{1\},\{2,3\}\})$



Cette conjecture a été vérifiée jusqu'à $n=5$.

Corollaire : la formule de Matthieu $\frac{1}{n+1}\binom{n+1}{k+1}(k+1)^{n-k-1}(n-k)^k$; les paires de faces de dim $(k, n-k)$ seraient alors en bijection avec les arbres bipartitionnés avec une taille fixée de la bipartition.

À montrer : 1) ϕ bien définie → la paire de partitions sous-jacente est une paire de
2) ϕ injective → si on permute les blocs, on quitte la diag.
3) ϕ surj.
↳ comment retrouver l'ordre des blocs

par edge $\Delta \sim \Delta^{op}$

pair diag	représentants bien choisis !	ordre lexico × lexico^{op}
1|2|3 × 123	$I=\{1,2,3\}$ $J=\{2\}$	1|2|3| × 123
12|3 × 2|13	$I=\{1,3\}$ $J=\{1,2\}$	12|3 × 2|13
2|13 × 23|1	$I=\{2,3\}$ $J=\{2,1\}$	2|13 × 23|1
	ça ne marche pas	ne marche pas !
3|14|2 × 34|12	$I=\{3,1,2\}$ $J=\{4,2\}$	

① $n=1$: 1×1

② $n=2$: 1|2×12

12×2|1

Quel ordre sur les blocs ?

⑧ $n=3$: 1|2|3×123

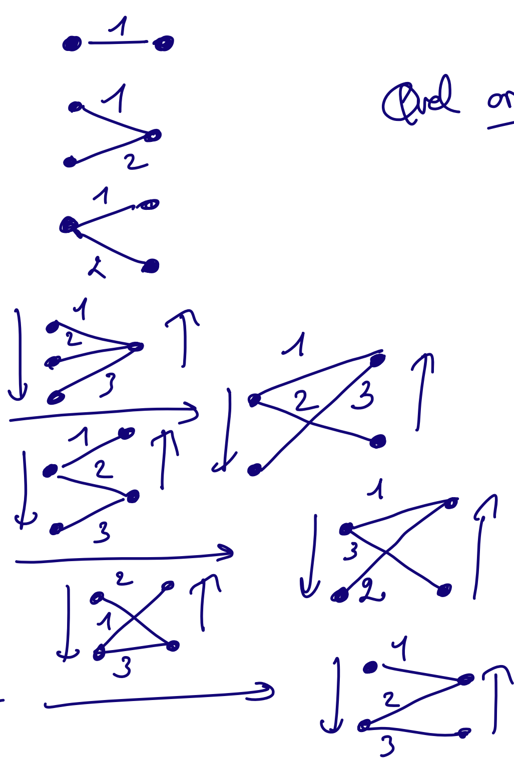
12|3×2|13

12|3 × 23|1

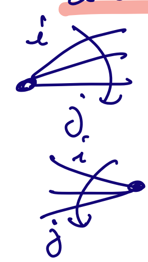
13|2 × 3|12

2|13×23|1

1|23 × 3|12

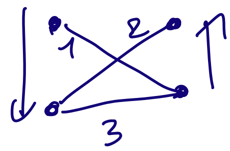


ordre autour d'un sommet

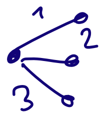


$i<j$

$1|23 \times 13|2 \longrightarrow$



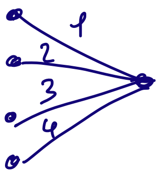
$123 \times 3|2|1$

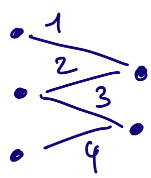


50

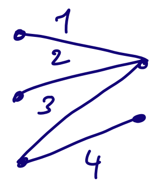
$n = 4$

$$
\begin{aligned}
(1234) = {} & 1|2|3|4 \times 1234 \cup 1234 \times 4|3|2|1 \cup 12|3|4 \times 2|134 \cup 134|2 \times 4|3|12 \\
\cup {} & 12|3|4 \times 23|14 \cup 14|23 \times 4|3|12 \cup 2|13|4 \times 23|14 \cup 14|23 \times 4|13|2 \\
\cup {} & 13|2|4 \times 3|124 \cup 124|3 \times 4|2|13 \cup 1|23|4 \times 3|124 \cup 124|3 \times 4|23|1 \\
\cup {} & 1|2|34 \times 124|3 \cup 3|124 \times 34|2|1 \cup 1|3|24 \times 134|2 \cup 2|134 \times 24|3|1 \\
\cup {} & 1|23|4 \times 134|2 \cup 2|134 \times 4|23|1 \cup 2|3|14 \times 234|1 \cup 1|234 \times 14|3|2 \\
\cup {} & 2|13|4 \times 234|1 \cup 1|234 \times 4|13|2 \cup 12|3|4 \times 234|1 \cup 1|234 \times 4|3|12 \\
\cup {} & 1|24|3 \times 14|23 \cup 23|14 \times 3|24|1 \cup 1|2|34 \times 14|23 \cup 23|14 \times 34|2|1 \\
\cup {} & 1|23|4 \times 13|24 \cup 24|13 \times 4|23|1 \cup 14|2|3 \times 4|123 \cup 123|4 \times 3|2|14 \\
\cup {} & 1|24|3 \times 4|123 \cup 123|4 \times 3|24|1 \cup 1|2|34 \times 4|123 \cup 123|4 \times 34|2|1 \\
\cup {} & 3|14|2 \times 34|12 \cup 12|34 \times 2|14|3 \cup 1|3|24 \times 34|12 \cup 12|34 \times 24|3|1 \\
\cup {} & 13|4|2 \times 34|12 \cup 12|34 \times 2|4|13 \cup 1|23|4 \times 34|12 \cup 12|34 \times 4|23|1 \\
\cup {} & 2|14|3 \times 24|13 \cup 13|24 \times 3|14|2 \cup 12|4|3 \times 24|13 \cup 13|24 \times 3|4|12 \\
\cup {} & 1|2|34 \times 24|13 \cup 13|24 \times 34|2|1
\end{aligned}
$$

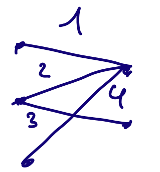




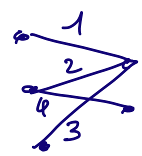
$1|23|4 \times 34|12$



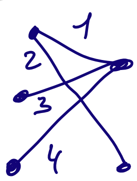
$1|2|34 \times 4|123$



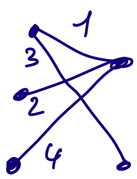
$1|23|4 \times 3|124$



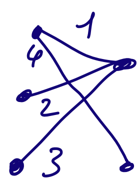
$1|24|3 \times 4|123$



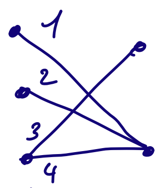
$12|3|4 \times 2|134$



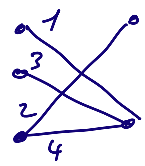
$13|2|4 \times 3|124$



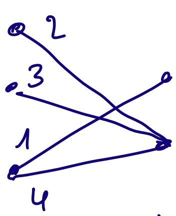
$14|2|3 \times 4|123$



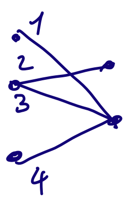
$1|2|34 \times 124|3$



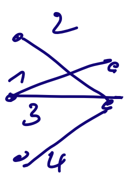
$1|3|24 \times 134|2$



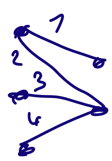
$2|3|14 \times 234|1$



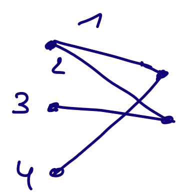
$1|23|4 \times 134|2$



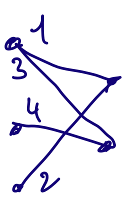
$2|13|4 \times 234|1$



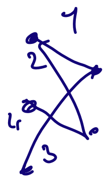
$12|3|4 \times 234|1$



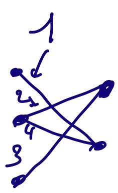
$12|3|4 \times 23|14$



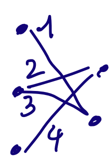
$13|4|2 \times 34|12$



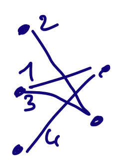
$12|4|3 \times 24|13$



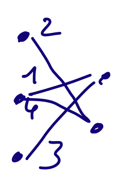
$1|24|3 \times 14|23$



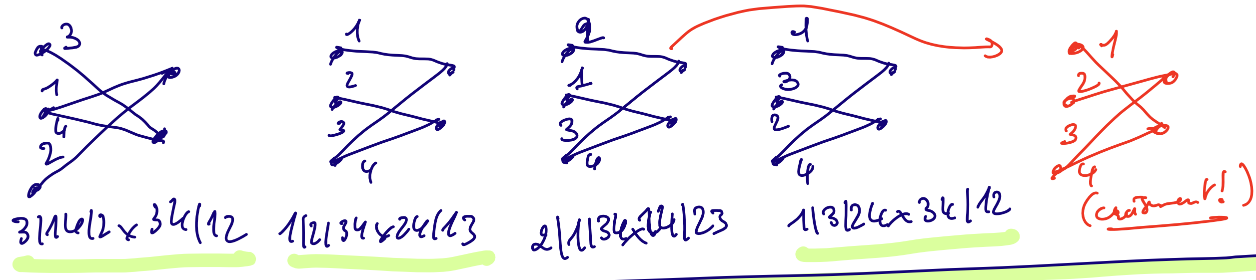
$1|23|4 \times 13|24$



$2|13|4 \times 23|14$

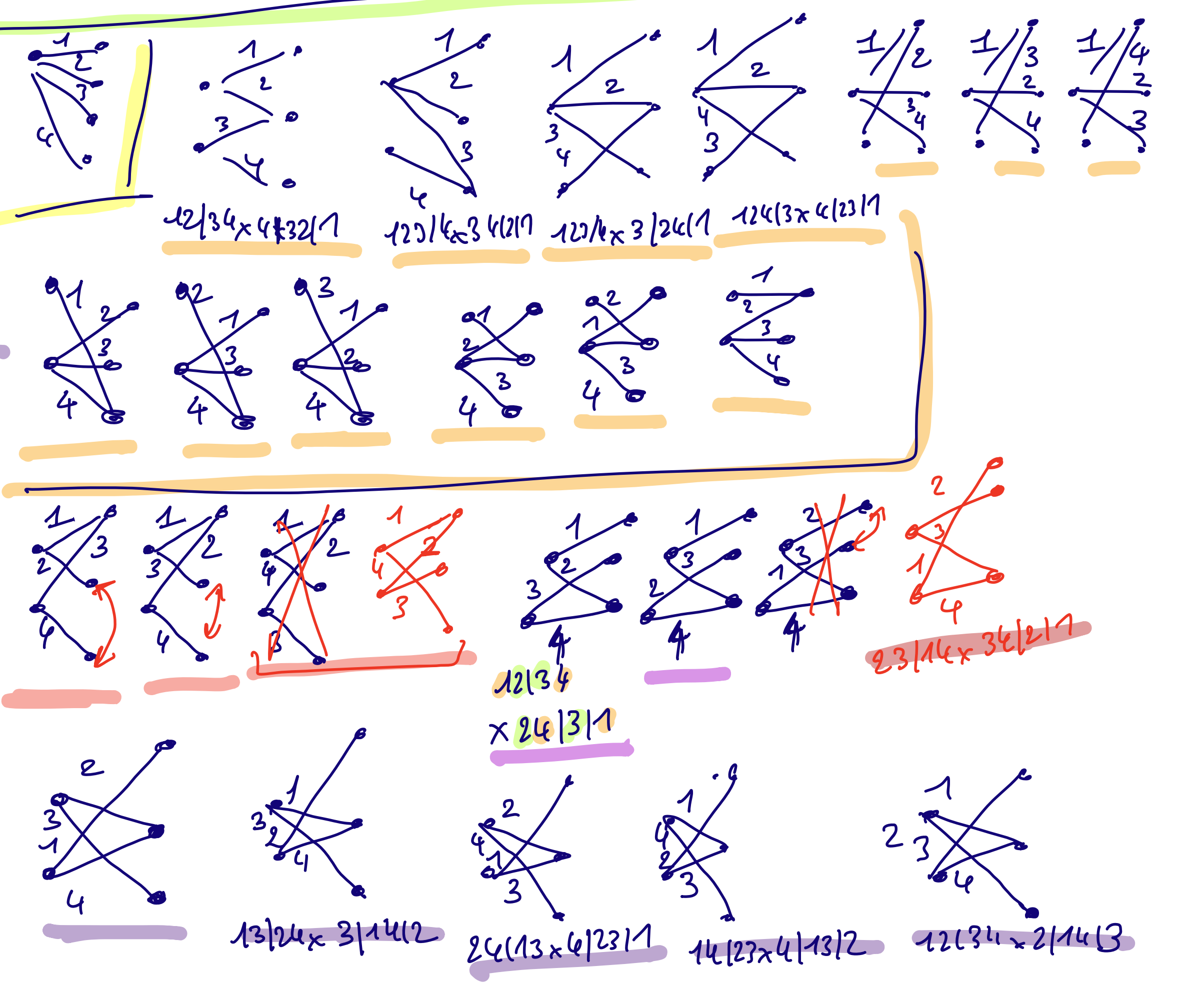


$2|14|3 \times 24|13$



1234 × 4|3|2|1
14|23 × 4|3|12
124|3 × 4|2|13
3|124 × 34|2|1
2|134 × 4|23|1
1|234 × 4|13|2
23|14 × 3|24|1
24|13 × 4|23|1
123|4 × 3|24|1
12|34 × 2|14|3
12|34 × 2|4|13
13|24 × 3|14|2
13|24 × 34|2|1

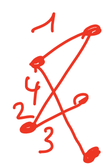
134|2 × 4|3|12
14|23 × 4|13|2
124|3 × 4|23|1
2|134 × 24|3|1
1|234 × 14|3|2
1|234 × 4|3|12
23|14 × 34|2|1
123|4 × 3|2|14
123|4 × 34|2|1
12|34 × 24|3|1
12|34 × 4|23|1
13|24 × 3|4|12

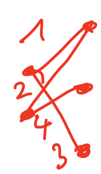


Pourquoi 14|23 × 4|3|12

mais

13|24 × 3|4|12

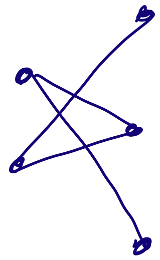




?

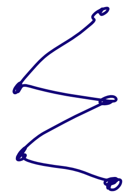
• Peut-on montrer les prop de ↗ ? { C'est une manière de rendre l'arbre biparti planaire ↳ plus que ça !

$(F,G) \in \mathrm{Im}\,\Delta_{(P,\vec{v})} \iff \forall (I,J) \in D(n-1), \exists N \in \mathcal{N}, |N \cap I| > |N \cap J|$ *or* $\exists N' \in \mathcal{N}', |N' \cap I| < |N' \cap J|$.

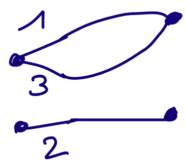


?

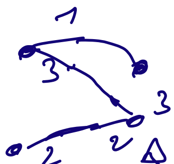
ns



$13|2 \times 2|3 \quad 1 \notin \Delta$

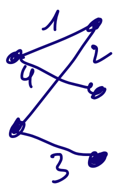
↳ 

ns pas connectés ni acycliques

$13|2 \times 23|1$ 

ns ne vérifie pas la ↗ !

↳ $I=\{2\}$ $J=\{3\}$ en effet là où le pb

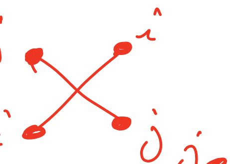


→ $14|23 \times 3|4|12$

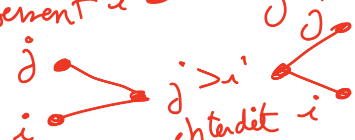
Règle pour $|I|=|J|=1$

j •

i • ⟹ croisement

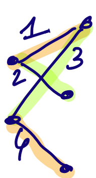


$j>i$

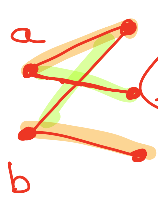
Notamment 

interdit

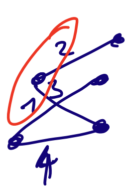
↳ croissance à droite et à gauche



$12|34 \times 4|2|13$ $I=\{1,4\}$ $J=\{2,3\}$



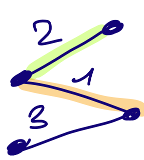
interdit



croisent !

$23|14 \times 34|1|2$

• Croissance à gauche

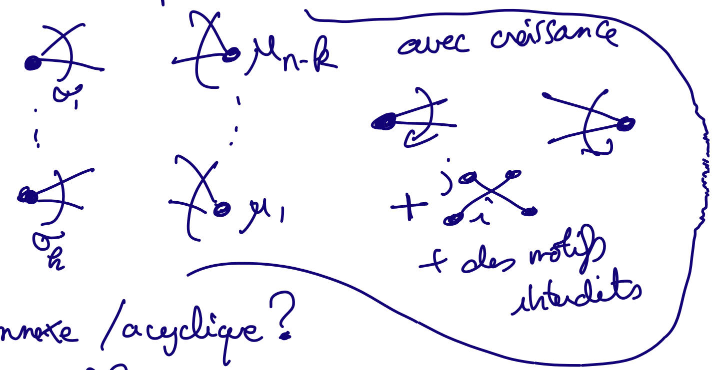


ns $2|13 \times 13|2$

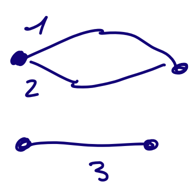
pas très essentielles

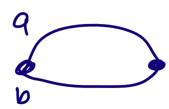
• $(\sigma,\mu)\in\Delta \overset{\varphi}{\underset{\sim}{\longmapsto}}$ arbre plan $\overset{\psi}{\longmapsto}$ arbres bipartis avec croissance

$\sigma = \sigma_1|\dots|\sigma_k \qquad \mu = \mu_1|\dots|\mu_{n-k}$



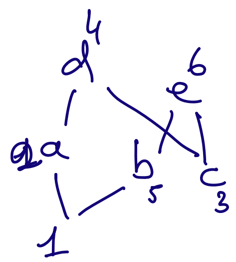
1) * Pourquoi est-ce bien connexe / acyclique ?
2) * φ/ψ injectif ?
3) * φ/ψ surjectif ?
4) * Quels motifs interdits ?
↳ lien Poisson

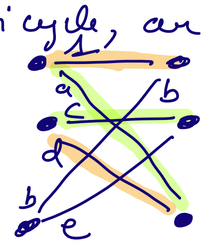
1) 

$12|3 \times 3|12 \Rightarrow$ 

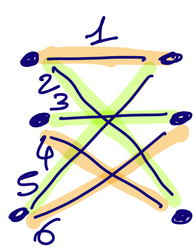
interdit car $I=\{a\}$, $J=\{b\}$ vide la condition

$\Rightarrow$ si cycle, au $\ominus$ 3 sommets (4 car bipartis)





no potentiellement moche



$12|34|56 \times 24|36|15$

↳ On colorie les arêtes avec alternance I et J en commençant par un min

alors en chaque sommet, autant de I que de J
↳ n'est pas ds la diagonale

$\Rightarrow$ acyclique (et c'est un arbre)

2) φ clairement injectif car on relit σ et μ sur l'arbre

3) φ surjectif car les motifs interdits correspondent aux poissons

4) Motifs interdits

Motif poisson

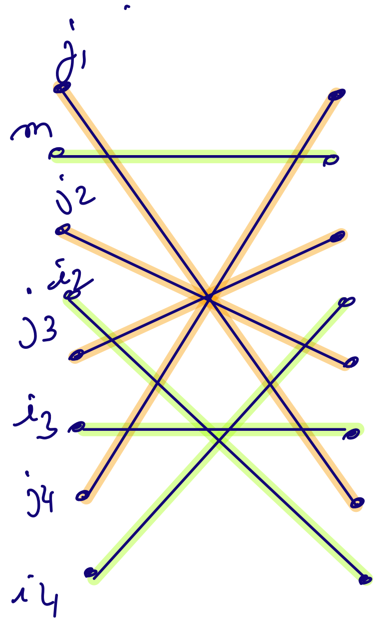
↳ $|I| = |J| = n$ $\min I < \min J$ ($i_1 = m$)

+ motifs $j_1\, m\, j_2\, i_2 \cdots j_n\, i_n$ dans σ_1

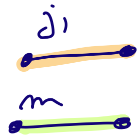
et $i_2\, j_1\, i_3\, j_2 \cdots i_n\, j_{n-1}\, m\, j_n$ dans σ_2

feuille : Poisson

$i\, j\, j\, i\, j\, i \ldots j\, i$ ⟺ pas de diag

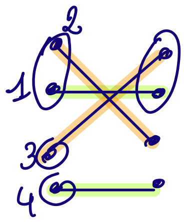


$n=1$



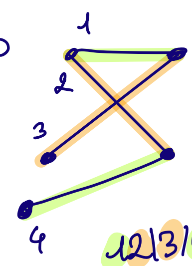
$m < j_1$

$n=2$



+ union de noeuds

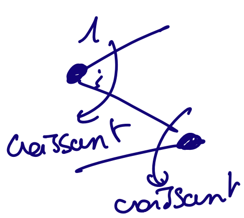
exple ms

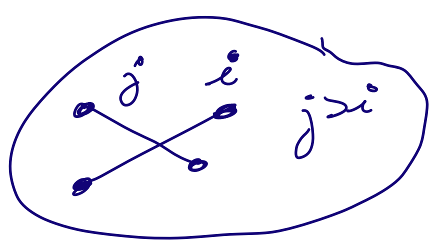


12|3|4 × 24|13

Que dire de ψ ?

ψ^{-1} : on plonge l'arbre dans le plan





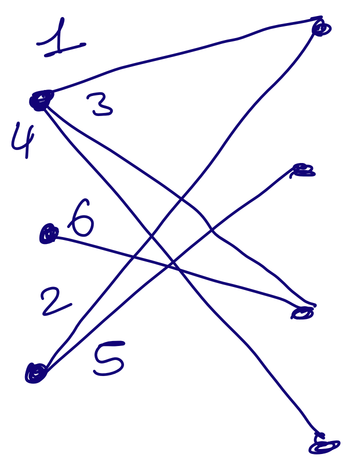
- À priori, ambiguïté :

↳ Mq 1 et 1 seul des plongements ne vérifie pas le motif poisson

(revient à mq seul un ordre sur les parts des partitions complémentaires essentielles convient)

Réunion avec Kurt, Guillaume et Matthieu

$F \quad 134|6|25 \times \underbrace{4|36|5|12}_{G}$



$(F_1, G) \quad F_1 = 1346|25 \longrightarrow$

$$M \begin{cases} \left.\begin{matrix} 101101 \\ 010010 \end{matrix}\right\} F_1 \\ \left.\begin{matrix} 000100 \\ 001001 \\ 000010 \\ 110000 \end{matrix}\right\} G \end{cases} \qquad X = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

⇝ solve $MX = 0$

$(F_2, G) \quad F_2 = 134|256$

$$\begin{matrix} 101100 \\ 010011 \\ 000100 \\ 001001 \\ 000010 \\ 110000 \end{matrix} \rightarrow d = e = 0 \qquad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

⇝ Système d'équations

(nonzero coordinate of a boolean vector

$x = {}^t(a \; b \; c \; d \; e \; f)$

$$Mx = 0 \iff \begin{cases} a+c+d+f=0 \\ b+e=0 \quad \underline{e=a} \\ d=0 \\ c+f=0 \to c=-f \\ e=0 \\ a+b=0 \to \underline{b=-a} \end{cases}$$

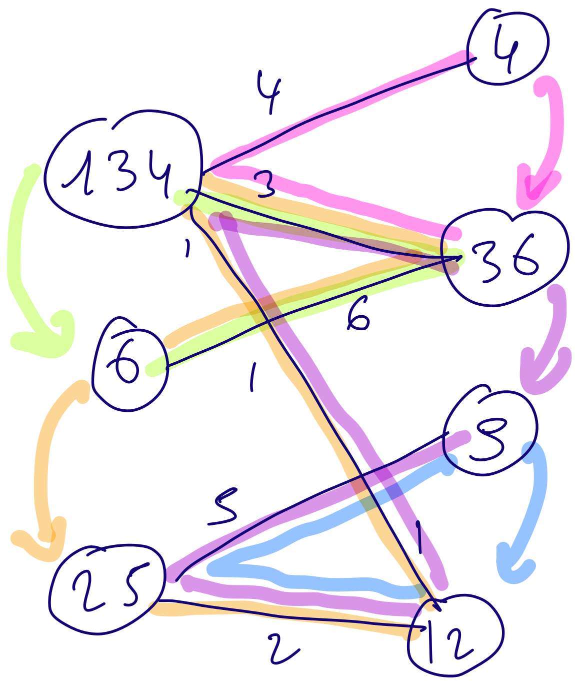
(F, G_1) 134|6|25 × 346|5|12

$$\begin{matrix} 1&0&1&1&0&0 \\ 0&0&0&0&0&1 \\ 0&1&0&0&1&0 \\ 0&0&1&1&0&1 \\ 0&0&0&0&1&0 \\ 1&1&0&0&0&0 \end{matrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

(F, G_2) 134|6|25 × 4|256|12

101100

(F, G_3) $134|6|25 \times 4|36|1|25$



Claim: the selects above are in biject with paths between consecutive blocks

these selects are the condit's (I, J) that Guillaume obtained

$1 \rightsquigarrow I$
$-1 \rightsquigarrow J$

Thm: Guillaum $\to$ Quantifies on some I and J $\rightsquigarrow$ given by paths
$\rightsquigarrow$ minimal I and J needed for any pairs

Maximal pair $\to n(I, J)$

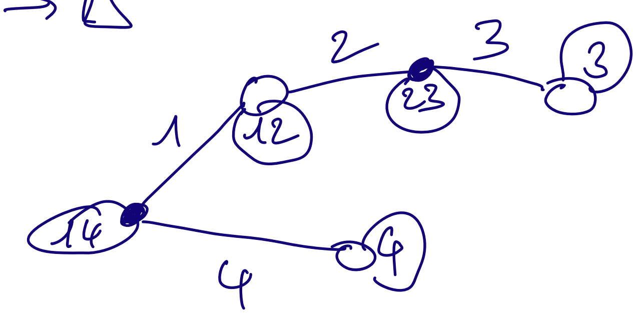
$\not\nearrow$ dimension

- Prove that the solu° of the system is in bijection with paths

$\mu : \Delta \to C$

forget the order
and prove that we have a bijectio with bipartite trees (Comme j'ai fait)

$\sigma : C \to \Delta$



$I = \{1\}$ $J = \{2\}$

$6 \to 25\{\overset{I}{6}, 3, 1, \underset{J}{2}\}$ I edges $\to$ J edges $\leftarrow$

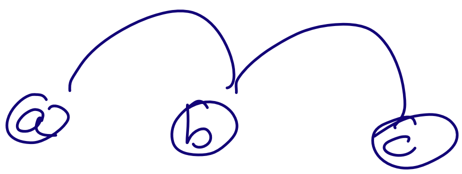
smaller of path from left to right

to complete :

↳ prove that no dhunDn condit° needed

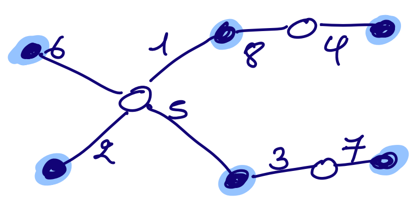
• Mq ordre transitif → fait!

↳ $a < b$, $b < c$



→ Union des chemins et différence symétrique

Exemple :



4 •

18 •

2 •

35 •

•

↳ Bravo!

Formule d'énumération :

$$\boxed{\frac{1}{n+1-k}\binom{n+1}{k+1}(n+1-k)^{k}(k+1)^{n-k}} \qquad 0\leq k\leq n$$

$n=0,\ k=0 \quad \frac{1}{2}\binom{2}{1}2^{0}1^{1}=1 \qquad \frac{1}{1}\binom{1}{1}(1)^{0}1^{0}=1$

$n=1,\ k=0$

$k=1 \longrightarrow \frac{1}{2-1}\binom{2}{2}(2-1)^{1}2^{1-1}=1$

$n=2,\ k=0 \longrightarrow \frac{1}{3}\binom{3}{1}3^{0}1^{2}=1$

$k=1 \longrightarrow \frac{1}{2}\binom{3}{2}2^{1}2^{1}=6$ OK !

$k=2 \longrightarrow \frac{1}{1}\binom{3}{3}1^{2}3^{0}=1$

Bijection : avec arbres bipartis avec $(k+1)$ sommets •

$(n-k+1)$ — o

et $(n+1)$ arêtes étiquetées par $1,\dots,n+1$

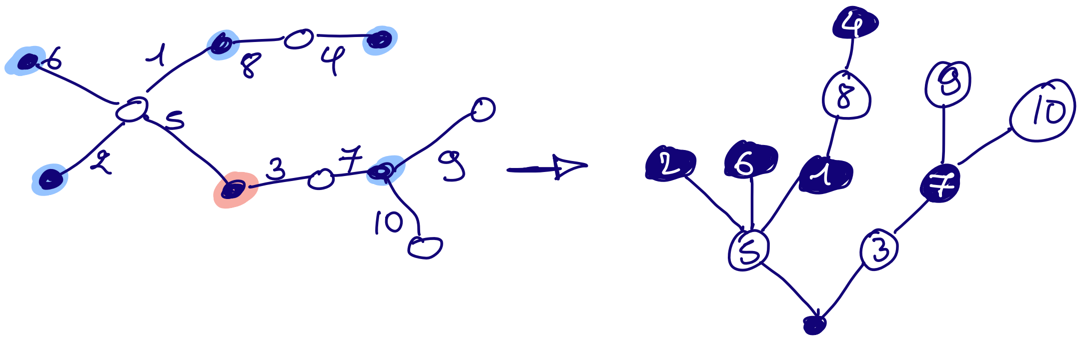
$$\frac{1}{n+1-k}\binom{n+1}{k+1}(n+1-k)^{k}(k+1)^{n-k}$$

$$=\frac{(n+1)!}{(k+1)!\,(n-k)!}(n+1-k)^{k-1}(k+1)^{n-k}$$

$$=\frac{1}{(k+1)}\times\frac{(n+1)!}{k!\,(n+1-k)!}(n+1-k)^{k}(k+1)^{n-k}$$

$$=\frac{1}{k+1}\binom{n+1}{k}(n+1-k)^{k}(k+1)^{n-k}$$

1 $\times(k+1)$ ↝ on enracine en un sommet noir



$\binom{n+1}{k}$

↳ choix des n^{os} des sommets noirs (et des sommets blancs du coup)

Code de Prüfer

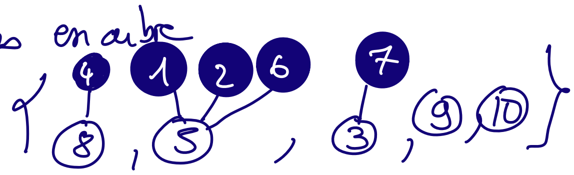
↳

$$\left.\begin{array}{l} 1 \to 5 \\ 2 \to 5 \\ 4 \to 8 \\ 6 \to 5 \\ 7 \to 3 \end{array}\right) (n-k+1) \text{ choix}$$

$\leadsto (n+1-k)^k$

Reste à assembler ces corolles en arbre

1 • 77

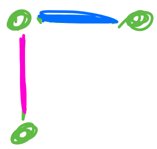


Comprendre les sommets de manière générale :

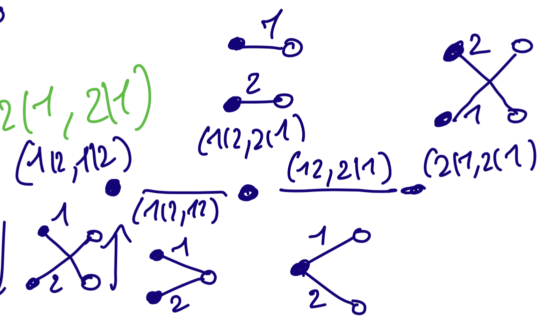
Exple

$n=0$ $(1,1) \rightsquigarrow 1$

$n=1$



$\rightsquigarrow \underset{(1|2,\,1|2)}{1} + \underset{(2|1,\,1|2)}{x} + y \to (2|1,\,2|1)$



$n=2$ $\rightsquigarrow$ $1 \longrightarrow (1|2|3,\ 1|2|3)$

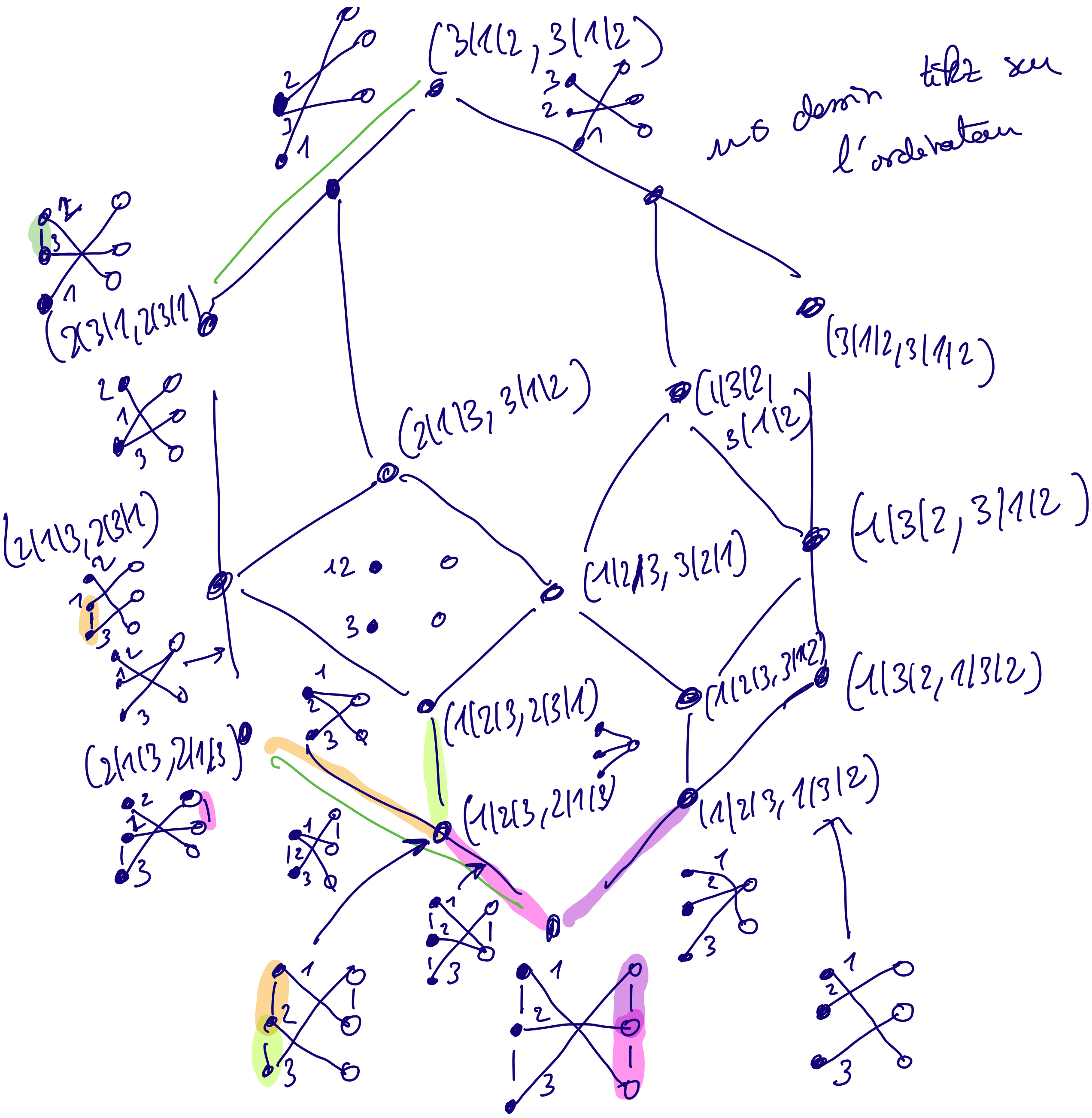
$+\,4x \longrightarrow (2|1|3,\ ")\ (2|3|1,\ ")\ (3|1|2,\ ")\ (1|3|2,\ ")$

$+\,4y \longrightarrow (1|2|3,\ 2|1|3)\ (",\ 2|3|1)\ (",\ 1|3|2)\ (",\ 3|1|2)$ (ou)

$+\,6xy \longrightarrow (2|1|3,\ 2|3|1),\ (2|1|3,\ 3|1|2)\ (2|3|1,\ 3|2|1)$
$(3|1|2,\ 3|2|1),\ (1|3|2,\ 3|1|2)\ (1|3|2,\ 3|1|2)$

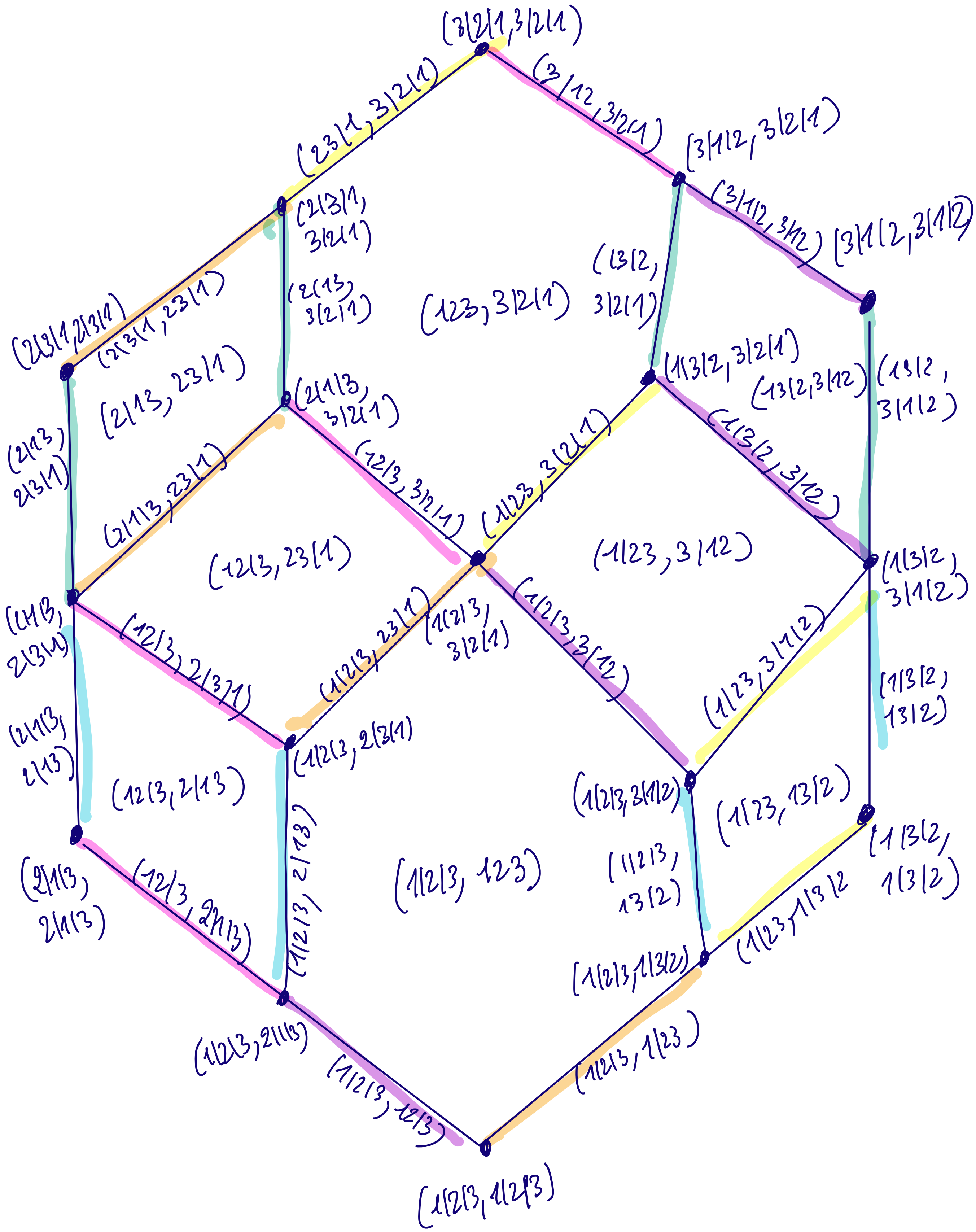
$+\,x^2 \longrightarrow (3|2|1,\ 3|2|1)$

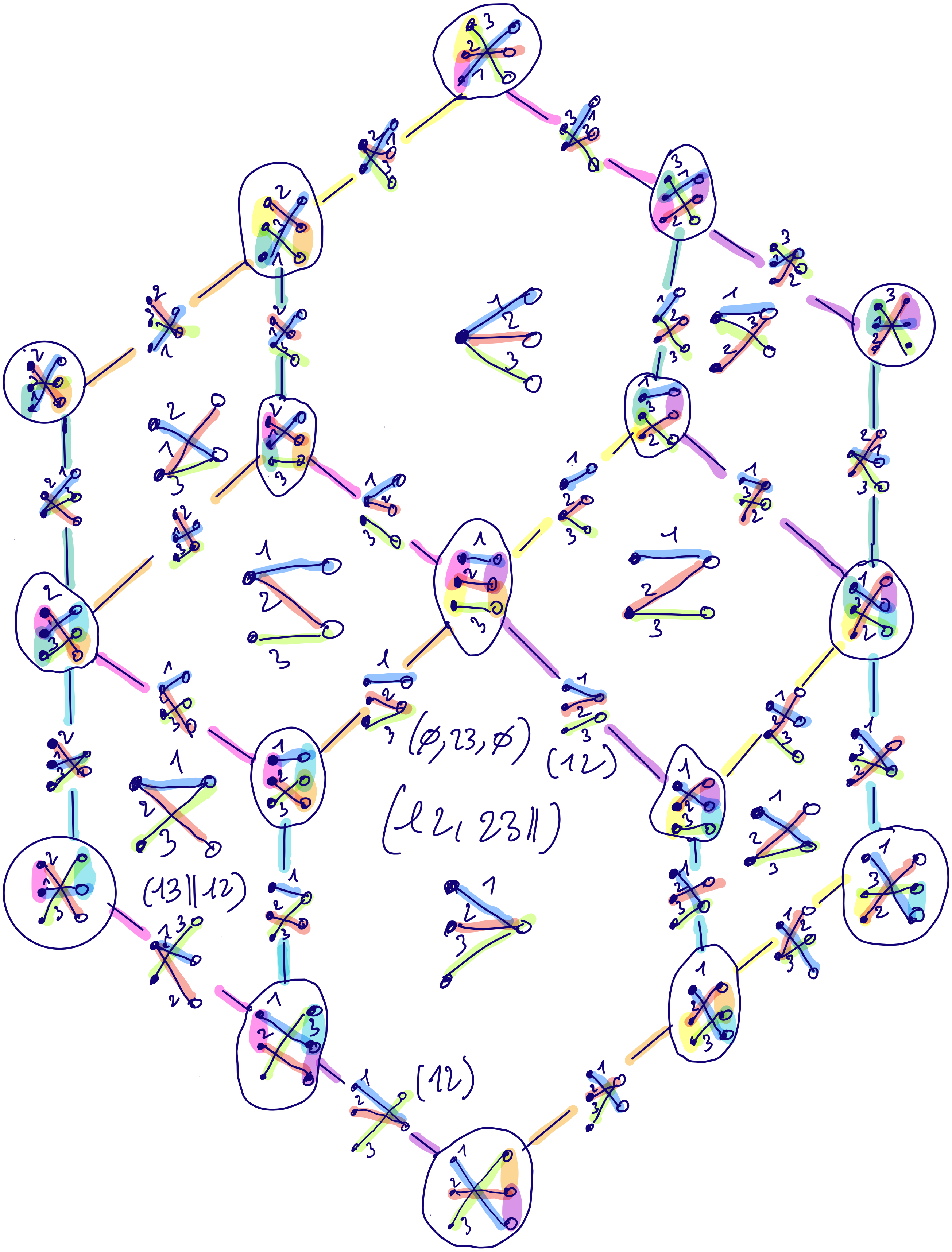
$+\,y^2 \longrightarrow (1|2|3,\ 3|2|1)$



⇝ dessin tikz sur l'ordinateur

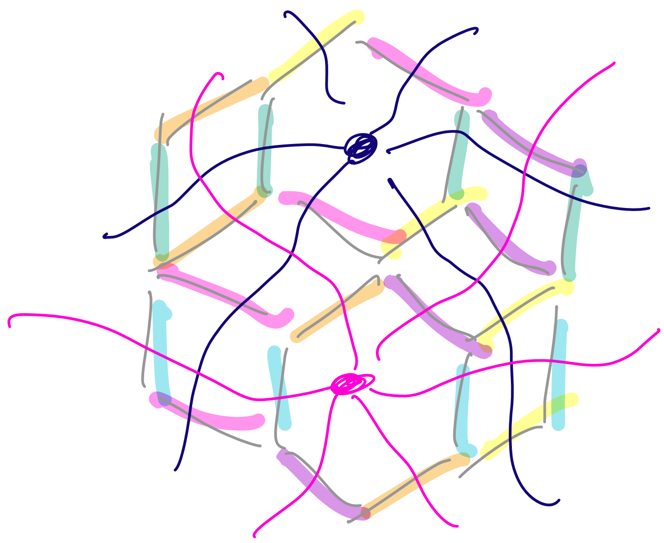
⇝ Forêts d'arbres bipartis


(3|2|1, 3|2|1)
(23|1, 3|2|1)
(3|12, 3|2|1)
(3|1|2, 3|2|1)
(2|3|1, 3|2|1)
(3|1|2, 3|12)
(3|1|2, 3|1|2)
(13|2, 3|2|1)
(123, 3|2|1)
(2|3|1, 2|3|1)
(2|3|1, 23|1)
(2|13, 3|2|1)
(1|3|2, 3|2|1)
(13|2, 3|12)
(13|2, 3|1|2)
(2|13, 23|1)
(2|1|3, 3|2|1)
(2|13, 2|3|1)
(12|3, 3|2|1)
(1|3|2, 3|12)
(2|1|3, 23|1)
(1|23, 3|2|1)
(12|3, 23|1)
(1|23, 3|12)
(1|3|2, 3|1|2)
(2|1|3, 2|3|1)
(12|3, 2|3|1)
(1|2|3, 23|1)
(1|2|3, 3|2|1)
(1|2|3, 3|12)
(2|1|3, 2|13)
(1|2|3, 2|3|1)
(1|23, 3|1|2)
(1|3|2, 13|2)
(12|3, 2|13)
(1|2|3, 3|1|2)
(1|23, 13|2)
(2|1|3, 2|1|3)
(12|3, 2|1|3)
(1|2|3, 2|13)
(1|2|3, 123)
(1|2|3, 13|2)
(1|3|2, 1|3|2)
(1|2|3, 1|3|2)
(1|23, 1|3|2)
(1|2|3, 2|1|3)
(1|2|3, 1|23)
(1|2|3, 12|3)
(1|2|3, 1|2|3)



(∅, 23, ∅)
(12)
(12, 23‖)
(13‖12)
(12)

P polytope $\qquad (P+P)/2$ here



$$P = \sum_{1 \le i < j \le n+1} [e_i - e_j, e_j - e_i]$$

À faire : → Étendre bijectⁿ à toutes les faces

↳ puis extension à l'associaèdre, le multiplicaèdre, ...

→ Link with Sanebl... et Umble

↳ → comprendre lien faces / facettes