THE COMBINATORICS OF THE PERMUTAHEDRON DIAGONALS

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Abstract. The purpose of this article is to provide a systematic combinatorial-geometric study of cellular diagonals on the permutahedra. First, we give general enumeration results for the faces of any geometric diagonal, using a theorem of Zavlasky on the Möbius function of hyperplane arrangements. The numbers obtained involve covering trees of bipartite graphs and Catalan families. Second, we prove that there are only two operadic families of diagonals on the permutahedra, and we give a bijection of their facets with planar bipartite trees and characterize their vertices as pattern-avoiding pairs of permutations. A corollary of our study is that the Saneblidze–Umble diagonal can be recovered by a choice of vectors in the fundamental hyperplane arrangements of the permutahedra, resolving a conjecture of the third-named author.

Contents

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Introduction

In this article, discrete geometry *informs* higher algebra.

Some purely combinatorial results, whose study is motivated by higher algebra: the combinatorially inclined reader can read the first part of this introduction, and go directly to Section 1.

Some higher algebraic results, obtained via discrete geometry: the algebraically inclined reader can read the second part of this introduction, and go directly to Section 2.

The reader who reads the entire article will start feeling the essence of what could be *higher discrete geometry*, a new field of mathematics whose purpose is to use the theory of polytopes in order to study higher algebraic structures and homotopy theory.

Combinatorics. The purpose of the first part of this article is to study the combinatorics of the cellular diagonal of the permutohedra, explained in more details in the second part of the introduction. To make this part more accessible to non-algebraists, we formulate it in terms of hyperplane arrangements, as *faces* of the cellular diagonal are in bijection with *regions* of the associated hyperplane arrangements. The hyperplane arrangement associated with the cellular diagonal is two generically translated copied of the braid arrangement. Motivated by the beautiful combinatorial formula obtained in our study, we generalize it to the study of ℓ generically translated copies of the braid arrangement, which would algebraically corresponds to some cellular ℓ-gonal. We compute the f-vector and the b-vector of ℓ-generically translated copies of the braid arrangement, using Zaslavsky's Theorem. We refine the enumeration of the vertices of the hyperplane arrangement using a combinatorial interpretation in terms of (ℓ, n)-trees and get the following multiplicative formula:

Theorem 0.1. *The number of vertices of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ (or equivalently of (ℓ, n)-trees) is*

$$f_0(\mathcal{B}_n^\ell) = \ell\big((\ell-1)n+1\big)^{n-2}.$$

We end this combinatorial section with a conjecture on edge-colored trees.

Higher Algebra. The purpose of this article is to study cellular diagonals on the permutahedra, which are cellular maps homotopic to the usual thin diagonal $\triangle : P \to P \times P, x \mapsto (x, x)$. Such diagonals, and in particular coherent families that we call *operadic* diagonals (see [DEF]), are of interest in algebraic geometry and topology: via the theory of Fulton–Sturmfels [FS97], they give explicit formulas for the cup product on Losev–Manin toric varieties [LM00]; they define universal tensor products of homotopy operads, and in particular universal tensor products of homotopy associative permutads (shuffle algebras) [Lap22]; they allow for the definition of a coproduct on permutahedral sets, which are used to models of two-fold loop spaces [SU04]; and their study is moreover needed to pursue the work of Baues aiming at defining explicit combinatorial models for higher iterated loop spaces [Bau80]. Moreover, using the canonical projections to the multiplihedra and the associahedra, they define universal tensor product of A_∞-algebras and A_∞-morphisms [LAM23]. Guillaume: lien avec les matroides?

The first cellular diagonal for the permutahedra was defined at the level of chains by S. Saneblidze and R. Umble in [SU04], we will call it the *SU diagonal*. Cellular diagonals for the associahedra and the multiplihedra were also defined there, via projection. The first topological map for the permutahedra was given in [Lap22] -we will call it the *LA diagonal*, where a general theory of cellular diagonals of polytopes was developed. The LA diagonal, however, is distinct from the SU diagonal at the cellular level [Lap22, Remark 3.19].

computer program for the SU diagonal [Vej07]; we give a much more effective computer program (however, without the signs)

corollary: le dernier article de SU! corollary: IJ-description of SU diagonal corollary: left and right shifts for LA, decomposition of the cube associated to LA diagonal

Conventions. Guillaume: Please make a new line for each sentence as to facilitate comparison between versions in GitHub

Guillaume: Would it be possible to avoid bold letters, use "slash emph{}", and also reduce to the maximum possible the use of indices; as soon as the context is clear, drop any extra index

1. Combinatorics

1.1. Combinatorics of generically translated copies of the braid arrangement.

1.1.1. *Recollection on hyperplane arrangements.* We first briefly recall classical results on the combinatorics of affine hyperplane arrangements, in particular the enumerative connection between their intersection posets and their face lattices due to T. Zaslavsky [Zas75].

Definition 1.1. A (finite affine real) *hyperplane arrangement* is a finite set $\mathcal{A}$ of affine hyperplanes in $\mathbb{R}^d$.

Definition 1.2. A *region* of $\mathcal{A}$ is a connected component of $\mathbb{R}^d \smallsetminus \bigcup_{H\in\mathcal{A}} H$. A *face* of $\mathcal{A}$ is the intersection of the closure of a region of $\mathcal{A}$ with an hyperplane of $\mathcal{A}$. The *face poset* of $\mathcal{A}$ is the poset $\mathsf{Fa}(\mathcal{A})$ of faces of $\mathcal{A}$ ordered by inclusion. The *f-polynomial* $\boldsymbol{f}_{\mathcal{A}}(x)$ and *b-polynomial* $\boldsymbol{b}_{\mathcal{A}}(x)$ of $\mathcal{A}$ are the polynomial

$$\boldsymbol{f}_{\mathcal{A}}(x) := \sum_{k=0}^{d} f_k(\mathcal{A})\, x^k \qquad \text{and} \qquad \boldsymbol{b}_{\mathcal{A}}(x) := \sum_{k=0}^{d} b_k(\mathcal{A})\, x^k$$

where $f_k(\mathcal{A})$ denotes the number of k-dimensional faces of $\mathcal{A}$, while $b_k(\mathcal{A})$ denotes the number of bounded k-dimensional faces of $\mathcal{A}$.

Definition 1.3. A *flat* of $\mathcal{A}$ is a non-empty affine subspace of $\mathbb{R}^d$ that can be obtained as the intersection of some hyperplanes of $\mathcal{A}$. The *flat poset* of $\mathcal{A}$ is poset $\mathsf{Fl}(\mathcal{A})$ of flats of $\mathcal{A}$ ordered by reverse inclusion.

Definition 1.4. The *Möbius polynomial* $\mu_{\mathcal{A}}(x,y)$ is the polynomial defined by

$$\boldsymbol{\mu}_{\mathcal{A}}(x,y) := \sum_{F\le G} \mu_{\mathsf{Fl}(\mathcal{A})}(F,G)\, x^{\dim(F)}\, y^{\dim(G)},$$

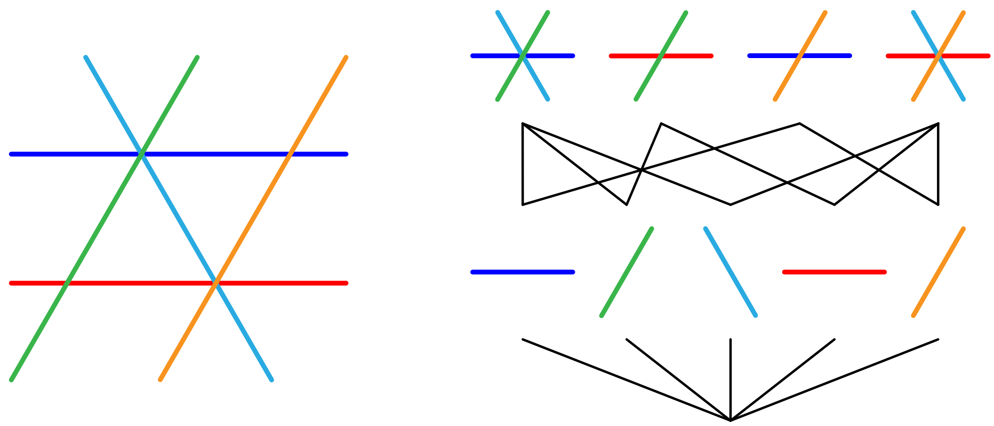


Figure 1. A hyperplane arrangement (left) and its intersection poset (right).

where $F \le G$ ranges over all intervals of the flat poset $\mathsf{Fl}(\mathcal{A})$, and $\mu_{\mathsf{Fl}(\mathcal{A})}(F, G)$ denotes the *Möbius function* on the flat poset $\mathsf{Fl}(\mathcal{A})$ defined as usual by

$$\mu_{\mathsf{Fl}(\mathcal{A})}(F, F) = 1 \qquad \text{and} \qquad \sum_{F \le G \le H} \mu_{\mathsf{Fl}(\mathcal{A})}(F, G) = 0$$

for all $F, G, H \in \mathsf{Fl}(\mathcal{A})$.

Remark 1.5. Two observations on Definition 1.4:

- The coefficient of x^d in the Möbius polynomial $\boldsymbol{\mu}_{\mathcal{A}}(x, y)$ gives the more classical *characteristic polynomial*
$$\boldsymbol{\chi}_{\mathcal{A}}(y) := [x^d]\, \boldsymbol{\mu}_{\mathcal{A}}(x, y) = \sum_{G} \mu_{\mathsf{Fl}(\mathcal{A})}(\mathbb{R}^d, G)\, y^{\dim(G)}.$$
- Our definition of the Möbius polynomial slightly differs from that of [Zas75] as we use the dimension of F instead of its codimension, in order for the next statement to be slightly simpler.

Theorem 1.6 ([Zas75, Thm. A])**.** *The f-polynomial, the b-polynomial, and the Möbius polynomial of an arrangement $\mathcal{A}$ are related by*

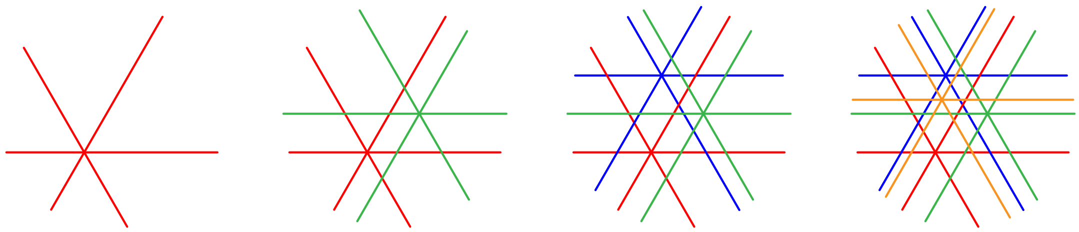
$$\boldsymbol{f}_{\mathcal{A}}(x) = \boldsymbol{\mu}_{\mathcal{A}}(-x, -1) \qquad \textit{and} \qquad \boldsymbol{b}_{\mathcal{A}}(x) = \boldsymbol{\mu}_{\mathcal{A}}(-x, 1).$$

Example 1.7. For the arrangement $\mathcal{A}$ of 5 hyperplanes of Figure 1, we have

$$\boldsymbol{\mu}_{\mathcal{A}}(x, y) = x^2y^2 - 5x^2y + 6x^2 + 5xy - 10x + 4$$

so that

$$\boldsymbol{f}_{\mathcal{A}}(x) = \boldsymbol{\mu}_{\mathcal{A}}(-x, -1) = 12\,x^2 + 15\,x + 4$$
$$\text{and} \qquad \boldsymbol{b}_{\mathcal{A}}(x) = \boldsymbol{\mu}_{\mathcal{A}}(-x, 1) = 2\,x^2 + 5\,x + 4.$$



$\ell = 1$ $\ell = 2$ $\ell = 3$ $\ell = 4$

FIGURE 2. The $(\ell, 3)$-braid arrangements for $\ell \in [4]$.

$\ell = 1$

$n \backslash k$	0	1	2	3	Σ
1	1				1
2	2	1			3
3	6	6	1		13
4	24	36	14	1	75

$n \backslash k$	0	1	2	3	Σ
1	1				1
2	0	1			1
3	0	0	1		1
4	0	0	0	1	1

$\ell = 2$

$n \backslash k$	0	1	2	3	Σ
1	1				1
2	3	2			5
3	17	24	8		49
4	149	324	226	50	749

$n \backslash k$	0	1	2	3	Σ
1	1				1
2	1	2			3
3	5	12	8		25
4	43	132	138	50	363

$\ell = 3$

$n \backslash k$	0	1	2	3	Σ
1	1				1
2	4	3			7
3	34	54	21		109
4	472	1152	924	243	2791

$n \backslash k$	0	1	2	3	Σ
1	1				1
2	2	3			5
3	16	36	21		73
4	224	684	702	243	1853

$\ell = 4$

$n \backslash k$	0	1	2	3	Σ
1	1				1
2	5	4			9
3	57	96	40		193
4	1089	2808	2396	676	6969

$n \backslash k$	0	1	2	3	Σ
1	1				1
2	3	4			7
3	33	72	40		145
4	639	1944	1980	676	5239

TABLE 1. The face numbers (top) and the bounded face numbers (bottom) of the (ℓ, n)-braid arrangements for $\ell, n \in [4]$.

1.1.2. *The (ℓ, n)-braid arrrangement.* We now focus on the following specific hyperplane arrangements, illustrated in Figure 2, and whose face numbers are given in Table 1.

Definition 1.8. Denote by $\mathbb{H}$ the hyperplane of $\mathbb{R}^n$ defined by $\sum_{i=1}^n x_i = 0$. For any integer $n \geq 1$, the *braid arrangement* $\mathcal{B}_n$ is the arrangement of the hyperplanes $\{\boldsymbol{x} \in \mathbb{H} \mid x_i = x_j\}$ for all $1 \leq i < j \leq n$. For any integers $\ell, n \geq 1$, the *(ℓ, n)-braid arrangement* $\mathcal{B}_n^\ell$ is the arrangement obtained as the union of ℓ generically translated copies of the braid arrangement. Vincent: We have to take a decision here: do we work in $\mathbb{R}^n$ or in $\mathbb{H}$. At the moment, I work in $\mathbb{H}$. Working in $\mathbb{R}^n$ changes vertices to rays, and multiplies all Möbius polynomials by a factor xy.

Remark 1.9. The combinatorics of the braid arrangement $\mathcal{B}_n$ is well-known. It has a k-dimensional flat $\{\boldsymbol{x} \in \mathbb{R}^n \mid x_i = x_j \text{ for all } i, j \in p \in \pi\}$ for each partition π of $[n]$ into $k+1$ parts. The flat poset $\mathsf{Fl}(\mathcal{B}_n)$ is thus the refinement poset Pa_n on partitions of $[n]$. Its Möbius function is given by

$$\mu_{\mathsf{Pa}_n}(\pi, \omega) = \prod_{p \in \omega} (-1)^{\#\pi[p]-1} (\#\pi[p] - 1)!$$

where $\pi[p]$ denotes the restriction of the partition π to the part p of the partition ω. The Möbius polynomial of the braid arrangement $\mathcal{B}_n$ is given by

$$\boldsymbol{\mu}_{\mathcal{B}_n}(x, y) = \sum_{k \in [n]} x^{k-1} P(n, k) \prod_{i \in [k-1]} (y - i)$$

where $P(n, k)$ denotes the Stirling number of second kind, *i.e.*, the number of partitions of n into k parts. For instance

$$\begin{aligned}
\boldsymbol{\mu}_{\mathcal{B}_1}(x, y) &= 1 \\
\boldsymbol{\mu}_{\mathcal{B}_2}(x, y) &= xy - x + 1 = x(y-1) + 1 \\
\boldsymbol{\mu}_{\mathcal{B}_3}(x, y) &= x^2y^2 - 3x^2y + 2x^2 + 3xy - 3x + 1 = x^2(y-1)(y-2) + 3x(y-1) + 1 \\
\boldsymbol{\mu}_{\mathcal{B}_4}(x, y) &= x^3y^3 - 6x^3y^2 + 11x^3y - 6x^3 + 6x^2y^2 - 18x^2y + 12x^2 + 7xy - 7x + 1 \\
&= x^3(y-1)(y-2)(y-3) + 6x^2(y-1)(y-2) + 7x(y-1) + 1.
\end{aligned}$$

Remark 1.10 (Combinatorics of the braid arrangement). The intersection poset of the braid arrangement $\mathcal{B}_n$ is the poset of partitions of (a set of) size n, ordered by refinement. A partition π is smaller than a partition π' if the parts of π are contained in parts of π'. It is well-known (see for instance [Bir95, Rot64]) that the Möbius number of the poset of partitions of size p is given by:

$$(-1)^{p-1}(p-1)!$$

The Möbius polynomial of the braid arrangement is then given by:

$$\mu_{\mathcal{B}_n}(x, y) = \sum_{\pi \leq \pi'} \mu(\pi, \pi') x^{|\pi|-1} y^{|\pi'|-1}, \tag{1.1}$$

where the sum runs over all the pair of comparable partitions and $|\pi|$ denotes the number of parts in the partition π.

$$\mu_{\mathcal{B}_n}(x, y) = \frac{1}{xy} \sum_{1 \leq k \leq n} \sum_{\ell_1, \ldots, \ell_k \geq 1} S(n, \ell) \frac{(\ell - 1)!}{\prod_{i=2}^k \left(\sum_{j=i}^k \ell_j \right)} (-x)^\ell (-y)^k, \tag{1.2}$$

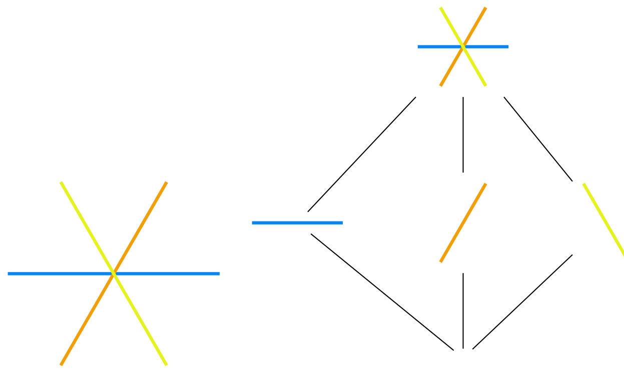


FIGURE 3. The braid arrangement $\mathcal{B}_3$ and its associated intersection poset.

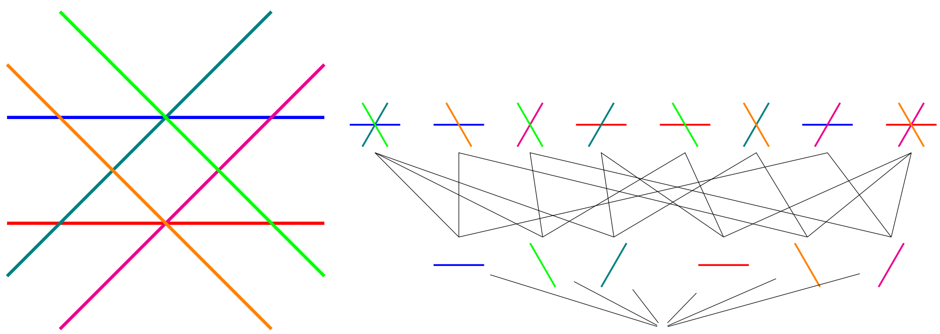


FIGURE 4. The $(2,3)$-braid arrangement $\mathcal{B}_3^2$ and its associated intersection poset.

where $\ell = \sum_{i=1}^{k} \ell_i$. Bérénice: Nicer formula ? We represent on Figure 3 the braid arrangement and its associated intersection poset. We generalize in this section results to ℓ copies of braid arrangements ; on figure 6 is represented the $(2,3)$-braid arrangement $\mathcal{B}_3^2$ and its associated intersection poset.

Definition 1.11. The *intersection hypergraph* $I(F_1, \dots, F_\ell)$ of a ℓ-tuple of partitions of $[n]$ is the ℓ-regular ℓ-partite hypergraph on all parts of all the partitions F_i, with an hyperedge for all $i \in [n]$ connecting the parts containing i. A *(ℓ, n)-forest* (resp. *(ℓ, n)-tree*) is a ℓ-tuple $\boldsymbol{F} := (F_1, \dots, F_\ell)$ of partitions of $[n]$ whose intersection hypergraph is a hyperforest (resp. hypertree). See Figure 5. The *(ℓ, n)-forest poset* is the poset Fo_n^ℓ on (ℓ, n)-forests ordered by componentwise refinement. In other words, Fo_n^ℓ is the subposet of the ℓth Cartesian power of the partition poset Pa_n induced by (ℓ, n)-forests. Note that the maximal elements of Fo_n^ℓ are the (ℓ, n)-trees.

Remark 1.12. Note that by rooting each connected component of the (ℓ, n) forest in its smallest label and splitting hyperedges in edges between each vertex of the hyperedge and

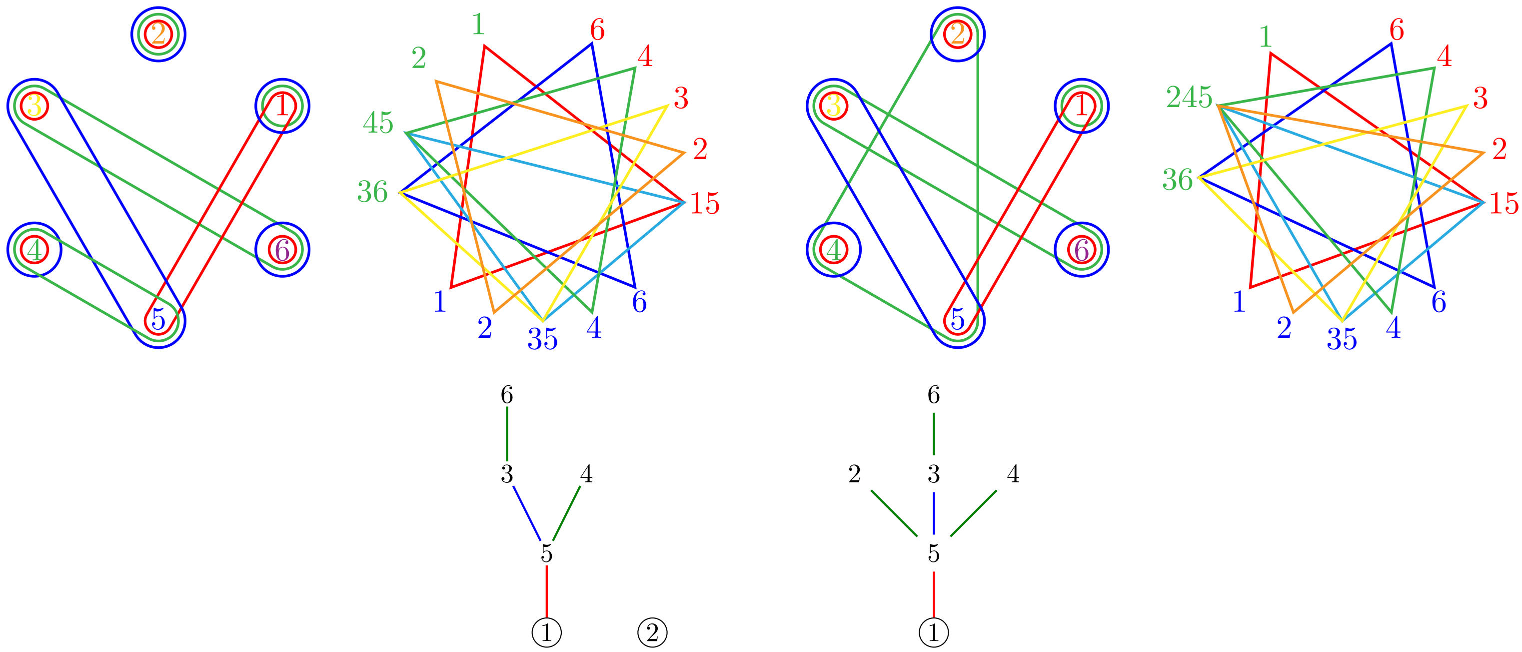


FIGURE 5. A $(3, 6)$-forest with its intersection hypergraph (left) and a $(3, 6)$-tree with its intersection hypertree (right). Bérénice: @Vincent : j'aimerais mettre mes figures avec les tiennes, pourrais-tu juste scinder "forests.pdf" en deux stp?

the closest vertex from the root one get the following alternative definitions for (ℓ, n)-forests and (ℓ, n)-trees: A (ℓ, n)-forest (resp. (ℓ, n)-tree) is a forest (resp. tree) in which:

- each root is the smallest in its connected component
- edges are colored by $\{1, \dots, \ell\}$
- if a vertex v is not a root, the unique path from v to the root starts with an edge e of color c: e is then the only edge adjacent to v of this color.

We represent this interpretation of (ℓ, n)-forests on Figure 5 and the according intersection poset of $\mathcal{B}_3^2$ on Figure 6.

Theorem 1.13. *The Möbius polynomial of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ is given by*

$$\boldsymbol{\mu}_{\mathcal{B}_n^\ell}(x, y) = x^{n-1-\ell n} y^{n-1-\ell n} \sum_{\boldsymbol{F} \le \boldsymbol{G}} \prod_{i \in [\ell]} x^{\#F_i} y^{\#G_i} \prod_{p \in G_i} (-1)^{\#F_i[p]-1} (\#F_i[p] - 1)! \,,$$

where $\boldsymbol{F} \le \boldsymbol{G}$ ranges over all intervals of the (ℓ, n)-forest poset Fo_n^ℓ, and $F_i[p]$ denotes the restriction of the partition F_i to the part p of G_i.

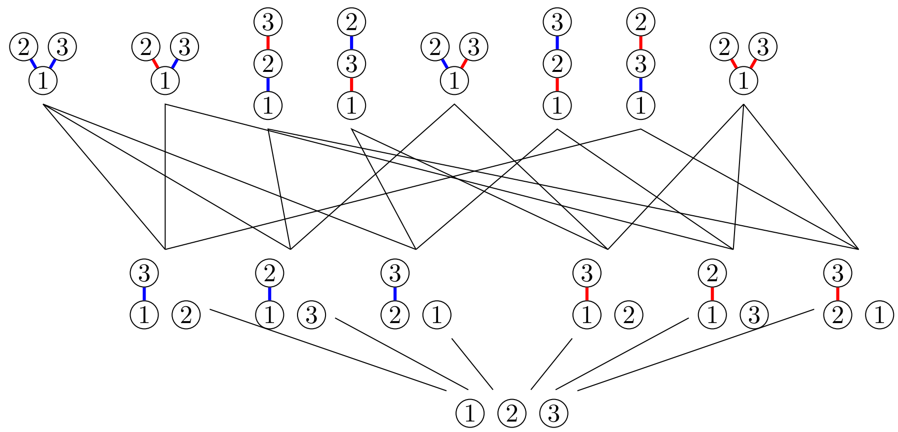


FIGURE 6. The intersection poset of $\mathcal{B}_3^2$ in terms of forests.

Proof. Observe that for $\boldsymbol{F} = (F_1, \dots, F_\ell)$ and $\boldsymbol{G} = (G_1, \dots, G_\ell)$ in Fo_n^ℓ, we have

$$[\boldsymbol{F}, \boldsymbol{G}] = \prod_{i \in [\ell]} [F_i, G_i] \simeq \prod_{i \in [\ell]} \prod_{p \in G_i} \mathsf{Pa}_{\#F_i[p]}.$$

As the Möbius function is multiplicative, that is, $\mu_{P \times Q}\big((p,q),(p',q')\big) = \mu_P(p,p') \cdot \mu_Q(q,q')$, we obtain that

$$\mu_{\mathsf{Fo}_n^\ell}(\boldsymbol{F}, \boldsymbol{G}) = \prod_{i \in [\ell]} \prod_{p \in G_i} (-1)^{\#F_i[p]-1} (\#F_i[p] - 1)!$$

Hence

$$\begin{aligned}\boldsymbol{\mu}_{\mathcal{B}_n^\ell}(x,y) &= \sum_{\boldsymbol{F} \le \boldsymbol{G}} \mu_{\mathsf{Fo}_n^\ell}(\boldsymbol{F}, \boldsymbol{G})\, x^{\dim(\boldsymbol{F})}\, y^{\dim(\boldsymbol{G})} \\ &= x^{n-1-\ell n} y^{n-1-\ell n} \sum_{\boldsymbol{F} \le \boldsymbol{G}} \prod_{i \in [\ell]} x^{\#F_i} y^{\#G_i} \prod_{p \in G_i} (-1)^{\#F_i[p]-1} (\#F_i[p] - 1)!. \qquad \square\end{aligned}$$

We thus obtain the following statament from Theorems 1.6 and 1.13.

Corollary 1.14. *The f- and b-polynomials of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ are given by*

$$\boldsymbol{f}_{\mathcal{B}_n^\ell}(x) = x^{n-1-\ell n} \sum_{\boldsymbol{F} \le \boldsymbol{G}} \prod_{i \in [\ell]} x^{\#F_i} \prod_{p \in G_i} (\#F_i[p] - 1)!$$

and $$\boldsymbol{b}_{\mathcal{B}_n^\ell}(x) = (-1)^\ell x^{n-1-\ell n} \sum_{\boldsymbol{F} \le \boldsymbol{G}} \prod_{i \in [\ell]} x^{\#F_i} \prod_{p \in G_i} -(\#F_i[p] - 1)!$$

The face numbers and bounded face numbers of $\mathcal{B}_n^\ell$ for $\ell, n \in [4]$ are gathered in Table 1.

Example 1.15. For $n = 1$, we have

$$\boldsymbol{\mu}_{\mathcal{B}_1^\ell}(x,y) = \boldsymbol{f}_{\mathcal{B}_1^\ell}(x) = \boldsymbol{b}_{\mathcal{B}_1^\ell}(x) = 1.$$

For $n = 2$, we have

$$\boldsymbol{\mu}_{\mathcal{B}_2^\ell}(x, y) = xy - \ell x + \ell, \quad \boldsymbol{f}_{\mathcal{B}_2^\ell}(x) = (\ell + 1)x + \ell \quad \text{and} \quad \boldsymbol{b}_{\mathcal{B}_2^\ell}(x) = (\ell - 1)x + \ell.$$

The case $n = 3$ is already more interesting. Consider the partitions $P := \big\{\{1\}, \{2\}, \{3\}\big\}$, $Q_i := \big\{\{i\}, [3] \smallsetminus \{i\}\big\}$, and $R := \big\{[3]\big\}$. Observe that the $(\ell, 3)$-forests are all of the form

$$\boldsymbol{F} := P^\ell, \quad \boldsymbol{G}_i^p := P^p Q_i P^{\ell-p-1}, \quad \boldsymbol{H}_{i,j}^{p,q} := P^p Q_i P^{\ell-p-q-2} Q_j P^q \ (i \neq j) \quad \text{or} \quad \boldsymbol{K}^p := P^p R P^{\ell-p-1}.$$

(where we write a tuple of partitions of [3] as a word on $\{P, Q_i, R\}$, and p and q are such that the total length is ℓ). Moreover, the cover relations in the $(\ell, 3)$-forest poset are precisely the relations

$$\begin{array}{ccc} & & \boldsymbol{H}_{i,j}^{p,q} \\ & \rotatebox{45}{$\leq$} & \\ \boldsymbol{F} \leq & \boldsymbol{G}_i^p & \leq \boldsymbol{K}^p \\ & \rotatebox{-45}{$\leq$} & \\ & & \boldsymbol{H}_{j,i}^{\ell-q-1,\ell-p-1} \end{array}$$

for $i \neq j$ and p, q such that $p + q \leq \ell - 2$. Hence, we have

$$\begin{aligned} \boldsymbol{\mu}_{\mathcal{B}_3^\ell}(x, y) &= x^2y^2 - 3\ell x^2 y + \ell(3\ell - 1)x^2 + 3\ell xy - 3\ell(2\ell - 1)x + \ell(3\ell - 2), \\ \boldsymbol{f}_{\mathcal{B}_3^\ell}(x) &= (3\ell^2 + 2\ell + 1)x^2 + 6\ell^2 x + \ell(3\ell - 2), \\ \text{and} \qquad \boldsymbol{b}_{\mathcal{B}_3^\ell}(x) &= (3\ell^2 - 4\ell + 1)x^2 + 6\ell(\ell - 1)x + \ell(3\ell - 2). \end{aligned}$$

Note that $3\ell^2 + 2\ell + 1$ is A056109, that $\ell(3\ell - 2)$ is A000567, and that $3\ell^2 - 4\ell + 1$ is A045944. Vincent: There is a weird connection between the first and the last. Namely, $3\ell^2 - 4\ell + 1 = 3(\ell - 1)^2 + 2(\ell - 1)$. Is there a bijective explanation on the arrangements?

Theorem 1.16. *The number of vertices of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ (or equivalently of (ℓ, n)-trees) is*

$$f_0(\mathcal{B}_n^\ell) = \ell\big((\ell - 1)n + 1\big)^{n-2}.$$

Proof. We adapt the proof from [CKSS04]. By rooting in the part of colour 1 containing 1, the number of vertices of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ is the same as the number of Cayley trees rooted in 1 with an edge-coloring by $\{1, \dots, \ell\}$ satisfying that the colour of an edge from a child is different from the colour of the edge between the parent and the child. A common reasoning for counting Cayley trees is the use of its Prüfer code defined by recursively pruning the smallest leaf and writing down the number of its parent. This bijection can be adapted to edge-coloured Cayley trees by writing down the number of the parent and the colour of the edge linking the leaf to its parent. We get a coloured word of length $n - 1$. Their are two possibilities:

- either the node we are removing is attached to the root labelled by 1 and there are ℓ possible different colours for the edge (ℓ)
- or it is attached to one of the $n - 1$ other nodes and the edge can have $\ell - 1$ different colours. $((n - 1)(\ell - 1))$

Note that the last letter in the Prüfer code (obtained by removing the last edge) is necessarily the root 1, with ℓ possible different colours. There are $(\ell + (n - 1)(\ell - 1))^{n-2} \times \ell$ such words, hence the result. $\square$

Conjecture 1. For any $\boldsymbol{k} \coloneqq (k_1, \dots, k_\ell)$ such that $0 \le k_i \le n-1$ for all $i \in [\ell]$ and $\sum_{i=1}^{\ell} k_i = n-1$, the number of vertices v of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ such that the smallest flat of the ith copy of $\mathcal{B}_n$ containing v has dimension $d_i = n - k_i - 1$ is given by

$$f_{\boldsymbol{d}}(\mathcal{B}_n^\ell) = n^{\ell-1} \binom{n-1}{k_1, \dots, k_l} \prod_{i=1}^{\ell} (n-k_i)^{k_i-1}.$$

Bérénice: Warning : k_i is the number of edges of colours i in the tree ! It has to be adapted to the dimension. Vincent: I hope I made it right. Bérénice: Isn't there rather a shift in k_i? Like a partition with k parts has dimension $k-1$. What about the following formula ?

$$f_{\boldsymbol{k}}(\mathcal{B}_n^\ell) = (n+1)^{\ell-1} \binom{n}{k_1+1, \dots, k_l+1} \prod_{i=1}^{\ell} (n-k_i)^{k_i}. \tag{1.3}$$

For any $\boldsymbol{k} \coloneqq (k_1, \dots, k_\ell)$ such that $0 \le k_i \le n-1$ for all $i \in [\ell]$ and $\sum_{i=1}^{\ell} k_i = n - \ell - 1$, the number of vertices v of the (ℓ, n)-braid arrangement $\mathcal{B}_n^\ell$ such that the smallest flat of the ith copy of $\mathcal{B}_n$ containing v has dimension k_i is given by

$$f_{\boldsymbol{k}}(\mathcal{B}_n^\ell) = n^{\ell-1} \binom{n-1}{k_1+1, \dots, k_l+1} \prod_{i=1}^{\ell} (n-1-k_i)^{k_i}. \tag{1.4}$$

Vincent: TODO: Check whether we can say something on the facets as well. I guess not, but I want to be sure.

Vincent: TODO: Discuss generalizations to ℓ copies of an arbitrary arrangement, or a graphical arrangement.

1.2. **Enumerative results for any diagonal.** We now specialize the results of the previous section to the case $\ell = 2$ to derive enumerative results on the diagonal of the permutahedron.

Note that

$$\begin{aligned}
\boldsymbol{\mu}_{\mathcal{B}_1^2}(x,y) &= 1 \\
\boldsymbol{\mu}_{\mathcal{B}_2^2}(x,y) &= xy - 2x + 2 \\
\boldsymbol{\mu}_{\mathcal{B}_3^2}(x,y) &= x^2y^2 - 6x^2y + 10x^2 + 6xy - 18x + 8 \\
\boldsymbol{\mu}_{\mathcal{B}_4^2}(x,y) &= x^3y^3 - 12x^3y^2 + 52x^3y - 84x^3 + 12x^2y^2 - 96x^2y + 216x^2 + 44xy - 182x + 50 \\
\boldsymbol{\mu}_{\mathcal{B}_5^2}(x,y) &= x^4y^4 - 20x^4y^3 + 160x^4y^2 - 620x^4y + 1008x^4 + 20x^3y^3 - 300x^3y^2 + 1640x^3y - 3360x^3 \\
&\quad + 140x^2y^2 - 1430x^2y + 4130x^2 + 410xy - 2210x + 432
\end{aligned}$$

which can be seen on matrices as

$$\begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 8 & -18 & 10 \\ 0 & 6 & -6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 50 & -182 & 216 & -84 \\ 0 & 44 & -96 & 52 \\ 0 & 0 & 12 & -12 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 432 & -2210 & 4130 & -3360 & 1008 \\ 0 & 410 & -1430 & 1640 & -620 \\ 0 & 0 & 140 & -300 & 160 \\ 0 & 0 & 0 & 20 & -20 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

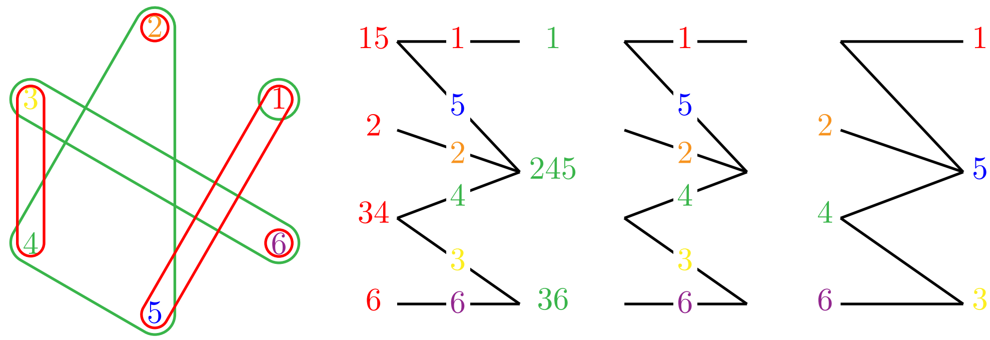


FIGURE 7. The bijection from rooted (ℓ, n)-trees (left) to spanning trees of K_{n+1} containing the edge $(0, 1)$ (right).

Theorem 1.17. *There are bijections between*

- *the vertices of the* $(2, n)$*-braid arrangement* $\mathcal{B}_n^2$*,*
- *the* $(2, n)$*-trees (i.e., pairs of partitions whose intersection graph is a tree),*
- *the spanning trees of the complete graph* K_{n+1} *on* $\{0, \ldots, n\}$ *containing the edge* $(0, 1)$*.*

These sets are counted by

$$f_0(\mathcal{B}_n^2) = 2(n+1)^{n-2}.$$

Proof. Note that the formula follows from Theorem 1.16. We give an alternative simple proof. The first two sets are in bijection by Corollary 1.14. Consider now a $(2, n)$-tree $\boldsymbol{F} := (F_1, F_2)$ (hence $\#F_1 + \#F_2 = n + 1$). Consider the intersection tree T of $\boldsymbol{F}$ with vertices labeled by the parts of F_1 and of F_2 and edges labeled by $[n]$, root T at the part of F_1 containing vertex 1, forget the vertex labels of T, and send each edge label of T to the next vertex away from the root, and label the root by 0. See Figure 7. The result is a spanning tree of the complete graph K_{n+1} on $\{0, \ldots, n\}$ which must contain the edge $(0, 1)$ (because we have chosen the root to be the part of F_1 containing 1). Finally, by double counting the pairs (T, e) where T is a spanning tree of K_{n+1} and e is an edge of T, we see that n times the number of spanning trees of K_{n+1} equals $\binom{n+1}{2}$ times the number of spanning trees of K_{n+1} containing $(0, 1)$. Hence, by Cayley's formula for spanning trees of K_{n+1}, we obtain that

$$f_0(\mathcal{B}_n^2) = \frac{2n}{n(n+1)}(n+1)^{n-1} = 2(n+1)^{n-2}. \qquad \square$$

Theorem 1.18. *For any* $0 \le k_1, k_2 \le n-1$ *with* $k_1 + k_2 = n - 1$*, the number of vertices* v *of the* $(2, n)$*-braid arrangement* $\mathcal{B}_n^2$ *obtained as the intersection of a* k_1*-flat of the first copy of the braid arrangement with a* k_2*-flat of the second copy of the braid arrangement is*

$$f_{k_1,k_2}(\mathcal{B}_n^2) = n\binom{n-1}{k_1}(n-k_1)^{k_1-1}(n-k_2)^{k_2-1}.$$

Proof. Note that the formula follows from Conjecture 1. We give an alternative simple proof. By Corollary 1.14, we want to count the number of $(2, n)$-trees $\boldsymbol{F} := (F_1, F_2)$ where F_1 has

k_1+1 parts while F_2 has k_2+1 parts. Consider the intersection tree T of $\boldsymbol{F}$ with vertices labeled by the parts of F_1 and of F_2 and edges labeled by $[n]$, root T at an arbitrary part of F_1, forget the vertex labels of T, and send each edge label of T to the next vertex away from the root. See Figure 7. Note here that in contrast to the proof of Theorem 1.17, we have rooted T here at an arbitrary part of F_1 to have more symmetry. The result is a spanning tree of the complete bipartite graph K_{k_1+1,k_2+1} together with a choice of which of the labels are on each part of this graph. Since the number of spanning trees of K_{k_1+1,k_2+1} is known to be $(k_1+1)^{k_2}(k_2+1)^{k_1}$, we obtain that

$$f_{k_1,k_2}(\mathcal{B}_n^2) = \underbrace{\frac{1}{k_1+1}}_{\text{root}} \underbrace{\binom{n}{k_1}}_{\text{labels}} \underbrace{(k_1+1)^{k_2}(k_2+1)^{k_1}}_{\#\text{ spanning trees}} = n\binom{n-1}{k_1}(n-k_1)^{k_1-1}(n-k_2)^{k_2-1}. \qquad \square$$

Vincent: Do we want to mention here the refinement counting spanning trees of K_{n+1} containing $(0,1)$ according to their number of nodes at even distance from the root?

Theorem 1.19. *The coefficient of x^{n-1} in the Möbius polynomial $\boldsymbol{\mu}_{\mathcal{B}_n^2}(x,y)$ of the $(2,n)$-braid arrangement $\mathcal{B}_n^2$ is given by*

$$[x^{n-1}]\,\boldsymbol{\mu}_{\mathcal{B}_n^2}(x,y) = \frac{(-1)^n n!}{y}\,[z^n]\,\exp\left(\sum_{m\ge1}\frac{-C_m\,y\,z^m}{m}\right).$$

where $C_m := \dfrac{1}{m+1}\dbinom{2m}{m}$ denotes the mth Catalan number.

Proof. We prove that the exponential generating function of the coefficient of x^{n-1} in the Möbius polynomial $\boldsymbol{\mu}_{\mathcal{B}_n^2}(x,y)$ is given by

$$1+\sum_{n\ge1}\frac{[x^{n-1}]\,\boldsymbol{\mu}_{\mathcal{B}_n^2}(x,y)\,y\,z^n}{(n+1)!} = \exp\left(\sum_{m\ge1}\frac{(-1)^{m-1}C_m\,y\,z^m}{m}\right).$$

Vincent: TODO. The proof from my email is in the comments. $\square$

Corollary 1.20. *The numbers of regions and of bounded regions of the $(2,n)$-braid arrangement $\mathcal{B}_n^2$ are given by*

$$f_{n-1}(\mathcal{B}_n^2) = n!\,[z^n]\exp\left(\sum_{m\ge1}\frac{C_m\,z^m}{m}\right)$$

and

$$b_{n-1}(\mathcal{B}_n^2) = (n-1)!\,[z^{n-1}]\exp\left(\sum_{m\ge1}C_m\,z^m\right).$$

The corresponding integer sequences are

$1, 3, 17, 149, 1809, 28399, 550297, 12732873, \ldots$ A213507,

and $1, 1, 5, 43, 529, 8501, 169021, 4010455, 110676833, \ldots$ A251568.

Vincent: TODO: improve presentation

Proof. As $\boldsymbol{f}_{\mathcal{B}_n^2}(x) = \boldsymbol{\mu}_{\mathcal{B}_n^2}(-x,-1)$ and $\boldsymbol{b}_{\mathcal{B}_n^2}(x) = \boldsymbol{\mu}_{\mathcal{B}_n^2}(-x,1)$ by Theorem 1.6, we obtain from Theorem 1.19 that

$$f_{n-1}(\mathcal{B}_n^2) = [x^{n-1}]\,\boldsymbol{f}_{\mathcal{B}_n^2}(x) = (-1)^{n-1}\,[x^{n-1}]\,\boldsymbol{\mu}_{\mathcal{B}_n^2}(-x,-1) = n!\,[z^n]\,\exp\Big(\sum_{m\ge1}\frac{C_m\,z^m}{m}\Big),$$

$$b_{n-1}(\mathcal{B}_n^2) = [x^{n-1}]\,\boldsymbol{b}_{\mathcal{B}_n^2}(x) = (-1)^{n-1}\,[x^{n-1}]\,\boldsymbol{\mu}_{\mathcal{B}_n^2}(-x,1) = -n!\,[z^n]\,\exp\Big(\sum_{m\ge1}\frac{-C_m\,z^m}{m}\Big).$$

To conclude, we thus just need to observe that $F(z) = \frac{\partial}{\partial z}G(z)$ where

$$F(z) := \exp\Big(\sum_{m\ge1} C_m\,z^m\Big) \qquad\text{and}\qquad G(z) := -\exp\Big(\sum_{m\ge1}\frac{-C_m\,z^m}{m}\Big).$$

For this, consider the generating functions

$$C(z) := \sum_{m\ge0} C_m\,z^m \qquad\text{and}\qquad D(z) := \sum_{m\ge1}\frac{C_m\,z^m}{m}.$$

Recall that $C(z)$ satisfies the functional equation $C(z) = 1 + z\,C(z)^2$. We thus obtain that $C'(z)\big(1-2\,z\,C(z)\big) = C(z)^2$ and $C(z)\big(1-2\,z\,C(z)\big) = 2 - C(z)$. Combining these two equations, we get

$$C(z)^3 = C'(z)\big(1-C(z)\big). \tag{1.5}$$

Observe now that $z\,D'(z) = C(z) - 1 = z\,C(z)^2$, so that $D''(z) = 2\,C(z)\,C'(z)$. Hence

$$F(z) = \exp\big(C(z)-1\big) = \exp\big(z\,D'(z)\big)$$

and

$$\frac{\partial}{\partial z}G(z) = \frac{\partial}{\partial z} - \exp\big(-D(z)\big) = D'(z)\exp\big(-D(z)\big)$$

Consider the function

$$H(z) = G'(z)/F(z) = D'(z)\exp\big(-D(z) - z\,D'(z)\big).$$

Clearly, $H(0) = 1$ and its derivative is

$$\begin{aligned} H'(z) &= \Big(D''(z)\big(1 - z\,D'(z)\big) - 2D'(z)^2\Big)\exp\big(-D(z) - z\,D'(z)\big) \\ &= 2\,C(z)\Big(C'(z)\big(2 - C(z)\big) - C(z)^3\Big)\exp\big(-D(z) - z\,D'(z)\big), \end{aligned}$$

which vanishes by (1.5).

Vincent: Can we make a bijective proof of the equality $F(z) = \frac{\partial}{\partial z}G(z)$? ☐

2. Higher Algebra

2.1. **Operadic diagonals.**

2.1.1. *Cellular diagonals.* The usual *thin diagonal* of a topological space X is the map $\Delta : X \to X \times X$ defined by $\Delta(x) := (x, x)$ for all $x \in X$.

Definition 2.1. A *cellular diagonal* of a polytope P is a continuous map $P \to P \times P$ such that

(1) its image is a union of $\dim P$-faces of $P \times P$ (i.e. it is *cellular*),
(2) it agrees with the thin diagonal on the vertices of P, and
(3) it is homotopic to the thin diagonal, relative to the image of the vertices.

A cellular diagonal is said to be *face-coherent* if its restriction to a face of P is itself a cellular diagonal for that face.

A powerful geometric technique to define face-coherent cellular diagonals on polytopes first appeared in [FS97], was presented in [MTTV21], and was fully developed in [Lap22]. The key idea is the following: any vector $\vec{v}$ in generic position with respect to P defines a cellular diagonal $\triangle_{(P,\vec{v})}$, via the following formula

$$\begin{array}{rccl} \triangle_{(P,\vec{v})} & : & P & \to & P \times P \\ & & z & \mapsto & \big(\min_{\vec{v}}(P \cap \rho_z P), \max_{\vec{v}}(P \cap \rho_z P)\big) \ . \end{array}$$

Here, $\rho_z P := 2z - P$ denotes the reflection of P with respect to the point z, and $\min_{\vec{v}}(P)$ denotes the unique vertex of P which minimizes the scalar product with $\vec{v}$. The diagonal $\triangle_{(P,\vec{v})}$ defines a canonical polytopal subdivision of P: it is by construction a tight coherent section of the projection $\pi : P \times P \to (P + P)/2$, and one just needs to draw the polytopes $(F + G)/2$, for all pairs of faces $(F, G) \in \operatorname{Im} \triangle_{(P,\vec{v})}$. For the rest of the paper, every time we will speak about a diagonal, *we will always mean such a map.*

Definition 2.2. A *cellular diagonal* or simply a *diagonal* $\triangle_{(P,\vec{v})}$ of the permutahedron P is a tight coherent section of the projection $P \times P \to P, (x, y) \mapsto (x + y)/2$.

Such a map can be seen as a topological map, or sometimes as a map of lattices; we shall not change notation and the context should make clear which We will sometimes denote a diagonal of the n-permutahedron by $\triangle_n$.

Definition 2.3. The *f-vector* of the diagonal $\triangle_{(P,\vec{v})}$ is the number of faces of $P \times P$ of given total dimension in its cellular image. Alternatively, it is the f-vector of the polytopal complex $\pi(\triangle_{(P,\vec{v})})$.

For the purpose of studying this f-vector, one can study the dual of the polytopal complex.

Proposition 2.4. *The f-vector of a cellular diagonal of the permutahedron $\triangle_{(P,\vec{v})}$ is given by the opposite of the f-vector of the hyperplane arrangement made of the braid arrangement together with a second copy of it, translated in the generic direction $\vec{v}$.*

Proof. This follows from [Lap22, Proposition 1.3]; [Lap22, Corollary 1.4] describes precisely the intersection poset of this hyperplane arrangement. □

The first part of the paper provides explicit formulas for this f-vector.

2.1.2. *The* LA *diagonal on the permutahedra.* Let us first set up some notations that will be of use throughout the paper. A set $\sigma_I := \bigcup_{i\in I}\sigma_i$ is a *partition* of $[n] := \{1,\dots,n\}$ if $\bigcup_{i\in I}\sigma_i = [n]$ and $\sigma_i \cap \sigma_j = \emptyset$ for $i \neq j$. The subsets σ_i are called *blocks*. We denote by $|\sigma| := |I|$ the size of the partition (its number of blocks). A partition is *ordered* if the indexing set I is equipped with a total order; in what follows we shall use $I = [k]$ for $k \in \mathbb{N}$. We use the shorthand $14|23$ to denote both the unordered partition $\{\{1,4\},\{2,3\}\}$, and also the ordered partition $(\{1,4\},\{2,3\})$ (when the order is clear from context).

Let us recall the combinatorial formula for the cellular approximation of the diagonal of the permutahedra from [Lap22, Theorem 3.16]. Let $n \geq 1$, and let us write

$$\mathrm{LA}(n) := \{(I,J) \mid I,J \subset \{1,\dots,n\}, |I| = |J|, I\cap J = \emptyset, \min(I\cup J) \in I\}.$$

Let $\vec{v} \in \mathbb{R}^n$ be such that $\forall (I,J) \in \mathrm{LA}(n)$, we have $\sum_{i\in I} v_i > \sum_{j\in J} v_j$, and let $P \subset \mathbb{R}^n$ denote the standard $(n-1)$-dimensional permutahedron. For any pair (σ,τ) of ordered partitions of $[n]$, we have

$$(\sigma,\tau) \in \operatorname{Im} \triangle_{(P,\vec{v})} \iff \begin{array}{l} \forall (I,J) \in \mathrm{LA}(n), \exists k \in [n], \left|\sigma_{[k]} \cap I\right| > \left|\sigma_{[k]} \cap J\right| \text{ or} \\ \exists l \in [n], \left|\tau_{[l]} \cap I\right| < \left|\tau_{[l]} \cap J\right| \ . \end{array}$$

We shall denote by $\triangle^{\mathrm{LA}}$ the set of pairs of ordered partitions of $[n]$ which satisfy the above condition. There is an equivalent description of $\triangle^{\mathrm{LA}}$ which has the following form:

Proposition 2.5. *For a two ordered partitions $\sigma,\tau \subset [n]$, we have*

$$(\sigma,\tau) \in \triangle^{\mathrm{LA}} \iff \begin{array}{l} \forall (I,J) \in \mathrm{LA}(\sigma,\tau), \exists k \in [n], \left|\sigma_{[k]} \cap I\right| > \left|\sigma_{[k]} \cap J\right| \text{ or} \\ \exists l \in [n], \left|\tau_{[l]} \cap I\right| < \left|\tau_{[l]} \cap J\right| \ . \end{array}$$

Here, $\mathrm{LA}(\sigma,\tau) \subset \mathrm{LA}(n)$ is a proper subset of $\mathrm{LA}(n)$ which depends on the choice of (σ,τ), and comes from the geometry of the situation, see [Lap22, Theorem 1.26] for more details. For our present purposes, it will be enough to restrict our attention to facets of LA, that is pairs (σ,τ) which satisfy $|\sigma| + |\tau| = n+1$. In this case, $\mathrm{LA}(\sigma,\tau)$ has $n-1$ elements, and admits the following description.

For any subset $\sigma_i \subset [n]$, let $\vec{\sigma}_i \in \mathbb{R}^n$ denote the boolean vector whose coordinates are given by 1 in position j if $j \in \sigma_i$ and 0 otherwise. Given a facet (σ,τ) of $\triangle^{\mathrm{LA}}$, one can consider the system of equations $\langle \vec{\sigma}_i, x\rangle = 0$, $\langle \vec{\tau}_j, x\rangle = 0$ given by the blocks of both partitions. For geometric reasons (see the proof of [Lap22, Theorem 1.26]), the solution of this system is $x = 0$. Now we will be interested in the solutions of the systems associated to the pairs (σ',τ) and (σ,τ') where σ' (resp. τ') has been obtained from σ (resp. τ) by merging two adjacent blocks.

Proposition 2.6. *There is a bijection between the set $\mathrm{LA}(\sigma,\tau)$ and the solutions to the systems of equations of the form (σ',τ) and (σ,τ').*

Guillaume: A revoir!! Il faut utiliser le fait que ce sont des facettes!!

Proof. For any $z \in (\mathring{\sigma}+\mathring{\tau})/2$, the face $\tau \cap \rho_z\sigma$ of $P \cap \rho_z P$ is a vertex of the polytope $P \cap \rho_z P$. The faces of the form $\tau \cap \rho_z \sigma'$ and $\tau' \cap \rho_z \sigma$ are the edges of $P \cap \rho_z P$ which are adjacent to the vertex $\tau \cap \rho_z \sigma$. By definition $D(\sigma,\tau)$ describes the directions of these edges, and the translation is made as follows: for a given pair (I, J), define the corresponding direction $\vec{d}$ by its coordinates $d_i := 1$ if $i \in I$, $d_j := -1$ if $j \in J$, and $d_k := 0$ otherwise. We refer to [Lap22, Section 1.5] for more details. □

We will sometimes refer to the elements of $\mathrm{LA}(\sigma, \tau)$ as the *minimal* (I, J)*-pairs*.

2.1.3. *Operadic diagonals on the permutahedra.* We consider the unordered sets

$$U(n) := \{\{I, J\} \mid I, J \subset [n], |I| = |J|, I \cap J = \emptyset\} \ .$$

Definition 2.7. The LA and SU orders on $U = \{U(n)\}$ are defined by

- $\mathrm{LA}(n) := \{(I, J) \mid \{I, J\} \in U(n),\ \min(I \cup J) = \min I\}$, and by
- $\mathrm{SU}(n) := \{(I, J) \mid \{I, J\} \in U(n),\ \max(I \cup J) = \max J\}$.

As we have seen in the preceding Section, the LA order defines the diagonal $\triangle^{\mathrm{LA}}$. Similarly, we will show in Guillaume: Corollary X that the SU order defines the Saneblidze–Umble diagonal from $\triangle^{\mathrm{SU}}$ [SU04]. Both of these diagonals have opposites $(\triangle^{\mathrm{LA}})^{\mathrm{op}}$ and $(\triangle^{\mathrm{SU}})^{\mathrm{op}}$, obtained by permuting the factors in every term; at the level of orderings they are obtained by permuting I and J in the definitions of $\mathrm{LA}(n)$ and $\mathrm{SU}(n)$. Geometrically, these opposite versions are given by taking $-\vec{v}$ instead of $\vec{v}$ as an orientation vector.

In general, an *ordering* O of U is a family of sets $\{O(n)\}_{n\geq 1}$ where each $O(n)$ has as elements ordered pairs (I, J) or (J, I), for each $\{I, J\} \in U(n)$. For an ordered pair (I, J) in $O(n)$, we denote by $\mathrm{std}(I, J)$ the standardisation function, e.g. $\mathrm{std}(\{5, 9, 10\}, \{6, 8, 12\}) = (\{1, 4, 5\}, \{2, 3, 6\})$.

Definition 2.8. An ordering $O := \{O(n)\}_{n\geq 1}$ of $U := \{U(n)\}_{n\geq 1}$ is *operadic* if for all $(I, J) \in O$, we have that $(I', J') \subsetneq (I, J) \implies \mathrm{std}(I', J') \in O$.

We will see Guillaume: in prop X that these choice of orderings induce operadic structure on the operahedra, multiplihedra and associahedra.

Lemma 2.9. *The* LA *and* SU *orderings are operadic, and are extensions of the following* (I, J) *pairs,*

- LA : *for all* $k \geq 1$ $(\{1, k+2, k+3, \ldots, 2k-1, 2k\}, \{2, 3, \ldots, k+1\})$, *and*
- SU : *for all* $k \geq 1$ $(\{k, k+1, \ldots, 2k-1\}, \{1, 2, 3, \ldots, k-1, 2k\})$.

The dual orders LA^{op} *and* SU^{op} *are also operadic, and extensions of the opposite pairs.*

Proof. We present the proof for the LA ordering, the proofs for the SU and opposite orders are similar. First, observe that if an ordered pair (I, J) can be written as $(I_a \sqcup I_b, J_a \sqcup J_b)$ with (I_a, J_a) and (I_b, J_b) in $\mathrm{LA}(|I|)$, then (I, J) is itself in LA. Second, it is apparent that $(I_k, J_k) := (\{1, k+2, k+3, \ldots, 2k-1, 2k\}, \{2, 3, \ldots, k+1\})$ is not a union of other LA pairs as 1 is the only element of I_k which is smaller than other elements of J_k. As such, if we try to decompose (I_k, J_k) as a non-trivial union, there is always one pair (I, J) in this union for which $1 \notin I$, so we have $\min(I \cup J) = \min J$, which implies that $(I, J) \notin \mathrm{LA}$. Third, we show that any pair (I, J) in LA which is not of the form (I_k, J_k) can be decomposed as a union of such pairs. In combination with the two previous observations, this will prove the Lemma.

Let (I,J) be in $\mathrm{LA}(k)$ and suppose that $(I,J) \neq (I_k, J_k)$, then there exists $i_2 \in I \smallsetminus \min I$ such that $i_2 < \max J$. This means that (I,J) can be decomposed as a union: if we write it as $(\{i_1, \dots, i_k\}, \{j_1, \dots, j_k\})$, where each set ordered smallest to largest, then we must have $1 = i_1 < i_2 < j_k$, in which case $(\{i_2\}, \{j_k\})$ and $(\{i_1, i_3, \dots, i_k\}, \{j_1, \dots, j_{k-1}\})$ are both smaller LA pairs. Then it must be the case that $\mathrm{std}((\{i_2\}, \{j_k\})) = (\{1\}, \{2\})$, and $\mathrm{std}((\{i_1, i_3, \dots, i_k\}, \{j_1, \dots, j_{k-1}\}))$ is either (I_{k-1}, J_{k-1}), or we can repeat this decomposition. This process must eventually terminate with the right-hand side reducing to (I_l, J_l) for some $1 \leq l \leq k-1$. In other words, any $(I,J) = (\{i_1, \dots, i_k\}, \{j_1, \dots, j_k\})$ decomposes as

$$(I,J) = (\{i_2\}, \{j_k\}) \sqcup (\{i_3\}, \{j_{k-1}\}) \sqcup \dots \sqcup (\{i_{l+1}\}, \{j_{k-l-1}\}) \sqcup (I', J')$$

where $\mathrm{std}((I', J')) = (I_l, J_l)$, and $1 \leq l \leq k$. □

We note that the decomposition in Lemma 2.9 is one of potentially many different decompositions of the pair (I,J). However, by definition of the LA order, for any decomposition $(I,J) = (\sqcup_{a \in A} I_a, \sqcup_{a \in A} J_a)$, we have that $\forall a \in A, \mathrm{std}(I_a, J_a) \in \mathrm{LA}$. As such all decompositions of a pair (I,J) order it the same way.

Proposition 2.10. *The only operadic orderings of $U = \{U(n)\}_{n \geq 1}$ are the* $\mathrm{LA}, \mathrm{SU}, \mathrm{LA}^{\mathrm{op}}$ *and* $\mathrm{SU}^{\mathrm{op}}$ *orderings.*

Proof. We first observe that there are precisely four ways to order the $U(n)$ pairs of size $k \leq 2$ such that the coherent extension of these orders do not collide. These are,

- $(\{1\}, \{2\})$ and $(\{1,4\}, \{2,3\})$ which corresponds to LA i.e. $\min(I \cup J) = \min I$;
- $(\{1\}, \{2\})$ and $(\{2,3\}, \{1,4\})$ which corresponds to SU i.e. $\max(I \cup J) = \max J$;
- $(\{2\}, \{1\})$ and $(\{2,3\}, \{1,4\})$ which corresponds to $\mathrm{LA}^{\mathrm{op}}$ i.e. $\min(I \cup J) = \min J$;
- $(\{2\}, \{1\})$ and $(\{1,4\}, \{2,3\})$ which corresponds to $\mathrm{SU}^{\mathrm{op}}$ i.e. $\max(I \cup J) = \max I$.

More specifically, we must first order the sole reduced pair $\{\{1\}, \{2\}\}$ of $U(1)$. Then, the sole reduced pair of $U(2)$ which must be ordered is $\{\{1,4\}, \{2,3\}\}$. There are clearly four ways to order these two pairs, and we can identify these (I,J) pairs as cases of Lemma 2.9.

We now show that once we have committed to one of these four orders we must follow through with it. We will show this through induction for the LA order, the SU and opposite orders proceeds similarly. Let $l \geq 2$ and suppose that for all $k \leq l$, we have ordered $I_k = \{1, k+2, k+3, \dots, 2k-1, 2k\} < J_k = \{2, 3, \dots, k+1\}$. Then from Lemma 2.9 we know that the only $\{I,J\}$ pair of order $l+1$ that will not decompose (and hence be specified by the already chosen conditions) is $\{I_{l+1}, J_{l+1}\}$. As such, the only way we can vary from LA is to order this element in the opposite direction i.e. choose (J_{l+1}, I_{l+1}). However, this choice leads to contradictions in the ordering of the $\{I,J\}$ pairs of order $|I| = |J| = l+2$. The following is a generalisation of Example 2.11. For $m = l+2$, the pair $\{I_m, J_m\}$ can be oriented in both directions. On the one side, we can write $\{I_m, J_m\} = \{I_a \cup I_b, J_a \cup J_b\}$ with $I_a := \{1, m+3, \dots, 2m\} > J_a := \{4, 5, \dots, m+2\}$ and $I_b := \{3\} > J_b := \{2\}$, which imply that $I_m > J_m$ by the first remark in the proof of Lemma 2.9. This decomposition makes use of the (reversed) order $J_{l+1} > I_{l+1}$ and the (non-reversed) order $I_1 < J_1$. On the other side, we can write $\{I_m, J_m\} = \{I_c \cup I_d, J_c \cup J_d\}$, where $I_c := \{1, m+3, \dots, 2m-1\} < J_c := \{2, 5, \dots, m+1\}$

and $I_d := \{3, 2m\} < J_d := \{4, m+2\}$, which imply that $I_m < J_m$ via the (non-reversed) orders $I_l < J_l$ and $I_2 < J_2$, a contradiction. □

Example 2.11. Suppose that the LA order holds for pairs of order 1 and 2, but is reversed for pairs of order 3, i.e. we have

$$\{1\} < \{2\}, \quad \{1,4\} < \{2,3\}, \text{ and } \{1,5,6\} > \{2,3,4\} \ .$$

Then $\{I, J\} = \{\{1,3,7,8\}, \{2,4,5,6\}\}$ admits two different orientations. In particular,

$$\begin{aligned} \{1,7,8\} > \{4,5,6\} \text{ and } \{3\} > \{2\} &\implies \{1,3,7,8\} > \{2,4,5,6\} \text{ and} \\ \{1,7\} < \{2,5\} \text{ and } \{3,8\} < \{4,6\} &\implies \{1,3,7,8\} < \{2,4,5,6\} \ . \end{aligned}$$

Now recall that each face $A_1|\ldots|A_k$ of the permutahedron $P_{|A_1|+\cdots+|A_k|-1}$ is isomorphic to the product $P_{|A_1|-1} \times \cdots \times P_{|A_k|-1}$ of lower dimensional permutahedra, via the isomorphism

$$\begin{array}{cccc} \Theta \ : & \mathbb{R}^{|A_1|} \times \cdots \times \mathbb{R}^{|A_k|} & \xrightarrow{\cong} & \mathbb{R}^{|A_1|+\cdots+|A_k|} \\ & (x_1, \ldots, x_{|A_1|}) \times \cdots \times (x_{|A_1|+\cdots+|A_{k-1}|+1}, \ldots, x_{|A_1|+\cdots+|A_k|}) & \mapsto & (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(|A_1|+\cdots+|A_k|)}) \end{array}$$

where σ is the $(|A_1|, \ldots, |A_k|)$-shuffle sending the increasingly ordered elements of $A_1 \cup \ldots \cup A_k$ to the block by block increasingly ordered elements of $A_1|\ldots|A_k$. Note that this map is a particular instance of the eponym map introduced in Point (5) of [Lap22, Prop. 2.3].

Remark 2.12. This fact underlies a *permutadic* structure [LR13, Mar20].

A *diagonal of the permutahedra* $\triangle := \{\triangle_n\}$ is a family of diagonals $\triangle_n : P_n \to P_n \times P_n$, for each n-permutahedron P_n, $n \geq 1$.

Definition 2.13. A diagonal of the permutahedra $\triangle$ is *operadic* if for every face $A_1|\ldots|A_k$ of the permutahedron $P_{|A_1|+\cdots+|A_k|-1}$, the map Θ induces a topological cellular isomorphism

$$\triangle(A_1) \times \ldots \times \triangle(A_k) \cong \triangle(A_1|\ldots|A_k) \ .$$

In other words, we require that the diagonal $\triangle$ commutes with the map Θ, see [Lap22, Section 4.2]. At the algebraic level, this property is called "comultiplicativity" in [SU04]. Note that in particular such an isomorphism respects the poset structures.

Proposition 2.14. *There are exactly four operadic diagonals on the permutahedra.*

Proof. First, by [Lap22, Theorem 3.9] any diagonal in the sense of Definition 2.2 is given by a choice of ordering of U. Second, every operadic diagonal satisfies [Lap22, Proposition 4.14], which amounts precisely to an operadic ordering of U in the sense of Definition 2.8. We conclude with Proposition 2.10. □

The orientation vectors $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ inducing each of these four diagonals are given by the conditions

$$\sum_{i \in I} v_i > \sum_{j \in J} v_j \ , \ \forall (I, J) \in O(n) \ ,$$

where $O(n)$ stands for either $\mathrm{LA}(n), \mathrm{LA}(n)^{\mathrm{op}}, \mathrm{SU}(n)$ or $\mathrm{SU}(n)^{\mathrm{op}}$.

Remark 2.15. This answers by the negative a conjecture regarding unicity of diagonals on the permutahedra, raised at the beginning of [SU04, Section 3], and could be seen as a "corrected" version of it. Guillaume: See the next section where we prove that SU diagonal is SU diagonal

The permutahedra are part of a more general family of polytopes called Loday realizations of the *operahedra* [Lap22, Def. 2.9] which encodes the notion of homotopy operad [Lap22, Def. 4.11]. For every planar tree t, there is a corresponding polytope P_t whose codimension k faces are in bijection with nestings of t with k non-trivial nests. Since the operahedra are generalized permutahedra [Lap22, Cor. 2.16], a choice of diagonal for the permuthedra induces a choice of diagonal for every operahedron [Lap22, Cor. 1.31]. Every face of an operahedron is isomorphic to a product of lower-dimensional operahedra, via an isomorphism Θ which generalizes the one above, see Point (5) of [Lap22, Prop. 2.3].

Definition 2.16. An *operadic diagonal* for the operahedra is a choice of diagonal $\triangle_t$ for each operahedron P_t, such that $\triangle := \{\triangle_t\}$ commutes with the map Θ, i.e. it satisfies [Lap22, Prop. 4.14].

Theorem 2.17 (Operadic structures on the operahedra)**.** *There are exactly*

(1) four operadic diagonals of the Loday operahedra, therefore exactly
(2) four colored topological cellular operad structures on the Loday operahedra, and incidentally exactly
(3) four universal tensor products of homotopy operads,

two of which (the LA *and* SU *diagonals) agree with the generalized Tamari order on fully nested trees.*

Proof. Let us first examine Point (1). By Proposition 2.14, we know that if one of the four choices $\triangle^{\mathrm{LA}}, (\triangle^{\mathrm{LA}})^{\mathrm{op}}, \triangle^{\mathrm{SU}}, (\triangle^{\mathrm{SU}})^{\mathrm{op}}$ is made on an operahedron P_t, one has to make the same choice on every lower-dimensional operahedron appearing in the decomposition $P_{t_1} \times \cdots \times P_{t_k} \cong F \subset P_t$ of a face F of P_t. Now suppose that one makes two distinct choices for two operahedra P_t and $P_{t'}$. It is easy to find a bigger tree t'' of which both t and t' are subtrees. Therefore, P_t and $P_{t'}$ appear as facets of $P_{t''}$ and by the preceding remark, any choice of diagonal for $P_{t''}$ will then contradict our initial two choices. Thus, these had to be the same from the start, which concludes the proof.

Point (2) then follows from fact that a choice of diagonal for the Loday realizations of the operahedra *forces* a unique topological cellular colored operad structure on them, see [Lap22, Theorem 4.18]. Since universal tensor products of homotopy operads are induced by a colored operad structure on the operahedra [Lap22, Cor. 4.24], we obtain Point (3). Finally, since only vectors with strictly descreasing coordinates induce the generalized Tamari order on the 1-skeleton of the operahedra [Lap22, Prop. 3.11], we get the last part of the statement. □

This answers the first question in [Lap22, Remark 3.14].

Two other important families of generalized permutahedra are the *Loday associahedra* and *Forcey–Loday multiplihedra*, which encode respectively A_∞-algebras and A_∞-morphisms

[LAM23, Prop. 4.9], as well as A_∞-categories and A_∞-functors [LAM23, Section 4.3]. Using the same techniques as in the case of the operahedra, they were endowed with diagonals and compatible operad and operadic bimodule structures in [LAM23, Theorem 1]. An analogous notion of *operadic diagonal* is available for these polytopes, see [LAM23, Proposition 2.14].

Theorem 2.18 (Operadic structures on the multiplihedra)**.** *There are exactly*

- *four operadic diagonals of the Forcey–Loday multiplihedra, therefore exactly*
- *four topological cellular operadic bimodule structures (over the Loday associahedra) on the Forcey–Loday multiplihedra, and incidentally exactly*
- *four compatible universal tensor products of* A_∞*-algebras and* A_∞*-morphisms,*

two of which (the LA *and* SU *diagonals) agree with the Tamari(-type) order on (2-colored) planar trees.*

Proof. The proof is similar to the one of Theorem 2.17, using the results of [LAM23]. □

We shall see now that these different operadic structures are related to one another in the strongest possible sense: they are all isomorphic as topological cellular operadic structures.

2.2. **Relating operadic diagonals.** Recall that the topological cellular operad structure on the operahedra [Lap22, Def. 4.17] is given by a family of partial composition maps

$$\circ_i^{\mathrm{LA}} \ : \ P_{t'} \times P_{t''} \xrightarrow{\mathrm{tr}\times\mathrm{id}} P_{(t',\omega)} \times P_{t''} \xhookrightarrow{\Theta} P_t \ .$$

Here, the map tr is the *unique* topological cellular map which commutes with the diagonal $\triangle^{\mathrm{LA}}$ [MTTV21, Prop. 7]. This partial composition $\circ_i^{\mathrm{LA}}$ is an isomorphism (in the category Poly [Lap22, Def. 4.13]) between the product $P_{t'} \times P_{t''}$ and the facet $t' \circ_i^{\mathrm{LA}} t''$ of P_t. Using the SU diagonal $\triangle^{\mathrm{SU}}$, one can define similarly a topological operad structure via the same formula, but with a different transition map tr, which commutes with $\triangle^{\mathrm{SU}}$.

Recall that a face F of P_t is represented by a nested tree $(t,\mathcal{N})$, which can be written uniquely as a sequence of substitution of trivially nested trees $(t,\mathcal{N}) = t_1 \circ_{i_1} t_2 \circ_{i_2} t_3 \cdots \circ_{i_k} t_{k+1}$. Note that we do not need to specify a parenthesization of this sequence of $\circ_i$ operations since these form an operad [Lap22, Def. 4.7]. At the geometric level, we have an isomorphism

$$\circ_{i_1}^{\mathrm{LA}} \circ_{i_2}^{\mathrm{LA}} \cdots \circ_{i_k}^{\mathrm{LA}} : P_{t_1} \times P_{t_2} \times \cdots \times P_{t_{k+1}} \xrightarrow{\cong} F \subset P_t$$

between a uniquely determined product of lower dimensional operahedra, and the face $F = (t,\mathcal{N})$ of P_t. Note that we can omit the parentheses in the sequence of $\circ_i^{\mathrm{LA}}$ operations since they define an operad structure [Lap22, Thm 4.18]. The same holds when taking the $\circ_i^{\mathrm{SU}}$ operations instead of the $\circ_i^{\mathrm{LA}}$.

Definition 2.19. For any operahedron P_t, the map $\Psi_t : P_t \to P_t$ is defined

- on the interior of the top face by the identity $\mathrm{id} : \mathring{P}_t \to \mathring{P}_t$, and
- on the interior of the face $F = t_1 \circ_{i_1} t_2 \circ_{i_2} \cdots \circ_{i_k} t_{k+1}$ of P_t by the composition of the isomorphisms

$$(\circ_{i_1}^{\mathrm{SU}} \circ_{i_2}^{\mathrm{SU}} \cdots \circ_{i_k}^{\mathrm{SU}})(\circ_{i_1}^{\mathrm{LA}} \circ_{i_2}^{\mathrm{LA}} \cdots \circ_{i_k}^{\mathrm{LA}})^{-1} : F \to F \ .$$

Proposition 2.20. *The map $\Psi := \{\Psi_t\}$ is an isomorphism of topological cellular symmetric colored operad in the category* Poly.

Proof. By definition, we have that Ψ is an isomorphism in the category Poly. It remains to show that it preserves the operad structures, i.e. that the following diagram commutes

$$\begin{tikzcd} P_{t'} \times P_{t''} \arrow[r, "\circ_i^{\mathrm{LA}}"] \arrow[d, "\Psi_{t'} \times \Psi_{t''}"] & P_t \arrow[d, "\Psi_t"] \\ P_{t'} \times P_{t''} \arrow[r, "\circ_i^{\mathrm{SU}}"] & P_t \ . \end{tikzcd}$$

For two interior points $(x, y) \in \mathring{P}_{t'} \times \mathring{P}_{t''}$, the diagram clearly commutes by definition, since $\Psi_{t'}$ $\Psi_{t''}$ are the identity in that case. If x is in a face $F = t_1 \circ_{i_1} t_2 \circ_{i_2} \cdots \circ_{i_k} t_{k+1}$ of the boundary of $P_{t'}$, then the lower composite is equal to $\circ_i^{\mathrm{SU}}(\circ_{i_1}^{\mathrm{SU}} \circ_{i_2}^{\mathrm{SU}} \cdots \circ_{i_k}^{\mathrm{SU}} \times \mathrm{id})(\circ_{i_1}^{\mathrm{LA}} \circ_{i_2}^{\mathrm{LA}} \cdots \circ_{i_k}^{\mathrm{LA}} \times \mathrm{id})^{-1}$, and so is the upper composite since Ψ_t starts with the inverse $(\circ_i^{\mathrm{LA}})^{-1}$ and the decomposition of F unto $P_{t_1} \times \cdots \times P_{t_{k+1}} \times P_{t''}$ is unique. The case when y is in the boundary of $P_{t''}$ is similar. Finally, the compatibility of Ψ with units and the symmetric group actions are straightfoward to check, see [Lap22, Def. 4.17 and Thm 4.18]. □

Note that Ψ is *not* a morphism of "Hopf" operads, i.e. it does not commute with the respective diagonals $\triangle^{\mathrm{LA}}$ and $\triangle^{\mathrm{SU}}$. Guillaume: Give explicit counterexample

Corollary 2.21. *The four topological operad structures on the operahedra are all isomorphic.*

Proof. It is clear that the proof of Proposition 2.20 does not depend on the specific choices of diagonals $\triangle^{\mathrm{LA}}$ and $\triangle^{\mathrm{SU}}$, it can therefore be applied to any pair of diagonals among $\mathrm{LA}, \mathrm{LA}^{\mathrm{op}}, \mathrm{SU}$ and $\mathrm{SU}^{\mathrm{op}}$. □

Guillaume: Find an iso which commutes??

Example 2.22. This isomorphism is a highly non-trivial map. For instance, it is not the one you would expect... [compare with the general symmetries]

Guillaume: faire direct l'iso topologique, puis traduire?

These isomorphisms can be seen at the combinatorial level, as ismorphisms between the different orderings of U

$$\begin{tikzcd} \mathrm{LA} \arrow[r, "t"] \arrow[d, "r"] & \mathrm{LA}^{\mathrm{op}} \arrow[d, "r"] \\ \mathrm{SU}^{\mathrm{op}} \arrow[r, "t"] & \mathrm{SU} \end{tikzcd}$$

where $t(I, J) := (J, I)$ is the transposition of the factors and r inverts the order on an (I, J) pair, mapping the smallest element to the largest element, then the next smallest to next largest and so on, e.g.

$$i(\{1, 7, 8\}, \{4, 5, 6\}) = (\{8, 2, 1\}, \{7, 6, 5\}) = (\{1, 2, 8\}, \{5, 6, 7\})$$

These bijections in orders induce a bijection between the diagonals.

Proposition 2.23. *There is a bijection between diagonals $\theta : \triangle^{\mathrm{LA}} \to \triangle^{\mathrm{SU}}$.*

Proof. For any pair (σ, τ) of ordered partitions of $[n]$, we have

$$(\sigma,\tau) \in \triangle^{\mathrm{LA}} \iff \forall (I,J) \in \mathrm{LA}(n), \exists k, \left|\bigcup_{i=1}^{k} \sigma_i \cap I\right| > \left|\bigcup_{1=1}^{k} \sigma_i \cap J\right|$$
$$\text{or } \exists l, \left|\bigcup_{i=1}^{l} \tau_k \cap I\right| < \left|\bigcup_{i=1}^{l} \tau_l \cap J\right|$$

and Kurt: Note the change! (not reversing each perm)

$$(\sigma,\tau) \in \triangle^{\mathrm{SU}} \iff \forall (I,J) \in \mathrm{SU}(n), \exists k, \left|\bigcup_{i=1}^{k} \tau_i \cap J\right| > \left|\bigcup_{i=1}^{k} \tau_i \cap I\right|$$
$$\text{or } \exists l, \left|\bigcup_{i=1}^{l} \sigma_i \cap J\right| < \left|\bigcup_{i=1}^{l} \sigma_i \cap I\right|$$

Let $\theta((\sigma,\tau)) := (\overline{\tau}, \overline{\sigma})$, where an ordered $[n]$ partition $\sigma_1|\ldots|\sigma_k$ is mapped to $\overline{\sigma} := \overline{\sigma_1}|\ldots|\overline{\sigma_k}$, where $\overline{\sigma_i}$ is the set $\{n-j+1 \mid j \in \sigma_i\}$. In words, the $\overline{\sigma}$ reverses the order of the blocks of σ, and reverses the order of its constituents i.e. $1 \mapsto n, 2 \mapsto n-1, \ldots, n \mapsto 1$. To see that θ is well defined, observe that under the bijection between $LA(n)$ and $SU(n)$ that $(I,J) \in LA(n)$ is mapped to $(J', I') := (i(J), i(I)) \in SU(n)$. As such,

$$(\sigma,\tau) \in \triangle^{LA} \iff \forall (J',I') \in SU(n), \exists k, \left|\bigcup_{i=1}^{k} \sigma'_i \cap I'\right| > \left|\bigcup_{i=1}^{k} \sigma'_i \cap J'\right|$$
$$\text{or } \exists l, \left|\bigcup_{i=1}^{l} \tau'_i \cap I'\right| < \left|\bigcup_{i=1}^{l} \tau'_i \cap J'\right|$$

i.e. we translate the the $\triangle^{LA}$ conditions imposed on (σ,τ) through θ. Hence, by definition $(\tau', \sigma') \in \triangle^{LA}$. From here, it is immediate that θ and its clear inverse are both injective, so we have a bijection. $\square$

As a quick example $\theta(15|7|234|6 \times 57|46|13|2) = 13|24|57|6 \times 37|1|456|2$, see Example 2.42 for a combinatorial interpretation of the change in the underlying I, J conditions for this particular example.

Furthermore, this bijection is a face poset isomorphism.

Lemma 2.24. *Given two faces $(\sigma,\tau), (\sigma',\tau') \in \triangle$, then (σ',τ') is a face of (σ,τ) iff $\mathcal{N}(\sigma,\tau) \subset \mathcal{N}(\sigma',\tau')$. Furthermore, (σ',τ') is a facet of (σ,τ) iff $\mathcal{N}(\sigma,\tau) \subset \mathcal{N}(\sigma',\tau')$ and $|\mathcal{N}(\sigma',\tau')| = |\mathcal{N}(\sigma,\tau)| + 1$.*

Proof. ... $\square$

The bijection between ordered partition (pairs) and (pairs of) nestings may be used to translate this lemma as follows. Kurt: What is the shortest path to this corollary that uses the least amount of nesting theory?

Corollary 2.25. *Given two faces $(\sigma, \tau), (\sigma', \tau') \in \triangle$ (either $\triangle^{LA}$ or $\triangle^{SU}$), then (σ', τ') is a facet of (σ, τ) iff*

$$\sigma = \sigma' \text{ and } \tau' = ref_{A,B}(\tau) \text{ or } \tau = \tau' \text{ and } \sigma' = ref_{A,B}(\sigma)$$

where if $\sigma = b_1|\ldots|b_k$ and b_i is partitioned by A, B then

$$ref_{A,B}(\sigma) = b_1|\ldots|b_{i-1}|A|B|b_{i+1}|\ldots|b_k$$

Lemma 2.26. *Given two faces $(\sigma, \tau), (\sigma', \tau') \in \triangle^{LA}$, (σ', τ') is a facet of (σ, τ) iff $\theta((\sigma', \tau'))$ is a facet of $\theta((\sigma, \tau))$.*

Proof. We prove the case when $\sigma = \sigma'$ and $\tau' = ref_{A,B}(\tau)$, the other case is similar. We directly compute,

$$\theta((\sigma, \tau)) = (\overline{\tau}, \overline{\sigma}) = (\overline{b_k}|\ldots|\overline{b_{i+1}}|\overline{b_i}|\overline{b_{i-1}}|\ldots|\overline{b_1}, \overline{\sigma})$$
$$\theta((\sigma', \tau')) = (\overline{\tau'}, \overline{\sigma'}) = (\overline{b_k}|\ldots|\overline{b_{i+1}}|\overline{B}|\overline{A}|\overline{b_{i-1}}|\ldots|\overline{b_1}, \overline{\sigma})$$

and as $\overline{B} \cup \overline{A} = \overline{b_i}$ it follows that $\theta((\sigma', \tau'))$ is a facet of $\theta((\sigma, \tau))$. □

As a face of face can be identified through a chain of facets, an immediate corollary of this lemma is that θ is an isomorphism of face posets, i.e,

Corollary 2.27. *Given two faces $(\sigma, \tau), (\sigma', \tau') \in \triangle^{LA}$, (σ', τ') is a face of (σ, τ) iff $\theta((\sigma', \tau'))$ is a face of $\theta((\sigma, \tau))$.*

2.3. Bijective results for operadic diagonals.

2.3.1. *Facets.* In this section we establish a bijection between the facets of $\triangle$ and a family of pairs of unordered partitions introduced and enumerated in a series of 3 papers [Che69, CG71, KUC82]. An intermediary bijection to a type of bipartite tree is of particular importance and provides [...]. In particular, we obtain that the number of facets in the image of the diagonal $\triangle_n$ of the n-dimensional permutahedron is $2(n+1)^{n-2}$ (A007334), and more precisely that the pairs of dimensions $(k, n-k)$ are counted by the formula $\frac{1}{k+1}\binom{n+1}{k}(k+1)^{n-k}(n+1-k)^k$. Guillaume: OEIS ref?

2.3.2. *Essential complementary partitions and bipartite trees.* Let us recall some basic definitions and results from the series of papers [Che69, CG71, KUC82].

Definition 2.28. A set of *distinct representatives* of a partition σ_I is a set $M \subset [n]$ such that $\forall i \in I, |\sigma_i \cap M| = 1$.

Definition 2.29. A pair of partitions (σ_L, τ_R) is said to be *complementary* if there exists $M \subset [n]$ and $m \in M$ such that M and $([n] \smallsetminus M) \cup \{m\}$ are distinct representatives of σ_L and τ_R, respectively. It is furthermore *essential* if there does not exist proper subsets $L' \subset L$, $R' \subset R$ and $N' \subset [n]$ such that $(\sigma_{L'}, \tau_{R'})$ is a complementary partition of N'.

Definition 2.30. A pair of partitions (σ_L, τ_R) is said to be

(1) *complimentary*, if $\forall l \in L, r \in R$ we have that $|\sigma_l \cap \tau_r| \leq 1$
(2) *essential*, if there are no proper subsets of $L' \subset L, R' \subset R$ such that $\cup_{l\in L}\sigma_l = \cup_{r\in R'}\tau_r$

Kurt: Maybe can immediately switch to Vincent's variant of def?

In cases where there is no ambiguity we will drop the subscripts L, R. We shall denote the set of all essential complementary pairs of partitions of $[n]$ by $\mathcal{E}$. Let us emphasize that the pairs of partitions of $\mathcal{E}$ are *unordered*.

Guillaume: Notations to be uniformized from here

Example 2.31. For $n = 2$, the two essential complementary partitions are $1|2 \times 12$ and $12 \times 1|2$. For $n = 3$, the eight essential complementary partitions are

$$1|2|3 \times 123, \quad 123 \times 1|2|3, \quad 1|23 \times 13|2, \quad 13|2 \times 1|23$$
$$1|23 \times 12|3, \quad 12|3 \times 1|23, \quad 13|2 \times 12|3, \quad 12|3 \times 13|2$$

For a larger example see Example 2.36.

The reader will have observed from this example, that many pairs of partitions are not essential and complementary. This can be checked directly from the definition, however, a graphical connection which we now introduce provides a simpler interpretation. A *tree* is a simply connected graph with no cycles. A *bipartite graph* is a graph whose vertices are partitioned into two sets such that vertices in one set are only adjacent to vertices in the other. We say a bipartite graph is *ordered* if one of the sets is considered smaller than the other and we denote the partition (V_L, V_R). We say a graph with n edges is *edge labelled* if there exists a bijection between the edges and $\{1, \ldots, n\}$.

Proposition 2.32 ([KUC82, Theorem 3])**.** *The set of essential complementary partitions $\mathcal{E}$ and the set edge labelled bipartite trees $\mathcal{B}$ are in bijection.*

Proof. Informally, given a partition pair (σ, τ), the corresponding graph $G((\sigma, \tau))$ has the subpartitions of each partition as its vertices, and two vertices are connected by an edge if the subpartitions share a common element. By convention we place the vertices of For an example of this bijection see Example 2.36. □

2.3.3. *Bijection with the facets of the diagonal.* In this section we denote by $\triangle$ be the set of pairs of ordered partitions of $[n]$ labeling *facets* of the diagonal $\triangle$. Bérénice: We should perhaps change the notation for $\triangle$. What about $\triangle_f$ or $\triangle_M$? Guillaume: I would keep the indices to the minimum

Theorem 2.33. *Facets of the diagonal and essential complementary partitions are in bijection through the inverse functions $u : \triangle \to \mathcal{E}$ and $o : \mathcal{E} \to \triangle$, where*

(1) The function u forgets the order of the ordered partition pair.
(2) The function o uniquely orders an essential complementary partition pair via the minimal (I, J)-pairs defining the diagonal.

An immediate corollary of this theorem, and an enumeration of labelled bipartite trees first done in [KUC82], is that there are $2(n + 1)^{n-2}$ facets $\triangle_n$. This enumeration will be refined in the next section. We shall prove this theorem by establishing the necessary total order, showing that the functions are well defined, and then showing that they are injective.

Lemma 2.34. *The function $u : \triangle \to \mathcal{E}$ that forgets the order in a pair of partitions is well defined.*

Proof. Let $P \in \triangle_n$. Then $G(u(P))$ is a graph with $l + r = n + 1$ vertices, and n edges. Furthermore, as no vertices can be isolated it must be the case that this graph is a tree. It is straightforward to verify that $G(u(P))$ must be labeled bipartite tree, but here is how we may explicitly produce the necessary distinct representatives using an algorithm of [KUC82, Theorem 2].

Let G' be a copy of $G(u(P))$. While there is a vertex of degree 1 in G' delete it and add the sole edge of that vertex as a distinct representative of the corresponding partition of that vertex. As G' is a tree this process can continue until there is a single edge connecting two vertices of degree 1. This edge specifies the element p of the distinct representatives. □

Construction 2.35. *For $P = (\sigma, \tau) = (\bigcup_{l\in L} \sigma_l, \bigcup_{r\in R} \tau_r) \in \mathcal{E}$ an essential complementary pair, we construct total orders on the blocks of σ and τ in three steps:*

(1) For $\sigma_l, \sigma_{l'} \in \sigma$ considered as vertices in $G(P)$, there exists a unique minimal path of even cardinality $p_{l,l'}$ connecting them. We partition $p_{l,l'} = I \cup J$ where I contains the minimal edge, and the path alternates between elements of I, J. Likewise, for $\tau_r, \tau_{r'} \in \tau$ there exists a path and decomposition $p_{r,r'} = I \cup J$.
(2) Direct each path so that all I point left to right, and all J point right to left.
(3) We say $\sigma_l < \sigma_{l'}$ (or $\tau_r < \tau_{r'}$) if the path is directed as $\sigma_l \to \sigma_{l'}$ ($\tau_r \to \tau_{r'}$).

Proof. We first show our binary relation is well defined before verifying that it defines a total order on $G(P)$ and hence P via the bijection of Proposition 2.32.

As $G(P)$ is a bipartite tree, every vertex is connected, and every path connecting two vertices on the same side must be of even length. As I and J partition the path in an alternating fashion i.e. $p = (i_1, j_1, i_2, j_2, \dots,)$, we can orient the path by forcing I to point left and J to point right. This order is clearly total, reflexive (by convention) and anti-symmetric, what remains to be checked is its transitivity.

Let p_{ab} denote the unique maximal path between two vertices a and b on the left of $G(P)$, that is two blocks of σ. Let I_{ab} denote the set of left-to-right edges in this path, and let J_{ab} denote its complement. Then, we have

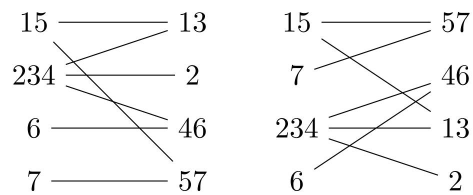
$$a < b \iff \min(I_{ab} \cup J_{ab}) = \min(I_{ab}) \ . \tag{2.1}$$

Suppose now that $a < b$ and $b < c$. Since $p_{ac} = (p_{ab} \cup p_{bc}) \smallsetminus (p_{ab} \cap p_{bc})$, we have

$$I_{ac} = (I_{ab} \cup I_{bc}) \smallsetminus (J_{ab} \cup J_{bc}) \text{ and } J_{ac} = (J_{ab} \cup J_{bc}) \smallsetminus (I_{ab} \cup I_{bc}) \ ,$$

and from the condition (2.1) above it is clear that $\min(I_{ac} \cup J_{ac}) = \min(I_{ac})$, which completes the proof of the transitivity for the total order on σ. The proof for τ is similar. □

Example 2.36. We now illustrate the ordering on the essential complementary partition $15|234|6|7 \times 13|2|46|57$. The fact that this is an essential complementary partition is easily visually verified by constructing its bipartite tree, and this tree, on the left, admits the ordering on the right.



Here are two examples paths providing order information.

$$7 <_L 234 \text{ as } 7 \xrightarrow{7} 57 \xrightarrow{5} 15 \xrightarrow{1=I_*} 13 \xrightarrow{3} 234$$

$$57 <_R 46 \text{ as } 57 \xrightarrow{5} 15 \xrightarrow{1=I_*} 13 \xrightarrow{3} 234 \xrightarrow{4} 46$$

This order far from being arbitrary provides the unique way to order an essential complementary partition pair into an ordered partition pair of $\triangle$, as we shall demonstrate next.

First we need a geometrical lemma. Kurt: notation clash...

Proposition 2.37. *The paths between adjacents vertices of P_L or P_R are in bijection with the minimal (I, J)-pairs.*

Proof. By Proposition 2.6, it suffices to show that the paths between adjacent vertices of P_L are in bijection with the solutions of the system of equations of the form (ρ^1, σ^2). To ease notation let us write ρ for ρ^1 and σ for σ^1. Suppose that ρ is obtained from σ by merging the two blocks σ_a and σ_b. The two equations $\langle \vec{\sigma}_a, x\rangle = 0$ and $\langle \vec{\sigma}_b, x\rangle = 0$ now become $\langle \vec{\sigma}_a + \vec{\sigma}_b, x\rangle = 0$; nothing else changes in the system. Since the solution to the system (σ^1, σ^2) was $x = 0$, now the solution is of dimension 1, and it is given precisely by the path between a and b in $G(P)$. Such a path is given by an alternating sequence of vertices and edges $\sigma_1 := \sigma_a, e_1, \sigma_2, e_2, \ldots, e_{k-1}, \sigma_k := \sigma_b$. Every edge $e_i \in \{1, \ldots, n\}$ is by definition the intersection $\sigma_i \cap \sigma_{i+1}$; thus it is the only common non-zero coordinate between $\vec{\sigma}_i$ and $\vec{\sigma}_{i+1}$. Thus the path encodes the series of equations $x_{e_1} + x_{e_{k-1}} = 0$, $x_{e_1} + x_{e_2} = 0$, $x_{e_2} + x_{e_3} = 0$, $\ldots$, $x_{e_{k-2}} + x_{e_{k-1}} = 0$. Thus, $x_{e_1} = 1$, $x_{e_2} = -1$, $x_{e_3} = 1$, $\ldots$, $x_{e_{k-2}} = 1$, $x_{e_{k-1}} = -1$ is a basis of one-dimensional space of solutions, and it gives the corresponding minimal (I, J)-pair. $\square$

Lemma 2.38. *The function $o : \mathcal{E} \to \triangle$ that orders an essential complementary pair is well defined.*

Proof. Let $P = (\sigma, \tau) = (\bigcup_{l \in L} \sigma_l, \bigcup_{r \in R} \tau_r) \in \mathcal{E}$ and consider $o(P)$. We first show that every (I, J)-condition, for $(I, J) \in D(n)$, which corresponds to a path between vertices is satisfied. In particular, this statement will be true for minimal (I, J)-pairs, which will be enough in virtue of Proposition 2.5. Suppose I, J corresponds to a path between two vertices on the left, i.e.

$$\sigma_l = \sigma_{l_1} \xrightarrow{i_1} \tau_{r_1} \xrightarrow{j_1} \sigma_{l_2} \xrightarrow{i_2} \ldots \xrightarrow{i_k} \tau_{r_{k-1}} \xrightarrow{j_k} \sigma_{l_k} = \sigma_{l'}$$

By construction we have that $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\} \in D(n)$ (note we are ordering I and J by the path, so it is not necessarily the case that $\min I = i_1$). Furthermore, each sub partition of τ either contains a single element of I and a single element of J, or it contains no elements of I and no elements of J. As such for any ordering of the sub-partitions of τ we have that Kurt: Could use [-] notation, but feels clearer to write out?

$$\forall m, \left| \bigcup_{1 \leq k \leq m} \tau_k \cap I \right| = \left| \bigcup_{1 \leq k \leq m} \tau_k \cap J \right|$$

Hence in order for this $D(n)$ condition to be satisfied it must be the case that for some ordering of the sub-partitions of σ we have

$$\exists m, \left| \bigcup_{1 \leq k \leq m} \sigma_k \cap I \right| > \left| \bigcup_{1 \leq k \leq m} \sigma_k \cap J \right|$$

Every sub-partition of σ excluding σ_l and $\sigma_{l'}$ either contains no elements of both I and J, or it contains a single element of I and a single element of J. So the only way for the condition to be satisfied is for σ_l to come before $\sigma_{l'}$, which is precisely what is required by the total order.

If I, J correspond to a path between two vertices on the right,

$$\tau_r = \tau_{r_1} \xrightarrow{j_1} \sigma_{l_1} \xrightarrow{i_1} \tau_{r_2} \xrightarrow{j_2} \dots \xrightarrow{j_k} \sigma_{l_{k-1}} \xrightarrow{1_k} \tau_{r_k} = \tau_{r'}$$

then a similar chain of logic implies we must have an ordering of the sub-partitions of τ such that

$$\exists m, | \bigcup_{1 \leq k \leq m} \tau_k \cap I| < | \bigcup_{1 \leq k \leq m} \tau_k \cap J|$$

and this can only happen if τ_r comes before $\tau_{r'}$. $\square$

Remark 2.39. It would be interesting to know if there is a geometrical interpretation of the paths that are not between adjacent vertices.

To complete the proof of Theorem 2.33, it remains to show that both $u : \triangle \to \mathcal{E}$ and $o : \mathcal{E} \to \triangle$ are injective, with the other function being their inverse.

Proof of Theorem 2.33. The forgetful function u is clearly the inverse to o as forgetting any assigned order will clearly return the original essential complementary partition pair. The ordering function o is the inverse to u as it returns the sole ordering of the sub-partitions which is compatible with the $D(n)$ conditions. $\square$

So the results of this section provide us with the follow characterisation of facets of the $\triangle^{\mathrm{LA}}$ diagonal.

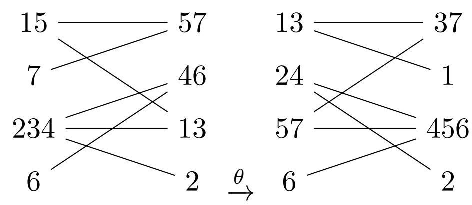
Proposition 2.40. *The facets of the $\triangle^{\mathrm{LA}}$ diagonal are ordered essential complementary partitions in which the minimal element of each path is traversed left to right.*

The bijection $\theta : \triangle^{\mathrm{SU}} \to \triangle^{\mathrm{LA}}$ may be used to characterise the facets of $\triangle^{\mathrm{SU}}$ as being ordered essential complementary partitions. The bijective ordering function $o' : \mathcal{E} \to \triangle^{\mathrm{SU}}$ is identical to Construction 2.35, but orients each path so that the maximal element points right to left, i.e.

Proposition 2.41. *The facets of the $\triangle^{\mathrm{SU}}$ diagonal are ordered essential complementary partitions in which the maximal element of each path is traversed right to left.*

We illustrate by example how the bijection θ immediately induces this result, and note that the result can also be obtained by simply altering the constructions of the section in the obvious manner for the $\triangle^{\mathrm{SU}}$ diagonal.

Example 2.42. We apply the bijection $\theta : \triangle^{\mathrm{LA}} \to \triangle^{\mathrm{SU}}$ to a previously encountered element of $\triangle^{\mathrm{LA}}$, Example 2.36.



Observe that the $\mathrm{LA}(n)$ conditions corresponding to paths, such as $(\{1,4,\},\{3,5\}) \in \mathrm{LA}(n)$

$$57 <_R 46 \text{ as } 57 \xrightarrow{5} 15 \xrightarrow{1=I_*} 13 \xrightarrow{3} 234 \xrightarrow{4} 46$$

are mapped to $\mathrm{SU}(n)$ conditions corresponding to paths, e.g. the prior I, J condition maps to $(\{3,5\},\{4,7\}) \in \mathrm{SU}(n)$, and

$$13 <_L 24 \text{ as } 13 \xrightarrow{3} 37 \xrightarrow{7=J^*} 57 \xrightarrow{5} 456 \xrightarrow{4} 24$$

Note the composite $\triangle^{\mathrm{LA}} \xrightarrow{f} \mathcal{E} \xrightarrow{o'} \triangle^{\mathrm{SU}}$ provides another bijection of facets, however this map is not equal to θ and is not defined on the other faces.

2.3.4. *Vertices.* We are now interested in characterizing the pairs of vertices that occur in the diagonal, that is pairs of permutations $(\sigma_1, \sigma_2) \in \triangle$.

Theorem 2.43. *There exists $(I, J) \in D(n)$ such that $\forall k, |\sigma_1^1 \cdots \sigma_1^k \cap I| \leq |\sigma_1^1 \cdots \sigma_1^k \cap J|$ and $\forall l, |\sigma_1^1 \cdots \sigma_1^l \cap I| \geq |\sigma_1^1 \cdots \sigma_1^l \cap J|$ (diagonal condition) if and only if $\exists (I', J') = (\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_m\}) \in D(m)$, $m \leq n$, such that*

$$\sigma_1 \cap (I' \cup J') = j_1 i_1 j_2 i_2 \cdots j_n i_n$$

and

$$\sigma_2 \cap (I' \cup J') = i_2 j_1 i_3 j_2 \cdots i_n j_{n-1} i_1 j_n \ ,$$

where $i_1 = \min(I' \cup J')$ (fish condition).

Proof.
- If a pair of permutations $(\sigma_1, \sigma_2) \in \mathfrak{S}_N^2$ satisfies the fish condition, then there exist two sets I and J of same cardinality such that $\min(I) < \min(J)$. Denoting σ_1 and σ_2 by two words of size N $\sigma_1^1 \ldots \sigma_1^N$ and $\sigma_2^1 \ldots \sigma_2^N$, then the pair $((\sigma_1, \sigma_2), (I, J))$ satisfies that for any k in $[\![1; N]\!]$, $|\sigma_1^1 \ldots \sigma_1^k \cap J| \geq |\sigma_1^1 \ldots \sigma_1^k \cap I|$ and $|\sigma_2^1 \ldots \sigma_2^k \cap I| \geq |\sigma_2^1 \ldots \sigma_2^k \cap J|$, hence the diagonal condition.
- We will now prove the converse. Let us presume that (σ_1, σ_2) is a pair of permutations satisfying the diagonal condition for a pair of sets $(I, J) \in D(n)$, minimal for the inclusion of sets.

Case $n = 1$:
If $|I| = |J| = 1$, then it follows directly from the diagonal condition above that $\sigma_{1|I\cup J} = j_1 i_1$ and $\sigma_{1|I\cup J} = i_1 j_1$, hence the fish condition is satisfied.

Case $n > 1$:
In this case, the proof is made by absurdum by considering the number of "well-placed" elements of I and J in σ_1 and σ_2. In what follows, for any set E, σ_i^E will stands for $(\sigma_i)_{|E}$. We write also $n_{i,k}^E$ for the number of elements of E in the k first letters of σ_i. The main argument in each of the small proofs below is the same: if the permutations do not satisfy the pattern described above, then it is possible to find an appropriate pair of elements $(i, j) \in I \times J$ such that $(I - i, J - j)$ satisfies the diagonal condition, hence contradicting the minimality of (I, J).

We first prove that the leftmost element of σ_1^I is i_1. Indeed, if it is not the case, we consider i, the leftmost element in σ_1^I and j the leftmost element in σ_2^J. The pair $(I - i, J - j)$ is in $D(n - 1)$, as i is different from i_1. Moreover, it is clear that the diagonal condition still holds for $((\sigma_1, \sigma_2), (I, J))$. As this would contradict the minimality of (I, J), the leftmost element of σ_1^I is i_1.

We then prove that $\sigma_1^{I\cup J}$ starts by $j_1 i_1$ and that this j_1 is exactly the leftmost element in σ_2^J. On that purpose, we suppose that either i_1 is preceeded by several elements of J or that the unique element of J is not the leftmost one in σ_2^J. We then adapt the previous argument by choosing i to be the leftmost element in $\sigma_1^{I-\{i_1\}}$ and j the leftmost element in σ_2^J. The pair $(I - i, J - j)$ is in $D(n - 1)$. Let us briefly explain while the diagonal condition would still be fulfilled in this case. If j is after i_1 in σ_1, then the difference $n_{1,k}^{J-j} - n_{1,k}^{I-i}$ is greater than $n_{1,k}^J - n_{1,k}^I$ for any k, hence is non negative. If j is before i_1 in σ_1, then by hypothesis, the difference $n_{1,k}^{J-j} - n_{1,k}^{I-i}$ is:

- strictly positive before i_1 an greater than 1 just before i_1
- non negative after i_1
- increase between i_1 and i
- is equal to $n_{1,k}^J - n_{1,k}^I$ after i,

hence is always non negative. Moreover, if i is after j in σ_2, the diagonal condition is clearly satisfied. If i is before j, then the difference $n_{2,k}^{I-i} - n_{1,k}^{J-j}$ is:

- strictly positive before j an greater than 1 just before j
- is equal to $n_{2,k}^I - n_{1,k}^J$ after j,

hence is always non negative. In short, if i_1 is preceeded by several elements of J or the unique element of J is not the leftmost one in σ_2^J, we obtain a contradiction with the minimality of (I, J).

Let us now consider the biggest $k \geq 1$ such that $\sigma_1^{I\cup J}$ begins with $j_1 i_1 j_2 i_2 \dots j_k i_k$ and $\sigma_2^{I\cup J}$ begins with $i_2 j_1 i_3 j_2 \dots i_k j_{k-1} w j_k$, where w is a word with letters in I. We want to show that $k = n$. Let us first remark that if $k = n$, $w = i_1$. If $1 \leq k < n$, then the sets $\tilde{I} = I - \{i_1, \dots, i_k\}$ and $\tilde{J} = J - \{j_1, \dots, j_k\}$ are non empty. Let us choose i_{k+1} to be the leftmost element in $\sigma_1^{\tilde{I}}$ and j_{k+1} the leftmost element in $\sigma_2^{\tilde{J}}$. We thus have $\sigma_1^{I\cup J} = j_1 i_1 j_2 i_2 \dots j_k i_k w' i_{k+1} \dots$, where w' is in J and

$\sigma_2^{I\cup J} = i_2 j_1 i_3 j_2 \ldots i_k j_{k-1} w j_k w'' j_{k+1} \ldots$, where w and w' are words with letters in I. The pair $(I - i_{k+1}, J - j_{k+1})$ is in $D(n-1)$. Following the study as in the previous case, σ_1 always satisfies the diagonal condition for $(I - i_{k+1}, J - j_{k+1})$ and σ_2 satisfies it if and only if $w \neq i$. By minimality of (I, J), we then have $w = i_{k+1}$. If $k+1 = n$, we are done as the only possible word in J is j_{k+1}, hence $w' = j_{k+1}$. Otherwise, we can choose i_{k+2} to be the leftmost element in $\sigma_1^{\tilde{I} - i_{k+1}}$. Using the same reasoning as above, we show that $((\sigma_1, \sigma_2), (I - i_{k+2}, J - j_{k+1}))$ satisfies the diagonal condition if and only if $w' \neq j_{k+1}$. To sum up, the only possibility for (I, J) to be minimal is to have $k = n$, which implies the fish condition.

□

Corollary 2.44. *For any pair of permutations $(\sigma_1, \sigma_2$, there exists $(I, J) \in D(n)$ such that $((\sigma_1, \sigma_2), (I, J))$ satisfies the diagonal condition if and only if there exists $(I', J') \in E(m)$, $m < n$ such that $((\sigma_1, \sigma_2), (I', J'))$ satisfies the fish condition, with*

$$E(m) = \left\{ (I, J) \in D(m) \;\middle|\; \begin{array}{l} \min(J) < \min(I - \min(I)), \\ |[\![1; k]\!] \cap J| > |[\![1; k]\!] \cap I| \\ \textit{if } |[\![1; k]\!] \cap J| \geq 2 \textit{ and } I \subsetneq [\![1; k]\!] \end{array} \right\}$$

Proof. It follows directly from the fish condition: if the fish condition is satisfied, as inversions of σ_1 are included in inversions of σ_2, we get $j_{k-1}, j_k < i_k$ for any $k > 1$. □

2.4. **The Saneblidze–Umble diagonal.** Here we prove that the diagonal $\triangle$ admits a combinatorial description completely analogous to that of the Saneblidze–Umble diagonal [SU04]. A direct corollary is that the two diagonals are in bijection, and moreover that the SU diagonal can be obtained from a certain choice of chambers in the fundamental hyperplane arrangements of the permutahedra, resolving a conjecture made in [Lap22]. In particular, this provides an alternative proof that all known diagonals on the associahedra agree [SU22].

Given any permutation, one can canonically produce a facet of the diagonal in the following way.

Definition 2.45. A pair of ordered partitions (σ, τ) is called *strong complementary* if σ is obtained from a permutation by merging the adjacent elements which are decreasing, and τ is obtained from the same permutation by merging the adjacent elements which are increasing.

An example is shown in Figure 8. The same argument as in Lemma 2.34 shows that the underlying pair of partitions is an essential complementary pair. Moreover we see directly that any path between adjacent vertices has length 2, and that its minimum is always traversed from left to right, hence all strong complementary pairs are in $\triangle$.

There is a natural partial order on the facets of $\triangle$. For two elements $x, y \in [n]$, we say that the set $\{x, y\}$ is an *inversion* of a facet (σ, τ) if we have that $x < y$, the element x appears before y in σ, and the element y appears before x in τ.

Definition 2.46. We say that two facets $(\sigma, \tau) \leq (\sigma', \tau')$ are comparable if the set of inversions of (σ, τ) is contained in the set of inversions of (σ', τ').

$$2|5|37|146 \times 257|36|4|1$$

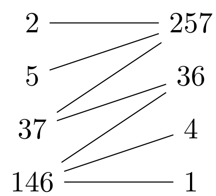


Figure 8. The strong complementary pair associated to the partition $2|5|7|3|6|4|1$.

It is immediate to see that this defines a poset. Guillaume: and even a lattice if a maximal and a minimal element are added?

Proposition 2.47. *The set of inversions of a facet is in bijection with its set of edge crossings. Moreover, the set of facets with no crossings is the set of strong complementary pairs.*

Proof. For the first part of the statement, it is clear that every inversion gives rise to a crossing. For the converse, one needs to check that an anti-inversion, where y appears before x in σ and x appears before y in τ, cannot occur in a facet of $\triangle$; this follows immediately from the (I, J)-conditions for $|I| = |J| = 1$. The second part of the statement follows from the fact that facets of the diagonal with no crossings are in bijection with permutations. As explained above, from a partition one obtains a strong complementary pair, which is in $\triangle$. In the other way around, given a strong complementary pair, one can read-off the partition in the associated tree, which has no crossings, by going along the edges from top to bottom, see Figure 8. □

We aim now at characterizing the cover relations of this poset. Let (σ, τ) be a facet of $\triangle$. We say that a pair of adjacent blocks $\sigma_i|\sigma_{i+1}$ of σ is *admissible* if there is an element $\rho \in (\sigma_{i+1} \smallsetminus \max \sigma_{i+1})$ such that $\rho < \min \sigma_i$ and $\rho < \min \gamma$ for γ the unique path between σ_i and σ_{i+1}. Such an element ρ is said to be *critical*. Let us define the *left shift operator* L, which takes an admissible pair $\sigma_i|\sigma_{i+1}$ in σ and creates a new ordered partition $L(\sigma)$ where the critical element ρ is shifted one block to the left: we have $L(\sigma)_i := \sigma_i \cup \rho$, $L(\sigma)_{i+1} := \sigma_{i+1} \smallsetminus \rho$, and $L(\sigma)_j := \sigma_j$ for all $j \neq i, i+1$.

In the same fashion, the *right shift operator* R sends an element $\rho \in (\tau_i \smallsetminus \max \tau_i)$ with $\rho < \min \tau_{i+1}$ and $\rho < \min \gamma$ one block to the right; the resulting ordered partition $R(\tau)$ is such that $R(\tau)_i := \tau_i \smallsetminus \rho$ and $R(\tau)_{i+1} := \tau_{i+1} \cup \rho$.

Definition 2.48. The facets of the *dual SU diagonal* are the pairs or ordered partitions that can be obtained from an strong complementary pair (σ, τ) by iterated applications of the left shift operator L on the first term σ, and iterated applications of the right shift operator R on the second term τ.

We shall show shortly that this definition recovers a dual of the SU diagonal on the permutahedra [SU04].

Lemma 2.49. *Applying the left or right shift to a pair $(\sigma, \tau) \in \triangle$ does not change the minima of the paths between adjacent vertices.*

Proof. We give the argument for the left shift, the one for the right one is similar. We observe that the paths in $(L(\sigma), \tau)$ are precisely the following: either they do not contain ρ, in which case they are paths of (σ, τ); or they do contain ρ, in which case they are obtained from paths in (σ, τ) by inserting at some place the unique path γ between σ_i and σ_{i+1}, or its inverse. Since $\rho < \min \gamma$, adding the path γ does not change the minima of the paths in $(L(\sigma), \tau)$, with respect to the ones in (σ, τ). □

Theorem 2.50. *The facets of the diagonal $\triangle$ and the facets of the dual SU diagonal are equal.*

Proof. Since we consider only the action of L on the first term of the pair (σ, τ) and the action of R on the second term, we will prove statements for L, the ones for R are similar.

First we show that every facet of $\triangle$ is a facet of the dual SU diagonal. Let (σ, τ) be a pair in $\triangle$ and suppose that it satisfies the following property: for all pairs of consecutive blocks $\sigma_i|\sigma_{i+1}$ in σ, if $\min \sigma_i < \max \sigma_{i+1}$, then the unique path γ between σ_i and σ_{i+1} has length 2 and is given by $\{\min \sigma_i, \max \sigma_{i+1}\}$. In particular, we have $\min \sigma_i \geq \min \gamma$. We claim that (σ, τ) must be a strong complementary pair. To see this, suppose that (σ, τ) is *not* a strong complementary pair; therefore by Proposition 2.71 there exists an edge crossing. This implies that there is a minimal crossing, i.e. a crossing between edges adjacent to two consecutive blocks $\sigma_i|\sigma_{i+1}$. Since edges that are incident to a block are always in increasing order from bottom to top (this is a direct consequence of the (I, J)-conditions for $|I| = |J| = 1$), there is a crossing between $\min \sigma_i$ and $\max \sigma_{i+1}$, which contradicts the property assumed above. So, there is no crossing in (σ, τ) and according to Proposition 2.71, we have that (σ, τ) is a strong complementary pair. The proof of the inclusion is now complete: if we are given a pair of facets (σ, τ) in $\triangle$ which has at least one crossing, then there is a pair of consecutive blocks $\sigma_i|\sigma_{i+1}$ such that $\min \sigma_i < \max \sigma_{i+1}$ and $\min \sigma_i < \min \gamma$, and one can apply the inverse operator L^{-1} shifting $\min \sigma_i$ to the right; and by induction we obtain a facet of the dual SU diagonal.

Second, we show that every facet of the dual SU diagonal is in $\triangle$. We already know that strong complementary pairs are in $\triangle$. Thus, it suffices to prove that if a dual SU pair (σ, τ) is in $\triangle$, then $(L(\sigma), \tau)$ is also in $\triangle$. Lemma 2.49 shows that the minima of paths between consecutive vertices in $(L(\sigma), \tau)$ are the same as the ones in (σ, τ). Thus, all minima of paths in $(L(\sigma), \tau)$ are traversed from left to right, and we have $(L(\sigma), \tau) \in \triangle$. □

Proposition 2.51. *The cover relations of the poset of facets are precisely the pairs of the form $(\sigma, \tau) \prec (L(\sigma), \tau)$ and $(\sigma, \tau) \prec (\sigma, R(\tau))$ for some L and R.*

Proof. From the proof of Theorem 2.50, we know that for any facet $(\sigma, \tau) \in \triangle$, we have that the pairs $(L(\sigma), \tau)$ and $(\sigma, R(\tau))$ are indeed facets of $\triangle$.

We show first that the left shift operator creates inversions, that is, for any $(\sigma, \tau) \in \triangle$, we have that $(\sigma, \tau) \leq (L(\sigma), \tau)$. Observe that shifting a critical element to the left in σ cannot delete any inversion: if $x < y$ and x precedes y in σ, then x must precede y also in $L(\sigma)$. Now, we need to show that for $x < y$, if either y comes before x in σ, or both x and

y are in the same block of σ, then y must come before x in τ. But this follows immediately from the (I, J)-condition for $I = \{x\}$ and $J = \{y\}$ defining $\triangle$. The result then follows, since ρ and $\max \sigma_{i+1}$ are in the same block of σ; thus $\max \sigma_{i+1}$ comes before ρ in τ and the pair $(\rho, \max \sigma_{i+1})$ is an inversion of $(L(\sigma), \tau)$, which is not an inversion of (σ, τ).

It remains to show that if $(\sigma, \tau) \leq (\sigma', \tau) \leq (L(\sigma), \tau)$, then we have either $\sigma = \sigma'$ or $\sigma' = L(\sigma)$. Indeed, if there is an inversion (x, y) of $(L(\sigma), \tau)$ which is not an inversion of (σ, τ), then it must be that $x = \rho$. To the contrary, we have $\sigma = \sigma'$, which completes the proof. □

Now we give the definition of the Saneblidze–Umble diagonal [SU04], following the description below Example 1 in [SU22], and replace "max" with "min" and the symbol "$>$" with the symbol "$<$".

Let (σ, τ) be a pair of ordered partitions. We say that a pair of adjacent blocks $\sigma_i|\sigma_{i+1}$ of σ is *SU admissible* if there is a non-empty subset $\rho \subset (\sigma_{i+1} \smallsetminus \max \sigma_{i+1})$ such that $\max \rho < \min \sigma_i$. Let us define the *subset left shift operator* L^i, which takes an SU admissible pair $\sigma_i|\sigma_{i+1}$ in σ and creates a new ordered partition $L^i(\sigma)$ where the subset ρ is shifted one block to the left: we have $L^i(\sigma)_i := \sigma_i \cup \rho$, $L^i(\sigma)_{i+1} := \sigma_{i+1} \smallsetminus \rho$, and $L^i(\sigma)_j := \sigma_j$ for all $j \neq i, i+1$. In the same fashion, the *subset right shift operator* R^i sends a subset $\rho \subset (\tau_i \smallsetminus \max \tau_i)$ with $\max \rho < \min \tau_{i+1}$ one block to the right; the resulting ordered partition $R^i(\tau)$ is such that $R^i(\tau)_i := \tau_i \smallsetminus \rho$ and $R^i(\tau)_{i+1} := \tau_{i+1} \cup \rho$.

Definition 2.52 (Dual SU diagonal, second definition)**.** The facets of the dual SU diagonal are the pairs of ordered partitions of the form $(L^{i_k} \cdots L^{i_1}(\sigma), R^{j_l} \cdots R^{j_1}(\tau))$ obtained from a strong complementary pair (σ, τ) by iterated application of subset left and right shifts operators, where moreover $i_1 > \cdots > i_k$ is decreasing and $j_1 < \cdots < j_l$ is increasing.

Lemma 2.53. *Applying the subset left or right shift to a pair (σ, τ) in the dual SU diagonal does not change the minima of the paths between adjacent vertices.*

Proof. We analyse the left shift operator, the case of the right shift is similar. First, we observe that any critical subset $\rho \subset L^{i_k} \cdots L^{i_1}(\sigma)_j, j \leq i_k$ satisfies $\max \rho < \min \gamma$, where γ is the unique path between $L^{i_k} \cdots L^{i_1}(\sigma)_{j-1} = \sigma_{j-1}$ and $L^{i_k} \cdots L^{i_1}(\sigma)_j$. Indeed, the path γ is the same as the path between σ_{j-1} and σ_j in (σ, τ), and is equal to $\{\min \sigma_{j-1}, \max \sigma_j\}$; but since ρ is critical we must have $\max \rho < \min \sigma_{j-1} = \min \gamma$. The rest of the proof is the same as for Lemma 2.49, with ρ replaced by $\max \rho$. □

Proposition 2.54. *The two definitions of the dual SU diagonal coincide.*

Proof. We analyse the left shift operator, the case of the right shift is similar. First, we observe that any subset left shift $L^i(\sigma)$ can be decomposed in a series of left shifts: since $\max \rho < \min \gamma$ by the proof of Lemma 2.53, we can first shift $\max \rho$ to the left, then $\max(\rho \smallsetminus \max \rho)$, and so on until the entire subset ρ has been shifted to the left. This shows that any facet in the second definition of the dual SU diagonal is also a facet in the first definition.

For the reverse inclusion, we show by induction on the number of left shifts that a pair (σ', τ) where σ' has been obtained by iterated left shifts can also be obtained by iterated subset left shifts, that is, can be written in the form $(L^{i_k}\cdots L^{i_1}(\sigma),\tau)$. The base case of a strong complementary pair is trivial. Suppose that we are given a pair (σ',τ) where σ' has been obtained by a sequence of $l+1$ left shifts. By the induction hypothesis, the first l left shifts can be rewritten as a sequence of subset left shifts, i.e. in the form $L^{i_k}\cdots L^{i_1}(\sigma)$ where $i_1 > \cdots > i_k$ are descreasing. Now suppose that the $(l+1)$-th left shift occurs between $L^{i_k}\cdots L^{i_1}(\sigma)_j$ and $L^{i_k}\cdots L^{i_1}(\sigma)_{j+1}$, with $j > i_k$ (otherwise, we are done!). Let us denote by γ the unique path between $L^{i_k}\cdots L^{i_1}(\sigma)_j$ and $L^{i_k}\cdots L^{i_1}(\sigma)_{j+1}$. By definition, we have that the critical element ρ satisfies $\rho < \min\gamma$. But by Lemma 2.53, this minimum is the same as the minimum of the path between σ_j and σ_{j+1}, which is just $\min\sigma_j$. Thus we have $\rho < \min\sigma_j$ (as well as $\rho < \min L^{i_k}\cdots L^{i_1}(\sigma)_j$, by criticality), and so ρ can be integrated in a new or an existing subset left shift of the family $L^{i_k}\cdots L^{i_1}$, which completes the proof. □

One can thus interpret the left and right shift operators for singletons introduced by Saneblidze–Umble as generators of the poset of facets of the diagonal, with minimal elements the strong complementary pairs.

Now, we end with two important consequences of the preceding results. Consider the symmetry s of the $(n-1)$-dimensional permutahedron, which consists in the reflection with respect to the hyperplane $x_1+x_n = x_2+x_{n-1}$. It sends an ordered partition $\sigma := \sigma_1|\ldots|\sigma_k$ to the ordered partition $s\sigma := n-\sigma_k+1|\ldots|n-\sigma_1+1$, where $n-\sigma_i+1$ is the set $\{n-j+1 \mid j\in\sigma_i\}$. Kurt: This is what the next two corollaries should be i.e. altered to put max IJ =max J. Should probably combine with above.

Corollary 2.55. *The terms of the diagonal* $\triangle$ *and the terms of the Saneblidze–Umble diagonal are in bijection through* $(\sigma,\tau)\mapsto(s\tau,s\sigma)$

Proof. According to Theorem 2.50, it suffices to show that the symmetry s sends the SU diagonal to the dual SU diagonal. □

This allows us to prove a conjecture made in [Lap22, Remark 3.19].

Corollary 2.56. *The Saneblidze–Umble diagonal is given by the following choice of chambers in the fundamental hyperplane arrangement of the permutahedron: any vector* $\vec{v}$ *with strictly decreasing coordinates and which satisfy* $\sum_{i\in I} v_i > \sum_{j\in J} v_j$ *for all* $I, J \subset \{1,\ldots,n\}$ *such that* $I\cap J=\emptyset, |I|=|J|\geq 2$ *and* $\max(I\cup J)\in J$ *induces the Saneblidze–Umble diagonal on the* $(n-1)$*-dimensional standard permutahedron.*

Proof. These orientation vectors are obtained precisely from the ones defining $\triangle$ (see **??**) via the symmetry described in Corollary 2.55 above [...]. □

This gives a geometric, alternative proof of the fact that all known diagonals of the associahedra agree [SU22].

Guillaume: Thus, (I,J)-description for the SU diagonal...

Guillaume: Now we have matrix representation for free!

Example 2.57. The permutation 5714632 corresponds to the strong complementary pair $(\sigma,\tau) := 5|17|4|236 \times 57|146|3|2$. We now illustrate all possible shifts of this strong complimentary pair. The possible left shifts (indicated by their corresponding critical element, and drawn to avoid crossings) are

$$\begin{tikzcd}
 & 5|17|4|236 \times \tau \arrow[dl, "p=3"'] \arrow[d, "p=1"] \arrow[dr, "p=2"] & \\
5|17|34|26 \times \tau \arrow[d, "p=2"] \arrow[dr, "p=1"] & 15|7|4|236 \times \tau \arrow[d, "p=3"] \arrow[dr, "p=2"] & 5|17|24|36 \times \tau \arrow[d, "p=1"] \\
5|17|234|6 \times \tau \arrow[d, "p=1"] & 15|7|34|26 \times \tau \arrow[dl, "p=2"] & 15|17|24|36 \times \tau \\
15|7|234|6 \times \tau & &
\end{tikzcd}$$

The possible right shifts are simply

$$\sigma \times 57|146|3|2 \xrightarrow{p=1} \sigma \times 57|46|13|2 \xrightarrow{p=1} 57|46|3|12$$

As the left and right shifts can be performed independently we could combine these diagrams. No other shifts are possible, observe for instance that we cannot perform the left shift $15|7|234|6 \times 57|46|13|2 \xrightarrow{p=2} 15|27|34|6 \times 57|46|13|2$ as the minimal path connecting 234 and 7 contains 1 which is smaller than 2 (see Example 2.36). The unique way of producing $15|7|234|6 \times 57|146|3|2$ by the second definition of the SU diagonal **????** is given by

$$5|17|4|236 \times 57|146|3|2 \xrightarrow{p=\{2,3\}} 5|17|234|6 \times 57|146|3|2 \xrightarrow{p=\{1\}} 15|7|234|6 \times 57|146|3|2$$

This corresponds to combining the top two arrows in the leftmost arrows of the diagram.

2.5. **The Saneblidze-Umble diagonal (Version 2).** Kurt: What do we want subscripts? superscripts? indices and/or sets?

In this section we show that the I, J description of $\triangle^{\mathrm{SU}}$, referred to as $\triangle^{\mathrm{SU}}$, is equivalent to the original description of Saneblidze-Umble diagonal [SU04], referred to as subset shift $\triangle^{\mathrm{SU}}$. We fully detail the equalities of the top row of the following commutative diagram, (Propositions 2.67 and 2.68)

$$\begin{tikzcd}
\text{Subset Shift } \triangle^{\mathrm{SU}} \arrow[r, leftrightarrow] & \text{Shift } \triangle^{\mathrm{SU}} \arrow[r, leftrightarrow] & \triangle^{\mathrm{SU}} \arrow[d, leftrightarrow] \\
\text{Subset Shift } \triangle^{\mathrm{LA}} \arrow[r, leftrightarrow, dotted] & \text{Shift } \triangle^{\mathrm{LA}} \arrow[r, leftrightarrow, dotted] & \triangle^{\mathrm{LA}}
\end{tikzcd}$$

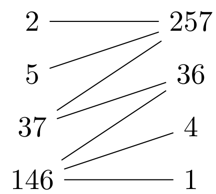
, and together with the bijection $\theta : \triangle^{\mathrm{LA}} \to \triangle^{\mathrm{SU}}$ (Proposition 2.23) which corresponds to the vertical arrow, this proves the equivalence of the only two diagonals. Finally we use the fact that θ is a face poset isomorphism, to induce shift descriptions of the $\triangle^{\mathrm{LA}}$ diagonal.

We first recall the necessary theory to describe the subset shift $\triangle^{\mathrm{SU}}$, which generates all facets by performing shifts on strong complementary partitions. We use similar notation to the recent paper [SU22].

Definition 2.58. A pair of ordered partitions (σ, τ) is called *strong complementary* if σ is obtained from a permutation by merging the adjacent elements which are decreasing, and τ is obtained from the same permutation by merging the adjacent elements which are increasing. Let SCP denote the set of all strong complementary pairs.

Example 2.59. The SCP associated to the permutation 2573641 is,

$$2|5|37|146 \times 257|36|4|1$$



Observe that the permutation can be read off the graph of the SCP by a vertical down slice through the edges of the graph.

As the underlying graph of an SCP is a bipartite tree, there is a unique path between any two vertices. We use $\sigma_{i,j}$ to denote the unique path between blocks σ_i and σ_j, and similarly $\tau_{i,j}$. We note this is a mild abuse of notation as the path $\sigma_{i,j}$ also depends on τ, but we will clarify our notation where needed. Kurt: maybe? We can immediately characterise the paths between adjacent blocks of SCPs.

Lemma 2.60. *All $(\sigma,\tau) \in SCP$ are in both $\triangle^{\mathrm{LA}}$ and $\triangle^{\mathrm{SU}}$, as a consequence of*

(1) $\sigma_{i,i+1} = (\min\sigma_i, \max\sigma_{i+1})$, *and* $\min\sigma_i < \max\sigma_{i+1}$.
(2) $\tau_{i,i+1} = (\max\tau_i, \min\tau_{i+1})$, *and* $\min\tau_{i+1} < \max\tau_i$.

Proof. The path description of (σ,τ) is a straightforward observation. The minima of these adjacent paths are traversed left to right, and the maxima right to left. As such, by Propositions 2.40 and 2.41, SCP is a subset of both $\triangle^{\mathrm{LA}}$ and $\triangle^{\mathrm{SU}}$. □

Kurt: better presentation of subset diagonal...

Definition 2.61. Let $(\sigma,\tau) = (\sigma_1|\dots|\sigma_k, \tau_1|\dots|\tau_l)$ be an ordered partition pair, and let $M_i \subset (\sigma_i \setminus \min\sigma_i)$ be a non-empty subset such that $\min M_i > \max\sigma_{i+1}$. For this set we define the *subset right shift operator* by

$$R_{M_i}(\sigma) := \sigma_1|\dots|\sigma_i \setminus M_i|\sigma_i \sqcup M_i|\dots|\sigma_k$$

Dually, let $M_{i+1} \subset (\tau_{i+1} \setminus \min\tau_{i+1})$ be a non-empty subset such that $\min M_{i+1} > \max\tau_i$. For this set we define the *subset left shift operator* by

$$L_{M_{i+1}}(\tau) := \tau_1|\dots|\tau_i \sqcup M_{i+1}|\tau_i \setminus M_{i+1}|\dots|\tau_l$$

For $l \geq 1$, let $\mathbf{M} = (M_{i_1},\dots,M_{i_l}) \in [n]^l$, be such that $1 \leq i_1 < i_2 \dots < i_l \leq k-1$, and each of the iterated compositions $R_{\mathbf{M}}(\sigma) := R_{M_{i_l}} \dots R_{M_{i_1}}(\sigma)$ are well-defined. Similarly, for $m \geq 1$ let $\mathbf{N} = (N_{i_m},\dots,N_{i_1}) \in [n]^m$ be such that $2 \leq i_1 < i_2 \dots < i_m \leq k$, and each of the iterated compositions $L_{\mathbf{N}}(\tau) := L_{N_{i_1}} \dots L_{N_{i_m}}(\tau)$ are well-defined. We define the case

$R_\emptyset\sigma := \sigma$ and $L_\emptyset\tau := \tau$. Then we denote,

$$A_\sigma \times B_\tau := \bigsqcup_{\mathbf{M},\mathbf{N}} \{R_\mathbf{M}(\sigma) \times L_\mathbf{N}(\tau)\}$$

where the union is over all valid $\mathbf{M},\mathbf{N}$ such that the iterated operators are well-defined.

See the bottom of Example 2.80 for an example of a subset shift. Observe that the left shift operator moves elements to the left, but acts on the right ordered partition. Dually, the right shift operator moves elements to the right but acts on the left ordered partition. We have departed from the notation of [SU22] in two ways. Firstly we have reversed the indexing of the left shift operator, and secondly we have omitted the case $M = \emptyset$ whose corresponding operator was the identity. As we have omitted the case $M = \emptyset$, we have altered our indexing set from $\{1, 2, \dots, d\}$ to the increasing sequence $i_1 < \dots < i_d$.

Proposition 2.62 (Theorem [SU04], Restated in [SU22])**.** *The subset-shift SU diagonal is generated by all valid subset-shifts of SCPs,*

$$\triangle^{\mathrm{SU}} = \bigsqcup_{(\sigma,\tau)\in SCP} A_\sigma \times B_\tau$$

We now make an interesting observation, whose proof is key to an equivalent definition of the subset shift $\triangle^{\mathrm{SU}}$ diagonal.

Lemma 2.63. *The* SU *subset shift operators conserve the maximal element of paths. In particular, for $(\sigma,\tau) \in SCP$ and a valid shift of it $(R_\mathbf{M}(\sigma), L_\mathbf{N}(\tau))$, then*

$$\max R_\mathbf{M}(\sigma)_{i,j} = \max \sigma_{i,j} \ \textit{and} \ \max L_\mathbf{N}(\tau)_{i,j} = \max \tau_{i,j}$$

and consequently,

$$\max R_\mathbf{M}(\sigma)_{i,i+1} = \max \sigma_{i+1} \ \textit{and} \ \max L_\mathbf{N}(\tau)_{i,i+1} = \max \tau_i$$

Proof. We consider the right subset shift operator, and the left subset shift operator proceeds similarly. As SCPs trivially meet these conditions, we will prove the result inductively by assuming the result holds for $(R_\mathbf{M}(\sigma), L_\mathbf{N}(\tau))$, and then showing that applying a valid operator R_{M_k} conserves the maximal elements. By the inductive hypothesis, we know that $\max R_\mathbf{M}(\sigma)_{k,k+1} = \max \sigma_{k+1}$. As R_{M_k} is a valid operator, we know two things. Firstly $\min M_k > \max R_\mathbf{M}(\sigma)_{k+1}$. Secondly, as k is greater than the maximal index used by $\mathbf{M}$, we have that $\max \sigma_{k+1} = \max R_\mathbf{M}(\sigma)_{k+1}$. So combining these we know,

$$\min M_k > \max R_\mathbf{M}(\sigma)_{k+1} = \max \sigma_{k+1} = \max R_\mathbf{M}(\sigma)_{k,k+1} \tag{2.2}$$

A key consequence of this inequality is that the corresponding graph of $(R_{M_k} R_\mathbf{M}(\sigma), L_\mathbf{N}(\tau))$ is a bipartite tree conditional on $(R_\mathbf{M}(\sigma), L_\mathbf{N}(\tau))$ being a bipartite tree (the shift will not disconnect the graph as none of the shifted elements are in the path $R_\mathbf{M}(\sigma)_{k,k+1}$). So, it is well-defined to speak of unique paths between blocks. We now explicitly explore how the shift operator R_{M_k} alters paths. Throughout the rest of this proof, we use the following shorthand. Let $\delta_{k,k+1} := R_\mathbf{M}(\sigma)_{k,k+1}$ (i.e. the path between the adjacent blocks in $(R_\mathbf{M}(\sigma), L_\mathbf{N}(\tau))$), and let $\delta_{k+1,k}$ be the same path reversed. Let γ be a path between two blocks on one side of $(R_\mathbf{M}(\sigma), L_\mathbf{N}(\tau))$, and γ' the path between the same blocks, by indices, in $(R_{M_k}R_{\mathbf{M}}(\sigma), L_{\mathbf{N}}(\tau))$. We assume for now that the start and end of the path γ are distinct from those of δ_k. There are four cases to consider.

(1) γ does not contain an element of M_k. It is thus unaffected by the shift, so $\gamma' = \gamma$.
(2) γ contains one element $m \in M_k$, i.e. $\gamma = \alpha m \beta$. Then there are two cases to consider
   (a) β does not use any steps of $\delta_{k,k+1}$, in which case $\gamma' = \alpha m \delta_{k+1,k}\beta$. This is a path in the tree of $(R_{M_k}R_{\mathbf{M}}(\sigma), L_{\mathbf{N}}(\tau))$ with no repeated steps, as such it must be the unique minimal path. (There cannot be another path as this would induce a cycle on the tree.)
   (b) β uses steps of $\delta_{k,k+1}$, in which case $\gamma' = \alpha m(\delta_{k+1,k} \setminus \beta)(\beta \setminus \delta_{k+1,k})$. This follows, as we know that β must follow the path $\delta_{k,k+1}$ for some time before diverging (β could also be a subset of $\delta_{k,k+1}$, in which case it will never diverge). As such, the path $(\delta_{k+1,k} \setminus \beta)$ reaches the point of divergence from $R_{\mathbf{M}}(\sigma)_{k+1}$ instead of $R_{\mathbf{M}}(\sigma)_k$, then the path $(\beta \setminus \delta_{k+1,k})$ completes the rest of the route unchanged.
(3) γ contains two elements of M_k. In which case we still have $\gamma' = \gamma$ (in path elements) but γ' will step through the $(k+1)$th block instead of γ stepping through the kth block.
(4) γ contains more than two elements of M_k. This is impossible, as γ would not be a minimal path on a tree.

We now consider the cases where γ is a path to/from at least one of $(R_{\mathbf{M}}(\sigma))_k, (R_{\mathbf{M}}(\sigma))_{k+1}$. We will use j to denote any other arbitrary index. We assume that γ contains a single element $m \in M_k$ (as all other cases lead to contradictions or are trivially affected by the shift in similar logic to the prior). Then γ is altered by the shift to γ' as follows,

- if γ connects $R_{\mathbf{M}}(\sigma)_j$ and $R_{\mathbf{M}}(\sigma)_k$, then $\gamma = \alpha m \mapsto \gamma' = \alpha m \delta_{k+1,k}$
- if γ connects $R_{\mathbf{M}}(\sigma)_j$ and $R_{\mathbf{M}}(\sigma)_{k+1}$, then $\gamma = \alpha m \delta_{k,k+1} \mapsto \gamma' = \alpha m$
- if γ connects $R_{\mathbf{M}}(\sigma)_k$ and $R_{\mathbf{M}}(\sigma)_{k+1}$, then $\gamma = \delta_{k,k+1}$ so this implies that $m \in \delta_{k,k+1}$, which is a contradiction as $m > \max \delta_{k,k+1}$.

Observe all (non-trivial or non-contradictory) paths γ' contain $m > \min M_k$ and either some addition or deletion by $\delta_{k,k+1}$. It thus follows from Equation (2.2) that $\max \gamma' = \max \gamma$, in each case, the maximal element will either be m, or in α, or in β. ☐

The key result of the inductive proof was the chain of inequalities which yielded $\min M_k > \max R_{\mathbf{M}}(\sigma)_{k,k+1}$ (Equation (2.2)). It turns out, if we assume the inequality holds, we can drop the assumption that k is greater than the maximal index used by $\mathbf{M}$, and the assumption that $\min M_k > \max R_{\mathbf{M}}(\sigma)_{k+1}$. Furthermore, once we have dropped the assumption that we need to perform shifts in an increasing order, we can generate all subset shifts by chains of singleton set shifts. This motivates the following definition.

Definition 2.64. Let (σ, τ) be a pair of ordered partitions, and let $\rho \in (\sigma_i \setminus \min \sigma_i)$ such that $\rho > \max \sigma_{i,i+1}$. Such an element ρ is said to be *critical*. We similarly define the critical elements of τ to be, $\rho \in (\tau_{i+1} \setminus \min \tau_{i+1})$ such that $\rho > \max \tau_{i,i+1}$. The right and left shift operators are defined on singleton sets, and we will denote these shift operators being applied to critical elements as R_ρ and L_ρ.

As we have assumed the necessary feature of the proof of Lemma 2.63, we have the immediate corollary.

Corollary 2.65. *The critical left and right shift operators conserve the maximal element of paths, and if (σ', τ') is generated from critical shifts of $(\sigma, \tau) \in SCP$ then,*

$$\max \sigma'_{i,i+1} = \max \sigma_{i+1} \text{ and } \max \tau'_{i,i+1} = \max \tau_i$$

Definition 2.66. The facets of the shift $\triangle^{\mathrm{SU}}$ are SCPs, and their image under iterated applications of critical left and right shift operators.

As one might suspect given the above corollaries.

Proposition 2.67. *The subset shift, and shift definitions of the $\triangle^{\mathrm{SU}}$ diagonal coincide.*

Proof. We analyse the right shift operator, the case of the left shift is similar. First, we observe that any subset right shift $R_{M_k}(\sigma)$ can be decomposed in a series of right shifts: since $\min M_k > \max \sigma_{k,k+1}$ by the proof of Lemma 2.63, we can first shift $\min M_k$ to the right, then $\min(M_k \smallsetminus \min M_k)$, and so on until the entire subset M_k has been shifted to the right. This shows that any facet of the subset shift $\triangle^{\mathrm{SU}}$ diagonal is also a facet in the shift $\triangle^{\mathrm{SU}}$ diagonal

For the reverse inclusion, we proceed by induction. We are required to show that if we apply a critical right shift to $(R_{\mathbf{M}}(\sigma), L_{\mathbf{N}}(\tau))$, say $(R_\rho R_{\mathbf{M}}(\sigma), L_{\mathbf{N}}(\tau))$, then this can be re-expressed as a well-defined subset shift operation i.e. $(R_{\mathbf{M}'}(\sigma), L_{\mathbf{N}}(\tau))$. Suppose that prior to the critical shift that ρ lives in block l, then we must have

$$1 \leq i_1 < \cdots < i_j \leq l < i_{j+1} < \cdots < i_d \leq k-1$$

for some $j \in [d]$. If $i_j < l$, then $R_\rho R_{\mathbf{M}}(\sigma) = R_{M_{i_d}} \dots R_{\{\rho\}_l} R_{M_{i_j}} \dots R_{M_{i_1}}(\sigma)$, and we are done. Otherwise, if $i_j = l$, let $M'_{i_j} = M_{i_j} \cup \{\rho\}$. It is clear that $R_\rho R_{\mathbf{M}}(\sigma) = R_{M_{i_d}} \dots R_{M'_{i_j}} \dots R_{M_{i_1}}(\sigma)$, however, we need to check that $R_{M'_{i_j}}$ is a well-defined subset operator. We know the subset shift operators will never move the minimal element, so $\rho > \min R_{\mathbf{M}}(\sigma)_{i_j} = \min(R_{M_{i_{j-1}}} \dots R_{M_{i_1}}(\sigma))_{i_j}$. Then from Corollary 2.65, we know that $\rho > \max \sigma_{i_j+1} = \max(R_{M_{i_{j-1}}} \dots R_{M_{i_1}}(\sigma))_{i_j+1}$, where the equality follows as $i_1 < \cdots < i_{j-1} < i_j$. This proves that $R_{M'_{i_j}}$ is well-defined, completing the inductive proof. □

The remainder of this section is devoted to the proof that the shift $\triangle^{\mathrm{SU}}$ diagonal is equal to the $\triangle^{\mathrm{SU}}$ diagonal. We present the proof before explicitly unpacking the needed lemmas and definitions. Kurt: maybe after is cleaner, had to decompose the second half too much

Proposition 2.68. *The facets of shift $\triangle^{\mathrm{SU}}$ and $\triangle^{\mathrm{SU}}$ are equal.*

Proof. We first note that SCPs are known elements of $\triangle^{\mathrm{SU}}$ and shift $\triangle^{\mathrm{SU}}$. The proof that every facet of shift $\triangle^{\mathrm{SU}}$ is in $\triangle^{\mathrm{SU}}$ follows from the closure of $\triangle^{\mathrm{SU}}$ under the shift operator (Lemma 2.69). The proof that every facet of $\triangle^{\mathrm{SU}}$ is in shift $\triangle^{\mathrm{SU}}$ follows from the closure of $\triangle^{\mathrm{SU}}$ under the inverse shift operator (Lemma 2.76) and an inductive argument that decomposes any facet into a SCP through inverse shift operators (Lemma 2.77). As shift operators are the inverse of the inverse shift operators (Lemma 2.74), for any given $\triangle^{\mathrm{SU}}$ facet, this provides a SCP and a sequence of shifts to form it, showing it is a shift $\triangle^{\mathrm{SU}}$ facet. □

Given our existing theory, the closure of the critical shift operator is easy to establish.

Lemma 2.69. *Let $(\sigma, \tau) \in \triangle^{\mathrm{SU}}$, if we apply a critical shift operator to it, then we remain in $\triangle^{\mathrm{SU}}$.*

Proof. We consider just the right shift, and the left proceeds similarly. Corollary 2.65 shows that the maxima of paths between consecutive vertices in $(R_\rho(\sigma), \tau)$ are the same as the ones in (σ, τ). Thus, all maxima of paths in $(R_\rho(\sigma), \tau)$ are traversed from right to left, and hence by Proposition 2.41, we have that $(R_\rho(\sigma), \tau) \in \triangle^{\mathrm{SU}}$. □

We now need to develop the theory of inverse shift operators.

Definition 2.70. Let (σ, τ) be a pair of ordered partitions $[n]$, we say that (x, y), where $x < y$, is an *inversion* if x appears before y in σ and y appears before x in τ.

Proposition 2.71. *The set of inversions of a $\triangle^{\mathrm{SU}}$ facet is in bijection with its set of edge crossings. Moreover, the set SCP is equal to the set of facets with no crossings.*

Proof. For the first part of the statement, it is clear that every inversion gives rise to a crossing. For the converse, one needs to check that an anti-inversion, where y appears before x in σ and x appears before y in τ, cannot occur in a facet of $\triangle^{\mathrm{SU}}$; this follows immediately from the (I, J)-conditions for $|I| = |J| = 1$. The second part of the statement follows from the fact that facets of the diagonal with no crossings are in bijection with permutations. As explained above, from a partition one obtains a strong complementary pair, which is in $\triangle^{\mathrm{SU}}$. In the other way around, given a strong complementary pair, one can read-off the partition in the associated tree, which has no crossings, by a vertical down-slice of edges, see Example 2.59. □

We say that an edge crossing is an *adjacent crossing* if the two crossing elements are in adjacent blocks (i.e. $\sigma_i|\sigma_{i+1}$ or $\tau_i|\tau_{i+1}$).

Lemma 2.72. *A facet $(\sigma, \tau) \in \triangle^{\mathrm{SU}}$ has a crossing if, and only if, it has an adjacent crossing.*

Proof. An adjacent crossing is clearly a crossing. In the other direction, suppose there is a crossing between an element of σ_i and an element of σ_j. If σ_i and σ_j are not adjacent, then the 'triangle' produced by the crossing elements encloses other σ_k such that $i < k < j$, and this produces other crossings. We may repeat this process until we find an adjacent crossing. □

We now define inverse critical shift operators.

Definition 2.73. Let (σ, τ) be a pair of ordered partitions. Let $\rho \in \sigma_{i+1}$ be such that $\rho > \max \sigma_{i,i+1}$ and (α, ρ) is an adjacent crossing, where α is an arbitrary element which is smaller than ρ. Similarly, let $\rho \in \tau_i$ such that $\rho > \max \tau_{i,i+1}$ and (α, ρ) is an adjacent crossing. Subject to these constraints, we informally define the inverse left/right shift operators as, R_ρ^{-1} returns ρ to σ_i, and L_ρ^{-1} returns ρ to τ_{i+1}

The theory of these inverse operators behaves as one would expect. We only develop the necessary results to prove the equality of the diagonals.

Lemma 2.74. *The shift operators are the inverses of the inverse shift operators.*

Proof. If we apply a well-defined inverse shift operator R_ρ^{-1} to (σ, τ), then we have $\alpha, \rho \in R_\rho^{-1}(\sigma)_i$ such that $\min R_\rho^{-1}(\sigma)_i \le \alpha < \rho$ and $\rho > \max R_\rho^{-1}(\sigma)_{i,i+1}$. As such, R_ρ is well-defined, and $R_\rho R_\rho^{-1} = id$. A similar result holds for the left operator. □

Lemma 2.75. *The inverse shift operators conserve the maximal elements of paths.*

Proof. This is an immediate corollary of the prior lemma, and the fact that the shift operators conserves maximal elements of paths (Lemma 2.63). □

Lemma 2.76. *Let $(\sigma, \tau) \in \triangle^{\mathrm{SU}}$, if we apply an inverse critical shift operator to it, then we remain in $\triangle^{\mathrm{SU}}$.*

Proof. As in Lemma 2.69, as the inverse shift operators conserve the maxima of paths, and these same maxima continue to be traversed from right to left, it follows by Proposition 2.41 that the output of an inverse shift operator will remain in $\triangle^{\mathrm{SU}}$. □

Lemma 2.77. *All $(\sigma, \tau) \in \triangle^{\mathrm{SU}}$ are mapped to a SCP by a finite number of inverse shifts.*

Proof. It is immediate that given any element, $(\sigma, \tau) \in \triangle^{\mathrm{SU}}$ then we will only be able to apply a finite chain of inverse shift operators to it before no further operations are available. (It is straightforward to show each inverse shift decreases the number of crossings. An even simpler argument is, there are only a finite number of elements to move by a finite number of blocks).

We now need to show that every facet which is not a SCP admits an inverse shift. Kurt: Still thinking on this...

- We know any non SCP facet must have adjacent crossings, (α, ρ) but we don't know that $\rho > \max \sigma_{i,i+1}$ (or $\rho > \max \tau_{i,i+1}$)
- Straightforward to show that ρ will cross the connecting path, so surely this provides the order information, but need to show this. Kurt: ...

□

2.5.1. *Shifts Under the Face Poset Isomorphism.*

Lemma 2.78. *The critical left and right shift operators send a facet of $\triangle^{\mathrm{SU}}$ to an adjacent facet of $\triangle^{\mathrm{SU}}$. This can be interpreted as stepping through a shared facet of the facets.*

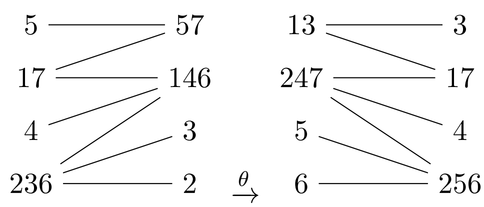
Proof. Let $(\sigma_1 | \dots | \sigma_k, \tau) \in \triangle^{\mathrm{SU}}$, and let $\rho \in \sigma_i$ be a critical element. Then, the shift operator R_ρ steps from the facet $(\sigma_1 | \dots | \sigma_i | \sigma_{i+1} | \dots | \sigma_k, \tau)$, to the facet $(\sigma_1 | \dots | \sigma_i \smallsetminus \{\rho\} | \sigma_{i+1} \sqcup \{\rho\} | \dots | \sigma_k, \tau)$, through the facets of facets $(\sigma_1 | \dots | \sigma_i \smallsetminus \{\rho\} | \rho | \sigma_{i+1} | \dots | \sigma_k, \tau)$. □

As such, an immediate corollary of $\theta : \triangle^{\mathrm{LA}} \to \triangle^{\mathrm{SU}}$ being a face poset isomorphism is that their are analogous critical shift definitions of $\triangle^{\mathrm{LA}}$ induced by θ^{-1}. We now explicitly state the critical shift $\triangle^{\mathrm{LA}}$ definition.

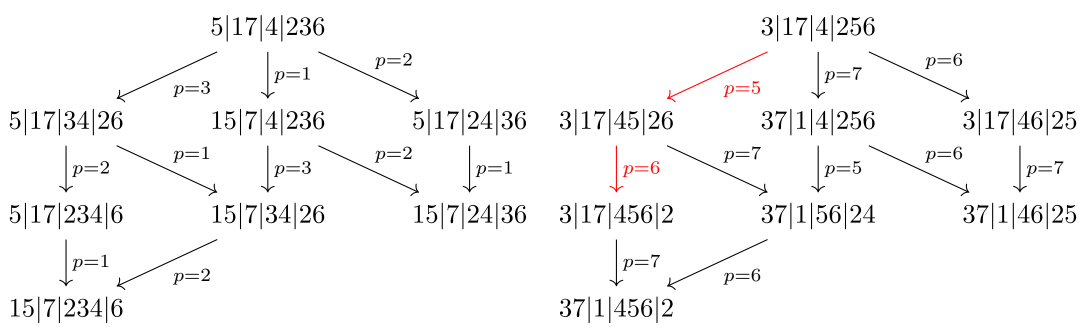
Definition 2.79. Let (σ, τ) be a pair of ordered partitions, and let $\rho \in (\sigma_{i+1} \setminus \max \sigma_{i+1})$ such that $\rho < \min \sigma_{i,i+1}$. Such an element ρ is said to be *critical*. We similarly define the critical elements of τ to be, $\rho \in (\tau_i \setminus \max \tau_i)$ such that $\rho < \min \tau_{i,i+1}$. We define the left shift operator L_ρ to act on σ, and the right shift R_ρ operator to act on τ.

We note that compared to shift $\triangle^{\mathrm{SU}}$, the left shift operator now acts on the left ordered partition, and the right shift operator on the right ordered partition, a distinct notational improvement. The correctness of this definition can be verified directly by obvious variants of the proofs of this section for the $\triangle^{\mathrm{LA}}$ diagonal. There is also a clear translation of the subset shift operator, which we will not write out in full. We now explore in example how the $\triangle^{\mathrm{LA}}$ shift structure is induced by θ.

Example 2.80. The *SCP*s $(\sigma, \tau) := 5|17|4|236 \times 57|146|3|2$, and $(\sigma', \tau') := 13|247|5|6 \times 3|17|4|156$, are in bijection through θ. Graphically this bijection has a clear symmetry,



We now illustrate all possible $\triangle^{\mathrm{LA}}$ shifts of the left *SCP* and all possible $\triangle^{\mathrm{SU}}$ shifts of the right *SCP*. We first display how the $\triangle^{\mathrm{LA}}$ left shifts act on σ, and the $\triangle^{\mathrm{SU}}$ left shifts act on τ'. We indicate shifts by their corresponding critical element, drawing them to avoid crossings, and so they align with their isomorphic shift under θ.



The possible $\triangle^{\mathrm{LA}}$ right shifts (acting on τ) are,

$$\sigma \times 57|146|3|2 \xrightarrow{p=1} \sigma \times 57|46|13|2 \xrightarrow{p=1} \sigma \times 57|46|3|12$$

The possible $\triangle^{\mathrm{SU}}$ right shifts (acting on σ') are,

$$13|247|5|6 \times \tau' \xrightarrow{p=7} 13|24|57|6 \times \tau' \xrightarrow{p=7} 13|24|5|67 \times \tau'$$

We note that as the left and right shifts (for both diagonals) can be performed independently, we could combine these lattices into a product of lattices. No other shifts are possible, observe for instance that we cannot perform the $\triangle^{\mathrm{LA}}$ left shift $15|7|234|6 \times 57|46|13|2 \xrightarrow{p=2} 15|27|34|6 \times 57|46|13|2$ as the minimal path connecting 234 and 7 contains 1 which is smaller than 2 (see Example 2.36). As an example of a $\triangle^{\mathrm{SU}}$ subset shift, and the bijection to critical shifts (Proposition 2.67), we observe that the unique way of producing $13|247|5|6 \times 37|1|456|2$ from subset shifts is given by

$$13|247|5|6 \times 3|17|4|256 \xrightarrow{R_{\{5,6\}}} 13|247|5|6 \times 3|17|456|2 \xrightarrow{R_{\{7\}}} 13|247|5|6 \times 37|1|456|2$$

This corresponds to combining the critical right shift operators of the $\triangle^{\mathrm{SU}}$ diagonal which have been indicated in red into a single subset. We note it is also possible to apply $\triangle^{\mathrm{SU}}$ shifts to (σ, τ) and $\triangle^{\mathrm{LA}}$ shifts to (σ', τ').

Kurt: Mention the LAD and SUD shifts generate the (same) facets of the hexagon in two different ways.

We have the following equivalent characterisations of the $\triangle^{\mathrm{LA}}$ and $\triangle^{\mathrm{SU}}$ diagonals. Kurt: Probably in the intro somewhere

- I, J conditions
- As orderings of essential complementary partitions
- Subset shift
- Critical element shift
- Matrix
- Cubical?

2.6. **Shuffle trees.**

3. Tables

In this section we present low dimensional computations of the enumeration results obtained above and we connect them to other known combinatorial objects.

dim	**0**
0	1

dim	**0**	**1**
0	3	1
1	1	

dim	**0**	**1**	**2**
0	17	12	1
1	12	6	
2	1		

dim	**0**	**1**	**2**	**3**
0	149	162	38	1
1	162	150	24	
2	38	24		
3	1			

Figure 9. Number of pairs of faces in the cellular image of the diagonal 0, 1, 2 and 3-dimensional permutahedra.

Guillaume: On garde?

dim	**0**	**1**	**2**	**3**	**4**
0	1809	2660	1080	110	1
1	2660	3540	1200	80	
2	1080	1200	270		
3	110	80			
4	1				

dim	**0**	**1**	**2**	**3**	**4**	**5**
0	28399	52635	30820	6165	302	1
1	52635	90870	67580	7785	240	
2	30820	47580	20480	2160		
3	6165	7785	2160			
4	302	240				
5	1					

FIGURE 10. Number of pairs of faces in the cellular image of the diagonal 4 and 5-dimensional permutahedra.

Pairs $(F, G) \in \operatorname{Im} \triangle_{(P,\vec{v})}$	**Polytopes**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	[OEI22]
$\dim F + \dim G = \dim P$	Associahedra	1	2	6	22	91	408	1938	A000139
	Multiplihedra	1	2	8	42	254	1678	11790	to appear
	Permutahedra	1	2	8	50	432	4802	65536	A007334
$\dim F = \dim G = 0$	Associahedra	1	3	13	68	399	2530	16965	A000260
	Multiplihedra	1	3	17	122	992	8721	80920	to appear
	Permutahedra	1	3	17	149	1809	28399	550297	A213507

FIGURE 11. Number of pairs of faces in the cellular image of the diagonal of the associahedra, multiplihedra and permutahedra of dimension $0 \leq \dim P \leq 6$, induced by any good orientation vector.

3.0.1. *Combinatorial formula for facets of the diagonal.* From Theorem 2.33, we can deduce a formula for the number of facets of the diagonal:

Proposition 3.1. *The number of pairs of ordered partitions of dimension $(k, n-k)$ which correspond to facets of the diagonal is given by:*

$$\frac{1}{k+1}\binom{n+1}{k}(k+1)^{n-k}(n+1-k)^k. \tag{3.1}$$

Proof. According to Theorem 2.33, pairs of ordered partitions of dimension $(k, n-k)$ which correspond to facets of the diagonal are in one-to-one correspondence with bipartite trees with $k+1$ black vertices, $n-k+1$ white vertices and $n+1$ edges labeled from 1 to $n+1$.

We do not prove exactly here the proposition but a slightly modified version: Rooted bipartite trees with $k+1$ black vertices and $n-k+1$ white vertices such that:

- a black vertex is distinguished and called *the root*
- the $n+1$ non-root vertices are labeled,
- every label between 1 and $n+1$ is used exactly once.

are counted by:

$$\binom{n+1}{k}(k+1)^{n-k}(n+1-k)^k. \tag{3.2}$$

Let us construct such a bipartite tree.

First, there are $\binom{n+1}{k}$ ways to choose the labels for black vertices (white vertices being labeled by the non-chosen labels). We denote by $\mathcal{B}$ this set of labels.

Moreover, the labeled black vertices are different from the root, hence they should have a white parent : there are $n+1-k$ ways to choose the parent of any labeled black vertex. We thus have $(n+1-k)^k$ ways to build corollas with labeled black leaves and a white root, called bi-colored corollas (or sometimes just corollas) in the sequel.

Finally, we arrange bi-colored corollas in a rooted bipartite tree by adapting the algorithm which convert a Prüfer code to a tree. Here what is called *Prüfer code* is a word of length $n-k$ over the alphabet $\mathcal{B} \cup \{\bullet\}$, where $\bullet$ stands for the non-labeled black vertex. Let us start with a word $c = c_1 \dots c_{n-k}\bullet$ of length $n-k+1$ and the set $\mathcal{T} = \mathcal{S} \cup \{\bullet\}$ of $n-k+2$ bi-colored corollas augmented with the unlabeled black vertex. We apply Algorithm 1. Let us first prove it termination and correctness. The equality $\text{length}(c) = \text{Card}(\mathcal{T}) - 1$ is a loop invariant for the While loop: indeed at each iteration of the loop, the length of c and the number of elements in $\mathcal{T}$ decrease exactly by one. It ensures the termination of the loop and the fact that $\mathcal{T}$ contains a unique element when exiting the loop. Moreover, the set of trees $\mathcal{T}$ contains at each steps exactly one unlabeled black vertex, k labeled black vertices and $n-k+1$ white vertices. Finally, when adding an edge between two trees, one can only get a tree. Moreover, as the edge is added between a white root and the label of a black vertex, the obtained tree is indeed bipartite.

Algorithm 1: Prüfer algorithm : from a word to a tree

Input: a word $c = c_1 \dots c_i$ and a set $\mathcal{T}$ of i bi-colored trees with white root and one bi-colored tree with an unlabeled black root

Output: a bipartite rooted tree

while $\text{length}(c) > 0$ **do**

  /* loop invariant: $\text{length}(c) = \text{Card}(\mathcal{T}) - 1$, at each iteration, the length of c decreases by 1 */

  t $\leftarrow \min\{a \in \mathcal{T} \mid$ none of the c_i is a label in $a\}$ // Here the order is given by the order on the labels of the root (as the tree with a black root does not satisfy the condition)

  p $\leftarrow$ tree of $\mathcal{T}$ to which belongs the first letter of c // Note that it cannot be t itself

  Remove t and p from $\mathcal{T}$ // Decrease the cardinality of $\mathcal{T}$ by two

  Add an edge between the root of t and the first letter of c and add the obtained tree to $\mathcal{T}$ // Increase the cardinality of $\mathcal{T}$ by one

  Remove the first letter of c // Decrease the length of c by one

Return the unique element of $\mathcal{T}$

To prove that this algorithm defines a bijection between the pairs of Prüfer code and set of bipartite rooted trees, let us give the algorithm which convert a rooted bipartite tree in such a pair in Algorithm 2.

This second algorithm terminates as the number of white vertices decreases strictly by one at each iterations. Moreover, every letter added to c is the label of a black vertices,

---

Algorithm 2: Prüfer algorithm : from a tree to a word

Input: a bipartite rooted tree A

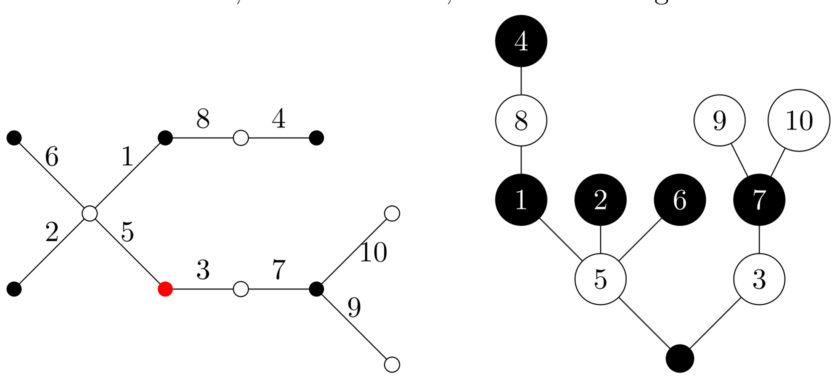
Output: a word $c = c_1 \dots c_i$ and a set $\mathcal{T}$ of i bi-colored trees with white root, except one bi-colored tree with an unlabeled black root

```
c ← empty word                                                         //
𝒯 ← empty set                                          // Initialization
while A has more than one vertex do
    /* At each iteration, the number of white vertices decreases by 1  */
    t ← min{w ∈ 𝒯 | w is a white vertex whose children are leaves }  // Here the order
        is the one on white labels
    c ← c concatenated with label of the parent of t   // This label is a black vertex
    Remove the edge between t and its parent: the root part goes in A and the
        corolla in 𝒯                             // Increase the cardinality of 𝒯 by one
Return the pair (c, 𝒯 ∪ A)
```

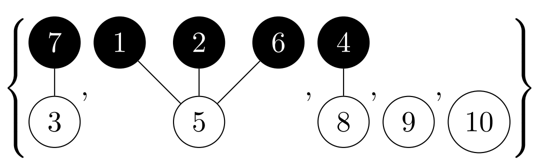
---

and every tree added to $\mathcal{T}$ is a bi-colored corolla or $\bullet$ (the tree with only the non-labeled black root). Finally, the cardinality of the set of bipartite trees is one more than the length of c. As this algorithm is the classical reverse algorithm of the first one, it ends the proof. Bérénice: Do I need to add more details ? □

Example 3.2. Let us apply Prüfer algorithm on an example. Consider the following rooted bipartite tree on the left below, with a red root, and a redrawing of it on the right:



Separating corollas with a Prüfer algorithm, we get the word $1 \bullet 77$ and the following set of corollas:



This set of corollas can be viewed as a function f from the set of labelled black vertices $\{1, 2, 4, 6, 7\}$ to the set of white vertices $\{3, 5, 8, 9, 10\}$ satisfying $f(1) = f(2) = f(6) = 5$, $f(7) = 3$ and $f(4) = 8$.

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