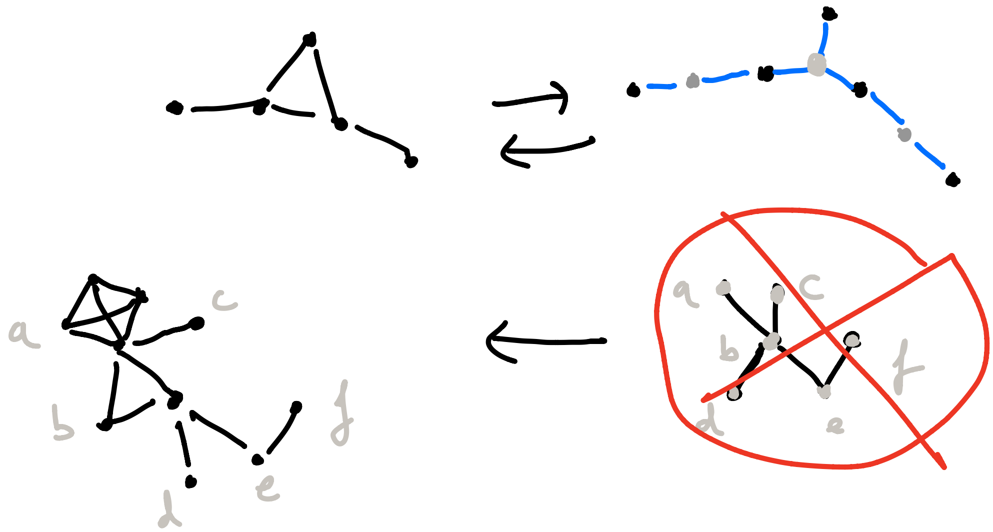
"G est un arbre de cliques"

1) mettre un blanc au milieu de chaque clique



"plongement dans un arbre"
bipartite

⊕ $\gamma(i,j)$ est ~~un~~ l'unique chemin géodésique (le + court entre 2 points)

[pas les seuls graphes avec cette propriété]

1) S sépare $X \cup Y$ mais ni X ni Y

2) $\cancel{X} \to X$ (cohérence avec la suite) X, Y, Z

3) "does not separate" non! faut être contraposée

4) vérif. d'épine : "pairwise separated"

Theorem 1.31. *Let G be a block graph. The spine fan $\mathcal{F}(G)$ is a complete simplicial fan.*

We will show that any linear order σ on $V(G)$ is a linear extension of a unique minimal spine S on G. Let $\sigma : V(G) \to \{1, \dots, n\}$ be a linear order on $V(G)$. We see it as a directed linear tree $\sigma(x_1) \prec \sigma(x_2) \prec \dots \prec \sigma(x_n)$.

Lemma 1.32. *Let G be a block graph, and let S be a ~~minimal~~ maximal G-spine. Then, we have*

$$x \prec y \text{ in } S \iff \begin{array}{l} \overline{s_{<}(y) \setminus x} \cup \underline{t^{>}(x) \setminus y} \text{ does not separate } x \text{ and } y, \text{ and} \\ \exists \text{ a linear extension } \sigma \text{ such that } \sigma(x) \prec \sigma(y). \end{array}$$

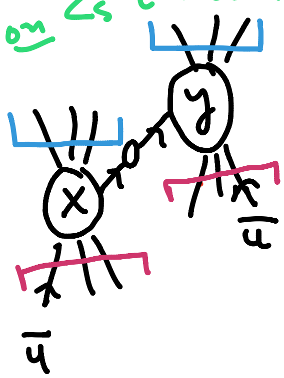
$x <_S y$ $\qquad = x <_\sigma y$

Proof. ($\Longrightarrow$) Suppose that $\overline{s_{<}(y) \setminus x} \cup \underline{t^{>}(x) \setminus y}$ separates x and y. Then, in virtue of Corollary 1.8, there is some $u \in \overline{s_{<}(y) \setminus x} \cup \underline{t^{>}(x) \setminus y}$ that separates x and y, and we must have $x \prec u \prec y$. The fact that $x \prec y$ implies the existence of σ such that $\sigma(x) \prec \sigma(y)$ follows from the fact that S is an oriented tree. ($\Longleftarrow$) If $y \lessdot x$, there is no linear extension σ such that $\sigma(x) \prec \sigma(y)$. If x and y are not comparable in S, there exists u in $\overline{s_{<}(y) \setminus x}$ or in $\underline{t^{>}(x) \setminus y}$ that separates x and y (je prends la premiére bifurcation). $\square$

[< doit être réservé aux entiers]

poset courbé on $<_S$

$\lessdot$ $\downarrow$ $\{\not\leq, \leq, \lessdot\}$ $\uparrow$ couverture



soit $x \not\leq y \Rightarrow x \not\lessdot y$

soit $x \leq y \Rightarrow \exists\, x < u < y$
$\Rightarrow x \not\lessdot y$

$$x \not\lessdot y \;\begin{cases} x \not\leq y \begin{cases} y \leq x \\ \text{pas comparable} \end{cases} \\ x \leq y \to \exists\; x < u < y \end{cases}$$

réécrire cette preuve

Lemma 1.32. *Let G be a block graph, and let S be a minimal G-spine. Then, we have*

$$x \prec y \text{ in } S \iff \begin{array}{l} \overline{s_{<}(y) \setminus x} \cup \underline{t^{>}(x) \setminus y} \text{ does not separate } x \text{ and } y, \text{ and} \\ \exists \text{ a linear extension } \sigma \text{ such that } \sigma(x) \prec \sigma(y). \end{array}$$

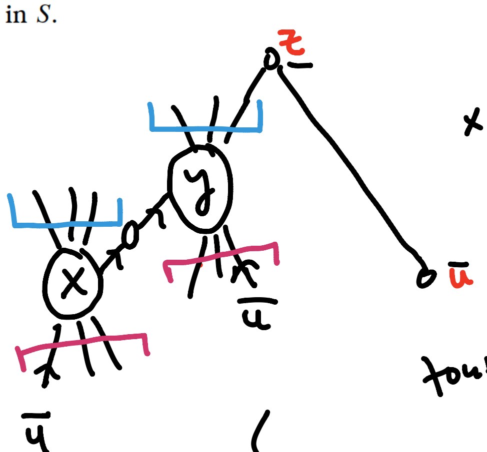
Lemma 1.33. *Let G be a block graph, and let S be a ~~minimal~~ max G-spine, σ ext. lin. Then,*

$$x \prec y \text{ in } S \iff \begin{array}{l} \cancel{\text{For every linear extension } \sigma \text{ of } S, \text{ we have}}\ \sigma(x) \cancel{<} \sigma(y), \\ \overline{\sigma^{-1}(\mathrm{s}_<(\sigma(y)) \setminus \sigma(x))} \cup \underline{\sigma^{-1}(\mathrm{t}^>(\sigma(x)) \setminus \sigma(y))} \text{ does not separate } x \text{ and } y, \text{ and} \\ \cancel{\exists \text{ a linear extension } \sigma' \text{ such that } \sigma'(x) < \sigma'(y).} \end{array}$$

$\Rightarrow$

Proof. ($\implies$) Suppose that $\overline{\sigma^{-1}(\mathrm{s}_<(\sigma(y)) \setminus \sigma(x))} \cup \underline{\sigma^{-1}(\mathrm{t}^>(\sigma(x)) \setminus \sigma(y))}$ separates x and y. Then, in virtue of Corollary 1.8, there is some $u \in \overline{\sigma^{-1}(\mathrm{s}_<(\sigma(y)) \setminus \sigma(x))} \cup \underline{\sigma^{-1}(\mathrm{t}^>(\sigma(x)) \setminus \sigma(y))}$ that separates x and y, and we must have $x < u < y$. The fact that $x \prec y$ implies the existence of σ' such that $\sigma'(x) \prec \sigma'(y)$ follows from the fact that S is an oriented tree.

($\impliedby$) There exists two linear extensions σ_1 and σ_2, such that $\overline{\sigma_1^{-1}(\mathrm{s}_<(\sigma_1(y)) \setminus \sigma_1(x))} = \overline{\mathrm{s}_<(y) \setminus x}$ and $\underline{\sigma_2^{-1}(\mathrm{t}^>(\sigma_2(x)) \setminus \sigma_2(y))} = \underline{\mathrm{t}^>(x) \setminus y}$. So neither $\overline{\mathrm{s}_<(y) \setminus x}$ nor $\underline{\mathrm{t}^>(x) \setminus y}$ separate x and y, and by Proposition 1.10 we have that $\overline{\mathrm{s}_<(y) \setminus x} \cup \underline{\mathrm{t}^>(x) \setminus y}$ does not separate x and y. Now, Lemma 1.32 implies that $x \prec y$ in S. □



x, y. $\leftarrow \underline{z} \rightarrow \bar{u}$

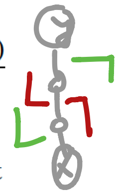
tout ce qui sépare x et y

Corollaire 1.8 $\left\{ \begin{array}{l} \text{est sur un chemin de } x \text{ à } y \\ x \;—\; z \;—\; \bar{u} \\ y \;—\; \end{array} \right.$

$t_<(x)$ $\quad$ $t^>(y)$

Proof of the theorem. Let $\sigma : V(G) \to \{1, \ldots, n\}$ be a linear order on $V(G)$. We define a poset S by the following covering relations

$$x \prec y \text{ in } S \iff \sigma(x) < \sigma(y) \text{ and } \overline{\sigma^{-1}(\mathrm{s}_<(\sigma(y)) \setminus \sigma(x))} \cup \underline{\sigma^{-1}(\mathrm{t}^>(\sigma(x)) \setminus \sigma(y))} \text{ does not separate } x \text{ and } y.$$



Using induction, we have the following characterization of S : $x' = y$ $y' = x$

$$x \le y \iff \sigma(x) \le \sigma(y) \text{ and } \exists x', y' \text{ with } \sigma(x) \le \sigma(x'), \sigma(y') \le \sigma(y) \text{ such that } \overline{\sigma^{-1}(\mathrm{s}_<(\sigma(x')) \setminus \sigma(x))} \cup \underline{\sigma^{-1}(\mathrm{t}^>(\sigma(y')) \setminus \sigma(y))} \text{ does not separate } x \text{ and } y.$$

(1) S is a poset. Reflexivity and antisymmetry are clear. For transitivity, suppose that we have $x \le y$ and $y \le z$. That means there exists x_1, y_1, y_2, z_2 with $\sigma(x) \le \sigma(x_1), \sigma(y_1) \le \sigma(y) \le \sigma(y_2), \sigma(z_2) \le \sigma(z)$ such that

$$\overline{\sigma^{-1}(\mathrm{s}_<(\sigma(x_1)) \setminus \sigma(x))} \cup \underline{\sigma^{-1}(\mathrm{t}^>(\sigma(y_1)) \setminus \sigma(y))} \quad \text{does not separate } x \text{ and } y, \text{ and}$$
$$\overline{\sigma^{-1}(\mathrm{s}_<(\sigma(y_2)) \setminus \sigma(y))} \cup \underline{\sigma^{-1}(\mathrm{t}^>(\sigma(z_2)) \setminus \sigma(z))} \quad \text{does not separate } y \text{ and } z.$$

Since $\sigma(x_1) \le \sigma(y_2)$ and $\sigma(y_1) \le \sigma(z_2)$, we have that

$$\overline{\sigma^{-1}(\mathrm{s}_<(\sigma(x_1)) \setminus \sigma(x))} \cup \underline{\sigma^{-1}(\mathrm{t}^>(\sigma(z_2)) \setminus \sigma(z))} \quad \text{does not separate } x \text{ and } z.$$

(2) S is a tree. Vincent? # rel. couverture = # sommets − 1

(3) S is a minimal spine. Let z be a vertex of S. It is enough to prove that the source set of each incoming edge of z lies in a distinct connected component of $G \setminus \underline{\{z\}}$; the proof for the outgoing edges is symmetric. Let $x, y \in V(G)$ be such that $x \prec z$ and $y \prec z$. Without loss of generality, we can suppose that $\sigma(x) < \sigma(y) < \sigma(z)$.

If $x \le y$, we have by definition that $\underline{\sigma^{-1}(\mathrm{t}^>(\sigma(y)))}$ does not separate x and y. Since $\sigma(z) \in \mathrm{t}^>(\sigma(y))$, we have in particular that $\underline{\{z\}}$ does not separate x and y.

Now we suppose that $x \prec z$ and $y \prec z$. If $\underline{\{z\}}$ does not separate x and y, using Proposition 1.10 we have that $\overline{\sigma^{-1}(\mathrm{s}_<(\sigma(z)) \setminus \{\sigma(x), \sigma(y)\})} \cup \underline{\sigma^{-1}(\mathrm{t}^>(\sigma(y)))}$ does not separate x and y either. Now let w be such that $\sigma(w) \prec \sigma(y)$. Then,

$$\overline{\sigma^{-1}(\mathrm{s}_<(\sigma(y)) \setminus \sigma(x))} \cup \underline{\sigma^{-1}(\mathrm{t}^>(\sigma(w)) \setminus \sigma(y))}$$

does not separate x and y, so we have $x \le y$, a contradiction.

(4) S is unique. Let S' be a minimal G-spine for which σ is a linear extension. From Lemma 1.33 we have that $x \prec y$ in S' implies $x \prec y$ in S. Since S and S' have the exact same number of covering relations (they are both oriented trees with $n - 1$ oriented edges), we also have that $x \prec y$ in S implies $x \prec y$ in S'. (Toute injection entre deux ensembles de même cardinalité est une bijection)

□

(2) montrer que les couples x,y qui vérifient [...] sont au plus au nombre de n−1

(1,3) → on peut les remplacer par l'arg. géométrique

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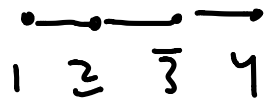
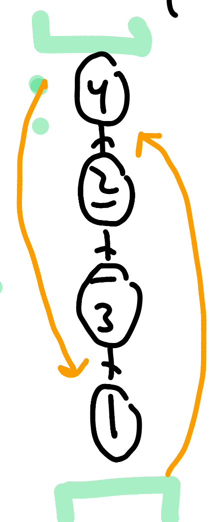


σ : $1 \to 1$

$\underline{2} \to 3$

$\bar{3} \to 2$

$4 \to 4$



$\rangle$ $\sigma(i) \leq \sigma(j)$ et i, j

sont ds m̄ cc de

$G \setminus (\overline{S_\sigma(i)} \cup \underline{t_\sigma(j)})$

1,4 ???