Optimization and Optimal Research
Correction

Résumé

During the exam, I just want you to pay attention to the redaction of you copy. I'm giving you an example of redaction in this correction. You don't need to write as much as I'm doing but i don't want to see only calculous on your copy.

Inner product and norms

Exercise 1 :

To proove that the following functions define an inner product, you have to check that they bilinear, symetric, positive and definite form. In other words, you have to check that :

- $f(\mathbf{x}, \mathbf{x}) \geq 0$,
- $f(\mathbf{x}, \mathbf{x}) = 0 \iff \mathbf{x} = 0$,
- $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$,
- $f(\mathbf{x} + \lambda\mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{z}) + \lambda f(\mathbf{y}, \mathbf{z})$.

- This first fonction define an inner product by definition
- This second function do not satisfy the positive definite character. Indeed, if we take $\mathbf{x} = (0, 0, 1)$, we have $f(\mathbf{x}, \mathbf{x}) = 0$ but $\mathbf{x} \neq 0$.
- You can easily check that this function is bilinear positive and definite.
- This last function is not bilinear due to the presence of the quadratic and cubc terms.

Exercise 2 : Frobenius' Norm

We set $A = (a_{ij})$ and $B = (b_{ij})$

- In this question, we need to check the four points as it was done in the previous exercise. Remember that $\langle A, B\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij}$.
  $\langle A, A\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2 \geq 0$ and this sum is equal to 0 if and only if $A = 0$.
  It is clearly symetric and for all $\lambda \in \mathbb{R}$ and $C \in \mathcal{M}_{n,m}(\mathbb{R})$, we have :

$$\langle A + \lambda C, B\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m}(a_{ij} + \lambda c_{ij})(b_{ij}) = \lambda \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}c_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij} = \lambda\langle A, C\rangle + \langle A, B\rangle.$$

- For the first part of the question, you just apply the definition on an inner product. We then check that this application defines a norm.
  It is obviously positive and definite for the same reason as before.
  For the scalability, we consider $\lambda \in \mathbb{R}$ and evaluate $\|\lambda A\|$:

$$\|\lambda A\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda^2 a_{ij}^2\right)^{\frac{1}{2}} = |\lambda| \left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2\right)^{\frac{1}{2}} = |\lambda|\|A\|_F.$$

Finally, the triangle inequality :

$$\begin{array}{rcl}
\|A+B\|_F^2 & = & Tr((A+B)^T(A+B)), \\
& = & Tr(A^TA + A^TB + B^TA + B^TB), \\
& = & Tr(A^TA) + Tr(A^TB) + Tr(B^TA) + Tr(B^TB), \\
& = & \|A\|_F^2 + \|B\|_F^2 + 2Tr(A^TB), \\
& \leq & \|A\|_F^2 + \|B\|_F^2 + 2\|A\|_F\|B\|_F, \\
& \leq & (\|A\|_F + \|B\|_F)^2 .
\end{array}$$

We conclude by taking the square root.

- We will simply apply the definition of norm to the vector Ax where $(Ax)_k = \sum_{j=1}^m x_j a_{kj}$. We then have :

$$\begin{array}{rcl}
\|Ax\|_2^2 & = & \displaystyle\sum_{k=1}^n \left(\sum_{j=1}^m x_j a_{kj}\right)^2, \\
& \leq & \displaystyle\sum_{j=1}^m x_j^2 \sum_{k=1}^n (a_{kj})^2, \\
& \leq & \displaystyle\left(\sum_{j=1}^m x_j^2\right)\left(\sum_{j=1}^m\sum_{k=1}^n a_{kj}^2\right), \\
& = & \|x\|_2^2\|A\|_F^2.
\end{array}$$

The first inequality comes from the follwing one : $a^2+b^2 > 2ab$. We also conclude by tacking the square root on both sides.

- We will use the definition of norm and apply the Cauchy-Scwharz inequality :

$$\begin{array}{rcl}
\|AB\|_F^2 & = & \displaystyle\sum_{i=1}^n\sum_{j=1}^p \left(\sum_{k=1}^m a_{ik}b_{kj}\right)^2, \\
& \leq & \displaystyle\sum_{i=1}^n\sum_{j=1}^p \left(\sum_{k=1}^m a_{ik}^2 \sum_{k=1}^m b_{kj}^2\right), \\
& \leq & \displaystyle\left(\sum_{i=1}^n\sum_{k=1}^m a_{ik}^2\right)\left(\sum_{j=1}^p\sum_{k=1}^m b_{kj}^2\right), \\
& \leq & \|A\|_F^2\|B\|_F^2.
\end{array}$$

- $$\|A\|_F = 2\sqrt{5} \quad and \quad \|A\|_F = \sqrt{53}.$$

Exercise 3 :

1) Let $0 < p, q < \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$

- This equality holds because $\ln(x^p) = p\ln(x)$ and $\ln(xy) = \ln(x) + \ln(y)$.
- We apply the exponential function to the previous equality and use the definition of convexity :

$$xy = \exp\left(\frac{\ln(x^p)}{p} + \frac{ln(y^q)}{q}\right) \leq \frac{\exp(\ln(x^p))}{p} + \frac{\exp(\ln(y^q))}{q} = \frac{x^p}{p} + \frac{y^q}{q}.$$

2) We want to proove now that : $\|xy\|_1 \leq \|x\|_p\|y\|_q$ (*Hölder's inquality*)

We consider $0 < p, q < \infty$ such that $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$

- We first apply *Young's Inequality* using $p = p'r$ and $q = q'r$ so that $1 = \dfrac{1}{p'} + \dfrac{1}{q'}$, we have :

$$|ab| \leq \frac{1}{p'}|a|^{p'} + \frac{1}{q'}|b|^{q'},$$

  where a and b are real numbers. We then set $a = x_i^r$ and $b = y_i^r$ to find the inequality.

- In the previous inequality we begin by using the hint and consider the special case where $r = 1$:

$$\frac{|x_i y_i|}{\|y\|_q^q\|x\|_p^p} \leq \frac{1}{p}\frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q}\frac{|y_i|^q}{\|y\|_q^q}.$$

  We then sum over the index i and we have :

$$\frac{\|xy\|_1}{\|y\|_q^q\|x\|_p^p} \leq \frac{\|x\|_p^p}{\|x\|_p^p} + \frac{1}{q}\frac{\|y\|_q^q}{\|y\|_q^q} \leq 1.$$

  Multiplying on both sides by $\|y\|_q^q\|x\|_p^p$ leads to the result.

3) The triangle inequality for the L^p norm.

- We begin by writing the left hand side, the triangle inequality and *Hölder's Inequality* with $\dfrac{1}{q} = \dfrac{p-1}{p}$

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{i=1}^n |x_i + y_i|^p, \\
&\leq \sum_{i=1}^n (|x_i| + |y_i|)|x_i + y_i|^{p-1}, \\
&\leq \sum_{i=1}^n |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i||x_i + y_i|^{p-1}, \\
&\leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^{(p-1)\times q}\right)^{\frac{1}{q}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^{(p-1)\times q}\right)^{\frac{1}{q}}, \\
&\leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^{p}\right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^{p}\right)^{\frac{p-1}{p}}, \\
&\leq \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |i_i|^p\right)^{\frac{1}{p}}\right] + \left(\sum_{i=1}^n |x_i + y_i|^{p}\right)^{\frac{p-1}{p}} \\
&\leq \left(\|x\|_p + \|y\|_p\right)\|x+y\|_p^{p-1}.
\end{aligned}$$

- The hardest point is to proove that the triangle inequality holds, but it has been done in the previous question.

Derivatives

Exercise 1 : Calculous

- $$\nabla f(x,y) \begin{pmatrix} 8x + y\exp(xy) & x\exp(xy) \end{pmatrix}.$$
$$Hf(x,y) \begin{pmatrix} 8 + y^2\exp(xy) & \exp(xy)(1+y) \\ \exp(xy)(1+y) & x^2\exp(xy) \end{pmatrix}$$
- $$\nabla f(x,y) \begin{pmatrix} 7y - \sin(x) + 2x & 7x + 8y \end{pmatrix}.$$
$$Hf(x,y) \begin{pmatrix} 2 - \cos(x) & 7 \\ 7 & 8 \end{pmatrix}$$
- $$\nabla f(x,y) \begin{pmatrix} 20x^3 - 20xy + 8x - 8y & -10x^2 + 18y - 8x \end{pmatrix}.$$
$$Hf(x,y) \begin{pmatrix} 60x^2 - 20y + 8 & -20x - 8 \\ 20x - 8 & 18 \end{pmatrix}$$
- $$\nabla f(x,y) \begin{pmatrix} 2x\exp(x^2+y^2) & 2y\exp(x^2+y^2) \end{pmatrix}.$$
$$Hf(x,y) \begin{pmatrix} \exp(x^2+y^2)(2+4x^2) & 4xy\exp(x^2+y^2) \\ 4xy\exp(x^2+y^2) & \exp(x^2+y^2)(2+4y^2) \end{pmatrix}$$

Exercise 2 : Schwarz Theorem : a counter example

- The two diffenretial are equal to :
$$\frac{\partial f}{\partial x}(x,y) = \frac{[x^2+y^2][y(x^2-y^2)+2x^2y] - 2x(xy)(x^2-y^2)}{(x^2+y^2)^2},$$
and
$$\frac{\partial f}{\partial y}(x,y) = \frac{[x^2+y^2][x(x^2-y^2)-2xy^2] - 2y(xy)(x^2-y^2)}{(x^2+y^2)^2}/$$
Evaluated at the given point, we have : $\frac{\partial f}{\partial x}(0,y) = -y$ and $\frac{\partial f}{\partial y}(x,0) = x$.
- According to the previous section : $\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0,0) = -1$ and $\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(0,0) = 1$
- The function f is not twice countinuously differentiable at at the origin, that is the reason why the theorem does not hold.

Convex set

Exercise 1 : Using definition

- To proove that C is convex, we need to show that for any point $x, y \in C$ and for all $t \in [0,1]$, the point $z(t)$ defined by $z(t) = tx + (1-t)y \in C$.
If x, y belong to C they belong in both C_1 and C_2, then using the convexity of C_1 and C_2, we can say that $z(t)$ belongs to C_1 and C_2 so it belongs to the intersection.

- 1) Suppose that C is covnex and denote by D any straight line. Then the intersection $I = C \cap D$ is convex using the previous question and the fact that a stragiht line is convex.
  2) Conversely suppose that the intersection $I = C \cap D$ is convex. So for all points x, y and all $t \in [0, 1]$ the point $z(t)$ defined by $z(t) = tx + (1-t)y \in I$. It means that $z(t)$ belongs to both C and D, so it belongs to C. We can conclude that C is convex.
- We have seen in class the case when $k = 2$ (i.e. taking the combination of two points). Let us now consider that the definition holds for a given $k = n - 1$ and let us show the definition holds for $k = n$. We consider any n points $x_1, \ldots, x_n \in C$ and the set of weights $\{\theta_1, \ldots, \theta_n \mid \sum_{i=1}^{n} \theta_i = 1\}$. We have to show that $y = \sum_{i=1}^{n} \theta_i x_i \in C$.
  Without loss of generality we can consider that $\theta_n \neq 1$. The point y can be rewritten as follow :
  $$y = \theta_n x_n + (1 - \theta_n)(\lambda_1 x_1 + \ldots + \lambda_{n-1} x_{n-1}),$$
  with $\lambda_i = \dfrac{\theta_i}{1 - \theta_n}$. We have $\sum_{i=1}^{n-1} \lambda_i = 1$ so the point $\lambda_1 x_1 + \ldots + \lambda_{n-1} x_{n-1}$ belongs to C using the fact that the definition holds for $k = n - 1$. And finally the point y belongs to C using the case $k = 2$.

Exercise 2 :

- We solve this exercise with different methods :
  1) Let $E(f)$ be the epigraph of the function f defined by $y = f(x) = 3x^2 - 6x + 2$. Then $E(f)$ is convex because f is convex. So the set C can be seen as the intersection of the line $y = 0$ with $E(f)$. These two sets are convex so C is convex.
  2) Solve the inequation $3x^2 - 6x + 2 \leq 0$ and the result is the segment $\left[1 - \dfrac{1}{\sqrt{3}}; 1 + \dfrac{1}{1 + \sqrt{3}}\right]$.
- We have previously shown that a set is convex if and only if its intersection with any line segment is convex. So let us set $x = u + tv$ where $t \in [0, 1]$ and $u, v \in \mathbb{R}^n$, we have :
  $$\begin{aligned} x^T Ax - b^T x + c &= (u + tv)^T A(u + tv) - b^T (u + tv) + c, \\ &= (v^T Av)t^2 + (2u^T Av - b^T v)t + (c - b^T u + u^T Au), \\ &= \alpha t^2 + \beta t + \gamma. \end{aligned}$$
  The set $\{t \in \mathbb{R} \mid \alpha t^2 + \beta t + \gamma \leq 0\}$ is convex if $\alpha \geq 0$ (using argument 1, as in the previous question). So this set is convex because A is PSD.

Exercise 3 :

- We first begin by prooving the hint using the convexity of the exponential function. For all $\theta \in [0, 1]$ and for all $x, y \in \mathbb{R}_{++}$ we have :
  $$x^\theta y^{1-\theta} = \exp(\theta \ln(x) + (1 - \theta) \ln(y)) \leq \theta \exp(\ln(x)) + (1 - \theta) \exp(\ln(y)) = \theta x + (1 - \theta)y.$$
  We now have to show that $\prod_{i=1}^{n} (\theta x_i + (1 - \theta)y_i) \geq 1$. The result is a consequence of the hint :
  $$\prod_{i=1}^{n} (\theta x_i + (1 - \theta)y_i) \geq \prod_{i=1}^{n} x_i^\theta y_i^{1-\theta} = \left(\prod_{i=1}^{n} x_i\right)^\theta \left(\prod_{i=1}^{n} y_i\right)^{1-\theta} \geq 1.$$

Convex function

Exercise 1 : Calculous

- It is convex as the sum of quadratic and convex functions.
- It is convex as the sum of quadratic and linear convex functions.
- It is convex as the sum of convex functions (indeed, for all $n \in \mathbb{N}$, x^{2n} is convex).
- You can compute the Trace and the determinant of the Hessian Matrix. These lasts are respectively equal to 22 and 84. So the function f is convex.
- We compute the Hessian Matrix H :

$$Hf(x,y)\begin{pmatrix} y^2\exp(xy) & (1+xy)\exp(xy) \\ (1+xy)\exp(xy) & x^2\exp(xy) \end{pmatrix}.$$

  The Trace is equal to $x^2+y^2>0$ and the Determinant is equal to $-1-2xy>0$ because $y<-1$ and $x>1$ so $2xy>2$.

Exercise 2 : Calculous

- Yes, it is sum of two convex terms, the first one is a quadratic convex term. The second term is convex as the composition of a convex function (linear and quadratic) with an increasing and convex function ($f(x)=x^2$).
- This function is also convex as the sum of two convex quadratic terms.
- This function is not convex because of the term $-1.05x^4$.
- This function is not convex because the function g defined by $g(x)=\sin(x)$ is not convex.
- Same reason as before with the function $\cos$.

Exercise 3 : A PSD matrix

- The aim is to show that the eigenvalues of this matrix are non-negative. Let us consider (λ,u) the couple eigenvalue, eigenvector of the matrix G such that $Gu=\lambda u$.
  We now want to show that $\lambda \geq 0$. The previous equation can be rewritten as follow :

$$Gu = XX^Tu = \lambda u.$$

  We then take the inner product with u on both sides :

$$u^TXX^Tu = (X^Tu)^T(X^Tu) = \|X^Tu\|^2 = \lambda\|u\|^2.$$

  This last equality, combining with the positiveness of the norm imply the non-negativity of the eigenvalue λ.

Optimization and Algorithm

Exercise 1 : A Quadratic function : Matyas function

You can notice that f is a quadratic function. In fact, f can be written as : $f(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle$ where A is the matrix $\begin{pmatrix} 0.52 & -0.48 \\ -0.48 & 0.52 \end{pmatrix}$ and b the vector : $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. It will be usefull in this exercise when it comes to calculate the first iteration of the algorithm.

1. The function f is convex. In fact, the Jacobian and Hessian matrices are given by :

$$J_{f(x,y)} = \begin{pmatrix} 0.52x - 0.48y \\ -0.48x + 0.52y \end{pmatrix} \quad H_{f(x,y)} = \begin{pmatrix} 0.52 & -0.48 \\ -0.48 & 0.52 \end{pmatrix}$$

   The trace of the Hessian Matrix is equal to 1.04 and the determinant is equal to $0.52^2 - 0.48^2$. Both of them are non-negative. So the Hessian Matrix is PSD so the function f is convex.

2. We have to solve the following linear system :

$$\begin{aligned} 0.52x - 0.48y &= 0, \\ -0.48x + 0.52y &= 0. \end{aligned}$$

   A solution of this system is given by $(x, y) = (0, 0)$.

3. Due to *Euler's Equation* and because of the convexity of the function. The function f reaches its minimum at $(x, y) = (0, 0)$ and the minimum is equal to 0.

4. We set $u_0 = (x, y)^{(0)} = (1, 1)$, the initial point of the gradient descent with the optimal learning rate (or optimal step)

   (a) • First we have to choose an initial point u_0 which is given here.
   • For $k = 0, 1, \dots$ we estimate $\nabla f(u_k)$
   • We solve the problem $\rho_k = \underset{\rho}{Argmin}\ f(u_k - \rho\nabla f(u_k))$. Inother words, we are looking for the optimal step.
   • We then set $u_{k+1} = u_k - \rho\nabla f(u_k)$.
   • Till $\|\nabla f(u_{k+1})\| \leq \varepsilon$.

   (b) We have to apply the previous process the estimate u_1 and u_2.
   • We have $\nabla f(u_0) = \begin{pmatrix} 0.04 \\ 0.04 \end{pmatrix}$. Acording to what we have seen during the course about quadratic function, the optimal step is given by :

$$\rho_0 = \frac{\|Au_0 - b\|^2}{\|Au_0 - b\|_A^2}.$$

   We have $Au_0 = \begin{pmatrix} 0.04 \\ 0.04 \end{pmatrix}$ (this is no more than the gradient of f) and $A(Au_0) = \begin{pmatrix} 0.04^2 \\ 0.04^2 \end{pmatrix}$. Then $\rho_0 = \dfrac{2 \times 0.04^2}{2 \times 0.04^3} = \dfrac{1}{0.04} = 25$.
   So $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 25 \begin{pmatrix} 0.04 \\ 0.04 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We have reached the global minimum
   • We have already reached the global minimum so $u_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Exercise 2 : The Rosenbrock function

- The function f is not convex. Indeed, the Jacobian and Hessian matrices are given by :

$$J_{f(x,y)} = \begin{pmatrix} 40x^3 - 40xy + 2x - 2 \\ -20x^2 + 20y \end{pmatrix} \quad H_{f(x,y)} = \begin{pmatrix} 120x^2 - 40y + 2 & -40x \\ -40x & 20 \end{pmatrix}$$

  The Trace of the Hessian matrix is negative for $x = 0$ and $y = 1$.
- By solving the linear system $\nabla f(x, y) = 0$, you will find one solution that is the point $(1, 1)$.
- The function f is non-negative (as the sum of two non-negative terms). Moreover, $f(1, 1) = 0$. So the global minimum of f is zero and it reaches its minimum at the point $(1, 1)$.
- First we have to choose an initial point u_0 which is given here.
  • For $k = 0, 1, \ldots$ we estimate $\nabla f(u_k)$
  • We then set $u_{k+1} = u_k - \rho \nabla f(u_k)$ where $\rho = 0.5$
  • Till $\|\nabla f(u_{k+1})\| \leq \varepsilon$.
- We have to apply the previous process the estimate u_1 and u_2.
  We have $\nabla f(u_0) = \begin{pmatrix} 162 \\ 60 \end{pmatrix}$. Because $\rho = 0.5$ we can immediatly set $u_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - 0.5 \begin{pmatrix} 162 \\ 60 \end{pmatrix} = \begin{pmatrix} -79 \\ -28 \end{pmatrix}$. You do the same thing to compute u_2. But the values are here to big, so do not do this. The learning is not well chosen for this function.

Exercise 3 : The Rastrigin function

1. The function f is not convex. In fact, the Jacobian and Hessian matrices are given by :

$$J_{f(x,y)} = \begin{pmatrix} 2x + 20\pi \sin(2\pi x) \\ 2y + 20\pi \sin(2\pi y) \end{pmatrix} \quad H_{f(x,y)} = \begin{pmatrix} 2 + 40\pi^2 \cos(2\pi x) & 0 \\ 0 & 2 + 40\pi^2 \cos(2\pi y) \end{pmatrix}$$

   The trace of the Hessian Matrix is equal to $4(1 + 20\pi^2(\cos(2\pi x) + \cos(2\pi y)))$. If we set $x = y = \frac{1}{2}$ then the trace is equal to $4(1 + 20\pi^2(-1 + (-1))) = 1 - 160\pi^2 < 0$
2. We know that $sin(0) = 0$, so a trivial solution of *Euler's Equation* is given by the couple $(x, y) = (0, 0)$.
3. We assume that the function f is non-negative, i.e. $f(x, y) \geq 0$. However, if we evaluate the function f at the point $(0, 0)$ we have $f(0, 0) = 0$. So 0 is the global minimum of f and this minimum is reached at $(0, 0)$.
4. We set $u_0 = (x, y)^{(0)} = (2, 2)$, the initial point of the gradient descent with a learning rate $\rho = 0.5$.
   (a) • First we have to choose an initial point u_0 which is given here.
   • For $k = 0, 1, \ldots$ we estimate $\nabla f(u_k)$
   • We then set $u_{k+1} = u_k - \rho \nabla f(u_k)$ where $\rho = 0.5$
   • Till $\|\nabla f(u_{k+1})\| \leq \varepsilon$.
   (b) We have to apply the previous process the estimate u_1 and u_2.
   • We have $\nabla f(u_0) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$. Because $\rho = 0.5$ we can immediatly set $u_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - 0.5 \begin{pmatrix} 4 \\ 4 \end{pmatrix} =$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We have reached the global minimum

- We have already reached the global minimum so $u_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

5. Because the function is not convex, given a random starting point u_0, we are not sure that the algorithm will reach the global solution.
6. Do the same question with $u_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\rho = 0.25$.

Exercise 4 : A quadratic function

1. The function f is convex. In fact, the Jacobian and Hessian matrices are given by :

$$J_{f(x,y)} = \begin{pmatrix} 8x - 5y + 2 \\ 14y - 5x - 7 \end{pmatrix} \quad H_{f(x,y)} = \begin{pmatrix} 8 & -5 \\ -5 & 14 \end{pmatrix}$$

The trace of the Hessian Matrix is equal to 22 and the determinant is equal to $8 \times 14 - 25 = 87$. Both of them are non-negative. So the Hessian Matrix is PSD so the function f is convex (f is strictly convex because the two quantities are both positive).

2. We have to solve the following linear system :

$$\begin{aligned} 8x - 5y + 2 &= 0, \\ 14y - 5x - 7 &= 0. \end{aligned}$$

The solution is given by $(x, y) = (\frac{7}{87}, \frac{46}{87})$.

3. Due to *Euler's Equation* and because of the convexity of the function. The function f reaches its minimum at $(x, y) = (\frac{7}{87}, \frac{46}{87})$ and the minimum is equal to 2308.
4. We set $u_0 = (x, y)^{(0)} = (1, 1)$, the initial point of the Newton's Method.
   (a) - First we have to choose an initial point u_0 which is given here.
       - For $k = 0, 1, \ldots$ we estimate $\nabla f(u_k)$.
       - We also calculate $\nabla^2 f(u_k)$ and we invert it.
       - We then set $u_{k+1} = u_k - (\nabla^2 f(u_k))^{-1} \nabla f(u_k)$.
       - Till $\|\nabla f(u_{k+1})\| \leq \varepsilon$.

   (b) We have to apply the previous process the estimate u_1 and u_2.
       - We have $\nabla f(u_0) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\nabla^2 f(u_0) = \begin{pmatrix} 8 & -5 \\ -5 & 14 \end{pmatrix}$. Notice that the Hessian matrix doesn't depend on x and y so it will be the same at each step of the algorithm.

       So $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{87} \begin{pmatrix} 14 & 5 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{7}{87} \\ \frac{46}{87} \end{pmatrix}$. We have reached the global minimum

       - We have already reached the global minimum so $u_2 = \begin{pmatrix} \frac{7}{87} \\ \frac{46}{87} \end{pmatrix}$

Exercise 5 : A last function

- The Hessian matrix is given by :

$$H_{f(x,y)} = \begin{pmatrix} 5x^4 - 12.6x^2 + 4 & 1 \\ 1 & 2 \end{pmatrix}$$

- To check if a 2×2 matrix is PSD, we have to compute its eigenvalues and show that these lasts are non-negative. Which is equivalent, in this particular case, to show that both Trace and determinant are non-negative.
  The Trace is equal to $5x^4 - 12.6x^2 + 6$ and the determinant is equal to $10x^4 - 25.2x^2 + 6$. These polynoms admit two roots in x^2 so both trace and determinant can take negative values. So the function f is not convex.
- We first compute the Jacobian of the function f :

$$J_{f(x,y)} = \begin{pmatrix} x^5 - 4.2x^3 + 4x + y & x + 2y \end{pmatrix}$$

  The results is immediate by replacing x and y by 0.
  You can show that the function f is non-negative by reformulating this last.
- Because f is non-negative and reaches the value 0 at the point $(0,0)$, its global minimum is reached at $(0,0)$ with 0 as minimum.
- Same as in the other exercises :
  (a) • First we have to choose an initial point u_0 which is given here.
  • For $k = 0, 1, \dots$ we estimate $\nabla f(u_k)$.
  • We then set $u_{k+1} = u_k - \rho \nabla f(u_k)$.
  • Till $\|\nabla f(u_{k+1})\| \leq \varepsilon$.
  (b)
- We have $\nabla f(u_0) = \begin{pmatrix} 1.8 \\ 3 \end{pmatrix}$. Because $\rho = 0.5$ we can immediatly set $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 0.5 \begin{pmatrix} 1.8 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.5 \end{pmatrix}$.
  We do the same to compute u_2.
  We have $\nabla f(u_1) = \begin{pmatrix} 10^{-5} - 4.2 \times 10^{-3} + 0.9 \simeq 0.9 \\ 1.1 \end{pmatrix}$. Because $\rho = 0.5$ we can immediatly set $u_2 = \begin{pmatrix} 0.1 \\ 0.5 \end{pmatrix} - 0.5 \begin{pmatrix} 0.9 \\ 1.1 \end{pmatrix} = \begin{pmatrix} -0.35 \\ -0.05 \end{pmatrix}$. The learning rate seems to be approrpriate. The norm of the gradient is decreasing at each iteration. But we can not be sure that the algorithm will converge (without theoretical analysis).

Exercise 6 : An Application of Newton's Method : The Logistic Regression

- We will simply the definition of the model :

$$\ln\left(\frac{p(1 \mid X)}{p(0 \mid X)}\right) = w_0 + w_1 x_1 + w_2 x_2 + \dots + w_d x_d.$$

We use the fact that $p(0 \mid X) = 1 - p(1 \mid X)$ and apply the exponential function to have :

$$\frac{p(1 \mid X)}{1 - p(1 \mid X)} = \exp(w_0 + w_1 x_1 + w_2 x_2 + ... + w_d x_d).$$

Finally, we have :

$$g(w) = p(1 \mid X) = \frac{1}{1 + \exp(-w_0 - w_1 x_1 + ... - w_d x_d)} = \frac{1}{1 + \exp\left(\sum_{i=0}^{d} w_i x_i\right)},$$

where $x_0 = 1$.

- The function g is continuously differentiable and for all $w \in \mathbb{R}$ we have :

$$\begin{aligned}
\nabla_w g(w) &= \frac{\exp(-w)}{(1 + \exp(-w))^2}, \\
&= \frac{1}{1 + \exp(-w)} \frac{-\exp(-w)}{1 + \exp(-w)}, \\
&= \frac{1}{1 + \exp(-w)} \left(1 - \frac{1}{1 + \exp(-w)}\right), \\
&= g(w)(1 - g(w)).
\end{aligned}$$

- The gradient of the function g is also continuously differentiable, because g is. For all $w \in \mathbb{R}$:

$$\begin{aligned}
\nabla_w^2 g(w) &= \nabla_w g(w)(1 - g(w)) - \nabla_w g(w) g(w), \\
&= (1 - g(w))^2 g(w) - g(w)^2 (1 - g(w)), \\
&= g(w)(1 - g(w))(1 - 2g(w)).
\end{aligned}$$

  So the function g is convex if $g(w) < 0.5$, i.e. if $w \leq 0$. If $w \geq 0$ the function g is concave.

- Because the function g is positive and its first order derivative too, the function $\ln(g(x))$ is twice continuously differentiable for all $w \in \mathbb{R}$ and we have :

$$\nabla_w \ln(g(w)) = \frac{\nabla_w g(w)}{g(w)} = 1 - g(w).$$

  The second order derivative is then given by :

$$\nabla_w^2 \ln(g(w)) = -\nabla_w g(w) = -g(w)(1 - g(w)).$$

  Recall that, for all $w \in \mathbb{R},\ 0 < g(w) < 1$. So the function $\ln(g)$ is concave.

In the following questions, for all indexes i, $x_i \in \mathbb{R}^d + 1$, where $x_{i,0} = 1$ for all $i = 1, ..., n$.

- The likelihood of the data is defined by :

$$L \prod_{i=1}^{n} P(y_i \mid x_i, w) = \prod_{i=1}^{n} g(w, x_i)^{y_i} (1 - g(w, x_i))^{1 - y_i}.$$

  Then the likelihood is given by :

$$\ell = \ln(L) = \sum_{i=1}^{n} y_i \ln(g(w, x_i)) + (1 - y_i) \ln(1 - g(w, x_i)).$$

  The expression $-\ell$ is then trivial.

- We will also compute the first and second order derivative of $-\ell$:

$$-\nabla_w \ell = -\sum_{i=1}^{n} y_i(1-g(w,x_i)) - (1-y_i)g(w,x_i) = -\sum_{i=1}^{n} (y_i - g(w,x_i))\, x_i.$$

  And the second order derivative is given by :

$$-\nabla_w^2 \ell = \sum_{i=1}^{n} g(w,x_i)(1-g(w,x_i))x_i x_i^T.$$

  You can show that $-\nabla_w^2 \ell$ is PSD using the exercise 3 of the Section **Convex Functions**.
- For this last question, let us denote $w^{(k)}$ the k$-th$ iterate of the studied algorithm. We then have :

$$w^{(k+1)} = w^{(k)} - \left(-\nabla_w^2 \ell\right)^{-1}(-\nabla_w \ell).$$

Exercise 7 : The Backtracking Line Search

- According to Taylor's Theorem, for all $x, y \in \mathbb{R}^n$ we have :

$$f(y) = f(x) + \langle \nabla f(x), y - x\rangle + R_2(y),$$

  where $R_2(y) = \mathcal{O}(y^2)$,i.e. the function R_2/y^2 tends to zero when y tends to x.
  A particular result in Analysis (The Taylor-Cauchy Formula) claims that, if the function f is twice differentiable (which is the case here), Then, it exists $z \in \mathbb{R}^n$ such that $x \leq z \leq y$ (or $y \leq z \leq x$) and :

$$R_2(y) = \frac{1}{2}\langle \nabla^2 f(y)(y-x), (y-z)\rangle.$$

  The first inequality can be rewritten in the following form :

$$f(y) = f(x) + \langle \nabla f(x), y-x\rangle + \frac{1}{2}\langle \nabla^2 f(y)(y-x), (y-z)\rangle.$$

  Note that for all z previously defined we have $\langle (y-x), (y-z)\rangle \leq \|x-y\|_2^2$ and using the strong convexity of f we have $\|\nabla^2 f(y)\|_2 \leq M$.
  These two upper bounds lead us to the result, you simly have to set $y = x + \rho d_x$.
- According to the previous question, using the existence of a constant $M > 0$ such that : $\nabla^2 f(x) \leq M\mathbf{I}$, we have :

$$f(x+\rho d_x) \leq f(x) + \rho\langle \nabla f(x), d_x\rangle + \frac{M}{2}\rho^2\langle d_x, d_x\rangle.$$

  Moreover, $f(x+\rho d_x) \leq f(x) + \alpha\rho\langle \nabla f(x), d_x\rangle$ if :

$$\rho(1-\alpha)\langle \nabla f(x), d_x\rangle + \frac{M}{2}\rho^2\langle d_x, d_x\rangle \leq 0.$$

  So the exit condition holds for all $0 < \rho \leq \rho_0$ such that :

$$\rho_0 = -2(1-\alpha)\frac{\langle \nabla f(x), d_x\rangle}{M\langle d_x, d_x\rangle} \geq -\frac{\langle \nabla f(x), d_x\rangle}{M\langle d_x, d_x\rangle}.$$

  Th equality comes from the previous inequality, we are simply looking for a value of ρ for which the inequality holds. The inequality uses the fact that $\alpha \in [0, 0.5]$. If you further assume that $\rho \leq 1$, then $\beta^k \rho \leq \beta^k < \rho_0$ if $k > \dfrac{\ln(1/\rho_0)}{\ln(1/\beta)}$.

Exercise 8 : Minimizing a quadratic-linear fractional function :

- A minimizer $x^\star$ of the function f is given by $\nabla f(x^\star) = 0$.

$$\begin{aligned}
\nabla f(x) &= \frac{2}{c^T x - d} A^T(Ax - b) - \frac{\|Ax - b\|_2^2}{c^T x - d} c, \\
&= \frac{1}{c^T x - d}\left(2A^T(Ax - b) - \frac{\|Ax - b\|_2^2}{c^T x - d} c\right), \\
&= \frac{1}{c^T x - d}\left(2A^T Ax - 2A^T b - \frac{\|Ax - b\|_2^2}{c^T x - d} c\right), \\
&= \frac{1}{c^T x - d}\left(2A^T Ax - 2A^T b - 2tc\right).
\end{aligned}$$

  Then, if $x = x^\star$, we have :

$$\begin{aligned}
\nabla f(x^\star) &= 0, \\
A^T Ax - A^T b - tc &= 0, \\
x^\star &= (A^T A)^{-1} A^T b + t(A^T A)^{-1} c.
\end{aligned}$$

- According to the definition of t, it should satisfy :

$$t = \frac{\|A(x_1 + tx_2) - b\|_2^2}{2(c^T(x_1 + tx_2) - d)}.$$

  Thus, the value of t must satisfy :

$$\begin{aligned}
2t^2 c^T x_2 + 2t(c^T x_1 - d) &= t^2\|Ax_2\|_2^2 + 2t(Ax_1 - b)^T Ax_2 + \|Ax_1 - b\|_2^2, \\
&= t^2 c^T x_2 + +\|Ax_1 - b\|_2^2,
\end{aligned}$$

  you can easily check that $(Ax_1 - b)^T Ax_2 = 0$ and $\|Ax_2\|_2^2 = c^T x_2$. You simply have to develop each expression.
  Finally, t is the positive root of the following quadratic equation :

$$t^2 c^T x_2 + 2t(c^T x_1 - d) - \|Ax_1 - b\|_2^2 = 0.$$

  You can now try to find the positive root of this equation and proove that solution satisfies $c^T(x_1 + tx_2) - d > 0$

Exercise 9 : The optimal step algorithm : illustration of convergence

- The function f reaches its minimum at the point $(0, 0)$ and this minimum is equal to 0.
- Just have a quick look in your slides. I just remind you that the optimal learning rate is given :

$$\rho = \frac{\|Ax - b\|_2^2}{\|Ax - b\|_A^2},$$

  where $A = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$ and b is the nul vector.

We initialize the algorithm at the point $x^{(0)} = (x_1^{(0)}, x_2^{(0)}) = (\gamma, 1)$

- We need first to compute the optimal learning rate ρ_0. Its value is equal to $\dfrac{2}{\gamma+1}$. Then $x^{(1)}$ is the vector :

$$x^{(1)} = \left(\left(\frac{\gamma-1}{\gamma+1} \right) \gamma, - \left(\frac{\gamma-1}{\gamma+1} \right) \right).$$

We apply the same process to compute the value of $x^{(2)}$. The optimal learning rate is given by $\rho_1 = \dfrac{2}{\gamma+1}$.Then $x^{(2)}$ is the vector :

$$x^{(2)} = \left(\left(\frac{\gamma-1}{\gamma+1} \right)^2 \gamma, \left(\frac{\gamma-1}{\gamma+1} \right)^2 \right).$$

- You have to proove it by induction.

The Initialization has been made in the previous question.

We will assume it exists $k \in \mathbb{N}$ such that the relation (i.e. the expression of $x^{(k)}$) is true. We will show that it is still true for $k+1$.

We will first compute the optimal learning rate ρ_k :

$$\rho_k = \frac{\|Ax^{(k)}\|_2^2}{\|Ax^{(k)}\|_A^2}.$$

We will compute the two quantities, we begin with the numerator. We have :
$Ax^{(k)} = \left(\gamma \left(\dfrac{\gamma-1}{\gamma+1} \right)^k, (-1)^k \gamma \left(\dfrac{\gamma-1}{\gamma+1} \right)^k \right)$. So $\|Ax^{(k)}\|_2^2 = 2\gamma^2 \left(\dfrac{\gamma-1}{\gamma+1} \right)^{2k}$.
Moreover $A(Ax^{(k)}) = \left(\gamma \left(\dfrac{\gamma-1}{\gamma+1} \right)^k, (-1)^k \gamma^2 \left(\dfrac{\gamma-1}{\gamma+1} \right)^k \right)$. So $\|Ax^{(k)}\|_A^2 = \left(\dfrac{\gamma-1}{\gamma+1} \right)^{2k} (\gamma^2 + \gamma^3)$.
We can finally compute the learning rate. This one is given by :

$$\rho_k = \frac{2}{1+\gamma}.$$

The value of the gradient $f_\gamma(x^{(k)})$ is equal to $Ax^{(k)}$ and has been previously published. It remains to compute $x^{(k+1)}$.
The value of $x_1^{(k+1)}$ is given by :

$$\begin{aligned}
x_1^{(k+1)} &= x_1^{(k)} - \rho_k A x_1^{(k)}, \\
&= \gamma \left(\frac{\gamma-1}{\gamma+1} \right)^k - \frac{2}{\gamma+1} \left(\frac{\gamma-1}{\gamma+1} \right)^k \gamma, \\
&= \gamma \left(\frac{\gamma-1}{\gamma+1} \right)^k \left(1 - \frac{2}{\gamma+1} \right), \\
&= \gamma \left(\frac{\gamma-1}{\gamma+1} \right)^k \left(\frac{\gamma-1}{\gamma+1} \right), \\
&= \gamma \left(\frac{\gamma-1}{\gamma+1} \right)^{k+1}.
\end{aligned}$$

of $x_2^{(k+1)}$ is given by :

$$\begin{aligned}
x_2^{(k+1)} &= x_2^{(k)} - \rho_k A x_2^{(k)}, \\
&= (-1)^{(k)} \left(\frac{\gamma-1}{\gamma+1}\right)^k - \frac{2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^k \gamma(-1)^{(k)}, \\
&= (-1)^{(k)} \left(\frac{\gamma-1}{\gamma+1}\right)^k \left(1 - \frac{2\gamma}{\gamma+1}\right), \\
&= (-1)^{(k)} \left(\frac{\gamma-1}{\gamma+1}\right)^k \left(-\frac{\gamma-1}{\gamma+1}\right), \\
&= (-1)^{(k+1)} \left(\frac{\gamma-1}{\gamma+1}\right)^{k+1}.
\end{aligned}$$

We conclude by induction and say that expressions are true for $k \in \mathbb{N}$.

- Using the expresion of $x^{(k)}$ and the fact that $f_\gamma(x^{(0)}) = \frac{1}{2}\left(\gamma^2 + \gamma\right)$ we have :

$$f_\gamma(x^{(k)}) = \frac{1}{2}\left(\left(\frac{\gamma-1}{\gamma+1}\right)^{2k} \gamma^2 + \gamma\left(\frac{\gamma-1}{\gamma+1}\right)^{2k}\right) = f_\gamma(x^{(0)})\left(\frac{\gamma-1}{\gamma+1}\right)^{2k}.$$

- The previous expression directly shows that the function $\underset{k\to\infty}{lim} f_\gamma(x^{(k)}) = 0$. Indeed, $\left|\frac{\gamma-1}{\gamma+1}\right| < 1$.

Exercise 10 : Convergence analysis of the gradient descent with optimal step