Asymptotic Index Recovery in Simplex Numbers

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Abstract

This note is expository and presents known results in a unified framework. We examine asymptotic properties of k-simplex numbers arising from their recursive polynomial structure. The cumulative summation defining the k-simplex numbers produces separation of polynomial degree, yielding stabilising density limits under natural normalisations. These limits follow from dominance of the leading term. Inversion of the leading term provides asymptotically accurate index recovery, while a corrected estimator yields eventual exact inversion after applying the floor function.

1. Introduction

The k-simplex numbers form a classical family of figurate numbers defined recursively by cumulative summation. For fixed $k \geq 1$, the k-simplex numbers are given explicitly by

$$S_k(n) = \binom{n+k-1}{k}$$

Equivalently, they may be generated recursively via

$$S_k(n) = \sum_{i=1}^{n} S_{k-1}(i)$$

with $S_1(n) = n$.

It is well known that $S_k(n)$ is a polynomial in n of degree k with leading coefficient $1/k!$. The purpose of this note is to collect standard consequences of this polynomial form and present them together, showing how the leading term determines density and inversion behaviour.

2. Polynomial Structure and Degree Separation

For each fixed k, the simplex number $S_k(n)$ admits a polynomial expansion

$$S_k(n) = \frac{n^k}{k!} + \frac{k(k-1)}{2k!} n^{k-1} + \text{lower-degree terms}$$

The recursive definition shows that each increase in dimension corresponds to summation of a degree-$k-1$ polynomial, yielding a degree-k polynomial. Since summation increases polynomial degree by one, $S_k(n)$ has degree k with leading coefficient $1/k!$.

Observation 2.1 (Degree dominance)

For fixed k,

$$S_k(n) = \frac{n^k}{k!} + O(n^{k-1})$$

Hence,

$$\lim_{n\to\infty} \frac{S_k(n)}{n^k} = \frac{1}{k!}$$

This follows from the fact that for polynomials, the highest-degree term determines growth.

3. Density Under Hypercubic Normalisation

Consider embedding $S_k(n)$ inside the k-dimensional hypercube of side length n. The total number of lattice points in the cube is n^k.

Define the density

$$\delta_k(n) = \frac{S_k(n)}{n^k}$$

Observation 3.1 (Hypercubic density limit)

$$\lim_{n\to\infty} \delta_k(n) = \frac{1}{k!}$$

This follows directly from the polynomial asymptotics above. Lower-degree boundary contributions vanish after normalisation by n^k.

Second-Order Density Correction

Using the explicit expansion in Section 2,

$$S_k(n) = \frac{n^k}{k!} + \frac{k(k-1)}{2k!} n^{k-1} + O(n^{k-2})$$

and dividing by n^k, we obtain

$$\frac{S_k(n)}{n^k} = \frac{1}{k!} + \frac{k(k-1)}{2k!}\frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

Equivalently,

$$\frac{S_k(n)}{n^k} = \frac{1}{k!} + \frac{k-1}{2(k-1)!}\,\frac{1}{n} + O(n^{-2})$$

Thus the ratio approaches 1/k! at rate 1/n, with the next term coming from the second coefficient of the polynomial.

4. Density Under Iterated Stacking Normalisation

Alternatively, consider normalisation relative to the previous simplex layer. Using the recursive definition,

$$S_k(n) = \sum_{i=1}^{n} S_{k-1}\,(i)$$

Define

$$\rho_k(n) = \frac{S_k(n)}{n \cdot S_{k-1}(n)}$$

Using leading-term asymptotics,

$$S_{k-1}(n) = \frac{n^{k-1}}{(k-1)!} + O(n^{k-2})$$

so

$$n \cdot S_{k-1}(n) = \frac{n^k}{(k-1)!} + O(n^{k-1})$$

Observation 4.1 (Stacking density limit)

$$\lim_{n\to\infty} \rho_k(n) = \frac{1}{k}$$

Indeed,

$$\frac{S_k(n)}{nS_{k-1}(n)} = \frac{\frac{n^k}{k!} + O(n^{k-1})}{\frac{n^k}{(k-1)!} + O(n^{k-1})} \longrightarrow \frac{1/k!}{1/(k-1)!} = \frac{1}{k}$$

5. Asymptotic Inversion

Since

$$S_k(n) = \frac{n^k}{k!} + O(n^{k-1})$$

inversion of the leading term yields

$$n \sim (k!\ S_k(n))^{1/k}$$

Observation 5.1 (Asymptotic index recovery)

If $m = S_k(n)$, then

$$n = (k!\,m)^{1/k} + O(1)$$

The error comes from lower-degree terms and stays bounded as m → ∞

Explicit Bounded Inversion Error

Using the more precise expansion

$$S_k(n) = \frac{n^k}{k!} + \frac{k(k-1)}{2k!} n^{k-1} + O(n^{k-2})$$

and solving for n via binomial expansion, one obtains the refined estimate

$$n = (k!\,m)^{1/k} - \frac{k-1}{2} + O\left(\frac{1}{n}\right)$$

In particular, there exists a constant C_k depending only on k such that

$$|\ n - (k!\,m)^{\frac{1}{k}}\ | \leq\ C_k \text{ for all } n$$

Thus the inversion error is uniformly bounded for fixed k, and the leading-term estimator determines the index up to an additive constant independent of n.

Observation 5.2 (Eventual Exact Inversion)

For each fixed $k \geq 1$, there exists N_k such that for all $n \geq N_k$,

$$n = \left\lfloor (k!\ S_k(n))^{1/k} - \frac{k-1}{2} \right\rceil$$

In particular, the corrected leading-term estimator

$$(k!\,m)^{1/k} - \frac{k-1}{2}$$

recovers the exact index after application of the floor function for all sufficiently large simplex numbers.

Justification

From the refined expansion,

$$(k!\ S_k(n))^{1/k} = n + \frac{k-1}{2} + O\left(\frac{1}{n}\right)$$

so

$$(k!\ S_k(n))^{1/k} - \frac{k-1}{2} = n + O\left(\frac{1}{n}\right)$$

Since the error term tends to zero as $n \to \infty$, there exists N_k such that the absolute value of the error is less than $1/2$ for all $n \geq N_k$. The stated identity then follows by taking floors.

6. Interpretation

Since $S_k(n)$ is a degree-k polynomial obtained by summation of a degree-$k-1$ polynomial, standard polynomial estimates give the density limits and inversion results above.

Thus:
- Hypercubic normalisation reflects the ratio of a degree-k polynomial to n^k.
- Iterated stacking normalisation reflects the ratio of two degree-k polynomials with different leading coefficients.
- Leading-term inversion provides asymptotically accurate index recovery, and a first-order correction yields eventual exact recovery after flooring.

All limits and inversion estimates follow directly from the polynomial form of $S_k(n)$.

References:

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