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Note to grader: all the code explanations and figures are included in this notebook

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import warnings
from scipy import interpolate
from scipy import integrate
%matplotlib inline
import matplotlib as mpl
mpl.rcParams['figure.dpi'] = 200
plt.rcParams.update({"text.usetex": True})
```

Q1

The function we have to integrate to get the electric field due to a shell of Radius R a distance z from the center is (up to a constant)

$$f(z) = \frac{z-z'}{(R^2-2zz'+z^2)^{3/2}} \tag{1}$$

So,

$$\boxed{E(z) = \int_{-R}^{R} \frac{z-z'}{(R^2-2zz'+z^2)^{3/2}}dz'} \tag{2}$$

Note that at $z=R$, the integral is

$$E(R) = \int_{-R}^{R} \frac{R-z'}{(2R^2-2Rz')^{3/2}}dz' \tag{3}$$

and we see that at $z'=R$, the integrand is not defined. This is a problem for an integrator like the one we wrote in class. Quad is able to work around that, though. In what follows I avoid the point $z=R$ for that reason.

In [2]:
```
# variable step size integrator adapted from the one written in class
def my_integrate(f,a,b,tol):
    # Integrating with 3 point and 5 point Simpson's rule
    x = np.linspace(a,b,5)
    y = f(x)
    int_3 = (x[-1]-x[0])/6*(y[0]+4*y[2]+y[4])
    int_5 = (x[-1]-x[0])/12*(y[0]+4*y[1]+2*y[2]+4*y[3]+y[4])
    # if approximate error < tol, return the result, else split the
    # integral in 2
    if np.abs(int_3-int_5)<tol:
        return int_5
    int_left = my_integrate(f,a,(a+b)/2,tol/2)
```

```
    int_right = my_integrate(f,(a+b)/2,b,tol/2)
    return int_left + int_right

# function to generate the function to be integrate (the electric field
# of charged rings)
def gen_f(z,R):
    def f(zprime):
        return (z-zprime)/((R**2-2*z*zprime+z**2)**(3/2))
    return f
```

Taking many values of z, fixing $R = 1$, and plotting the electric field as a function of z

In [3]:

```
R=1
z_array = np.linspace(-2*R,2*R,500)
E_array = np.zeros_like(z_array)
E_array_quad = np.zeros_like(z_array)
for i in range(500):
    f = gen_f(z_array[i],R)
    E_array[i] = my_integrate(f,-R,R,1e-5)
    E_array_quad[i] = integrate.quad(f,-R,R)[0]
```

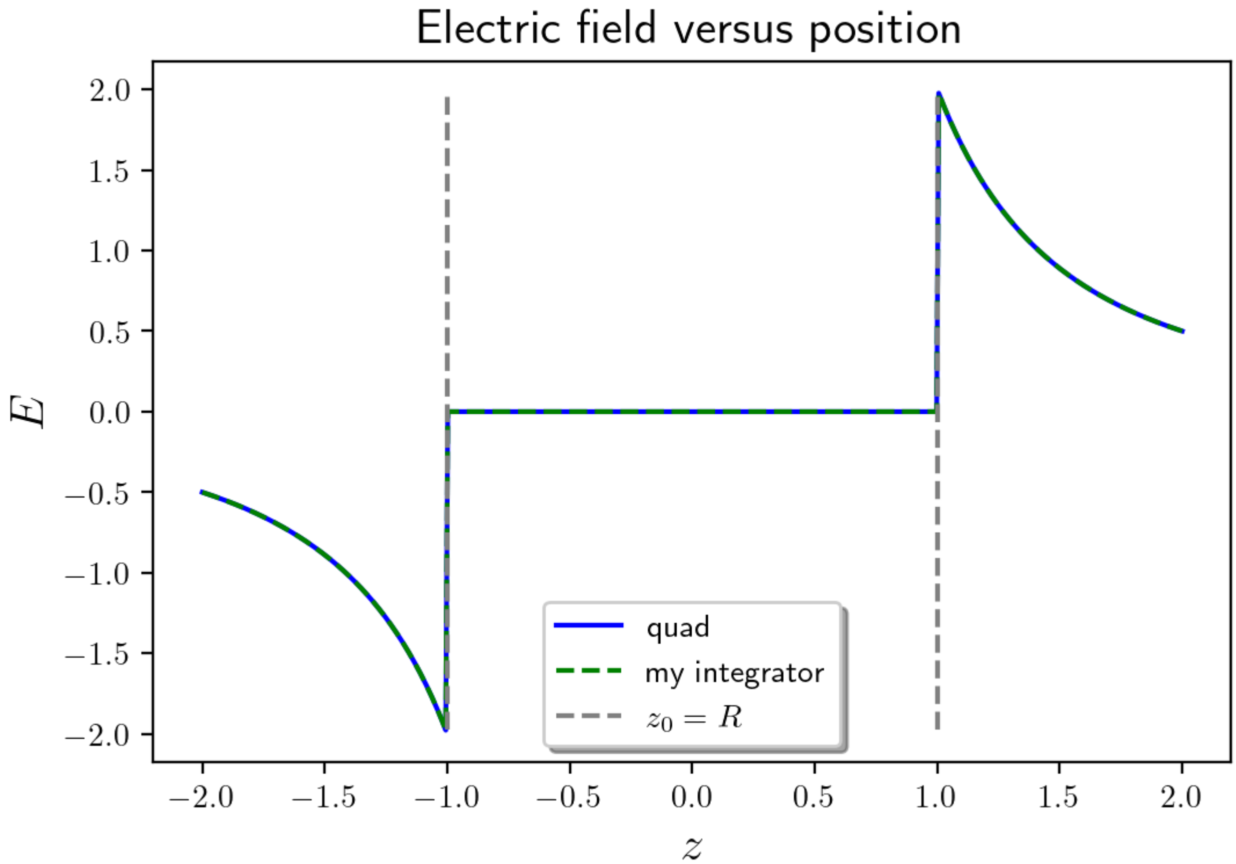
In [4]:

```
plt.plot(z_array,E_array_quad,color='blue',label='quad')
plt.plot(z_array,E_array,color='green',label='my integrator',ls='--')

plt.plot([-R,-R],[E_array.min(),E_array.max()],ls='--',color='gray',label='$z_0=R$')
plt.plot([R,R],[E_array.min(),E_array.max()],ls='--',color='gray')
plt.xlabel('$z$',fontsize=15)
plt.ylabel('$E$',fontsize=15)
plt.title('Electric field versus position',fontsize=15)
plt.legend(loc=0,frameon=True,shadow=True,fontsize=10)
```

Out[4]:

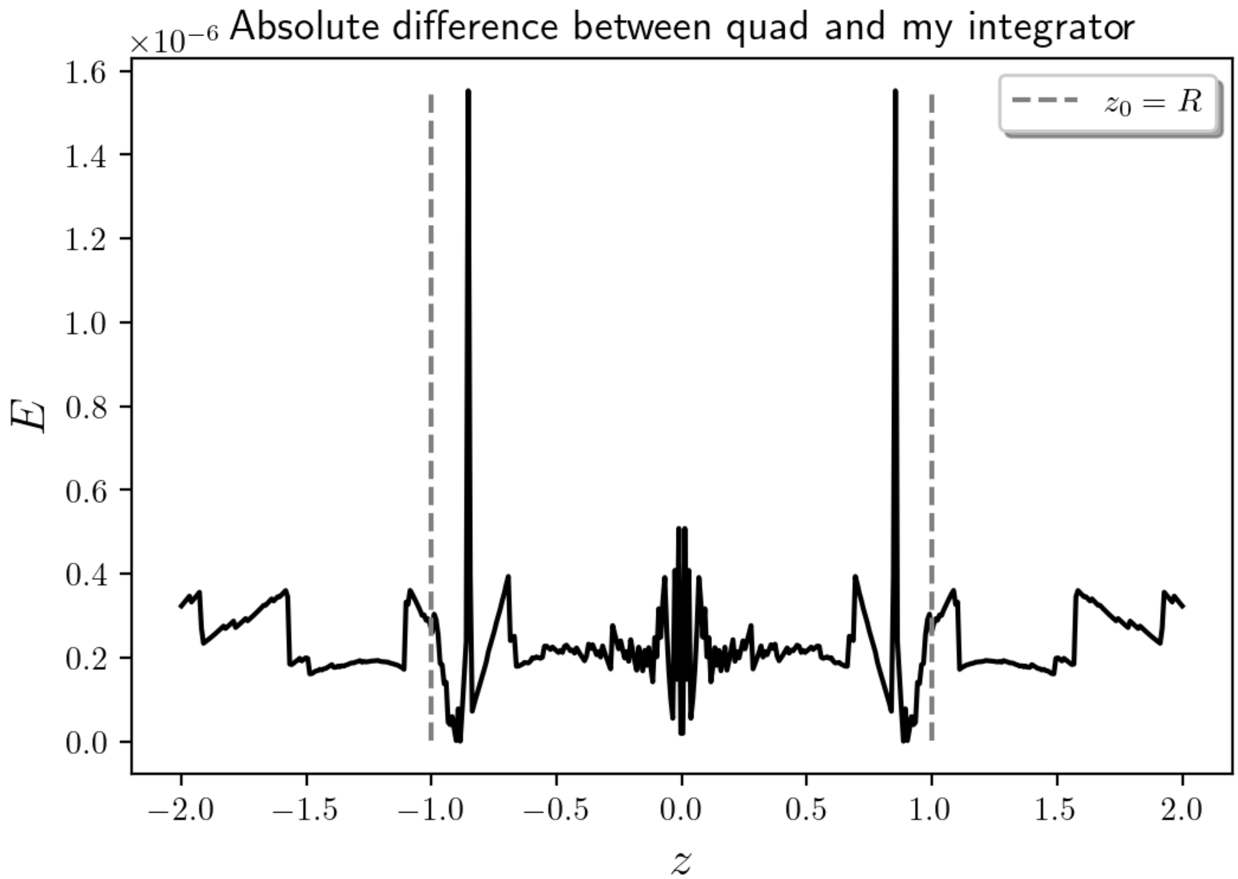
```
<matplotlib.legend.Legend at 0x2fa593e7c08>
```



In [5]:

```
plt.plot(z_array,np.abs(E_array_quad-E_array),color='black')
plt.plot([-R,-R],[np.abs(E_array_quad-E_array).min(),np.abs(E_array_quad-E_array).max()
plt.plot([R,R],[np.abs(E_array_quad-E_array).min(),np.abs(E_array_quad-E_array).max()],
plt.xlabel('$z$',fontsize=15)
plt.ylabel('$E$',fontsize=15)
plt.title('Absolute difference between quad and my integrator',fontsize=13)
plt.legend(loc=0,frameon=True,shadow=True,fontsize=10)
```

Out[5]: <matplotlib.legend.Legend at 0x2fa5a9ea1c8>



We see that quad and my integrator agree everywhere, and the absolute difference with respect to quad is less that 10^{-5} which is what I requested as tollerance. Now I check that my integrator indeed breaks down at $z = R$, and that quad manages to work around the singularity in the integrand

In [6]:

```
f = gen_f(R,R)

# Trying with my integrator
try:
    my_integrate(f,-R,R,1e-2)
except Exception as exception:
    print(exception)

# Trying with quad
print('Integral as calculated by quad:', integrate.quad(f,-R,R)[0])
```

```
maximum recursion depth exceeded while calling a Python object
Integral as calculated by quad: 0.9999999999999938
```

```
C:\Users\Guill\Anaconda3\envs\ml_pytorch\lib\site-packages\ipykernel_launcher.py:20: Run
timeWarning: invalid value encountered in true_divide
```

Now as a fix for this, I show that we can integrate not from $z' = -R$ to $z' = R$ but rather from z' very close to $-R$ to z' very close to R. So here I use my integrator in this way and recreate the integral value of ~1 obtained by quad.

In [7]:

```
f = gen_f(R,R)
```

```
print(my_integrate(f,-R+R*1e-10,R-R*1e-10,1e-5))
```

```
0.9999931992016304
```

This is accurate to ~10^{-5} which is what I requested as tolerance from my integrator.

Q2

I start by adapting the integrator from last question so that it keeps memory of every evaluation of f it does. Whether to make use of the memory is passed as an extra argument "use_memory". I then integrate $f(x) = \cos(x)$ from $-\pi/2$ to $\pi/2$ with a tolrance of 10^{-10} and observed how many function calls are saved

In [8]:

```
def my_integrate(f,a,b,tol,extra=None,use_memory=True):
    # If the memory is empty, initiate it, else retrieve it
    if not extra:
        x_memory = []
        y_memory = []
    else:
        x_memory = extra[0]
        y_memory = extra[1]

    # Integrating with 3 point and 5 point Simpson's rule
    x = np.linspace(a,b,5)
    y = np.zeros_like(x)

    # Using memory to get y values corresponding to this array of x,
    # and adding new evaluations to memory
    for i in range(len(x)):
        if x[i] in x_memory:
            y[i] = y_memory[x_memory.index(x[i])]
        else:
            y[i] = f(x[i])
            if use_memory:
                x_memory.append(x[i])
                y_memory.append(y[i])
            # if memory usage is off, still append a nan to the memory
            # because this is how I keep track of # of function calls
            else:
                x_memory.append(np.nan)
                y_memory.append(np.nan)

    # Integrating with 3 point and 5 point Simpson's rule
    int_3 = (x[-1]-x[0])/6*(y[0]+4*y[2]+y[4])
    int_5 = (x[-1]-x[0])/12*(y[0]+4*y[1]+2*y[2]+4*y[3]+y[4])

    # if approximate error < tol, return the result, else split the
    # integral in 2
    if np.abs(int_3-int_5)<tol:
        return int_5
    int_left = my_integrate(f,a,(a+b)/2,tol/2,[x_memory,y_memory],use_memory=use_memory
    int_right = my_integrate(f,(a+b)/2,b,tol/2,[x_memory,y_memory],use_memor

    # Printing number of function calls at the end
    if not extra:
```

```
        print('number of function calls:',len(x_memory))
    return (int_left + int_right)
```

In [9]:

```
f = np.cos
print(my_integrate(f,-np.pi/2,np.pi/2,1e-10,use_memory=True))
print(my_integrate(f,-np.pi/2,np.pi/2,1e-10,use_memory=False))
```

```
number of function calls: 1011
2.0000000000015197
number of function calls: 2335
2.0000000000015197
```

So we see that the number of calls was reduced in this case by a factor of ~ 2.3 which is great.

Q3

I fit a Chebyshev polynomial to $\log_2(x)$, 5000 points and 100 coefficients.

I use the fact that the Chevyshev polymials are bounded by -1 and 1 to say that the error on the Chebyshev polynomial evaluated at these 5000 points is less than the sum of coefficients which I cut out. For that to work it needs to be clear that the coefficients drop fast enough that their sum can be approximated by the 100 coefficients that will be cut off. Also this doesn't say anything about the error at points in between the 5000 selected points, but we'll assume that the interpolation is good.

So first I do the Chebyshev fit:

In [10]:

```
npts = 5000
order = 100
x = (np.linspace(0.5,1,npts)-0.75)*4
y = np.log2(np.linspace(0.5,1,npts))
coeffs = np.polynomial.chebyshev.chebfit(x, y, order)
```

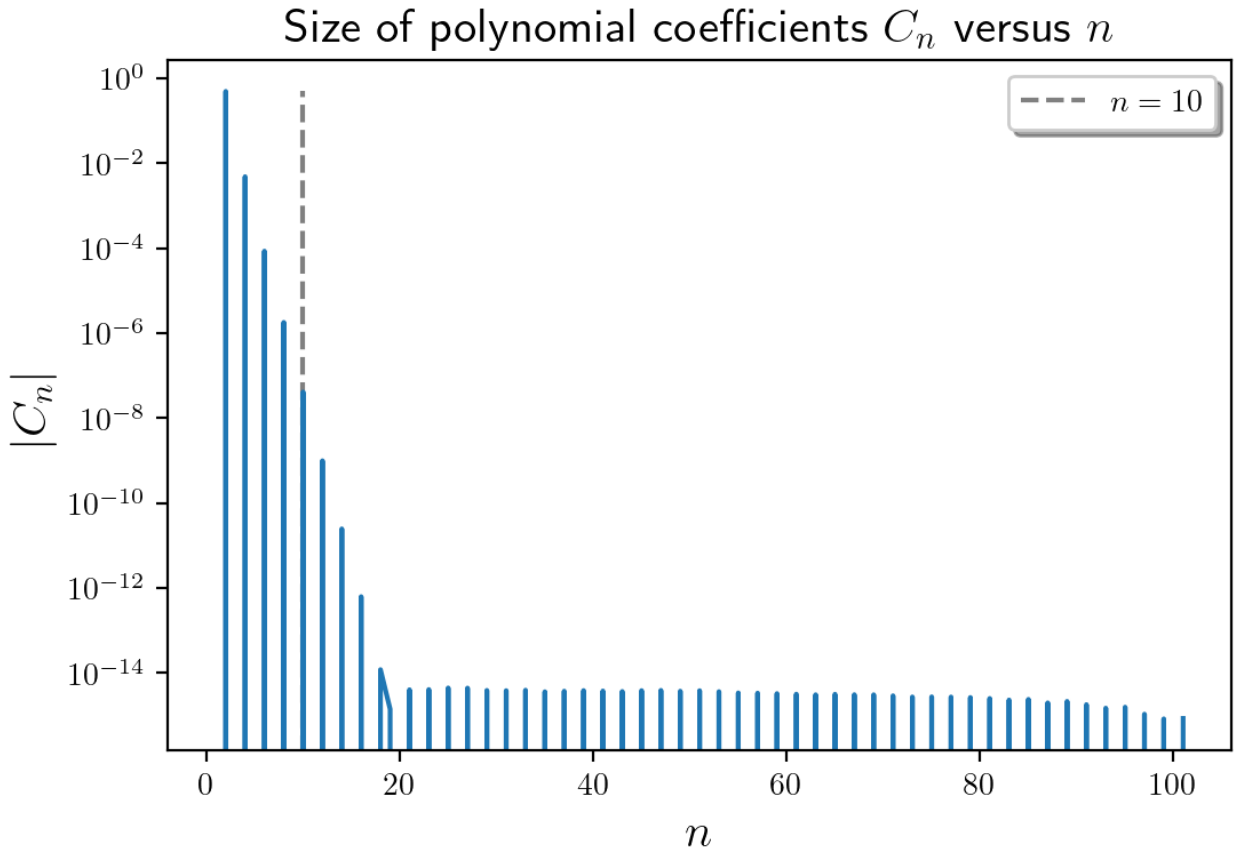
Now I plot the coefficients to see that they drop very quickly

In [11]:

```
plt.plot([10,10],[coeffs.min(),coeffs.max()],ls='--',color='gray',label='$n=10$')
plt.plot(np.arange(1,102),coeffs)
plt.xlabel('$n$',fontsize=15)
plt.ylabel('$|C_n|$',fontsize=15)
plt.title('Size of polynomial coefficients $C_n$ versus $n$',fontsize=15)
plt.yscale('log')
plt.legend(loc=0,frameon=True,shadow=True,fontsize=10)
```

Out[11]:

```
<matplotlib.legend.Legend at 0x2fa5abbae08>
```



If we cut off the coefficients C_{10} and beyond, we get that the sum of the cut off coefficients is less than the demanded tolerance of 10^{-6} (see line of code below), and we see from the plot above that the sum of the C_n from C_{10} to C_{100} should approximate the sum of all coefficients beyond C_{10} very well.

In [12]:
```
# Sum of C_n starting at C_10
np.sum(np.abs(coeffs[10:]))
```

Out[12]:
```
7.559071381489694e-09
```

So we can keep the coefficients C_0 to C_9 only.

Here I test the resulting 9th order Chebyshev polynomial by evaluating $log_2(0.75)$

In [13]:
```
x0 = 0.75
x0_ = (x0-0.75)*4

truth = np.log2(x0)
approx = np.polynomial.chebyshev.chebval(x0_,coeffs[0:10])
print('truth',truth)
print('approx',approx)
print('error',np.abs(truth-approx))
```

```
truth -0.4150374992788438
approx -0.4150375055043163
error 6.225472470866578e-09
```

The error is indeed less that 10^{-6}.

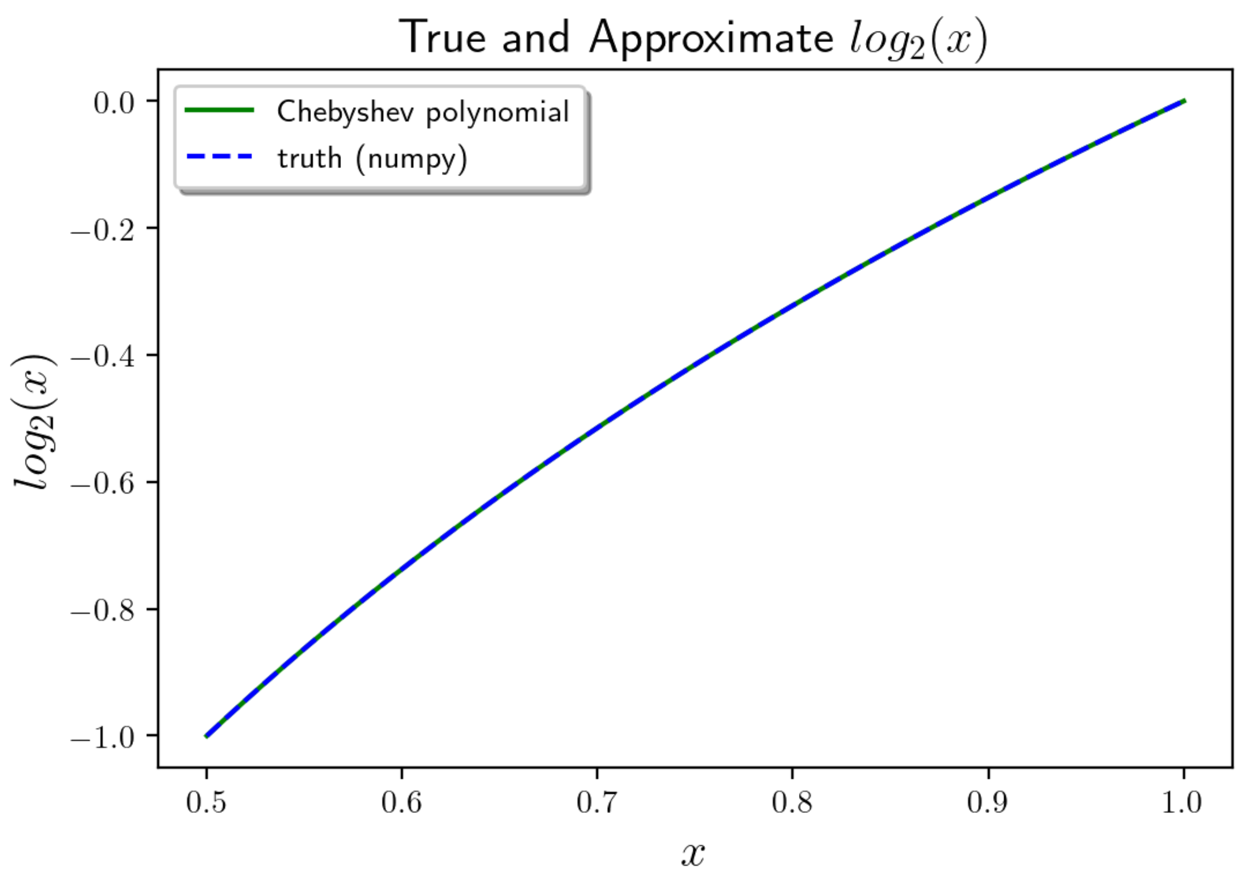
Now I also plot the approximated log_2 and the true log_2, as well as the error, over the entire approximated range, to make sure that this really worked

In [14]:
```
x = np.linspace(0.5,1,1000)
x_ = (x-0.75)*4

truth = np.log2(x)
approx = np.polynomial.chebyshev.chebval(x_,coeffs[0:10])
```

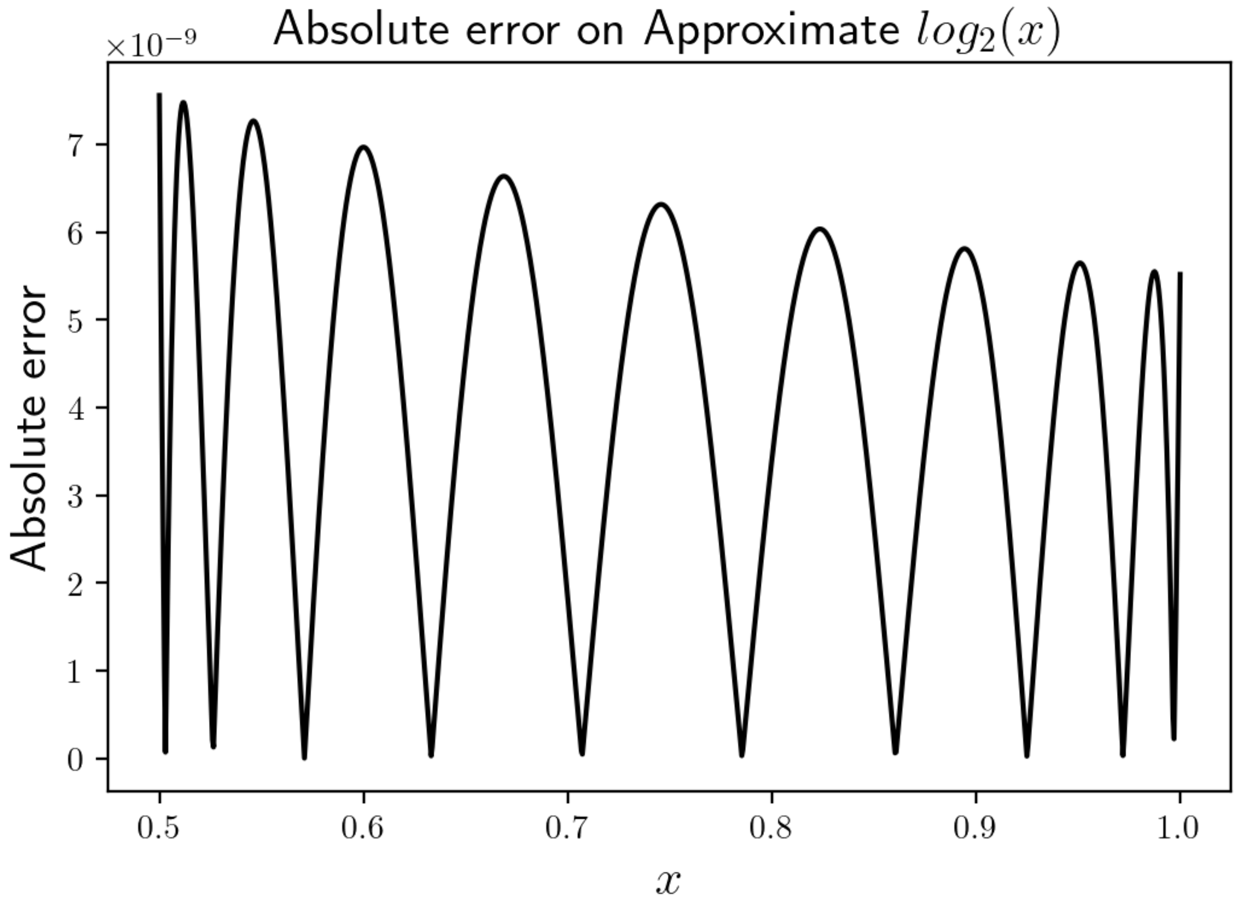
In [15]:
```
plt.plot(x,approx,color='green',label='Chebyshev polynomial')
plt.plot(x,truth,color='blue',label='truth (numpy)',ls='--')
plt.xlabel('$x$',fontsize=15)
plt.ylabel('$log_2(x)$',fontsize=15)
plt.title('True and Approximate $log_2(x)$',fontsize=15)
plt.legend(loc=0,frameon=True,shadow=True,fontsize=10)
```

Out[15]:
```
<matplotlib.legend.Legend at 0x2fa5aa88148>
```



In [16]:
```
plt.plot(x,np.abs(approx-truth),color='black')
plt.xlabel('$x$',fontsize=15)
plt.ylabel('Absolute error',fontsize=15)
plt.title('Absolute error on Approximate $log_2(x)$',fontsize=15)
```

Out[16]:
```
Text(0.5, 1.0, 'Absolute error on Approximate $log_2(x)$')
```



So we see that the error is indeed below 10^{-6} across the entire interval as desired.

Now we make a $\log_2(x)$ function that can take in any positive number x. The idea is to write x as it's mantissa and twos exponent decomposition, ie,

$$x = m2^n \tag{4}$$

where for a positive number the mantissa m satisfies $0.5 < m < 1$ and n is the exponent. But then

$$\begin{aligned}\log_2(x) &= \log_2(m) + n\log_2(2) &(5)\\ &= \log_2(m) + n &(6)\end{aligned}$$

and now we just have to use the log_2 chebyshev coefficients found before for $0.5 < x < 1$ and we get our result. In what follows I write a function that does this and test it against numpy's np.log2

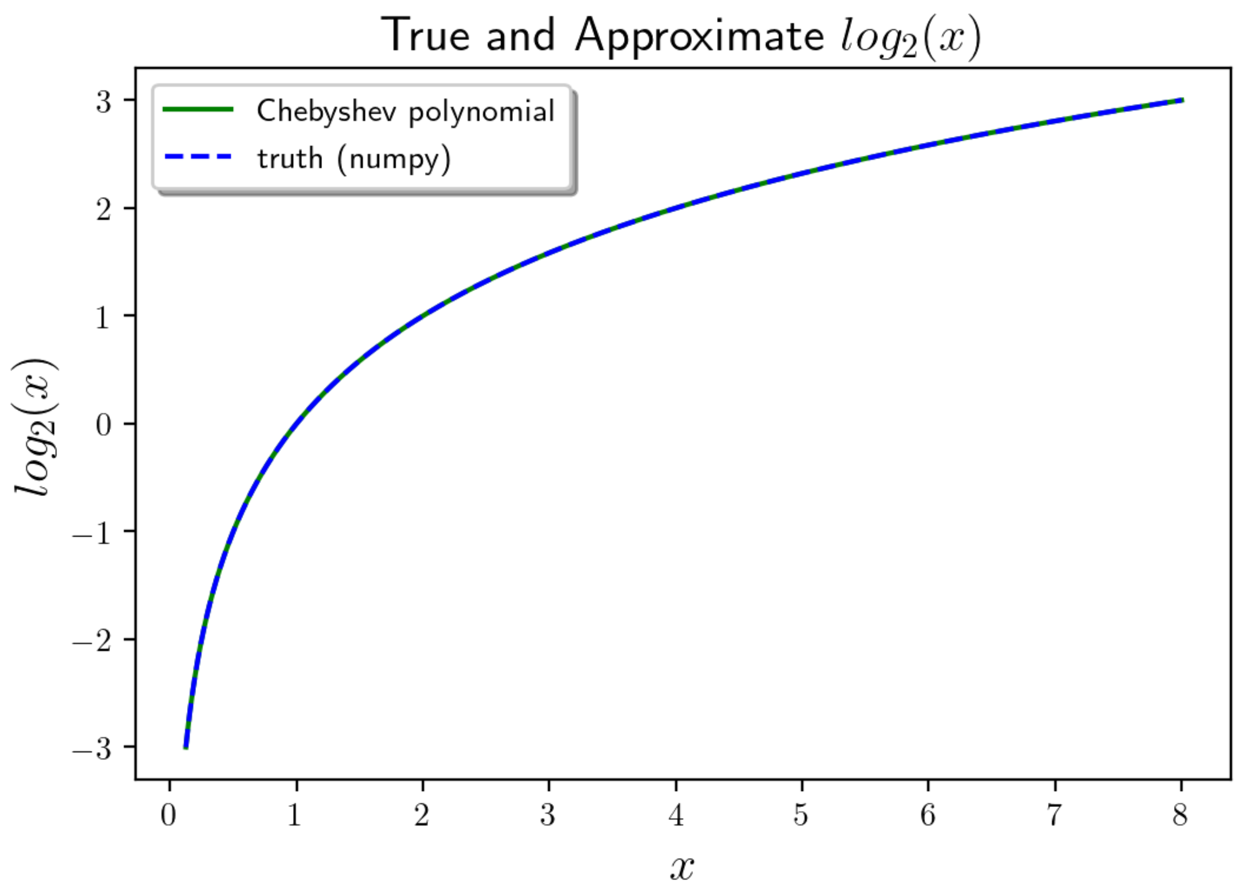
In [17]:
```
def gen_log2(cheb_coeffs):
    # generate a log2 function given cheb coefficients
    def mylog2(x):
        # decompose x into mantissa and exponent
        m,n = np.frexp(x)
        # map mantissa to the domain of cheb polynomials
        m_ = (m-0.75)*4
        # compute log2(x) using cheb coefficients
        y = np.polynomial.chebyshev.chebval(m_,coeffs[0:10])+n
        return y
    return mylog2
```

In [18]:
```
x = np.linspace(2**(-3),2**3,1000)
mylog2 = gen_log2(coeffs)

truth = np.log2(x)
approx = mylog2(x)
```

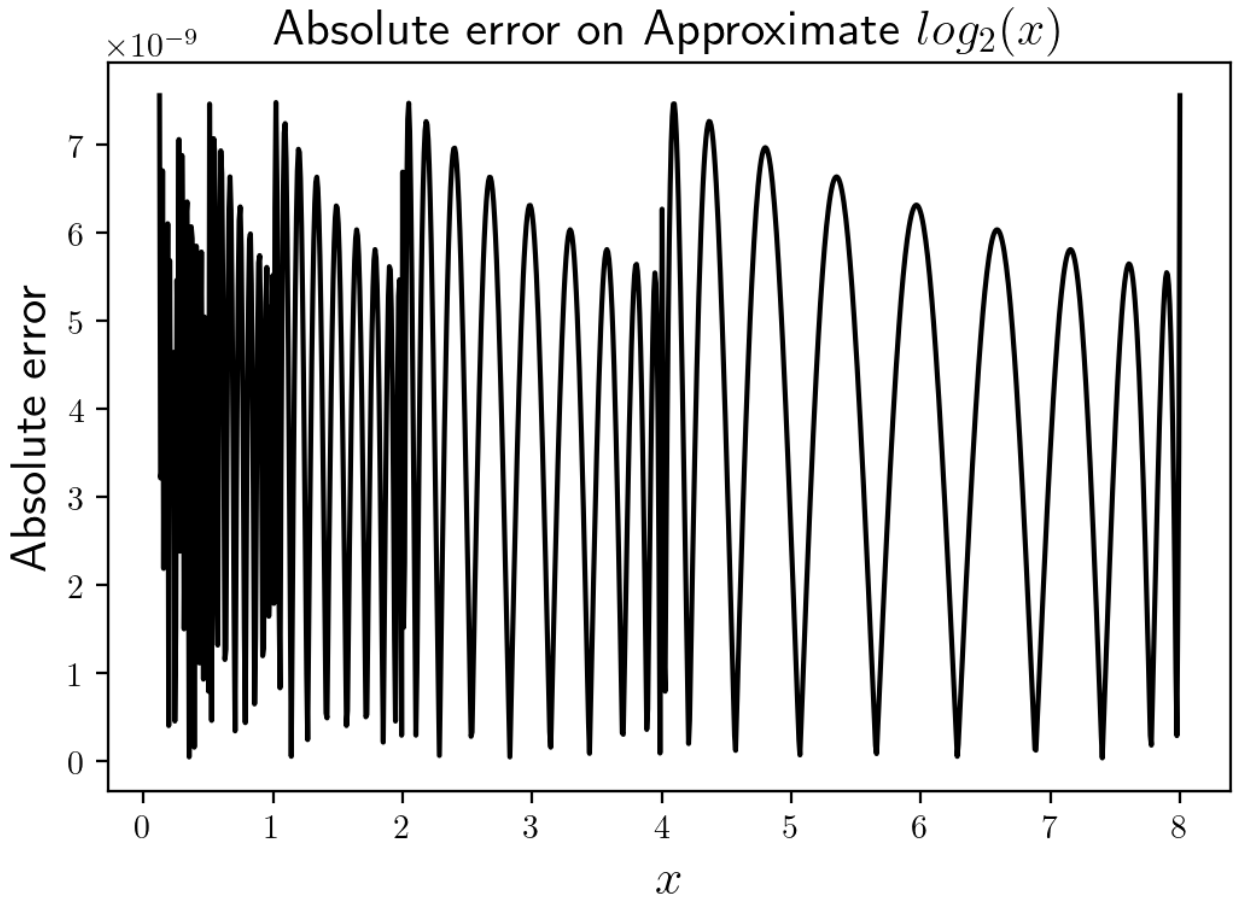
In [19]:
```
plt.plot(x,approx,color='green',label='Chebyshev polynomial')
plt.plot(x,truth,color='blue',label='truth (numpy)',ls='--')
plt.xlabel('$x$',fontsize=15)
plt.ylabel('$log_2(x)$',fontsize=15)
plt.title('True and Approximate $log_2(x)$',fontsize=15)
plt.legend(loc=0,frameon=True,shadow=True,fontsize=10)
```

Out[19]:
```
<matplotlib.legend.Legend at 0x2fa5aaf15c8>
```



In [20]:
```
plt.plot(x,np.abs(approx-truth),color='black')
plt.xlabel('$x$',fontsize=15)
plt.ylabel('Absolute error',fontsize=15)
plt.title('Absolute error on Approximate $log_2(x)$',fontsize=15)
```

Out[20]:
```
Text(0.5, 1.0, 'Absolute error on Approximate $log_2(x)$')
```



So we see that the error is indeed below 10^{-6} across the entire interval.