Sequential Monte Carlo Methods For Probabilistic Graphical Models

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5 - SMC Sampler

SMC algorithms are based on some sequential use of Importance Sampling.

- The aim is to deal with an untractable distribution in high dimension: $\pi_d(x_{1:d}) = \frac{\gamma_d(x_{1:d})}{Z}$.
- SMC sampler can be used for instance to sample from $\pi_d(x_{1:d})$ or to estimate the partition function Z.
- We **sample iteratively from conditional proposals in lower dimension**, corresponding to cliques of the graph.
- SMC methods include a **re-sampling** step to avoid exponential grow in the weights.

Perform each step for $i = 1, \ldots, N$

begin

  Sample $X^i_{\mathcal{L}_1} \sim r_1(.)$

  Set $w^i_1 = \gamma_1(X^i_{\mathcal{L}_1}) / r_1(X^i_{\mathcal{L}_1})$

  for $k = 2, \ldots, K$ **do**

    Sample a^i_k according to $\mathbb{P}(a^i_k = j) = \frac{\nu^j_{k-1} w^j_{k-1}}{\sum_l \nu^l_{k-1} w^l_{k-1}}$

    Sample $\xi^i_k \sim r_k(.|X^{a^i_k}_{\mathcal{L}_{k-1}})$

    Set $X^i_{\mathcal{L}_k} = X^{a^i_k}_{\mathcal{L}_{k-1}} \cup \xi^i_k$

    Set $w^i_k = W_k(X^i_{\mathcal{L}_k})$

  end

end

Algorithm 1: Sequential Monte Carlo

6 - Partition function

The **partition function** - normalizing constant - is a very interesting quantity in many applications (see [1]), such as:

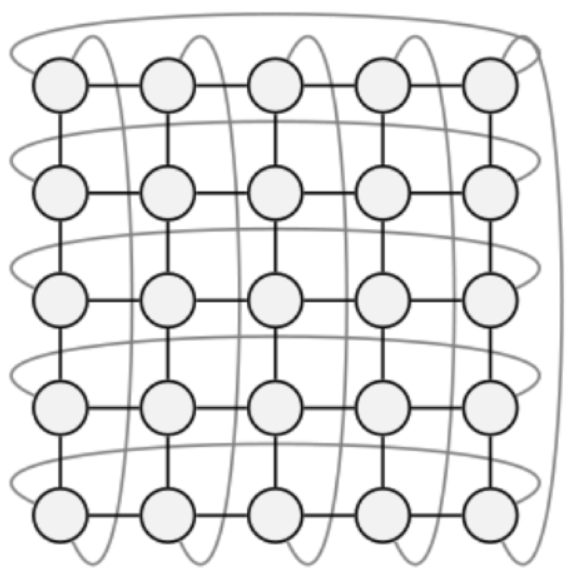
- likelihood-based learning of parameters of the PGM
- derivation of free energy of a system of objects in mechanics

Our Sequential Monte Carlo approach provides an **unbiased estimator** of this partition function, given by:

$$\hat{Z}^N_k = \left(\frac{1}{N} \sum_{i=1}^{N} \omega^i_k \right) \left\{ \prod_{l=1}^{k-1} \frac{1}{N} \sum_{i=1}^{N} \nu^i_l \omega^i_l \right\}$$

This result is not straightforward. Therefore, the proof is reported in our final report.

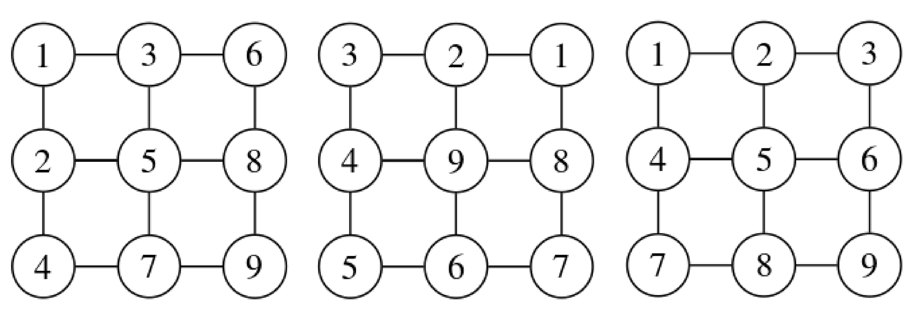
7 - Application to Classical XY model



- **XY Model**: A 2-dimensional d-square lattice model, extension of Ising model
- Set of cliques: pairs of adjacent vertices x_i and x_j
- Joint probability distribution:

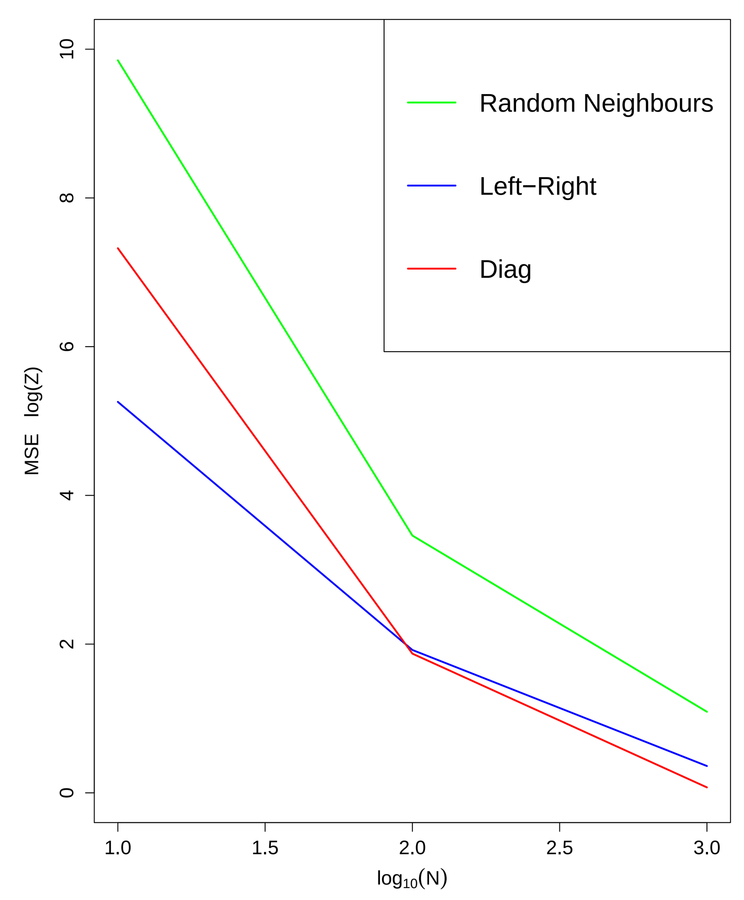
$$p(X_{\mathcal{V}}) \propto \exp \left\{ \beta \sum_{i \sim j} \cos(x_i - x_j) \right\}$$

Several decompositions are used to sequence the graph : left-right decomposition, random decomposition...

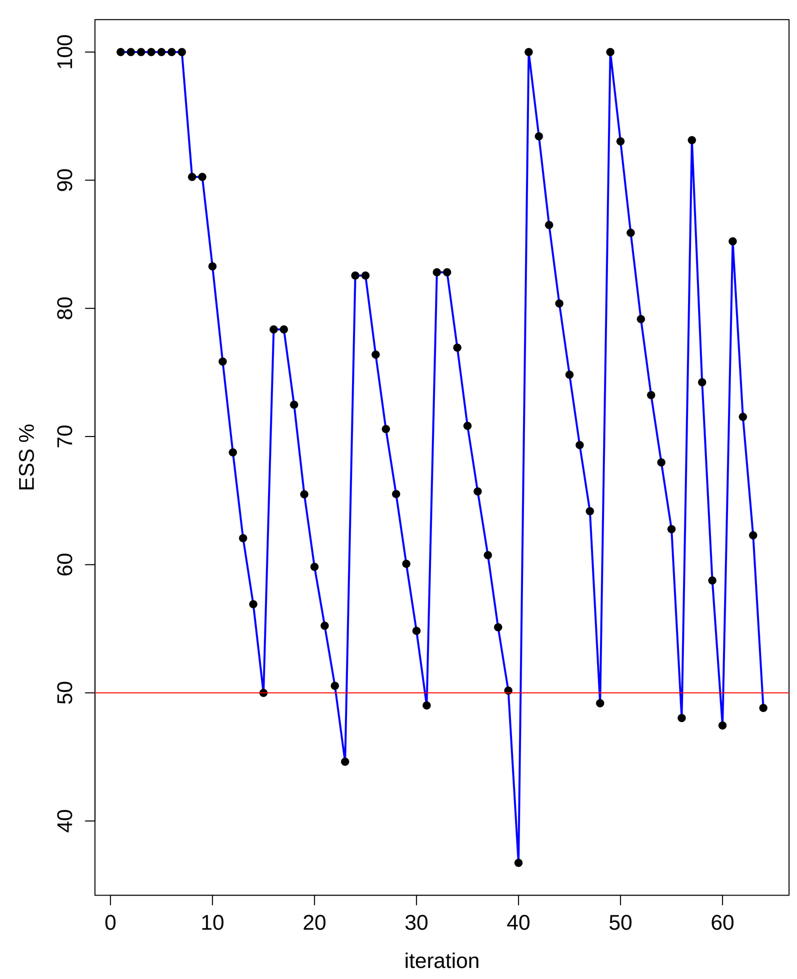


(a) Diagonal (b) Spiral (c) Left-Right

We observe in practice that **the ordering has an important impact on the performance** of the SMC sampler.



Orderings and M.S.E. evolution



Evolution of E.S.S. measure (re-sampling under 50%)

8 - To go further

We presented a new framework for inference in PGM using Sequential Monte Carlo.

Although the NIPS 2014 paper is not the first tentative on this topic, previous work were:

- only designed for Gaussian or discrete interactions between variables [Hamze and De Freitas (2005); Everitt (2012)]
- and-or did not develop such sequential decomposition of the graph [Isard, 2003; Frank *et al.*,2009]

Besides XY model, the authors also introduced two others applications for their SMC sampler:

- a simple toy **Gaussian Markov Random Field** model to incorporate **Particle MCMC** sampling
- a **Latent Dirichlet Allocation** - directed - PGM, to perform likelihood estimation in topic models

We provide a quick review of these applications in the final report.

Source Code

All implementations have been made with R. The entire source code of this project will be available on the following GitHub link:

`https://github.com/GuillaumeSalha/ParticleFiltering`

References

[1] Naesseth, C. A., Lindsten, F., & Schon, T. B. (2014). Sequential Monte Carlo for graphical models. *In Advances in Neural Information Processing Systems*, pp. 1862-1870.

[2] Doucet, A., & Johansen, A. M. (2009). A tutorial on particle filtering and smoothing: Fifteen years later. *Handbook of Nonlinear Filtering*, 12, 656-704.

[3] Robert, C., Casella, G. (2004). Monte Carlo statistical methods. *Springer Science Business Media*.

[4] Chaikin, P. M., Lubensky, T. C. (2000). Principles of condensed matter physics. *Cambridge: Cambridge university press*.