NOTES ON NORMALISED FREQUENCY SPECTRA

The Discrete and Fast Fourier Transforms (DFT, FFT)

- The Fast-Fourier Transform (FFT) is an algorithm designed for the efficient computation of the Discrete Fourier Transform (DFT). The DFT calculates the discrete frequency spectrum of a discrete time signal.
- The DFT requires $O(N^2)$ computations. In particular, N complex multiplications and N-1 complex additions per frequency bin k; each *multiply-accumulate calculation* (MAC) counts as one operation. The FFT requires $O(N\log_2 N)$ computations.
- The DFT is defined by the following formula

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \quad k = 0, \dots, N-1$$

  where $x[n]$ is the discrete time signal, $X[k]$ is the discrete frequency signal, n and k are the sample indices in time and frequency, and N is the number of elements in $X[k]$ and $x[n]$.
- $X[k]$ is a *discrete, periodic* sequence containing N *complex* samples extending from frequency $f = 0$ Hz (DC, $k = 0$) to $f = f_s$ ($k = N - 1$, where f_s is the sampling rate).
- $x[n]$ is a *discrete*, *periodic* sequence containing N *real* samples.
- The units in $X[k]$ are the same as those in $x[n]$.
- The DFT assumes *periodicity in time*. Infinite periodic signals do not exist in practice, and DFT will be typically be applied to a *portion of a non-periodic*, discrete time signal.
- For $x[n]$ real, $X^*[k] = X[-k]$, with * being the *complex conjugate*. In other words, the magnitude and phase of the DFT of a real time signal are positive symmetric and negative symmetric with respect to $k = 0$, respectively.
- $X^*[k]X[k] = |X_k|^2$, where $|X_k|$ is the magnitude of the DFT.
- Multiplication in the discrete time domain is equivalent to applying circular convolution in the discrete frequency domain. In particular, $x_{n,1} \cdot x_{n,2} = \left(X_{k,1} \otimes X_{k,2}\right)/N$, where $\otimes$ indicates circular convolution over the N frequency samples. *Windowing* in the time domain will produce *leakage* in the frequency domain due to the convolution of spectrum of the signal with the window, characterised by a wider main lobe accompanied by secondary lobes.
- Multiplication in the discrete frequency domain is equivalent to convolution in the discrete time domain. In particular, $x_{n,1} \otimes x_{n,2} = \left(X_{k,1} \cdot X_{k,2}\right)$. An example of application of this property is the calculation of the pressure waveform from the digital audio signal and the system's sensitivity response. This is typically done by calculating the inverse discrete Fourier transform (IDFT) of the product of the spectral responses of the signal and system chain.

Frequency Spectra and Their Relation to the Signal's Energy

The DFT provides valuable information about the phase and magnitude of the frequency components of a discrete time signal. However, the DFT does not directly represent the energy or power of the signal at specific frequency bands. The DFT spectrum must be normalised in a certain way according to the *parameter* (energy, power, exposure) and *banding* (per FFT bin, per Hz) that we want to visualise.

In the first section of this chapter, the Parseval Theorem is presented, a property of the Fourier Transform that relates the spectrum to the energy of the signal. The following sections discuss the different ways of representing the DFT spectrum using the Parseval theorem. Depending on the *parameter* and *banding*, there are six common ways of representing the discrete Fourier spectrum: *energy spectrum* ES, *power spectrum* PS, *exposure spectrum* XS, *energy spectral density* ESD, *power spectral density* PSD, and *exposure spectral density* XSD.

The Parseval Theorem

- The *Parseval theorem* states that the integral of the square of a function is equal to the integral of the square of its transform.
- From the Parseval theorem, a relation can be established between the energy of a signal in the time and frequency domains. The energy E_x of a discrete periodic signal $x[n]$ of length N is given by the Parseval relation for the DFT:

$$E_x = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

The Mean Value

- The mean $\bar{x}$ of a discrete periodic signal $x[n]$ of length N is given by the two equivalent formulas for the time and frequency domains:

$$\bar{x} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] = \frac{X[1]}{N}$$

Energy Spectrum (ES)

- The *energy spectrum* (ES) of a signal $x[n]$ is the signal's auto-spectrum expressed as energy per DFT bin. The "energy per FFT bin" refers to the energy within the band covered by the DFT bin.
- The *energy spectrum* of a signal $x[n]$ of length N is given by:

$$E_{xx}[k] = \frac{|X[k]|^2}{N}$$

- The units of $E_{xx}[k]$ are those of $x^2[n]$. For example, for $x[n]$ in Pa, $E_{xx}[k]$ is expressed in Pa^2.
- $E_{xx}[k]$ decreases as N increases, since the energy E_x is shared by a larger number of DFT bins.
- From the Parseval relation, the energy E_x can be calculated as the sum of the samples in the energy spectrum $E_{xx}[k]$.

$$E_x = \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} E_{xx}[k]$$

- The *energy spectrum* allows to identify the energy of narrowband components directly from the spectrum. This type of representation is best suited for *coherent* processes with dominant *narrowband* components.
- Energy is a metric best suited for *transient* signals.
- For our purposes, more useful than the energy spectrum is the *exposure spectrum* (XS).

Power Spectrum (PS)

- The *power spectrum* (PS) of a signal $x[n]$ is the signal's auto-spectrum expressed as power per DFT bin. The "power per DFT bin" refers to the power within the band covered by the DFT bin.
- The *power spectrum* of a signal $x[n]$ of length N is given by:

$$P_{xx}[k] = \frac{|X[k]|^2}{N^2}$$

- The units of the $P_{xx}[k]$ are those of $x^2[n]$. For example, for $x[n]$ in Pa, P_{xx} is expressed in Pa^2.
- $P_{xx}[k]$ decreases as N increases, since the power P_x is shared by a larger number of DFT bins.
- From the Parseval relation, the power P_x or *mean-square value* of $x[n]$ can be calculated as the sum of the samples in the power spectrum $P_{xx}[k]$.

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 = \sum_{k=0}^{N-1} P_{xx}[k]$$

- The root-mean-square value (RMS) of $x[n]$ is $\sqrt{P_x}$.
- The *power spectrum* $P_{xx}[k]$ allows to identify the energy of narrowband components directly from the spectrum. This type of representation is best suited for *coherent* processes with dominant *narrowband* components.
- Power is a metric best suited for long *continuous* signals.

Exposure Spectrum (XS)

- The *exposure spectrum* (XS) of a signal $x[n]$ is the signal's auto-spectrum expressed as exposure per DFT bin. The "exposure per DFT bin" refers to the energy within the band covered by the DFT bin.
- The *exposure spectrum* of a signal $x[n]$ of length N, duration T and sampling frequency f_s is given by:

$$\Xi_{xx}[k] = P_{xx}[k] \cdot T = \frac{|X[k]|^2}{N f_s}$$

- The units of the $\Xi_{xx}[k]$ are those of $x^2[n]$ multiplied by the time unit. For example, for $x[n]$ in Pa, Ξ_{xx} is expressed in Pa^{2}s.
- The exposure spectrum $\Xi_{xx}[k]$ is a variant of the energy spectrum $\mathrm{E}_{xx}[k]$ for representing *sound exposure levels* (SEL), as opposed to the power spectrum, which can be used for representing *sound pressure levels* ($\mathrm{SPL}_{\mathrm{rms}}$).
- $\Xi_{xx}[k]$ decreases as N increases, since the exposure Ξ_x is shared by a larger number of DFT bins.

- From the Parseval relation, the exposure Ξ_x can be calculated as the sum of the samples in the exposure spectrum $\Xi_{xx}[k]$.

$$\Xi_x = P_x \cdot T = \frac{1}{f_s}\sum_{n=0}^{N-1}|x[n]|^2 = \sum_{k=0}^{N-1}\Xi_{xx}[k]$$

- The *exposure spectrum* $\Xi_{xx}[k]$ allows to identify the exposure of narrowband components directly from the spectrum. This type of representation is best suited for *coherent* processes with dominant *narrowband* components.
- Exposure is a metric suited for both *transient* and long *continuous* signals.

<u>Energy Spectral Density (ESD)</u>

- The *energy spectral density* (ESD) of a signal $x[n]$ is the signal's auto-spectrum expressed as energy per Hz.
- The *energy spectral density* of a signal $x[n]$ of duration N is given by:

$$\mathcal{E}_{xx}[k] = \frac{E_{xx}[k]}{f_{bin}} = \frac{|X[k]|^2}{f_s}$$

  where $f_{bin} = f_s/N$ is the bandwidth of each DFT bin.
- The units of $\mathcal{E}_{xx}[k]$ are those of $x^2[n]$ divided by the frequency unit. For example, for $x[n]$ in Pa, $\mathcal{E}_{xx}[k]$ is expressed in Pa^2/Hz.
- $\mathcal{E}_{xx}[k]$ is independent of N.
- From the Parseval relation, the energy E_x can be calculated as the sum of the samples in the energy spectral density $\mathcal{E}_{xx}[k]$ multiplied by f_{bin}.

$$E_x = f_{bin}\sum_{k=0}^{N-1}\mathcal{E}_{xx}[k]$$

- The *energy spectral density* $\mathcal{E}_{xx}[k]$ depicts the energy per 1-Hz band. This type of representation is best suited for *broadband, uncorrelated processes* that have no dominant narrowband components.
- Energy is a metric best suited for *transient* signals.
- For our purposes, more useful than the energy spectral density is the *exposure spectral density* (XSD).

<u>Power Spectral Density (PSD)</u>

- The power spectral density (PSD) of a signal $x[n]$ is the signal's auto-spectrum expressed as power per Hz.
- The *power spectral density* of a signal $x[n]$ of duration N is given by:

$$\wp_{xx}[k] = \frac{P_{xx}[k]}{f_{bin}} = \frac{|X[k]|^2}{N f_s}$$

  where $f_{bin} = f_s/N$ is the bandwidth of each DFT bin.
- The units of $\wp_{xx}[k]$ are those of $x^2[n]$ divided by the frequency unit. For example, for $x[n]$ in Pa, $\wp_{xx}[k]$ is expressed in Pa^2/Hz.

- $\wp_{xx}[k]$ decreases as N increases, since the power P_x is shared by a larger number of DFT bins.
- From the Parseval relation, the power P_x can be calculated as the sum of the samples in the power spectral density $\wp_{xx}[k]$ multiplied by f_{bin}.

$$P_x = f_{bin} \sum_{k=0}^{N-1} \wp_{xx}[k]$$

- The *power spectral density* $\wp_{xx}[k]$ depicts the energy per 1-Hz band. This type of representation is best suited for *broadband, uncorrelated processes* that have no dominant narrowband components.
- Power is a metric best suited for long *continuous* signals.

Exposure Spectral Density (XSD)

- The *exposure spectral density* (XSD) of a signal $x[n]$ is the signal's auto-spectrum expressed as exposure per Hz.
- The *exposure spectral density* ξ_{xx} of a signal $x[n]$ of duration N is given by:

$$\xi_{xx} = \frac{\Xi_{xx}[k]}{f_{bin}} = \frac{|X[k]|^2}{f_s^2}$$

where $f_{bin} = f_s/N$ is the bandwidth of each DFT bin.

- The exposure spectral density ξ_{xx} is a variant of the energy spectral density E_{xx} for representing *sound exposure levels* (SEL), as opposed to the power spectral density, which can be used for representing *sound pressure levels* (SPL_{rms}).
- $\xi_{xx}[k]$ is independent of N.
- From the Parseval relation, the exposure Ξ_x can be calculated as the sum of the samples in the exposure spectral density $\xi_{xx}[k]$ multiplied by f_{bin}.

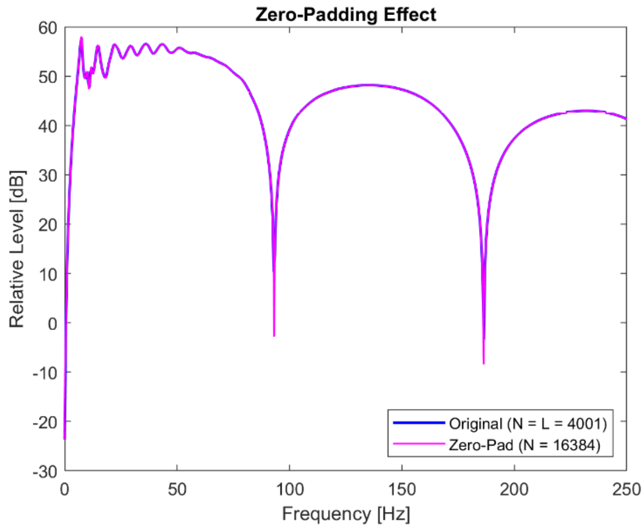
$$\Xi_x = f_{bin} \sum_{k=0}^{N-1} \xi_{xx}[k]$$

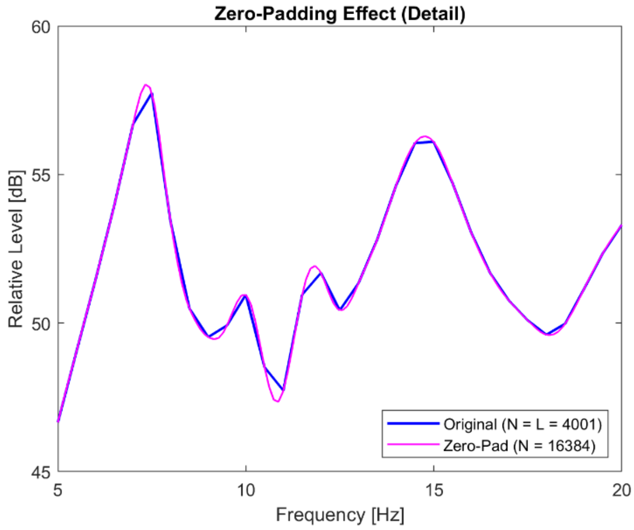
- The units of $\xi_{xx}[k]$ are those of $x^2[n]$ multiplied by the time unit and divided by the frequency unit. For example, for $x[n]$ in Pa, $\xi_{xx}[k]$ is expressed in Pa^2s/Hz.
- The *exposure spectral density* $\xi_{xx}[k]$ depicts the energy per 1-Hz band. This type of representation is best suited for *broadband, uncorrelated processes* that have no dominant narrowband components.
- Exposure is a metric suited for both *transient* and long *continuous* signals.

Correction for the Number of FFT Points: *Zero-Padding* and *Wrapping*

<u>Zero-Padding</u>

- *Zero-padding* is a technique used to increase the resolution of the frequency spectrum $X[k]$ by appending zeroes to the time-domain signal $x[n]$.
- The DFT represents the complex amplitude at each frequency point. Increasing the number of samples in $x[n]$ from its original length L to N by appending zeroes will not alter the frequency response of x_n, it will only improve its *frequency resolution*.
- Zero-padding is also useful for avoiding issues related to delays introduced by band-pass and high-pass filtering. Appending to a short signal $x[n]$ a number of zeros larger than the expected delay of the filter will prevent the filtered signal from drifting outside the processing window. The zero-padding will not alter the response of the filtered signal in any way other than increasing its frequency resolution.
- FFT algorithms are programmed to efficiently calculate the frequency response of a sequence of length equal to a power of 2. It is common practice to zero-pad $x[n]$ to a length N equal to the closest (but larger than) power of 2 of the original signal length L (e.g. $N = 4096$ for $L = 4000$).
- For $N > L$, MATLAB's function *fft.m* will apply zero-padding.
- For $N < L$, *fft.m* will truncate the signal. This will result in loss of potentially useful information which may noticeably affect the spectral response. To avoid information loss when $N < L$, we can use time *wrapping* (see next section).

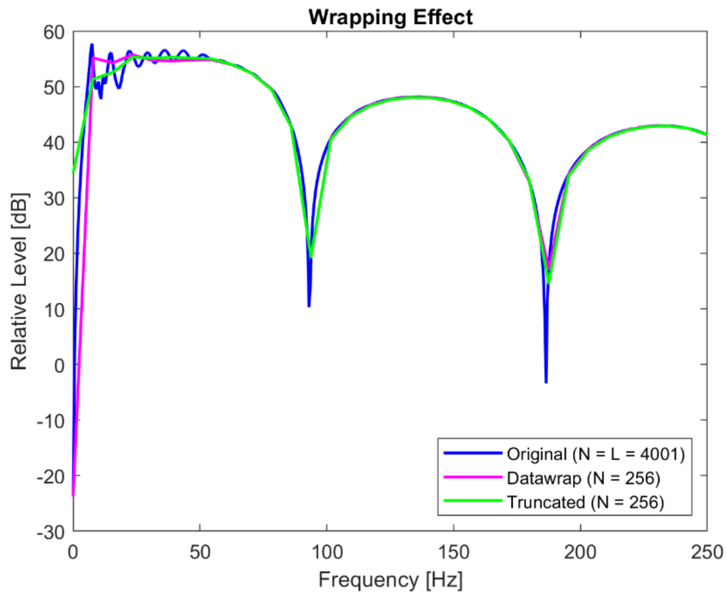




<u>Wrapping</u>

- *Wrapping* consists in dividing a discrete time-domain signal $x[n]$ of length L into multiple segments of length N and then adding all segments together. The last segment is padded with zeroes.
- A signal $x[n]$ is wrapped to make its frequency response $X[k]$ independent of the number of DFT points N. *Wrapping* is an alternative to truncation that conserves the frequency response of the original signal $x[n]$. Both wrapping and truncation result in lower frequency resolution ($N < L$).

- In the same way we use *zero-padding* to increase the resolution of the spectrum without altering its shape and amplitude, we use *wrapping* to reduce the resolution of the spectrum without altering it.
- MATLAB *datawrap.m* can be used for wrapping discrete time signals.
- In the figure below, the spectrum of the wrapped signal (purple) is identical to that of the original signal (blue) at the spectral points, whereas the spectrum of the truncated signal (green) deviates from it.



<u>Correction for N</u>

- The overall amplitude of some types of normalised frequency responses (ES, PS, XS, PSD) depends on the number of DFT points N. Performing the DFT on a signal $x[n]$ with a number of bins N *larger* (zero-padding) or *smaller* (wrapping) than the signal's original length L will spread the energy/power over a *larger* or *smaller* number of bins, effectively *reducing* or *increasing* the overall amplitude of the spectral curve.
- However, in the same way $|X[k]|$ is independent of N when using zero-padding and wrapping, we would also want the normalised frequency spectra to be independent of N. Using a number of DFT bins different to the length of the signal will affect the duration of the signal T and the resolution of its spectrum f_{bin}. These are undesirable effects, since at the end of the day, we want to obtain a univocal and consistent frequency response for the original signal $x[n]$.
- The solution is simple: replacing N with L on the normalised spectra (ES, PS, XS, PSD).
- In order to calculate the energy, power or exposure from the corrected normalised spectra, the correction will have to be reversed to obtain the correct spectral amplitude at each DFT bin. For example, $E'_{xx}[k] = |X[k]|^2/L$ will have to be multiplied by L/N before their values can be added up to obtain E_x. A rectangular window is assumed ($W = 1$); for window gain corrections see following chapter.
- The mean is calculated directly from the zero-sample of the corrected normalised spectra. For example, $\bar{x} = \sqrt{P'_{xx}[1]} = |X[1]|/L$. A rectangular window is assumed ($W = 1$).

Correction of the Window Gain

- *Windowing* a signal in time gives better control on the way narrowband features are represented in the spectrum.
- All windows have a spectral response characterised by a main lobe followed by secondary lobes of lower amplitude. Multiplying the time signal by the window is equivalent to convoluting the spectra of the signal and the window. Calculating the DFT of a signal $x[n]$ delimited by a window of finite length will produce a degree of smoothing in frequency, known as *leakage*.
- The *degree* and *behaviour* of the spectral leakage will depend on the *length* and *shape* of the window.
- *Longer windows* result in narrower lobes and less leakage, thus improving the representation of narrowband features. That is why it is preferrable to process the DFT over longer sequences. However, this is not always possible as longer signals (e.g., continuous noise) and high sampling rates are not always available.
- *Smoother windows* are characterised by a wider main lobe and larger attenuation on the secondary lobes. Sharp windows, like the *rectangular window*, are best suited for signals with well differentiated narrowband components (e.g., engine noise, sonar pulse), due to their narrow main lobe. Smooth windows, like the *hanning window*, are best suited for broadband random signals (e.g. ambient noise), due to the low amplitude of their secondary lobes, which minimises contamination from nearby frequencies.
- When a window is applied to a signal in time, its mean and variance are modified. The effect of the window will have to be compensated to obtain a true read of the normalised spectrum (ES, PS, XS, ESD, PSD, XSD), and an exact calculation of the energy, power, exposure and mean. The correction to be applied will depend on the type of signal (coherent, random, transient), spectrum type (energy, density), and parameter (energy, power, exposure, mean).
- A window should always be applied to the original signal $x[n]$ of length L, before zero-padding or wrapping to N DFT bins. Therefore, the window should always be length L.

<u>Correction for Broadband Random Signals</u>

- *Broadband random signals* are characterised by a relatively uniform broadband spectrum with no dominant narrowband components.
- To obtain the correct spectral amplitude, the broadband random signal $x[n]$ must be divided by the root-mean-square (RMS) of the window $w[n]$. This correction factor is known as the *noise window gain* W_{NG}:

$$W_{NG} = \mathrm{RMS}(w[n]) = \sqrt{\frac{1}{L}\sum_{n=0}^{N-1} w^2[n]}$$

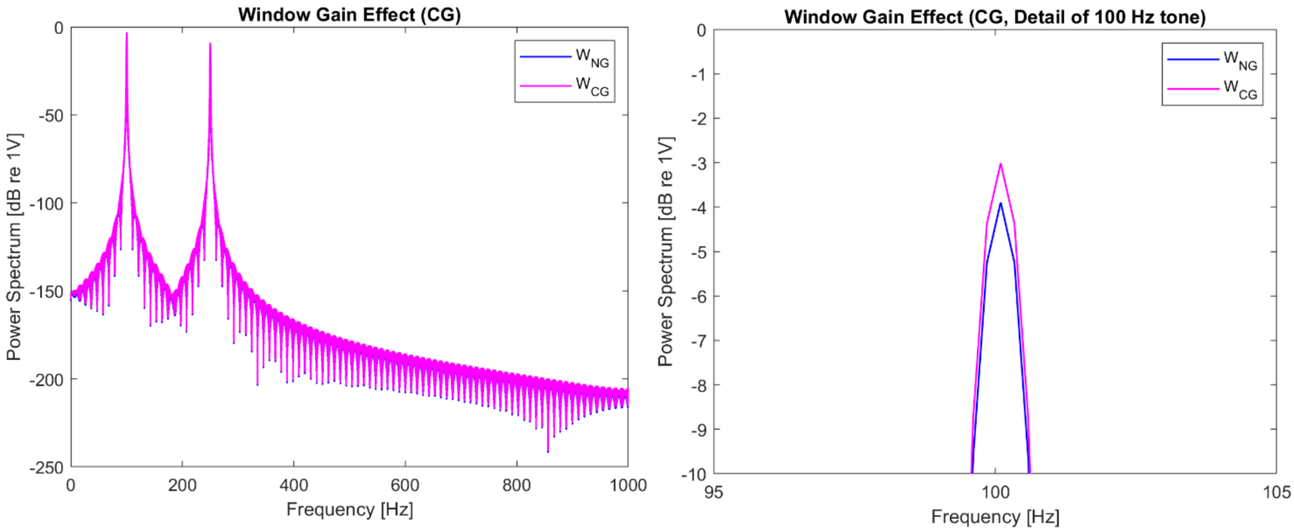
- The frequency response of broadband random signals is most commonly represented as a *density spectrum* (ESD, PSD or XSD). Since there are no dominant individual frequency components, the amplitude per Hz band is more relevant.
- MATLAB's function *periodogram.m* corrects for W_{NG} by default when the 'psd' option is selected. In other words, the function assumes that the input is a broadband random signal, as is typically the case when representing the power spectral density (PSD). If a continuous coherent signal is used instead, the result must be multiplied by W_{CG}/W_{NG}.

Correction for Narrowband Coherent Signals

- *Narrowband coherent signals* are characterised by dominant narrowband components.
- To obtain the correct spectral amplitude, the narrowband coherent signal $x[n]$ must be divided by the mean of the window $w[n]$. This correction factor is known as *coherent window gain* W_{CG}:

$$W_{CG} = \overline{w}[n] = \frac{1}{L}\sum_{n=0}^{N-1} w[n]$$

- The frequency response of narrowband coherent signals is most commonly represented as an *energy* or *power spectrum* (ES, PD). An energy or power spectrum will be more relevant, as it allows for a direct read of the energy or power of dominant frequency components.
- MATLAB's function *periodogram.m* corrects for W_{CG} by default when the 'power' option is selected. In other words, the function assumes that the input is a narrowband coherent signal, as is typically the case when representing the power spectrum (PS). If a broadband random signal is used instead, the result must be multiplied by W_{NG}/W_{CG}.
- In the Figure below is represented the power spectrum (PS) of a signal with two tones of peak amplitudes 1 and 0.5 V, corresponding to RMS levels of -3 dBV and -9 dBV. Note that only by normalising by W_{CG} the spectrum will show the exact power of the tonal components.



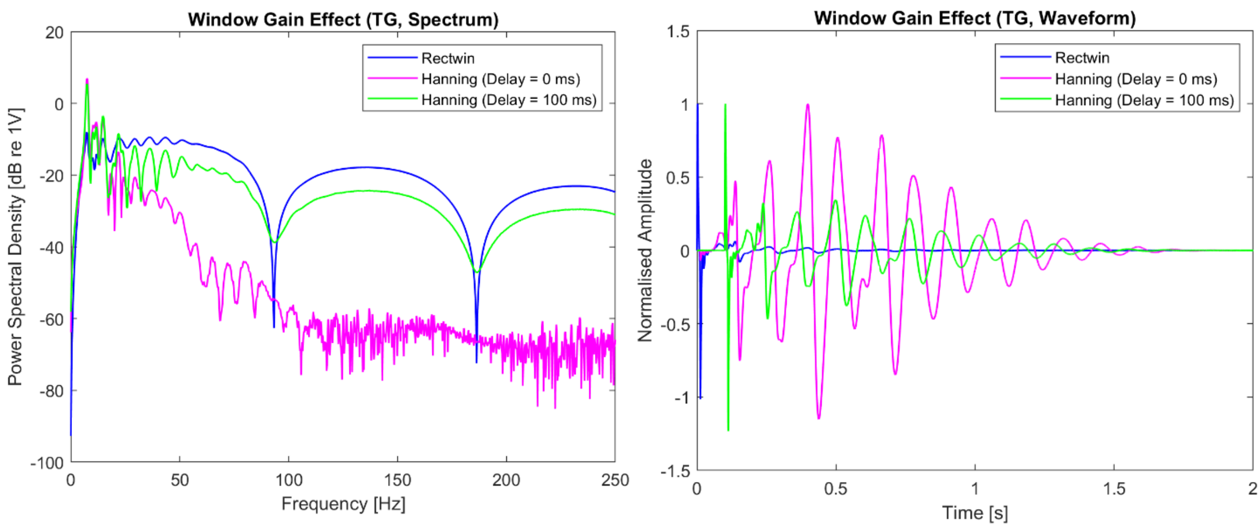
Correction for Transient Signals

- *Transient signals* are characterised by a broadband spectrum, with or without narrowband components.
- *Transient signals* are a special case. The reason is that transient signals have not reached a steady state where the waveform can be univocally characterised at any point in time. As a result, a transient signal triggered at the start of the window will be weighted in a completely different way to that same signal trigged at the centre of the window. This has two effects: 1) The overall attenuation effect from windowing cannot be predicted from the window itself, and most importantly 2) Some frequency components may be weighted more strongly than others due to the time-dependence of the frequency response.
- It is strongly recommended to use a non-attenuating window such as the *rectangular window*. All other windows should only be used when the frequency content of the signal is steady across its entire length, for example, when processing the 90% energy window (T90).

- To obtain the correct spectral amplitude, the transient signal $x[n]$ must be divided by a factor that depends on the window $w[n]$ and the signal itself. I call this correction factor the *transient window gain* W_{TG}:

$$W_{TG} = \sqrt{\frac{\sum_{n=0}^{N-1} x^2[n]w^2[n]}{\sum_{n=0}^{N-1} x^2[n]}} = \frac{\mathrm{RMS}(x[n]w[n])}{\mathrm{RMS}(x[n])}$$

- For a random process $x[n]$, $W_{TG} = W_{NG}$.
- The frequency response of transient signals may be represented as an *energy* or *power spectrum* (ES, PS) or *density spectrum* (ESD, PSD) depending on its frequency content. For general transient signals (e.g. airgun, piling), a density spectrum will be more relevant; for transient signals with dominant tonal content (e.g. sonar), a power or energy spectrum may be more appropriate.
- In the figure below is represented the power spectral density (PSD) of an airgun signature processed with a rectangular window (blue) and a hanning window (purple, green). a delay of 100 ms was applied to $x[n]$ for the calculation of the green line. It can be observed that for the hanning window with 0 ms delay (purple) most of the high frequency content, which is known to occur at the start of the airgun pulse, is cancelled. The PSD of the delayed airgun signature with hanning window (green) is similar to the PSD of the original signal with rectangular window, since the main transient experiences lower attenuation. It becomes evident that highly attenuating windows will alter the frequency content of transient signals. To avoid a spectral behaviour that is dependent on the position of the signal within the processing window, a minimum attenuation window such as *rectangular* or *hamming* must be used.



<u>Correction for Power, Energy, and Exposure Calculations</u>

- For calculating the power, energy or exposure of a signal $x[n]$, it must be divided by W_{TG}.
- Note that W_{TG} works for any type of signal, including transients and steady state signals. For the latter, $W_{TG} = W_{NG}$.

<u>Correction for DC Offset Calculation</u>

- For calculating the mean of a signal $x[n]$, it must be divided by W_{CG}.

Summary of Corrected Frequency Responses

Table 1 Formulas for the calculation of the normalised energy, power, and exposure spectra. The standard spectrum should only be applied to DFTs with number of bins equal to the length of the signal ($N = L$). The standard and corrected spectra are weighted by the appropriate window gain W.

	ES ($E_{xx}[k]$)	**PS** ($P_{xx}[k]$)	**XS** ($\Xi_{xx}[k]$)
Standard Spectrum ($S_{xx}[k]$)	$\frac{\lvert X[k]\rvert^2}{W^2 N}$	$\frac{\lvert X[k]\rvert^2}{W^2 N^2}$	$\frac{\lvert X[k]\rvert^2}{W^2 N f_s}$
Corrected Spectrum ($S'_{xx}[k]$)	$\frac{\lvert X[k]\rvert^2}{W^2 L}$	$\frac{\lvert X[k]\rvert^2}{W^2 L^2}$	$\frac{\lvert X[k]\rvert^2}{W^2 L f_s}$
Correction Factor (K)	$\frac{N}{L}$	$\frac{N^2}{L^2}$	$\frac{N}{L}$
Units	Pa^2	Pa^2	$\mathrm{Pa}^2\mathrm{s}$

Table 2 Formulas for the calculation of the normalised energy, power, and exposure spectral densities. The standard spectrum should only be applied to DFTs with number of bins equal to the length of the signal ($N = L$). The standard and corrected spectra are weighted by the appropriate window gain W.

	ESD ($\mathcal{E}_{xx}[k]$)	**PSD** ($\wp_{xx}[k]$)	**XSD** ($\xi_{xx}[k]$)
Standard Spectrum ($S_{xx}[k]$)	$\frac{\lvert X[k]\rvert^2}{W^2 f_s}$	$\frac{\lvert X[k]\rvert^2}{W^2 N f_s}$	$\frac{\lvert X[k]\rvert^2}{W^2 f_s^2}$
Corrected Spectrum ($S'_{xx}[k]$)	$\frac{\lvert X[k]\rvert^2}{W^2 f_s}$	$\frac{\lvert X[k]\rvert^2}{W^2 L f_s}$	$\frac{\lvert X[k]\rvert^2}{W^2 f_s^2}$
Correction Factor (K)	1	$\frac{N}{L}$	1
Units	$\mathrm{Pa}^2/\mathrm{Hz}$	$\mathrm{Pa}^2/\mathrm{Hz}$	$\mathrm{Pa}^2\mathrm{s}/\mathrm{Hz}$

Table 3 Formulas for the calculation of the energy, power, exposure and mean of a signal $x[n]$ from its waveform and from its different normalised standard spectra (ES, PS, XS).

	Time	**ES** ($E_{xx}[k]$)	**PS** ($P_{xx}[k]$)	**XS** ($\Xi_{xx}[k]$)
Energy (E_x)	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\sum_{k=0}^{N-1} E_{xx}[k]$	$N \sum_{k=0}^{N-1} P_{xx}[k]$	$f_s \sum_{k=0}^{N-1} \Xi_{xx}[k]$
Power (P_x)	$\frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{1}{N} \sum_{k=0}^{N-1} E_{xx}[k]$	$\sum_{k=0}^{N-1} P_{xx}[k]$	$\frac{f_s}{N} \sum_{k=0}^{N-1} \Xi_{xx}[k]$
Expos. (Ξ_x)	$\frac{1}{f_s} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{1}{f_s} \sum_{k=0}^{N-1} E_{xx}[k]$	$\frac{N}{f_s} \sum_{k=0}^{N-1} P_{xx}[k]$	$\sum_{k=0}^{N-1} \Xi_{xx}[k]$
Mean ($\bar{x}$)	$\frac{1}{N} \sum_{n=0}^{N-1} x[n]$	$\frac{W\sqrt{N}}{W_{CG}L} \sqrt{E_{xx}[1]}$	$\frac{WN}{W_{CG}L} \sqrt{P_{xx}[1]}$	$\frac{W\sqrt{f_s N}}{W_{CG}L} \sqrt{P_{xx}[1]}$

Table 4 Formulas for the calculation of the energy, power, exposure and mean of a signal $x[n]$ from its waveform and from its different normalised standard spectral densities (ES, PS, XS).

	Time	**ESD** ($\mathcal{E}_{xx}[k]$)	**PSD** ($\wp_{xx}[k]$)	**XSD** ($\xi_{xx}[k]$)
Energy (E_x)	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{f_s}{N} \sum_{k=0}^{N-1} \mathcal{E}_{xx}[k]$	$f_s \sum_{k=0}^{N-1} \wp_{xx}[k]$	$\frac{f_s}{T} \sum_{k=0}^{N-1} \mathcal{E}_{xx}[k]$
Power (P_x)	$\frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{f_s}{N^2} \sum_{k=0}^{N-1} \mathcal{E}_{xx}[k]$	$\frac{f_s}{N} \sum_{k=0}^{N-1} \wp_{xx}[k]$	$\frac{f_s^2}{N^2} \sum_{k=0}^{N-1} \mathcal{E}_{xx}[k]$
Expos. (Ξ_x)	$\frac{1}{f_s} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{1}{N} \sum_{k=0}^{N-1} E_{xx}[k]$	$\sum_{k=0}^{N-1} \wp_{xx}[k]$	$\frac{f_s}{N} \sum_{k=0}^{N-1} \mathcal{E}_{xx}[k]$
Mean ($\bar{x}$)	$\frac{1}{N} \sum_{n=0}^{N-1} x[n]$	$\frac{W\sqrt{f_s}}{W_{CG}L} \sqrt{\mathcal{E}_{xx}[1]}$	$\frac{W\sqrt{f_s N}}{W_{CG}L} \sqrt{\wp_{xx}[1]}$	$\frac{W f_s}{W_{CG}L} \sqrt{\xi_{xx}[1]}$

Calculating the energy, power, exposure, or mean from a normalised spectrum (ES, PS, XS, ESD, PSD, XSD) is straightforward. However, the spectral curve alone is not sufficient and the following should be known:

- Spectral values $S_{xx}[k]$ or $S'_{xx}[k]$
- Duration of original signal T
- Sampling rate f_s
- Window.
- Window normalisation (NG, CG).
- Was duration normalisation applied?

Code

```
% Window Signal
xwin = x.*win;

% Calculate Window Gain
switch signalType
    case 'random'
        winGain = rms(win);
    case 'coherent'
        winGain = mean(win);
    case 'transient'
        winGain = rms(xwin)/rms(x);
end

% Calculate Normalising Factor
switch specType % Spectrum type
    case 'ene' % Energy Spectrum (ES)
        normFactor = nSamples;
    case 'pow' % Power Spectrum (PS)
        normFactor = nSamples*nSamples;
    case 'exp' % Exposure Spectrum (XS)
        normFactor = nSamples*fs;
    case 'esd' % Energy Spectral Density (ESD)
        normFactor = fs;
    case 'psd' % Power Spectral Density (PSD)
        normFactor = nSamples*fs;
    case 'xsd' % Exposure Spectral Density (XSD)
        normFactor = fs*fs;
end

% Normalised Spectrum (Two Sided)
xwin = datawrap(xwin,nfft); % time wrapping
xfft = fft(xwin)/winGain; % DFT spectrum (window gain corrected)
X = xfft.*conj(xfft)/normFactor; % normalised two-sided spectrum
f = (0:fs/nfft:fs-fs/nfft); % frequency vector (0 to fs)
end

% Normalised Spectrum (Centered)
xwin = datawrap(xwin,nfft); % time wrapping
xfft = fft(xwin)/winGain; % DFT spectrum (window gain corrected)
X = xfft.*conj(xfft)/normFactor; % normalised two-sided spectrum
X = fftshift(X); % centre spectrum
f = (0:fs/nfft:fs-fs/nfft); % frequency vector (positive)
f = f - fs/2 + fs/nfft; % frequency vector (-fs/2 to fs/2)

% Normalised Spectrum (One Sided)
halfPoint = ceil((nfft + 1)/2); % end sample of one-sided spectrum
xwin = datawrap(xwin,nfft); % time wrapping
xfft = fft(xwin)/winGain; % DFT spectrum (window gain corrected)
xfft = xfft(1:halfPoint); % one-sided DFT spectrum
X = xfft.*conj(xfft)/normFactor; % normalised two-sided spectrum
X(2:end,:) = 2*X(2:end,:); % double amplitude of non-DC frequencies
f = (0:fs/nfft:fs/2); % frequency vector (0 to fs/2)
```