Tesi doctoral en Matemàtiques

HIGHER LIMITS VIA HOMOTOPICAL ALGEBRA

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INTRODUCTION

Derived functors of limits and colimits, also known as higher limits and higher colimits, are powerful tools that arise naturally in various problems related to homotopy theory, homological algebra, and combinatorics. They can be seen as a generalised version of cohomology (resp. homology) with twisted coefficients in a functor F for a small category $\mathcal{C}$,

$$R^i(\lim(-))(F) = H^i(\mathcal{C}; F), \qquad L_i(\operatorname{colim}(-))(F) = H_i(\mathcal{C}; F).$$

In the realm of homotopy theory, higher limits have proven to be an invaluable tool for studying the cohomology groups of spaces that are constructed piece-wise. Bousfield and Kan's seminal work [BK72] established a deep connection between algebraic topology and simplicial methods in homological algebra by describing a spectral sequence that relates the cohomology groups of the homotopy colimit of a functor with the limit of the cohomology groups of the functor. The initial page of these spectral sequences can be expressed in terms of higher limits:

$$H^i(\mathcal{C}; H^j(F)) \Longrightarrow H^{i+j}(\operatorname{hocolim}_{\mathcal{C}} F).$$

This implies that the lim-acyclicity of the functor $H^j(F)$, for every $j \geqslant 0$, guarantees that cohomology and homotopy colimit commute.

This spectral sequence yields an interesting obstruction theory to determine the existence and uniqueness of maps from a homotopy colimit. This dates back to Wojtkowiak's work [Woj87]. In detail, let $F\colon \mathcal{C} \to \mathrm{Top}$ be a functor over a small category $\mathcal{C}$. The restriction morphism gives rise to a map:

$$\left[\operatorname*{hocolim}_{c\in\mathcal{C}} F(c), X\right] \longrightarrow \lim_{c\in\mathcal{C}}[F(c), X], \tag{1}$$

that decomposes a continuous map $f\colon \operatorname{hocolim} F \to X$ into a compatible tuple, up to homotopy, $(f_c\colon F(c) \to X)_{c\in\mathcal{C}}$ of continuous maps. Then, it is natural to ask if a compatible tuple of maps (f_c) lifts to a map $f\colon \operatorname{hocolim} F \to X$. This question has a positive answer if the $(n+1)$-st higher limit of the functor $\alpha_n\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$, described by $\alpha_n(c) = \pi_n(\operatorname{map}(F(c), X), f(c))$, vanish, that is,

$$H^{n+1}(\mathcal{C}, \alpha_n) = 0, \text{ for all } n > 0.$$

Moreover, the uniqueness of the map f is related to the vanishing of the n-th higher limit of the same functor.

This fundamental concept has been well studied and used successfully in the literature; see Jackowski, McClure, and Oliver's survey [JMO94]. There are more recent examples of related problems, such as the obstruction for the existence and uniqueness of the classifying space of a fusion system, as presented in Broto, Levi, and Oliver's work [BLO03], or the existence of homotopy representations for compact p-local groups, shown by Cantarero and Castellana [CC17].

These examples demonstrate the fundamental need to provide conditions of acyclicity for functors or, at least, establish vanishing bounds for their higher limits. The most significant results for the lim-acyclicity of functors over orbit categories are associated with Mackey functors [JM92, DRP15], and Lambda functors [JMO92a, JMO92b]. Concerning posets, the Mittag-Leffler conditions are widely recognised [Wei94], and there are also conditions related to projectivity [DR09] or with lower factoring sections [KL22]. Futhermore, higher limits over a category can be reduced to higher limits over posets through a spectral sequence [Sło01].

In sheaf theory, higher limits over posets play a central role since they compute the sheaf cohomology. Note that finite T_0-spaces can be identified with finite posets via the *Alexandroff topology*, and by McCord's theorem [McC66], every simplicial complex has a finite model, up to *weakly homotopy equivalence*. Therefore, sheaves over finite $T0$-spaces or simplicial complexes can be seen as sheaves over posets.

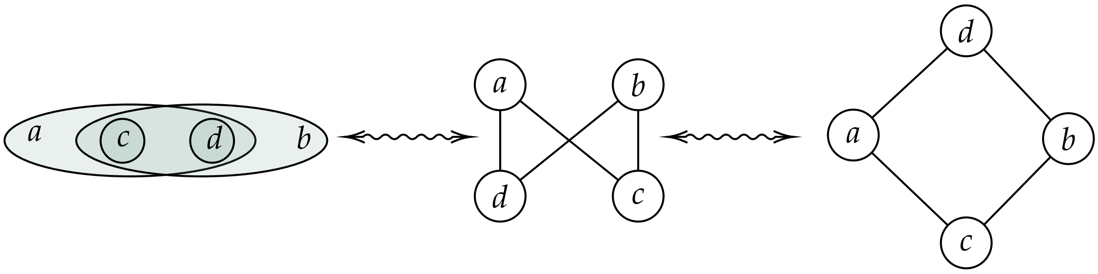

Figure 1: Finite model of the sphere S^1.

Under this identification, if $\mathcal{P}$ is a finite T_0-space, the cohomology of a sheaf $F\colon \mathrm{Open}(\mathcal{P})^{\mathrm{op}} \to \mathrm{Mod}_R$ is isomorphic to the higher limits of $\widehat{F}\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$, where $\widehat{F}$ is the composite of restriction of F to the minimal open sets $\mathcal{P}_{\leqslant p}$ followed by the projection $\mathcal{P}_{\leqslant p} \mapsto p$.

$$H^*(\mathrm{Open}(\mathcal{P}); F) \cong H^*(\mathcal{P}; \widehat{F}).$$

This manifests that it is not only fundamental to have vanishing bounds of higher limits but also to be able to compute them explicitly. Some examples of computational techniques are provided by Everitt-Turner for cellular posets [ET15] and for geometric lattices [ET22b]. Additionally, Curry's thesis [Cur14] has shown how sheaf cohomology (higher limits) over finite posets is an extraordinary tool in data analysis and engineering.

In this thesis, we develop a new technique to compute higher limits and colimits using tools from homotopy theory, replacing the classical injective (resp. projective) resolution [Wei94, Chapter 2] that produces the cochain complex associated with the functor [AKO11, Section III.5.1] with a fibrant (resp. cofibrant) replacement that we construct explicitly.

Categories that naturally emerge in homotopy theory or group theory are categories in which every endomorphism is an isomorphism, and most of them are equipped with a $\mathbb{N}$-filtration; we name these categories *filtered EI-categories*, see [Lüc89]. By Berger and Moerdijk's work [BM11], filtered EI-categories admit a generalised Reedy structure. Given a filtered EI-category, a commutative ring with unit R is said to be $\mathcal{C}$-bijective if and only if $|\operatorname{Aut}(c)|$ is invertible in R for every $c \in \mathcal{C}$. From now on, $\mathcal{C}$ will be a filtered EI-category and R a $\mathcal{C}$-bijective ring.

The first step to setting up our homotopy theoretical framework is to identify the category of functors $\operatorname{Fun}(\mathcal{C}, \operatorname{Mod}_R)$ with the full subcategory of those functors in $\operatorname{Fun}(\mathcal{C}, \operatorname{Ch}(R))$ which take values concentrated in degree 0. Next, using the generalised Reedy structure on $\mathcal{C}$ and the $\mathcal{C}$-bijectiveness of R, we introduce a model category structure on the category of functors $\operatorname{Fun}(\mathcal{C}, \operatorname{Ch}(R))$, which we named the direct model category structure, see Proposition 5.1.8, in which higher colimits can be computed by a cofibrant replacement.

Proposition 5.1.15 *Let $\mathcal{C}$ be a filtered EI-category, and R be a $\mathcal{C}$-bijective ring. Given a functor $F\colon \mathcal{C} \to \operatorname{Mod}_R$, then,*

$$H_i(F;\mathcal{C}) = H_i(\operatorname{colim} \mathbf{Q}F)$$

where $\mathbf{Q}F\colon \mathcal{C} \to \operatorname{Ch}(R)$ is a cofibrant replacement of F in the direct model category structure.

As a direct corollary, we obtain that every cofibrant functor is colim-acyclic. A functor $F\colon \mathcal{P} \to \operatorname{Mod}_R$ over a filtered poset is cofibrant in the direct model category if and only if the natural morphism

$$\operatorname*{colim}_{q<p} F(q) \to F(p)$$

is injective for every $p \in \mathcal{P}$.

One of the consequences of this result is that we characterise pseudo-projective functors over filtered posets [DR09] as cofibrant functors in this model category structure.

The functor F is said to be *pseudo-projective* if for every $p \in \mathcal{P}$ and every element $\oplus_{q<p} x_q \in \bigoplus_{q<p} F(q)$, the condition:

$$\sum_{q<p} F(q < p)(x_q) = 0$$

implies that $x_q \in \mathrm{Im}_F(q) = \sum_{k<q} \mathrm{Im}\, F(k < q)$ for every $q \in \max\{q < p \mid x_q \neq 0\}$.

Theorem 6.10 *Let $\mathcal{P}$ be a filtered poset, and $F\colon \mathcal{P} \to \mathrm{Ab}$ be a functor. Then F is cofibrant if and only if it is pseudo-projective.*

Moreover, we define Mackey functors over posets inspired by the classical Mackey functors over the orbit category of a group but restricted to the subcategory of normal subgroups. Finally, define weak Mackey functors by dropping the contravariant functoriality and the meet-semilattice constraint. With this definition, we can now prove the following theorem.

Theorem 6.8 *Let $\mathcal{P}$ be a filtered poset and $F\colon \mathcal{P} \to \mathrm{Ab}$ be a weak Mackey functor with a quasi-unit. Then, F is pseudo-projective, and hence, it is* colim*-acyclic.*

Dualising, we jump to the category of contravariant functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}(R))$ where the inverse model category structure Proposition 5.1.3 allows us to compute higher limits via fibrant replacement, see Proposition 5.1.12. As a direct corollary, we obtain that given a functor F, a bound for the *height* of its fibrant replacement $\mathbf{R}F$ induces a vanishing bound for its higher limits.

In general, computing fibrant replacement is not an easy task. Nevertheless, one of the main advantages of our method is that, by the combinatorial structure of the generalised Reedy category, we construct a fibrant replacement of a given functor inductively. Roughly speaking, given a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_R$, constructing a fibrant replacement becomes, for every $c \in \mathcal{C}$, a factorisation problem. The method consists of choosing by induction on the filtration, for every object $c \in \mathcal{C}$, a cochain complex $\mathbf{R}F(c)$ together with two morphisms, an epimorphism

$\mathbf{R}F(c) \to \lim_{s<c} \mathbf{R}F(s)$, and a morphism that induces isomorphism in cohomology $F(c) \to \mathbf{R}F(c)$, such that the following square commutes:

$$\begin{tikzcd}
F(c) \arrow[r] \arrow[d, dashed, "\sim"] \arrow[dr, dashed] & \lim_{s\to c} F(s) \arrow[d] \\
\mathbf{R}F(c) \arrow[r, two heads] & \lim_{s\to c} \mathbf{R}F(s).
\end{tikzcd}$$

The mapping cocylinder of the composite, see Definition 5.2.3, provides a standard choice. However, this choice is not always the best for providing vanishing bounds since it increases the height of the functor in each step. To solve this inconvenience, we provide a *truncated version* of the mapping cocylinder that does not increase the height of the functor. As one can expect, this truncated version cannot always be used. But, when possible, it is the key in the proof of the vanishing bounds that we describe next, and it is based on the following result.

Corollary 5.1.14 *Let $\mathcal{C}$ be a filtered EI-category, R be a $\mathcal{C}$-bijective ring, and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a functor. If $\mathbf{R}F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}(R)$ is a fibrant replacement of F such that $h(\mathbf{R}F) = n$, then*

$$H^i(\mathcal{C}; F) = 0$$

for every $i > n$.

The first vanishing bound presented in this thesis is about the combinatorial aspects of the given poset. The strategy consists of labelling the poset with a function that indicates the possibility of using the truncated version of the mapping cocylinder. The labelling of a poset $\mathcal{P}$ is a function $B\colon \mathcal{P} \to \mathbb{N}$ that coincides with the degree for objects of degree 0 and 1 and, inductively, for an object $p \in \mathcal{P}$, let $n = \max_{q<p} B(q)$. If p *closes a circuit* using only the objects with labels n and $n-1$, then we label $B(p) = n+1$; otherwise, we label $B(p) = n$; see Figure 2.

Theorem 7.1.4 *Let $\mathcal{P}$ be a filtered poset, and $B\colon \mathcal{P} \to \mathbb{N}$ its associated labelling function. For every functor $F : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$,*

$$H^i(\mathcal{P}; F) = 0,$$

if $i > \sup B$.

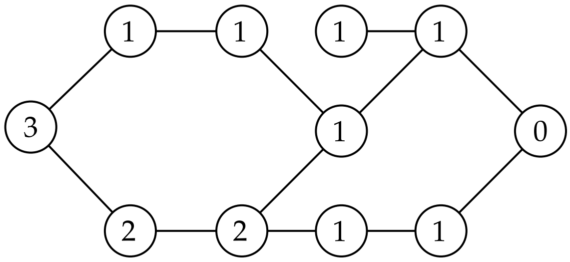

Figure 2: A labelled poset.

A direct application of this result is that we can produce a vanishing bound for a poset $\mathcal{P}$ that depends on a filtration $\mathcal{P}_0 \subset \cdots \subset \mathcal{P}_n = \mathcal{P}$, where $\mathcal{P}_0$ is a maximal tree of $\mathcal{P}$ and , fixed a degree n_i, $\mathcal{P}_i$ is obtained by adding to $\mathcal{P}_{i-1}$ every missing arrows whose codomain have degree n_i; see Figure 3 where the degrees are indicated vertically. Thus, the number of elements in this filtration induces a

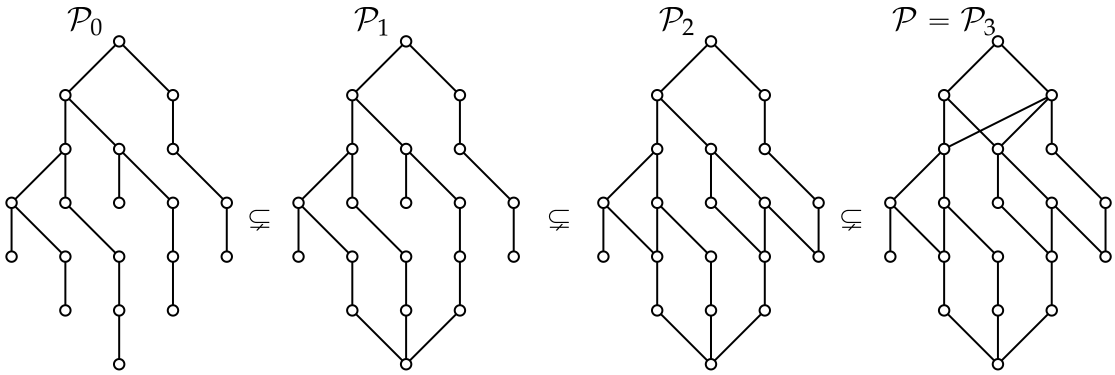

Figure 3

vanishing bound for every functor $\mathcal{P} \to \mathrm{Ab}$.

Theorem 7.1.7 *Let $\mathcal{P}$ be a filtered poset with degree function $\mathrm{d}\colon \mathcal{P} \to \mathbb{N}$, and $\mathcal{T}$ be a maximal tree of $\mathcal{H}$, the Hasse diagram of $\mathcal{P}$. Let $D(\mathcal{T}) = \#\{\mathrm{d}(q) \mid p \to q \in \mathcal{H} \setminus \mathcal{T}\}$. Then, for every functor $F : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$:*

$$H^i(\mathcal{P}; F) = 0$$

for every $i > 2D(\mathcal{T}) + 1$.

This technique also produces a *local to global bound*, describing the vanishing of the higher limits of a functor $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ in terms of the sub-functors $F|_{\mathcal{P}_{<p}}$ for every $p \in \mathcal{P}$.

Theorem 7.2.1 *Let $\mathcal{P}$ be a filtered poset, and $F : \mathcal{P}^{\mathrm{op}} \to \mathsf{Ab}$ be a functor. If for every $p \in \mathcal{P}$, $H^k(\mathcal{P}_{<p}; F) = 0$ for every $k \geqslant n$, then:*

$$H^k(\mathcal{P}; F) = 0$$

for every $k > n$.

The last vanishing bound presented in this text relates the higher limits of a functor $F \colon \mathcal{P}^{\mathrm{op}} \to \mathsf{Ab}$ with the ordinary cohomology with coefficients in abelian groups of a family of subposet of $\mathcal{P}$.

A frequent method in the literature is filtering a functor by subfunctors such that their successive quotients take the value zero except on one object; see, for example, [JMO92b, Proposition 5.6] or [BLO03, Corollary 3.4]. These are called atomic functors. In the case of posets, given a filtered poset $\mathcal{P}$ and an abelian group A, the *atomic functor* of A at $p_0 \in \mathcal{P}$, denoted by $\mathcal{A}(A, p_0) : \mathcal{P}^{\mathrm{op}} \to \mathsf{Ab}$, is the functor defined by:

$$\mathcal{A}(A, p_0)(p) = \begin{cases} A & \text{if } p = p_0 \\ 0 & \text{otherwise.} \end{cases}$$

We first show that the higher limits of an atomic functor $\mathcal{A}(A; p_0)$ are isomorphic to the reduced cohomology of the nerve of $\mathcal{P}_{>p_0}$ shifted by 1, i.e,

$$H^i(\mathcal{P}; \mathcal{A}(A, p_0)) \cong \widetilde{H}^{i-1}(|\mathcal{P}_{>p_0}|; A).$$

Now, using the technique of filtering a functor $F \colon \mathcal{P}^{\mathrm{op}} \to \mathsf{Ab}$ into subfunctors

$$F^0 \subset \cdots \subset F^n = F$$

such that its successive quotients are isomorphic to a direct sum of atomic functors

$$F^k / F^{k-1} \cong \bigoplus_{\mathrm{d}(p) = k} \mathcal{A}(F(p); p)$$

we prove the following result.

Theorem 7.3.4 *Let $\mathcal{P}$ be a finite filtered poset and $F : \mathcal{P}^{\text{op}} \to \text{Ab}$ be a functor. If there exists $k > 0$ such that, for every $p \in \mathcal{P}$, $\widetilde{H}^n(|\mathcal{P}_{>p}|; F(p)) = 0$, for $n > k$; then*

$$H^n(\mathcal{P}; F) = 0 \textit{ for } n > k.$$

The last contribution of this thesis is about the higher limits of functors over dual CL-shellable posets. By the geometric interpretation of this context, we discuss this problem in terms of sheaves and sheaf cohomology.

Roughly speaking, a bounded, finite and pure poset $\mathcal{P}$ is said to be dual CL-shellable if, for every irreducible chain,

$$c_0 \prec c_1 \prec c_2 \prec \ldots c_{n-1} \prec \hat{1},$$

where $\hat{1}$ is the maximum of $\mathcal{P}$, there exists a linear order $\ll_c$ in the set of elements that are covered by c_0, $\mathcal{P}_{\prec c_0}$, that is compatible with the order induced by the chain obtained from c by omitting c_0 and provides some connectivity properties. This notion was originally presented by Björner and Wachs [BW82] to show that Bruhat order of Coxeter groups is a shellable poset.

One of the most relevant properties of shellable posets is that they have the homotopy type of a wedge of k $(\dim|\mathcal{P}|)$-spheres for some k that depends on $\mathcal{P}$. Therefore, the constant sheaf on R over a CL-shellable poset, $\underline{R}\colon \mathcal{P}^{\text{op}} \to \text{Mod}_R$, verifies:

$$H^i(\mathcal{P}\backslash\{\hat{0},\hat{1}\}; \underline{R}) \cong \begin{cases} R^k & \text{if } i = \dim|\mathcal{P}|, \\ R & \text{if } i = 0, \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

Motivated by this example, we abstract the essential property of the constant sheaf. If $F\colon \mathcal{P}^{\text{op}} \to \text{Mod}_R$ is a sheaf over a dual CL-shellable poset, one says that F has the *stability property* at codegree n, if for every object $p \in \mathcal{P}$ of codegree n, and every $Q \subset \mathcal{P}_{\prec p}$ satisfying some mild compatibility properties, the natural morphism

$$F(p) \to \lim_{\langle Q \rangle} F$$

is an epimorphism.

Theorem 8.2 *Let $\mathcal{P}$ be a dual CL-shellable poset of degree $d \geqslant 2$, $n \in \mathbb{N}$ such that $1 \leqslant n \leqslant d-1$, and $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a sheaf. If the pair $(\mathcal{P}, F)$ has the stability property in codegree n, then*

$$H^n(\mathcal{P} \backslash \{\hat{1}\}; F) = 0.$$

Examples of sheaves verifying this property are provided by the i-linear forms sheaves in a hyperplane arrangement. Let V be a finite-dimensional k-vector space and $\mathcal{H}$ be a finite set of hyperplanes of V. The arrangement lattice of $\mathcal{H}$, denoted by $L_{\mathcal{H}}$, is the intersection lattice generated by the set $\mathcal{H}$ with the empty intersection being V, ordered by inclusion. The lattice structure is given by:

$$\hat{0} = \bigcap_{h \in \mathcal{H}} h, \qquad \hat{1} = V, \qquad \mathrm{d}(x) = \dim(x) - \dim(\hat{0}),$$

$$x \wedge y = x \cap y, \quad x \vee y = \bigcap \{z \in L_{\mathcal{H}} \mid x \cup y \subset z\}.$$

For every $i \geqslant 1$, we define the *i-linear forms sheaf* on $L_{\mathcal{H}}$ to be the sheaf $\Lambda^i(-)^*\colon L_{\mathcal{H}}^{\mathrm{op}} \to \mathrm{Vect}_k$ that sends every $W \in L_{\mathcal{H}}$ to the i-linear forms of W, i.e., the i-th exterior product:

$$\Lambda^i \operatorname{Hom}(W, k) = \Lambda^i W^*,$$

and $W' < W$ to the restriction $\Lambda^i W^* \twoheadrightarrow \Lambda^i (W')^*$.

Theorem 8.2.1 *Let V be a finite-dimensional vector space, $\mathcal{H}$ be a finite set of hyperplanes of V, and $L_{\mathcal{H}}$ the arrangement lattice of $\mathcal{H}$. For every $j < \mathrm{d}(L_{\mathcal{H}}) - i - 2$, the j-th cohomology of the i-linear forms sheaf on $L_{\mathcal{H}} \backslash \{\hat{1}\}$ vanishes, that is:*

$$H^j(L_{\mathcal{H}} \backslash \{\hat{1}\}; \Lambda^i(-)^*) = 0.$$

OUTLINE OF THIS THESIS– This thesis is divided into two parts. In Part I, we summarise the results about posets and EI-categories, including some proofs about hyperplane arrangements, Chapter 1; limits, colimits and Kan extensions, Chapter 2; model categories, Chapter 3; and Reedy structures, Chapter 4.

In Part II, we expose the results of the thesis, and it is divided into 4 chapters. In Chapter 5, we describe two model categories for the category of functors that are useful to compute higher limits and colimits and essential for the rest of the thesis. In Chapter 6, we define Mackey functors for posets, and we show that, in the case of Mackey functors with quasi-unit, their covariant parts are cofibrant functors. We characterise pseudo-projective functors as cofibrant objects. In Chapter 7, we introduce some vanishing bounds: the first is provided just by the combinatorics aspect of the poset; the second one by a local-to-global method; and the last one by comparing the functor with the higher limits of a family of atomic functors. Finally, in Chapter 8, mimicking the constant functor, we abstract a condition that implies that the higher limits of a functor vanish in non-extreme dimensions. We conclude that chapter by showing that the family of i-linear forms in an arrangement lattice satisfy the stability property described in this chapter.

Part I

PRELIMINARIES

Chapter 1

POSETS AND EI-CATEGORIES

This chapter summarises the basic results about posets and EI-categories that we will need in the following chapters. We assume familiarity with the basic notions of Category Theory, such as categories, functors, natural transformation, and adjointness. If not, we refer the reader to Mac Lane's book [ML98] or Leinster's book [Lei14].

A *partially ordered set* (poset for short) is a set $\mathcal{P}$ equipped with a binary relation $\leqslant$ that is:

REFLEXIVE: $p \leqslant p$ for all $p \in \mathcal{P}$,

TRANSITIVE: if $p \leqslant q$ and $q \leqslant r$, then $p \leqslant r$, and

ANTISYMMETRIC: if $p \leqslant q$ and $q \leqslant p$, then $p = q$.

Given a poset $\mathcal{P}$, we can construct a category whose objects are the elements in $\mathcal{P}$, and there is a single morphism $p \to q$ if and only if $p \leqslant q$. We abuse notation, and we denote by $\mathcal{P}$ either the poset as a set or as a category.

A natural generalisation of posets are categories in which every endomorphism is an isomorphism. Following [Lüc89], a category $\mathcal{C}$ with this property, that is,

$$\mathrm{End}_{\mathcal{C}}(c) = \mathrm{Aut}_{\mathcal{C}}(c) \text{ for all } c \in \mathcal{C}$$

is said to be an EI-category. Examples of EI-categories naturally appear in group theory as shown next.

Example 1.1. Let G be a finite group, the orbit category of G, denoted by $\mathcal{O}(G)$, is the category whose objects are the homogeneous G-sets G/H, for every $H \leqslant G$, and whose morphism sets are G-equivariant maps $G/H \to G/K$. The orbit category $\mathcal{O}(G)$ is an EI-category.

Example 1.2. Let G be a group and $H \leqslant G$ be a subgroup, not necessarily normal. Let $\mathcal{C}$ be the category with two objects 0 and 1, and the hom-set is given by:

$$\begin{aligned} \mathrm{Hom}(0,0) &= G & \mathrm{Hom}(0,1) &= \varnothing \\ \mathrm{Hom}(1,1) &= \{1\} & \mathrm{Hom}(1,0) &= G/H. \end{aligned}$$

The composition is given by the product in G and the left action of G on G/H. By conctruction, $\mathcal{C}$ is an EI-category.

Example 1.3. Let p be a prime number and S be a finite p-subgroup. A *fusion system* over S is a category $\mathcal{F}$ whose objects are the set of all subgroups of S and the hom-set verifies the following two properties for all $P, Q \leqslant S$:

1. $\mathrm{Hom}_S(P,Q) \subset \mathrm{Hom}_{\mathcal{F}}(P,Q) \subset \mathrm{Inj}(P,Q)$, where $\mathrm{Hom}_S(P,Q)$ is the set of homomorphisms given by a conjugation in S, and $\mathrm{Inj}(P,Q)$ are the injective homomorphisms; and

2. each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion.

Every fusion system is an EI-category.

Proposition 1.4. *Let $\mathcal{C}$ be an EI-category. There exists a poset $[\mathcal{C}]$ whose elements are the isomorphism classes of objects and given $[x], [y] \in [\mathcal{C}]$,*

$$[x] \leqslant [y] \iff \mathrm{Hom}_{\mathcal{C}}(x,y) \neq \varnothing.$$

We call $[\mathcal{C}]$ the isomorphism poset of $\mathcal{C}$.

Proof. Reflexive and transitive properties are a consequence of the existence of the identity and composition, respectively; it remains to check the antisymmetric one. This property holds because if $[x] \leqslant [y]$ and $[y] \leqslant [x]$, then there exists a pair of morphism $f\colon x \to y$ and $g\colon y \to x$, then the composites $f \circ g \in \mathrm{Aut}_{\mathcal{C}}(y)$, and $g \circ f \in \mathrm{Aut}_{\mathcal{C}}(x)$ are invertible morphism, therefore, $[x] = [y]$. □

Definition 1.5. A filtered EI-category is a pair $(\mathcal{C}, d)$ where $\mathcal{C}$ is an EI-category and $\mathrm{d}\colon [\mathcal{C}] \to \mathbb{N}$ is an order-preserving map. This map is called filtration or degree function.

In this thesis, we primarily focus on functors indexed in posets, although we work in the more general context of EI-categories in Chapter 5. The main motivation for this restriction is that it simplifies the computation while still allowing it to handle interesting problems in homotopy theory and group theory. Moreover, following Słomińska's work [Sło01], the computation of higher limits over an EI-category can be reduced to the computation of higher limits over posets and groups.

Example 1.6. Some examples of posets.

1. The set of natural numbers $\mathbb{N}$ with the standard order relation $\leqslant$ is a poset.
2. The set of positive integers $\mathbb{Z}^+$ equipped with the divisibility relation, i.e., $x \leqslant y$ if $y \mid x$ is a poset.
3. The set of non-zero integers $\mathbb{Z} \backslash \{0\}$ with the divisibility relation is not a poset because the relation is not antisymmetric
4. Given a simplicial complex K, its *face poset* $\mathcal{P}(K)$ is the set of simplices of K ordered by the inclusion. For practical reasons, we include K as the top element of $\mathcal{P}(K)$; see Figure 4.

Given a simplicial complex, we can construct a poset that collects all its homotopical information. Reciprocally, Alexandroff [Ale37] shows that it is possible to go backwards. The *order complex* of $\mathcal{P}$, denoted $\Delta(\mathcal{P})$, is the simplicial complex whose n-faces $\Delta_n(\mathcal{P})$ are the chains of length $n+1$ in $\mathcal{P}$,

$$p_0 < p_1 < \cdots < p_n.$$

In addition, we denote by d_i the i-face map defined by deleting the i-th element in a chain,

$$d_i(p_0 < \cdots < p_i < \cdots < p_n) = p_0 < \cdots < p_{i-1} < p_{i+1} < \cdots < p_n.$$

Given a chain $c \in \Delta_n(\mathcal{P})$, we denote by c_k the element of $\mathcal{P}$ at position k for $0 \leqslant k \leqslant n$. Also, given $p \in \mathcal{P}$, we denote by $\Delta_p(\mathcal{P})$, or by Δ_p if the poset $\mathcal{P}$ is understood, the set of all chains $c \in \Delta_{\mathrm{d}(p)}(\mathcal{P})$ such that $c_{\mathrm{d}(p)} = p$.

The *cover relation* in a poset $\mathcal{P}$, denoted by $\prec$, is defined by $p \prec q$ if and only if $p < q$ and

$$\forall r \in \mathcal{P}, p \leqslant r \leqslant q \Rightarrow p = r \text{ or } q = r.$$

The Hasse diagram of a poset $\mathcal{P}$ is a directed graph whose vertices are the elements of $\mathcal{P}$, and there is a directed edge between x and $y \in \mathcal{P}$ if and only if $x \prec y$. In this text, we represent the cover relation as an upward edge in the Hasse diagram.

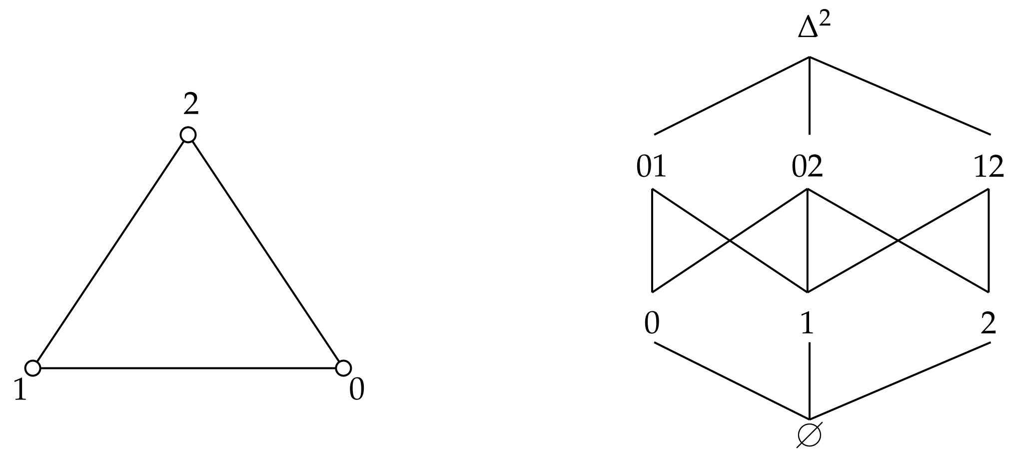

Figure 4: The face poset of $\Delta[2]$.

We say that a chain c between two comparable elements $p \leqslant q$ is unrefinable if given two consecutive elements c_i, c_{i+1} we have $c_i \prec c_{i+1}$, i.e, the chain can be written

$$p \prec c_1 \prec c_2 \prec \cdots \prec c_{n-1} \prec q.$$

For every $p \in \mathcal{P}$, we denote by $\mathcal{P}_{\leqslant p}$ the sub-poset of $\mathcal{P}$ consisting of those $q \in \mathcal{P}$ such that $q \leqslant p$.

$$\mathcal{P}_{\leqslant p} := \{q \in \mathcal{P} \mid q \leqslant p\}.$$

We can similarly define $\mathcal{P}_{\geqslant p}$, $\mathcal{P}_{>p}$, $\mathcal{P}_{<p}$ and $\mathcal{P}_{\prec p}$. If $p, q \in \mathcal{P}$, we denote by $[p, q]$ the subposet of $\mathcal{P}$ given by:

$$[p, q] = \{r \in \mathcal{P} \mid p \leqslant r \leqslant q\} = \mathcal{P}_{\geqslant p} \cap \mathcal{P}_{\leqslant q}.$$

For Q a subset of $\mathcal{P}$, we denote by $\langle Q\rangle$ the sub-poset of $\mathcal{P}$ whose elements are lesser than some element in Q; that is,

$$\langle Q\rangle := \{p \in \mathcal{P} \mid p \leqslant q \text{ for some } q \in Q\} = \bigcup_{q\in Q} \mathcal{P}_{\leqslant q}.$$

If $\mathcal{P}$ is a linear order, we say that a subset $S \subset \mathcal{P}$ is an initial segment if it can be expressed as $S = \langle p\rangle = \mathcal{P}_{\leqslant p}$ for some $p \in \mathcal{P}$. Given J, K a pair of subsets of $\mathcal{P}$, we write $J \leqslant K$ if, for every $j \in J$, there exits $k \in K$ such that $j \leqslant k$. In the same way, we define $J < K$, $J \prec K$, $J > K$ and $J \geqslant K$. A subset Q of a poset $\mathcal{P}$ is *upper convex* if $\mathcal{P}_{\geqslant x} \subset Q$ for every $x \in Q$.

We say that a poset $\mathcal{P}$ is *bounded* if it has a maximum $\hat{1}$ and a minimum $\hat{0}$. For a bounded poset $\mathcal{P}$ we denote by $\overline{\mathcal{P}}$ the subposet $\mathcal{P}\backslash\{\hat{0},\hat{1}\}$, and for a non-necessarily bounded poset $\mathcal{P}$, we let $\hat{\mathcal{P}}$ denote the (unique) minimal bounded poset such that $\mathcal{P} \subset \hat{\mathcal{P}}$. If $\mathcal{P}$ is bounded then $\hat{\mathcal{P}} = \mathcal{P}$. A finite poset is said to be *pure* if all maximal chains have the same length. Notice that a pure poset satisfies the Jordan-Dedekind condition:

All unrefinable chains between two comparable elements have the same length.

A poset is *graded* if it is pure and bounded. A graded poset $\mathcal{P}$ admits a canonical filtration $\mathrm{d}\colon \mathcal{P} \to \mathbb{N}$ defined by $\mathrm{d}(p)$ equal to the (common) length of an unrefinable chain from $\hat{0}$ to p. This filtration verifies the following property:

$$p \prec q \quad\Longrightarrow\quad \mathrm{d}(q) = \mathrm{d}(p) + 1.$$

The *length* of a filtered poset is the supremum of its filtration:

$$\operatorname{length}(\mathcal{P}) = \sup\{\mathrm{d}(p) \mid p \in \mathcal{P}\}.$$

If $\mathcal{P}$ is a graded poset, its length coincides with the value of d at $\hat{1}$, this is, $\operatorname{length}(\mathcal{P}) = \mathrm{d}(\hat{1})$.

Example 1.7. Given a simplicial complex K, its face poset $\mathcal{P}(K)$ is a graded poset.

A poset $\mathcal{P}$ has the *descending chain condition* if there are no infinite descending chains, i.e., for every descending chain:

$$a_0 \geqslant a_1 \geqslant \cdots \geqslant a_{n-1} \geqslant a_n \geqslant \ldots,$$

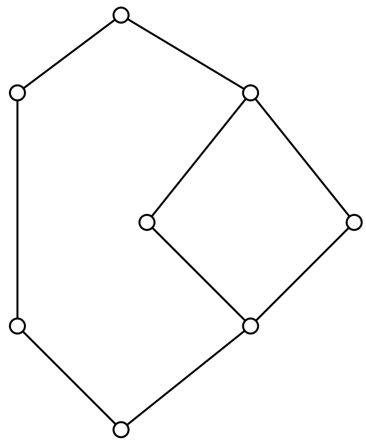

Figure 5: A filtered but not pure poset.

there exits $N \in \mathbb{N}$ such that $a_n = a_{n+1}$ for every $n \geqslant N$. For short, if $\mathcal{P}$ has the descending chain condition, we say that $\mathcal{P}$ is a DCC poset.

Example 1.8. Notice that a filtered poset is a DCC poset, but the converse is not true. Let $\mathcal{P}$ be a subset of $\mathbb{N} \times \mathbb{N}$ defined by:

$$\mathcal{P} = \{(n, r) \in \mathbb{N} \times \mathbb{N} \mid n = r = 0 \text{ or } n \geqslant r > 0\}$$

equipped with the order relation

$$(n, r) \leqslant (n', r') \text{ iff } \begin{cases} n' = r' = 0 \\ n = n' \text{ and } r' \leqslant r. \end{cases}$$

See Figure 6. An order-preserving map $\mathrm{d}\colon \mathcal{P} \to \mathbb{N}$ implies a bound in the set of natural numbers. Therefore, $\mathcal{P}$ is a DCC poset that is not filtered.

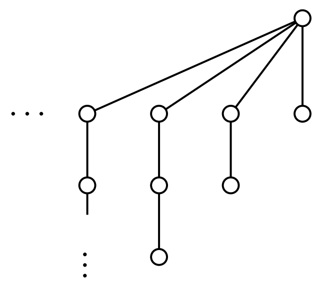

Figure 6: Hasse diagram of the poset described in Example 1.8.

These three conditions verify the following inclusions:

$$\text{Graded} \subset \text{Filtered} \subset \text{DCC}.$$

1.1 SHELLABLE POSETS

Shellable complexes are of interest in several areas of mathematics as algebraic topology, commutative algebra, and combinatorics. They were popularised by Björner [Bjö80] to study Stanley-Reisner rings since shellable complexes are Cohen-Macaulay. Moreover, being shellable implies some important algebraic, topological, and combinatorial properties for the complex, see [BW82, Bjö84].

A simplicial complex is said to be pure if it is finite and its maximal faces (facets, for short) have the same dimension.

Definition 1.1.1. A *shelling* for a pure simplicial complex K is a linear order on its facets $F_1 \ll F_2 \ll \cdots \ll F_n$ such that the simplicial complex

$$F_k \cap \left(\bigcup_{i=1}^{k-1} F_i \right)$$

is a non-empty union of facets of ∂F_k, for every $k = 2 \ldots, n$. A simplicial complex is said to be shellable if it admits a shelling. A poset $\mathcal{P}$ is shellable if its order complex $\Delta(\mathcal{P})$ is a shellable complex.

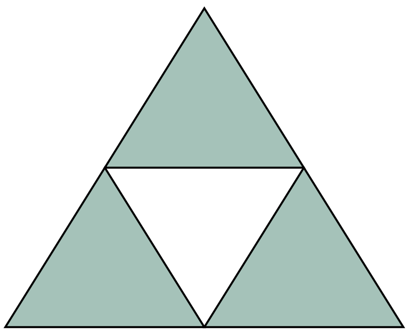

Figure 7: A non-example of a shellable complex.

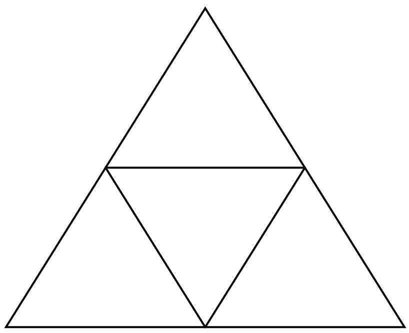

Figure 8: A shellable complex of dimension 1.

Example 1.1.2 ([Bjö92]). Let K be the simplicial complex of vertices $\{a, b, c, d, e\}$ and facets

$$A = \{b, d, e\}, \qquad B = \{c, e, d\}, \qquad C = \{c, b, e\}$$
$$D = \{a, c, d\}, \qquad E = \{a, b, c\}, \qquad D = \{a, c, d\},$$
$$F = \{c, b, d\}.$$

see Figure 9. Then, the ordering $A \ll B \ll C \ll D \ll E \ll F$ is a shelling for K.

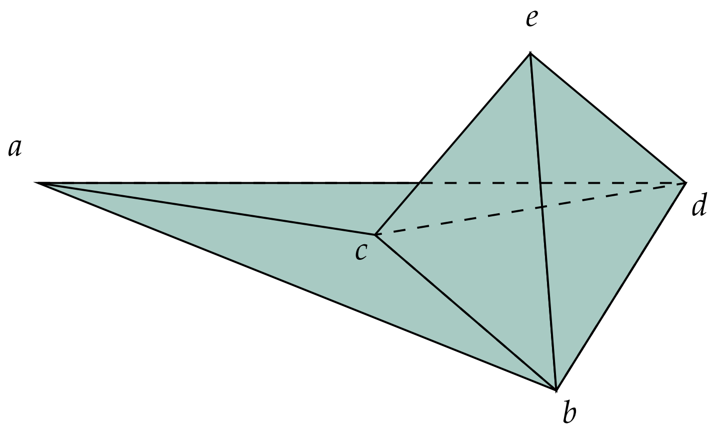

Figure 9: A shellable complex [Bjö92, Figure 1].

Example 1.1.3 (Bruhat order). A Coxeter group is a pair (W, S) where W is a group and S is a distinguished set of generators of W such that

$$W = \langle S \mid s_i^2 = 1,\ (s_i s_j)^{m_{i,j}} = 1 \text{ for } i \neq j \rangle$$

where $m_{i,j} = 2, 3, \ldots \infty$ for $i \neq j$. Finite Coxeter groups arise as Weyl groups of root systems and the symmetry groups of regular polytopes and tessellations.

The elements of W can be expressed as words in S, i.e., for any $w \in W$,

$$w = s_1 s_2 \ldots s_k \qquad s_i \in S.$$

If k is the shortest possible length for such an expression of w, then it is defined as the *length* of w, denoted by $l(w) = k$. The set of reflections of the Coxeter group is defined as the set of conjugates of S:

$$T = \{wsw^{-1} \in W \mid s \in S,\ w \in W\}.$$

The group W admits a partial order, the *Bruhat order*, given by $w \leqslant w'$ if there exists reflections $t_1, t_2, \ldots, t_m \in T$ such that $w' = wt_1 t_2 \ldots t_m$ and

$$l(wt_1 t_2 \ldots t_{i-1}) < l(wt_1 t_2 \ldots t_i) \qquad \text{for } i = 1, 2 \ldots, m.$$

Then the order complex of $(W, \leqslant)$ is shellable. We refer the reader to Björner-Brenti's Book [BB05], and Björner-Wachs' paper [BW82] for further details.

Example 1.1.4 ([BM71]). The boundary of a convex polytope is shellable.

The most relevant homotopical property of shellable complexes is that they have the homotopy type of a wedge of spheres.

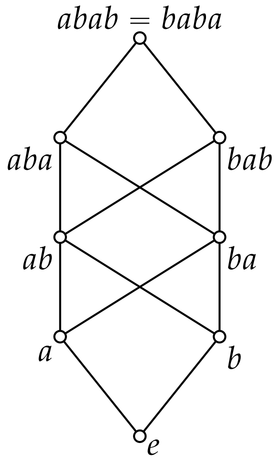

Figure 10: Bruhat order of $I_2(4) = \langle a, b | a^2 = b^2 = e,\ (ab)^4 = e \rangle$.

Theorem 1.1.5 ([Bjö84, Theorem 1.3]). *Let K be a shellable d-dimensional complex. Then, there exists $h \in \mathbb{N}$ such that $|K|$ has the homotopy type of a wedge of h d-dimensional spheres,*

$$|K| \simeq \bigvee_{i=0}^{h} S^d.$$

Deciding if a poset is shellable involves two problems. First, we do not have an intrinsic notion of a shellable poset; we need to compute the order complex of the given poset. The second problem is that *deciding if a pure d-dimensional complex is shellable is NP-complete* for $d \geqslant 3$ [GPP+19]. Despite this problem, in the 80s, notions that imply the shellability of posets were presented by Björner [Bjö80] and Björner-Wachs [BW82]. However, Goaoc et al., proved in the same paper [GPP+19] that *CL-shellability of a poset*, one of the notions presented by Björner-Wachs (see Definition 1.1.7), is also NP-hard for $d \geqslant 4$.

Let $\mathcal{P}$ be a graded poset of length n and $\mathcal{E}(\mathcal{P}) \subset \mathcal{P} \times \mathcal{P}$ be the set of relations induced by the covering relation,

$$\mathcal{E}(\mathcal{P}) := \{(x, y) \in \mathcal{P} \times \mathcal{P} \mid x \prec y\}.$$

An *edge labelling* of $\mathcal{P}$ is map $\lambda\colon \mathcal{E}(\mathcal{P}) \to \mathbb{Z}$. Given an edge labelling λ, each unrefinable chain $c = (c_0 \prec c_1 \prec \cdots \prec c_k)$ can be associated with a k-tuple:

$$\sigma(c) = (\lambda(c_0, c_1), \lambda(c_1, c_2), \ldots, \lambda(c_{k-1}, c_k)).$$

We say that c is an λ-*increasing chain* if the k-tuple $\sigma(c)$ is increasing; this is,

$$\lambda(c_0, c_1) \leqslant \lambda(c_1, c_2) \leqslant \cdots \leqslant \lambda(c_{k-1}, c_k)$$

This edge labelling allows us to order the maximal chains by the lexicographic order induced by σ. We denote this order relation by $\ll_\lambda$ or just $\ll$ if the edge labelling is understood.

Definition 1.1.6. An edge labelling is called an *EL-labelling* (edge lexicographical) if for every interval $[x, y]$ in $\mathcal{P}$,

1. there is a unique increasing maximal chain c in $[x, y]$, and
2. $c \ll c'$ for all other maximal chains c' in $[x, y]$.

A graded poset that admits an EL-labelling is said to be *EL-shellable*.

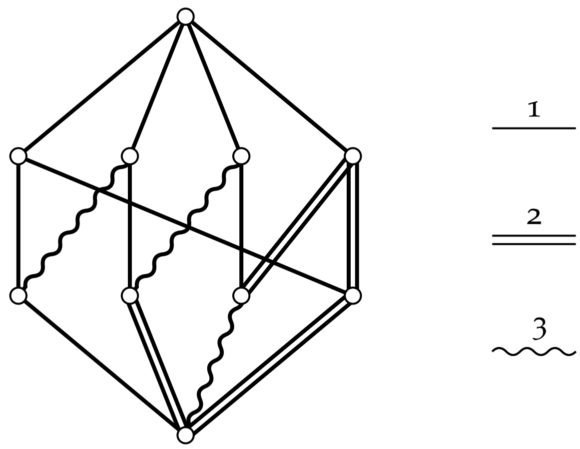

Figure 11: EL-labeling of the face lattice of a square [BW83, Figure 2.1].

Björner-Wachs [BW82] presents another notion that implies the shellability of the poset. For a graded poset $\mathcal{P}$ of length n, we define $\mathcal{E}^*(\mathcal{P})$ to be the set of edges of maximal chains of $\mathcal{P}$,

$$\mathcal{E}^*(\mathcal{P}) := \{(c, x, y) \mid x \prec y, c \in \Delta_n(\mathcal{P}), x, y \in c\}.$$

A *chain-edge labelling* of $\mathcal{P}$ is a map $\lambda \colon \mathcal{E}^*(\mathcal{P}) \to \mathbb{Z}$ that satisfies the following condition: If two maximal chains

$$c = (\hat{0} \prec c_1 \prec \cdots \prec c_{n-1} \prec \hat{1}) \text{ and } c' = (\hat{0} \prec c'_1 \prec \cdots \prec c'_{n-1} \prec \hat{1})$$

coincide along their first d edges, then their labels also coincide along these edges, i.e., if $c_i = c'_i$ for $i = 0, \dots, d$, then:

$$\lambda(c, c_{i-1}, c_i) = \lambda(c, c'_{i-1}, c'_i)$$

Let λ be a chain-edge labelling of $\mathcal{P}$. Each maximal chain

$$c = (\hat{0} \prec c_1 \prec \cdots \prec c_{n-1} \prec \hat{1})$$

of $\mathcal{P}$ can be associated with a unique n-tuple:

$$\sigma(c) = (\lambda(c, \hat{0}, c_1), \lambda(c, c_1, c_2), \dots, \lambda(c, c_{n-1}, \hat{1})).$$

Given $x \leqslant y \in \mathcal{P}$ and r a unrefinable chain from $\hat{0}$ to x, we say that the pair $([x, y], r)$ is a *rooted interval* with *root* r and we denote it $[x, y]_r$. If c is any maximal chain of $[x, y]$, then $r \cup c$ is a maximal chain of $[\hat{0}, y]$. For a maximal chain c in a rooted interval $[x, y]_r$ has a unique k-tuple

$$\sigma_r(c) = (\lambda(c', x, c_1), \lambda(c', c_1, c_2), \dots, \lambda(c', c_{k-1}, y))$$

where c' is any maximal chain from $\hat{0}$ to $\hat{1}$ that contains $r \cup c$. We say that a maximal chain c in a rooted interval $[x, y]_r$ is *increasing* if the k-tuple $\sigma_r(c)$ is increasing. If c_1 and c_2 are maximal chains of $[x, y]_r$ then c_1 is said to *lexicographically precede* c_2 in $[x, y]_r$ if $\sigma_r(c_1)$ lexicographically precedes $\sigma_r(c_2)$, and we denote it by $c_1 \ll c_2$ in $[x, y]_r$.

Definition 1.1.7. A chain-edge labelling λ is called a *CL-labelling* if for every rooted interval $[x, y]_r$ in $\mathcal{P}$,

1. there is a unique increasing maximal chain c in $[x, y]_r$, and
2. $c \ll c'$ for all other maximal chains c' in $[x, y]_r$.

A graded poset is said to be *CL-shellable* if it admits a CL-labelling. A graded poset $\mathcal{P}$ is said to be *dual CL-shellable* if $\mathcal{P}^{\text{op}}$, the poset obtained from $\mathcal{P}$ by the reverse order is CL-shellable.

The relations between these notions can be summarised in the following diagram:

$$\text{EL-shellable} \Rightarrow \text{CL-shellable} \Rightarrow \text{shellable}.$$

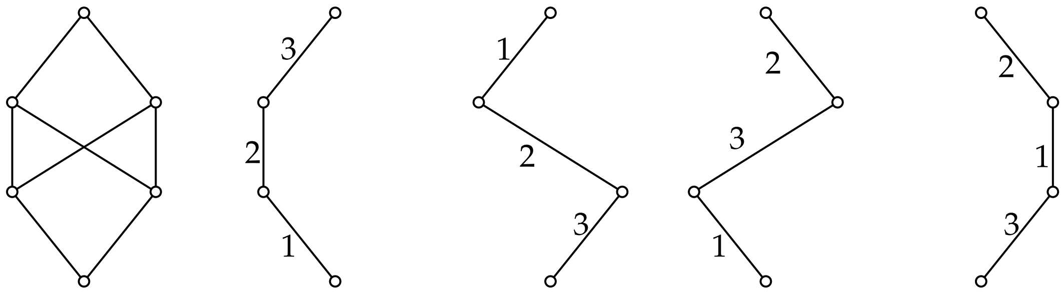

Figure 12: A CL-labeling [BW83, Figure 2.2].

These inclusions are strict since there are examples of CL-shellable posets that are not EL-shellable [Li19] and results that imply the existence of shellable posets that are not CL-shellable [Wal85]. However, being CL-shellable and shellable are close conditions.

Theorem 1.1.8 (Björner-Wachs [BW83, Teorem 4.3])**.** *Let K be a pure polyhedral complex. The face poset $\mathcal{P}(K)$ is dual CL-shellable if and only if K is a shellable complex.*

Later, Björner-Wachs characterise CL-shellable posets in terms of their atoms [BW83]. However, since we are interested in the opposite category of a CL-shellable poset, we present here a characterisation of dual CL-shellable posets. An *atom* a of a graded poset $\mathcal{P}$ is an element that covers the minimum $\hat{0} \prec a$. Dually, a *coatom* h is an element covered by the maximum $h \prec \hat{1}$.

Let $\mathcal{P}$ be a graded poset, $p \in \mathcal{P}$ and $\ll$ be a linear order in $\mathcal{P}_{\prec p}$. Given $h \prec p$, we denote by $C_{\ll}(h)$ to be the elements of $\mathcal{P}_{\prec h}$ that are covered by some $h' \ll h$, this is,

$$C_{\ll}(h) := \{x \prec h \mid x \prec h' \text{ for some } h' \ll h\}.$$

Definition 1.1.9. Let $\mathcal{P}$ be a graded poset. A *recursive coatom ordering* for $\mathcal{P}$ is, for every unrefinable chain c that ends in $\hat{1}$, a linear order $\ll_c$ on the set of coatoms of $\mathcal{P}_{\leqslant c_0}$, such that:

CL1 if $c_0 \neq \hat{1}$, the set $C_{\ll_{d_0(c)}}(c_0)$ is an initial segment of the linear ordered set $(\mathcal{P}_{\prec c_0}, \ll_c)$, i.e., if $x \in C_{\ll_{d_0(c)}}(c_0)$, and $y \in \mathcal{P}_{\prec c_0} \backslash C_{\ll_{d_0(c)}}(c_0)$, then $x \ll_c y$; and

CL2 for every pair of coatoms $h \ll_c h' \in \mathcal{P}_{\prec c_0}$, if there exists $y \in \mathcal{P}_{\leqslant c_0}$ such that $y \leqslant h, h'$, then there is a coatom $h'' \ll h$ of $\mathcal{P}_{\leqslant c_0}$ and an element $z \in \mathcal{P}_{\leqslant c_0}$ such that $y \leqslant z \prec h'', h'$.

By abuse of notation, we say that $\ll$ is a recursive coatom ordering if the family $\{(\mathcal{P}_{\prec c_0}, \ll_c)\}_c$ is a recursive coatom ordering where c ranges over the family of all unrefinable chains that end in $\hat{1}$. Moreover, if the chain c is just the trivial chain, we denote $\ll$, the linear order $\ll_c$.

Originally, Björner-Wachs introduced the property of *admitting a recursive coatom ordering* instead of defining what a recursive coatom ordering is. However, since we introduce techniques that use the recursive coatom ordering explicitly, we prefer to define it properly.

Remark 1.1.10. Let $\ll$ be a recursive coatom ordering, c be an unrefinable chain that ends in $\hat{1}$ and $h \prec c_0$. we denote by $C_c(h)$ the set $C_{\ll_c}(h)$. If the chain c is understood, we denote it by $C(h)$.

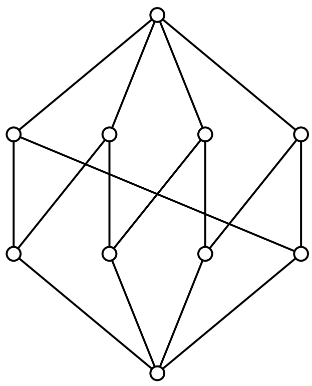

Figure 13: An example of a CL-shellable poset; see [BW83, Figure 3.1].

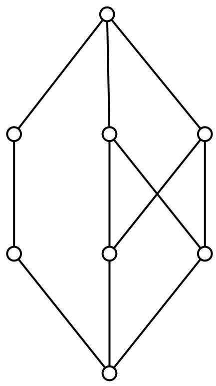

Figure 14: An example of a non CL-shellable poset; see [BW83, Figure 3.1].

Proposition 1.1.11 ([BW82]). *A graded poset $\mathcal{P}$ is dual CL-shellable if and only if there exists a recursive coatom ordering for $\mathcal{P}$.*

In general, given a dual CL-shellable poset $\mathcal{P}$ with recursive coatom ordering $\ll$, and two chains c, c' both ending in $\hat{1}$ and with the same source, the linear orders $\ll_c$ and $\ll_{c'}$ may not coincide. Li [Li20] characterise EL-shellable poset as the ones for which these orders coincide.

Proposition 1.1.12 ([Li20, Proposition 2.1.1]). *Let $\mathcal{P}$ be a dual CL-shellable poset. Then $\mathcal{P}$ is dual EL-shellable if and only if there exists a recursive coatom ordering $\ll$ such that for every $p \in \mathcal{P}$, the linear ordering $(\mathcal{P}_{\prec p}, \ll_c)$ is independent of the chain $c = p \prec \cdots \prec \hat{1}$.*

Next, we introduce some properties of CL-shellable poset that we will need later.

Lemma 1.1.13. *Let $\mathcal{P}$ be a dual CL-shellable poset with recursive coatom ordering $\ll$, and $\mathcal{Q}$ be the subposet of $\mathcal{P}$ consisting of the objects of degree $\mathrm{d}(\mathcal{P}) - 1$ and $\mathrm{d}(\mathcal{P}) - 2$. Let $x_1 \ll x_0$ be a pair of coatoms of $\mathcal{P}$ such that there exists $p \in \mathcal{P}_{<x_1} \cap \mathcal{P}_{<x_0}$. Then, there exists a sequence of coatoms $\{z_n\}_{n \in \mathbb{N}}$ such that:*

1. *the sequence is connected in $\mathcal{Q} \cap \mathcal{P}_{\geqslant p}$,*
2. *it starts in x_0, this is, $z_0 = x_0$, and*
3. *there exists $N \in \mathbb{N}$, such that $z_n = z_{n-1} \ll x_1$ for all $n > N$.*

Proof. We define this sequence recursively. First, we set $z_0 := x_0$ and given $\{z_k\}_{k=0}^{n-1}$ verifying (1) and (2), we define z_n as follows:

If $z_{n-1} \mathrel{\underline{\ll}} x_1$, then we define $z_n := z_{n-1}$. Otherwise, by the induction hypothesis, $p < x_1, z_{n-1}$. Therefore, By **CL2** (see Definition 1.1.9), there exists at least a coatom y and $q_{n-1} \in \mathcal{Q}$ such that $p \leqslant q_{n-1} \prec y, z_{n-1}$ with $y \ll z_{n-1}$. We define z_n to be any element y with this property. By construction, z_n is connected in $\mathcal{Q} \cap \mathcal{P}_{\geqslant p}$ with z_{n-1} by q_{n-1}. By the induction hypothesis, there exists a zigzag that connects z_n with z_0 in $\mathcal{Q} \cap \mathcal{P}_{\geqslant p}$. So, we can extend this zigzag, obtaining a connected sequence $\{z_k\}_{k=0}^{n}$:

$$\begin{tikzcd}
z_0 & & z_1 & & \cdots & & z_{n-1} & & z_n \\
& q_0 \arrow[ul] \arrow[ur] & & \cdots \arrow[ul] & & q_{n-2} \arrow[ul] \arrow[ur] & & q_{n-1} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Finally, since the set of coatoms between x_1 and x_0 is finite, there exists $N \in \mathbb{N}$ such that $z_n \mathrel{\underline{\ll}} x_1$ for all $n > N$. □

Lemma 1.1.14. *Let $\mathcal{P}$ be a dual CL-shellable poset, and $\mathcal{Q}$ be the subposet of $\mathcal{P}$ consisting of the objects of degree $\mathrm{d}(\mathcal{P}) - 1$ or $\mathrm{d}(\mathcal{P}) - 2$. Then, $\mathcal{Q}$ is final in $\mathcal{P} \backslash \{\hat{1}\}$ and in $\overline{\mathcal{P}}$.*

Proof. If $\mathrm{d}(\mathcal{P}) = 0$ the condition is empty and for $\mathrm{d}(\mathcal{P}) = 1, 2$ we have $\mathcal{Q} = \mathcal{P} \backslash \{\hat{1}\}$. Then, we assume that $\mathrm{d}(\mathcal{P}) > 2$. We will only prove that $\mathcal{Q}$ is final in $\overline{\mathcal{P}}$; the other case is analogous.

Let $\ll$ be a recursive coatom ordering for $\mathcal{P}$. First, notice that for every $p \in \overline{\mathcal{P}}$, the comma category $(p/\mathcal{Q})$, see Definition 2.3.2, is non-empty. This holds since $p \leqslant h$ for at least a coatom h, and by definition, $\mathcal{Q}$ contains every coatom of $\mathcal{P}$.

Next, we check that $p/\mathcal{Q}$ is connected. Let $x_0, x_1 \in \mathcal{Q}$ such that $p < x_0, x_1$. We can assume without loss of generality that x_1, x_0 are coatoms and $x_1 \ll x_0$. Now, we construct a connected sequence $\{x_n\}$ of coatoms that connects x_1 with x_0.

Given x_{n-1} and x_{n-2} we construct x_n as follows: If $x_{n-1} = x_{n-2}$, then we define $x_n = x_{n-1}$. Otherwise, we apply Lemma 1.1.13 to x_{n-1} and x_{n-2} obtaining a sequence of coatoms $\{z_n\}$ such that is connected in $\mathcal{Q} \cap \mathcal{P}_{\geqslant p}$, it starts in x_{n-2}, and a natural number $N \in \mathbb{N}$ such that $z_m = z_{m-1} \ll x_{n-1}$ for all $m > N$. Then we define $x_n := z_m$ for any $m > N$. By finiteness of the coatoms set, this sequence eventually ends; this is, for some $n_0 \in \mathbb{N}$, $x_m = x_{m+1}$ for all $m > n_0$.

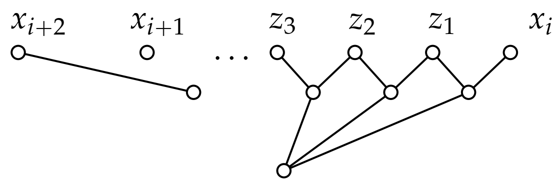

Figure 15

By construction of the sequence, the odd elements x_{2n+1} are connected with x_1, and the even ones are connected with x_0 (see Figure 15), and as we said before, there is some n such that $x_n = x_{n+1}$, that means that both subsequences are connected. $\square$

Definition 1.1.15. Let $\mathcal{P}$ be a dual CL-shellable poset with recursive coatom ordering $\ll$, and c be an unrefinable chain that ends in $\hat{1}$. Given Q a subset of coatoms of $\mathcal{P}_{\leqslant c_0}$, we say that c *makes* Q *compatible with the recursive coatom ordering* $\ll$ if Q is an initial segment of the linear ordered set $(\mathcal{P}_{\prec c_0}, \ll_c)$. A subset of coatoms $Q \subset \mathcal{P}_{\prec c_0}$ is said to be *compatible with the recursive coatom ordering* $\ll$ if there exists a chain $c = (c_0 \prec c_1 \prec \cdots \prec \hat{1})$ that makes Q compatible with $\ll$.

Lemma 1.1.16. *Let $\mathcal{P}$ be a dual CL-shellable poset with recursive coatom ordering $\ll$, p be an object of $\mathcal{P}$ and Q be a subset of coatoms of $\mathcal{P}_{\leqslant p}$ compatible with the recursive coatom ordering. Then, $\langle Q\rangle \cup \{p\}$ is a dual CL-shellable poset of length* $\mathrm{d}(p)$.

Proof. Let c be an unrefinable chain from p to $\hat{1}$ that makes Q compatible with $\ll$. Then, the family $\{(\langle Q\rangle_{\prec c'_0}, \ll'_{c'})\}_{c'}$ is a recursive coatom ordering, where c' ranges in the family of all unrefinable chains c' in $\langle Q\rangle$ that ends in p and $(\langle Q\rangle_{\prec c'_0}, \ll'_{c'})$ is the linear ordered set $(\mathcal{P}_{\prec c'_0}, \ll_{c' \prec d_0(c)})$. $\square$

1.1.1 Intersection lattice of a hyperplane arrangement

A lattice is a poset such that any two elements x and y have a maximum, $x \vee y$, and a minimum, $x \wedge y$. A geometric lattice is a graded lattice such that the degree function verifies:

$$d(x \wedge y) + d(x \vee y) \leqslant d(x) + d(y). \tag{2}$$

If $\mathcal{P}$ is a geometric lattice and $p \in \mathcal{P}$, then $\mathcal{P}_{\leqslant p}$ is again a geometric lattice because if $x, y \in \mathcal{P}_{\leqslant p}$, then $x \vee y \leqslant p$, so Equation (2) holds in $\mathcal{P}_{\leqslant p}$. Every linear ordering on the atoms of a geometric lattice induces a recursive atom ordering [BW83, Theorem 5.1]. Therefore, every geometric lattice is CL-shellable. For practical reasons, we consider the reverse order relation in the geometric lattices to obtain dual CL-shellable lattices. An example of a geometric lattice is the intersection lattice of a hyperplane arrangement.

Let V be a finite-dimensional vector space over a field k, let $\mathcal{H} = \{h_1, \dots, h_n\}$ be a finite set of linear hyperplanes in V. The *arrangement lattice* $L_{\mathcal{H}}$ has elements all possible intersections of hyperplanes in $\mathcal{H}$ and is ordered by the inclusion relation. The graded lattice structure is given by:

$$\hat{0} = \bigcap_{h \in \mathcal{H}} h, \qquad \hat{1} = V, \qquad d(x) = \dim(x) - \dim(\hat{0}),$$

$$x \wedge y = x \cap y, \quad x \vee y = \bigcap \{z \in L_{\mathcal{H}} \mid x \cup y \subset z\}.$$

Remark 1.1.17. Generally, $\hat{0}$ is not the trivial vector space, $\hat{0} \neq 0$.

With this lattice structure, $L_{\mathcal{H}}$ is a dual CL-shellable lattice. In a geometric lattice L, a collection of coatoms S is said to be *independent* if for every $T \subsetneq S$, we have

$$\bigwedge S < \bigwedge T;$$

otherwise, we say that S is *dependent*. A *basis* of coatoms is a maximal independent subset.

Example 1.1.18. Consider in $\mathbb{R}^3$ the following hyperplanes $h_1 \equiv 0 = y - z$, $h_2 \equiv 0 = z$, $h_3 \equiv 0 = y + z$ and $h_4 \equiv 0 = x$, and let $L = L_{\mathcal{H}}$ the arrangement lattice of $\mathcal{H} = \{h_1, h_2, h_3, h_4\}$. The set $\{h_1, h_2, h_3\}$ is dependent in L since $h_1 \cap h_2 = h_1 \cap h_2 \cap h_3$, and $\{h_1, h_2, h_4\}$ is a basis, see Figure 16.

Lemma 1.1.19. *Let V be a finite-dimensional k-vector space and $\mathcal{H}$ be a finite set of hyperplanes. A subset $\{h_1, h_2, \dots, h_n\}$ of coatoms of $L_{\mathcal{H}}$ is independent if and only if the*

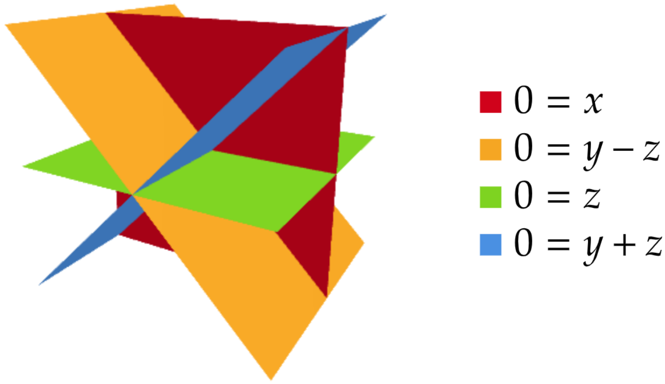

Figure 16

vectors $f_1, f_2, \dots, f_n$ are independent in V^*, where f_i is a generator of the annihilator of h_i, this is, $\operatorname{an}(h_i) = \{g \in V^* \mid g(v) = 0$ *for all* $v \in h_i\}$.

Proof. We prove one implication; the other one is analogous. Assume by contradiction that $\{h_1, h_2, \dots, h_n\}$ is an independent set of coatoms and there exists $\lambda_1, \lambda_2, \dots, \lambda_n \in k$ with some $\lambda_i \neq 0$ such that $\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n = 0$. Without loss of generality, assume that $\lambda_1 = 1$. Then, we have $f_1 = -(\lambda_2 f_2 + \cdots + \lambda_n f_n)$. In particular,

$$\ker f_1 = \ker -(\lambda_2 f_2 + \cdots + \lambda_n f_n) \supset \bigcap_{i=2}^{n} \ker f_i.$$

This implies that

$$\bigwedge_{i=1}^{n} h_i = h_1 \wedge \left(\bigwedge_{i=2}^{n} h_n \right) = \ker f_1 \cap \left(\bigcap_{i=2}^{n} \ker f_i \right) = \bigcap_{i=2}^{n} \ker f_i = \bigwedge_{i=2}^{n} h_i,$$

which contradicts that $\{h_1, \dots, h_n\}$ is an independent set of coatoms. □

Corollary 1.1.20. *Let V be a finite-dimensional vector space and $\mathcal{H}$ be a finite set of hyperplanes. If $B \subset \mathcal{H}$ is an independent set of coatoms of $L_{\mathcal{H}}$, then $\#B \leqslant \mathrm{d}(V)$. Moreover, $\#B = \mathrm{d}(V)$ if and only if B is a basis.*

Proof. This follows directly from Lemma 1.1.19 □

Lemma 1.1.21. *Let V be a finite-dimensional vector space, $\mathcal{H}$ be a finite set of hyperplanes, and L be the arrangement lattice of $\mathcal{H}$. Let B be a basis of L and $h_0 \in \mathcal{H} \backslash B$. Then, there exists a subset B_0 of B such that,*

1. *$B_0 \cup \{h_0\}$ is a basis of L; and*

2. *the set* $B' = \{h \cap h_0 \mid h \in B_0\}$ *is a basis of the geometric lattice* $L_{\leqslant h_0}$.

Proof. By Lemma 1.1.19, (1) is reduced to the well-known Steinitz's Lemma about finding a base of a vector space in a set of generators.

Notice that (2) is equivalent to

(2′) B' is an independent set and $\#B' = \#B_0$.

To prove statement (2′), we follow the next linear algebra argument. Let $h_1, \dots, h_n$ be the elements in B_0, and, for every $i = 0, 1, \dots, n$, let f_i be a generator of $\mathrm{an}(h_i)$. Since B_0 is an independent set of coatoms, we can assume without loss of generality that $\{f_0, f_1, \dots, f_n\}$ is a basis of V^*. If not, consider the quotient $V^*/\hat{0}$.

First, notice that $(h_0)^*$ can be identified with the quotient $V^*/\langle f_0 \rangle$, and a basis of this quotient is provided by the set $\{f_1 + \langle f_0 \rangle, f_2 + \langle f_0 \rangle, \dots f_n + \langle f_0 \rangle\}$. Next, by direct computation, we have $\mathrm{an}(h_0 \cap h_i) = \langle f_0 + f_i \rangle$. By Lemma 1.1.19 under the identification $(h_0)^* \cong V^*/\langle f_0 \rangle$, the set $\{h_0 \cap h_i \mid h_i \in B_0\}$ is an independent set and it has exactly $|B_0|$ elements. □

Lemma 1.1.22. *Let V be a finite-dimensional vector space, and let $\mathcal{H}$ be a finite set of hyperplanes. For every $W \in L_{\mathcal{H}}$, the meet of the hyperplanes of W is the minimum of $L_{\mathcal{H}}$, i.e.,*

$$\bigwedge_{\omega \prec W} \omega = \hat{0}.$$

Proof. $L_{\mathcal{H}}$ is an arrangement lattice, and hence we can write W as a finite intersection of hyperplanes in $\mathcal{H} = \{h_1, h_2, \dots, h_m\}$. We assume without loss of generality that $W = \bigwedge_{i=1}^{k} h_i$. Note that for every $j = k+1 \dots m$, $W \wedge h_j \leqslant W$. Then,

$$\bigwedge_{i=k+1}^{m} (W \wedge h_i) = W \wedge (\bigwedge_{i=k+1}^{m} h_i) = (\bigwedge_{i=1}^{k} h_i) \wedge (\bigwedge_{i=k+1}^{m} h_i) = \bigwedge_{i=1}^{m} h_i = \hat{0}$$ □

Definition 1.1.23. Let L be a geometric lattice. A *basis-like recursive coatom ordering* for L is a recursive coatom ordering $\ll$ for L such that:

CL3 For every unrefinable chain c that ends in $\hat{1}$, an initial segment of $(\mathcal{P}_{\prec c_0}, \ll_c)$ is a basis of the geometric lattice $\mathcal{P}_{\leqslant c_0}$, this is, there exists $h \in \mathcal{P}_{\prec c_0}$ such that the set $\{h' \in \mathcal{P}_{\prec c_0} \mid h' \underline{\ll}_c h\}$ is a basis of $\mathcal{P}_{\leqslant c_0}$.

Lemma 1.1.24. *Let V be a finite-dimensional vector space and $\mathcal{H}$ be a finite set of hyperplanes of V. Then, $L_{\mathcal{H}}$ admits a basis-like recursive coatom ordering.*

Proof. For short, let L be the arrangement lattice of $\mathcal{H}$. First, **CL2**, see Definition 1.1.9, in an arrangement lattice is trivial. For every $c_0 \in L$, every $h, h' \prec c_0$ with the property that there exists $y \leqslant h, h'$, we have: $y \leqslant h \wedge h' \prec h, h'$.

So, we define for every unrefinable chain c that ends in $\hat{1}$, a linear ordered set $(L_{\prec c_0}, \ll_c)$ that verifies **CL1** and **CL3**. We proceed by induction on the length of c. For $c = (\hat{1})$, the trivial chain, we set on the coatoms of L any linear order with the property that an initial segment is a basis. This is, we choose a basis B of L, and we define $\ll_c$ to be any linear order for $L_{\prec c_0}$ with the property:

$$x \in B \text{ and } y \in L_{\prec c_0} \backslash B, \text{ then } x \ll_c y.$$

Now, assume that $\ll_c$ is defined for every unrefinable chain of length less than n, and it verifies **CL1** and **CL3**. Let c be an unrefinable chain of length n. To shorten the notation, we rename $h = c_0$ and $c' := d_0(c)$.

We have two options either $\bigwedge C_{c'}(h) = \hat{0}$ or $\bigwedge C_{c'}(h) > \hat{0}$. In the first case, there exists a basis B of $L_{\leqslant h}$ whose elements belong to $C_{c'}(h)$. Then, we define $\ll_c$ to be any order such that both B and $C_{c'}(h)$ are initial segments of $(L_{\prec h}, \ll_c)$, i.e.,

- for every $x \in B$ and every $y \in C_{c'}(h) \backslash B$, $x \leqslant y$; and
- for every $x \in C_{c'}(h)$ and every $y \in \mathcal{P}_{\prec h} \backslash C_{c'}(h)$, $x \leqslant y$.

Otherwise, we first show that $C_{c'}(h)$ is an independent set of coatoms. Let T be a proper subset of $C_{c'}(h)$. By definition of $C_{c'}(h)$, every $\omega \in C_c(h)$ can be expressed as $\omega = h \wedge h'$ for some $h' \ll_{c'} h$. Let $S = \{h' \in \mathcal{P}_{\prec c_1} \mid h' \underline{\ll}_{c'} h\}$ and $T' = \{h' \in S \mid h' \wedge h \in T\} \cup \{h\}$. Therefore we have:

$$\bigwedge T = \bigcap_{(h \cap h') \in T} (h \cap h') = \left(\bigcap_{(h \cap h') \in T} h' \right) \cap h = \bigcap_{h' \in T'} h' = \bigwedge T'.$$

$$\bigwedge C_{c'}(h) = \bigcap_{(h \cap h') \in C_{c'}(h)} (h \cap h') = \left(\bigcap_{(h \cap h') \in C_{c'}(h)} h' \right) \cap h = \bigcap_{h' \in S} h' = \bigwedge S.$$

Finally, by the induction hypothesis, an initial segment of $\left(L_{\prec c_1}, \ll_{c'}\right)$ is a basis, and S is an initial segment with $\bigwedge S > 0$. So, S is independent, and $T' \subsetneq S$, then we conclude:

$$\bigwedge T = \bigwedge T' < \bigwedge S = \bigwedge C_c(h).$$

Therefore, $C_{c'}(h)$ is independent. In this case, $\bigwedge C_{c'}(h) > \hat{0}$, we extend $C_{c'}(h)$ to a basis B of $L_{\leqslant h}$. We define $\ll_c$ to be any order that both B and $C_{c'}(h)$ are initial segments, this is:

- for every $x \in C_{c'}(h)$ and every $y \in B \backslash C_{c'}(h)$, $x \leqslant y$; and
- for every $x \in B$ and every $y \in \mathcal{P}_{\prec h} \backslash B$, $x \leqslant y$.

Therefore, $\ll$ is a basis-like recursive coatom ordering. $\square$

Chapter 2

LIMITS AND COLIMITS

To motivate the notion of categorical limit, we present an intentionally non-rigorous example that covers the intuition of this concept. In Calculus, the limit of a sequence $\{x_n\}$ seems to be the *last* element of this sequence. For example, if we consider the sequence $\{1/n\}_{n\in\mathbb{Z}^+}$, our intuition tells us that the *last* element of this sequence should be 0. In some sense, if we consider $\mathbb{Z}^+$ and $\mathbb{R}$ as a category induced by the standard order, this sequence defines a (contravariant) functor, $x\colon (\mathbb{Z}^+)^{\mathrm{op}} \to \mathbb{R}$, that we can represent as the following diagram:

$$\cdots \to 1/n \to \cdots \to 1/4 \to 1/3 \to 1/2 \to 1$$

This way, 0 is the "closer" element from the left to this sequence; this is, for every $r \in \mathbb{R}$ such that $r < 1/n$ for all $n \in \mathbb{Z}^+$, we have that $r \leqslant 0 < 1/n$, with more technical words, 0 is the *terminal* object with this property. This is the concept of categorical limits.

In this chapter, we recall the definition of limit from a theoretical point of view, and we give some examples of limits and computation tools for functors taking values in Mod_R. Later, we dualise these concepts to introduce the notion of colimit. Finally, we define the concept of a Kan extension and how limits and colimits help to compute them.

2.1 LIMIT

Let $\mathcal{C}$ be a small category, $\mathcal{M}$ be a category non-necessarily small, and $M \in \mathcal{M}$. We denote by $\underline{M}$ the constant functor on $\mathcal{C}$, this is, the functor $\underline{M}\colon \mathcal{C} \to \mathcal{M}$, that $\underline{M}(c) = M$, for all $c \in \mathcal{C}$; and $\underline{M}(c \to c') = \mathrm{Id}_M$, for all $(c \to c') \in \mathcal{C}$.

Definition 2.1.1. Let $\mathcal{M}$ be a category, $\mathcal{C}$ be a small category and $F\colon \mathcal{C} \to \mathcal{M}$ be a functor. A *cone* over F with *vertex* $M \in \mathcal{M}$ is a natural transformation $\eta\colon \underline{M} \to F$. That is, a cone over F is an object $M \in \mathcal{M}$, the vertex, together with a family of morphism in $\mathcal{M}$, $\{\eta_i\colon M \to F(i)\}_{i \in \mathcal{C}}$, such that for every map $u\colon i \to j$ in $\mathcal{C}$, the triangle

$$\begin{tikzcd} & F(i) \arrow[d, "F(u)"] \\ M \arrow[ur, "\eta_i"] \arrow[dr, "\eta_j"'] & \\ & F(j) \end{tikzcd}$$

commutes. A *limit* for F, if it exists, is the terminal object in the category of cones over F.

Theorem 2.1.2. *Let $\mathcal{M}$ be a category, $\mathcal{C}$ be an small category, and $F\colon \mathcal{C} \to \mathcal{M}$ be a functor. A limit for F, if it exists, is unique.*

Proof. This holds directly by the universal property of terminal objects. □

We denote the vertex of the limit for a functor $F\colon \mathcal{C} \to \mathcal{M}$ by $\lim_{\mathcal{C}} F$ or just $\lim F$ if the index category is understood. It is common to use the term *the limit of F* to refer to $\lim F$ itself rather than to the vertex of the limit.

Definition 2.1.3. Let $\mathcal{M}$ be a category. We say that $\mathcal{M}$ is *complete* or that $\mathcal{M}$ *has all small limits* if, for every small category $\mathcal{C}$ and every functor $F\colon \mathcal{C} \to \mathcal{M}$, the limit of F exists.

Example 2.1.4. The categories of sets Set, topological spaces Top, and R-modules Mod_R are complete.

Definition 2.1.5. Let $\mathcal{M}$ be a category and $\mathcal{I}$ be a discrete category. The limit of a functor $X\colon \mathcal{I} \to \mathcal{M}$, if it exists, is called the *product* of the family $\{X_i\}_{i\in\mathcal{I}}$, and it is denoted by:

$$\prod_{i\in\mathcal{I}} X_i := \lim_{\mathcal{I}} X.$$

Example 2.1.6. In some categories, products do not always exist. For example, in the case where $\mathcal{M}$ is a discrete category, there does not exist any non-trivial product in $\mathcal{M}$.

Example 2.1.7. In the categories of sets, topological spaces, and R-modules, the set-theoretical product is a categorical product. More precisely, for any small discrete category $\mathcal{I}$ and a functor $F\colon \mathcal{I} \to \mathrm{Set}$, $\mathcal{I} \to \mathrm{Top}$, or $\mathcal{I} \to \mathrm{Mod}_R$, the set-theoretical product $\prod_{i\in\mathcal{I}} F(i)$ together with the projection maps $\pi_j\colon \prod_{i\in\mathcal{I}} F(i) \to F(j)$ form a categorical product. The projections are part of their respective categories. The product topology is used to define the product of topological spaces, while the algebraic structure is given component-wise for R-modules. In both cases, the projections belong to their respective categories.

Example 2.1.8. Let $\mathcal{P}$ be a poset, and $x, y \in \mathcal{P}$. The product of x and y, if it exists, is the *greatest lower bound* or *meet*. We say that $\mathcal{P}$ is a *meet-semilattice* if products always exit in $\mathcal{P}$.

Definition 2.1.9. Let $\mathcal{M}$ be a category, $\mathcal{P}$ be the poset with three objects $b \leqslant a \geqslant c$, and $X\colon \mathcal{P} \to \mathcal{M}$ be a functor. The limit of X, if it exists, is called the *pullback* of the diagram:

$$\begin{tikzcd} & X_c \arrow[d] \\ X_b \arrow[r] & X_a, \end{tikzcd}$$

and it is denoted by:

$$X_c \times_{X_a} X_b := \lim_{\mathcal{P}} X.$$

We finish this section by showing some computation techniques.

Example 2.1.10. Let R be a commutative ring with identity, and let $\mathcal{C}$ be a small category. Given a functor $F\colon \mathcal{C} \to \mathrm{Mod}_R$, we form the product $\prod_{d\in\mathcal{C}} F(d)$, which is the set of tuples (x_c), where each $x_c \in F(c)$. For an object $c \in \mathcal{C}$, we have a projection map $\pi_c\colon \prod_{d\in\mathcal{C}} F(d) \to F(c)$, defined by $\pi_c((x_d)) = x_c$. However, the collection of projection maps $\{\pi_c\}_{c\in\mathcal{C}}$ does not form a cone unless $\mathcal{C}$ is a discrete category.

To construct a cone over F, we define a sub-module $L \leqslant \prod_{d\in\mathcal{C}} F(d)$ consisting of tuples (x_c) that are compatible in the sense that for every morphism $c \to c'$ in $\mathcal{C}$, the morphism $F(c \to c')$ sends x_c to $x_{c'}$, i.e.,

$$L := \left\{ (x_d) \in \prod_{d\in\mathcal{C}} F(d) \mid \text{for every } c \to c' \in \mathcal{C}, F(c \to c')(x_c) = x_{c'} \right\}.$$

The vertex L together with the projections $\{\pi_c \colon L \to F(c)\}$ form a cone over F, so it only remains to show that it is the terminal cone.

Let $v \colon \underline{M} \to F$ be a cone with vertex M. Then, there exists a unique R-linear map $\phi \colon M \to \prod_{d\in\mathcal{C}} F(d)$ such that $v_c = \pi_c \circ \phi$ for every object $c \in \mathcal{C}$.

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[dr, "v_c"'] & \prod_{d\in\mathcal{C}} F(d) \arrow[d, "\pi_c"] \\
& F(c).
\end{tikzcd}$$

Since v is a natural transformation, it follows that ϕ preserves the compatibility of tuples. Hence, $\phi(M) \leqslant L$, and we have a factorisation $M \to \phi(M) \to \prod_{d\in\mathcal{C}} F(d)$ of the cone v. Therefore, L is the limit of F.

Example 2.1.11. Let R be a commutative ring with identity, and let $\mathcal{C}$ be a small category. Consider a functor $F \colon \mathcal{C} \to \mathrm{Ch}(R)$, where $\mathrm{Ch}(R)$ is the category of unbounded cochain complexes of R-modules.

The limit of F can be computed degree-wise. That is, the limit of F can be written as follows:

$$\lim F \colon \qquad \dots \longrightarrow \lim F^{i-1} \longrightarrow \lim F^{i} \longrightarrow \lim F^{i+1} \longrightarrow \dots$$

where $\lim F^i$ is the limit of the functor $F^i \colon \mathcal{C} \to \mathrm{Mod}_R$.

Moreover, the differentials $\lim F^{i-1} \to \lim F^i$ are obtained by factoring the cone induced by the composite of the projection followed by the differential through $\lim F^i$,

$$\begin{tikzcd}
\lim F^i \arrow[r] & F^i(c) \\
\lim F^{i-1} \arrow[r] \arrow[u, dashed] & F^{i-1}(c). \arrow[u]
\end{tikzcd}$$

Definition 2.1.12. Let $\mathcal{C}$ be an small category, and $F\colon \mathcal{C} \to \mathrm{Mod}_R$ be a functor. The support of F, denoted by $\mathrm{supp}(F)$, is the full subcategory of $\mathcal{C}$ spanned by the objects $c \in \mathcal{C}$ such that $F(c) \neq 0$, i.e.,

$$\mathrm{supp}(F) = \{c \in \mathcal{C} \mid F(c) \neq 0\}.$$

Proposition 2.1.13. *Let $\mathcal{P}$ be a poset, and $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a functor. Let $\mathcal{B}$ be a subposet of $\mathcal{P}$ containing the support of F. If $\mathcal{B}$ is upper convex, then*

$$\lim_{\mathcal{B}} F \cong \lim_{\mathcal{P}} F.$$

Proof. We prove this isomorphism directly by the definition of limit as a universal object. Let $\eta\colon \underline{\lim_{\mathcal{B}} F} \to F|_{\mathcal{B}}$ be the limiting cone of $F|_{\mathcal{B}}$. Because the support of F is contained in $\mathcal{B}$, the limit $\lim_{\mathcal{B}} F$ is the vertex of a cone $\vartheta\colon \underline{\lim_{\mathcal{B}} F} \to F$ over F, defined as follows:

For $p \in \mathcal{P}$, we have

$$\vartheta_p(x) := \begin{cases} \eta_p(x) & \text{if } p \in \mathcal{B} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the restriction morphism induces a morphism $r\colon \lim_{\mathcal{P}} F \to \lim_{\mathcal{B}} F$ such that the following diagram commutes:

$$\begin{tikzcd} \underline{\lim_{\mathcal{P}} F} \arrow[d, "r"'] \arrow[dr] & \\ \underline{\lim_{\mathcal{B}} F} \arrow[r] & F. \end{tikzcd}$$

Thus, by the universal property of limit, we conclude that r is an isomorphism. $\square$

2.2 COLIMIT

Definition 2.2.1. Let $\mathcal{M}$ be a category, $\mathcal{C}$ be a small category and $F\colon \mathcal{C} \to \mathcal{M}$ be a functor. A *cocone* over F with vertex $M \in \mathcal{M}$ is a natural transformation

$\eta\colon F \to \underline{M}$, i.e., an object $M \in \mathcal{M}$, the *vertex*, together with a family of morphism in $\mathcal{M}$, $\{\eta^i\colon F(i) \to M\}_{i\in\mathcal{C}}$, such that for every map $u\colon i \to j$ in $\mathcal{C}$, the triangle

$$\begin{tikzcd} F(i) \arrow[dr, "\eta^i"] \arrow[dd, "F(u)"'] & \\ & M \\ F(j) \arrow[ur, "\eta^j"'] & \end{tikzcd}$$

commutes. A *colimit* for F, if it exists, is the initial object in the category of cocones over F.

Theorem 2.2.2. *Let $\mathcal{M}$ be category, $\mathcal{C}$ be an small category, and $F\colon \mathcal{C} \to \mathcal{M}$ be a functor. A colimit for F, if it exists, is unique.*

Proof. This holds by the universal property of initial objects. □

If it exists, we denote the colimit of $F\colon \mathcal{C} \to \mathcal{M}$ by $\operatorname{colim}_{\mathcal{C}} F$ or just $\operatorname{colim} F$ if the index category is understood.

Definition 2.2.3. Let $\mathcal{M}$ be a category. We say that $\mathcal{M}$ is *cocomplete* or $\mathcal{M}$ *has all small colimits* if, for every small category $\mathcal{C}$ and every functor $F\colon \mathcal{C} \to \mathcal{M}$, the colimit of F exists. We say that $\mathcal{M}$ is *bicomplete* if $\mathcal{M}$ is both complete and cocomplete.

Example 2.2.4. The categories of R-modules, sets, and topological spaces are bicompletes.

Definition 2.2.5. Let $\mathcal{M}$ be a category and $\mathcal{I}$ be a discrete category. The colimit of a functor $X\colon \mathcal{I} \to \mathcal{M}$, if it exists, is called the *coproduct* of the family $\{X_i\}_{i\in\mathcal{I}}$, and it is denoted by:

$$\coprod_{i\in\mathcal{I}} X_i := \operatorname*{colim}_{\mathcal{I}} X.$$

Example 2.2.6. The coproduct in the category of R-modules is the direct sum, where the algebraic structure is performed component-wise. However, in the categories of topological spaces and sets, the coproduct is the disjoint union. For topological spaces, the disjoint union is equipped with the weak topology.

Example 2.2.7. Let $\mathcal{P}$ be a poset, and $x, y \in \mathcal{P}$. The coproduct of x and y, if it exists, is the *least upper bound* or *join*. We say that $\mathcal{P}$ is a *join-semilattice* if coproducts always exit in $\mathcal{P}$.

Definition 2.2.8. Let $\mathcal{M}$ be a category, $\mathcal{P}$ be the poset with three objects $b \geqslant a \leqslant c$, and $X\colon \mathcal{P} \to \mathcal{M}$ be a functor. The colimit of X, if it exists, is called the *pushout* of the diagram:

$$\begin{tikzcd} X_a \arrow[r] \arrow[d] & X_c \\ X_b, & \end{tikzcd}$$

and it is denoted by

$$X_c \sqcup_{X_a} X_b := \operatorname*{colim}_{\mathcal{P}} X.$$

Example 2.2.9. Let R be a commutative ring with identity, $\mathcal{C}$ be a small category, and $F\colon \mathcal{C} \to \mathrm{Mod}_R$ be a functor.

We form the coproduct $\bigoplus_{d\in\mathcal{C}} F(d)$, which is, as we said before, the direct sum of the R-modules. Given an object $c \in \mathcal{C}$, we have an inclusion map $\iota_c\colon F(c) \to \bigoplus_{d\in\mathcal{C}} F(d)$, defined by

$$(\iota_c(x))_d = \begin{cases} x & \text{if } c = d \\ 0 & \text{otherwise.} \end{cases}$$

Analogous to the limit case, the collection of inclusion maps $\{\iota_c\}_{c\in\mathcal{C}}$ does not form a cocone unless $\mathcal{C}$ is a discrete category. Now, to construct a cocone over F, we define a quotient R-module

$$C = \left(\bigoplus_{d\in\mathcal{C}} F(d)\right) /R$$

where R is the submodule generated by elements of the form $x_c - F(c \to c')(x_c)$, for all morphisms $c \to c'$ in $\mathcal{C}$ and all $x_c \in F(c)$. Then, we have a canonical projection map $\rho_c\colon F(c) \to C$, defined by $\rho_c(x) = [\iota_c(x)]$, where $[-]$ denotes the equivalence class in C. The collection of projection maps $\{\rho_c\}_{c\in\mathcal{C}}$ form a cocone over F.

Moreover, if $v\colon F \to \underline{M}$ is another cocone with vertex M, then there exists a unique R-linear map $\phi\colon \oplus_{d\in\mathcal{C}} F(d) \to M$ such that $v_c = \phi \circ \iota_c$ for every object $c \in \mathcal{C}$,

$$\begin{tikzcd} F(c) \arrow[d, "\iota_c"] \arrow[dr, "v_c"] & \\ \bigoplus_{d\in\mathcal{C}} F(d) \arrow[r, "\phi"] & M. \end{tikzcd}$$

Since v is a natural transformation, it follows that ϕ preserves the equivalence classes. This implies that ϕ factorizes through C, and hence, C is the colimit of F.

Example 2.2.10. In a similar way to the situation with limits, we can compute colimits in the category of cochain complexes degree-wise; this is, the colimit of a functor $F : \mathcal{C} \to \mathrm{Ch}(R)$ is given degree-wise by the colimit of the functors $F^i : \mathcal{C} \to \mathrm{Mod}_R$. Furthermore, the differentials in the colimit are induced by the differentials in each object $c \in \mathcal{C}$ of the cochain complex $F(c)$.

2.3 KAN EXTENSION

Kan extensions are a way to *extend* functors through another one. Let $\mathcal{D}$ be a category, $\mathcal{C}$ be a subcategory of $\mathcal{D}$ and $i \colon \mathcal{C} \hookrightarrow \mathcal{D}$ be the inclusion functor. Given a functor $F \colon \mathcal{C} \to \mathcal{M}$, a Kan extension, if it exists, is an extension of F in $\mathcal{D}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, hook, "i"] & \mathcal{M} \\ \mathcal{D}. \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

That is, a Kan extension should be a kind of *inverse* for the restriction functor:

$$i^* \colon \mathrm{Fun}(\mathcal{D}, \mathcal{M}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{M}).$$

Definition 2.3.1. Let $p \colon \mathcal{C} \to \mathcal{D}$ be a functor, and $\mathcal{M}$ be a category. The *left Kan extension* along p, if it exists, is a left adjoint of the functor

$$p^* \colon \mathrm{Fun}(\mathcal{D}, \mathcal{M}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{M}).$$

It is typically denoted by $p_!$ or Lan_p. Dually, the *right Kan extension* along p, if it exists, is a right adjoint of p^*, denoted by p_* or Ran_p.

Right and left Kan extensions can be characterised in terms of limits and colimits, and comma categories.

Definition 2.3.2. Let $\mathcal{E}, \mathcal{D}$ and $\mathcal{C}$ be categories and $T \colon \mathcal{E} \to \mathcal{C}$ and $S \colon \mathcal{D} \to \mathcal{E}$ be a pair of functors. The *comma category* (T/S) is a category whose objects are triples (e, d, f), where $e \in \mathcal{E}$, $d \in \mathcal{D}$ and $f \colon T(e) \to S(d)$, and whose morphisms are pairs of morphisms (h, k) from (e, d, f) to (e', d', f') such that $h\colon e \to e'$ is a morphism in $\mathcal{E}$, $k\colon d \to d'$ is a morphism in $\mathcal{D}$ and the following square commutes

$$\begin{tikzcd} T(e) \arrow[r, "T(h)"] \arrow[d, "f"] & T(e') \arrow[d, "g"] \\ S(d) \arrow[r, "T(k)"] & S(d'). \end{tikzcd}$$

If T or S is the constant functor $c \in \mathcal{C}$, we use the notation (c/S) or (T/c), respectively.

Proposition 2.3.3 ([ML98, Theorem X.3.1]). *Let $\mathcal{D}$ be a small category, and $\mathcal{M}$ be a bicomplete category. Then:*

1. *the right Kan extension of a functor $F\colon \mathcal{D} \to \mathcal{M}$ along a functor $p\colon \mathcal{C} \to \mathcal{D}$ exists and given $c \in \mathcal{C}$:*

$$(\mathrm{Ran}_p F)(c) = \lim \Big((c/p) \to \mathcal{D} \xrightarrow{F} \mathcal{M} \Big);$$

2. *the left Kan extension of F along p exists and given $c \in \mathcal{C}$:*

$$(\mathrm{Lan}_p F)(c) = \operatorname{colim} \Big((p/c) \to \mathcal{D} \xrightarrow{F} \mathcal{M} \Big).$$

In both cases, given $g\colon c \to c'$, the value of the Kan extension for g is given by the induced map between the (co)limits.

Chapter 3

MODEL CATEGORIES

Quillen presents in his seminal work [Qui67] the notion of model category, or «a category of models for a homotopy theory», as a category $\mathcal{M}$, endowed with three distinguished families of morphisms called weak equivalences, cofibrations and fibrations satisfying certain axioms, the most important being the following two: the first one is the **LIFTING** axiom, given a commutative square:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

where i is a cofibration, p is a fibration, and either i or p is also a weak equivalence, there exists a morphism $h\colon B \to X$ such that $h \circ i = f$ and $p \circ h = g$. The next axiom is about **FACTORISATION**, every map f can be factored both, as $f = p \circ i$ and as $f = p' \circ i'$ where p, p' are fibrations, i, i' are cofibrations and p, i' are also weak equivalences.

This notion sets a very general framework to do homotopy theory without having topological spaces involved. The main references of this section are Dwyer-Spalinski's survey [DS95], Hirschhorn's book [Hir03], Hovey's [Hov99], Balchin's [Bal21] and Riehl's [Rie14].

3.1 DEFINITION AND EXAMPLES

As we say in the motivation, one of the main objectives for defining model categories is about *lifting*. Given $i\colon A \to B$ and $p\colon X \to Y$ two morphism in a category $\mathcal{C}$, a *lifting problem* between i and p is a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y. \end{tikzcd} \tag{3}$$

A *solution of the lifting problem* or just a *lift*, is a morphism $h\colon B \to X$ such that the resulting diagram with five arrows commutes, this is, $h \circ i = f$ and $p \circ h = g$. A morphism $i\colon A \to B$ is said to have the *left lifting property* (LLP for short) with respect $p\colon X \to Y$ and p is said to have the *right lifting property* (RLP for short) with respect to i if every lifting problem between i and p has a solution.

Definition 3.1.1. A *model category structure* on a bicomplete category $\mathcal{M}$ is a triple of classes of morphisms $(\mathrm{Weak}_{\mathcal{M}}, \mathrm{Fib}_{\mathcal{M}}, \mathrm{Cof}_{\mathcal{M}})$:

$\xrightarrow{\sim}$ $\mathrm{Weak}_{\mathcal{M}}$, the weak equivalences;

$\hookrightarrow$ $\mathrm{Fib}_{\mathcal{M}}$, the fibrations; and

$\twoheadrightarrow$ $\mathrm{Cof}_{\mathcal{M}}$, the cofibrations.

These distinguished classes must satisfy the following axioms:

CLOSED Each of these classes is closed under composition and contains every identity morphism.

LIFTING Every lifting problem between i and p, see Diagram (3), has (as least) a solution if i is a cofibration and p is a fibration and either i or p is a weak equivalence.

FACTORISATION Each morphism f in $\mathcal{M}$ can be factored in two ways: as a cofibration followed by a fibration that is a weak equivalence and as a cofibration that is a weak equivalence followed by a fibration

$$\begin{tikzcd} & A \arrow[dr, two heads] & \\ X \arrow[ur, hook, "\sim"] \arrow[rr, two heads, "f"] \arrow[dr, hook] & & Y. \\ & B \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

2-OUT-3 Given two composable morphism f and g, if two of f, g and $g \circ f$ are weak equivalences, then so is the third.

RETRACTS The three classes are closed under retracts; this is, if f, g are morphisms in $\mathcal{M}$ such that

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "g"] \arrow[rr, bend left, "\mathrm{Id}_A"] & X \arrow[r, "r"] \arrow[d, "f"] & A \arrow[d, "g"] \\ B \arrow[r, "i'"] \arrow[rr, bend right, "\mathrm{Id}_B"'] & Y \arrow[r, "r'"] & B. \end{tikzcd}$$

If f is a weak equivalence (resp. fibration, cofibration), then g is a weak equivalence (resp. fibration, cofibration).

We say that a morphism that is both a fibration and a weak equivalence is an *acyclic fibration*. Similarly, a morphism that is a cofibration and a weak equivalence is an *acylic cofibration*.

By abuse of notation and only if there is no confusion, we say that $\mathcal{M}$ is a model category instead of $\mathcal{M}$ is a category equipped with the three distinguished classes of morphisms. Sometimes, if in a category we work with many model category structures, we name it by the sub-index.

Definition 3.1.2. Let $\mathcal{M}$ be a model category. An object $X \in \mathcal{M}$ is said to be:

1. *fibrant* if the unique morphism $X \to *$ is a fibration; and

2. *cofibrant* if the unique morphism $\varnothing \to X$ is a cofibration.

Now, we present some classical examples of model category structures.

Example 3.1.3 (Strøm). The category of topological spaces Top can be equipped with a model category structure defining a continuous map $f : X \to Y$ to be:

$\xrightarrow{\sim}$ a weak equivalence if it is a homotopy equivalence;

$\hookrightarrow$ a cofibration if it is a closed Hurewicz cofibration; and

$\twoheadrightarrow$ a fibration if it is a Hurewicz fibration.

This model category structure is named the Strøm model category on Top, for more details we refer the reader to Strøm's original papers [Str72]

Example 3.1.4. There is a more useful model category structure for the category of topological spaces Top, the classical model category structure, in which a continuous map $f\colon X \to Y$ is defined to be:

$\xrightarrow{\sim}$ a weak equivalence if it is a weak homotopy equivalence;

$\hookrightarrow$ a cofibration if it is a retract of a map $X \to Y'$ in which Y' is obtained from X by attaching cells; and

$\twoheadrightarrow$ a fibration if it has the right lifting property with respect to all inclusions of the form
$$D^n \to D^n \times \{0\} \hookrightarrow D^n \times I.$$

This model category structure is due to Quillen [Qui67].

Example 3.1.5 ([Qui67]). Let R be a commutative ring with unit, and $\mathrm{Ch}(R)$ be the category of (unbounded) cochain complexes. There is a model category structure on $\mathrm{Ch}(R)$ in which a map $f\colon C \to D$ is:

$\xrightarrow{\sim}$ a weak equivalence if it induces an isomorphism in cohomology;

$\hookrightarrow$ a cofibration if it is a degreewise monomorphism with degreewise cofibrant cokernel; and

$\twoheadrightarrow$ a fibration if it is a degreewise epimorphism.

This model category structure is called the *projective model category structure*; see [MP12, Section 18.5] .

Example 3.1.6 ([Qui67]). Let R be a commutative ring with unit, and $\mathrm{Ch}(R)$ be the category of (unbounded) cochain complexes. There is a model category structure on $\mathrm{Ch}(R)$ in which a map $f\colon C \to D$ is a

$\xrightarrow{\sim}$ a weak equivalence if it induces an isomorphism in cohomology;

$\hookrightarrow$ a cofibration if it is a degreewise monomorphism; and

$\twoheadrightarrow$ a fibration if it is a degreewise epimorphism with a degreewise fibrant kernel.

This model category structure is called the *injective model category structure.*

Remark 3.1.7. In this text, we do not distinguish between cochain complexes and chain complexes since we are not imposing any bounded conditions on them. However, if we have a cochain complex C with non-zero values concentrated in non-positive degree, $C^i = 0$ for every $i > 0$, we will denote it as a chain complex by using the notation $C_i = C^{-i}$.

Proposition 3.1.8 ([MP12, Proposition 18.5.2]). *Let* $\mathrm{Ch}(R)$ *be the category of cochain complexes equipped with the projective model category structure, and* $C \in \mathrm{Ch}(R)$.

1. $0 \to C$ *is an acyclic cofibration if and only if* C *is a projective object in* $\mathrm{Ch}(R)$.
2. *If* C *is cofibrant, then* C *is degreewise projective.*
3. *If* C *is bounded above and degreewise projective, then* C *is cofibrant*

Proposition 3.1.9 ([MP12, Proposition 18.5.4]). *Let* $\mathrm{Ch}(R)$ *be the category of cochain complexes equipped with the injective model category structure, and* $C \in \mathrm{Ch}(R)$.

1. $C \to 0$ *is an acyclic fibration if and only if* C *is an injective object in* $\mathrm{Ch}(R)$.
2. *If* C *is fibrant, then* C *is degreewise injective.*
3. *If* C *is bounded below and degreewise injective, then* C *is fibrant*

Fibrant and cofibrant objects play a fundamental role in model category theory as they allow us to define and study homotopy theory in a general context. If an object is not fibrant or cofibrant, by the **FACTORISATION** axiom, there are nice substitutes for them that are fibrant or cofibrant.

Definition 3.1.10. Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}$.

1. A *fibrant replacement* of X is a fibrant object $\mathbf{R}X$ equipped with a weak equivalence $X \xrightarrow{\sim} \mathbf{R}X$.
2. A *cofibrant replacement* of X is a cofibrant object $\mathbf{Q}X$ equipped with a weak equivalence $\mathbf{Q}X \xrightarrow{\sim} X$.

3.2 HOMOTOPY BETWEEN MAPS

The first step for defining the homotopy category is to define homotopies between morphisms. In the context of topological spaces, we have two ways for defining a homotopy between two maps $f, g\colon X \to Y$: The first one is by the *cylinder* $X \times I$, a homotopy between f and g is a map $H\colon X \times I \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$; the second one by the *paths* of Y, this is the mapping space $\operatorname{map}(I, Y)$ with the compact-open topology, a homotopy between f and g is a map $H\colon X \to \operatorname{Path}(Y)$, such that for every $x \in X$, $H(x)(0) = f(x)$ and $H(x)(1) = g(x)$. In topological spaces, these definitions are equivalent, but in a model category, these definitions need not be equivalent.

Definition 3.2.1. Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}$. A *cylinder object* for X is an object $\operatorname{cyl}(X)$ together with a diagram

$$X \sqcup X \xrightarrow{i_1 + i_2} \operatorname{cyl}(X) \xrightarrow{\sim} X$$

which factors the folding map $\mathrm{Id}_X + \mathrm{Id}_X$.

Definition 3.2.2. Two morphism $f, g\colon X \to Y$ in $\mathcal{M}$ are said to be *left homotopic*, and we denote it by $f \overset{l}{\sim} g$, if there exists a cylinder object $\operatorname{cyl}(X)$ for X such that the sum $f + g\colon X \sqcup X \to Y$ can be extended to a map $H\colon \operatorname{cyl}(X) \to Y$

$$\begin{tikzcd} X \sqcup X \arrow[r, "f+g"] \arrow[d] & Y \\ \operatorname{cyl}(X). \arrow[ur, dashed, "H"] & \end{tikzcd}$$

Example 3.2.3. In the category of topological spaces, equipped with the classical model category structure, $X \times I$ is a cylinder for X and a left homotopy between f and g is a map $H\colon X \times I \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$.

Lemma 3.2.4 ([DS95, Lemma 4.7])**.** *Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}$. If X is a cofibrant object, then $\overset{l}{\sim}$ is an equivalence relation on* $\operatorname{Hom}_{\mathcal{M}}(X, Y)$.

To define right homotopic maps, we must define the *paths* in a model category.

Definition 3.2.5. Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}$. A *cocylinder object* or *path object* for X is an object $\operatorname{cocyl}(X)$ together with a diagram:

$$X \xrightarrow{\sim} \operatorname{cocyl}(X) \xrightarrow{p} X \times X$$

which factors the diagonal map $(\mathrm{Id}_X, \mathrm{Id}_X)\colon X \to X \times X$.

Definition 3.2.6. Two morphism $f, g\colon X \to Y$ in $\mathcal{M}$ are said to be *right homotopic*, we denote it by $f \overset{r}{\sim} g$, if there exists a cocylinder object $\mathrm{cocyl}(Y)$ for Y such that the product map $(f, g)\colon X \to Y \times Y$ lifts to a map $H\colon X \to \mathrm{cocyl}(Y)$

$$\begin{tikzcd} & \mathrm{cocyl}(Y) \arrow[d] \\ X \arrow[r, "(f,g)"'] \arrow[ur, dashed, "H"] & Y \times Y. \end{tikzcd}$$

Example 3.2.7. In the category of topological spaces, equipped with the classical model category structure, $\mathrm{Path}(Y)$ is a cocylinder object for Y, and a right homotopy between f and g is a map $H\colon X \to \mathrm{Path}(Y)$ such that

$$H(-)(0) = f(-) \text{ and } H(-)(1) = g(-).$$

Lemma 3.2.8 ([DS95, Lemma 4.15]). *Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}$. If Y is a fibrant object, then $\overset{r}{\sim}$ is an equivalence relation on* $\mathrm{Hom}_{\mathcal{M}}(X, Y)$

Definition 3.2.9. Let $\mathcal{M}$ be a model category and $f, g\colon X \to Y$ be a pair of maps. If $f \overset{l}{\sim} g$ and $f \overset{r}{\sim} g$, then we say that f is *homotopic* to g and we denote it by $f \sim g$.

Proposition 3.2.10 ([DS95, Lemma 4.21]). *Let $f, g\colon X \to Y$ be a pair of maps in a model category $\mathcal{M}$. Then,*

1. *if X is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$; and*
2. *if Y is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.*

Proposition 3.2.11 ([DS95, Lemma 4.24]). *Let $f\colon X \to Y$ be a map in a model category $\mathcal{M}$ between objects that are both fibrant and cofibrant. Then f is a weak equivalence if and only if there exists $g\colon Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are homotopic to the respective identity maps.*

Example 3.2.12. In the classical model category structure on Top, every object is fibrant, and cofibrant objects are the ones that are retracts of *generalised CW-complex*. Then, Whitehead's theorem, see [Hat02, Theorem 4.5], is a corollary of the proposition above.

Definition 3.2.13. Let $\mathcal{M}$ be a model category. The *homotopy category* $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$ is the category with the same object as $\mathcal{M}$ and, given $X, Y \in \mathcal{M}$, the hom-set:

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y) = \mathrm{Hom}(\mathbf{Q}X, \mathbf{R}Y)/\sim.$$

The homotopy category of a model category can be characterised by its weak equivalences by formally inverting them.

Definition 3.2.14. Let $\mathcal{M}$ be a category and W be a class of morphism. A functor $F\colon \mathcal{M} \to \mathcal{D}$ is said to be a *localisation* of $\mathcal{M}$ with respect to W if:

(i) $F(f)$ is an isomorphism for each $f \in W$; and

(ii) If $G\colon \mathcal{M} \to \mathcal{D}'$ is a functor verifying (i), then there exists a unique functor $G'\colon \mathcal{D} \to \mathcal{D}'$ such that $G' \circ F = G$.

Given a model category $\mathcal{M}$, the homotopy category $\mathrm{Ho}(\mathcal{M})$ can be characterised as a localisation of $\mathcal{M}$ with respect to the class of weak equivalences.

Theorem 3.2.15 ([Qui67, Chapter I]). *Let $\mathcal{M}$ be a model category. There exists a localisation functor $\gamma\colon \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ that is the identity on objects and sends weak equivalences into isomorphisms in $\mathrm{Ho}(\mathcal{M})$.*

3.3 DERIVED FUNCTORS

A functor $F\colon \mathcal{M} \to \mathcal{D}$ from a model category $\mathcal{M}$ does not always induce a functor from the homotopy category. However, the left- and right-derived functors play the role of *the best approximation* of a hypothetical functor in the homotopy category

$$\tilde{F}\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{M}$$

by the respective side.

Definition 3.3.1. Let $\mathcal{M}$ be a model category, $\gamma\colon \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ be the localisation functor, and $F\colon \mathcal{M} \to \mathcal{D}$ be a functor.

(a) A *left-derived functor* of F, if it exists, is the right Kan extension of F along γ, this is, it is a functor $LF\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{D}$ together with a natural transformation $t\colon LF \circ \gamma \to F$ such that for every other functor $G\colon \mathrm{Ho}(\mathcal{M}) \to \mathcal{D}$ and other natural transformation $s\colon G \circ \gamma \to F$, there exists a natural transformation $s'\colon F \to LF$ such that the following diagram commutes:

$$\begin{tikzcd} G \circ \gamma \arrow[r, "s' \circ \gamma"] \arrow[dr, "s"'] & LF \circ \gamma \arrow[d, "t"] \\ & F. \end{tikzcd}$$

(b) A *right-derived functor* of F, if it exists, is the left Kan extension of F along γ, this is, it is a functor $RF\colon \operatorname{Ho}(\mathcal{M}) \to D$ together with a natural transformation $t\colon F \to RF \circ \gamma$ such that for every other functor $G\colon \operatorname{Ho}(\mathcal{M}) \to \mathcal{D}$ and other natural transformation $s\colon F \to G \circ \gamma$, there exists a natural transformation $s'\colon RF \to G$ that such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[d, "t"] \arrow[dr, "s"] & \\ LF \circ \gamma \arrow[r, "s' \circ \gamma"] & G \circ \gamma. \end{tikzcd}$$

Definition 3.3.2. Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between model categories, and $\gamma\colon \mathcal{N} \to \operatorname{Ho}(\mathcal{N})$ be localisation functor. A *total left-derived functor* $\mathbb{L}F$ for F is a functor

$$\mathbb{L}F\colon \operatorname{Ho}(\mathcal{M}) \to \operatorname{Ho}(\mathcal{N})$$

which is a left-derived functor for the composite $\gamma \circ F\colon \mathcal{M} \to \operatorname{Ho}(\mathcal{N})$.

Similarly, a *total right-derived functor* $\mathbb{R}F$ for F is a functor

$$\mathbb{R}F\colon \operatorname{Ho}(\mathcal{M}) \to \operatorname{Ho}(\mathcal{N})$$

which is a right-derived functor for the composite $\gamma \circ F$.

The following example shows that (co)limits do not preserve weak equivalences.

Example 3.3.3. Let $j_n\colon S^{n-1} \to D^n$ be the inclusion of the $(n-1)$-sphere as the boundary of the n-disk. Then, consider the following diagram

$$\begin{tikzcd} D^n \arrow[d] & S^{n-1} \arrow[l, "j_n"'] \arrow[r, "j_n"] \arrow[d, "\mathrm{Id}"] & D^n \arrow[d] \\ * & S^{n-1} \arrow[l] \arrow[r] & *. \end{tikzcd}$$

Despite the horizontal arrows being weak homotopy equivalences, the pushouts of the horizontal arrows are not homotopy equivalents.

$$\begin{tikzcd} S^{n-1} \arrow[r, "j_n"] \arrow[d, "j_n"'] & D^n \arrow[d, dashed] \\ D^n \arrow[r, dashed] & S^n \end{tikzcd} \qquad \begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d, dashed] \\ * \arrow[r, dashed] & *. \end{tikzcd}$$

Definition 3.3.4. Let $\mathcal{M}$ be a model category, $\mathcal{C}$ be a small category such that the category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ admits a model category structure in which weak equivalences are object-wise weak equivalences, and $F \colon \mathcal{C} \to \mathcal{M}$ be a functor. The *homotopy colimit* of F is the total left-derived functor of colim evaluated on F, this is,

$$\mathrm{hocolim}\, F = (\mathbb{L}\, \mathrm{colim}(-))(F).$$

The *homotopy limit* of F is the total right-derived functor of lim evaluated on F, this is,

$$\mathrm{holim}\, F = (\mathbb{R} \lim(-))(F).$$

In the context of homological algebra, the existence of a derived functor is given by the exactness of F and projective or injective resolution [Wei94]. Since we work in the framework of *homotopical algebra*, this existence is provided by Quillen pair.

Definition 3.3.5. Let $\mathcal{M}$ and $\mathcal{N}$ be two model categories, and $L \colon \mathcal{M} \rightleftarrows \mathcal{N} \colon R$ be a pair of adjoint functors. We say that (L, R) is a *Quillen pair* if the following equivalent conditions are satisfied:

1. L preserves cofibrations and acyclic cofibrations;
2. R preserves fibrations and acyclic fibrations;
3. L preserves cofibrations and R preserves fibrations; and
4. L preserves acyclic cofibrations and R preserves acyclic fibrations.

Lemma 3.3.6 (K. Brown [Hov99, Lemma 1.1.12])**.** *Let $L \colon \mathcal{M} \rightleftarrows \mathcal{N} \colon R$ be a Quillen pair, then:*

1. *the left adjoint L preserves weak equivalences between cofibrant objects; and*
2. *the right adjoint R preserves weak equivalences between fibrant objects.*

Corollary 3.3.7. *Let $\mathcal{M}$ be a model category, and $\mathcal{C}$ be a small category such that the category of functors* $\mathrm{Fun}(\mathcal{C}, \mathcal{M})$ *admits a model category in which weak equivalences are object-wise weak equivalences. If* $\lim \colon \mathrm{Fun}(\mathcal{C}; \mathcal{M}) \rightleftarrows \mathcal{M} \colon \Delta$*, where Δ is the diagonal functor, form a Quillen pair then:*

$$\mathrm{holim}\, F \cong \lim \mathbf{R}F,$$

where $\mathbf{R}F \colon \mathcal{C} \to \mathcal{M}$ is a fibrant replacement for F. Dually, If $\Delta \colon \mathcal{M} \rightleftarrows$ $\mathrm{Fun}(\mathcal{C}; \mathcal{M})$ *is a Quillen pair, then:*

$$\operatorname{hocolim} F \cong \operatorname{colim} \mathbf{Q}F,$$

where $\mathbf{Q}F$ *is a cofibrant replacement for* F.

Chapter 4

REEDY STRUCTURE

A Reedy structure on a category $\mathcal{R}$ is a powerful tool to induce a model category structure on the category of functors $\mathrm{Fun}(\mathcal{R}, \mathcal{M})$ when $\mathcal{M}$ is a model category. Examples of Reedy categories are the simplex category Δ and its dual Δ^{op}. Roughly speaking, a Reedy structure on a category $\mathcal{R}$ is a degree function $\mathrm{Ob}(\mathcal{R}) \to \mathbb{N}$ together with two wide subcategories $\overrightarrow{\mathcal{R}}$, the direct category, and $\overleftarrow{\mathcal{R}}$, the inverse one, satisfying certain compatibility axioms. The main disadvantage of a Reedy structure is that it does not allow the underlying category to have non-trivial automorphism. To solve this problem, we will work with generalised Reedy categories, following Berger-Moerdjick's work [BM11]. However, we also refer the reader to Hirschornn's book [Hir03, Chapter 15] or Reedy's unpublished work *Homotopy Theory of Model Categories* [Ree] to read about classical Reedy categories.

Definition 4.1. A *generalised Reedy structure* on a small category $\mathcal{R}$ consist of two wide subcategories: $\overrightarrow{\mathcal{R}}$, the direct category, and $\overleftarrow{\mathcal{R}}$, the inverse one; and a degree-function $\mathrm{d}\colon \mathrm{Ob}(\mathcal{R}) \to \mathbb{N}$ satisfying the following axioms:

1. non-invertible morphism in $\overrightarrow{\mathcal{R}}$ (resp. $\overleftarrow{\mathcal{R}}$) raise (resp. lower) the degree; isomorphism in $\mathcal{R}$ preserve the degree;
2. $\overrightarrow{\mathcal{R}} \cap \overleftarrow{\mathcal{R}} = \mathrm{Iso}(\mathcal{R})$;
3. every morphism f of $\mathcal{R}$ factors as $f = g \circ h$ with $g \in \overrightarrow{\mathcal{R}}$ and $h \in \overleftarrow{\mathcal{R}}$, and this factorisation is unique up to isomorphism;
4. if $\theta \circ f = f$ for $\theta \in \mathrm{Iso}(\mathcal{R})$ and $f \in \overrightarrow{\mathcal{R}}$, then θ is an identity; and
5. if $f \circ \theta = f$ for $\theta \in \mathrm{Iso}(\mathcal{R})$ and $f \in \overleftarrow{\mathcal{R}}$, then θ is an identity.

A *generalised Reedy category* is a small category equipped with a generalised Reedy structure.

Remark 4.2. In Berger and Moerdijk's original work [BM11], the 5-th axiom corresponds with the notion of dualisable generalised Reedy structure. However, we include it in the definition because every category that appears in this work trivially satisfies this axiom.

Example 4.3. If $\mathcal{R}$ is a generalised Reedy category, then $\mathcal{R}^{\mathrm{op}}$ is also a generalised Reedy category.

Example 4.4. If $\mathcal{P}$ is a filtered poset with degree function d, then it is a generalised Reedy category with $\overrightarrow{\mathcal{P}} = \mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ being the discrete category of objects in $\mathcal{P}$. In fact, this is a (classical) Reedy structure; see [Hir03, Chapter 15]

Example 4.5. Given a finite group G, the *orbit category* $\mathcal{O}(G)$ admits a generalised Reedy category structure setting $\overrightarrow{\mathcal{O}(G)} = \mathcal{O}(G)$, $\overleftarrow{\mathcal{O}(G)} = \mathrm{Iso}(G)$, with a degree function $\mathrm{d}(G/H) = \#H$.

Example 4.6. More generally, if $\mathcal{C}$ is a filtered EI-category with degree function d, then it is a generalised Reedy category with $\overrightarrow{\mathcal{C}} = \mathcal{C}$, and $\overleftarrow{\mathcal{C}} = \mathrm{Iso}(\mathcal{C})$.

4.1 FUNCTORS AND NATURAL TRANSFORMATION

A generalised Reedy category is not just *a good tool* to induce a model category structure in the category of functors, as we will see in the next section. The combinatorics of the generalised Reedy structure also provides a powerful tool to construct functors and natural transformations. To simplify the discussion, we treat the EI-category $\mathcal{C}$ as a poset. Later, we will see how this intuitive idea translates to the more general case.

To shorten notation, given a generalised Reedy category $\mathcal{C}$ with degree-function d, we denote $\mathcal{C}^n$ the full subcategory of $\mathcal{C}$ spanned by the objects $c \in \mathcal{C}$ of degree $\mathrm{d}(c) \leqslant n$.

Notice that $\mathcal{C}^0$ contains no non-identity morphisms; thus, define a functor $F\colon \mathcal{C}^0 \to \mathcal{M}$ is to choose for every $c \in \mathcal{C}^0$ an object $F(c)$.

Now, we turn on the inductive machinery of the generalised Reedy structure on $\mathcal{C}$. Given a functor $F\colon \mathcal{C}^n \to \mathcal{M}$, our purpose is to *extend* F to $\mathcal{C}^{n+1}$. Let $c \in \mathcal{C}$ be an object of degree $n+1$ and let $F(c) \in \mathcal{M}$. For each object $d \in \mathcal{C}^n$ with $d < c$ we need to choose a morphism $F(d) \to F(c)$ with the additional property that for every $d' \in \mathcal{C}^n$ $d' < d \leqslant c$ the following diagram commutes

$$\begin{tikzcd} F(d') \arrow[d] \arrow[dr] & \\ & F(c). \\ F(d) \arrow[ur] & \end{tikzcd}$$

This is, for every object $c \in \mathcal{C}$ of degree $n+1$, choose an object $F(c) \in \mathcal{M}$ together with a morphism:

$$\operatorname*{colim}_{d<c} F(d) \to F(c).$$

Similarly, given two functors $F, G\colon \mathcal{C} \to \mathcal{M}$, we can construct a natural transformation $\eta\colon F \to G$ by induction. We start by choosing for every $c \in \mathcal{C}^0$ a morphism in $\mathcal{M}$

$$\eta_c\colon F(c) \to G(c).$$

Then, given a natural transformation $\eta\colon (F|_{\mathcal{C}^n}) \to (G|_{\mathcal{C}^n})$, extend it to $\mathcal{C}^{n+1}$ is to choose for every object $c \in \mathcal{C}$ of degree $\mathrm{d}(c) = n+1$, a morphism in $\mathcal{M}$, $\eta_c\colon F(c) \to G(c)$ such that the following diagram commutes:

$$\begin{tikzcd} \operatorname{colim}_{d<c} F(d) \arrow[r] \arrow[d] & F(c) \arrow[d, dashed, "\eta_c"] \\ \operatorname{colim}_{d<c} G(d) \arrow[r] & G(c). \end{tikzcd}$$

Dually, if we have a functor $F\colon (\mathcal{C}^n)^{\mathrm{op}} \to \mathcal{M}$, extend it to a functor in $\mathcal{C}^{n+1}$ is equivalent to choose for every $c \in \mathcal{C}$ of degree $\mathrm{d}(c) = n+1$, an object $F(c) \in \mathcal{M}$ together with a morphism

$$F(c) \to \lim_{d<c} F(d).$$

Similarly, given two functors $F, G\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{M}$, and a natural transformation $\eta\colon (F|_{\mathcal{C}^n}) \to (G|_{\mathcal{C}^n})$, extend η to $\mathcal{C}^{n+1}$ is to choose for every object $c \in \mathcal{C}$ of degree $n+1$, a morphism in $\mathcal{M}$, $\eta_c\colon F(c) \to G(c)$ such that the following diagram commutes:

$$\begin{tikzcd} F(c) \arrow[r] \arrow[d, dashed, "\eta_c"] & \lim_{d<c} F(d) \arrow[d] \\ G(c) \arrow[r] & \lim_{d<c} G(d). \end{tikzcd}$$

We will devote the rest of this section to formalising these notions and extending them to the case of EI-categories.

Let $\mathcal{R}$ be a generalised Reedy category. For every $n \in \mathbb{N}$, the category $\mathbb{G}_n(\mathcal{R})$ denotes the full subgroupoid of $\mathcal{R}$ spanned by the objects of degree n; the category $\overrightarrow{\mathcal{R}}(n)$ has objects the non-invertible arrows $u\colon s \to r$ in $\overrightarrow{\mathcal{R}}$ such that $\mathrm{d}(r) = n$, and as morphism $\alpha\colon (s \xrightarrow{u} r) \to (s' \xrightarrow{u'} r')$ the commutative squares:

$$\begin{tikzcd} s \arrow[r, "\alpha_0"] \arrow[d, "u"] & s' \arrow[d, "u'"] \\ r \arrow[r, "\alpha_1"] & r' \end{tikzcd}$$

such that $\alpha_0 \in \overrightarrow{\mathcal{R}}$ and $\alpha_1 \in \mathbb{G}_n(\mathcal{R})$.

Dually, the category $\overleftarrow{\mathcal{R}}(n)$ has objects the non-invertible arrows $u\colon r \to s$ in $\overleftarrow{\mathcal{R}}$ such that $\mathrm{d}(r) = n$, and as morphisms $\alpha\colon (r \xrightarrow{u} s) \to (r' \xrightarrow{u'} s')$ the commutative squares:

$$\begin{tikzcd} r \arrow[r, "\alpha_0"] \arrow[d, "u"] & r' \arrow[d, "u'"] \\ s \arrow[r, "\alpha_1"] & s' \end{tikzcd}$$

such that $\alpha_1 \in \overleftarrow{\mathcal{R}}$ and $\alpha_0 \in \mathbb{G}_n(\mathcal{R})$.

We denote by $s_n\colon \overrightarrow{\mathcal{R}}(n) \to \mathcal{R}$, and $s^n\colon \overleftarrow{\mathcal{R}}(n) \to \mathbb{G}_n(\mathcal{R})$ the respective domain-functors, and by $t_n\colon \overleftarrow{\mathcal{R}}(n) \to \mathbb{G}_n(\mathcal{R})$, and $t^n\colon \overleftarrow{\mathcal{R}}(c) \to \mathcal{R}$ the respective codomain-functors.

Definition 4.1.1. Let $\mathcal{R}$ be a generalised Reedy category, $\mathcal{M}$ be a cocomplete category and $F\colon \mathcal{R} \to \mathcal{M}$ be a functor. We define the *latching object* of F at $r \in \mathcal{R}$ to be

$$L_r F := (s_n)_!(t_n)^*(F)(r) = \operatorname*{colim}_{s \to r} X_s$$

where the limit is taken over the full subcategory of $\overrightarrow{\mathcal{R}}/r$ of non-invertible morphism.

Proposition 4.1.2. *Let $\mathcal{P}$ be a filtered poset, and $F\colon \mathcal{P} \to \mathrm{Mod}_R$ be a functor. For every $p \in \mathcal{P}$, the latching object of F at p is given by the following formula:*

$$L_pF = \operatorname*{colim}_{\mathcal{P}_{<p}} F.$$

Proof. Given a poset $\mathcal{P}$, and $p \in \mathcal{P}$, the category $\overrightarrow{\mathcal{P}}/p$ is equivalent to the category of arrows $q \to p$, i.e., $q < p$, and whose morphism are commutative triangles:

$$\begin{tikzcd} q \arrow[d] \arrow[dr] & \\ q' \arrow[r] & p. \end{tikzcd}$$

But this is just the category $\mathcal{P}_{<p}$. □

Then, for an EI-category $\mathcal{C}$, the inductive step described at the beginning of the section can be reformulated to choose an object $F(c) \in \mathcal{M}$ together with a morphism

$$L_cF \to F(c).$$

Definition 4.1.3. Let $\mathcal{R}$ be a generalised Reedy category, $\mathcal{M}$ be a complete category and $F\colon \mathcal{R} \to \mathcal{M}$ be a functor. We define the *matching object* of F at $r \in \mathcal{R}$ to be

$$M_rF := (s_n)_*(t_n)^*(F)(r) = \lim_{r \to s} X_s$$

where the limit is taken over the full subcategory of $r/\overleftarrow{\mathcal{R}}$ of non-invertible morphism.

Proposition 4.1.4. *Let $\mathcal{P}$ be a filtered poset, and $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a functor. For every $p \in \mathcal{P}$, the matching object of F at p is given by the following formula:*

$$M_pF = \lim_{\mathcal{P}_{<p}} F.$$

Proof. This result holds by dualising the argument in the proof of Proposition 4.1.2. □

4.2 MODEL CATEGORY STRUCTURE

Originally, Reedy categories were introduced to produce a new model category structure in the category of *simplicial objects* in a model category $\mathcal{M}$. In this document, we use the notion of generalised Reedy categories to produce a model category in the category of functors that allows describing higher limits in terms of fibrant replacements; see Corollary 3.3.7.

Definition 4.2.1. Let $\mathcal{R}$ be a generalised Reedy category, $\mathcal{M}$ be a bicomplete category, and $\eta\colon F \to G$ be a natural transformation between functors in $\mathrm{Fun}(\mathcal{R}, \mathcal{M})$. For every $r \in \mathcal{R}$ we define:

- the *relative latching map* to be the morphism induced by the pushout:

$$F(r) \coprod_{L_r F} L_r G \to G(r).$$

- the *relative matching map* to be the morphism induced by the pullback

$$F(r) \to M_r F \times_{M_r G} G(r).$$

Given a group G, we define by $\mathcal{M}^G$ the category of objects in $\mathcal{M}$ equipped with a G-action. This is equivalent to the category of functors from the category consisting of a single object $*$ and $\mathrm{Aut}(*) = G$. If $\mathcal{M}$ is equipped with a cofibrantly generated model category, see [Hir03, Section 11.6], the category $\mathcal{M}^G$ carries a projective model structure; this is, a G-equivariant morphism $f\colon X \to Y$ is a weak equivalence or a fibration if it is, respectively, a weak equivalence or a fibration by forgetting the G-action.

Definition 4.2.2. Let $\mathcal{R}$ be a generalised Reedy category and $\mathcal{M}$ be a model category. One says that $\mathcal{M}$ is $\mathcal{R}$-projective if for each object $r \in \mathcal{R}$, the category $\mathcal{M}^{\mathrm{Aut}(r)}$ admits a projective model structure. Moreover, it is said to be $\mathcal{R}$-bijective if, for every $r \in \mathcal{R}$, the forgetful functor $\mathcal{U}\colon \mathcal{M}^{\mathrm{Aut}(r)} \to \mathcal{M}$ also detects cofibrations, i.e., a morphism $f\colon m \to m'$ is a cofibration in $\mathcal{M}^{\mathrm{Aut}(r)}$ if and only if $\mathcal{U}(f)$ is a cofibration in $\mathcal{M}$.

Theorem 4.2.3 ([BM11, Theorem 1.6]). *Let $\mathcal{R}$ be a generalised Reedy category, and $\mathcal{M}$ be a $\mathcal{R}$-projective model category. There is a model category structure on* $\mathrm{Fun}(\mathcal{R}, \mathcal{M})$ *in which a natural transformation $\eta\colon F \to G$ is:*

- $\xrightarrow{\sim}$ *a weak equivalence if, for every $r \in \mathcal{R}$, the morphism $\eta_r\colon F(r) \to G(r)$ is a weak equivalence in $\mathcal{M}$;*
- $\hookrightarrow$ *a cofibration if, for every $r \in \mathcal{R}$, the relative latching morphism*

$$F(r) \sqcup_{L_r F} L_r G \to G(r)$$

  is a cofibration in $\mathcal{M}^{\mathrm{Aut}(r)}$; and
- $\twoheadrightarrow$ *a fibration if, for every $r \in \mathcal{R}$, the relative matching morphism*

$$F(r) \to M_r F \times_{M_r G} G(r)$$

  is a fibration in $\mathcal{M}$.

Remark 4.2.4. If $\mathcal{M}$ is a $\mathcal{R}$-bijective model category, we can replace $\mathcal{M}^{\mathrm{Aut}(r)}$ by $\mathcal{M}$ in the definition of cofibrations.

Part II

THESIS RESULTS

Chapter 5

HIGHER LIMITS

Let R be a commutative ring with unit, and $\mathcal{C}$ be a small category. The category of R-modules has all limits and colimits; thus, these constructions define respective functors:

$$\lim\colon \operatorname{Fun}(\mathcal{C}, \operatorname{Mod}_R) \to \operatorname{Mod}_R, \qquad \operatorname{colim}\colon \operatorname{Fun}(\mathcal{C}, \operatorname{Mod}_R) \to \operatorname{Mod}_R\,.$$

The limit functor is left exact functor but is not right exact. Since the category of R-modules has enough injectives, the right-derived functor of lim exists, see [Wei94, Chapter 2]. Given a functor F, there is an injective resolution of functors:

$$F \to I^0 \to I^1 \to \cdots \to I^n \to \ldots$$

Higher limits of F are the derived functors of lim evaluated in F, i.e., the cohomology of the cochain complex obtained from I^* by applying lim:

$$H^*(F;\mathcal{C}) := R^* \lim(-)(F) = H^*(\lim I^*).$$

Dually, colim is a right exact functor that is not left exact. Then, there exists a projective resolution:

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to F.$$

Higher colimits of F are the homology of the chain complex obtained from P_* by applying colim:

$$H_*(\mathcal{C}; F) = L_* \operatorname{colim}(-)(F) = H_*(\operatorname{colim} P_*).$$

Classically, higher (co)limits are computed as the cohomology of a cochain complex associated with the functor; we refer the reader to Grodal's thesis [Gro02, Section 2] or Aschbacher, Kessar and Oliver's book [AKO11, Subsection III.5.1].

In this chapter, we describe higher limits of functor indexed over filtered EI-categories using techniques from *homotopical algebra* instead of homological algebra.

5.1 HOMOTOPY THEORETICAL APPROACH

From now on, $\mathcal{C}$ will be a filtered EI-category with degree function $\mathrm{d}\colon \mathcal{C} \to \mathbb{N}$. A functor $F\colon \mathcal{C} \to \mathrm{Mod}_R$ is considered as an object of $\mathrm{Fun}(\mathcal{C}, \mathrm{Ch}(R))$ by setting: for every $c \in \mathcal{C}$, $F(c)$ to be a cochain complex concentrated in degree 0.

To describe a model category structure in the category of functor $\mathrm{Fun}(\mathcal{C}, \mathrm{Ch}(R))$, we use the generalised Reedy structure described in Example 4.6 for filtered EI-categories, that is, $\overleftarrow{\mathcal{C}} = \mathrm{Iso}(\mathcal{C})$ and $\overrightarrow{\mathcal{C}} = \mathcal{C}$ with degree function d.

For practical reasons, we fix the convention that higher colimits are computed in the category of covariant functors $\mathrm{Fun}(\mathcal{C}, \mathrm{Ch}(R))$ and higher limits in the category of contravariant functor $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}(R))$. This is not a strong assumption because the notion of the generalised Reedy category, see Example 4.3, is self-dual. Therefore, if $F\colon \mathcal{C} \to \mathrm{Mod}_R$ is a covariant functor indexed in a generalised Reedy category, then $F\colon (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Mod}_R$ is a contravariant functor indexed in the generalised Reedy category $\mathcal{C}^{\mathrm{op}}$.

Theorem 5.1.1. *Let $\mathcal{C}$ be a filtered EI-category, R be a ring such that $|\operatorname{Aut}(c)|$ is invertible in R for every $c \in \mathcal{C}$, and $\operatorname{Ch}(R)$ be the category of unbounded cochain complexes equipped with a cofibrantly generated model category. Then, $\operatorname{Ch}(R)$ is $\mathcal{C}$-bijective; see Definition 4.2.2.*

Proof. As $\operatorname{Ch}(R)$ is equipped with a cofibrantly generated model category structure, for every $c \in \mathcal{C}$, there is a model category structure on $\operatorname{Ch}(R)^{\operatorname{Aut}(c)}$, see [Bal21, Section 4.5], in which an $\operatorname{Aut}(c)$-equivariant morphism $f\colon C \to D$ is defined to be:

$\xrightarrow{\sim}$ a weak equivalence if f is a weak equivalence on $\operatorname{Ch}(R)$;

$\hookrightarrow$ a cofibration if f verifies LLP with respect to acyclic fibrations; and

$\twoheadrightarrow$ a fibration if f is a fibration on $\operatorname{Ch}(R)$.

We will prove that under the assumption that $|\operatorname{Aut}(c)|$ is invertible in R, a morphism in $\operatorname{Ch}(R)^{\operatorname{Aut}(c)}$ is a cofibration if and only if it is a cofibration in $\operatorname{Ch}(R)$ by forgetting the $\operatorname{Aut}(c)$-action.

Let $i : X \to Y$ be a cofibration in $\operatorname{Ch}(R)^{\operatorname{Aut}(c)}$. For every acyclic fibration in $\operatorname{Ch}(R)^{\operatorname{Aut}(c)}$, $p : A \twoheadrightarrow B$, and every $\operatorname{Aut}(c)$-equivariant commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"] & A \arrow[d, two heads, "p", "\sim"'] \\ Y \arrow[r, "f'"] & B, \end{tikzcd}$$

there exists an $\operatorname{Aut}(c)$-equivariant map $h : Y \to A$ such that $h \circ i = f$ and $p \circ h = f'$. Since the class of acyclic fibrations in $\operatorname{Ch}(R)^{\operatorname{Aut}(c)}$ and in $\operatorname{Ch}(R)$ are the same, we see that i verifies LLP with respect to every acyclic fibration in $\operatorname{Ch}(R)$. Thus, we conclude that i is a cofibration in $\operatorname{Ch}(R)$.

Conversely, let $i : X \to B$ be an $\operatorname{Aut}(c)$-equivariant map which is also a cofibration in $\operatorname{Ch}(R)$, and $p : A \to B$ be an $\operatorname{Aut}(c)$-equivariant acyclic fibration. Given an $\operatorname{Aut}(c)$-equivariant commutative diagram,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"] & A \arrow[d, two heads, "p", "\sim"'] \\ Y \arrow[r, "f'"] & B, \end{tikzcd} \tag{4}$$

the existence of a morphism $h\colon Y \to A$ between cochain complex such that the $h \circ i = f$ and $p \circ h = f'$ follows since the class of acyclic fibrations in $\mathrm{Ch}(R)$ and $\mathrm{Ch}(R)^{\mathrm{Aut}(c)}$ are the same. The map h does not need to be $\mathrm{Aut}(c)$-equivariant map. If not, we define $\widetilde{h} : Y \to A$ by $\widetilde{h}(y) = |\operatorname{Aut}(c)|^{-1} \sum_{g \in \mathrm{Aut}(c)} gh(g^{-1}y)$. One can check that $\widetilde{h}$ is $\mathrm{Aut}(c)$-equivariant. It remains to show that $\widetilde{h}$ is a lift for Diagram (4):

$$\begin{aligned} p \circ \widetilde{h}(y) =& p(|\operatorname{Aut}(r)| \sum_g gh(g^{-1}y)) = |\operatorname{Aut}(r)|^{-1} \sum_g gph(g^{-1}y) = \\ &= |\operatorname{Aut}(r)|^{-1} \sum_g gf'(g^{-1}y) = \widetilde{f}'(y) = f'(y). \end{aligned}$$

$$\begin{aligned} \widetilde{h} \circ i(x) =& |\operatorname{Aut}(r)| \sum_g gh(g^{-1}i(x))) = |\operatorname{Aut}(r)|^{-1} \sum_g gh \circ i(g^{-1}y) = \\ &= |\operatorname{Aut}(r)|^{-1} \sum_g gf(g^{-1}x) = \widetilde{f}(x) = f(x). \end{aligned}$$

Then, i is a cofibration in $\mathrm{Ch}(R)^{\mathrm{Aut}(c)}$. Therefore, $\mathrm{Ch}(R)$ is $\mathcal{C}$-bijective. □

Remark 5.1.2. We only require the model category in $\mathrm{Ch}(R)$ to be cofibrantly generated, and, in this thesis, we only consider the projective and injective model category in $\mathrm{Ch}(R)$. Therefore, in the following, we abuse notation saying that R is a $\mathcal{C}$-bijective ring instead of $\mathrm{Ch}(R)$ is a $\mathcal{C}$-bijective model category, whenever $\mathrm{Ch}(R)$ is equipped with the projective or injective model category.

Now, we present the model category that we will use to compute higher limits.

Proposition 5.1.3. *Let $\mathcal{C}$ a filtered EI-category, R be a $\mathcal{C}$-bijective ring. Then, there is a model category structure on the category of functors* $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}(R))$ *in which a natural transformation $\eta\colon X \to Y$, is a*

$\xrightarrow{\sim}$ *weak equivalence if, for every object $c \in \mathcal{C}$, the morphism $\eta_c\colon X(c) \to Y(c)$ induces an isomorphism in cohomology;*

$\hookrightarrow$ *cofibration if, for every object $c \in \mathcal{C}$, the morphism $\eta_c\colon X(c) \to Y(c)$ is a split monomorphism with cofibrant cokernel; and*

$\twoheadrightarrow$ *fibration if, for every object $c \in \mathcal{C}$, the relative matching morphism,*

$$X(c) \to Y(c) \times_{M_c Y} M_c X,$$

is an epimorphism.

We call this model category structure the inverse model category.

Proof. We combine the model category structure for the category of functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}(R))$ described in Theorem 4.2.3 with the projective model category structure, see Example 3.1.5. Then, since $|\mathrm{Aut}(c)|$ is invertible in R for every $c \in \mathcal{C}$, we apply Theorem 5.1.1 to describe the cofibrations. $\square$

Corollary 5.1.4. *Given a filtered EI-category $\mathcal{C}$, and a $\mathcal{C}$-bijective ring R. A functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_R$ is fibrant if for every $c \in \mathcal{C}$, the matching morphism:*

$$F(c) \to M_c F$$

is an epimorphism.

Proof. This holds directly by the definition of a fibrant object; see Definition 3.1.2. Consider the unique natural transformation $F \to 0$. Then, the relative matching morphism at $c \in \mathcal{C}$, described in Proposition 5.1.3, becomes $F(c) \to M_c F$. Therefore, the natural transformation $F \to 0$ is a fibration if and only if, for every $c \in \mathcal{C}$, the morphism $F(c) \to M_c F$ is an epimorphism. $\square$

Now, we present some examples of fibrant functors.

Example 5.1.5. Let $\mathcal{P}$ be a filtered poset with an initial object, and M be an R-module. Then, the constant functor $\underline{M} \colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ is a fibrant functor.

Example 5.1.6. Let $\mathcal{P}$ be the face poset of a shellable complex. Let $F \colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Vect}_k$ be the functor defined on objects by $\sigma \mapsto k[\sigma]$, where $k[\sigma]$ is the free k-vector space generated by the vertices of σ; and the image of an inclusion $\sigma \subset \tau$ by F is the epimorphism induced by:

$$s \in \sigma \mapsto \begin{cases} s & \text{if } s \in \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Then, F is a fibrant functor.

Example 5.1.7. Let G be a group and $H \leqslant G$ be a subgroup. Let $\mathcal{I}$ be the category with two objects 0 and 1 and:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{I}}(0,0) &= G & \qquad \mathrm{Hom}_{\mathcal{I}}(0,1) &= \varnothing \\ \mathrm{Hom}_{\mathcal{I}}(1,1) &= \{1\} & \qquad \mathrm{Hom}_{\mathcal{I}}(1,0) &= G/H, \end{aligned}$$

and the composition is described by the product in G and by the left action of G on G/H; see Example 1.2. If $F\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Mod}_{\mathbb{Q}}$ is a functor, then, by Corollary 5.1.4, F is fibrant iff the matching morphism at 1,

$$F(1) \to M_1 F,$$

is an epimorphism. A direct computation shows that $M_1 F = F(0)^H$. Then we can conclude that a functor is fibrant iff the natural morphism $\varepsilon\colon F(1) \to F(0)^H$ is an epimorphism. This category was presented in Aguadé's paper [Agu89] in which he realises various polynomial algebras over $\mathbb{F}_p$ as the cohomology rings of spaces constructed as homotopy colimits.

Dually, to compute higher colimits, we present the following model category structure in the category of covariant functor.

Proposition 5.1.8. *Let $\mathcal{C}$ a filtered EI-category, R be a $\mathcal{C}$-bijective ring. Then there is a model category structure on the category of functors* $\mathrm{Fun}(\mathcal{C}, \mathrm{Ch}(R))$ *in which a natural transformation $\eta\colon X \to Y$, is a:*

$\xrightarrow{\sim}$ *weak equivalence if, for every object $c \in \mathcal{C}$, the morphism $\eta_c\colon X(c) \to Y(c)$ induces an isomorphism in cohomology;*

$\hookrightarrow$ *cofibration if, for every object $c \in \mathcal{C}$, the relative latching morphism,*

$$X(c) \sqcup_{L_c X} L_c Y \to Y(c),$$

is a monomorphism; and

$\twoheadrightarrow$ *fibration if, for every object $c \in \mathcal{C}$, the morphism $X(c) \to Y(c)$ is a split epimorphism with fibrant kernel.*

We call this model category structure the direct model category.

Proof. As in Proposition 5.1.3, we combine the model category structure for the category of functors $\mathrm{Fun}(\mathcal{C}, \mathrm{Ch}(R))$ described in Theorem 4.2.3 with the injective model category structure, see Example 3.1.6. Then, since $|\mathrm{Aut}(c)|$ is invertible in R for every $c \in \mathcal{C}$, we can apply Theorem 5.1.1 to describe the cofibrations. □

Corollary 5.1.9. *Given a filtered EI-category $\mathcal{C}$, and a $\mathcal{C}$-bijective ring R. A functor $F\colon \mathcal{C} \to \mathrm{Mod}_R$ is cofibrant if for every $c \in \mathcal{C}$, the latching morphisms:*

$$L_c F \to F(c)$$

is a monomorphism.

Proof. This holds directly by dualising Corollary 5.1.4 □

Example 5.1.10. Let $\mathcal{P}$ be a filtered poset with an initial object, and M be an R-module. Then, the constant functor $\underline{M} \colon \mathcal{P} \to \mathrm{Mod}_R$ is a cofibrant functor.

Example 5.1.11. Let $\mathcal{P}$ be the face poset of a simplicial complex, and (D, I) be a twin pair of functors. That is, $D \colon \mathcal{P} \to \mathrm{Ab}$ is a covariant functor and $I \colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ is a contravariant functor, such that:

1. for every $\sigma \in \mathcal{P}$, $D(\sigma) = I(\sigma)$, and

2. every pullback diagram:

$$\begin{tikzcd} \sigma \cap \sigma' \arrow[r, "i"] \arrow[d, "i'"] & \sigma \arrow[d, "j"] \\ \sigma' \arrow[r, "j'"] & \tau, \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} I(\sigma \cap \sigma') \arrow[r, "D(i)"] & I(\sigma) \\ I(\sigma') \arrow[u, "I(i')"] \arrow[r, "D(j')"] & I(\tau). \arrow[u, "I(j)"'] \end{tikzcd}$$

Then, D is cofibrant in the direct model category, and I is fibrant in the inverse model category. For more details, we refer the reader to Notbohm-Ray [NR05].

The following results describe how to compute higher limits via fibrant replacement.

Proposition 5.1.12. *Let $\mathcal{C}$ be a filtered EI-category, and R be a $\mathcal{C}$-bijective ring. Given a functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_R$, then*

$$H^i(F; \mathcal{C}) = H^i(\lim \mathbf{R}F)$$

where $\mathbf{R}F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}(R)$ is a fibrant replacement of F in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ch}(R))$ with the inverse model category; see Proposition 5.1.3.

Proof. By Lemma 3.3.6, the pair of functors $\Delta \colon \mathrm{Ch}(R) \leftrightarrows \mathrm{Fun}(\mathcal{P}^{\mathrm{op}}, \mathrm{Ch}(R)) \colon \lim$ is a Quillen pair because the diagonal functor Δ sends weak equivalences and cofibrations into weak equivalences and cofibrations. Therefore, by Corollary 3.3.7, homotopy limits can be computed by a fibrant replacement. Then, we are done by the fact that higher limits are the cohomology of the homotopy limit of a functor concentrated in degree 0, see [Wei94, Corollary 10.5.7],

$$H^i(F; \mathcal{C}) = H^i(\operatorname{holim} F) = H^i(\lim \mathbf{R}F).$$

□

Thanks to this result, we extract directly vanishing bounds from the *height* of a fibrant replacement.

Definition 5.1.13. Let C be a bounded cochain complex. The *height* of C, denoted by $h(C)$, is defined to be the integer n such that $C^k = 0$ for all $k > n$ and $C^n \neq 0$. For a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}(R)$ we define the *height* of F to be the supremum of the heights, i.e.,

$$h(F) = \sup\{h(F(c)) \mid c \in \mathcal{C}\}.$$

Corollary 5.1.14. *Let $\mathcal{C}$ be a filtered EI-category, and R be a $\mathcal{C}$-bijective ring. Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a functor. If $\mathbf{R}F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}(R)$ is a fibrant replacement of F such that $h(\mathbf{R}F) = n$, then*

$$H^i(\mathcal{C}; F) = 0$$

for every $i > n$.

Proof. This follows directly from Proposition 5.1.12. □

Dually, we can compute higher colimits by a cofibrant replacement

Proposition 5.1.15. *Let $\mathcal{C}$ be a filtered EI-category, and R be a $\mathcal{C}$-bijective ring. Given a functor $F\colon \mathcal{C} \to \mathrm{Mod}_R$, then*

$$H_i(F; \mathcal{C}) = H_i(\operatorname{colim} \mathbf{Q}F)$$

where $\mathbf{Q}F\colon \mathcal{C} \to \mathrm{Ch}(R)$ is a cofibrant replacement of F in $\mathrm{Fun}(\mathcal{C}, \mathrm{Ch}(R))$ with the direct model category; see Proposition 5.1.8.

Proof. This result holds by dualising the proof of Proposition 5.1.12. □

As before, we can obtain vanishing bounds from the depth of a cofibrant replacement.

Definition 5.1.16. Let C be a bounded chain complex. The *depth* of C, denoted by $\mathrm{depth}(C)$, is the integer n such that $C_k = 0$ for all $k > n$ and $C_n \neq 0$. For a functor $F\colon \mathcal{C} \to \mathrm{Ch}(R)$ we define the *depth* of F to be the supremum of the heights, i.e.,

$$\mathrm{depth}(F) = \sup\{\mathrm{depth}(F(c)) \mid c \in \mathcal{C}\}.$$

Corollary 5.1.17. *Let $\mathcal{C}$ be a filtered EI-category, and R be a $\mathcal{C}$-bijective ring. Let $F\colon \mathcal{C} \to \mathrm{Mod}_R$ be a functor. If $\mathbf{Q}F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}(R)$ is a cofibrant replacement of F such that $\mathrm{depth}(\mathbf{Q}F) = n$, then*

$$H_i(\mathcal{C}; F) = 0$$

for every $i > n$.

Proof. This follows directly from Proposition 5.1.15. □

5.2 ON FIBRANT REPLACEMENT CONSTRUCTIONS

The idea underlying the preceding results is that higher limits can be computed by means of a fibrant replacement rather than an injective resolution. Consequently, the goal of this section is to present a systematic approach for computing a fibrant replacement of a given functor. To describe it explicitly, we need the following notions.

Definition 5.2.1. Let $\mathcal{C}$ be a filtered EI-category, and R be a $\mathcal{C}$-bijective ring. A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_R$ is said to be *locally fibrant* at $c \in \mathcal{C}$, if for every d such that either there is a non-invertible arrow $d \to c$ or $d = c$, the matching map:

$$F(d) \to M_d F$$

is an epimorphism.

Remark 5.2.2. A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}(R)$ is fibrant if and only if it is locally fibrant at c for all $c \in \mathcal{C}$.

Our desired fibrant replacement will not change the functor whenever it is already fibrant. The method we present proceeds by induction on the degree of the objects. At each step, if the object is not locally fibrant, we need to transform the functor using the following construction.

First, we review the notion of the mapping cocylinder and its factorisation property.

Definition 5.2.3. Let $f\colon C \to D$ be a map between unbounded cochain complexes. The mapping cocylinder of f is the cochain complex $\mathrm{cocyl}(f)$ whose degree n part is

$$\mathrm{cocyl}(f)^n := C^n \times D^{n-1} \times D^n,$$

and differential is given by the formula

$$\partial(c, d, d') = (\partial c, d' - f(c) - \partial d, \partial d').$$

Proposition 5.2.4 ([Wei94, Section 1.5]). *Let $f\colon C \to D$ be a morphism between cochain complexes. Then, there is a factorisation of f:*

$$\begin{array}{rcccl} C & \xrightarrow{i} & \mathrm{cocyl}(f) & \xrightarrow{\pi} & D \\ c & \longmapsto & (c, 0, f(c)) & & \\ & & (c, d, d') & \longmapsto & d' \end{array}$$

where i is a monomorphism inducing an isomorphism in cohomology, and π is a split epimorphism.

Next, we introduce a new concept that provides us with a factorisation property similar to the mapping cocylinder.

Definition 5.2.5. A morphism $f\colon C \to D$ between cochain complexes is said to be *truncatable* if $h(C) < h(D)$, and the differential $D^{h(D)-1} \to D^{h(D)}$ is onto.

Definition 5.2.6. Let $f\colon C \to D$ be a truncatable morphism between cochain complexes with $h(D) = n + 1$.

1. The *truncation* of D is the cochain complex $\mathbf{T}D$ whose degree k part is

$$(\mathbf{T}D)^k := \begin{cases} D^k & \text{if } k \leqslant n \\ 0 & \text{if } k > n, \end{cases}$$

and the *truncation* of f is the morphism $\mathbf{T}f\colon C \to \mathbf{T}D$, defined by

$$(\mathbf{T}f)^k := \begin{cases} f^k & \text{if } k \leqslant n \\ 0 & \text{if } k > n. \end{cases}$$

2. The *truncated mapping cocylinder* of f, denoted by $\mathrm{cocyl}_{\mathbf{T}}(f)$, is the mapping cocylinder of the morphism $\mathbf{T}(f)$

$$\mathrm{cocyl}_{\mathbf{T}}(f) := \mathrm{cocyl}\left(C \overset{\mathbf{T}(f)}{\to} \mathbf{T}D\right).$$

Proposition 5.2.7. *Let $f\colon C \to D$ be a truncatable morphism between cochain complexes. Then, f factors through the truncated mapping cocylinder as a weak equivalence followed by an epimorphism,*

$$C \xrightarrow{\sim} \mathrm{cocyl}_{\mathbf{T}}(f) \to D.$$

Proof. The inclusion $i\colon C \to \mathrm{cocyl}_{\mathbf{T}}(f)$ is the one given by the mapping cocylinder, see Proposition 5.2.4. The morphism $\pi\colon \mathrm{cocyl}_{\mathbf{T}}(f) \twoheadrightarrow D$ is given by:

$$\pi^k := \begin{cases} 0 & \text{if } k > n+1 \\ \partial_D : D^n \to D^{n+1} & \text{if } k = n+1 \\ \pi_{D^k} : C^k \times D^{k-1} \times D^k \to D^k & \text{if } k \leqslant n, \end{cases}$$

where π_{D^k} is the projection and ∂_D is the differential of D. □

Now, we have all the ingredients to describe how to construct fibrant replacements for a given functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_R$. The generalised Reedy structure on $\mathcal{C}$ allows us to follow an inductive strategy.

In the full subcategory of $\mathcal{C}$ spanned by the objects of degree 0, there are no morphisms between different objects. So, for every $c \in \mathcal{C}$ of degree 0, the matching object $M_cF = 0$, thus, the first step is to define

$$\mathbf{R}F(c) := F(c).$$

Assume that $\mathbf{R}F$ is already defined in the full subcategory of objects of degree less than n. In order to define $\mathbf{R}F$ on $c \in \mathcal{C}$ of degree n, we need to choose a factorisation of the composite $\varepsilon_c\colon F(c) \to M_cF \to M_c\mathbf{R}F$ as a weak equivalence followed by an epimorphism. That is, to choose an object $\mathbf{R}F(c)$ together with a weak equivalence $F(c) \to \mathbf{R}F(c)$ and an epimorphism $\mathbf{R}F(c) \to M_c\mathbf{R}F$ such that the following diagram commutes:

$$\begin{tikzcd} F(c) \arrow[r] \arrow[d, dashed, "\sim"] & M_cF \arrow[d] \\ \mathbf{R}F(c) \arrow[r, dashed, two heads] & M_c\mathbf{R}F. \end{tikzcd} \tag{5}$$

To construct this factorisation properly, we follow the next rules:

RULE 1: If $M_c\mathbf{R}F$ is concentrated in degree 0 and ε_c is surjective. Then, define $\mathbf{R}F(c)$ as $F(c)$ and the trivial factorisation of ε_c,

$$F(c) \xrightarrow{\mathrm{Id}_{F(c)}} \mathbf{R}F(c) = F(c) \xrightarrow{\varepsilon_c} M_c\mathbf{R}F.$$

RULE 2: If the composite ε_c is truncatable. Then, define $\mathbf{R}F(c)$ as the truncated mapping cocylinder $\mathrm{cocyl}_{\mathbf{T}}(\varepsilon_c)$ and the factorisation described in Proposition 5.2.7.

$$F(c) \to \mathrm{cocyl}_{\mathbf{T}}(\varepsilon_c) \to M_c\mathbf{R}F.$$

RULE 0: In general, define $\mathbf{R}F(c)$ as the mapping cocylinder $\mathrm{cocyl}(\varepsilon_c)$ and the factorisation described in Proposition 5.2.4,

$$F(c) \to \mathrm{cocyl}(\varepsilon_c) \to M_c\mathbf{R}F.$$

Note that **RULE 1** and **RULE 2** can be considered guidelines that *the user* can omit since one can always apply **RULE 0** and, in each case, the transformation becomes a fibrant replacement. Thus, by choosing each step to follow every rule or just **RULE 0**, we can produce a *custom* fibrant replacement that is optimal for our purpose. For example, in Chapter 7, the fibrant replacement applying all rules when it is possible will play a central role. But in Chapter 8, for practical reasons, we will work with the fibrant replacement constructed just by applying only **RULE 0**. Thus, to simplify the notation, we give a name to both extreme cases.

Definition 5.2.8. Let $\mathcal{C}$ be a filtered EI-category, R be a $\mathcal{C}$-bijective ring and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a functor. The functor $\mathbf{T}F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ch}(R)$ denote the fibrant replacement of F constructed inductively following the next rules:

1. if F is locally fibrant at c, $\mathbf{T}F(c) := F(c)$;
2. if $\varepsilon_c\colon F(c) \to M_c\mathbf{T}F$ is truncatable, $\mathbf{T}F(c) = \mathrm{cocyl}_{\mathbf{T}}(\varepsilon_c)$; and
3. otherwise, $\mathbf{T}F(c) = \mathrm{cocyl}(\varepsilon_c)$.

Notice that the truncability of the morphism $\varepsilon_c\colon F(c) \to M_c\mathbf{T}F$ implies that $h(M_c\mathbf{T}F) > 0$. Thus, if ε_c is truncatable, then F cannot be locally fibrant at c.

Definition 5.2.9. Let $\mathcal{C}$ be a filtered EI-category, R be a $\mathcal{C}$-bijective ring, and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a functor. The *cocylinder* of F, denoted by $\mathbf{cocyl}\,(F)$, is the fibrant replacement of F constructed inductively by choosing for every $c \in \mathcal{C}$,

$$\mathbf{cocyl}\,(F)\,(c) := \begin{cases} F(c) & \text{if } \mathrm{d}(c) = 0, \\ \mathrm{cocyl}(\varepsilon_c) & \text{otherwise.} \end{cases}$$

We end this section with an example that illustrates the difference between both fibrant replacements and shows how $\mathbf{T}F$ could induce vanishing bounds directly.

Example 5.2.10. Let $\mathbb{N}$ be the poset of natural numbers with the trivial filtration $\mathrm{Id}_{\mathbb{N}}\colon \mathbb{N} \to \mathbb{N}$, and $F\colon \mathbb{N}^{\mathrm{op}} \to \mathrm{Ab}$ be a functor. Notice that the matching object at $n > 0$, is the value of the functor at $n-1$ and the matching morphism is just $F(n-1 < n)$, this is,

$$F(n) \overset{F(n-1<n)}{\longrightarrow} M_nF = F(n-1).$$

Therefore, F is locally fibrant at n iff for every $k \leqslant n$, $F(k-1<k)\colon F(k) \to F(k-1)$ is an epimorphism. Assume the morphisms $F(0 < 1)$ is not surjective. This implies that F is not locally fibrant at 1; thus, we need to define $\mathbf{T}F(1)$ to be the mapping cocylinder of $F(0<1)$.

$$\begin{tikzcd} x_1 - F(0<1)(x_0) & & F(1) \\ (x_0, x_1) \arrow[u, mapsto] & & F(0)\times F(1) \arrow[u, two heads] \arrow[r] & F(0) \\ & & (x_0,x_1) \arrow[r, mapsto] & x_0. \end{tikzcd}$$

Now, observe that the differential of $\mathbf{T}F(1)$ is an epimorphism, so the map $\varepsilon_2\colon F(2) \to M_2\mathbf{T}F = \mathbf{T}F(1)$ is truncatable, thus we define $\mathbf{T}F(2)$ to be the truncated mapping cocylinder of ε_2. More precisely, $\mathbf{T}F(2)$ is the cochain complex whose degree n part is:

$$\mathbf{T}F(2)^n := \begin{cases} F(0)\times F(1) & \text{if } n = 1 \\ F(0)\times F(1)\times F(2) & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and the non-zero differential, $\partial\colon F(0)\times F(1)\times F(2) \to F(0)\times F(1)$, is given by the formula

$$\partial(x_0, x_1, x_2) = (x_0 - F(0<2)(x_2), x_1 - F(1<2)(x_2)).$$

The matching map $\mathbf{T}F(2) \to M_2\mathbf{T}F = \mathbf{T}F(1)$ is given by

$$\begin{aligned} F(0)\times F(1) &\to F(0) & \qquad F(0)\times F(1)\times F(2) &\to F(0)\times F(1) \\ (x_0,x_1) &\mapsto x_1 - F(0<1)(x_0) & (x_0,x_1,x_2) &\mapsto (x_0,x_1). \end{aligned}$$

A short computation proves that we can reiterate this construction and define $\mathbf{T}F(n)$ to be: $F(0)$, if $n = 0$; cocyl$(F(0 < 1))$, if $n = 1$; otherwise:

$$\begin{array}{ccc} (x_i - \sum_{i<n} F(i < n)(x_n)) & & \prod_{i<n} F(i) \\ \uparrow & & \uparrow \\ (x_i) & & \prod_{i \leqslant n} F(i). \end{array}$$

Note that for every $n \in \mathbb{N}$, $h(\mathbf{T}F(n)) \leqslant 1$. Therefore, it follows from Corollary 5.1.14 that higher limits vanish for degrees greater than 1,

$$H^n(\mathbb{N}; F) = H^n(\lim \mathbf{T}F) = 0 \text{ if } n > 1.$$

However, for every $n \in \mathbb{N}$, $h(\mathbf{cocyl}\,(F)\,(n)) = n$, so this fibrant replacement does not give directly a vanishing bound applying Corollary 5.1.14.

The fibrant replacement $\mathbf{T}F$ can be used to prove directly that the well-known Mittag-Leffler condition for the vanishing of higher limits on towers, see [Wei94, Section 3.5], implies $H^1(\mathbb{N}; F) = 0$.

5.2.1 Some properties of the cocylinder

Let $\mathcal{P}$ be a graded poset and $F\colon \mathcal{P}^{\text{op}} \to \text{Mod}_R$ be a functor. The higher limits of F restricted to $\mathcal{P}\backslash\{\hat{1}\}$ are the cohomology R-modules of the matching object of a fibrant replacement of F at $\hat{1}$. In particular, it is for the cocylinder of F,

$$H^i(\mathcal{P}\backslash\{\hat{1}\}; F) \cong H^i(M_{\hat{1}}\mathbf{cocyl}\,(F)).$$

Moreover, an explicit description of the last non-trivial differential of the matching object $M_p\mathbf{cocyl}\,(F)$ could help us to know if we can apply RULE 2 at $p \in \mathcal{P}$. Thus, we devote this subsection to prove the following result about this differential.

Theorem 5.2.11. *Let $\mathcal{P}$ be graded poset, and let $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a functor. Given $p \in \mathcal{P}$ with $\mathrm{d}(p) = n \geqslant 1$, the matching object $M_p\mathbf{cocyl}(F)$ at cohomological height $n-1$ satisfies,*

$$(M_p\mathbf{cocyl}(F))^{n-1} = \prod_{c\in\Delta_p} F(c_0).$$

Moreover, if $n \geqslant 2$, the matching object $M_p\mathbf{cocyl}(F)$ at cohomological height $n-2$ satisfies,

$$(M_p\mathbf{cocyl}(F))^{n-2} = \prod_{i=0}^{n-1} \prod_{c\in d_i(\Delta_p)} F(c_0),$$

and the differential $(M_p\mathbf{cocyl}(F))^{n-2} \xrightarrow{\partial} (M_p\mathbf{cocyl}(F))^{n-1}$ is defined as follows, for $c \in \Delta_p$ and $x \in (M_p\mathbf{cocyl}(F))^{n-2}$,

$$\partial(x)_c = (-1)^{n-1}F(c_1 \to c_0)(x_{d_0(c)}) + \sum_{i=1}^{n-1}(-1)^{n-i-1}x_{d_i(c)}.$$

As an immediate consequence, we obtain, by Definition 5.2.9, the following two results.

Lemma 5.2.12. *Let $\mathcal{P}$ be a graded poset, and $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be functor. Given $p \in \mathcal{P}$ with $\mathrm{d}(p) = n \geqslant 1$, the cocylinder $\mathbf{cocyl}(F)$ at cohomological heights n and $n-1$, $\mathbf{cocyl}(F)^{n-1}(p) \xrightarrow{\partial} \mathbf{cocyl}(F)^{n}(p)$, satisfies,*

$$\mathbf{cocyl}(F)^{n}(p) = \prod_{c\in\Delta_p} F(c_0),$$

$$\mathbf{cocyl}(F)^{n-1}(p) = \prod_{c\in\Delta_p} F(c_0) \times \prod_{i=0}^{n-1} \prod_{c\in d_i(\Delta_p)} F(c_0),$$

and the differential ∂ is defined as follows, for $c \in \Delta_p$ and $x \in \mathbf{cocyl}(F)^{n-1}(p)$,

$$\partial_p(x)_c = x_c + (-1)^n F(c_1 \to c_0)(x_{d_0(c)}) + \sum_{i=1}^{n-1}(-1)^{n-i}x_{d_i(c)}.$$

Proof. For $p \in \mathcal{P}$ of degree greater than 2, the result holds by Definition 5.2.9. If $p \in \mathcal{P}$ has degree 1 the result hold since $\prod_{i=0}^{0}\prod_{c\in d_i(\Delta_p)} F(c_0) = F(p)$. $\square$

Lemma 5.2.13. *Let $\mathcal{P}$ be a graded poset, and $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a functor. Given $p \in \mathcal{P}$ with $\mathrm{d}(p) = n \geqslant 2$ and $q \prec p$, the cocylinder homomorphism*

$$\mathbf{cocyl}\,(F)^{n-1}\,(q < p)\colon \mathbf{cocyl}\,(F)^{n-1}\,(p) \to \mathbf{cocyl}\,(F)^{n-1}\,(q)$$

satisfies,

$$\mathbf{cocyl}\,(F)^{n-1}\,(p) = \prod_{c\in\Delta_p} F(c_0) \times \prod_{i=0}^{n-1} \prod_{c\in d_i(\Delta_p)} F(c_0),$$
$$\mathbf{cocyl}\,(F)^{n-1}\,(q) = \prod_{c\in\Delta_q} F(c_0),$$

and for all $c \in \Delta_q$ and $x \in \mathbf{cocyl}\,(F)^{n-1}\,(p)$,

$$\mathbf{cocyl}\,(F)^{n-1}\,(q < p)(x)_c = x_{c<p}.$$

Proof of Theorem 5.2.11. By Definition 5.2.9, the result is true for $\mathrm{d}(p) = 1$, and we proceed by induction on the degree $n = \mathrm{d}(p)$ of $p \in \mathcal{P}$. If $q < p$, then $\mathrm{d}(q) \leqslant n-1$, and we have that $\mathbf{cocyl}\,(F)^{n-1}\,(q) = 0$ unless $q \prec p$. Then, by the induction hypothesis and Lemma 5.2.12,

$$(M_p\mathbf{cocyl}\,(F))^{n-1} = \prod_{q\prec p} \prod_{c\in\Delta_q} F(c_0) = \prod_{c\in\Delta_p} F(c_0).$$

For the description of $(M_p\mathbf{cocyl}\,(F))^{n-2}$, notice that, for each $q < p$, the cocylinder of F verifies $\mathbf{cocyl}\,(F)^{n-2}\,(q) = 0$ unless $n-1 \leqslant \mathrm{d}(q) \leqslant n-2$. By the induction hypothesis and Lemma 5.2.13, for $\mathrm{d}(q) = n-1$ and $\mathrm{d}(r) = n-2$, $r, q < p$, we have

$$\mathbf{cocyl}\,(F)^{n-2}\,(q) = \prod_{c\in\Delta_q} F(c_0) \times \prod_{i=0}^{n-2} \prod_{c\in d_i(\Delta_q)} F(c_0) \text{ and } \mathbf{cocyl}\,(F)^{n-2}\,(r) = \prod_{c\in\Delta_r} F(c_0),$$

and $x \in \mathbf{cocyl}\,(F)^{n-2}\,(q)$ is mapped by the homomorphism $\mathbf{cocyl}\,(F)^{n-2}\,(r < q)$ to the tuple of $\mathbf{cocyl}\,(F)^{n-2}\,(r)$ with value $x_{c<q}$ at the chain $c \in \Delta_r$. Thus, the inverse limit $(M_p\mathbf{cocyl}\,(F))^{n-2}$ is given by

$$(M_p\mathbf{cocyl}\,(F))^{n-2} = L \times L',$$

where

$$L = \prod_{q\prec p} \prod_{i=0}^{n-2} \prod_{c\in d_i(\Delta_q)} F(c_0) = \prod_{i=0}^{n-2} \prod_{c\in d_i(\Delta_p)} F(c_0),$$

and where L' consists of the tuples

$$(x_q)_{q \prec p} \in \prod_{q \prec p} \prod_{c \in \Delta_q} F(c_0)$$

such that, if $r < q_1, q_2 < p$, $\mathrm{d}(q_1) = \mathrm{d}(q_2) = n-1$, $\mathrm{d}(r) = n-2$, $c \in \Delta_r$, then

$$(x_{q_1})_{c<q_1} = (x_{q_2})_{c<q_2}.$$

From this description, it turns out easily that $L' = \prod_{c \in d_{n-1}(\Delta_p)} F(c_0)$, and hence we get the description of $(M_p\mathbf{cocyl}\,(F))^{n-2}$ in the statement.

To prove that the formula for the differential holds, it is enough to check that, for every $q \prec p$, the following diagram commutes,

$$\begin{tikzcd}
\prod_{c \in \Delta_p} F(c_0) \arrow[rr, "\eta^{n-1}"] & & \prod_{c \in \Delta_q} F(c_0) \\
\prod_{i=0}^{n-1} \prod_{c \in d_i(\Delta_p)} F(c_0) \arrow[u, "\partial"] \arrow[r, "\eta^{n-2}"] & \prod_{c \in \Delta_q} F(c_0) \times \prod_{i=0}^{n-2} \prod_{c \in d_i(\Delta_q)} F(c_0), \arrow[ur, "\partial_q"] &
\end{tikzcd}$$

where ∂ is given as in the statement, ∂_q is given by induction, and Lemma 5.2.12, and the horizontal arrows were constructed in the previous part of this proof as follows,

$$\eta^{n-1}(x)_c = x_{c<p}, \text{ for } c \in \Delta_q,\ x \in (M_p\mathbf{cocyl}\,(F))^{n-1},$$

$$\eta^{n-2}(x)_c = \begin{cases} x_{c<p}, \text{ for } i = 0, \dots, n-2,\ c \in d_i(\Delta_q),\ x \in (M_p\mathbf{cocyl}\,(F))^{n-2}, \\ x_{d_{n-1}(c<p)}, \text{ for } c \in \Delta_q,\ x \in (M_p\mathbf{cocyl}\,(F))^{n-2}. \end{cases}$$

Now consider $x \in (M_p\mathbf{cocyl}\,(F))^{n-2}$ and $c \in \Delta_q$. Then the following computation finishes the proof, where we write $c' = c < p$ for simplicity,

$$\begin{aligned}
\eta_{n-1}(\partial(x))_c &= (\partial(x))_{c<p} \\
&= (-1)^{n-1} F(c_0 < c_1)(x_{d_0(c')}) + \sum_{i=1}^{n-1} (-1)^{n-i-1} x_{d_i(c')} \\
&= (-1)^{n-1} F(c_0 < c_1)(x_{d_0(c')}) + \sum_{i=1}^{n-2} (-1)^{n-i-1} x_{d_i(c')} + x_{d_{n-1}(c')} \\
&= \partial_q(\eta^{n-2}(x))_c.
\end{aligned}$$

□

5.3 A COFIBRANT REPLACEMENT CONSTRUCTION.

This thesis focuses on the description of higher limits. As such, detailed instructions for constructing fibrant replacements were presented in the preceding section. Nevertheless, it is imperative not to miss the opportunity to provide a method for computing higher colimits by giving at least one strategy to construct cofibrant replacements. To produce a cofibrant replacement of a given functor, we need the notion of mapping cylinder.

Definition 5.3.1. Let $f\colon C \to D$ be a map between unbounded cochain complexes. The mapping cylinder of f is the chain complex $\mathrm{cyl}(f)$ whose degree n part is:

$$\mathrm{cyl}(f)_n := C_n \times C_{n-1} \times D_n,$$

and whose differential is given by the formula:

$$\partial(c, c', d) = (\partial c + c', -\partial c', \partial d - f(c')).$$

Proposition 5.3.2 ([Wei94, Section 1.5]). *Let $f\colon C \to D$ be a morphism between chain complexes. Then f factors through the mapping cylinder:*

$$\begin{array}{ccccc} C & \hookrightarrow & \mathrm{cyl}(f) & \xrightarrow{\sim} \!\!\!\!\rightarrow & D \\ c & \longmapsto & (c,0,0) & & \\ & & (c,c',d) & \longmapsto & f(c)+d \end{array}$$

as a split monomorphism followed by an epimorphism that induces an isomorphism in cohomology.

Let $\mathcal{C}$ be a filtered EI-category and R be a $\mathcal{C}$-bijective ring. We construct a cofibrant replacement of a functor $F\colon \mathcal{C} \to \mathrm{Mod}_R$ by induction on the filtration. For every object $c \in \mathcal{C}$ of degree 0, we have that $M_c F = 0$. Therefore, we start by defining

$$\mathbf{Q}F(c) := F(c).$$

Assume that $\mathbf{Q}F$ is defined in the full subcategory of objects of degree less than n. To define $\mathbf{Q}F$ on $c \in \mathcal{C}$ of degree n, we need to choose a factorisation of the composite $\varepsilon_c\colon L_c\mathbf{Q}F \to L_cF \to F(c)$ as a monomorphism followed by weak equivalence. That is, to choose an object $\mathbf{Q}F(c)$ together with a monomorphism

$L_c\mathbf{Q}F(c) \to \mathbf{Q}F(c)$ and a weak equivalence $\mathbf{Q}F(c) \to F(c)$ such that the following diagram commutes:

$$\begin{tikzcd} L_c\mathbf{Q}F \arrow[r, hook, dashed] \arrow[d] & \mathbf{Q}F(c) \arrow[d, dashed, "\sim"'] \\ L_cF \arrow[r] & F(c). \end{tikzcd} \tag{6}$$

Proposition 5.3.2 provides such a factorisation of ε_c using the mapping cylinder,

$$L_c\mathbf{Q}F \to \mathbf{Q}F(c) = \mathrm{cyl}(\varepsilon_c) \to F(c).$$

Chapter 6

ACYCLICITY OF MACKEY FUNCTORS FOR POSETS

Mackey functors are a generalisation of group representations. They abstract how representations of subgroups of a given group can be combined to form representations of the entire group in a more general context. Mackey functors naturally appear in algebraic topology, homological algebra, and, obviously, representation theory.

In particular, Mackey functors are related to lim- and colim-acyclicity. In this chapter, we introduce the notion of Mackey functors for posets inspired by the classical one [Web00]. We also show how Mackey functors with quasi-unit are related to the pseudo-projectivity condition introduced by Díaz [DR09]. In the case of the underlying poset being filtered, these notions agree with the cofibrant functors.

A *Mackey functor* for a group G over a commutative ring with unit R is a pair of functor (M_*, M^*) from the category of G-sets to the category of R-modules such that M_* is covariant, M^* is contravariant, both coincide on objects, and other axioms that we are not going to cite here, for more details see Webb's paper

[Weboo]. Given $J, K \leqslant H$ subgroups of G, the action of the Mackey functors in the homogeneous G-sets, G/H, G/J and G/K, are related by the Mackey formula

$$M^*([\iota_J^H]) \circ M_*[\iota_K^H]) = \sum_{x \in [J \backslash H/K]} M_*([\iota_{J \cap {}^xK}^J]) \circ M_*([c_x]) \circ M^*([\iota_{J^x \cap K}^K]) \tag{7}$$

where c_x is the conjugation by x morphism, $\iota_\cdot^\cdot$ is the inclusion and $[J\backslash H/K]$ is a set of representatives in G for the double cosets $J\backslash H/K$.

If we restrict these functors to the meet-semilattice of central subgroups of G, with meet defined as the intersection, the Mackey formula (7) becomes

$$M^*([\iota_J^H]) \circ M_*([\iota_K^H]) = \left(\sum_{x \in [J \backslash H/K]} M_*([c_x]) \right) \circ M_*([\iota_{J \cap K}^J]) \circ M^*([\iota_{J \cap K}^K])$$

To define Mackey functors for posets mimicking this formula, we substitute the element $\left(\sum_{x \in [J \backslash H/K]} M_*([c_x]) \right)$ by a certain kind of endomorphisms that commute with the composite equivalent to $M_*([\iota_{J \cap K}^J]) \circ M^*([\iota_{J \cap K}^K])$.

Definition 6.1. Let $\mathcal{P}$ be a poset, R a commutative ring with unit, and $F\colon \mathcal{P} \to \mathcal{C}$ a functor. Given $p \in \mathcal{P}$, a endomorphism of R-modules $\alpha \in \mathrm{End}_R(F(p))$ (or automorphism $\alpha \in \mathrm{Aut}_R(F(p))$) is said to be *F-linear* if, for every $q < p$, the following condition holds:

$$\alpha \circ F(q < p) = F(q < p) \circ \beta,$$

where $\beta \in \mathrm{End}_R(F(q))$ (or $\beta \in \mathrm{Aut}_R(F(q))$). We use the notation $\mathrm{End}_R^F(p)$ (or $\mathrm{Aut}_R^F(p)$) to denote the submonoid (or subgroup) of F-linear endomorphisms (or automorphisms) of $F(p)$.

Example 6.2. Given $r \in R$, the homothety $x \mapsto rx$ is an example of F-linear endomorphism. In particular, every identity is F-linear.

Definition 6.3. Let $\mathcal{P}$ be a filtered meet-semilattice and R be a commutative ring with unit. A pair of functors (F, G) is said to be a *Mackey functor* if $F\colon \mathcal{P} \to \mathrm{Mod}_R$ is covariant, $G\colon \mathcal{P}^{op} \to \mathrm{Mod}_R$ is contravariant, $F(p) = G(p)$ for all $p \in \mathcal{P}$, and for all $q < p, k < p$ there exist $\alpha(p, q, k) \in \mathrm{End}_R^F(q)$ such that

$$G(q < p) \circ F(k < p) = \alpha(p, q, k) \circ F(k \wedge q < q) \circ G(k \wedge q < k).$$

We say that (F, G) has a *quasi-unit* if, for every $q < p$, the endomorphism $\alpha(p, q, q)$ is an automorphism.

The term *quasi-unit* is borrowed from [JM92, 5.7 Definition], and it is related to acyclicity.

Example 6.4. Recall that a twin functor over the face poset of a simplicial complex $\mathcal{P}$ is a pair of functors (D, I) from $\mathcal{P}$ to Ab such that:

1. $D\colon \mathcal{P} \to \mathrm{Ab}$ is covariant and $I\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ is contravariant,
2. for every $\sigma \in \mathcal{P}$, $D(\sigma) = I(\sigma)$, and
3. every pullback diagram:

$$\begin{tikzcd} \sigma \cap \sigma' \arrow[r, "i"] \arrow[d, "i'"] & \sigma \arrow[d, "j"] \\ \sigma' \arrow[r, "j'"] & \tau, \end{tikzcd} \quad \text{induces a commutative diagram} \quad \begin{tikzcd} I(\sigma \cap \sigma') \arrow[r, "D(i)"] & I(\sigma) \\ I(\sigma') \arrow[u, "I(i')"] \arrow[r, "D(j')"] & I(\tau). \arrow[u, "I(j)"] \end{tikzcd}$$

Twin functors are a particular case of Mackey functor with $\alpha = \mathrm{Id}$.

We also define *weak Mackey functor* by dropping the contravariant functoriality and the meet-semilattice constraint.

Definition 6.5. Let $\mathcal{P}$ be a poset. A functor $F\colon \mathcal{P} \to \mathrm{Mod}_R$ is a *weak Mackey functor* if for every pair $q < p$, there exists a morphism in Mod_R, $G(q < p)\colon F(p) \to F(q)$ such that the composite

$$F(q) \xrightarrow{F(q<p)} F(p) \xrightarrow{G(q<p)} F(q)$$

is an F-linear endomorphism $\alpha(p,q) \in \mathrm{End}_R^F(q)$, and for $k < p$ such that $q \nleqslant k$,

$$\mathrm{Im}(G(q < p) \circ F(k < p)) \subseteq \mathrm{Im}_F(q).$$

The functor F is said to have a *quasi-unit* if, for every $p < q$, $\alpha(p,q) \in \mathrm{Aut}_R^F(q)$.

Remark 6.6. Notice that the covariant part of a Mackey functor is a weak Mackey functor.

Another condition related to colim-acyclicity is the pseudo-projectivity of functors in DCC posets.

Definition 6.7. A functor $F\colon \mathcal{P} \to \mathrm{Ab}$ over a DCC poset is *pseudo-projective* at $p \in \mathcal{P}$ if, for every finite subset $Q \subset \mathcal{P}_{\leqslant p}$ and every element $\oplus_{q\in Q} x_q \in \bigoplus_{q\in Q} F(q)$, the condition:

$$\sum_{q\in Q} F(q < p)(x_q) = 0$$

implies that $x_q \in \mathrm{Im}_F(q) = \sum_{k<q} \mathrm{Im}\, F(k < q)$ for every $q \in \max Q$. We say that F is *pseudo-projective* if it is pseudo-projective at p for every $p \in \mathcal{P}$.

First, we show that a weak Mackey functor with quasi-unit over a DCC poset is pseudo-projective.

Theorem 6.8. *Let $\mathcal{P}$ be a DCC poset and $F\colon \mathcal{P} \to \mathrm{Ab}$ be a weak Mackey functor with a quasi-unit. Then, F is pseudo-projective, and hence, it is* colim*-acyclic.*

Proof. Let $F\colon \mathcal{P} \to \mathrm{Mod}_R$ be a weak Makey functor with quasi-unit. Let $p \in \mathcal{P}$, Q be a finite subset of $\mathcal{P}_{\leqslant p}$, and $\oplus_{q\in Q} x_q \in \bigoplus_{q\in Q} F(q)$ such that:

$$\sum_{q\in Q} F(q < p)(x_q) = 0.$$

For $k \in \max Q$, we prove that $x_k \in \mathrm{Im}_F(k)$. One can assume without loss of generality that $p \notin Q$, then apply the morphism $G(k < p)$ to the equation above:

$$\begin{aligned} 0 &= G(k<p)\Big(\sum_{q\in Q} F(q<p)(x_q)\Big) = \sum_{q\in Q} (G(k<p)\circ F(q<p))(x_q) \\ &= (G(k<p)\circ F(k<p)(x_k) + \sum_{\substack{q\in Q\\ q\neq k}} (G(k<p)\circ F(q<p))(x_q), \end{aligned} \tag{8}$$

First, we show that the second addend in Equation (8) belongs to $\mathrm{Im}_F(k)$. We can split this addend as:

$$\sum_{\substack{q\in Q\\ q<k}} (G(k<p)\circ F(q<p))(x_q) + \sum_{\substack{q\in Q\\ q\nless k}} (G(k<p)\circ F(q<p))(x_q).$$

If $q < k < p$, we have $F(q < p)(x_q) = F(k < p) \circ F(q < k)(x_q)$. Then, there exists $\alpha \in \mathrm{Aut}_R^F(k)$ an F-linear automorphism such that the composite can be written as

$$G(k<p)\circ F(k<p)\circ F(q<k)(x_q) = \alpha \circ F(q<k)(x_q).$$

Since α is F-linear, there exists $\beta \in \mathrm{Aut}_R(F(q))$ such that

$$G(k<p)\circ F(k<p)\circ F(q<k)(x_q) = \alpha\circ F(q<k)(x_q) = F(q<k)\circ\beta(x_q).$$

Thus, we conclude that the element $G(k < p) \circ F(q < p)(x_q) \in \mathrm{Im}_F(k)$. In the other case, $q \nleq k$, we are done after applying the definition of weak Mackey functor,

$$(G(k < p) \circ F(q < p))(x_q) \in \mathrm{Im}_F(k).$$

Next, apply the definition of Mackey functor again to the first addend of Equation (8), $G(k < p) \circ F(k < p)(x_k) = \alpha(x_k)$ for some $\alpha \in \mathrm{Aut}_r^F(k)$. Then, we solve this term in the same equation obtaining

$$\alpha(x_k) = -\sum_{\substack{q \in Q \\ q \neq k}} (G(k < p) \circ F(q < p))(x_q).$$

As the right side of this identity belongs to $\mathrm{Im}_F(k)$, there exists finite many elements $y_l \in F(l)$ for $l < k$ such that

$$\alpha(x_k) = -\sum_{\substack{q \in Q \\ q \neq k}} (G(k < p) \circ F(q < p))(x_q) = \sum_{l < k} F(l < k)(y_l).$$

As α is invertible and F-linear, we can solve for x_k as follows,

$$x_k = \sum_{l < k} (\alpha^{-1} \circ F(l < k))(y_l) = \sum_{l < k} (F(l < k) \circ \beta_l)(y_l)$$

for some automorphisms $\beta_l \in \mathrm{Aut}_R(F(l))$. This implies that $x_k \in \mathrm{Im}_F(k)$, and we are done. $\square$

In the case of a filtered poset, we show how a functor $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ is pseudo-projective if and only if it is cofibrant. However, for just a DCC poset $\mathcal{P}$, the functor can be shown to be pseudo-projective if and only if it satisfies the same condition as cofibrant functors but without requiring the poset to be filtered.

Definition 6.9. Let $\mathcal{P}$ be a poset and $F\colon \mathcal{P} \to \mathrm{Ab}$ be a functor. The functor F is *locally injective at* $p \in \mathcal{P}$ if the natural map induced by colimit

$$\operatorname*{colim}_{\mathcal{P}_{<p}} F \to F(p)$$

is a monomorphism. The functor F is said to be *locally injective* if it is locally injective at p for every $p \in \mathcal{P}$.

Note that in a filtered poset, locally injective functors are the cofibrant ones.

Theorem 6.10. *Let $\mathcal{P}$ be a DCC poset, and $F\colon \mathcal{P} \to \mathsf{Ab}$ be a functor. Then F is locally injective if and only if it is pseudo-projective.*

In the case of $\mathcal{P}$ being a filtered poset, Theorem 6.10 says that pseudo-projective functors are cofibrant; therefore, we reprove Díaz's result about the colim-acyclicity of pseudo-projective functors [DR09]. We summarise this chapter in the following diagram.

$$\begin{tikzcd}
\boxed{\text{Mackey (poset)}} \arrow[r, Rightarrow] & \boxed{\text{Weak Mackey}} \arrow[r, Rightarrow] & \boxed{\text{Pseudo-projective}} \arrow[r, Leftrightarrow] \arrow[dr, Rightarrow, "{[\text{DR09}]}"'] & \boxed{\text{Cofibrant}} \arrow[d, Rightarrow] \\
& & & \boxed{\text{colim-acyclic}}
\end{tikzcd}$$

We devote the rest of the chapter to the proof of Theorem 6.10. The proof is divided into several lemmas.

Lemma 6.11 ([DR09, 2.6])**.** *Let $\mathcal{P}$ be a DCC poset, and $\{Q^n\}$ be a sequence of subsets of $\mathcal{P}$ such that $Q^n < Q^{n-1}$. Then, there exits $N \in \mathbb{N}$ such that $Q^n = \varnothing$ for every $n > N$.*

Lemma 6.12. *Let $\mathcal{P}$ be a DCC poset, $p \in \mathcal{P}$ and $F : \mathcal{P} \to \mathsf{Ab}$ be a functor such that F is pseudo-projective at p. Let $x = \oplus_{q<p} x_q \in \bigoplus_{q<p} F(q)$ satisfy*

$$\sum_{q<p} F(q<p)(x_q) = 0.$$

Then, there is a sequence $\{x^n\}_{n\geqslant 0}$, $x^n = \bigoplus_{q<p} x^n_q \in \bigoplus_{q<p} F(q)$, with $x^0 = x$,

$$\sum_{q<p} F(q<p)(x^n_q) = 0,\ x^{n+1} - x^n = \sum_{k<q\in \max \operatorname{supp}(x^n)} y_{k,q} \oplus -F(k<q)(y_{k,q}),$$

$[x^{n+1}] = [x^n]$ in $\operatorname{colim}_{\mathcal{P}_{<p}} F$, and $\operatorname{supp}(x^{n+1}) < \operatorname{supp}(x^n)$, for any $n \geqslant 0$, where $y_{k,q} \in F(k)$, In addition, there exists $N > 0$ such that $x^n_q = 0$ for all $q < p$ if $n \geqslant N$.

Proof. This is a finer reformulation of [DR09, Lemma 2.3], and we provide details. We define $x^{-1} = 0$ and work by induction on $n \geqslant 0$, assuming that x^n has already been constructed satisfying the properties in the statement. Then, as

$\sum_{q<p} F(q<p)(x_q^n) = 0$ and F is pseudo-projective at p, for every $q \in \max \operatorname{supp}(x^n)$ we have that $x_q^n \in \operatorname{Im}_F(q)$, i.e., there exists $\oplus_{k<q} y_{k,q} \in \bigoplus_{k<q} F(k)$ such that

$$x_q^n = \sum_{k<q} F(k<q)(y_{k,q}).$$

For every pair (k,q) with $k<q<p$, we set

$$x_{k,q} = \begin{cases} y_{k,q} & \text{if } k<q \in \max \operatorname{supp}(x^n), \\ x_q^n & \text{if } k=q \notin \max \operatorname{supp}(x^n), \\ 0 & \text{otherwise,} \end{cases}$$

and we define $x^{n+1} = \bigoplus_{q<p} x_q^{n+1}$ by

$$x_q^{n+1} = \sum_{k \geqslant q} x_{q,k}.$$

Then

$$\sum_{q<p} F(q<p)(x_q^{n+1}) = \sum_{q<p} F(q<p)\big(\sum_{k\geqslant q} x_{q,k}\big) = \sum_{q\leqslant k<p} F(q<p)(x_{q,k}). \tag{9}$$

In this last sum, if $q=k$ for $k \notin \max \operatorname{supp}(x^n)$, the corresponding addend is $F(q<p)(x_q^n)$. The rest of the addends can be reordered as follows,

$$\begin{aligned} \sum_{\substack{q<k<p \\ k\in\max\operatorname{supp}(x^n)}} F(q<p)(y_{q,k}) &= \sum_{\substack{k<p \\ k\in\max\operatorname{supp}(x^n)}} F(k<p)\big(\sum_{q<k} F(q<k)(y_{q,k})\big) \\ &= \sum_{\substack{k<p \\ k\in\max\operatorname{supp}(x^n)}} F(k<p)(x_k^n). \end{aligned}$$

Hence the sum in Equation (9) equals $\sum_{q<p} F(q<p)(x_q^n)$, and this is 0 by hypothesis. From the construction above, it easily follows that

$$x^{n+1} - x^n = \sum_{k<q\in\max\operatorname{supp}(x^n)} y_{k,q} \oplus -F(k<q)(y_{k,q}),$$

and, from here, it is clear that $[x^{n+1}] = [x^n]$ in $\operatorname{colim}_{\mathcal{P}_{<p}} F$ and that the supports are related by $\operatorname{supp}(x^{n+1}) < \operatorname{supp}(x^n)$. From this latter condition and Lemma 6.11, we obtain $N>0$ with the stated property. $\square$

Lemma 6.13. *Let $\mathcal{P}$ be a DCC poset, $F\colon \mathcal{P} \to \mathrm{Ab}$, and $p \in \mathcal{P}$. If F is pseudo-projective at p, then F is locally injective at p.*

Proof. Let $\varepsilon\colon \operatorname{colim}_{\mathcal{P}_{<p}} F \to F(p)$ be the corresponding natural map and consider $[x] \in \ker(\varepsilon)$ with $x \in \bigoplus_{q<p} F(q)$. By Lemma 6.12, there exists a sequence $\{x^n\}_{n \geqslant 0}$ with $x^n \in \bigoplus_{q<p} F(q)$ such that $x^0 = x$, $[x^{n+1}] = [x^n]$ and $x^N = 0$ for N big enough. Hence $[x^0] = [x^N] = [0] = 0$ and the Lemma is proven. $\square$

Lemma 6.14. *Let $\mathcal{P}$ be a DCC poset, $F\colon \mathcal{P} \to \mathrm{Ab}$, and $p \in \mathcal{P}$. If F is locally injective at q for every $q \leqslant p$, then F is pseudo-projective at q for every $q \leqslant p$.*

Proof. Since $\mathcal{P}_{\leqslant p}$ is a DCC poset, we proceed by induction. If $q \leqslant p$ is minimal in $\mathcal{P}$, then F is pseudo-projective at q by definition. Thus, consider now $q \leqslant p$ such that F is pseudo-projective at k for all $k < q$. We show that F is pseudo-projective at q too. So let $x = \oplus_{k<q} x_k \in \bigoplus_{k<q} F(k)$ be such that

$$\sum_{k<q} F(k<q)(x_k) = 0.$$

This is equivalent to that $\varepsilon([x]) = 0$ for the natural map $\varepsilon\colon \operatorname{colim}_{k<q} F \to F(q)$. By hypothesis, F is cofibrant at q, and hence $[x] = 0$. In turn, this equality is equivalent to the existence of elements $y_{l,k} \in F(l)$ for $l < k < q$ such that finitely many of them are different from zero and with

$$x = \oplus_{k<q} x_k = \sum_{l<k<q} y_{l,k} \oplus -F(l<k)(y_{l,k}), \tag{10}$$

which implies that, for any $k < q$,

$$x_k = \sum_{k<l} y_{k,l} - \sum_{l<k} F(l<k)(y_{l,k}). \tag{11}$$

Let $K = \{k \in \mathcal{P}_{<q} \mid \exists\, l < k \text{ with } y_{l,k} \neq 0\}$. We are about to show that we can choose the elements $y_{l,k}$'s appearing in (10) subject to the constraint that $\max K \subseteq \operatorname{supp}(x)$. Thus let $m \in \max K \backslash \operatorname{supp}(x)$. Then

$$x_m = 0 = -\sum_{l<m} F(l \to m)(y_{l,m}). \tag{12}$$

We can rewrite Equation (10) as follows,

$$\begin{aligned} x &= \sum_{l<m<q} y_{l,m} \oplus -F(l<m)(y_{l,m}) + \sum_{\substack{l<k<q \\ k\neq m}} y_{l,k} \oplus -F(l<k)(y_{l,k}) \\ &= \Big(\oplus_{l<m}\, y_{l,m} \Big) - \Big(\oplus_m \sum_{l<m} F(l<m)(y_{l,m}) \Big) + \sum_{\substack{l<k<q \\ k\neq m}} y_{l,k} \oplus -F(l<k)(x_{l,k}). \end{aligned}$$

which, by Equation (12), we can simplify to

$$x = y + \sum_{\substack{l<k<q \\ k\neq m}} y_{l,k} \oplus -F(l<k)(y_{l,k}), \tag{13}$$

where $y = \oplus_{l<m} y_{l,m} \in \oplus_{l<m} F(l)$. As F is pseudo-projective at $m<q$ by induction hypothesis, we apply Lemma 6.12 to the element y to obtain a sequence of elements $\{y^n\}_{n\geqslant 0}$ such that $y^0 = y$, $[y^{n+1}] = [y^n]$ for all $n \geqslant 0$, and $y^N = 0$ for N big enough. In addition, as $\operatorname{supp}(y^{n+1}) < \operatorname{supp}(y^n)$ and $\operatorname{supp}(y) < \{m\}$, we obtain that

$$y^n - y^{n+1} = \sum_{\substack{l<k \\ k\in\max\operatorname{supp}(y^n)}} z_{l,k} \oplus -F(l<k)(z_{l,k}) = \sum_{l<k<m} z_{l,k} \oplus -F(l<k)(z_{l,k}) \tag{14}$$

for elements $z_{l,k} \in F(l)$. Define $y^0_{l,k} = y_{l,k}$, assume by induction that

$$x = y^n + \sum_{\substack{l<k<q \\ k\neq m}} y^n_{l,k} \oplus -F(l<k)(y^n_{l,k}), \tag{15}$$

for elements $y^n_{l,k} \in F(l)$, and note that this holds for $n = 0$ by Equation (13). For the induction step, we may write

$$\begin{aligned} x &= y^{n+1} - y^{n+1} + y^n + \sum_{\substack{l<k<q \\ k\neq m}} y^n_{l,k} \oplus -F(l<k)(y^n_{l,k}) \\ &= y^{n+1} + \sum_{\substack{l<k<q \\ k\neq m}} y^{n+1}_{l,k} \oplus -F(l<k)(y^{n+1}_{l,k}), \end{aligned}$$

for elements $y^{n+1}_{l,k} \in F(l)$, where in the last equality we have employed Equation (14). Hence, for $n = N$, Equation (15) simplifies to

$$x = \sum_{\substack{l<k<q \\ k \neq m}} y^N_{l,k} \oplus -F(l < k)(y^N_{l,k}).$$

Repeating this process for every element $m \in \max K \backslash \operatorname{supp}(x)$ we find a decomposition similar to Equation (10),

$$x = \sum_{l<k<q} y'_{l,k} \oplus -F(l < k)(y'_{l,k}),$$

and satisfying that, for $K' = \{k \in \mathcal{P}_{<q} \mid \exists\, l < k \text{ with } y'_{l,k} \neq 0\}$, we have

$$\max K' \backslash \operatorname{supp}(x) < \max K \backslash \operatorname{supp}(x).$$

Iterating this procedure we obtain a sequence of sets $\{K^n\}_{n \geqslant 0}$ and decompositions similar to Equation (10) with $K^0 = K$, $K^1 = K'$, and such that

$$\max K^{n+1} \backslash \operatorname{supp}(x) < \max K^n \backslash \operatorname{supp}(x).$$

Setting $Q^n = \max K^n \backslash \operatorname{supp}(x)$ and applying Lemma 6.11 we find N such that $Q^N = \varnothing$, i.e., $\max K^N \subseteq \operatorname{supp}(x)$. For the corresponding decomposition,

$$x = \sum_{l<k<q} \hat{y}_{l,k} \oplus -F(l < k)(\hat{y}_{l,k}),$$

let k belong to $\max \operatorname{supp}(x)$ so that we have, similarly to Equation (11),

$$x_k = \sum_{k<l} \hat{y}_{k,l} - \sum_{l<k} F(l < k)(\hat{y}_{l,k}).$$

If $\sum_{k<l} \hat{y}_{k,l} \neq 0$, there exists some $l > k$ such that $\hat{y}_{l,k} \neq 0$, which is a contradiction with $\max K^N \subseteq \operatorname{supp}(x)$. Hence, $\sum_{k<l} \hat{y}_{k,l} = 0$ and $x_k \in \operatorname{Im}_F(k)$. $\square$

Chapter 7

VANISHING BOUNDS

Given a functor $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ over a filtered poset, in Example 5.2.10, we show how $\mathbf{T}F$ could help us to provide a vanishing bound for the higher limits of F. The goal of this chapter is to provide different conditions that imply the natural morphism described in the construction of $\mathbf{T}F$,

$$\varepsilon_p\colon F(p) \to M_p\mathbf{T}F,$$

is truncatable. For short, we say that an object $p \in \mathcal{P}$ is *F-truncatable* if the natural map $\varepsilon_p\colon F(p) \to M_p\mathbf{T}F$ is truncatable.

Notice that for every $p \in \mathcal{P}$, $F(p)$ is a cochain complex concentrated in degree 0. Then, p is F-truncatable if and only if $h(M_p\mathbf{T}F) > 0$, and the last non-trivial differential of $M_p\mathbf{T}F$,

$$(M_c\mathbf{T}F)^{h(M_c\mathbf{T}F)-1} \to (M_c\mathbf{T}F)^{h(M_c\mathbf{T}F)},$$

is an epimorphism.

7.1 COMBINATORIAL VANISHING BOUND

Given a functor $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$, in the inductive construction of a fibrant replacement $\mathbf{R}F$, once the first mapping cocylinder has been done, one only needs to check that the last non-trivial differential of $M_p\mathbf{R}F$ is an epimorphism. This condition holds if the matching object of $\mathbf{R}F$ at p is a direct sum of mapping cocylinders. However, this condition is strong because we only have to check $M_p\mathbf{R}F$ at its last non-zero dimensions. In this section, we introduce a method to construct a labelling for a given filtered poset $\mathcal{P}$ that controls the height of any functor $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$.

The intuitive idea behind the labelling we will present in this section is as follows. Let $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be a functor on a filtered poset, $p \in \mathcal{P}$, and n be the height of $\mathbf{T}F(p)$. If the support of $\mathbf{T}F^n$ and $\mathbf{T}F^{n-1}$ in $\mathcal{P}_{<p}$ has as many maximal elements as connected components, then $M_p\mathbf{T}F$ will behave like a sum of cocylinders at heights n and $n-1$.

Definition 7.1.1. Let $\mathcal{P}$ be a filtered poset, $p \in \mathcal{P}$ and S be a subposet of $\mathcal{P}_{<p}$. We say that p *closes a circuit in* S if

$$\#\mathrm{Max}(S) > \#\mathrm{Conex}(S),$$

where $\mathrm{Max}(S)$ denotes the maximal objects in S and $\mathrm{Conex}(S)$ its connected components.

In a combinatorial point of view, if "p closes a circuit in S", then there is an undirected circuit that contains p in the Hasse diagram of $S \cup \{p\}$.

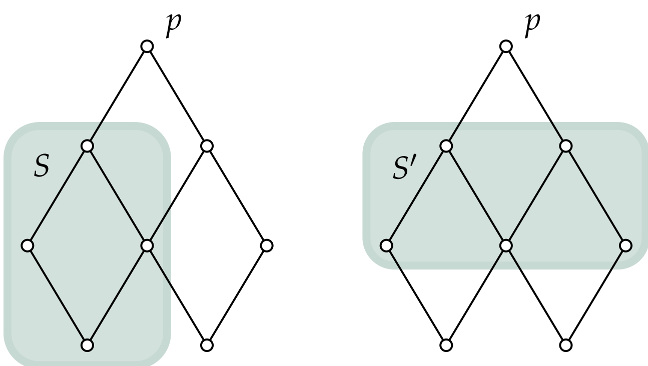

Figure 17: p closes a circuit in S' but not in S.

Definition 7.1.2. Let $\mathcal{P}$ be a filtered poset. We define the *labelling function* of $\mathcal{P}$ to be the map $B\colon \mathcal{P} \to \mathbb{N}$ defined inductively as follows: For objects of degree 0 or 1, we assign the values 0 or 1 respectively. Next, we assume that B has been defined for every object of degree less than n. For a fixed $p \in \mathcal{P}$ of degree n, we define m as the maximum label among objects strictly below p, and $\mathcal{B}_p$ as the full subposet of $\mathcal{P}_{<p}$ containing all the objects $s < p$ whose label is either $m-1$ or m. That is,

$$\mathcal{B}_p = \{s \in \mathcal{P}_{<p} \mid m-1 \leqslant B(s) \leqslant m\}.$$

We define the label of p by the following rule:

$$B(p) = \begin{cases} m+1 & \text{if } p \text{ closes a circuit in } \mathcal{B}_p, \\ m & \text{otherwise.} \end{cases}$$

Example 7.1.3. Consider the poset generated by the following Hasse diagram,

$$\begin{tikzcd}
& & p_4 \arrow[dl] & p_1 \arrow[l] & \\
& p_7 \arrow[dl] & & & \\
p_8 & & p_5 \arrow[ul] & p_2 \arrow[l] & p_0. \arrow[l] \arrow[uul] \arrow[dl] \\
& p_6 \arrow[ul] & & p_3 \arrow[ll] &
\end{tikzcd}$$

We will show how to define its labelling function inductively. First, by definition $B(p_0) = 0$ and $B(p_1) = B(p_2) = B(p_3) = 1$,

$$\begin{tikzcd}
& & B(p_4) \arrow[dl] & 1 \arrow[l] & \\
& B(p_7) \arrow[dl] & & & \\
B(p_8) & & B(p_5) \arrow[ul] & 1 \arrow[l] & 0. \arrow[l, dash] \arrow[uul, dash] \arrow[dl, dash] \\
& B(p_6) \arrow[ul] & & 1 \arrow[ll] &
\end{tikzcd}$$

Now, we focus on the object p_4. We have to check if p_4 *closes a circuit in* $\mathcal{B}_{p_4}$. This subposet has only two objects, p_1 and p_0. Then $\mathcal{B}_{p_4}$ has a maximum and a connected component; therefore, $B(p_4) = 1$. Similarly, $B(p_5) = B(p_6) = 1$,

$$
\begin{tikzcd}
 & & 1 \arrow[r, dash] \arrow[dl] & 1 \arrow[ddr, dash] & \\
 & B(p_7) \arrow[dl] & & & \\
 B(p_8) & & 1 \arrow[r, dash] \arrow[ul] & 1 \arrow[r, dash] & 0. \\
 & 1 \arrow[rr, dash] \arrow[ul] & & 1 \arrow[ur, dash] &
\end{tikzcd}
$$

Next, p_7 closes a circuit using the last two labels, i.e., $\mathcal{B}_{p_7}$ contains two maximal objects, p_4 and p_5 but only a connected component, thus $B(p_7) = 2$. To finish the labelling, note that $\mathcal{B}_{p_8}$ has two connected component and two maximal objects, p_7 and p_6,

$$
\begin{tikzcd}
 & & 1 \arrow[r, dash] \arrow[dl, dash] & 1 \arrow[ddr, dash] & \\
 & 2 \arrow[dl] \arrow[dr, dash] & & & \\
 B(p_8) & & 1 \arrow[r, dash] & 1 \arrow[r, dash] & 0. \\
 & 1 \arrow[rr, dash] \arrow[ul] & & 1 \arrow[ur, dash] &
\end{tikzcd}
$$

In spite of p_8 closing a circuit in $\mathcal{P}$, it does not close any circuit in $\mathcal{B}_{p_8}$. Then, $B(p_8) = 2$, and the labelling function in $\mathcal{P}$ is given by the following diagram:

$$
\begin{tikzcd}
 & & 1 \arrow[r, dash] \arrow[dl, dash] & 1 \arrow[ddr, dash] & \\
 & 2 \arrow[dl, dash] \arrow[dr, dash] & & & \\
 2 & & 1 \arrow[r, dash] & 1 \arrow[r, dash] & 0. \\
 & 1 \arrow[rr, dash] \arrow[ul, dash] & & 1 \arrow[ur, dash] &
\end{tikzcd}
$$

The main goal of this section is to show how the labelling functions gives a vanishing bound for the higher limits of any functor.

Theorem 7.1.4. *Let $\mathcal{P}$ be a filtered poset, and $B\colon \mathcal{P} \to \mathbb{N}$ its associated labelling function. For every functor $F : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$,*

$$H^i(\mathcal{P}; F) = 0,$$

if $i > \sup B$.

To prove this theorem, we need the following technical lemma.

Lemma 7.1.5. *Let $F : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be a functor over a filtered poset, $B\colon \mathcal{P} \to \mathbb{N}$ the labelling function associated to $\mathcal{P}$, and $p \in \mathcal{P}$ be an object such that:*

(a) For every $s < p$, $h(\mathbf{T}F(s)) \leqslant B(s)$;

(b) $h(M_p\mathbf{T}F) = \max\{B(s) \mid s < p\} \geqslant 1$; and

(c) $B(p) = \max\{B(s) \mid s < p\}$.

Then p is truncatable.

Proof. For short, name $m := \max\{B(s) \mid s < p\}$. By (b), $h(M_p\mathbf{T}F) = m$, thus p is F-truncatable iff the differential $\partial : (M_p\mathbf{T}F)^{m-1} \to (M_p\mathbf{T}F)^m$ is surjective. In addition, by (c), p does not close any circuit in $\mathcal{B}_p$. This implies that:

$$\lim_{\mathcal{B}_p} \mathbf{T}F = \bigoplus_{s \in \max \mathcal{B}_p} \mathbf{T}F(s).$$

Notice that, by (a), every $\mathbf{T}F(s)$ is a mapping cocylinder of height less or equal than m, so the differential

$$(\lim_{\mathcal{B}_p} \mathbf{T}F)^{m-1} \longrightarrow (\lim_{\mathcal{B}} \mathbf{T}F)^m$$

is an epimorphism. So, to prove the lemma, it is enough to check that the horizontal morphism, induced by the restriction, in the following diagram are isomorphisms:

$$\begin{tikzcd} (\lim_{\mathcal{P}_{<p}} \mathbf{T}F)^m \arrow[r] & (\lim_{\mathcal{B}_p} \mathbf{T}F)^m \\ (\lim_{\mathcal{P}_{<p}} \mathbf{T}F)^{m-1} \arrow[u] \arrow[r] & (\lim_{\mathcal{B}_p} \mathbf{T}F)^{m-1}. \arrow[u] \end{tikzcd}$$

By (a), if $s \notin \mathcal{B}_p$, then $h(\mathbf{T}F(s)) \leqslant B(s) \leqslant m-2$, so for every $s \notin \mathcal{B}_p$, $\mathbf{T}F^i(s) = 0$ for $i = m-1, m$; and $\mathcal{B}_p$ is upper convex; thus, by Proposition 2.1.13, we conclude that the horizontal arrows are isomorphism and we are done. $\square$

Now, we prove the main theorem of this section.

Proof of Theorem 7.1.4. As stated in Proposition 5.1.12, it suffices to demonstrate that $h(\mathbf{T}F(p)) \leqslant B(p)$.

We proceed by induction on the degree of the objects. If $p \in \mathcal{P}$ has degree 0 or 1 the result holds by definition.

Next, let $p \in \mathcal{P}$ be an object of degree n, and assume that $h(\mathbf{T}F(q)) \leqslant B(q)$ for every $q \in \mathcal{P}$ of degree less than n. By the induction hypothesis,

$$h(M_p\mathbf{T}F) \leqslant \max\{h(\mathbf{T}F(s)) \mid s < p\} \leqslant \max\{B(s) \mid s < p\} =: m.$$

If $h(M_p\mathbf{T}F) < m$ or $B(p) = m+1$, the result hold because:

$$h(\mathbf{T}F(p)) \leqslant h(M_p\mathbf{T}F) + 1 \leqslant m \leqslant B(p).$$

Therefore, we only need to prove the result when $h(M_p\mathbf{T}F) = m$. There are two options regarding $B(p)$: either $B(p) = m+1$, or $B(p) = m$.

In the first case the result holds by the following inequality:

$$h(\mathbf{T}F(p)) \leqslant h(M_p\mathbf{T}F) + 1 = m + 1.$$

Otherwise, $B(p) = m$, we apply Lemma 7.1.5 to conclude that p is truncatable, and hence

$$h(\mathbf{T}F(p)) = h(M_p\mathbf{T}F) = m. \qquad \square$$

Example 7.1.6. Let K be a simplicial complex of finite dimension d, and $\mathcal{P}$ be its face poset. The labelling function at $\mathcal{P}$ coincides with the dimension of every simplex. For 0-simplices and 1-simplices, the result is true by definition. Now, assume that for every $k < n$, and every σ k-simplex, $B(\sigma) = k$. Let τ be an n-simplex, and let σ_1, σ_2 two maximal faces of τ. Then we have the following subposet of $\mathcal{P}$:

$$\begin{tikzcd} & \tau & \\ \sigma_1 \arrow[ur] & & \sigma_2 \arrow[ul] \\ & \sigma_1 \cap \sigma_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

By the induction hypothesis, $B(\sigma_1) = B(\sigma_2) = n-1$ and $B(\sigma_1 \cap \sigma_2) = n-2$, then $\mathcal{B}_\tau$ has at least a connected component with two maximal objects. Therefore $B(\tau) = n$.

$$\begin{tikzcd}
 & B(\tau) & \\
n-1 \arrow[ur] & & n-1 \arrow[ul] \\
 & n-2 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Let A be an abelian group and $\underline{A}\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be the constant functor with value A. In that case, the conclusion of Theorem 7.1.4 is the well-known fact that $H^i(\mathcal{P}; \underline{A}) \cong H^i(|K|; A) = 0$ for $i > d$.

7.1.1 Maximal tree

Here we present a direct application of Theorem 7.1.4. A filtered tree is a filtered poset $\mathcal{P}$ whose Hasse diagram contains no undirected cycles. If $\mathcal{P}$ is a filtered poset, a maximal tree of $\mathcal{P}$ is a filtered tree $\mathcal{T}$ such that $\mathcal{T}$ contains every object of $\mathcal{P}$. Then, if $\mathcal{H}$ the Hasse diagram of $\mathcal{P}$, and $\mathcal{T}$ is a maximal tree of $\mathcal{P}$, we will show how the labelling function of $\mathcal{P}$ is controlled by the number of missing arrows in $\mathcal{T}$ whose codomains have different degrees. That is,

$$\#\{\mathrm{d}(q) \mid p \to q \in \mathcal{P} \backslash \mathcal{H}\}.$$

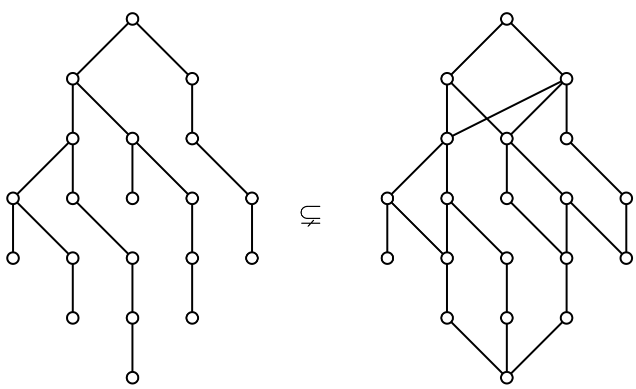

Figure 18: Maximal tree of a given poset.

Theorem 7.1.7. *Let $\mathcal{P}$ be a filtered poset, and $\mathcal{T}$ be a maximal tree of $\mathcal{H}$, the Hasse diagram of $\mathcal{P}$. Let $D(\mathcal{T}) = \#\{\mathrm{d}(q) \mid p \to q \in \mathcal{H}\backslash\mathcal{T}\}$. Then, for every functor $F : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$:*

$$H^i(\mathcal{P}; F) = 0$$

for every $i > 2D(\mathcal{T}) + 1$.

We divide the proof of this theorem into two lemmas.

Lemma 7.1.8. *Let $\mathcal{P}$ be a filtered poset. If there exists a maximal tree $\mathcal{T}$ of the Hasse diagram $\mathcal{H}$ of $\mathcal{P}$ in which the target of every missing arrow in $\mathcal{H}\backslash\mathcal{T}$ have the same degree, i.e.,*

$$\#\{\mathrm{d}(q) \mid p \prec q \in \mathcal{H}\backslash\mathcal{T}\} = 1,$$

then $\sup B \leqslant 3$.

Proof. Let $\mathcal{H}$ be the Hasse diagram of $\mathcal{P}$, and $\mathcal{T}$ be a maximal tree of $\mathcal{H}$ with the desired property. Let $\mathcal{Q}$ be the family of the target of the missing arrows,

$$\mathcal{Q} := \{q \in \mathcal{P} \mid p \prec q \in \mathcal{H}\backslash\mathcal{T}\}.$$

First, we prove that, for every $p \in \bigcup_{q\in\mathcal{Q}} \mathcal{P}_{<q}$, we have $B(p) \leqslant 1$. Fixed $p \in \bigcup_{q\in\mathcal{Q}} \mathcal{P}_{<q}$, there are no $q \in \mathcal{Q}$ such that $q \in \mathcal{P}_{\leqslant p}$, because every $q \in \mathcal{Q}$ has the same degree. Then, every covering relation in $\mathcal{P}_{\leqslant p}$ is in the maximal tree $\mathcal{T}$; thus, we conclude that there are no circuits in $\mathcal{P}_{\leqslant p}$. This implies that, for every $p \in \bigcup_{q\in\mathcal{Q}} \mathcal{P}_{<q}$, we have $B(p) \leqslant 1$.

Next, assume by contradiction that there exists $s \in \mathcal{P}$ such that $B(s) = 4$. Choosing a minimal s such that $B(s) = 4$, we have that $\max\{B(s') \mid s' < s\} = 3$. By Definition 7.1.2, s closes a circuit in $\mathcal{B}_s$, and this implies that there is a circuit $s \to \cdots \leftarrow s_i \to \cdots \leftarrow s$ in $\mathcal{H}$ with $2 \leqslant B(s_i) \leqslant 3$. But for every $q \in \mathcal{Q}$ and every cover relation $p \to q \notin \mathcal{H}\backslash\mathcal{T}$, we have that $B(p) \leqslant 1$. Therefore, there are no edges $p \to q$, with $q \in Q$ in the circuit $s \to \cdots \leftarrow s_i \to \cdots \leftarrow s$, so it is in $\mathcal{T}$, which contradicts that $\mathcal{T}$ is a tree. $\square$

Lemma 7.1.9. *Let $\mathcal{P}$ be a filtered poset. If $\mathcal{T}$ is a maximal tree of the Hasse diagram $\mathcal{H}$ of $\mathcal{P}$ with the property that*

$$\#\{\mathrm{d}(q) \mid p \to q \in \mathcal{H}\backslash\mathcal{T}\} = n,$$

then, $\sup B \leqslant 2n + 1$.

$$\begin{tikzcd}
0 \arrow[rr] & & 1 \arrow[dr] & \\
0 \arrow[r] \arrow[dr] & 1 \arrow[ur, dashed] \arrow[dr, dashed] & & 3 \\
& 1 \arrow[r] & 2 \arrow[ur] &
\end{tikzcd}$$

Figure 19: A poset in which the bound described at Lemma 7.1.8 is optimal.

Proof. As in Lemma 7.1.8, we set $\mathcal{Q} = \{q \in \mathcal{P} \mid p \to q \in \mathcal{H} \backslash \mathcal{T}\}$, and let $i_0 < \cdots < i_j < \cdots < i_n$ be the sequence of degrees of $q \in \mathcal{Q}$. Let $\mathcal{P}_j$ be the wide subposet of $\mathcal{P}$ generated by the covering relations in

$$\mathcal{T} \cup \{p \to q \mid p \to q \notin \mathcal{T},\ \mathrm{d}(q) \leqslant i_j\}.$$

Let $B_j : \mathcal{P}_j \to \mathbb{N}$ be the labelling function of $\mathcal{P}_j$. A direct computation shows that for every $p \in \mathcal{P}$, $B_j(p) \leqslant B_k(p)$ for every $j \leqslant k$; moreover, if $\mathrm{d}(p) \leqslant i_j$, then $B_j(p) = B_k(p)$.

We prove by induction that $\sup B_j = 2j + 1$. For $i = 0$, the result holds by Lemma 7.1.8. Now, assume that the result holds for i_j and consider $\mathcal{P}_{j+1}$.

Assume by contradiction that there exists $s \in \mathcal{P}_{j+1}$ such that $B_{j+1}(s) = 2j + 4$, and choose s a minimal with this property, i.e., we have

$$\max\{B(s') \mid s' < s\} = 2j + 3.$$

By Definition 7.1.2, s closes a circuit in $\mathcal{B}_s$, this implies the existence of a circuit $s \to \cdots \leftarrow s_i \to \cdots \leftarrow s$ in $\mathcal{H}$ with $2j + 2 \leqslant B(s_i) \leqslant 2j + 3$. But, by induction the hypothesis, for every $q \in \mathcal{Q}$ with $\mathrm{d}(q) = i_{j+1}$ and every $p \leqslant q$, $B_{j+1}(p) \leqslant 2j + 1$. Then the circuit $s \to \cdots \leftarrow s_i \to \cdots \leftarrow s$ is in $\mathcal{T}$ which is a contradiction. □

It is possible to show that the bound introduced in Lemma 7.1.9 is optimal by reiterating the poset in Figure 19; see Figure 20.

Proof of Theorem 7.1.7. This holds by applying Lemma 7.1.9 to Theorem 7.1.4. □

We obtain, as a direct corollary, the following result.

Corollary 7.1.10. *Let $\mathcal{P}$ be a filtered tree, and $F : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be a functor. Then $H^i(\mathcal{P}; F) = 0$ if $i > 1$.*

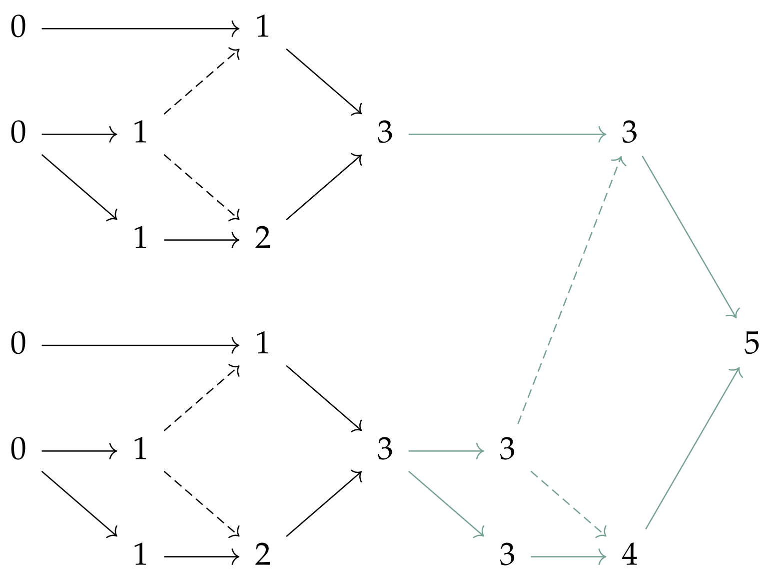

Figure 20

7.2 INDUCTIVE VANISHING BOUND

Given a functor $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$, the big deal with the F-truncability of an object $p \in \mathcal{P}$ is that it is expressed in terms of $\mathbf{T}F$, more precisely in terms of $M_p\mathbf{T}F$. However, $(\mathbf{T}F)\,|_{\mathcal{P}_{<p}}$ is a fibrant replacement of $F\,|_{\mathcal{P}_{<p}}$, and by Proposition 4.1.4,

$$H^m(\mathcal{P}_{<p}; F) \cong H^m(M_p\mathbf{R}F). \tag{16}$$

In the particular case of $m = h(M_p\mathbf{R}F)$, the truncability of the morphism $\varepsilon_p\colon F(p) \to M_p\mathbf{R}F$, is determined by the m-th cohomology group of $M_p\mathbf{R}F$, that is, ε_p is truncatable if

$$H^m(M_p\mathbf{R}F) = \operatorname{coker}(M_p\mathbf{R}F^{m-1} \xrightarrow{\partial} M_p\mathbf{R}F^m) = 0.$$

In this section, we prove that for every $p \in \mathcal{P}$, *a homogeneous bound* for the higher limit of $F|_{\mathcal{P}_{<p}}$ induces a bound in the higher limits of F.

Theorem 7.2.1. *Let $\mathcal{P}$ be a filtered poset, and $F : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be a functor. If for every $p \in \mathcal{P}$, $H^k(\mathcal{P}_{<p}; F) = 0$ for every $k \geqslant n$, then:*

$$H^k(\mathcal{P}; F) = 0$$

for every $k > n$.

We divide the proof of this theorem into several lemmas.

Lemma 7.2.2. *Let $\mathcal{P}$ be a filtered poset and $F : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be a functor. If $h(\mathbf{T}F) = m$, then, for every $0 \leqslant n \leqslant m$, there exist $p \in \mathcal{P}$ such that $h(\mathbf{T}F(p)) = n$. Furthermore, if $n > 0$, we can select $p \in \mathcal{P}$ such that $h(M_p\mathbf{T}F) = n - 1$.*

Proof. If $n = 0$, by definition of $\mathbf{T}F$ (see Definition 5.2.8), every $p \in \mathcal{P}$ of degree 0 verifies $h(\mathbf{T}F(p)) = 0$. Thus, assume that $0 < n \leqslant m$.

We proceed by contradiction. Let $n \in \mathbb{N}$ with $0 < n < m$ such that $h(\mathbf{R}F(p)) \neq n$ for all $p \in \mathcal{P}$, and let $\mathcal{S}$ be the sub-poset of $\mathcal{P}$ given by:

$$\mathcal{S} := \{s \in \mathcal{P} \mid h((\mathbf{T}F)_s) > n\}.$$

By hypothesis $\mathcal{S}$ is non-empty because $h(\mathbf{T}F) = m > n$, so there exists some $p \in \mathcal{P}$ such that $h(\mathbf{T}F(p)) = m$. Moreover, by Definition 5.2.8, S does not contain any locally injective object because $n > 0$.

Now, let s be a minimal object in $\mathcal{S}$, that is, $h(\mathbf{T}F(s)) > n$, and for every $t < s$, $h(\mathbf{T}F(t)) < n$. This implies that $h(M_s\mathbf{T}F) < n$ because

$$h(M_s\mathbf{T}F) = h(\lim_{\mathcal{P}_{<s}} \mathbf{T}F) \leqslant \max\{h(\mathbf{T}F(t)) \mid t < s\} < n.$$

If s is F-truncatable, we obtain a contradiction:

$$n < h(\mathbf{T}F(s)) = h(\mathrm{cocyl}_{\mathbf{T}}(\varepsilon_s : F_s \to M_s\mathbf{T}F)) = h(M_s\mathbf{T}F) < n.$$

Then s is not F-truncatable, this implies the following inequalities:

$$n < h(\mathbf{T}F(s)) = h(\mathrm{cocyl}(\varepsilon_s : F_s \to M_s\mathbf{T}F)) = h(M_s\mathbf{T}F) + 1 \leqslant n$$

leading again to a contradiction. This implies that there exists at least a $p \in \mathcal{P}$ such that $h(\mathbf{T}F(p)) = n$.

To show that we can choose it with the additional property of $h(M_p\mathbf{T}F) = n-1$, let $p \in \mathcal{P}$ with $h(\mathbf{T}F(p)) = n > 0$, minimal with this property, this is, for every $s < p$, $h(\mathbf{T}F(p)) < n$. Since p is minimal, $h(M_p\mathbf{T}F) < n$; therefore, by Definition 5.2.8, s must be non F-truncatable and:

$$n = h(\mathbf{T}F(s)) = h(\mathrm{cocyl}(F(s) \to M_s\mathbf{T}F)) = h(M_s\mathbf{T}F) + 1$$ □

Now we know that the fibrant replacement $\mathbf{T}F$ does not jump any height, the next lemma tells us that the higher limits of the functor F restricted to the subcategory $\mathcal{P}_{<p}$ control the F-truncability of p.

Proposition 7.2.3. *Let $\mathcal{P}$ be a filtered poset, $F : \mathcal{P}^{\mathrm{op}} \to \mathsf{Ab}$ be a functor, and n be a positive integer. The following are equivalent:*

1. $H^k(\mathcal{P}_{<p}; F) = 0$, *for every $p \in \mathcal{P}$ and $k \geqslant n$.*
2. $H^n(\mathcal{P}_{<p}; F) = 0$, *for every $p \in \mathcal{P}$.*
3. *The fibrant replacement $\mathbf{T}F$ has height $h(\mathbf{T}F) \leqslant n$.*

Proof. By the inductive construction of the fibrant replacement and Equation (16), for every $p \in \mathcal{P}$, it is verified that:

$$H^k(\mathcal{P}_{<p}; F) = H^k(\lim_{\mathcal{P}_{<p}} \mathbf{T}F) = H^k(M_p\mathbf{T}F). \tag{17}$$

The implication 1⇒ 2 is clear. To show that 2⇒3, assume by contradiction that $h(\mathbf{T}F) > n > 0$. By Lemma 7.2.2, there exists an object $p \in \mathcal{P}$ such that $h(\mathbf{T}F(p)) = n+1$ and $h(M_p\mathbf{T}F) = n$. Then, by Equation (17):

$$H^n(\mathcal{P}_{<p}; F) = H^n(M_p\mathbf{T}F) = \mathrm{coker}(\partial_{n-1} \colon (M_p\mathbf{T}F)^{n-1} \to (M_p\mathbf{T}F)^n).$$

However, $H^n(\mathcal{P}_{<p}; F) = 0$ if and only if ∂_{n-1} is an epimorphism if and only if $\varepsilon_p : F(p) \to M_p\mathbf{T}F$ is truncatable so, by definition, $\mathbf{T}F(p) = \mathrm{cocyl}_{\mathbf{T}}(\varepsilon_p)$. Therefore, $h(\mathbf{T}F(p)) = n \neq n+1$ which is a contradiction.

Finally, we need to prove 3⇒1. First, notice that from Equation (17) follows that 1 is equivalent to that, for every $p \in \mathcal{P}$, and $k \geqslant n$, $H^k(M_p\mathbf{T}F) = 0$. Fix $p \in \mathcal{P}$, since $h(\mathbf{T}F(p)) \leqslant h(\mathbf{T}F) \leqslant n$, it is enough to show that $H^n(M_p\mathbf{T}F) = 0$.

There is no loss of generality in assuming that $h(M_p\mathbf{T}F) = n$ and $h(\mathbf{T}F(p)) = n$. Therefore, $\varepsilon_p \colon F(p) \to M_p\mathbf{T}F$ must be truncatable, and hence the differential

$\partial_{n-1}\colon (M_p\mathbf{T}F)^{n-1} \to (M_p\mathbf{T}F)^n$ is an epimorphism. Thus, from Equation (17), we conclude

$$H^n(\mathcal{P}_{<p}; F) = H^n(M_p\mathbf{T}F) = \operatorname{coker}(\partial_{n-1}\colon (M_p\mathbf{T}F)^{n-1} \to (M_p\mathbf{T}F)^n) = 0. \qquad \square$$

In the case of $\mathcal{P}$ being a filtered poset of finite length, we prove can relax the bound in the hypothesis of Theorem 7.2.1 for a partial bound for the higher limits of $F|_{\mathcal{P}_{<p}}$ for p of certain consecutive degrees.

Theorem 7.2.4. *Let $\mathcal{P}$ be a filtered poset of finite length, and $F : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be a functor. If there exists $m \leqslant \operatorname{length}(\mathcal{P})$ and $n \in \mathbb{N}$ such that, for every $p \in \mathcal{P}$ with $\mathrm{d}(p) < m$, we have:*

$$H^k(\mathcal{P}_{<p}; F) = 0 \quad \textit{for } n \leqslant k.$$

Then, $H^k(\mathcal{P}; F) = 0$ for $n + \operatorname{length}(\mathcal{P}) - m < k$.

Proof. Let $\mathbf{R}F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ch}(\mathrm{Ab})$ be the functor defined by:

$$\mathbf{R}F(p) := \begin{cases} \mathbf{T}F(p) & \text{if } \mathrm{d}(p) \leqslant m, \\ \operatorname{cocyl}(F(p) \to M_p\mathbf{R}F) & \text{Otherwise.} \end{cases}$$

Note that this is a fibrant replacement of F. By Proposition 7.2.3, for every $p \in \mathcal{P}$ with $\mathrm{d}(p) \leqslant m$, we have:

$$h(\mathbf{R}F(p)) = h(\mathbf{T}F(p)) \leqslant n$$

Now, we prove by induction that, for every object $p \in \mathcal{P}$ with $\mathrm{d}(p) = m + k$ for some $k \geqslant 0$, we have

$$h(\mathbf{R}F) \leqslant n + k.$$

The basis case is done. Now, assume the result is true for $k - 1$, and let $p \in \mathcal{P}$ with $\mathrm{d}(p) = k$. Then,

$$\begin{aligned} h(\mathbf{R}F(p)) =& h(\operatorname{cocyl}(F(p) \to M_p\mathbf{R}F)) \\ \leqslant& h(M_p\mathbf{R}F) + 1 \leqslant \max\{h(\mathbf{R}F(q)) \mid q \leqslant p\} + 1 \\ \leqslant& n + k - 1 + 1 = n + k. \end{aligned}$$

Then, $h(\mathbf{R}F) \leqslant n + \operatorname{length}(\mathcal{P}) - m$ which implies the desired vanishing bound. $\square$

7.3 VANISHING BOUND BY ATOMIC FUNCTORS

A strategy that has been used in the literature is filtering a functor by subfunctors such that their successive quotients take the value zero except on one object. These last functors are called atomic functors. More precisely, given a filtered poset $\mathcal{P}$ and an abelian group A, the *atomic functor* of A at $p_0 \in \mathcal{P}$, denoted by $\mathcal{A}(A, p_0) : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$, is the functor defined by:

$$\mathcal{A}(A, p_0)(p) = \begin{cases} A & \text{if } p = p_0 \\ 0 & \text{otherwise.} \end{cases}$$

The goal of this section is to describe higher limits of atomic functors via ordinary cohomology of the nerve of a subposet, inspired by the next example.

Example 7.3.1. Let $\mathcal{P}$ be a functor and R be a commutative ring with unit. Then, the higher limits of the constant functor $\underline{R}\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ are isomorphic to the ordinary cohomology of the geometric realisation of $\mathcal{P}$ with coefficients in R, see [AKO11, III.5.4],

$$H^*(\mathcal{P}; \underline{R}) \cong H^*(|\mathcal{P}|; R).$$

First, we characterise the higher limits of atomic functors in terms of (reduced) ordinary cohomology.

Theorem 7.3.2. *Let $\mathcal{P}$ be a filtered poset and $p_0 \in \mathcal{P}$. For every abelian group A:*

$$H^i(\mathcal{P}; \mathcal{A}(A, p_0)) \cong \widetilde{H}^{i-1}(|\mathcal{P}_{>p_0}|; A).$$

Proof. Let $\mathcal{A}(A, p_0)\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be the atomic functor at $p_0 \in \mathcal{P}$, and $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be the extension by zero of the constant functor $\underline{A}\colon \mathcal{P}^{\mathrm{op}}_{>p_0} \to \mathrm{Ab}$, that is, $F|_{\mathcal{P}_{>p_0}} = \underline{A}$ and $F(p) = 0$ for all $p \not> p_0$.

Notice that $H^i(\mathcal{P}; F) \cong H^i(|\mathcal{P}_{>p_0}|; A)$. Thus, it is enough to prove that:

$$\mathbf{cocyl}\,(\mathcal{A})^0 = \underline{A} \text{ and, } \mathbf{cocyl}\,(\mathcal{A})^i = \mathbf{cocyl}\,(F)^{i-1} \text{ for } i > 0. \tag{18}$$

The supports of both functors are contained in $\mathcal{P}_{\geqslant p_0}$, and it is an upper convex poset. By Proposition 2.1.13, there is no loss of generality into assuming that p_0 is the initial object of $\mathcal{P}$.

We prove Equation (18) by induction on the degree of the objects. For p_0, the only object of degree 0, the result trivially holds. Next, assume that the result holds for every object of degree less than n and let $p \in \mathcal{P}$ of degree n.

Limits in the category of cochain complex are computed degree-wise; therefore, it is verified that $M_p\mathbf{cocyl}(\mathcal{A})^0 = A$, and $M_p\mathbf{cocyl}(\mathcal{A})^i = M_p\mathbf{cocyl}(F)^{i-1}$ for $i > 0$. A short computation shows that $\mathrm{cocyl}(0 \to M_p\mathbf{cocyl}(\mathcal{A}))^0 = A$, and

$$\mathrm{cocyl}(0 \to M_p\mathbf{cocyl}(\mathcal{A}))^i = \mathrm{cocyl}(A \to M_p\mathbf{cocyl}(F))^{i-1}$$

for each $i > 0$. $\square$

Lemma 7.3.3. *Let $\mathcal{P}$ be a filtered poset, $\{p_i\}_{i\in\mathcal{I}}$ be a collection of objects of $\mathcal{P}$ with the same degree, and $\{A_i\}_{i\in\mathcal{I}}$ be a collection of abelian groups then:*

$$H^k(\mathcal{P}; \bigoplus_{i\in\mathcal{I}} \mathcal{A}(A_i, p_i)) = \bigoplus_{i\in\mathcal{I}} H^k(\mathcal{P}; \mathcal{A}(A_i, p_i)).$$

Proof. This lemma holds since the direct sum of fibrant replacements, constructed as in Chapter 5, is a fibrant replacement of the direct sum. $\square$

Now, we are able to prove the main theorem of this section.

Theorem 7.3.4. *Let $\mathcal{P}$ be a finite filtered poset and $F : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be a functor. If there exists $k > 0$ such that, for every $p \in \mathcal{P}$, $\widetilde{H}^n(|\mathcal{P}_{>p}|; F(p)) = 0$, for $n > k$; then*

$$H^n(\mathcal{P}; F) = 0 \text{ for } n > k.$$

By Theorem 7.3.2 this previous result is equivalent to the following lemma.

Lemma 7.3.5. *Let $\mathcal{P}$ be a finite filtered poset and $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ be a functor. If there exists $n > 0$ such that $H^i(\mathcal{P}; \mathcal{A}(F(p), p)) = 0$ for every $p \in \mathcal{P}$ and every $i > n$, then*

$$H^k(\mathcal{P}; F) = 0$$

for every $k > n$.

Proof. Let m be the length of $\mathcal{P}$. Given $0 \leqslant k \leqslant m$, we define $F^k : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ to be the subfunctor of F defined by $F^k(p) = F(p)$ if p has degree less or equal to k; and $F^k(p) = 0$, otherwise. By construction, there is a chain of natural transformation of functors:

$$F^0 \to F^1 \to \cdots \to F^{m-1} \to F^m = F.$$

Given $0 < k \leqslant m$, we define $F^{(k)} := \text{coker}(F^{k-1} \to F^k)$ the cokernel of the inclusion. Notice that by construction, for every $0 < k \leqslant m$ $F^{(k)} = \bigoplus_{\mathrm{d}(p)=k} \mathcal{A}(F(p), p)$, and, by hypothesis and Lemma 7.3.3,

$$H^i(\mathcal{P}; F) \cong H^i(\mathcal{P}; \bigoplus_{\mathrm{d}(p)=k} \mathcal{A}(F(p), p)) \cong \bigoplus_{\mathrm{d}(p)=k} H^i(\mathcal{P}; \mathcal{A}(F(p), p)) = 0$$

for every $i > n$.

Moreover, we have a short exact sequence of functors:

$$0 \to F^{k-1} \to F^k \to F^{(k)} \to 0$$

inducing a long exact sequence:

$$\begin{aligned} 0 \to H^0(\mathcal{P}; F^{k-1}) \to H^0(\mathcal{P}; F^k) \to H^0(\mathcal{P}; F^{(k)}) \to H^1(\mathcal{P}; F^{k-1}) \to \dots \\ \dots \to H^i(\mathcal{P}; F^{k-1}) \to H^i(\mathcal{P}; F^k) \to H^i(\mathcal{P}; F^{(k)}) \to H^{i+1}(\mathcal{P}; F^{k-1}) \to \dots \end{aligned}$$

Now, we proceed by induction. For $k = 1$,

$$\dots \to H^i(\mathcal{P}; F^0) \to H^i(\mathcal{P}; F^1) \to H^i(\mathcal{P}; F^{(1)}) \to \dots$$

By Lemma 7.3.3, this exact sequence becomes

$$0 \to \bigoplus_{\mathrm{d}(p)=0} H^i(\mathcal{P}; \mathcal{A}(F(p), p)) \to H^i(\mathcal{P}; F^1) \to \bigoplus_{\mathrm{d}(p)=1} H^i(\mathcal{P}; \mathcal{A}(F(p), p)) \to \dots$$

By hypothesis, for every $p \in \mathcal{P}$, $H^i(\mathcal{P}; \mathcal{A}(F(p), p)) = 0$ if $i > n$, then $H^i(\mathcal{P}; F^1) = 0$ if $i > n$.

Assume that $H^i(\mathcal{P}; F^j) = 0$ for ever $i > n$ and every $j < k$. Then the long exact sequence specialises to:

$$\to 0 \to H^{n+1}(\mathcal{P}; F^k) \to H^{n+1}(\mathcal{P}; F^{(k)}) \to 0 \to H^{n+2}(\mathcal{P}; F^k) \to \dots$$

and, by hypothesis, $H^i(\mathcal{P}; F^{(k)}) = \bigoplus_{\mathrm{d}(p)=k} H^i(\mathcal{P}; \mathcal{A}(F(p), p)) = 0$, if $i > n$. Then $H^i(\mathcal{P}; F^k) = 0$ for $i > n$ and every $0 \leqslant k \leqslant m$. So, we conclude that $H^i(\mathcal{P}; F) = 0$ for $i > n$. $\square$

Example 7.3.6. Let (W, S) be a Coxeter group, $J \subset S$, $\leqslant$ the Bruhat order on W. Let W^J be the subposet of W defined by

$$W^J = \{w \in W \mid ws > w \text{ for all } s \in J\}.$$

Let $\mathcal{P}$ be the subposet of W^J obtained by removing the top element. Let $p \in \mathcal{P}$ and $n = \mathrm{d}(\hat{1}) - \mathrm{d}(p) - 2$. By [BW82, Theorem 5.4], the geometric realisation of $\mathcal{P}_{>p}$ is:

1. a n-sphere if $\mathcal{P}_{>p} = W_{>p} \backslash \{\hat{1}\}$, or
2. contractible in another case.

Let $p \in \mathcal{P}$ minimal with the property that the geometric realisation of $\mathcal{P}_{>p}$ is not contractible and let $n = \mathrm{d}(\hat{1}) - \mathrm{d}(p) - 2$. Then, for every functor $F \colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$, the homology groups verifies

$$\widetilde{H}^i(|\mathcal{P}_{>p}|; F(p)) = 0 \text{ for every } i > n.$$

Thus, by Theorem 7.3.4 we conclude

$$H^i(\mathcal{P}; F) = 0 \text{ for every } i > n.$$

Chapter 8

SHEAF COHOMOLOGY OF CL-SHELLABLE POSETS

Sheaf cohomology is an important tool in algebraic geometry, topology, and combinatorics, and it has been extensively studied by researchers in these and other fields. For example, Everitt and Turner show how Khovanov homology [ET14] can be described as the cellular cohomology of a certain sheaf [ET15]. Recently, they have computed the cohomology of a certain sheaf in arrangement lattices [ET22a] by the *deletion-restriction* method that they introduce in a previous work [ET22b].

In this chapter, we focus on the sheaf cohomology of CL-shellable posets. We begin by considering finite or Alexandroff T_0-spaces, where sheaves can be identified with functors and sheaf cohomology can be understood as their higher limits.

First, notice that Alexandroff T_0-spaces and posets are the same thing. Let $\mathcal{P}$ be an Alexandroff T_0-space, and $x \in \mathcal{P}$. Let U_x denote the minimal open set that contains x. Then, the binary relation $\leqslant$, defined below, is a partial order for $\mathcal{P}$.

$$x \leqslant y \text{ if } x \in U_y.$$

Conversely, given a poset $\mathcal{P}$, the family $\{\mathcal{P}_{\leqslant p}\}_{p\in\mathcal{P}}$ defines a basis for the *Alexandroff* topology of $\mathcal{P}$. With this topology, $\mathcal{P}$ is an Alexandroff T_0-space.

When $\mathcal{P}$ is an Alexandroff T_0 topological space, every functor $F\colon \mathrm{Open}(\mathcal{P})^{\mathrm{op}} \to \mathrm{Mod}_R$ verifies the **SHEAF AXIOM**:

SHEAF AXIOM: Let $\{U_i\}$ be an open cover of an open subset $U \in \mathrm{Open}(\mathcal{P})$. If $(f_i) \in \prod_i F(U_i)$ verifies that for every i, j,

$$F(U_i \cap U_j \subset U_i)(f_i) = F(U_i \cap U_j)(f_j);$$

then, there exists a unique $f \in F(U)$ such that it is mapped to every f_i under $F(U_i \subset U)$; see [Wei94].

Moreover, the cohomology of a sheaf $F\colon \mathrm{Open}(\mathcal{P})^{\mathrm{op}} \to \mathrm{Mod}_R$ is isomorphic to the higher limits of the functor $\hat{F}\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ under the identification $\mathcal{P}_{\leqslant p} \mapsto p$. Thus, every functor $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ can be interpreted as sheaf over $\mathcal{P}$.

In the case of K being a shellable complex, since they have the homotopy type of a wedge of k spheres of dimension $\dim K$ [Bjö80], we have a description of the sheaf cohomology of the constant sheaf $\underline{R}\colon \mathcal{P}(K)^{\mathrm{op}} \to \mathrm{Mod}_R$:

$$H^i(\mathcal{P}(K)\backslash\{\hat{1},\hat{0}\}; \underline{R}) \cong \begin{cases} R^k & \text{if } i = \dim K, \\ R & \text{if } i = 0, \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

We take inspiration from the constant sheaf to abstract the elementary condition that implies the sheaf cohomology vanishes in the non-extreme dimensions.

Let K be a shellable complex of dimension n, and L be the face lattice of K. Let R be a commutative ring with unit and $F\colon L^{\mathrm{op}} \to \mathrm{Mod}_R$ the extension by 0 of the constant functor $\underline{R}\colon \left(L\backslash\{\hat{0}\}\right)^{\mathrm{op}} \to \mathrm{Mod}_R$. The functor F only fails to be fibrant in objects of dimension 1, that is, for every $\sigma \in K$ of dimension $\dim\sigma \neq 1$, the natural map

$$F(\sigma) \to M_\sigma F$$

is surjective. This holds since for every $\tau, \tau' \prec \sigma$, the respective copies of R in τ and τ' are identified under the maps $\tau > \tau' \cap \tau < \tau'$. Moreover, given $Q \subset L_{\prec\sigma}$, the composite

$$F(\sigma) \to M_\sigma F \to \lim_{\langle Q\rangle} F$$

is also an epimorphism if and only if $\dim \sigma \neq 1$. In some sense, the surjectivity of the natural map $F(\sigma) \to M_\sigma F$ is stable under taking restrictions. We generalise this property in the following definition.

Definition 8.1. Let $\mathcal{P}$ be a dual CL-shellable poset of length $n \geqslant 2$ equipped with a recursive coatom ordering $\ll$, and $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a sheaf. We say that $(\mathcal{P}, F)$ has the *stability property in codegree* $i \in \mathbb{N}$, for $0 \leqslant i \leqslant n$, if, for every object p of degree $\mathrm{d}(p) = n - i$, and every $Q \subset \mathcal{P}_{\prec p}$ compatible with the recursive coatom ordering, the natural map

$$F(p) \to \lim_{\langle Q \rangle} F$$

is an epimorphism.

This property will be fundamental in this section since it implies the nullity of the respective cohomology module of the sheaf.

Theorem 8.2. *Let $\mathcal{P}$ be a dual CL-shellable poset of length $n \geqslant 2$, $i \in \mathbb{N}$ such that $1 \leqslant i \leqslant n-1$, and $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a sheaf. If the pair $(\mathcal{P}, F)$ has the stability property in codegree i, then*

$$H^i(\mathcal{P} \backslash \{\hat{1}\}; F) = 0.$$

We prove this theorem in the next section.

Example 8.3. Let K be a shellable complex of finite dimension n, and $\mathcal{P}$ be the face poset of K. By Theorem 1.1.5, $|K|$ has the homotopy type of a wedge of n-spheres. Let k be the number of spheres in the wedge,

$$|K| \cong \bigvee_{j=1}^{k} S^n.$$

Furthermore, by Theorem 1.1.8, $\mathcal{P}$ is dual CL-shellable of length $n+2$. Let R be a commutative ring with unit and $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ to be the extension by 0 of the constant functor $\underline{R}\colon (\mathcal{P} \backslash \{\hat{0}\})^{\mathrm{op}} \to \mathrm{Mod}_R$. Then, the pair $(\mathcal{P}, F)$ has the stability property in codegree $i \neq n = \mathrm{d}(\mathcal{P}) - 2$. From Theorem 8.2 follows that

$$H^i(\mathcal{P} \backslash \{\hat{1}\}; F) = 0,$$

for $i \neq 0, \mathrm{d}(\mathcal{P}) - 2 = n$. In addition, $H^0(\mathcal{P}; F) = \lim F = R$ and, by Theorem 5.2.11 and a short computation, $H^n(\mathcal{P}; F) = R^k$ for some k. Therefore, we recover the well-known result:

$$H^i(|K|; R) \cong H^i(\mathcal{P}; F) = \begin{cases} R^k & \text{if } i = \dim K \\ 0 & \text{if } 0 < i < \dim K \\ R & \text{if } i = 0 \end{cases}$$

8.1 COMBINATORIAL PROPERTIES OF THE COCYLINDER

Let $\mathcal{P}$ be a poset, and $\mathcal{Q}$ be a subposet of $\mathcal{P}$. Given $F : \mathcal{P} \to \mathrm{Mod}_R$, the *restriction morphism* induced by the inclusion $\mathcal{Q} \to \mathcal{P}$ is denoted by

$$\mathrm{Res}^{\mathcal{P}}_{\mathcal{Q}} \colon \lim_{\mathcal{P}} F \to \lim_{\mathcal{Q}} F.$$

First, we need to prove the following technical lemma which will be essential for the inductive step.

Lemma 8.1.1. *Let $\mathcal{P}$ be a dual CL-shellable poset, and $F \colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a sheaf. Let $p \in \mathcal{P}$, $\ll$ be a recursive coatom ordering in $\mathcal{P}_{\leqslant p}$, $Q \subsetneq \mathcal{P}_{\prec p}$ be a non-empty subset of coatoms compatible with the recursive coatom ordering. Let r be the first coatom of $\mathcal{P}_{\leqslant p} \backslash Q$, this is, the element $r \in \mathcal{P}_{\prec p}$ with the property that if $s \in \mathcal{P}_{\prec p}$ with $s \ll r$, then $s \in Q$. Then, for $i \geqslant 0$, if the composite*

$$\mathbf{cocyl}\,(F)^i\,(r) \to (M_r\mathbf{cocyl}\,(F))^i \to \lim_{\langle C(r)\rangle} \mathbf{cocyl}\,(F)^i$$

admits a section, then the restriction

$$\mathrm{Res}^{\langle Q\cup\{r\}\rangle}_{\langle Q\rangle} \colon \lim_{\langle Q\cup\{r\}\rangle} \mathbf{cocyl}\,(F)^i \to \lim_{\langle Q\rangle} \mathbf{cocyl}\,(F)^i,$$

also admits a section.

Proof. To simplify the notation, we denote by Q_r the union $Q \cup \{r\}$. Let $i \geqslant 0$. By hypothesis, there exists a section $s \colon \lim_{\langle C(r)\rangle} \mathbf{cocyl}\,(F)^i \to \mathbf{cocyl}\,(F)^i\,(r)$ of the composite morphism

$$\mathbf{cocyl}\,(F)^i\,(r) \to (M_r\mathbf{cocyl}\,(F))^i \to \lim_{\langle C(r)\rangle} \mathbf{cocyl}\,(F)^i.$$

Given $x = (x_q) \in \lim_{\langle Q \rangle} \mathbf{cocyl}\,(F)^i \leqslant \prod_{q \in Q} \mathbf{cocyl}\,(F)^i\,(q)$, we claim that the tuple

$$(s(\mathrm{Res}^{\langle Q \rangle}_{\langle C(r) \rangle}(x)), x) \in \mathbf{cocyl}\,(F)^i\,(r) \times \prod_{q \in Q} \mathbf{cocyl}\,(F)^i\,(q)$$

defines an element of $\lim_{\langle Q_r \rangle} \mathbf{cocyl}\,(F)^i$. Since Q is compatible with the coatom ordering and r is the first element in $\mathcal{P}_{\leqslant p} \backslash Q$, it follows that Q_r is compatible with the recursive coatom ordering. By Lemma 1.1.16, $\langle Q_r \rangle \cup \{p\}$ is a CL-shellable poset of length $\mathrm{d}(p)$, and, by Lemma 1.1.14, the full subcategory of $\langle Q_r \rangle$ whose objects are those of degree $\mathrm{d}(p) - 1$ and $\mathrm{d}(p) - 2$ is a final subcategory of $\langle Q_r \rangle$. Therefore, it is enough to check that, for every $q \in Q$ and every $t \in \mathcal{P}_{\prec r} \cap \mathcal{P}_{\prec q}$, we have $\mathbf{cocyl}\,(F)\,(t < q)(x_q) = \mathbf{cocyl}\,(F)\,(t < r)(s(\mathrm{Res}^{\langle Q \rangle}_{\langle C(r) \rangle}(x)))$.
Since $x \in \lim_{\langle Q \rangle} \mathbf{cocyl}\,(F)^i$, the element

$$\mathrm{Res}^{\langle Q \rangle}_{\langle C(r) \rangle}(x) \in \lim_{\langle C(r) \rangle} \mathbf{cocyl}\,(F)^i \leqslant \prod_{t \in C(r)} \mathbf{cocyl}\,(F)^i\,(t)$$

has coordinates $\mathrm{Res}^{\langle Q \rangle}_{\langle C(r) \rangle}(x)_t = \mathbf{cocyl}\,(F)\,(t < q)(x_q)$, where q is any element in Q such that $t < q$. Moreover, since s is a section for $\mathrm{Res}^{\langle Q \rangle}_{\langle C(r) \rangle}$ the composite,

$$\lim_{\langle C(r) \rangle} \mathbf{cocyl}\,(F)^i \xrightarrow{s} \mathbf{cocyl}\,(F)^i\,(r) \to \mathbf{cocyl}\,(F)^i\,(t)$$

is the projection to the t-coordinate

$$\begin{tikzcd}
\mathbf{cocyl}\,(F)^i\,(r) \arrow[rr] & & M_r\mathbf{cocyl}\,(F)^i \arrow[d] & \\
\lim_{\langle C(r) \rangle} \mathbf{cocyl}\,(F)^i \arrow[u] \arrow[rr, dashed, "\mathrm{Id}_{\lim_{\langle C(r) \rangle} \mathbf{cocyl}(F)^i}"] & & \lim_{\langle C(r) \rangle} \mathbf{cocyl}\,(F)^i \arrow[r] & \mathbf{cocyl}\,(F)^i\,(t)
\end{tikzcd}$$

Thus, we obtain the desired equality. $\square$

Next, we prove that the cocylinder of a sheaf verifies the stability property in any codegree.

Lemma 8.1.2. *Let $\mathcal{P}$ be a dual CL-shellable poset, and let $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a sheaf. Then, for every $p \in \mathcal{P}$, every recursive coatom ordering $\ll$ in $\mathcal{P}_{\leqslant p}$ and every non-empty subset $Q \subset \mathcal{P}_{\prec p}$ compatible with the recursive coatom ordering, the restriction morphism*

$$\mathrm{Res}^{\mathcal{P}_{<p}}_{\langle Q \rangle}\colon M_p\mathbf{cocyl}\,(F) \to \lim_{\langle Q \rangle} \mathbf{cocyl}\,(F)$$

is a degreewise split epimorphism.

Proof. Let $i \geqslant 0$. We denote by $\mathrm{Res}^{\mathcal{P}_{<p}}_{\langle Q \rangle} \colon (M_p \mathbf{cocyl}(F))^i \to \lim_{\langle Q \rangle} \mathbf{cocyl}(F)^i$ the restriction morphism at the fixed degree i. We proceed by induction on the degree of the object $p \in \mathcal{P}$. If p is of degree 1, then the explicit description of the morphism gives the result (see Theorem 5.2.11 with $n = 1$).

We assume the statement is true for every object of degree less than n, and let $p \in \mathcal{P}$ with $\mathrm{d}(p) = n$. If $Q = \mathcal{P}_{\prec p}$, the restriction morphism is just the identity, and we are done. Otherwise, we proceed as follows. Let r be the first element in $r \in \mathcal{P}_{\prec p} \backslash Q$. To shorten the notation, again, we denote by $Q_r = Q \cup \{r\}$.

We start by constructing a section of the restriction:

$$\mathrm{Res}^{\langle Q_r \rangle}_{\langle Q \rangle} \colon \lim_{\langle Q_r \rangle} \mathbf{cocyl}(F)^i \to \lim_{\langle Q \rangle} \mathbf{cocyl}(F)^i$$

by using Lemma 8.1.1. That is, we have to prove that there exists a section for the composite:

$$\mathbf{cocyl}(F)^i(r) \to M_r \mathbf{cocyl}(F)^i \to \lim_{\langle C(r) \rangle} \mathbf{cocyl}(F)^i .$$

We will do it by constructing a section of each morphism in the composition. First, we prove the existence of a section s_0 for the morphism

$$M_r \mathbf{cocyl}(F)^i \to \lim_{\langle C(r) \rangle} \mathbf{cocyl}(F)^i .$$

There are two options regarding $C(r)$. Either $C(r) = \mathcal{P}_{\prec r}$ or $C(r) \subsetneq \mathcal{P}_{\prec r}$. In the first case, s_0 is the identity. In the second one, we need to apply an induction argument. By Definition 1.1.9, for the chain $c = (r \prec p)$, we have a linear order for $\mathcal{P}_{\prec r}$ in which $C(r)$ is an initial segment. Therefore, by Lemma 1.1.16, $\langle C(r) \rangle \cup \{r\}$ is a CL-shellable poset of length $\mathrm{d}(r) = n - 1 < n$. Then, the section s_0 exists by induction hypothesis.

Next, by Proposition 5.2.4, the matching morphism $\mathbf{cocyl}(F)(r) \to M_r \mathbf{cocyl}(F)$ is a split epimorphism, so there exists a section $s_1 \colon M_r \mathbf{cocyl}(F) \to \mathbf{cocyl}(F)(r)$. Then $s_1 \circ s_0$ is a section of $\mathbf{cocyl}(F)^i(r) \to \lim_{\langle C(r) \rangle} \mathbf{cocyl}(F)^i$.

Finally, repeating this argument finitely many times with the remaining objects in $\mathcal{P}_{\prec p} \backslash Q$ we construct the desired section. $\square$

The second step of the proof consists in showing how the stability property in a sheaf F is translated in a weaker stability property for the subfunctor of its cocylinder $\mathbf{cocyl}(F)$ obtained by taking object-wise the kernel of the differential. More formally, given a sheaf $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$, we denote by $\ker \mathbf{cocyl}\,(F)^i : \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ the sub-functor of $\mathbf{cocyl}\,(F)^i : \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ defined on objects by

$$\ker \mathbf{cocyl}\,(F)^i\,(p) = \ker(\partial_p\colon \mathbf{cocyl}\,(F)^i\,(p) \to \mathbf{cocyl}\,(F)^{i+1}\,(p)).$$

Given $p \in \mathcal{P}$, $i \geqslant 0$ and $Q \subset \mathcal{P}_{\prec p}$ compatible with some recursive coatom ordering, by Lemma 8.1.2, we can consider $\lim_{\langle Q\rangle} \mathbf{cocyl}\,(F)^i$ as a subcochain complex of $M_p\mathbf{cocyl}\,(F)^i$; therefore, by abuse of notation, we denote by π the composite of the projection given by the cocylinder, $\pi\colon \mathbf{cocyl}\,(F)^i \to M_p\mathbf{cocyl}\,(F)^i$, followed by the restriction, $\mathrm{Res}^{\mathcal{P}_{<p}}_{\langle Q\rangle}\colon M_p\mathbf{cocyl}\,(F)^i \to \lim_{\langle Q\rangle} \mathbf{cocyl}\,(F)^i$:

$$\pi\colon \mathbf{cocyl}\,(F)^i\,(p) \to \lim_{\langle Q\rangle} \mathbf{cocyl}\,(F)^i\,.$$

Indeed, this projection is a cochain complex morphism. Therefore, it can be restricted to the kernel of the differential. We denote by π_0 this restriction:

$$\pi_0\colon\ \ker \mathbf{cocyl}\,(F)^i\,(p) \to \lim_{\langle Q\rangle} \ker \mathbf{cocyl}\,(F)^i\,.$$

If $\mathcal{P}$ is a dual CL-shellable poset, and $h_0 \ll h_1 \ll \cdots \ll h_n$ is a recursive coatom ordering for $\mathcal{P}$, we denote by $\mathcal{P}_k$ to be the subposet of $\mathcal{P}$ generated by the first $k-1$ coatoms, i.e.,

$$\mathcal{P}_k = \langle h_0, \ldots h_{k-1}\rangle.$$

Lemma 8.1.3. *Let $\mathcal{P}$ be a dual CL-shellable poset, and $\ll$ be a recursive coatom ordering for $\mathcal{P}$. Let $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a sheaf and $i > 0$. If, for every coatom h, the morphism:*

$$\pi_0\colon\ \ker \mathbf{cocyl}\,(F)^{i-1}\,(h) \to \lim_{\langle C(h)\rangle} \ker \mathbf{cocyl}\,(F)^{i-1}$$

is an epimorphism, then $H^i(\mathcal{P}\backslash\{\hat{1}\}; F) = 0$.

Proof. We assume without loss of generality that $\mathrm{d}(\mathcal{P}) > 2$; otherwise, there is nothing to prove. Let $\mathcal{P}$ be a dual CL-shellable poset of length $n > 2$ with recursive coatom ordering $\ll$, and let $h_0 \ll h_1 \cdots \ll h_m$ be the coatom of $\mathcal{P}$ ordered by $\ll$. By Proposition 5.1.12, $H^i(\mathcal{P}; F) = H^i(\lim \mathbf{cocyl}\,(F))$. Then, $H^i(\mathcal{P}; F) = 0$ if and only if the sequence

$$\lim_{\mathcal{P}\backslash\{\hat{1}\}} \mathbf{cocyl}\,(F)^{i-1} \xrightarrow{\partial} \lim_{\mathcal{P}\backslash\{\hat{1}\}} \mathbf{cocyl}\,(F)^{i} \xrightarrow{\partial} \lim_{\mathcal{P}\backslash\{\hat{1}\}} \mathbf{cocyl}\,(F)^{i+1}$$

is exact. Let $\alpha = (\alpha_j) \in \lim \mathbf{cocyl}(F)^i \leqslant \prod_{j=0}^m \mathbf{cocyl}(F)^i(h_j)$ such that $\partial\alpha = 0$. We construct, by induction on the ordering $h_1, h_2, \ldots, h_m$, an element β in the limit $\lim \mathbf{cocyl}(F)^{i-1}$ such that $\partial\beta = \alpha$.

Since $\alpha \in \lim \mathbf{cocyl}(F)^i$ with $\partial\alpha = 0$, the projection on $\mathbf{cocyl}(F)(h_0)$ is a cocycle i.e., $\partial\alpha_0 = 0$. By Definition 5.2.9, $H^i(\mathbf{cocyl}(F)(h_0)) = 0$ because $i > 0$. Then, there exists $\beta_0 \in \mathbf{cocyl}(F)^{i-1}(C_0)$ such that $\partial\beta_0 = \alpha_0$. Now, assuming that, there exists a tuple $(\beta_0, \ldots, \beta_{k-1}) \in \lim_{\mathcal{P}_k} \mathbf{cocyl}(F)^{i-1}$ such that, for every $j = 0, \ldots, k-1$, $\partial\beta_j = \alpha_j$, we construct $\beta_k \in \mathbf{cocyl}(F)^{i-1}(h_k)$ with the analogous conditions, this is, $\partial\beta_k = \alpha_k$ and $(\beta_0, \ldots, \beta_k) \in \lim_{\mathcal{P}_{k+1}} \mathbf{cocyl}(F)^{i-1}$ as follows:

As in the case of h_0, since $\partial\alpha_k = 0$, we choose any $\tilde{\beta} \in \mathbf{cocyl}(F)^{i-1}(h_k)$ such that $\partial\tilde{\beta} = \alpha_k$. If $(\beta_0, \ldots, \beta_{k-1}, \tilde{\beta})$ defines an element in the limit $\lim_{\mathcal{P}_{k+1}} \mathbf{cocyl}(F)^{i-1}$, we are done. Otherwise, consider $\gamma = (\gamma_r) \in \prod_{r \in C(h_k)} \mathbf{cocyl}(F)^{i-1}(r)$, described by:

$$\gamma_r = \mathbf{cocyl}(F)(r < h_j)(\beta_j) - \mathbf{cocyl}(F)(r < h_k)(\tilde{\beta}) \in \mathbf{cocyl}(F)^{i-1}(r).$$

Note that given $r \prec h_k$, it is possible having more than one h with the property that $h_j \ll h_k$ and $r \prec h_j, h_k$. But, since $(\beta_1, \ldots, \beta_{k-1})$ defines an element in $\lim_{\mathcal{P}_k} \mathbf{cocyl}(F)^{i-1}$, if $h_j, h_l \ll h_k$ have the property that $r \prec h_j, h_l$, then

$$\mathbf{cocyl}(F)(r < h_j)(\beta_j) = \mathbf{cocyl}(F)(r < h_l)(\beta_l).$$

We claim that $\gamma \in \lim_{\langle C(h_k)\rangle} \ker \mathbf{cocyl}(F)^{i-1}$. First, we prove that γ belongs to the limit $\lim_{\langle C(h_k)\rangle} \mathbf{cocyl}(F)^{i-1}$. But this is true because γ can be written as a sum of the image through the projection $\mathbf{cocyl}(F)^{i-1} \to \lim_{\langle C(h_k)\rangle} \mathbf{cocyl}(F)^{i-1}$ of $-\tilde{\beta}$, and the image through the restriction $\lim_{\mathcal{P}_k} \mathbf{cocyl}(F)^{i-1} \to \lim_{\langle C(h_k)\rangle} \mathbf{cocyl}(F)^{i-1}$ of $(\beta_0, \ldots, \beta_{k-1})$. Then we only need to check that $\partial\gamma = 0$. Notice that $\mathbf{cocyl}(F)$ is a cochain complex-valued functor; in particular, for every $p < q$, the morphism $\mathbf{cocyl}(F)(p < q)$ commutes with the differential. Therefore, for every $j = 0, \ldots, k-1$:

$$\begin{aligned}
\partial\gamma_r &= \partial(\mathbf{cocyl}(F)(r < h_j)(\beta_j) - \mathbf{cocyl}(F)(r < h_k)(\tilde{\beta})) \\
&= \partial(\mathbf{cocyl}(F)(r < h_j)(\beta_j)) - \partial(\mathbf{cocyl}(F)(r < h_k)(\tilde{\beta})) \\
&= \mathbf{cocyl}(F)(r < h_j)(\partial\beta_j) - \mathbf{cocyl}(F)(r < h_k)(\partial\tilde{\beta}) \\
&= \mathbf{cocyl}(F)(r < h_j)(\alpha_j) - \mathbf{cocyl}(F)(r < h_k)(\alpha_k) \\
&= 0.
\end{aligned}$$

By hypothesis, $\pi_0\colon \ker \mathbf{cocyl}(F)^{i-1}(h_k) \to \lim_{\langle C(h_k)\rangle} \ker \mathbf{cocyl}(F)^{i-1}$ is an epimorphism, therefore there exists an element $\omega \in \ker \mathbf{cocyl}(F)^{i-1}(h_k)$ such that $\pi_0(\omega) = \gamma$. Then, we define $\beta_k = \tilde{\beta} + \omega$.

Now, we have to check that β_k verifies the conditions described at the beginning. By definition, $\partial\beta_k = \partial\tilde{\beta} - \partial\omega = \partial\tilde{\beta} = \alpha_k$. To check that the tuple $(\beta_1, \beta_2, \dots, \beta_k)$ defines a compatible tuple, we proceed as follows. By Lemma 1.1.16, $\mathcal{P}_{k+1} \cup \{\hat{1}\}$ is dual CL-shellable of the same degree, and, by Lemma 1.1.14, the full subcategory whose objects are the ones of degree $n-1$ and $n-2$ is final in $\mathcal{P}_{k+1}$. We are thus left to prove that, for every $j = 0, \dots, k-1$,

$$\mathbf{cocyl}(F)(r < h_j)(\beta_j) = \mathbf{cocyl}(F)(r < h_k)(\beta_k).$$

But this is clear by the definition of β_k.

Finally, repeating this process for every coatom of $\mathcal{P}$, we obtain the desired $\beta \in \lim_{\mathcal{P}\setminus\{\hat{1}\}} \mathbf{cocyl}(F)^{i-1}$ such that $\partial\beta = \alpha$. $\square$

Finally, we are able to prove the main theorem.

Theorem 8.2. *Let $\mathcal{P}$ be a dual CL-shellable poset of length $n \geqslant 2$, and let $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a sheaf. Given $1 \leqslant i \leqslant n-1$, if the pair $(\mathcal{P}, F)$ has the stability property in codegree i, then*

$$H^i(\mathcal{P}\setminus\{\hat{1}\}; F) = 0.$$

Proof. We proceed by induction on the codegree of the stability property of the pair.

Let $F\colon \mathcal{P}^{\mathrm{op}} \to \mathrm{Mod}_R$ be a sheaf over a dual CL-shellable poset with coatom ordering $\ll$. Assume that the pair $(\mathcal{P}, F)$ has the stability property in codegree 1. In particular, for every coatom h, the natural map $F(h) \to \lim_{\langle C(h)\rangle} F$ is an epimorphism because $C(h)$ is compatible with the recursive coatom ordering. By Lemma 8.1.3, it is enough to check that, for every coatom h, the morphism

$$\pi_0\colon \ker \mathbf{cocyl}(F)^0(h) \to \lim_{\langle C(h)\rangle} \ker \mathbf{cocyl}(F)^0$$

is an epimorphism.

If $C(h) = \varnothing$ there is nothing to prove; otherwise, as there is a natural isomorphism between $\ker \mathbf{cocyl}(F)^0$ and F, we have the following commutative diagram:

$$\begin{tikzcd}
\ker \mathbf{cocyl}(F)^0(h) \arrow[r] \arrow[d, "\cong"] & \lim_{\langle C(h)\rangle} \ker \mathbf{cocyl}(F)^0 \arrow[d, "\cong"] \\
F(h) \arrow[r] & \lim_{\langle C(h)\rangle} F.
\end{tikzcd}$$

By hypothesis, the natural morphism $F(h) \to \lim_{\langle C(h)\rangle} F$ is an epimorphism. So we conclude that $\ker \mathbf{cocyl}(F)^0(h) \to \lim_{\langle C(h)\rangle} \ker \mathbf{cocyl}(F)^0$ is an epimorphism too.

Now, assume that, for every $i < n$, and every pair $(\mathcal{P}', F')$ with the stability property in codegree i, we have $H^i(\mathcal{P}' \backslash \{\hat{1}\}; F') = 0$. Consider a pair $(\mathcal{P}, F)$ with the stability property in codegree n, and $\ll$ a recursive coatom ordering for $\mathcal{P}$. Again, by applying Lemma 8.1.3, we have to check that, for every coatom h the morphism

$$\pi_0 \colon \ker \mathbf{cocyl}(F)^{n-1}(h) \to \lim_{\langle C(h)\rangle} \ker \mathbf{cocyl}(F)^{n-1}$$

is an epimorphism. We assume without loss of generality that $C(h) \neq \varnothing$. Notice that the following diagram is commutative:

$$\begin{tikzcd}
\ker \mathbf{cocyl}(F)^{n-1}(h) \arrow[rr, "\pi_0"] & & \lim_{\langle C(h)\rangle} \ker \mathbf{cocyl}(F)^{n-1} \\
\mathbf{cocyl}(F)^{n-2}(h) \arrow[r] \arrow[u] & M_h \mathbf{cocyl}(F)^{n-2} \arrow[r] & \lim_{\langle C(h)\rangle} \mathbf{cocyl}(F)^{n-2}, \arrow[u]
\end{tikzcd}$$

where the vertical morphisms are the respective differentials and the bottom horizontal morphism is the composition of the projection given by the mapping cocylinder $\mathbf{cocyl}(F)^{n-2}(h) \to M_h \mathbf{cocyl}(F)^{n-2}$ followed by the restriction morphism $M_h \mathbf{cocyl}(F)^{n-2} \to \lim_{\langle C(h)\rangle} \mathbf{cocyl}(F)^{n-2}$. We prove that π_0 is an epimorphism by proving that every other morphism in the diagram is onto.

By exactness of $\mathbf{cocyl}(F)(h)$, the image of its differential at height $n-2$, $\mathbf{cocyl}(F)^{n-2}(h) \to \mathbf{cocyl}(F)^{n-1}(h)$ is just $\ker \mathbf{cocyl}(F)^{n-1}(h)$, thus the first vertical map is an epimorphism. Since $\mathbf{cocyl}(F)$ is a fibrant functor, the morphism $\mathbf{cocyl}(F)^{n-2}(h) \to M_h \mathbf{cocyl}(F)^{n-2}$ is an epimorphism and, by Lemma 8.1.2, the restriction map $M_h \mathbf{cocyl}(F)^{n-2} \to \lim_{\langle C(h)\rangle} \mathbf{cocyl}(F)^{n-2}$ is also an epimorphism.

Finally, notice that $C(h)$ is compatible with the recursive coatom ordering. Therefore, by Lemma 1.1.16, the poset $\langle C(h)\rangle \cup \{h\}$ is a dual CL-shellable poset of length $\mathrm{d}(h) = \mathrm{d}(\mathcal{P}) - 1 = d - 1$, and the pair $(\langle C(h) \rangle \cup \{h\}, F\,|_{C(h)\cup\{h\}})$ has the stability property at codegree $n-1$; this is clear since

$$\mathrm{d}(\mathcal{P}) - n = \mathrm{d}(\langle C(h) \rangle) \cup \{h\}) + 1 - n = \mathrm{d}(h) + 1 - n = \mathrm{d}(h) - (n-1).$$

Then, by hypothesis induction $H^{n-1}(\langle C(h) \rangle; F) = 0$. So, we conclude that the differential

$$\lim_{\langle C(h) \rangle} \mathbf{cocyl}\,(F)^{n-2} \to \lim_{\langle C(h) \rangle} \ker \mathbf{cocyl}\,(F)^{n-1}$$

is an epimorphism. □

8.2 COHOMOLOGY OF THE i-LINEAR FORMS SHEAF

Let V be a finite-dimensional k-vector space, $\mathcal{H}$ be a finite set of hyperplanes of V, and $L_{\mathcal{H}}$ the arrangement lattice of $\mathcal{H}$. For every $i \geqslant 1$, we define the *i-linear forms sheaf* on $L_{\mathcal{H}}$ to be the sheaf $\Lambda^i(-)^* \colon L_{\mathcal{H}}^{\mathrm{op}} \to \mathrm{Vect}_k$ that sends every $W \in L$ to the i-linear forms of W, i.e.,

$$\Lambda^i \operatorname{Hom}(W, k) = \Lambda^i W^*,$$

and $W' < W$ to the restriction $\Lambda^i W^* \twoheadrightarrow \Lambda^i (W')^*$.

Theorem 8.2.1 ([ET22b]). *Let V be a finite-dimensional vector space, $\mathcal{H}$ be a finite set of hyperplanes of V, and $L_{\mathcal{H}}$ the arrangement lattice of $\mathcal{H}$. For every $j < \mathrm{d}(L_{\mathcal{H}}) - i - 2$, the j-th cohomology of the i-linear forms sheaf on $L_{\mathcal{H}} \backslash \{\hat{1}\}$ vanishes, this is:*

$$H^j(L_{\mathcal{H}} \backslash \{\hat{1}\}; \Lambda^i(-)^*) = 0.$$

Recently this result has been proven by Everitt-Turner [ET22b] using a deletion-restriction method [ET22a]. Here, we prove it by showing that i-linear form sheaf verifies the stability property on the required codegree. In order to prove it, we will need the following lemmas.

Lemma 8.2.2. *Let V be a vector space of dimension n, $\mathcal{H}$ be a finite set of hyperplanes, and $Q \subset \mathcal{H}$ be a set of independent hyperplanes of V. Let $L_{\mathcal{H}}$ be the arrangement lattice of $\mathcal{H}$. Then, for every $i \leqslant n-1$, the natural morphism induced by the restriction:*

$$\Lambda^i V^* \to \lim_{\langle Q \rangle} \Lambda^i(-)^*$$

is an epimorphism. Moreover, if $|Q| \geqslant i+1$, it is an isomorphism.

Proof. Let $h_1, \dots, h_r$ be the hyperplanes in Q. By the independence of Q and using an argument similar to the proof of Lemma 1.1.19, there exists a basis $B = \{e_1, \dots, e_n\}$ of V^* such that the set

$$B_j = \left\{ (e_1|_{h_j}), (e_2|_{h_j}), \dots, \widehat{(e_j|_{h_j})}, \dots, (e_n|_{h_j}) \right\}$$

is a basis of h_j^*. By abuse of notation, we denote just by e_s the 1-form restricted to h, $e_s|_h$, for every $h \in Q$.

Now, for $i \leqslant \dim(h_j) = n-1$, a basis of $\Lambda^i h_j^*$ is given by:

$$\Lambda^i B_j := \left\{ e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_i} \mid 1 \leqslant k_1 < k_2 < \cdots < k_i \leqslant n,\ j \neq k_1, \dots, k_i \right\}.$$

By Lemma 1.1.14, an element $(x_j) \in \lim_{\langle Q \rangle} \Lambda^i(-)^*$ is an element of the product $\prod_{j=1}^r \Lambda^i h_j^*$ with the property that, for every $j \neq j'$ and every tuple $(k_1, k_2, \dots, k_i)$ with $1 \leqslant k_1 < k_2 < \cdots < k_i \leqslant n$ and $k_1, k_2, \dots, k_i \notin \{j, j'\}$, the $(k_1, \dots, k_i)$-coordinate of x_j and $x_{j'}$ coincides:

$$(x_j)_{(k_1, k_2, \dots, k_i)} = (x_{j'})_{(k_1, k_2, \dots, k_i)}.$$

Therefore, a basis of $\lim_{\langle Q \rangle} \Lambda^i(-)^*$ is the union of the basis $\Lambda^i B_j$, and this is a subset of the basis of $\Lambda^i V^*$ induced by the basis B,

$$\Lambda^i B := \left\{ e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_i} \mid 1 \leqslant k_1 < k_2 < \cdots < k_i \leqslant n \right\}.$$

Moreover, if $r \geqslant i+1$ for every tuple $(k_1, \dots, k_i)$ such that $1 \leqslant k_1 < \cdots < k_i \leqslant n$, the element $e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_{r-1}} \in \Lambda^i B$ is in the basis $\Lambda^i B_j$ of $\Lambda^i h_j^*$, where j is any element of the set $\{1, \dots, r\} \backslash \{k_1, k_2, \dots k_i\}$. □

As a direct corollary, we obtain the following result.

Corollary 8.2.3. *Let V be a vector space of dimension n, $\mathcal{H}$ be a finite set of hyperplanes, and L the arrangement lattice of $\mathcal{H}$. Let $Q \subset \mathcal{H}$ be a basis of L. Then, if $i < \mathrm{d}(V)$, the natural morphism induced by the restriction:*

$$\Lambda^i V^* \to \lim_{\langle Q \rangle} \Lambda^i(-)^*$$

is an isomorphism

Proof. Notice that $i < \mathrm{d}(V) = \dim V - \dim \hat{0} \leqslant \dim V$. Moreover, Q is basis, in particular, an independent set of coatoms and, by Corollary 1.1.20, $|Q| = \mathrm{d}(V) > i$. We are done after applying Lemma 8.2.2. ☐

Lemma 8.2.4. *Let V be a finite-dimensional vector space and $\mathcal{H}$ be a finite set of hyperplanes in V. Let L be the arrangement lattice of $\mathcal{H}$, and $\ll$ be a basis-like recursive coatom ordering for L. Suppose $W \in L$, and Q is a subset of coatoms of $L_{\leqslant W}$ compatible with the recursive coatom ordering. Then, the natural morphism induced by the restriction map,*

$$\varepsilon \colon \Lambda^i W^* \to \lim_{\langle Q \rangle} \Lambda^i(-)^*,$$

is an epimorphism if $\dim W > \dim \hat{0} + i + 2$.

Proof. Let c be an irreducible chain from W to $\hat{1}$ that makes Q compatible with the recursive coatom ordering, and B be the basis of the geometric lattice $L_{\leqslant W}$ given by the first coatoms in $(L_{\prec W}, \ll_c)$. By the compatibility of Q with the recursive coatom ordering, then, either $Q \subseteq B$ or $B \subsetneq Q$.

In the first case, $Q \subseteq B$ is an independent set of coatoms of the geometric lattice $L_{\leqslant W}$ with $i < i + 2 + \dim \hat{0} < \dim W$. Then, according to Lemma 8.2.2, the natural morphism

$$\varepsilon \colon \Lambda^i W^* \longrightarrow \lim_{\langle Q \rangle} \Lambda^i(-)^*$$

is an epimorphism. Notice that it is an isomorphism if $Q = B$; see Corollary 8.2.3.

In the second case, where $B \subsetneq Q$, we prove that the natural morphism

$$\varepsilon \colon \Lambda^i W^* \longrightarrow \lim_{\omega \in \langle Q \rangle} \Lambda^i \omega^*$$

is an isomorphism. The composite

$$\Lambda^i W^* \xrightarrow{\varepsilon} \lim_{\omega \in \langle Q \rangle} \Lambda^i \omega^* \xrightarrow{\mathrm{Res}^Q_B} \lim_{\omega \in \langle B \rangle} \Lambda^i \omega^*,$$

is the morphism induced by the restriction. This is the extremal case of $Q = B$, and we have shown that it is an isomorphism. Thus, to show that ε is an isomorphism, it is enough to prove that the restriction morphism:

$$\mathrm{Res}_B^Q\colon \lim_{\langle Q\rangle} \Lambda^i(-)^* \longrightarrow \lim_{\langle B\rangle} \Lambda^i(-)^*$$

is also an isomorphism. Let $\omega_1, \omega_2, \ldots, \omega_{|Q|}$ be the elements of Q ordered by $\ll_c$.

We begin by proving the injectivity of the restriction morphism. Suppose, by contradiction, that there exists $\alpha = (\alpha_k) \in \lim_{\langle Q\rangle} \Lambda^i(-)^* \leqslant \prod_{k=1}^{|Q|} \Lambda^i \omega_k^*$ such that $\alpha \neq 0$ and $\mathrm{Res}_{\langle B\rangle}^{\langle Q\rangle}(\alpha) = 0$. Therefore, there exists $\omega_j \in Q \backslash B$ such that $\alpha_j \neq 0$.

By Lemma 1.1.21, there exists $B_0 \subset B$ such that $B_0 \cup \{\omega_j\}$ is a basis of $L_{\leqslant W}$, and $B' = \{\omega_j \cap \omega_k \mid \omega_k \in B_0\} \subset C(\omega_j)$ is a basis. By functoriality, for every $\omega_k \cap \omega_j \in B'$,

$$\alpha_j|_{\omega_k \cap \omega_j} = \alpha_k|_{\omega_k \cap \omega_j},$$

and, by hypothesis, $\alpha_k = 0$, so $\alpha_j|_{\omega_k \cap \omega_j} = 0$. The set B' is a basis of the geometric lattice $L_{\leqslant \omega_j}$, and $\mathrm{d}(\omega_j) = \mathrm{d}(W) - 1 > i + 2 - 1 = i - 1$. Then, by Corollary 8.2.3, the natural morphism induced by the restriction

$$\varphi\colon \Lambda^i \omega_j^* \to \lim_{\langle B'\rangle} \Lambda^i(-)^*$$

is an isomorphism. but, $\varphi(\alpha_j) = (\alpha_j|_{\omega_j \cap \omega_k}) \in \prod_{\omega_j \cap \omega_k \in B'} \Lambda^i(\omega_j \cap \omega_k)^*$ which is 0; this contradicts that $\alpha_j \neq 0$.

Next, we show that the morphism is onto. Let $R_j = \{\omega_0, \omega_1, \ldots, \omega_j\}$ be the set of the first j coatoms of $L_{\prec W}$. For every $j = \mathrm{d}(W), \mathrm{d}(W) + 1, \ldots, |Q| - 1$, we prove that the restriction morphism,

$$\mathrm{Res}_{R_j}^{R_{j+1}}\colon \lim_{\langle R_{j+1}\rangle} \Lambda^i(-)^* \longrightarrow \lim_{\langle R_j\rangle} \Lambda^i(-)^*,$$

is an epimorphism.

Let α be any element of the limit $\lim_{\langle R_j\rangle} \Lambda^i(-)^*$ of coordinates $(\alpha_s) \in \prod_{s=1}^{j} \Lambda^i \omega_s^*$, and let $D \subset C(\omega_{j+1})$ a basis of the geometric lattice $L_{\leqslant \omega_{j+1}}$. As before, we apply Corollary 8.2.3, because D is a basis of the geometric lattice $L_{\leqslant \omega_{j+1}}$ and $\mathrm{d}(\omega_{j+1}) > i + 1$, obtaining that the natural map induced by the restriction

$$\varphi\colon \Lambda^i \omega_{j+1}^* \to \lim_{\langle D\rangle} \Lambda^i(-)^*$$

is an isomorphism. Then, We claim that the element $\alpha_{j+1} \in \Lambda^i \omega_{j+1}^*$ defined by

$$\alpha_{j+1} = \varphi^{-1}(\mathrm{Res}_D^{R_j}(\alpha))$$

verifies that the tuple $(\alpha_1, \alpha_2, \dots, \alpha_j, \alpha_{j+1}) \in \prod_{s=1}^{j+1} \Lambda^i \omega_s^*$ belongs to $\lim_{\langle R_{j+1}\rangle} \Lambda^i(-)^*$.

By Lemma 1.1.14, it is enough to prove, for every $s < j+1$, that the following identity holds:

$$\alpha_s|_{\omega_s \cap \omega_{j+1}} = \alpha_{j+1}|_{\omega_s \cap \omega_{j+1}}. \tag{19}$$

If $\omega_s \cap \omega_{j+1} \in D$, then the identity holds by definition.

Otherwise, by Lemma 1.1.21, there exists $D_0 \subsetneq D$ such that $D_0 \cup \{\omega_s \cap \omega_{j+1}\}$ is a basis of $L_{\leqslant \omega_{j+1}}$, the set

$$D' = \{\omega_k \cap \omega_s \cap \omega_{j+1} \mid \omega_k \in D_0\}$$

is a basis of $L_{\leqslant(\omega_s \cap \omega_{j+1})}$.

As α is an element of the limit $\lim_{R_j} \Lambda^i(-)^*$, for every $\omega_s \cap \omega_{j+1} \cap \omega_k \in D'$, it is verified

$$\alpha_s|_{\omega_s \cap \omega_{j+1} \cap \omega_k} = \alpha_k|_{\omega_s \cap \omega_{j+1} \cap \omega_k}.$$

Moreover, by definition of α_{j+1}, it coincides with $\alpha_{j+1}|_{\omega_s \cap \omega_{j+1} \cap \omega_k}$.

Next, the morphism induced by the restriction

$$\varphi' \colon \Lambda^i(\omega_s \cap \omega_{j+1})^* \to \lim_{\langle D'\rangle} \Lambda^i(-)^*$$

is an isomorphism because $\mathrm{d}(\omega_s \cap \omega_{j+1}) = \mathrm{d}(W) - 2 > i + 2 - 2 > i$, and D' is basis of $L_{\leqslant \omega_s \cap \omega_{j+1}}$, see Corollary 8.2.3. In particular, it is injective and,

$$\varphi'(\alpha_{j+1}|_{\omega_s \cap \omega_{j+1}}) = (\alpha_k|_{\omega_s \cap \omega_{j+1} \cap \omega_k}) \in \prod_{\omega_s \cap \omega_{j+1} \cap \omega_k \in D'} \Lambda^i(\omega_s \cap \omega_{j+1} \cap \omega_k)^*.$$

This implies Equation (19) holds. □

Proof of Theorem 8.2.1. Let $L = L_{\mathcal{H}}$ and $\ll$ be a basis-like recursive coatom ordering for L. By Theorem 8.2, it is enough to check that the pair $(L, \Lambda^i(-)^*)$ has the stability property at codegree $j < \mathrm{d}(L) - i - 2$.

So, let $W \in L$ of degree $\mathrm{d}(W) = \mathrm{d}(L) - j > i + 2$ and Q be a subset of coatoms of $L_{\leqslant W}$ compatible with the recursive coatom ordering. Then, by Lemma 8.2.4, the natural map induced by the restriction map

$$\Lambda^i W^* \to \lim_{\langle Q \rangle} \Lambda^i(-)^*$$

is an epimorphism, and this concludes the proof. $\square$

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