# Problem Set 1
# ECE 685D Fall 2020

**Instructor: Vahid Tarokh**
ECE Department, Duke University

Due: 8:59:59 a.m. EST on Sept. 8, 2020

**Important: You are only allowed to use the Python built in function for generating uniform random variables.**

## Problem 1: Exponential distribution

Let $X$ denote an exponentially distributed random variable with parameter $\lambda > 0$, which we denote by $X \sim \text{Exp}(\lambda)$. Recall that the probability density function (PDF) and the cumulative distribution function (CDF) of $X$ are respectively given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad \text{and} \quad F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Let $U$ be a uniformly distributed random variable on $(0, 1)$, i.e., $U \sim \text{Unif}(0, 1)$. Then, it can be shown that the random variable

$$F_X^{-1}(U) = -\frac{1}{\lambda}\ln(1 - U) \sim \text{Exp}(\lambda).$$

Write a Python program for generating exponentially distributed random variable with parameter $\lambda$. Plot the histogram of $10^5$ random samples with bin width .01 for three different values of $\lambda \in \{.1, 1, 10\}$.

## Problem 2: Gamma distribution

Let $X_1 \sim \text{Exp}(\lambda)$ and $X_2 \sim \text{Exp}(\lambda)$. Define a new random variable $Y = X_1 + X_2$. The PDF of $Y$ can be computed via the following convolution. For $y \geq 0$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X_1}(t) f_{X_2}(y - t) dt \\ &= \int_0^y \lambda^2 e^{-\lambda t} e^{-\lambda(y-t)} dt \\ &= \lambda^2 e^{-\lambda y} \int_0^y dt \\ &= y\lambda^2 e^{-\lambda y}. \end{aligned}$$

Observe that $Y$ is Gamma distributed random variable with parameters 2 and $1/\lambda$, i.e., $Y \sim \text{Gam}(2, 1/\lambda)$.

Now, let $X_3 \sim \text{Exp}(\lambda)$. The PDF of the random variable defined as $Y = X_1 + X_2 + X_3$ can be computed similarly as above. For $y \geq 0$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X_1+X_2}(t) f_{X_3}(y-t)dt \\ &= \int_0^y \lambda^3 t e^{-\lambda t} e^{-\lambda(y-t)} dt \\ &= \lambda^3 e^{-\lambda y} \int_0^y t dt \\ &= \frac{y^2 \lambda^3}{2} e^{-\lambda y}, \end{aligned}$$

which is the PDF of $\text{Gam}(3, 1/\lambda)$. In the general case, it can be shown that the PDF of the random variable $Y = \sum_{i=1}^K X_i$ where $X_i \sim \text{Exp}(\lambda)$ for $i = 1, \dots, K$ is given by

$$f_Y(y) = \begin{cases} \frac{y^{K-1}\lambda^K}{\Gamma(K)} e^{-\lambda y} & y \geq 0 \\ 0 & y < 0 \end{cases}, \quad \Gamma(K) = (K-1)!,$$

where $K$ is a positive integer. Write a Python program for generating Gamma distributed random variable following $\text{Gam}(K, \beta)$. Plot the histogram of $10^5$ random samples with bin width .01 for $K = 5$ and $\beta \in \{.1, 1, 10\}$.

## Problem 3: Beta distribution

Let $\mu$ be a Beta distributed random variable with parameters $\alpha_1 > 0$ and $\alpha_2 > 0$, i.e., $\mu \sim \text{Beta}(\alpha_1, \alpha_2)$. The PDF of $\mu$ is given by

$$f_\mu(\mu) = \begin{cases} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \mu^{\alpha_1-1}(1-\mu)^{\alpha_2-1} & \mu \in (0,1) \\ 0 & \text{elsewhere,} \end{cases} \quad \text{with} \quad \Gamma(t) = \int_0^\infty t^{u-1} e^{-u} du,\ t > 0.$$

Recall from the lecture notes that the PDF of the $k$-th order statistic $U_{(k)}$ of $n$ i.i.d. uniform random variables $U_1, \dots, U_n \sim \text{Unif}(0,1)$ follows $\text{Beta}(k, n-k+1)$. Write a Python program for generating Beta distributed random variable with positive integer parameters $\alpha_1$ and $\alpha_2$. Plot the histogram of $10^5$ random samples with bin width .01 for $(\alpha_1, \alpha_2) = (5, 16)$ and $(\alpha_1, \alpha_2) = (10, 11)$.

## Problem 4: Dirichlet distribution

Recall from the lecture notes that the PDF of a Dirichlet random vector $\boldsymbol{\mu}$ distributed over the $N$-dimensional simplex with parameters $\alpha_i > 0$ for $i = 1, \dots, N$ with $N \geq 2$ is given by

$$f_{\boldsymbol{\mu}}(\boldsymbol{\mu}) = \begin{cases} \frac{\Gamma\left(\sum_{i=1}^N \alpha_i\right)}{\prod_{i=1}^N \Gamma(\alpha_i)} \prod_{i=1}^N \mu_i^{\alpha_i - 1} & \mu_i \in (0,1) \text{ and } \sum_{i=1}^N \mu_i = 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let $Y_i \sim \text{Gam}(\alpha_i, \beta)$ for $i = 1, \dots, N$. It can be shown that the random vector

$$\left( \frac{Y_1}{\sum_{i=1}^N Y_i}, \dots, \frac{Y_N}{\sum_{i=1}^N Y_i} \right) \sim \text{Dir}(\alpha_1, \dots, \alpha_N).$$

Write a Python program for generating Dirichlet distributed random vector with positive integer parameters $\alpha_i$ for $i = 1, \dots, N$. Plot the 2-dimensional histogram of $10^5$ random samples with bin width .01 for $N = 3$ and $(\alpha_1, \alpha_2, \alpha_3) = (10, 10, 10)$.

The Beta distribution is a special case of the Dirichlet distribution for $N = 2$. Write a Python program for generating Dirichlet distributed random vector using the stick method explained in the lecture notes. Plot the 2-dimensional histogram of $10^5$ random samples with bin width .01 for $N = 3$ and $(\alpha_1, \alpha_2, \alpha_3) = (10, 10, 10)$.