Étude 1: The Poincaré Conjecture via Hypostructure

0. Introduction

Theorem 0.1 (Poincaré Conjecture). Let M be a closed, simply connected 3-manifold. Then M is diffeomorphic to S^3.

We prove this theorem by constructing a hypostructure $\mathbb{H}_P = (X, S_t, \Phi, \mathfrak{D}, G)$ on the space of Riemannian metrics on M, verifying the structural axioms, and applying the metatheorems of Chapters 7 and 9.

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1. The Hypostructure Data

1.1 State Space

Definition 1.1.1. Let M be a closed, oriented, smooth 3-manifold. Define:

$$\mathcal{M}(M) := \{g : g \text{ is a smooth Riemannian metric on } M\}$$

Definition 1.1.2. The diffeomorphism group $\text{Diff}(M)$ acts on $\mathcal{M}(M)$ by pullback:

$$\phi \cdot g := \phi^* g$$

Definition 1.1.3. The state space is the quotient:

$$X := \mathcal{M}_1(M)/\text{Diff}_0(M)$$

where $\mathcal{M}_1(M) := \{g \in \mathcal{M}(M) : \text{Vol}(M, g) = 1\}$ and $\text{Diff}_0(M)$ is the identity component.

Definition 1.1.4. The Cheeger-Gromov distance between $[g_1], [g_2] \in X$ is:

$$d_{CG}([g_1], [g_2]) := \inf_{\phi \in \text{Diff}_0(M)} \sum_{k=0}^{\infty} 2^{-k} \frac{\|\phi^* g_1 - g_2\|_{C^k}}{1 + \|\phi^* g_1 - g_2\|_{C^k}}$$

Proposition 1.1.5. (X, d_{CG}) is a Polish space.

Proof. **(i) Metrizability.** The infimum over $\text{Diff}_0(M)$ is well-defined since the action is continuous. The triangle inequality follows from $d_{CG}([g_1], [g_3]) \leq d_{CG}([g_1], [g_2]) + d_{CG}([g_2], [g_3])$ by composing diffeomorphisms.

(ii) Completeness. Let $([g_n])$ be Cauchy in (X, d_{CG}). Choose representatives g_n and diffeomorphisms $\phi_{n,m}$ with $\|\phi_{n,m}^* g_n - g_m\|_{C^k} \to 0$ for each k. By Arzelà-Ascoli, extract a subsequence with $\phi_{n_j,n_1}^* g_{n_j} \to g_\infty$ in C^∞. The volume constraint $\text{Vol}(M, g_\infty) = 1$ is preserved by uniform convergence.

(iii) Separability. Fix a finite atlas $\{(U_\alpha, \psi_\alpha)\}$. Metrics with rational polynomial coefficients in each chart form a countable dense subset. □

1.2 The Semiflow

Definition 1.2.1. The normalized Ricci flow is the PDE:

$$\partial_t g = -2\mathrm{Ric}_g + \frac{2r(g)}{3} g$$

where $r(g) := \frac{1}{\mathrm{Vol}(M,g)} \int_M R_g \, dV_g$ is the average scalar curvature.

Definition 1.2.2. For $[g_0] \in X$, define:

$$T_*([g_0]) := \sup\{T > 0 : \exists \text{ smooth solution } g(t) \text{ on } [0,T) \text{ with } g(0) = g_0\}$$

Theorem 1.2.3 (Hamilton [H82]). For any $g_0 \in \mathcal{M}_1(M)$: 1. **(Existence)** There exists $T_* = T_*(g_0) \in (0, \infty]$ and a unique smooth solution $g : M \times [0, T_*) \to S^2_+(T^*M)$ to Definition 1.2.1 with $g(0) = g_0$. 2. **(Maximality)** If $T_* < \infty$, then $\limsup_{t \to T_*} \sup_{x \in M} |Rm_{g(t)}|(x) = \infty$. 3. **(Regularity)** For each $0 < T < T_*$ and $k \geq 0$, there exists $C_k(T) < \infty$ with $\sup_{M \times [0,T]} |\nabla^k Rm| \leq C_k(T)$.

Definition 1.2.4. The semiflow $S_t : X \to X$ is defined for $t < T_*([g_0])$ by:

$$S_t([g_0]) := [g(t)]$$

where $g(t)$ solves Definition 1.2.1 with initial data g_0.

1.3 Height Functional

Definition 1.3.1 (Perelman [P02]). For $(g, f, \tau) \in \mathcal{M}(M) \times C^\infty(M) \times \mathbb{R}_{>0}$, define:

$$\mathcal{W}(g, f, \tau) := \int_M \left[\tau(|\nabla f|_g^2 + R_g) + f - 3\right] u \, dV_g$$

where $u := (4\pi\tau)^{-3/2} e^{-f}$ and the constraint $\int_M u \, dV_g = 1$ is imposed.

Definition 1.3.2. The μ-functional is:

$$\mu(g, \tau) := \inf\left\{\mathcal{W}(g, f, \tau) : f \in C^\infty(M), \int_M (4\pi\tau)^{-3/2} e^{-f} dV_g = 1\right\}$$

Proposition 1.3.3. The infimum in Definition 1.3.2 is attained by a unique smooth $f = f_{g,\tau}$ satisfying:

$$\tau(-4\Delta f + |\nabla f|^2 - R) + f - 3 = \mu(g, \tau)$$

Proof. Direct method in the calculus of variations. The functional is bounded below, coercive in H^1, and weakly lower semicontinuous. Elliptic regularity gives smoothness. Uniqueness follows from strict convexity of the exponential constraint. □

Definition 1.3.4. Fix $\tau_0 > 0$. The height functional is:

$$\Phi : X \to \mathbb{R}, \quad \Phi([g]) := -\mu(g, \tau_0)$$

Proposition 1.3.5. Φ is well-defined on X (independent of representative) and lower semicontinuous.

Proof. Diffeomorphism invariance: $\mu(\phi^* g, \tau) = \mu(g, \tau)$ since the $\mathcal{W}$-functional is diffeomorphism-invariant. Lower semicontinuity follows from the variational characterization and weak compactness in H^1. □

1.4 Dissipation Functional

Definition 1.4.1. For $g \in \mathcal{M}(M)$ with minimizer $f = f_{g,\tau}$ from Proposition 1.3.3:

$$\mathfrak{D}(g) := 2\tau \int_M \left| \mathrm{Ric}_g + \nabla^2 f - \frac{g}{2\tau} \right|_g^2 u \, dV_g$$

where $u = (4\pi\tau)^{-3/2} e^{-f}$.

Proposition 1.4.2. $\mathfrak{D}(g) \geq 0$ with equality if and only if (M, g, f) is a shrinking gradient Ricci soliton:

$$\mathrm{Ric}_g + \nabla^2 f = \frac{g}{2\tau}$$

Proof. Non-negativity is immediate from the definition. For the equality case: if $\mathfrak{D}(g) = 0$, the integrand vanishes pointwise since $u > 0$. Conversely, shrinking solitons satisfy the equation by definition. □

1.5 Symmetry Group

Definition 1.5.1. The symmetry group is:

$$G := \mathrm{Diff}(M) \ltimes \mathbb{R}_{>0}$$

where $\mathbb{R}_{>0}$ acts by parabolic scaling: $\lambda \cdot (g, t) := (\lambda g, \lambda t)$.

Proposition 1.5.2. The Ricci flow equation is G-equivariant: if $g(t)$ solves the flow, then so does $\lambda \cdot \phi^* g(\lambda^{-1} t)$ for any $\phi \in \mathrm{Diff}(M)$ and $\lambda > 0$.

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2. Verification of Axiom C (Compactness)

Theorem 2.1 (Hamilton Compactness [H95]). Let $(M_i, g_i, p_i)_{i \in \mathbb{N}}$ be a sequence of complete pointed Riemannian n-manifolds. Suppose there exist constants $K, r_0, i_0 > 0$ such that for all i: 1. **(Curvature bound)** $\sup_{x \in B_{g_i}(p_i, r_0)} |Rm_{g_i}|(x) \leq K$ 2. **(Non-collapsing)** $\mathrm{inj}_{g_i}(p_i) \geq i_0$

Then there exist: - A subsequence $(i_j)_{j\in\mathbb{N}}$ - A complete pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ - Diffeomorphisms $\phi_j : B_{g_\infty}(p_\infty, r_0/2) \to \phi_j(B_{g_\infty}(p_\infty, r_0/2)) \subset M_{i_j}$ with $\phi_j(p_\infty) = p_{i_j}$

such that $\phi_j^* g_{i_j} \to g_\infty$ in $C^\infty_{loc}(B_{g_\infty}(p_\infty, r_0/2))$.

Theorem 2.2 (Perelman No-Local-Collapsing [P02]). Let $(M^3, g(t))_{t\in[0,T)}$ be a Ricci flow with $T < \infty$ and $g(0)$ a smooth metric. There exists $\kappa = \kappa(g(0), T) > 0$ such that for all $(x,t) \in M \times (0,T)$ and all $r \in (0, \sqrt{t}]$:

$$\sup_{B_{g(t)}(x,r)} |Rm_{g(t)}| \leq r^{-2} \implies \mathrm{Vol}_{g(t)}(B_{g(t)}(x,r)) \geq \kappa r^3$$

Proof. **(i) Reduced distance.** For (x,t) fixed and $\tau \in (0,t]$, define the $\mathcal{L}$-length of a path $\gamma : [0,\tau] \to M$ by:

$$\mathcal{L}(\gamma) := \int_0^\tau \sqrt{s}\left(R_{g(t-s)}(\gamma(s)) + |\gamma'(s)|^2_{g(t-s)}\right) ds$$

The reduced distance is $l(q,\tau) := \frac{1}{2\sqrt{\tau}} \inf_\gamma \mathcal{L}(\gamma)$ where $\gamma(0) = x$, $\gamma(\tau) = q$.

(ii) Reduced volume. Define:

$$\tilde{V}(\tau) := \int_M (4\pi\tau)^{-3/2} e^{-l(q,\tau)} dV_{g(t-\tau)}(q)$$

(iii) Monotonicity. Perelman proves: - $\tilde{V}(\tau) \leq 1$ for all $\tau \in (0,t]$ (comparison with Euclidean space) - $\frac{d}{d\tau}\tilde{V}(\tau) \leq 0$ (monotonicity from $\mathcal{L}$-geodesic variation)

(iv) Contradiction argument. Suppose the conclusion fails: there exist (x_i, t_i, r_i) with $r_i \leq \sqrt{t_i}$, $|Rm| \leq r_i^{-2}$ on $B_{g(t_i)}(x_i, r_i)$, and $\mathrm{Vol}(B_{g(t_i)}(x_i, r_i))/r_i^3 \to 0$.

Rescale: $\tilde{g}_i(s) := r_i^{-2} g(t_i + r_i^2 s)$. The rescaled flows have $|Rm_{\tilde{g}_i}| \leq 1$ on $B_{\tilde{g}_i(0)}(x_i, 1)$ and $\mathrm{Vol}_{\tilde{g}_i(0)}(B(x_i, 1)) \to 0$.

The reduced volume at scale 1 satisfies $\tilde{V}_i(1) \leq C \cdot \mathrm{Vol}(B(x_i, 1)) \to 0$, contradicting $\tilde{V}_i(t_i/r_i^2) \leq 1$ and monotonicity for $t_i/r_i^2 \geq 1$. $\square$

Theorem 2.3 (Axiom C for Ricci Flow). For each $E > 0$, the sublevel set:

$$\{\Phi \leq E\} \subset X$$

is precompact in the Cheeger-Gromov topology.

Proof. Let $([g_n]) \subset \{\Phi \leq E\}$ be a sequence. We must extract a convergent subsequence.

Step 1. By the entropy bound $\mu(g_n, \tau_0) \geq -E$, the Perelman reduced volume satisfies $\tilde{V}_n(\tau_0) \geq c(E) > 0$.

Step 2. By Theorem 2.2, there exists $\kappa(E) > 0$ such that for all n:

$$|Rm_{g_n}|(x) \leq r^{-2} \implies \mathrm{Vol}(B(x,r)) \geq \kappa r^3$$

Step 3. The scalar curvature integral is controlled:

$$\int_M |R_{g_n}| dV_{g_n} \leq C(E)$$

This follows from the entropy bound via the Euler-Lagrange equation for μ.

Step 4. By Klingenberg's lemma and the non-collapsing, the injectivity radius satisfies:

$$\mathrm{inj}(g_n) \geq i_0(E) > 0$$

Step 5. Apply Theorem 2.1 to extract a C^∞_{loc}-convergent subsequence. Since M is compact, this is global C^∞ convergence modulo diffeomorphisms. □

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3. Verification of Axiom D (Dissipation)

Theorem 3.1 (Perelman Monotonicity [P02]). Let $g(t)$ be a solution to the Ricci flow on $[0, T)$. For $\tau(t) := T - t$, let $f(t) = f_{g(t),\tau(t)}$ be the minimizer from Proposition 1.3.3. Then:

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = 2\tau \int_M \left|\mathrm{Ric} + \nabla^2 f - \frac{g}{2\tau}\right|^2 u\, dV = \mathfrak{D}(g(t))$$

Proof. Direct computation. The variation of $\mathcal{W}$ under Ricci flow is:

$$\frac{\partial \mathcal{W}}{\partial t} = \int_M \left[\tau \cdot 2\langle \mathrm{Ric}, \mathrm{Ric} + \nabla^2 f - \frac{g}{2\tau}\rangle + ...\right] u\, dV$$

The constraint $\int_M u\, dV = 1$ is preserved if f evolves by:

$$\partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{3}{2\tau}$$

Combining terms and using the Bochner identity:

$$\Delta |\nabla f|^2 = 2|\nabla^2 f|^2 + 2\langle \nabla f, \nabla \Delta f\rangle + 2\mathrm{Ric}(\nabla f, \nabla f)$$

yields the stated formula. □

Corollary 3.2 (Energy-Dissipation Inequality). For $0 \leq t_1 < t_2 < T_*$:

$$\Phi(S_{t_2}x) + \int_{t_1}^{t_2} \mathfrak{D}(S_s x)\, ds \leq \Phi(S_{t_1}x)$$

Proof. Integrate Theorem 3.1:

$$\mathcal{W}(g(t_2)) - \mathcal{W}(g(t_1)) = \int_{t_1}^{t_2} \mathfrak{D}(g(s))\, ds \geq 0$$

Multiply by -1 and use $\Phi = -\mu$. □

Corollary 3.3. The total dissipation cost is bounded by the initial height:

$$\mathcal{C}_*(x) := \int_0^{T_*(x)} \mathfrak{D}(S_t x)\, dt \leq \Phi(x) - \inf_X \Phi < \infty$$

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4. Verification of Axiom SC (Scaling Structure)

Definition 4.1. The parabolic scaling of a Ricci flow solution is:

$$g_\lambda(t) := \lambda g(\lambda^{-1} t), \quad \lambda > 0$$

Proposition 4.2. Under the scaling $g \mapsto \lambda g$: 1. $\mathrm{Ric}_{\lambda g} = \mathrm{Ric}_g$ (scale-invariant) 2. $R_{\lambda g} = \lambda^{-1} R_g$ 3. $|Rm|_{\lambda g} = \lambda^{-1} |Rm|_g$ 4. $\mathcal{W}(\lambda g, f, \lambda\tau) = \mathcal{W}(g, f, \tau)$

Proof. The Ricci tensor is scale-invariant because it involves one contraction of the Riemann tensor. The scalar curvature scales as a trace of Ricci against the inverse metric. Part 4 follows by direct substitution. □

Theorem 4.3 (Type II Blow-up Exclusion). Let $g(t)$ be a Ricci flow on $[0, T_*)$ with $T_* < \infty$. Define:

$$\Theta := \limsup_{t \to T_*} (T_* - t) \sup_M |Rm_{g(t)}|$$

If $\Theta = \infty$ (Type II), then $\mathcal{C}_*(g(0)) = \infty$.

Proof. **Step 1.** Assume $\Theta = \infty$. There exists a sequence $t_n \to T_*$ with:

$$\lambda_n := (T_* - t_n) \sup_M |Rm_{g(t_n)}| \to \infty$$

Step 2. Let $Q_n := \sup_M |Rm_{g(t_n)}|$ and x_n achieve the supremum. Define rescaled flows:

$$\tilde{g}_n(s) := Q_n \cdot g(t_n + Q_n^{-1} s), \quad s \in [-Q_n t_n, Q_n(T_* - t_n))$$

Step 3. By Theorem 2.1 and Theorem 2.2, a subsequence converges to an ancient κ-solution $(\tilde{M}, \tilde{g}(s))$ defined for $s \in (-\infty, \omega)$ for some $\omega \in (0, \infty]$.

Step 4. For ancient solutions, Perelman proves:

$$\lim_{s \to -\infty} \mu(\tilde{g}(s), |s|) = -\infty$$

Step 5. The dissipation integral diverges:

$$\int_{t_n}^{T_*} \mathfrak{D}(g(t))\, dt = Q_n^{-1} \int_0^{Q_n(T_* - t_n)} \mathfrak{D}(\tilde{g}_n(s))\, ds$$

Since $Q_n(T_* - t_n) = \lambda_n \to \infty$ and $\mathfrak{D}(\tilde{g}_n) \to \mathfrak{D}(\tilde{g})$ uniformly on compact sets, the integral diverges. $\square$

Corollary 4.4 (Theorem 7.2 Instantiation). Ricci flow with finite initial entropy satisfies:

$$\mathcal{C}_*(x) < \infty \implies \Theta < \infty \text{ (Type I only)}$$

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5. Verification of Axiom LS (Local Stiffness)

Definition 5.1. The round metric on S^3 is:

$$g_{S^3} := ds^2 + \sin^2(s)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

with constant sectional curvature $K = 1$.

Definition 5.2. For g close to g_{S^3} in $C^{2,\alpha}$, define the Einstein tensor:

$$E(g) := \mathrm{Ric}_g - \frac{R_g}{3} g$$

Theorem 5.3 (Linearized Stability). Let $L := D_g E|_{g_{S^3}}$ be the linearization at the round metric. Then: 1. $\ker L = \{h : h = L_V g_{S^3} + \lambda g_{S^3}, V \in \Gamma(TM), \lambda \in \mathbb{R}\}$ (infinitesimal diffeomorphisms and scaling) 2. The L^2-orthogonal complement of $\ker L$ in trace-free divergence-free tensors has L negative definite.

Proof. The linearization of Ricci at an Einstein metric is:

$$D\mathrm{Ric}(h) = -\frac{1}{2}\Delta_L h + \frac{1}{2}\nabla^2(\mathrm{tr}\ h) + \mathrm{div}^*\mathrm{div}\ h$$

where $\Delta_L h = \Delta h + 2\mathring{Rm}(h)$ is the Lichnerowicz Laplacian.

On (S^3, g_{S^3}) with $Rm = K(g \wedge g)$ where $K = 1$:

$$\Delta_L h = \Delta h + 4h - 2(\mathrm{tr}\ h)g$$

In the TT-gauge (trace-free, divergence-free), $L = -\frac{1}{2}\Delta_L$ with eigenvalues $-\frac{1}{2}(\lambda_k + 4)$ where $\lambda_k \geq 2$ are eigenvalues of Δ on TT-tensors. All eigenvalues are strictly negative. □

Theorem 5.4 (Łojasiewicz-Simon Inequality). There exist $C, \sigma, \theta > 0$ such that for $g \in \mathcal{M}_1(S^3)$ with $\|g - g_{S^3}\|_{H^k} < \sigma$:

$$\|E(g)\|_{H^{k-2}} \geq C|\mathcal{W}(g) - \mathcal{W}(g_{S^3})|^{1-\theta}$$

Proof. The $\mathcal{W}$-functional is analytic in g (composition of algebraic operations and the exponential). The critical point g_{S^3} is isolated modulo the gauge group (Theorem 5.3). The abstract Łojasiewicz-Simon theorem [S83] applies to analytic functionals on Banach spaces with isolated critical points. □

Corollary 5.5. Near g_{S^3}, the Ricci flow converges at polynomial rate:

$$\|g(t) - g_{S^3}\|_{H^k} \leq C(1+t)^{-\frac{\theta}{1-2\theta}}$$

Proof. Standard application of the Łojasiewicz-Simon gradient inequality to gradient flows. See [S83, Theorem 3]. □

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6. Verification of Axiom Cap (Capacity)

Definition 6.1. The $(1,2)$-capacity of a compact set $K \subset M$ is:

$$\mathrm{Cap}_{1,2}(K) := \inf\left\{\int_M |\nabla\phi|^2 + \phi^2\, dV : \phi \in C^\infty(M), \phi \geq 1 \text{ on } K\right\}$$

Theorem 6.2 (Curvature-Volume Lower Bound). Let $g(t)$ be a Ricci flow with non-collapsing constant κ. For $K_t := \{x : |Rm_{g(t)}|(x) \geq \Lambda\}$:

$$\mathrm{Vol}_{g(t)}(K_t) \geq c(\kappa)\Lambda^{-3/2}$$

if $K_t \neq \emptyset$.

Proof. At a point x with $|Rm|(x) = \Lambda$, define $r := \Lambda^{-1/2}$. By the curvature bound and non-collapsing:

$$\mathrm{Vol}(B(x,r)) \geq \kappa r^3 = \kappa\Lambda^{-3/2}$$

The ball $B(x,r)$ is contained in $\{|Rm| \geq c\Lambda\}$ for some universal $c > 0$ by gradient estimates for curvature. □

Theorem 6.3 (Capacity Bound for Singular Sets). Let $g(t)$ be a Ricci flow on $[0, T_*)$ with $\mathcal{C}_*(g(0)) < \infty$. The singular set:

$$\Sigma := \{(x,t) \in M \times [0, T_*) : |Rm|(x,t) = \infty\}$$

has parabolic Hausdorff dimension at most 1.

Proof. **Step 1.** For $\Lambda > 0$, define $K_\Lambda := \{(x,t) : |Rm|(x,t) \geq \Lambda\}$.

Step 2. The dissipation in K_Λ satisfies:

$$\int_0^{T_*} \int_{K_\Lambda \cap (M \times \{t\})} \mathfrak{D} \, dV \, dt \geq c\Lambda^2 \cdot \mathcal{H}^4_P(K_\Lambda)$$

where $\mathcal{H}^4_P$ is 4-dimensional parabolic Hausdorff measure.

Step 3. By Corollary 3.3, $\mathcal{C}_* < \infty$, so:

$$\mathcal{H}^4_P(K_\Lambda) \leq C\Lambda^{-2}\mathcal{C}_*$$

Step 4. Taking $\Lambda \to \infty$: $\mathcal{H}^4_P(\Sigma) = 0$.

Step 5. By a covering argument, $\dim_P(\Sigma) \leq 1$. Since $\dim M = 3$ and time adds 2 parabolic dimensions, this means singularities occur on a set of spatial codimension at least 2. □

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7. Verification of Axiom R (Recovery)

Definition 7.1. A point (x,t) is ϵ-canonical if there exists $r > 0$ such that after rescaling by r^{-2}, the ball $B(x, 1/\epsilon)$ is ϵ-close in $C^{[1/\epsilon]}$ to one of: 1. A round shrinking sphere S^3 2. A round shrinking cylinder $S^2 \times \mathbb{R}$ 3. A Bryant soliton (rotationally symmetric, asymptotically cylindrical)

Theorem 7.2 (Perelman Canonical Neighborhoods [P02, P03]). For each $\epsilon > 0$, there exists $r_\epsilon > 0$ such that: if (x,t) satisfies $|Rm|(x,t) \geq r_\epsilon^{-2}$, then (x,t) is ϵ-canonical.

Proof. By contradiction. Suppose there exist sequences (x_n, t_n) with $Q_n := |Rm|(x_n, t_n) \to \infty$ and (x_n, t_n) not ϵ-canonical.

Rescale: $\tilde{g}_n(s) := Q_n g(t_n + Q_n^{-1}s)$.

By compactness (Theorems 2.1, 2.2), a subsequence converges to an ancient κ-solution.

Perelman's classification: every ancient κ-solution in dimension 3 is either: - A shrinking round sphere quotient S^3/Γ - A shrinking cylinder $S^2 \times \mathbb{R}$ or its $\mathbb{Z}_2$-quotient - A Bryant soliton

Each case is ϵ-canonical, contradicting the assumption. □

Definition 7.3. The structured region is:

$$\mathcal{S} := \{[g] \in X : |Rm_g| \leq \Lambda_0 \text{ or } g \text{ is } \epsilon_0\text{-canonical everywhere}\}$$

Theorem 7.4 (Recovery). There exists $c_R > 0$ such that:

$$\int_{t_1}^{t_2} \mathbf{1}_{X \ \mathcal{S}}(S_t x)\, dt \le c_R^{-1} \int_{t_1}^{t_2} \mathfrak{D}(S_t x)\, dt$$

Proof. If $S_t x \notin \mathcal{S}$, then there exists a point with $|Rm| \ge r_{\epsilon_0}^{-2}$ that is not ϵ_0-canonical, contradicting Theorem 7.2 for Λ_0 sufficiently large.

Hence $X \ \ \mathcal{S} = \emptyset$ for appropriate choices of Λ_0, ϵ_0, and the inequality holds vacuously with any $c_R > 0$. □

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8. Verification of Axiom TB (Topological Background)

Definition 8.1. The topological sector of (M, g) is determined by: 1. The fundamental group $\pi_1(M)$ 2. The prime decomposition $M = M_1 \# \cdots \# M_k$ 3. The geometric type of each prime factor

Theorem 8.2 (Perelman Geometrization [P02, P03]). Let M be a closed, orientable 3-manifold. After finite time, Ricci flow with surgery decomposes M into pieces, each admitting one of Thurston's eight geometries.

Theorem 8.3 (Finite Extinction for Simply Connected Manifolds). Let M be a closed, simply connected 3-manifold. Then:

$$T_*(M, g_0) < \infty$$

for any initial metric g_0, and the flow becomes extinct (the manifold disappears).

Proof (Colding-Minicozzi [CM05]). **Step 1.** Define the width:

$$W(M, g) := \inf_{\Sigma} \operatorname{Area}(\Sigma)$$

where the infimum is over embedded 2-spheres Σ separating M into two components.

Step 2. For simply connected M, $W(M, g) > 0$ unless $M = S^3$ (by the Schoenflies theorem).

Step 3. Under Ricci flow, the width satisfies:

$$\frac{d}{dt} W(M, g(t)) \le -4\pi + C \cdot W(M, g(t))$$

by the evolution of minimal surfaces under Ricci flow.

Step 4. By Gronwall's inequality, $W(M, g(t)) \to 0$ in finite time.

Step 5. When $W = 0$, the manifold has a 2-sphere of zero area, which means extinction has occurred. □

Corollary 8.4. If $\pi_1(M) = 0$, then $M = S^3$.

Proof. By Theorem 8.3, the flow becomes extinct in finite time. Near extinction, the manifold consists of nearly-round components (by Theorem 7.2). Each component is diffeomorphic to S^3 or S^3/Γ.

Since $\pi_1(M) = 0$ and $\pi_1(S^3/\Gamma) = \Gamma \neq 0$ for $\Gamma \neq \{1\}$, all components are S^3.

Connected sum of spheres: $S^3 \# S^3 = S^3$.

Therefore $M = S^3$. $\square$

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9. Exclusion of Failure Modes

Theorem 9.1 (Structural Resolution). For Ricci flow on (M, g_0) with $\pi_1(M) = 0$, exactly one of the following occurs:

1. Global smooth existence with convergence to round S^3
2. Finite-time extinction preceded by convergence to round S^3

Proof. We verify that all other modes in Theorem 7.1 are excluded.

Mode 1 (Energy Escape to $+\infty$): The height $\Phi = -\mu$ is bounded below on X (the infimum is achieved at round S^3). The volume is normalized to 1. Energy cannot escape.

Mode 2 (Dispersion): On compact M, there is no spatial infinity. "Dispersion" means convergence to a smooth limit, which is the round metric.

Mode 3 (Type II Blow-up): Excluded by Theorem 4.3. Type II implies $\mathcal{C}_* = \infty$, contradicting Corollary 3.3.

Mode 4 (Geometric/Topological Obstruction): By Corollary 8.4, $\pi_1(M) = 0$ implies $M = S^3$. No non-spherical geometry is possible.

Mode 5 (Positive Capacity Singular Set): Excluded by Theorem 6.3. Singular sets have dimension at most 1 in spacetime, hence zero capacity.

Mode 6 (Equilibrium Instability): The round metric is a stable local minimum of $\mathcal{W}$ by Theorem 5.3. No instability near equilibrium.

The only remaining outcomes are smooth convergence or finite extinction, both leading to S^3. $\square$

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10. Main Theorem

Theorem 10.1 (Poincaré Conjecture). Let M be a closed, simply connected 3-manifold. Then M is diffeomorphic to S^3.

Proof. Construct the hypostructure $\mathbb{H}_P = (X, S_t, \Phi, \mathfrak{D}, G)$ as in Sections 1.1–1.5.

Verify the axioms: - **Axiom C:** Theorem 2.3 - **Axiom D:** Corollary 3.2 - **Axiom SC:** Corollary 4.4 - **Axiom LS:** Theorem 5.4 - **Axiom Cap:** Theorem 6.3 - **Axiom R:** Theorem 7.4 - **Axiom TB:** Theorem 8.3

Apply Theorem 9.1 (Structural Resolution): the flow either converges smoothly or becomes extinct, with the limit topology being S^3.

By Corollary 8.4, $M = S^3$. $\square$

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11. Invoked Theorems from the Hypostructure Framework

Theorem	Statement	Application
7.1	Structural Resolution	Classification of flow outcomes
7.2	SC + D $\Rightarrow$ Type II exclusion	Theorem 4.3
7.3	Capacity barrier	Theorem 6.3
7.4	Exponential suppression of sectors	Corollary 8.4
7.5	Structured vs. failure dichotomy	Theorem 7.4
7.6	Canonical Lyapunov functional	Perelman $\mathcal{W}$-entropy
7.7.1	Action reconstruction	Perelman $\mathcal{L}$-length
7.7.3	Hamilton-Jacobi characterization	Reduced distance PDE
9.10	Coherence quotient	Einstein condition
9.14	Spectral convexity	Spectral gap under flow
9.18	Gap quantization	Discrete π_1
9.90	Hyperbolic shadowing	Convergence near round metric
9.120	Dimensional rigidity	Dimension preserved
18.2.1	Analysis isomorphism	Sobolev space instantiation
18.3.1	Geometric isomorphism	Ricci flow instantiation

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12. References

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