Étude 2: Navier-Stokes Regularity via Hypostructure

0. Introduction

Problem 0.1 (Navier-Stokes Millennium Problem). Let $u_0 \in C^\infty(\mathbb{R}^3)$ be a divergence-free vector field with $|D^\alpha u_0(x)| \leq C_{\alpha,K}(1+|x|)^{-K}$ for all α, K. Does there exist a smooth solution $u : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ to the Navier-Stokes equations with $u(0) = u_0$?

We construct a hypostructure $\mathbb{H}_{NS} = (X, S_t, \Phi, \mathfrak{D}, G)$ and identify which axioms are verified, which remain open, and how the metatheorems constrain possible singularity formation.

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1. The Navier-Stokes Equations

1.1 The PDE System

Definition 1.1.1. The incompressible Navier-Stokes equations on $\mathbb{R}^3$ are:

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u$$

$$\nabla \cdot u = 0$$

where $u : \mathbb{R}^3 \times [0,T) \to \mathbb{R}^3$ is the velocity field, $p : \mathbb{R}^3 \times [0,T) \to \mathbb{R}$ is the pressure, and $\nu > 0$ is the kinematic viscosity.

Proposition 1.1.2 (Pressure Recovery). Given $u \in L^2_\sigma(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $q > 3$, the pressure p is uniquely determined (up to constant) by:

$$-\Delta p = \partial_i \partial_j (u_i u_j) = \mathrm{tr}(\nabla u \cdot \nabla u^T)$$

with $p(x) \to 0$ as $|x| \to \infty$. The solution is:

$$p = \sum_{i,j=1}^{3} R_i R_j (u_i u_j)$$

where $R_i := \partial_i (-\Delta)^{-1/2}$ is the i-th Riesz transform, satisfying $\|R_i f\|_{L^p} \leq C_p \|f\|_{L^p}$ for $1 < p < \infty$.

Proof. Apply $\nabla \cdot$ to the momentum equation and use $\nabla \cdot u = 0$:

$$\nabla \cdot \partial_t u = 0, \quad \nabla \cdot (\nu \Delta u) = \nu \Delta (\nabla \cdot u) = 0$$

$$\nabla \cdot \nabla p = \Delta p, \quad \nabla \cdot ((u \cdot \nabla)u) = \partial_i (u_j \partial_j u_i) = \partial_i \partial_j (u_i u_j)$$

where we used $\partial_j u_j = 0$. Inverting $-\Delta$ via the Newtonian potential and taking derivatives gives the Riesz transform representation. □

Definition 1.1.3. The Leray projector $\mathbb{P} : L^2(\mathbb{R}^3)^3 \to L^2_\sigma(\mathbb{R}^3)$ onto divergence-free fields is:

$$\mathbb{P} = I + \nabla(-\Delta)^{-1}\nabla\cdot$$

In Fourier space: $\widehat{\mathbb{P}f}(\xi) = (I - \frac{\xi\otimes\xi}{|\xi|^2})\hat{f}(\xi)$.

Definition 1.1.4. The projected Navier-Stokes equation is:

$$\partial_t u = \nu\Delta u - \mathbb{P}((u\cdot\nabla)u) =: \nu\Delta u - B(u,u)$$

where $B(u,v) := \mathbb{P}((u\cdot\nabla)v)$ is the bilinear form.

1.2 Function Spaces

Definition 1.2.1. The energy space is:

$$L^2_\sigma(\mathbb{R}^3) := \overline{\{u \in C_c^\infty(\mathbb{R}^3)^3 : \nabla\cdot u = 0\}}^{L^2}$$

Definition 1.2.2. The homogeneous Sobolev spaces are:

$$\dot{H}^s(\mathbb{R}^3) := \{f \in \mathcal{S}'(\mathbb{R}^3) : |\xi|^s\hat{f} \in L^2(\mathbb{R}^3)\}$$

with norm $\|f\|_{\dot{H}^s} := \||\xi|^s\hat{f}\|_{L^2}$.

Definition 1.2.3. The critical space for Navier-Stokes is $\dot{H}^{1/2}(\mathbb{R}^3)$, characterized by scale-invariance: if $u(x,t)$ solves NS, then so does:

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t)$$

and $\|u_\lambda(\cdot,0)\|_{\dot{H}^{1/2}} = \|u(\cdot,0)\|_{\dot{H}^{1/2}}$.

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2. The Hypostructure Data

2.1 State Space

Definition 2.1.1. The state space is:

$$X := L^2_\sigma(\mathbb{R}^3) \cap \dot{H}^{1/2}(\mathbb{R}^3)$$

with norm $\|u\|_X := \|u\|_{L^2} + \|u\|_{\dot{H}^{1/2}}$.

Proposition 2.1.2. $(X, \|\cdot\|_X)$ is a separable Banach space, hence Polish.

2.2 The Semiflow

Definition 2.2.1. For $u_0 \in X$, the maximal existence time is:

$$T_*(u_0) := \sup\{T > 0 : \exists \text{ mild solution } u \in C([0,T); X) \cap L^2(0,T; \dot{H}^{3/2})\}$$

Theorem 2.2.2 (Kato [K84]). For each $u_0 \in X$: 1. **(Local existence)** There exists $T_* = T_*(u_0) \in (0, \infty]$ and a unique function $u \in C([0, T_*); X) \cap L^2_{loc}([0, T_*); \dot{H}^{3/2})$ satisfying the integral equation:

$$u(t) = e^{\nu t \Delta} u_0 - \int_0^t e^{\nu(t-s)\Delta} B(u(s), u(s))\, ds$$

2. **(Continuous dependence)** The map $u_0 \mapsto u(t)$ is continuous from X to $C([0, T]; X)$ for $T < T_*(u_0)$. 3. **(Lower bound on existence time)** There exists $c > 0$ depending only on ν such that $T_* \geq c / \|u_0\|^4_{\dot{H}^{1/2}}$.

Proof sketch. Define $\Psi(u)(t) := e^{\nu t \Delta} u_0 - \int_0^t e^{\nu(t-s)\Delta} B(u, u)(s)\, ds$. Using the heat kernel estimates $\|e^{\nu t \Delta} f\|_{\dot{H}^{s+\alpha}} \leq C t^{-\alpha/2} \|f\|_{\dot{H}^s}$ and the bilinear estimate $\|B(u, v)\|_{\dot{H}^{-1/2}} \leq C \|u\|_{\dot{H}^{1/2}} \|v\|_{\dot{H}^{1/2}}$, one shows Ψ is a contraction on a ball in $C([0, T]; \dot{H}^{1/2})$ for T sufficiently small. □

Theorem 2.2.3 (Blow-up Criterion). If $T_* = T_*(u_0) < \infty$, then:

$$\lim_{t \nearrow T_*} \|u(t)\|_{\dot{H}^{1/2}} = \infty$$

Equivalently, the enstrophy integral diverges: $\int_0^{T_*} \|\nabla u(t)\|^2_{L^2}\, dt = \infty$.

Definition 2.2.4. The semiflow $S_t : X \to X$ is defined for $t < T_*(u_0)$ by:

$$S_t(u_0) := u(t)$$

2.3 Height Functional (Energy)

Definition 2.3.1. The kinetic energy is:

$$E(u) := \frac{1}{2} \|u\|^2_{L^2} = \frac{1}{2} \int_{\mathbb{R}^3} |u(x)|^2\, dx$$

Theorem 2.3.2 (Energy Inequality). For Leray-Hopf weak solutions:

$$E(u(t)) + \nu \int_0^t \|\nabla u(s)\|^2_{L^2}\, ds \leq E(u_0)$$

Definition 2.3.3. The height functional is $\Phi := E : X \to [0, \infty)$.

2.4 Dissipation Functional (Enstrophy)

Definition 2.4.1. The enstrophy (dissipation rate) is:

$$\mathfrak{D}(u) := \nu \|\nabla u\|^2_{L^2} = \nu \|\omega\|^2_{L^2}$$

where $\omega := \nabla \times u$ is the vorticity.

Proposition 2.4.2. For smooth solutions:

$$\frac{d}{dt}E(u(t)) = -\mathfrak{D}(u(t))$$

Proof. Multiply the Navier-Stokes equation by u and integrate:

$$\int u \cdot \partial_t u = \int u \cdot (\nu \Delta u) - \int u \cdot \nabla p - \int u \cdot (u \cdot \nabla) u$$

The pressure term vanishes: $\int u \cdot \nabla p = -\int p \nabla \cdot u = 0$.

The nonlinear term vanishes: $\int u \cdot (u \cdot \nabla) u = \frac{1}{2} \int (u \cdot \nabla)|u|^2 = -\frac{1}{2} \int |u|^2 \nabla \cdot u = 0$.

The viscous term: $\int u \cdot \Delta u = -\int |\nabla u|^2 = -\nu^{-1} \mathfrak{D}(u)$. □

2.5 Symmetry Group

Definition 2.5.1. The Navier-Stokes symmetry group is:

$$G := \mathbb{R}^3 \rtimes (SO(3) \times \mathbb{R}_{>0})$$

acting by: - Translation: $(\tau_a u)(x) := u(x-a)$ - Rotation: $(R_\theta u)(x) := R_\theta u(R_\theta^{-1} x)$ - Scaling: $(\sigma_\lambda u)(x,t) := \lambda u(\lambda x, \lambda^2 t)$

Proposition 2.5.2. The Navier-Stokes equations are G-equivariant.

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3. Verification of Axiom C (Compactness)

Theorem 3.1 (Rellich-Kondrachov). For bounded $\Omega \subset \mathbb{R}^3$:

$$H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), \quad 1 \le q < 6$$

Theorem 3.2 (Concentration-Compactness for NS). Let $(u_n) \subset X$ with $\sup_n E(u_n) \le E_0$. Then there exist: 1. A subsequence (still denoted u_n) 2. Sequences $(x_n^j)_{j\ge1} \subset \mathbb{R}^3$ and $(\lambda_n^j)_{j\ge1} \subset \mathbb{R}_{>0}$ 3. Profiles $(U^j)_{j\ge1} \subset X$

such that:

$$u_n = \sum_{j=1}^{J} (\lambda_n^j)^{1/2} U^j((\lambda_n^j)(\cdot - x_n^j)) + w_n^J$$

where $\|w_n^J\|_{L^q} \to 0$ as $n \to \infty$ then $J \to \infty$ for $2 < q < 6$.

Proof. Apply the profile decomposition of Gérard [G98] adapted to the NS scaling. The critical Sobolev embedding $\dot{H}^{1/2} \hookrightarrow L^3$ fails to be compact, but concentration at isolated scales/locations is captured by the profiles. □

Proposition 3.3 (Axiom C: Partial). On bounded subsets of X with:

$$\sup_n \|u_n\|_{L^2} \leq M, \quad \sup_n \|u_n\|_{\dot{H}^1} \leq M$$

the sequence (u_n) is precompact in L^2_{loc}.

Proof. The $\dot{H}^1$ bound gives compactness in L^2_{loc} by Rellich-Kondrachov. □

Remark 3.4. Full Axiom C (global precompactness in X) is not available due to the critical nature of $\dot{H}^{1/2}$ and non-compactness of $\mathbb{R}^3$.

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4. Verification of Axiom D (Dissipation)

Theorem 4.1 (Energy-Dissipation Identity). For smooth solutions on $[0, T]$:

$$E(u(T)) + \int_0^T \mathfrak{D}(u(t))\, dt = E(u(0))$$

Proof. Integrate Proposition 2.4.2. □

Corollary 4.2. Axiom D holds with $C = 0$:

$$\Phi(S_t u_0) + \int_0^t \mathfrak{D}(S_s u_0)\, ds = \Phi(u_0)$$

Corollary 4.3. The total dissipation cost is bounded:

$$\mathcal{C}_*(u_0) := \int_0^{T_*} \mathfrak{D}(u(t))\, dt \leq E(u_0) < \infty$$

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5. Verification of Axiom SC (Scaling Structure)

Definition 5.1. The scaling dimensions for Navier-Stokes are: - $[u] = -1$ (velocity scales as λ^{-1}) - $[t] = -2$ (time scales as λ^{-2}) - $[\nabla] = 1$ - $[E] = -1$ (energy scales as λ^{-1} in 3D) - $[\mathfrak{D}] = 1$ (enstrophy scales as λ)

Proposition 5.2. Under the scaling $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$:

$$E(u_\lambda(0)) = \lambda^{-1} E(u(0))$$

$$\int_0^{T/\lambda^2} \mathfrak{D}(u_\lambda(t))\, dt = \lambda^{-1} \int_0^T \mathfrak{D}(u(t))\, dt$$

Proof. Direct computation:

$$E(u_\lambda) = \frac{1}{2} \int |\lambda u(\lambda x)|^2 dx = \frac{\lambda^2}{2} \int |u(\lambda x)|^2 dx = \frac{\lambda^2}{2\lambda^3} \int |u(y)|^2 dy = \lambda^{-1} E(u)$$

Similarly for the dissipation integral. □

Theorem 5.3 (Criticality). The Navier-Stokes equations are critical: $\alpha = \beta$ where: - $\alpha = 1$ is the scaling exponent of energy - $\beta = 1$ is the scaling exponent of dissipation cost

Proof. Both E and $\mathcal{C}_*$ scale as λ^{-1}. □

Corollary 5.4 (Theorem 7.2 Inapplicable). The condition $\alpha > \beta$ for Type II exclusion is not satisfied. Theorem 7.2 does not exclude Type II blow-up for Navier-Stokes.

Remark 5.5. This is the fundamental obstruction. For supercritical problems ($\alpha > \beta$), Type II blow-up is excluded by scaling. For critical problems ($\alpha = \beta$), both Type I and Type II remain possible a priori.

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6. Critical Norms and Blow-up Criteria

6.1 Scaling-Invariant Norms

Definition 6.1.1. A norm $\|\cdot\|_Y$ is critical for NS if:

$$\|u_\lambda\|_Y = \|u\|_Y$$

for all $\lambda > 0$.

Proposition 6.1.2. The following norms are critical: - $\|u\|_{L^3(\mathbb{R}^3)}$ - $\|u\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$ - $\|u\|_{\dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)}$ for $3 < p < \infty$ - $\|u\|_{BMO^{-1}(\mathbb{R}^3)}$

6.2 Blow-up Criteria

Theorem 6.2.1 (Escauriaza-Seregin-Šverák [ESS03]). If $T_* < \infty$, then:

$$\limsup_{t \to T_*} \|u(t)\|_{L^3(\mathbb{R}^3)} = \infty$$

Theorem 6.2.2 (Ladyzhenskaya-Prodi-Serrin). The solution is regular on $[0, T]$ if:

$$u \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \leq \infty$$

Theorem 6.2.3 (Beale-Kato-Majda [BKM84]). For Euler equations ($\nu = 0$), blow-up requires:

$$\int_0^{T_*} \|\omega(t)\|_{L^\infty} \, dt = \infty$$

For Navier-Stokes, this remains a blow-up criterion but is not known to be necessary.

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7. Partial Verification of Axiom LS (Local Stiffness)

Definition 7.1. The zero solution $u \equiv 0$ is the unique equilibrium for Navier-Stokes on $\mathbb{R}^3$ with finite energy.

Theorem 7.2 (Stability of Zero). For $\|u_0\|_{\dot{H}^{1/2}}$ sufficiently small, the solution exists globally and:

$$\|u(t)\|_{\dot{H}^{1/2}} \leq C\|u_0\|_{\dot{H}^{1/2}} e^{-c\nu t}$$

Proof. Small data global existence in $\dot{H}^{1/2}$ follows from Kato's theorem with a contraction argument. The exponential decay follows from the spectral gap of the Stokes operator. □

Proposition 7.3 (Łojasiewicz Inequality at Zero). Near $u = 0$:

$$\mathfrak{D}(u) = \nu\|\nabla u\|_{L^2}^2 \geq c\|u\|_{L^2}^2 = 2c \cdot E(u)$$

by Poincaré inequality (on bounded domains) or Hardy inequality.

Remark 7.4. Axiom LS holds at the equilibrium $u = 0$. The open question is whether non-zero steady states or time-periodic solutions exist that could serve as alternative attractors.

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8. Partial Verification of Axiom Cap (Capacity)

Definition 8.0 (Suitable Weak Solution). A pair (u, p) is a *suitable weak solution* on $\mathbb{R}^3 \times (0, T)$ if: 1. $u \in L^\infty(0, T; L^2) \cap L^2(0, T; \dot{H}^1)$ and $p \in L^{5/3}_{loc}(\mathbb{R}^3 \times (0, T))$ 2. (u, p) satisfies NS in the sense of distributions 3. The local energy inequality holds: for a.e. t and all non-negative $\phi \in C_c^\infty(\mathbb{R}^3 \times (0, T))$:

$$\int |u|^2 \phi \, dx\Big|_t + 2\nu \int_0^t \int |\nabla u|^2 \phi \leq \int_0^t \int |u|^2(\partial_t \phi + \nu \Delta \phi) + \int_0^t \int (|u|^2 + 2p)(u \cdot \nabla \phi)$$

Theorem 8.1 (Caffarelli-Kohn-Nirenberg [CKN82]). Let (u, p) be a suitable weak solution on $\mathbb{R}^3 \times (0, T)$. Define the singular set:

$$\Sigma := \{(x, t) \in \mathbb{R}^3 \times (0, T) : u \notin L^\infty(B_r(x) \times (t - r^2, t)) \text{ for all } r > 0\}$$

Then the 1-dimensional parabolic Hausdorff measure vanishes: $\mathcal{P}^1(\Sigma) = 0$.

Proof. **(i) Scaled quantities.** For $(x_0, t_0) \in \mathbb{R}^3 \times (0, T)$ and $r > 0$ with $t_0 - r^2 > 0$, define:

$$A(r) := \sup_{t_0 - r^2 < t < t_0} \frac{1}{r} \int_{B_r(x_0)} |u|^2 \, dx$$

$$C(r) := \frac{1}{r^2} \int_{t_0 - r^2}^{t_0} \int_{B_r(x_0)} |u|^3 \, dx \, dt$$

$$D(r) := \frac{1}{r^2} \int_{t_0 - r^2}^{t_0} \int_{B_r(x_0)} |p - p_{B_r}|^{3/2} \, dx \, dt$$

$$E(r) := \frac{1}{r} \int_{t_0 - r^2}^{t_0} \int_{B_r(x_0)} |\nabla u|^2 \, dx \, dt$$

(ii) Regularity criterion. There exists $\epsilon_0 > 0$ universal such that if $\limsup_{r \to 0}(C(r) + D(r)) < \epsilon_0$, then (x_0, t_0) is a regular point: $u \in L^\infty$ near (x_0, t_0).

(iii) Energy control. From the local energy inequality and Hölder estimates:

$$C(r) + D(r) \leq C(E(r)^{3/4} A(r)^{1/4} + E(r)^{3/2})$$

(iv) Covering argument. Let $\Sigma_\epsilon := \{(x,t) : C(r) + D(r) \geq \epsilon_0 \text{ for all } r \leq \epsilon\}$. Cover Σ_ϵ by parabolic cylinders Q_{r_i} with $r_i \leq \epsilon$ and $C(r_i) + D(r_i) \geq \epsilon_0$. The Vitali covering lemma gives:

$$\sum_i r_i \leq C\epsilon_0^{-1} \int_0^T \int |\nabla u|^2 \leq C\epsilon_0^{-1} E(u_0)$$

(v) Conclusion. $\mathcal{P}^1(\Sigma_\epsilon) \leq C\epsilon_0^{-1} E(u_0)$ independent of ϵ. Since $\Sigma = \bigcap_{\epsilon > 0} \Sigma_\epsilon$ and the bound is uniform, $\mathcal{P}^1(\Sigma) = 0$. □

Corollary 8.2. The spatial singular set at any time has Hausdorff dimension at most 1:

$$\dim_H(\Sigma_t) \leq 1$$

Proposition 8.3 (Axiom Cap: Partial). Singularities cannot fill positive-capacity sets. Specifically:

$$\mathrm{Cap}_{1,2}(\Sigma_t) = 0$$

Proof. Sets of Hausdorff dimension ≤ 1 in $\mathbb{R}^3$ have zero $(1,2)$-capacity. □

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9. The Regularity Gap

9.1 What Is Known

Theorem 9.1 (Summary of Verified Axioms).

Axiom	Status	Reference
C (Compactness)	Partial (local, with extra derivative)	Theorem 3.3
D (Dissipation)	Verified	Theorem 4.1
SC (Scaling)	Critical ($\alpha = \beta$)	Theorem 5.3

Axiom	Status	Reference
LS (Local Stiffness)	Verified at $u = 0$	Theorem 7.2
Cap (Capacity)	Partial ($\dim \Sigma \leq 1$)	Theorem 8.1
R (Recovery)	Open	—
TB (Topological)	N/A (contractible state space)	—

9.2 What Is Missing

Open Problem 9.2. Verify Axiom R (Recovery) for Navier-Stokes: show that trajectories spending time in "wild" regions (high enstrophy) must dissipate proportionally.

Conjecture 9.3 (Axiom R for NS). There exists $c_R > 0$ such that:

$$\int_0^T \mathbf{1}_{\{\|\omega(t)\|_{L^\infty} > \Lambda\}} \, dt \leq c_R^{-1} \Lambda^{-\gamma} \int_0^T \mathfrak{D}(u(t)) \, dt$$

for some $\gamma > 0$.

Remark 9.4. If Conjecture 9.3 holds, combined with the CKN partial regularity, it would imply global regularity.

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10. Application of Metatheorems

10.1 Theorem 7.1 (Structural Resolution)

Application. Every finite-energy trajectory either: 1. Exists globally and decays to zero 2. Blows up at finite time $T_* < \infty$

The dichotomy is established; the question is which alternative occurs.

10.2 Theorem 7.3 (Capacity Barrier)

Application. By CKN (Theorem 8.1), any blow-up occurs on a set of dimension ≤ 1. This is the capacity barrier in action: high-dimensional blow-up sets are excluded.

Corollary 10.1. If blow-up occurs, it is necessarily of "sparse" type—concentrated on thin space-time filaments.

10.3 Theorem 9.10 (Coherence Quotient)

Definition 10.2. The coherence quotient for NS:

$$\mathcal{Q}(u) := \frac{\|u \otimes u - \frac{1}{3}|u|^2 I\|_{L^{3/2}}}{\|u\|_{L^3}^2}$$

measures deviation from isotropic turbulence.

Conjecture 10.3. Near blow-up, $\mathcal{Q}(u(t)) \to 0$ (flow becomes increasingly aligned/coherent).

10.4 Theorem 9.14 (Spectral Convexity)

Application. The energy spectrum $E(k,t) := \frac{1}{2}\int_{|\xi|=k} |\hat{u}(\xi,t)|^2 dS(\xi)$ satisfies convexity properties that constrain possible blow-up scenarios.

10.5 Theorem 9.90 (Hyperbolic Shadowing)

Application. Near the stable equilibrium $u = 0$, small perturbations decay exponentially. This is the shadowing property in the dissipative regime.

10.6 Theorem 9.120 (Dimensional Rigidity)

Application. Blow-up cannot change the "effective dimension" of the solution. Self-similar blow-up profiles must respect the 3D structure.

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11. Self-Similar Blow-up Analysis

11.1 Self-Similar Ansatz

Definition 11.1. A Type I blow-up at $(0, T_*)$ has the form:

$$u(x,t) = \frac{1}{\sqrt{T_* - t}} U\left(\frac{x}{\sqrt{T_* - t}}\right)$$

where $U : \mathbb{R}^3 \to \mathbb{R}^3$ is the blow-up profile.

Proposition 11.2. The profile U satisfies:

$$\nu\Delta U - \frac{1}{2}U - \frac{1}{2}(y \cdot \nabla)U - (U \cdot \nabla)U + \nabla P = 0$$

$$\nabla \cdot U = 0$$

11.2 Exclusion Results

Theorem 11.3 (Nečas-Růžička-Šverák [NRS96]). There is no non-trivial self-similar blow-up with $U \in L^3(\mathbb{R}^3)$.

Proof. Multiply the profile equation by U and integrate. Use the criticality of L^3 and Sobolev inequalities to derive a contradiction unless $U = 0$. □

Theorem 11.4 (Tsai [T98]). There is no non-trivial self-similar blow-up with $U \in L^p(\mathbb{R}^3)$ for any $p > 3$.

Remark 11.5. These results exclude "nice" self-similar blow-up but leave open singular self-similar profiles or non-self-similar blow-up.

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12. Enstrophy Evolution

12.1 The Enstrophy Equation

Theorem 12.1. For smooth solutions, the enstrophy $\Omega := \frac{1}{2}\|\omega\|_{L^2}^2$ satisfies:

$$\frac{d\Omega}{dt} = -\nu\|\nabla\omega\|_{L^2}^2 + \int_{\mathbb{R}^3} \omega \cdot (\omega \cdot \nabla)u \, dx$$

Proof. The vorticity equation is:

$$\partial_t\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu\Delta\omega$$

Multiply by ω and integrate:

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 = \nu \int \omega \cdot \Delta\omega + \int \omega \cdot (\omega \cdot \nabla)u$$

The transport term vanishes: $\int \omega \cdot (u \cdot \nabla)\omega = 0$. □

12.2 The Vortex Stretching Term

Definition 12.2. The vortex stretching term is:

$$\mathcal{S}(\omega, u) := \int_{\mathbb{R}^3} \omega \cdot (\omega \cdot \nabla)u \, dx = \int_{\mathbb{R}^3} \omega_i\omega_j S_{ij} \, dx$$

where $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the strain tensor.

Proposition 12.3 (Constantin-Fefferman [CF93]). The stretching term satisfies:

$$|\mathcal{S}(\omega, u)| \le C\|\omega\|_{L^2}^{3/2}\|\nabla\omega\|_{L^2}^{3/2}$$

Corollary 12.4. By Young's inequality:

$$\frac{d\Omega}{dt} \le -\frac{\nu}{2}\|\nabla\omega\|_{L^2}^2 + C\nu^{-3}\Omega^3$$

This shows enstrophy can grow at most doubly-exponentially in time.

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13. The Critical Threshold

13.1 Known Conditional Results

Theorem 13.1 (Regularity Below Critical Threshold). There exists $\epsilon_* > 0$ such that if:

$$\|u_0\|_{\dot{H}^{1/2}} < \epsilon_*$$

then the solution exists globally and decays.

Theorem 13.2 (Gallagher-Koch-Planchon [GKP16]). Global regularity holds if:

$$\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} < c\nu$$

where $\dot{B}^{-1}_{\infty,\infty}$ is a critical Besov space.

13.2 The Gap

Open Problem 13.3. Does there exist $u_0 \in X$ with $\|u_0\|_{\dot{H}^{1/2}} < \infty$ such that $T_*(u_0) < \infty$?

Remark 13.4. The hypostructure framework identifies this as a question about: 1. Whether Axiom R holds (recovery from high-enstrophy regions) 2. Whether the capacity barrier (CKN) can be strengthened to $\dim \Sigma = -\infty$ (no singularities)

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14. Conclusion

Theorem 14.1 (Summary). The Navier-Stokes equations form a hypostructure $\mathbb{H}_{NS}$ with:

Component	Instantiation
State space X	$L^2_\sigma \cap \dot{H}^{1/2}$
Height Φ	Kinetic energy $E(u)$
Dissipation $\mathfrak{D}$	Enstrophy $\nu\|\nabla u\|^2$
Symmetry G	Translations, rotations, scaling
Axiom D	Verified (energy equality)
Axiom SC	Critical ($\alpha = \beta = 1$)
Axiom LS	Verified at $u = 0$
Axiom Cap	Partial (CKN: $\dim \Sigma \leq 1$)

Corollary 14.2. By the metatheorems: 1. Any blow-up is confined to dimension ≤ 1 (Theorem 7.3) 2. Self-similar blow-up in L^3 is excluded (Theorem 11.3) 3. Small data gives global regularity (Theorem 7.2 at criticality)

Open. Full verification of Axioms C and R, which would imply global regularity.

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