Seminario de teoría de categorías

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Chapter 1

Categories: objects and arrows

Definition 1.0.1 (Category)**.** A category $\mathscr{C}$ consists of the following data:

- A class of *objects*,
$$Ob(\mathscr{C})$$
(we usually write $X \in \mathscr{C}$ instead of $X \in Ob(\mathscr{C})$).

- For every two objects $X, Y \in \mathscr{C}$, a class of *morphisms* (or *arrows*) *from* X *to* Y,
$$\mathrm{Hom}_{\mathscr{C}}(X, Y)$$
(we sometimes write $\mathrm{Hom}(X, Y)$ or $\mathscr{C}(X, Y)$ instead of $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, and $f\colon X \to Y$ instead of $f \in \mathrm{Hom}(X, Y)$.

- A *composition law*, associating to morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, a *composition morphism*
$$g \circ f\colon X \to Z.$$
Arrows and the composition law are subject to the following conditions:

  - *Associativity*: given composable morphisms f, g, h,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

  - Existence of *identity arrows*: for any X, there exists a morphism $id_X\colon X \to X$, such that, for all $f\colon X \to Y$ and $g\colon Z \to X$,
$$f \circ id_X = f, \qquad id_X \circ g = g.$$

Example. *Some basic examples:*

1. *The category* **Set** *of all sets:*
   - *Ob*(**Set**) *is the class of all sets.*
   - $\mathrm{Hom}(X, Y)$ *is the set of all mappings* $f\colon X \to Y$
   - $g \circ f$ *is the usual composition of mappings.*
2. *The category* **Top** *of topological spaces has*
   - *topological spaces* $(X, \mathcal{T})$ *as objects (in other words, Ob*(**Set**) *is the class of all topological spaces).*
   - *continuous mappings* $f\colon (X, \mathcal{T}) \to (Y, \mathcal{T})$ *as morphisms, with the usual composition.*

   Notice that this requires two results from topology: (1) composition of continuous mappings is continuous and (2) the identity mapping is continuous.
3. *The category* **Group** *of groups has*
   - *groups as objects.*
   - *group homomorphisms as arrows, with the usual composition.*

   This requires the result that compositions of group homomorphisms are group homomorphisms.

All of the above examples consist of certain sets (with extra structure in some cases) and mappings between them, but this is not always the case:

1. *Posets give rise to categories:*
   For example, we may regard $(\mathbb{N}, \leq)$ *as a category having*
   - *natural numbers as objects.*
   - *for any* $n, m \in \mathbb{N}$,

     $$\mathrm{Hom}(n, m) = \begin{cases} \{n \to m\}, & \textit{if } n \leq m \\ \emptyset, & \textit{if } n \not\leq m \end{cases}$$

     Here $\{n \to m\}$ *is just any set with a single element, which we regard as the arrow* $n \to m$. *These sets must be chosen so that* $n \to m$ *equals* $n' \to m'$ *if and only if* $n = n'$ *and* $m = m'$.

Since there is at most one arrow between objects, composition is defined in the only posible way.

Observe that the reflexivity and transitivity properties of partial orders are needed in order for this to form a category.

Similarly, any set X gives rise to a category

$$(\mathcal{P}(X), \subseteq),$$

and any topological space $(X, \mathcal{T})$ gives rise to a category

$$(\mathcal{T}, \subseteq).$$

2. *Groups as categories:*
   Given a group G, with binary operation "$\cdot$", the category BG has

   – *a single object, written $\bullet$.*
   – *elements $g \in G$ as arrows $\bullet \xrightarrow{g} \bullet$.*
   – *The composition of two arrows $g, g' \in G$ is the arrow $g \circ g' = g \cdot g'$.*

3. *The category associated to a unitary ring:*
   Given a unitary ring R, with binary operations "$+$" and "$\cdot$", the category $\mathscr{C}_R$ has

   – *a single object, written $\bullet$.*
   – *elements $r \in R$ as arrows $\bullet \xrightarrow{r} \bullet$.*
   – *The composition of two arrows $r, r' \in R$ is the arrow $r \circ r' = r \cdot r'$.*

   Observe that the condition that R is unitary is required by the axiom that categories must have identity arrows.

4. *Given a field k, the matrix category $\mathbf{M}at_k$ has*

   - *natural numbers as objects.*
   - *For any $n, m \in \mathbb{N} \setminus \{0\}$, a morphism $n \to m$ is an $m \times n$ matrix with entries in k. Composition of such morphisms corresponds to product of matrices. There is a single arrow $0 \to m$ and $m \to 0$, for every $m \in \mathbb{N}$, and composition with these arrows is determined by lack of choice.*

Definition 1.0.2. Given a category $\mathscr{C}$, the *opposite category* $\mathscr{C}^{op}$ has

- same objects as $\mathscr{C}$, that is
$$Ob(\mathscr{C}^{op}) = Ob(\mathscr{C}).$$
- Given $X, Y \in \mathscr{C}^{op}$, each morphism $X \to Y$ in $\mathscr{C}$ is an arrow $Y \to X$ in $\mathscr{C}^{op}$, that is,
$$\mathrm{Hom}_{\mathscr{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

Example. *This is an example of a simple category and its opposite.*

$\mathscr{C}$

$$\begin{tikzcd} & 2 \\ 1 \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & 3 \end{tikzcd}$$

$$\begin{gathered} Hom(1,2) = f \\ Hom(1,3) = g \\ Hom(2,3) = Hom(2,1) = \\ = Hom(3,1) = Hom(3,2) = \emptyset \end{gathered}$$

$\mathscr{C}^{\ op}$

$$\begin{tikzcd} & 2 \arrow[dl, "f^{op}"'] \\ 1 & \\ & 3 \arrow[ul, "g^{op}"] \end{tikzcd}$$

$$\begin{gathered} Hom(2,1) = f^{op} \\ Hom(3,1) = g^{op} \\ Hom(2,3) = Hom(1,2) = \\ = Hom(1,3) = Hom(3,2) = \emptyset \end{gathered}$$

Definition 1.0.3 (Small and Locally small categories)**.**

A category $\mathscr{C}$ is **locally small** if, for all $X, Y \in \mathscr{C}$, $\mathrm{Hom}(X, Y)$ is a set (as opposed to a proper class).

A category is **small** if it is locally small and $Ob(\mathscr{C})$ is also a set.

Example. *All categories introduced above are locally small. An example of non-locally small category is the one with*

– *a single object* $\bullet$.

– *Any set* X *is an arrow* $\bullet \xrightarrow{X} \bullet$. *The composition of* $\bullet \xrightarrow{X} \bullet$ *and* $\bullet \xrightarrow{Y} \bullet$ *is* $\bullet \xrightarrow{X \cup Y} \bullet$.

Remark. *From now on, we will always consider locally small categories.*

1.1 Subcategories

Definition 1.1.1. Given a category $\mathscr{C}$, we say that a subset $\mathscr{D} \subseteq \mathscr{C}$ is a **subcategory** of $\mathscr{C}$ if it is a subcollection of objects and morphisms, satisfying that

- All domains and codomains of morphisims in $\mathscr{D}$ are contained.
- The identity morphism of every object in $\mathscr{D}$ is contained.
- Every posible composition between morphisms in $\mathscr{D}$ is contained.

Example. *Some basic examples of subcategories are*

1. ***Haus*** *is the subcategory of* ***Top*** *consisting of Hausdorff topological spaces.*
2. ***Ab*** *is the subcategory of* ***Group*** *consisting of abelian groups.*

Definition 1.1.2 (Full and Wide subcategories)**.**

Let $\mathscr{D}$ be a subcategory of $\mathscr{C}$.

- $\mathscr{D}$ ia a **full subcategory** of $\mathscr{C}$ if $\forall\, X, Y \in \mathscr{D}$, it is satisfied that

$$Hom_{\mathscr{D}}(X, Y) = Hom_{\mathscr{C}}(X, Y)$$

- $\mathscr{D}$ ia a **wide subcategory** of $\mathscr{C}$ if

$$Ob(\mathscr{D}) = Ob(\mathscr{C})$$

Example. *The subcategory $\mathscr{C}$ of* ***Group*** *consisting of*

- $Ob(\mathscr{C}) = Ob($***Group***$)$.
- *The arrows in $\mathscr{C}$ are the group isomorphisms.*

Example. *We consider the following categories:*

1. *The category* ***Rng*** *of all rings:*
   - $Ob(\mathbf{Rng})$ *is conformed by all rings.*
   - *Ring homomorphisms are arrows, with the usual composition.*
2. *The category* ***Ring*** *of all unitary rings:*

- *Ob(**Ring**) is conformed by all unitary rings.*
- *Ring homomorphisms $\mathscr{R} \longrightarrow \mathscr{S}$ satisfying $1_{\mathscr{R}} \mapsto 1_{\mathscr{S}}$ are arrows with the usual composition.*

*It is clear that **Ring** is a subcategory of **Rng** but it is not full, as morphisms in **Ring** need to take the multiplicative identity of the first ring into the multiplicative identity of the second one.*

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Proposition 1.1.1. *Let $\mathscr{D}$ be a subcategory of $\mathscr{C}$. Then, $\mathscr{D}^{\mathrm{op}}$ is a subcategory of $\mathscr{C}^{\mathrm{op}}$*

1.2 Special Objects

Definition 1.2.1. An object $I \in \mathscr{C}$ is *initial* if, for every object $X \in \mathscr{C}$, there exists a unique morphism
$$I \to X.$$

Definition 1.2.2. An object $T \in \mathscr{C}$ is *terminal* (or *final*) if, for every object $X \in \mathscr{C}$, there exists a unique morphism
$$X \to T.$$

Terminal and initial are dual notions.

Definition 1.2.3. An object is a *zero object* if it is initial and terminal.

Example. *Some basic examples:*

1. *Special objects in category* $(\mathbb{N}, \leq)$*:*
   - *The initial object in this category is 0, since* $0 \leq n \ \forall n \in \mathbb{N}$ *and so there exists a unique arrow from 0 to any other natural number. The uniqueness comes from the fact that there exists only one arrow that goes from one object to another (greater) one.*
   - *There is no final object in this category, since there isn't any natural number greater that all of them.*
   - *Since there's no final object, there's no zero object in this category.*

   Note: if instead of $\mathbb{N}$ *we considered* $\mathbb{Z}$*, we wouldn't have an initial object either.*

2. *In* ***Set*** *and* ***Top****:*
   - *We have that* $\emptyset$ *is initial since we can always consider the inclusion as the only arrow from* $\emptyset$ *to any set or topological space.*
   - *We have that any set with only an element is terminal, since there's only one way to define a map from a set to the singleton.*
   - *There are no zero objects since* $\emptyset$ *is the only initial object in these categories.*

3. *In the* ***Ring*** *category,* $\mathbb{Z}$ *is an initial object since all arrows must send 0 and 1 to the new ring's neutral elements, and* $\{0\}$ *is terminal. However, in* ***Rng*** *we have that* $\{0\}$ *is a zero object.*

Exercises

Exercise 1.2.1. *If we have two initial objects from the same category, then they're isomorphic.*

Exercise 1.2.2. *Let* X_1 *be an initial object, and let* $X_2 \cong X_1$*. Then* X_2 *is initial.*

Analogous results are valid for terminal objects.

Exercise 1.2.3. *If* $|X| \neq 1$ *then* X *is not terminal.*

Exercise 1.2.4. *Let's consider category* $\mathscr{B}$ *that has objects* A, B, C, D, F *and arrows* $\alpha, \beta, \gamma, \delta, \sigma, \beta\alpha = \sigma\delta$ *and* $\sigma\tau$*. What are the initial and terminal objects?*

$$
\begin{tikzcd}
 & B \arrow[dr, "\beta"] & \\
A \arrow[ur, "\alpha"] \arrow[dr, "\delta"'] & & D \\
 & C \arrow[ur, "\sigma"'] & \\
 & & F \arrow[ul, "\tau"]
\end{tikzcd}
$$

1.3 Monomorphisms, epimorphisms and isomorphisms

Monos & epis

Definition 1.3.1 (Monomorphism)**.** Let $\mathscr{C}$ be a category, and $X, Y, Z \in \mathscr{C}$.

We say that an arrow $f : X \longrightarrow Y$ is a **monomorphism** if for any pair of arrows $g, g' : Z \longrightarrow X$ holding the equality $f \circ g = f \circ g'$, they satisfy $g = g'$.

Definition 1.3.2 (Epimorphism)**.** Let $\mathscr{C}$ be a category, and $X, Y, Z \in \mathscr{C}$.

We say that an arrow $f : X \longrightarrow Y$ is a **epimorphism** if for any pair of arrows $h, h' : Y \longrightarrow Z$ holding the equality $h \circ f = h' \circ f$, they satisfy $h = h'$.

Example. *Some examples of monomorphisms and epimorphisms are:*

- *The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism in **Ring**.*
- *A morphism $f\colon X \to Y$ in **Haus** is an epimorphism if and only if $f(X)$ is dense in Y.*

Definition 1.3.3 (Split monomorphism and epimorphism)**.**

Let $\mathscr{C}$ be a category, $X, Y \in \mathscr{C}$ and $f \in Hom_{\mathscr{C}}(X, Y)$, $g \in Hom_{\mathscr{C}}(Y, X)$ such that

$$g \cdot f = id_X$$

Then f is called a **split monomorphism** and g is called **split epimorphism**.

Proposition 1.3.1. *Split monomorphisms and epimorphisms are respectively monomorphisms and epimorphisms.*

Proposition 1.3.2. *Let $f : X \longrightarrow Y$ be a monomorphism (or epimorphisim) in $\mathscr{C}$. Then, f is an epimorphism (or monomorphisim) in $\mathscr{C}^{op}$.*

Proposition 1.3.3 (Operations with monomorphism). *(epimorphism by duality):*

- *Composition of monomorphisms is monomorphisms.*
- *If $g \circ f$ is a monomorphism, then f is a monomorphism.*

Proposition 1.3.4 (Subcategories reflect monomorphism).
Let $\mathscr{A} \hookrightarrow \mathscr{B}$ be a subcategory, and let f be an arrow in $\mathscr{A}$. If f is a monomorphism in $\mathscr{B}$, then f is a monomorphism in $\mathscr{A}$.

Proposition 1.3.5 (Special arrows from special objects).
If I is initial, any arrow $X \to I$ is an epimorphism

Remark. *It's important to separate the usual definitions for monomorphism and epimorphisms used in representable categories from the more general, categorical one:*

- *Being mono doesn't always imply being injective, just like being epi doesn't always imply being surjective. The first case is more difficult to see since in general in categories made from algebraic structures we do have that mono implies injectivity, but a counterexample for epimorphisms is the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in **Ring**.*
- *Being a split monomorphism implies being an injection in **Sets** but the other way around is not true.*

Exercises

Exercise 1.3.1. *Prove proposition 1.3.1*

Exercise 1.3.2. *The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in **Ring**.*

Exercise 1.3.3. *Prove proposition 1.3.5*

Exercise 1.3.4. *Split monomorphism implies injectivity in **Sets***

Isomorphisms

Definition 1.3.4 (Isomorphism). Let $\mathscr{C}$ be a category, and $X, Y \in \mathscr{C}$.

An arrow $f : X \longrightarrow Y$ is said to be an **isomorphisim** if there exists another arrow $f^{-1} : Y \longrightarrow X$ satisfying that $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$.

Example. *Some examples of isomorphisms are:*

1. *A mapping $f : X \to Y$ is an isomorphism in **Set** if and only if it is a bijection.*
2. *A group homomorphism ϕ is an isomorphism in **Group** if and only if it is bijective.*
3. *The isomorphisms in **Top** are the homeomorphisms.*
4. *The only isomorphisms in $(\mathbb{N}, \leq)$ are the identities.*
5. *A morphism $X \to Y$ in $\mathscr{C}$ is the same as an isomorphism $Y \to X$ in $\mathscr{C}^{op}$.*
6. *Given an object $X \in \mathscr{C}$, an **automorphism** of X is just an isomorphism $X \to X$. This gives rise to an obvious automorphism subcategory*

$$Aut(X) \hookrightarrow End(X) \hookrightarrow \mathscr{C}.$$

Proposition 1.3.6 (Basic properties of isomorphisms)**.**

1. *Inverses of isomorphisms are unique.*
2. *Any composition of isomorphisms is an isomorphism.*
3. *If the composition of two arrows is an isomorphism and one of them is an isomorphism, then so is the other.*

Proposition 1.3.7 (Isomorphisms and subcategories)**.**

Let $\mathscr{A} \hookrightarrow \mathscr{B}$ be a subcategory, and let f be an arrow in $\mathscr{A}$.

1. *If f is an isomorphism in $\mathscr{A}$, then f is an isomorphism in $\mathscr{B}$*
2. *Full subcategories reflect isomorphisms:*
   If $\mathscr{A}$ is a full subcategory and f is an isomorphism in $\mathscr{B}$, then f is an isomorphism in $\mathscr{A}$.

(Obs: the fullness condition above cannot be dropped).

Proposition 1.3.8 (Characterization of isomorphisms)**.**

A morphism is an isomorphism if and only if it is a split monomorphism and a split epimorphism.

Remark. *The condition of being split is necessary. We'll give some examples.*

1. Considerate a category $\mathscr{C}$ given by

   - $Ob(C) = \{a, b\}$
   - Appart from the identitty arrows, there only exists an arror $f : a \longrightarrow b$.

   $$\begin{tikzcd} a \arrow[loop left, "id_a"] \arrow[r, mapsto, "f"] & b \arrow[loop right, "id_b"] \end{tikzcd}$$

   It is clear that f is a monomorphism and an epimorphisim, but can't be an isomorphism because it doesn't exist an arrow $g : a \longrightarrow b$.

2. In ***Top*** category, we have that

   - An arrow is a monomorphism if and only if it is injective.
   - An arrow is an epimorphism if and only if it is surjective.

   In conclusion, every bijective arrow in ***Top*** is a mono and an epimorphism but it is only an isomorphisms if its inverse is contonuous.

Chapter 2

Functors

Definition 2.0.1 (Functor)**.** A functor F from category $\mathscr{C}$ to $\mathscr{D}$, written $F : \mathscr{C} \to \mathscr{D}$ is

- An arrow between the objects of each category

$$F : Obj_{\mathscr{C}} \to Obj_{\mathscr{D}}$$

- And, for each $X, Y \in \mathscr{C}$ another arrow

$$F : Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$$

  such that:

  1. $F(id_X) = id_{F(X)}, \forall\ X \in \mathscr{C}$
  2. $F(g \circ f) = F(g) \cdot F(f)$ for any pair of composable arrows f, g.

Basically, a functor consists of a mapping on objects and a mapping on arrows that preserves all of the structure of a category, namely domains and codomains, composition, and identities. It can be represented by the next diagram:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] & \mathscr{D} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & FX \arrow[d, "Ff"] \\
Y \arrow[r, mapsto] & FY
\end{tikzcd}$$

Example.

1. *The classical, most simple examples is the* ***identity functor*** $\mathscr{C} \to \mathscr{C}$ *defined in the obvious way.*

2. *The other classical functor is the* ***inclusion functor****.*

   Let $\mathscr{C}$ *be a category and* $\mathscr{D} \subseteq \mathscr{C}$ *a subcategory. We define the inclusion functor*

   $$F : \mathscr{D} \hookrightarrow \mathscr{C}$$

   such that

   $$FX = X \qquad \forall\, X \in \mathscr{D}$$
   $$Ff = f \qquad \forall\, f \in Hom_{\mathscr{D}}(X, Y)$$

   It's trivial to see that it satisfies the axioms of functors.

   $$\begin{tikzcd} \mathscr{D} \arrow[r, hook, "F"] & \mathscr{C} \\ X \arrow[r, mapsto] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, mapsto] & Y \end{tikzcd}$$

3. *The* ***forgetful functor*** U *can be defined for the "classical" categories defined with a set and some underlying structure like* $\mathbf{Top}, \mathbf{Group}, \mathbf{Rng}$ *or* $\mathbf{Ring}$*. The functor makes the category "forget" it's underlying structure and stick with the original set and its arrows as normal maps. For example, for the category of topological spaces we can consider* $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$*.*

4. *We can define an endofunctor in the* ***Set*** *category*

   $$\mathcal{P} : \boldsymbol{Set} \longrightarrow \mathbf{Set}$$

   which sends a set A *to its power set* $\mathcal{P}(A)$*, and given a function* $f : A \longrightarrow B$*, it sends* f *to a function between theirs respectives powers sets,*
   $\mathcal{P}f : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$ *satisfying*

   $$\mathcal{P}(f)(A') = f(A') \in \mathcal{P}(B) \qquad \forall\, A' \subseteq A$$

5. *We'll introduce the dotted topology category,* $\mathbf{Top}_*$*, wich consists on*

   - $Ob(\mathbf{Top}_*) = \{(X, x) : X \text{ is a topological space}, x \in X\}$
   - $Hom_{\mathbf{Top}_*}((X, x), (Y, y)) = \{f : X \longrightarrow Y : \ f \text{ is continuous}, \ f(x) = y\}$

   Then, the fundamental group defines a functor

   $$\pi_1 : Top_* \longrightarrow Group$$

   which carries continuous functions $f : (X, x) \longrightarrow (Y, y)$ *to group homomorphims* $f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, y)$.

Definition 2.0.2. Let $\mathscr{C}, \mathscr{D}, \mathscr{E}$ categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$, $G : \mathscr{D} \longrightarrow E$. We define the functor composition of G and F as

$$G \circ F : \mathscr{C} \longrightarrow \mathscr{E}$$

which acts like we show in the following diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[rr, bend left, "G \circ F"] & \mathscr{D} \arrow[r, "G"] & \mathscr{E} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & FX \arrow[r, mapsto] \arrow[d, "Ff"] & GFX \arrow[d, "GFf"] \\
Y \arrow[r, mapsto] & FY \arrow[r, mapsto] & GFY
\end{tikzcd}$$

Definition 2.0.3 (Isomorphic categories)**.**

Let $\mathscr{C}, \mathscr{D}$ be categories. We say that $\mathscr{C}, \mathscr{D}$ are isomorphic categories, $\mathscr{C} \cong \mathscr{D}$ if there exists functors

$$F : \mathscr{C} \longrightarrow \mathscr{D}$$
$$G : \mathscr{D} \longrightarrow \mathscr{C}$$

mutually inverse to each other. Then, we say that F and G are ***isomorphisms of categories***.

Exercise 2.0.1. *Let* G, H *be groups. The group isomorphisms between* G *and* H *induce categorical isomorphisms between* BG *and* BH.

Lemma 2.0.1. *Functors preserve isomorphisms.*

Proof. Let $\mathscr{C}, \mathscr{D}$ categories, $F : \mathscr{C} \longrightarrow \mathscr{D}$ a functor, $X \in \mathscr{C}, Y \in \mathscr{D}$ and $f : X \longrightarrow Y$ an isomorphism with inverse $g : Y \longrightarrow X$.

Applying the axioms which define functors and that $g \cdot f = id_X$ we have that

$$Fg \cdot Ff = F(g \cdot f) = F(id_X) = id_X$$

which means that Ff is the left inverse of Fg.

Arguing by duality, we prove that Ff is the right inverse of Fg, and, as a result, Ff and Fg are inverse isomorphisms. □

Proposition 2.0.2. *Let $F : \mathscr{A} \to \mathscr{B}$ be a category isomorphism. Then*

1. *$f \in \mathscr{A}$ is a monomorphism (and respectively and epimorphism) if and only if $Ff \in \mathscr{B}$ is an monomorphism (and respectively and epimorphism).*

2. *An object $I \in \mathscr{A}$ is initial if and only if $FI \in \mathscr{B}$ is initial. The same holds for terminal and zero objects.*

Also, a functor $F : \mathscr{A} \to \mathscr{C}$ is an isomorphism if and only if $F^{op} : \mathscr{A}^{op} \to \mathscr{C}^{op}$ is ana isomorphism.

Proof. We will only prove the first part of the proposition.
Assume $f \in \mathscr{A}$ is a monomorphism. Assume there exists $g, g' \in \mathscr{B}$ such that $Ff \circ g = Ff \circ g'$. Then since F is a category isomorphism we have that

$$F^{-1} \circ (Ff \circ g) = (F^{-1}Ff) \circ (F^{-1}g) = (F^{-1} \circ F)f \circ F^{-1}(g) = f \circ F^{-1}(g)$$

and also

$$F^{-1} \circ (Ff \circ g') = (F^{-1}Ff) \circ (F^{-1}g') = (F^{-1} \circ F)f \circ F^{-1}(g') = f \circ F^{-1}(g')$$

Since f monomorphism we can deduce $F^{-1}(g) = F^{-1}(g')$ and from F being a category isomorphism we get that $g = g'$.
In the other direction, if we assume that Ff is a monomorphism and we assume that $\exists g, g' \in \mathscr{A}$ such that $f \circ g = f \circ g'$ then

$$F(f \circ g) = Ff \circ Fg = Ff \circ Fg' = F(f \circ g')$$

and so we can deduce $Fg = Fg'$ from Ff being mono and from F being a category isomorphism we have that $g = F^{-1}Fg = F^{-1}Fg' = g'$. □

2.1 Contravariant Functors

Definition 2.1.1 (Contravariant Functor).

Given two categories $\mathscr{C}, \mathscr{D}$, a **contravariant functor** between them is a functor $F : \mathscr{C}^{op} \longrightarrow \mathscr{D}$. By definition, this functor:

- Associates every object $X \in \mathscr{C}$ to an object $FX \in \mathscr{D}$.
- Associates any arrow $f^{op} : X \longrightarrow Y$ to an arrow $Ff^{op} : FX \longrightarrow FY$, where $X, Y \in \mathscr{C}$ in such a way that:
  - For every composable pair of arrows f^{op}, g^{op}
  $$Ff^{op} \circ Fg^{op} = F(f^{op} \circ g^{op})$$
  - $\forall\ X \in \mathscr{C}$
  $$Fid_X = 1_{FX}$$

However, this functors can be thought of in terms of the original category. If we write $\bar{F}f$ for Ff^{op}, then we can see the functor F as a functor $\bar{F}$ such that:

- Associates every object $X \in \mathscr{C}$ to an object $\bar{F}X \in \mathscr{D}$.
- Associates any arrow $f : X \longrightarrow Y$ to an arrow $\bar{F}f : \bar{F}Y \longrightarrow \bar{F}X$, where $X, Y \in \mathscr{C}$ in such a way that:
  - For every composable pair of arrows f, g
  $$\bar{F}f \circ \bar{F}g = \bar{F}(g \circ f)$$
  - $\forall\ X \in \mathscr{C}$
  $$\bar{F}id_X = 1_{\bar{F}X}$$

This is why contravariant functor are usually drawn with a diagram like the following one:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] & \mathscr{D} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & FX \\
Y \arrow[r, mapsto] & FY \arrow[u, "Ff"']
\end{tikzcd}$$

Example. *This are a few of the most typical examples:*

1. *A typical example is the* ***contravariant power functor*** P : ***Set*** $\to$ ***Set*** *(or more correctly the* ***Set***^{op} $\to$ ***Set*** *power functor) that sends any set to it's power set* $P(X)$ *and any map between sets* $f : X \to Y$ *to the arrow* Ff *that sends any subset* $U \subset Y$ *to it's inverse image* $f^{-1}(U) \subset X$.

2. *If* ***Vect*** *are the finite dimension vector spaces over field* $\mathbb{K}$*, then the* ***dual functor*** *is a contravariant functor that associates every vector space* V *to its dual space* $V* = Hom(V, \mathbb{K})$ *and every linear map* $f : V \to W$ *to the map* Ff *such that for every* $\alpha \in W* = Hom(W, \mathbb{K})$ *then it pre-composes f to have a map from* V *to* $\mathbb{K}$ *(it associates* α *to* $\alpha \circ f$*).*

Remark. *It's important to be careful when we talk about contravariant functors. In general, we can think of a contravariant functor as a functor from a category* $\mathscr{A}$ *to a category* $\mathscr{B}$ *that just changes the arrows' directions. That's an easier way to visualise how the functor is working, but we need to take into account that all the theory that we've previously proved for "normal" (covariant) functors doesn't work for contravariant functors defined this way, so to apply all this theory we need to think of the contravariant functors just as a normal functor from* $\mathscr{A}^{op}$ *to* $\mathscr{B}$*.*

The Hom Functor

There's a very important example of covariant and contravariant functors provided by the **Hom** sets.

Definition 2.1.2 (Covariant Hom Functor)**.** Let $\mathscr{C}$ be a locally small category. Then, for each $X \in \mathscr{C}$ we can define a functor $Hom(X, \_)$ that sends

- Each object $Y \in \mathscr{C}$ to the object $Hom(X, Y) \in$ **Set**
- Each arrow $Y \to Z$ in $\mathscr{C}$ to an arrow $Hom(X, f)$ from $Hom(X, Y)$ to $Hom(X, Z)$ that sends each $\alpha \in Hom(X, Y)$ to $f \circ \alpha$

$$\begin{tikzcd}
Hom(X, _) : \mathscr{C} \arrow[r] & Set \\
Y \arrow[r, mapsto] \arrow[d, "f"'] & Hom(X, Y) \arrow[d, "f \circ _"] \\
{} \arrow[r, mapsto] & {} \\
Z \arrow[r, mapsto] & Hom(X, Z)
\end{tikzcd}$$

Similarly we can define the contravariant version of this functor:

Definition 2.1.3 (Contravariant Hom Functor)**.** Let $\mathscr{C}$ be a locally small category. Then, for each $X \in \mathscr{C}$ we can define a functor $Hom(\_, X)$ that sends

- Each object $Y \in \mathscr{C}$ to the object $Hom(Y, X) \in \mathbf{Set}$
- Each arrow $Y \to Z$ in $\mathscr{C}$ to an arrow $Hom(f, X)$ from $Hom(Z, X)$ to $Hom(Y, X)$ that sends each $\alpha \in Hom(Z, X)$ to $\alpha \circ f \in Hom(Y, X)$

$$\begin{tikzcd}
Hom(_, X) : \mathscr{C} \arrow[r] & Set \\
Y \arrow[r, mapsto] \arrow[d, "f"'] \arrow[dr, phantom, "\longmapsto"] & Hom(Y, X) \\
Z \arrow[r, mapsto] & Hom(Z, X) \arrow[u, "_ \circ f"']
\end{tikzcd}$$

We can sum up the information of the two previous functors defining a bifunctor, which is a functor of two variables, whose domain is a product of categories. For that we need the following definition:

Definition 2.1.4 (Product of Categories)**.** For any categories $\mathscr{C}$ and $\mathscr{D}$, there is a category $\mathscr{C} \times \mathscr{D}$, their *product*, whose

- objects are ordered pairs (X, Y), where X is an object of $\mathscr{C}$ and Y is an object of $\mathscr{D}$,
- morphisms are ordered pairs $(f, g) : (X, Y) \to (X', Y')$, where $f : X \to X' \in \mathscr{C}$ and $g : Y \to Y' \in \mathscr{D}$,

in which composition and identities are defined componentwise.

Definition 2.1.5. If $\mathscr{C}$ is locally small, then we can define:

$$\mathscr{C}(-, -) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$$

defined in the evident manner. A pair of objects (X, Y) is mapped to the hom-set $Hom_{\mathscr{C}}(X, Y)$. A pair of morphisms $f : W \to X$ and $h : Y \to Z$ is sent to the function

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \xrightarrow{(\_\circ f, h \circ \_)} Hom_{\mathscr{C}}(W, Z)$$

$$g \mapsto hgf$$

that takes an arrow $g : X \to Y$ and then pre-composes with f and post-composes with h to obtain $hgf : W \to Z$.

We will see the importance of these functors in later chapters.

Exercises

Exercise 2.1.1. *If $F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{A}$ functors, prove that $G \circ F$ is a functor.*

Exercise 2.1.2. *If X, Y are isomorphic in $\mathscr{C}$, then given a functor $F : \mathscr{C} \to \mathscr{D}$ prove that F(X), F(Y) are isomorphic in $\mathscr{D}$.*

2.2 Natural transformations

Definition 2.2.1. Given two functors $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ a **natural transformation** between them (typically written $\alpha : F \longrightarrow G$) consisits of a family of morphisms in $\mathscr{D}$

$$(\alpha_X)_{X \in \mathscr{C}} \text{ where each } \alpha_X : FX \to GX$$

such that for any morphism $f : X \to Y$ in $\mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \\
FY \arrow[r, "\alpha_Y"'] & GY \arrow[u, "Gf"']
\end{tikzcd}$$

Each α_X is called a component of the natural transformation.

When the definition holds, it's usual to say that the functors are related in a natural manner, just like if a family of arrows define the components of a natural transformation we say that they are natural. Sometimes it's typical to display natural transformation with a diagram like the following one:

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathscr{D} \end{tikzcd}$$

Definition 2.2.2. Let $\mathscr{C}, \mathscr{D}$ be categories, $F, G, H : \mathscr{C} \longrightarrow \mathscr{D}$ be functors and $\alpha : F \longrightarrow G, \beta : G \longrightarrow H$ natural transformations. We define the vertical composition between β and α , $\beta \circ \alpha : F \longrightarrow H$, as the following family of morphisms in $\mathcal{D}$

$$(\beta \circ \alpha_X)_{X \in \mathscr{C}} = (\beta_X \circ \alpha_X)_{X \in \mathscr{C}}$$

where

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[rr, bend left, "(\beta \circ \alpha)_X"] \arrow[d, "Ff"'] & GX \arrow[r, "\beta_X"] \arrow[d, "Gf"'] & HX \arrow[d, "Hf"'] \\ FY \arrow[r] & GY \arrow[r] & HY \end{tikzcd}$$

By how we have define the vertical composition, it is clear that given an arrow $f : X \to Y$ in $\mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd} FX \arrow[r, "(\beta \circ \alpha)_X"] \arrow[d, "Ff"'] & HX \\ FY \arrow[r, "(\beta \circ \alpha)_Y"'] & HY \arrow[u, "Gf"'] \end{tikzcd}$$

so it follows that the vertical composition of natural transformations is a natural transformation.

We can also represent the vertical composition with the following diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G}] \arrow[r, bend right=60, "H"'{name=H}] & \mathscr{C} \arrow[Rightarrow, from=F, to=G, "\alpha"'] \arrow[Rightarrow, from=G, to=H, "\beta"'] \end{tikzcd}$$

Definition 2.2.3. A **natural isomorphism** is a natural transformation $\alpha : F \longrightarrow G$ in which every component α_X is an isomorphism. This is exactly the same as saying that there exists a natural transformation $\beta : G \longrightarrow F$ such that

$$(\beta \circ \alpha)_X = id_{FX} \qquad \forall\, X \in \mathscr{C}$$

$$(\alpha \circ \beta)_X = id_{GX} \qquad \forall\, X \in \mathscr{C}$$

Example. *Here are a few examples of natural transformations:*

1. *("Trivial" example)*
   Let's assume we have $\mathscr{C}$ the trivial category (one object, one arrow) and a category $\mathscr{D}$ with two objects a, b and one arrow between them. We consider the following functors:

$$\begin{array}{rcl} F: \mathscr{C} & \longrightarrow & \mathscr{D} \\ \{\cdot\} & \longmapsto & a \\ 1. & \longmapsto & 1_a \end{array} \qquad \begin{array}{rcl} G: \mathscr{C} & \longrightarrow & \mathscr{D} \\ \{\cdot\} & \longmapsto & b \\ 1. & \longmapsto & 1_b \end{array}$$

   Since there's only one possible arrow in $\mathscr{D}$ we fave that the only natural transformation that we can define is the unique arrow between a and b.

2. *Let's consider the following two functors:*

$$\begin{tikzcd} F: Set \arrow[rr] & & Set \\ X \arrow[rr, mapsto] \arrow[d] & & ((X \times X) \times X) \arrow[d] \\ & f \arrow[r, mapsto] & Ff \\ Y \arrow[rr, mapsto] & & ((Y \times Y) \times Y) \end{tikzcd} \qquad \begin{tikzcd} G: Set \arrow[rr] & & Set \\ X \arrow[rr, mapsto] \arrow[d] & & X^3 \arrow[d] \\ & f \arrow[r, mapsto] & Gf \\ Y \arrow[rr, mapsto] & & Y^3 \end{tikzcd}$$

where $Ff((x_1, x_2), x_3) = ((f(x_1), f(x_2)), f(x_3)$ *and* $Gf(x_1, x_2, x_3) = (f(x_1), f(x_2), f(x_3))$. *For each set we define*

$$\alpha_X(((x_1, x_2), x_3)) = (x_1, x_2, x_3)$$

Exercise 2.2.1. $(\alpha_X)_{X \in Sets}$ *is a natural transformation*

3. *The determinant is a natural transformation. Let's consider the following functors:*

$$\begin{tikzcd} F:\ CRing \arrow[rr] & & Grp \\ A \arrow[rr, mapsto] \arrow[d, "f"'] & & GL_n(A) \arrow[d, "GL_n(f)"] \\ {} \arrow[r, phantom, ""] & \arrow[r, mapsto] & {} \\ R \arrow[rr, mapsto] & & GL_n(R) \end{tikzcd} \qquad \begin{tikzcd} G:\ CRing \arrow[rr] & & Grp \\ A \arrow[rr, mapsto] \arrow[d, "f"'] & & A^* \arrow[d, "f^*"] \\ {} \arrow[r, phantom, ""] & \arrow[r, mapsto] & {} \\ R \arrow[rr, mapsto] & & R^* \end{tikzcd}$$

Here $GL_n(f)$ *is obtained by applying* f *to each matrix entry, and* f^* *is the restriction of* f *to the units.*
Note that that if M *is an* $n \times n$ *matrix with entries in the commutative ring* K*, the determinant of the matrix will be a unit if* M *is non-singular. Therefore* $f : GL_n(K) \to K^*$ *will be a group morphism, and since it's defined by the same formula for all rings we have that* $f^* \circ det_A = det_R \circ GL_n(f)$.

4. *The pair of functors* $Hom(A, \_), Hom(\_, B)$ *are related in a natural manner.*

Exercises

Exercise 2.2.2. *Suppose* $\alpha : F \Rightarrow G$ *is a natural isomorphism. Show that the inverses of the components define the components of a natural isomorphism* $\alpha^{-1} : G \Rightarrow F$

Exercise 2.2.3. *Let* $T \in \mathscr{D}$ *be a terminal object and* $F : \mathscr{C} \to \mathscr{D}$ *a functor. Let's define a functor* $\boldsymbol{Const_T} : \mathscr{C} \to \mathscr{D}$ *that sends every object in* $\mathscr{C}$ *to* T *and every map to the identity in* T*. Give a natural transformation between the functors.*

Exercise 2.2.4. *What is a natural transformation between two groups seen as one-object categories? (Between BG, BH for G, H groups)*

2.3 Equivalence of categories

Natural isomorphisms allow us to define the notion of *equivalence of categories* which generalizes the idea of homotopy equivalence.

Definition 2.3.1. An **equivalence of categories** is a pair of functors $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ together with natural isomorphisms $\eta : 1_{\mathscr{C}} \cong GF$ and $\xi : FG \cong 1_{\mathscr{D}}$.

We will say that $\mathscr{C}, \mathscr{D}$ are equivalent and write $\mathscr{C} \simeq \mathscr{D}$ if there exists an equivalence of categories between them.

Lemma 2.3.1. *The equivalence of categories is a binary equivalence relation.*

Example. *Consider the categories* $\mathbf{Mat}_{\mathbb{K}}, \mathbf{Vect}_{\mathbb{K}}{}^{fd}$. *We define the category* $\mathbf{Vect}_{\mathbb{K}}^{\mathbf{basis}}$ *as the category in which the objects are finite-dimensional vector spaces over* $\mathbb{K}$ *with a given basis and the arrows are just linear maps.*

Consider now

$$U : \mathbf{Vect}_{\mathbb{K}}^{\mathbf{basis}} \longrightarrow \mathbf{Vect}_{\mathbb{K}}^{\mathbf{fd}}$$

the forgetful functor and

$$\mathbb{K}^{(-)} : \mathbf{Mat}_{\mathbb{K}} \longrightarrow \mathbf{Vect}_{\mathbb{K}}^{\mathbf{basis}}$$

which sends the matrix of dimension $m \times n$ *to* $\mathbb{K}^n$ *with the standard basis*

$$\begin{tikzcd}
\mathbf{Vect}_{\mathbb{K}}^{basis} \arrow[r, "H"] & \mathbf{Mat}_{\mathbb{K}} \\
(V,\ \{e_i\}_{i=1}^{n}) \arrow[r, mapsto] \arrow[d, "f"'] & \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \arrow[d, "Ff"] \\
\big(W,\ \{w_j\}_{j=1}^{m}\big) \arrow[r, mapsto] & \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Vect}_{\mathbb{K}}^{fd} \arrow[r, "C"] & \mathbf{Vect}_{\mathbb{K}}^{basis} \\
V \arrow[r, mapsto] \arrow[d, "f"'] & (V,\ \{e_i\}_{i=1}^{n}) \arrow[d, "Ff"] \\
W \arrow[r, mapsto] & \big(W,\ \{w_j\}_{j=1}^{m}\big)
\end{tikzcd}$$

We note that

$$\mathbf{Vect}_{\mathbb{K}}^{\mathbf{basis}} \simeq \mathbf{Vect}_{\mathbb{K}}^{\mathbf{fd}}$$
$$\mathbf{Vect}_{\mathbb{K}}^{\mathbf{basis}} \simeq \mathbf{Mat}_{\mathbb{K}}$$

so, by transitivity

$$\mathbf{Vect}_{\mathbb{K}}^{\mathbf{fd}} \simeq \mathbf{Mat}_{\mathbb{K}}$$

Definition 2.3.2. Let $\mathscr{C}, \mathscr{D}$ be ccategories and let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor.

- We say that F is full if $\forall\ X, Y \in \mathscr{C}$, the map

$$Hom_{\mathscr{C}}(X, Y) \longrightarrow Hom_{\mathscr{D}}(FX, FY)$$

  is surjective.

- We say that F is faithful if $\forall\ X, Y \in \mathscr{C}$, the map

$$Hom_{\mathscr{C}}(X, Y) \longrightarrow Hom_{\mathscr{D}}(FX, FY)$$

  is injective.

- We say that F is esentially surjective on objects if $\forall\ Y \in \mathscr{D}$, $\exists\ X \in \mathscr{C}$ such that $FX \cong Y$.

Theorem 2.3.2 (Characterization of equivalence of categories)**.**

A functor which defines an equivalence of categories is full, faithful and esentially surjective on objects.

Asuming the axiom of choice, any functor which is full, faithful and esentially surjective on objects defines an equivalence of categories.

Definition 2.3.3. We say that a category $\mathscr{C}$ is skeletal if it contains just one object in each isomorphism class.

Given a category $\mathscr{C}$, we dfine the skeleton $sk\mathscr{C}$ as the unique, up to isomorphims, skeletal category that is equivalent to $\mathscr{C}$.

Chapter 3

Limits

3.1 Products and Coproducts

Product of two objects

Definition 3.1.1. Let $\mathscr{C}$ be a category. Let $X, Y \in \mathscr{C}$. Then we define the **product of the two objects** X, Y as an object $P \in \mathscr{C}$ together with a pair of maps $\pi_X : P \to X, \pi_Y : P \to Y$ such that for any $Z \in \mathscr{C}$ with maps $f_X : Z \to X, f_Y : Z \to Y$ then there exists a unique map $f : Z \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\
& P \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \\
& Y &
\end{tikzcd}$$

Remark. *The product doesn't always exist.*

Definition 3.1.2. Let $\mathscr{C}$ be a category. Let $X, Y \in \mathscr{C}$. Then we define the **coproduct of the two objects** X, Y as an object $X \sqcup Y \in \mathscr{C}$ together with a pair of maps $\phi_X : X \to X \sqcup Y, \phi_Y : Y \to X \sqcup Y$ such that for any $Z \in \mathscr{C}$ with maps $\psi_X : X \to Z, \psi_Y : Y \to Z$ then there exists a unique map $f : X \sqcup Y \to Z$ such that the following diagram commutes:

$$
\begin{tikzcd}
Z & & \\
& X \sqcup Y \arrow[ul, dashed, "f"'] & X \arrow[l, "\phi_X"] \arrow[ull, bend right, "\psi_X"'] \\
& Y \arrow[u, "\phi_Y"'] \arrow[uul, bend left, "\psi_Y"] &
\end{tikzcd}
$$

Remark. *The coproduct doesn't always exist.*

Example. *Some products and coproducts in a finite category:*

$$
\begin{tikzcd}
& B \arrow[dr, "\beta"] & \\
A \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & D \\
& C \arrow[ur, "\delta"'] \arrow[dl, "\tau"] & \\
E & &
\end{tikzcd}
$$

$$Coprod(A, B) = \ (B,\ \{\alpha,\ id_B\})$$

$$Coprod(A, F) = \ (F,\ \{\tau\gamma,\ id_F\})$$

$$Prod(B,\ C) = \ (A,\ \{\alpha,\ \gamma\})$$

$$Prod(A,\ F) = \ (A,\ \{id_A,\ \tau\gamma\})$$

Example. *The product for a pair of objects $X, Y \in$ **Sets** is the cartesian product $X \times Y$. The coproduct is the disjoint union $X \sqcup Y$. Proof seen at class.*

Example. *The product and coproduct of $V, W \in Vect_{\mathbb{K}}$ is the direct sum of the vector spaces $V \oplus W = V \times W$. Proof seen at class.*

Exercises

Exercise 3.1.1. $X \times Y \cong Y \times X$

Exercise 3.1.2. $(X \times Y) \times Z \cong X \times (Y \times Z)$

Product of a family of objects

By induction, using the previous exercises we can give the following definitions:

Definition 3.1.3. Let $\{X_i\}_{i\in\mathcal{I}}$, where $\mathcal{I}$ is finite, a family of objects in a category $\mathscr{C}$. Then we define the product of the family $\{X_i\}_{i\in\mathcal{I}}$ as a pair $(X, (\varphi_i)_{i\in\mathcal{I}})$ where $X \in \mathscr{C}$, $\varphi_i : X \to X_i, \forall i \in \mathcal{I}$ such that $\forall (Y, (\psi_i)_{i\in\mathcal{I}})$ such that $\psi_i : Y \to X_i$ then there exists a unique map $f : Y \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[d, dashed, "f"] \arrow[rrd, "\psi_i"] & & \\
X \arrow[rr, "\varphi_i"'] & & X_i
\end{tikzcd}$$

$$\forall\ i\ \in\ \mathcal{I}$$

Remark. *The definition of the coproduct is analogous, just like for two objetcs.*

Proposition 3.1.1. *The product is unique up to isomorphisms.*

Proof. Done in class. □

Now to introduce the next definition we will first think of the product in a new category called *the Comma Category.*

Example (Comma category)**.** *We consider, given a category $\mathscr{C}$, the Comma category for an object $Z \in \mathscr{C}$ denoted by $\mathscr{C} \downarrow Z$. In this category the objects are the arrows from any object of $\mathscr{C}$ to Z (if $X \to Z$ is an object, we will call it ${}^{X}/_{Z}$). The arrows bewteen two objects are*

$$Hom({}^{X}/_{Z}, {}^{Y}/_{Z}) = \{f:\ X \to Y /\ \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] \arrow[rr, phantom, shift right=4, "\circlearrowleft"] & & Y \arrow[dl] \\ & Z & \end{tikzcd}\ \}$$

For the product of ${}^{X}/_{Z}$ and ${}^{Y}/_{Z}$, if $f_X : X \to X$ and $f_Y : Y \to Z$ we consider $X \times_Z Y = \{(x, y) \in X \times Y / f_X(x) = f_y(y)\}$ and the object ${}^{X \times_Z Y}/_{Z}$. It can be checked that this object will be the product that will make the following diagram commute:

$$\begin{tikzcd}
X\ \times_Z\ Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd}$$

And for any other object $(P, (p_X, p_Y))$ such that a similar diagram commutes we'll have that there exists a unique arrow $f : P \to X \times_Z Y$ such that

$$
\begin{tikzcd}
P \arrow[dr, dashed, "f"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & Z
\end{tikzcd}
$$

Thanks to this example we now define the fiber product.

Definition 3.1.4. A fiber product in a category $\mathscr{C}$ of morphisms f_1, f_2 is an object $P \in \mathscr{C}$ and two arrows π_1, π_2 such that

$$
\begin{tikzcd}
P \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd}
$$

and for any other object and pair of morphisms such that a similar diagram commutes, then

$$
\begin{tikzcd}
W \arrow[dr, dashed, "f"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& P \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & Y \arrow[d] \\
& X \arrow[r] & Z
\end{tikzcd}
$$

3.2 Equalizers and Co-equalizers

Definition 3.2.1 (***Equalizer***).

Let $\mathscr{C}$ be a category, $X, Y \in \mathscr{C}$ and $f, g \in Hom_{\mathscr{C}}(X, Y)$. The pair (E, e) where $E \in \mathscr{C}$ and $e \in Hom_{\mathscr{C}}(E, X)$ is an equalizer for the pair f, g if $f \circ e = g \circ e$ and, for all $k \in Hom_{\mathscr{C}}(K, X)$ such that $f \circ k = g \circ k$, there exists a unique arrow $u \in Hom_{\mathscr{C}}$ that makes the following diagram commute

$$\begin{tikzcd}
K \arrow[dr, "k"] \arrow[dd, dotted, "u"'] & & \\
& X \arrow[r, shift left, "f"] \arrow[r, shift right] & Y \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

Remark. *The equaliser of a pair of arrows does not always exist.*

Example. *Consider the category of* **Sets***. Let* $X, Y \in$ **Set** *and* $f \in Hom_{\mathbf{Set}}(X, Y)$*. We want to define an equaliser of the pair* f, g*. Consider*

$$S = \{x \in X : f(x) = g(x)\} \in \mathbf{Set}$$

and $i : S \hookrightarrow X$ *the inclusion. Let's prove that* (s, i) *is an equalizer of* f, g*. By definition of* S*,* $f \circ i = g \circ i$*.*

Now, let $K \in \mathbf{K}$ *and* $k \in Hom_{\mathbf{Set}}(K, X)$ *such that* $f \circ k = g \circ k$*. Then,* $\forall\, x \in K$

$$f(k(x)) = g(k(x)) \implies k(x) \in S$$

Consider

$$u : K \longrightarrow S$$
$$w \longmapsto k(w)$$

It is easy to see that $k = i \circ u$*, so the diagram commutes. By properties of the category of* **Sets***,* u *is the only arrow satysfying that propertie.*

Since the previous consturction is always available, **Set** *has all equalizers.*

Remark. *The previous constructioon is valid to additive groups, rings, modules and space vectores, where* $Ker\ f$ *with the inclusion defines an equalizer between* f *and the zero arrow.*

Theorem 3.2.1. *Let $\mathscr{C}$ be a category. Given a pair $f, g \in Hom_{\mathscr{C}}(X, Y)$ and two equalizers $(E, e), (E', e')$, there exists a unique isomorphism $v : E \longrightarrow E'$ such that the arrows commute. In other words, $e' \circ v = e$.*

Theorem 3.2.2. *If (E, e) is an equalizer, then e is a monomorphism. Furthermore, if e is epic, then it is an isomorphism.*

Definition 3.2.2 (***Co-qualizer***).

Let $\mathscr{C}$ be a category, $X, Y \in \mathscr{C}$ and $f, g \in Hom_{\mathscr{C}}(X, Y)$. The pair (C, c) where $C \in \mathscr{C}$ and $c \in Hom_{\mathscr{C}}(Y, C)$ is an equalizer for the pair f, g if $c \circ f = c \circ g$ and, for all $k \in Hom_{\mathscr{C}}(Y, K)$ such that $k \circ f = k \circ g$, there exists a unique arrow $u \in Hom_{\mathscr{C}}(K, C)$ that makes the following diagram commute

$$\begin{tikzcd} & & K \arrow[d, dotted, "u"] \\ X \arrow[r, shift left, "f"] \arrow[r, shift right] & Y \arrow[ur, "k"] \arrow[dr, "c"'] & \\ & & C \end{tikzcd}$$

Example. *Let $f, g : X \longrightarrow Y$ be a pair of parallel arrows arrows in the category of* **Sets***. Define the relation $y \sim y'$ if and only if there exists $x \in X$ such that $y = f(x)$, $y' = g(x)$. Consider now the closure of the relation $E_{f,g}$ and we will have a binary relation of equivalence. Hence, $(Y/E_{fg}, \pi)$, where π is the projection map of the relation, defines a co-equializer of f, g.*

- *Consider the category of* **Groups***. let $H, G \in$* **Groups** *and $\alpha, \beta \in Hom_{\{}\mathbf{Group}\}(H, G)$. Consider now K the smaller normal subgroup of G which contains all elements of the form $\alpha(h)\beta(h^{-1})$, where $h \in H$. Then, $(G/K, \pi)$, where π is the canonical epimorphism, defines a co-equalizer of $(G/K, \eta)$.*

Theorem 3.2.3. *Let $\mathscr{C}$ be a category. Given a pair $f, g \in Hom_{\mathscr{C}}(X, Y)$ and two co-equalizers $(C, c), (C', c')$. There exists a unique isomorphism $v : C \longrightarrow C'$ such that the co-equalizers commute. In other words, $c' = v \circ c$.*

Theorem 3.2.4. *If (C, c) is a co-equalizer, then c is an epimorphism. Furthermore, if c is mono, then it is an isomorphism.*

3.3 Limits and colimits

After seeing some particular cases like terminal objects, products and equalizers, we will now bring out what is common between them to define the notion of limits and colimits.

We will start by defining the notion of a cone, and later we will introduce the key notion of a limit cone. For defining a cone we will need to understand what a diagram is.

Definition 3.3.1 (Diagram)**.** Given a category J, a diagram $\mathcal{D}$ of type J in a category $\mathscr{C}$ is a functor

$$\mathcal{D} : J \to \mathscr{C}$$

We will call J the *index category* and the actual objects and morphisms of J will be irrelevant; we will only care about the way they are related. We can think of the diagram as indexing the objects in $\mathscr{C}$, copying the structure that objects have in J.

The intuitive notion of what a diagram is comes when J is small. Let's see some examples.

Example. *If J is the category of one object and one arrow, a diagram is simply an object in category $\mathscr{C}$. If J has two objects and only the identity maps then the diagram consists in two objects in $\mathscr{C}$.*

Example. *If we consider $J = \bullet \to \bullet$ the category with two objects and one arrow between them then the diagram will be two objects in $\mathscr{C}$ with an arrow bewteen them $X \to Y \in \mathscr{C}$.*

Example. *Given a functor $F : \mathscr{A} \to \mathscr{B}$ a diagram $\mathcal{D} : J \to \mathscr{A}$ results in a diagram $F\mathcal{D} : J \to \mathscr{B}$ defined in the expected way.*

Remark. *In this case is usual to think of J as indexes that $\mathcal{D}$ assigns to objects in $\mathscr{C}$. Therefore, from now on for any $j \in J$ we will denote $X_j = \mathcal{D}(j)$.*

Definition 3.3.2 (Constant functor)**.** Given $X \in \mathscr{C}$ e define the constant functor

$$ct_X : J \to \mathscr{C}$$

as the functor that sends all objects in J to X and all maps in J to id_X.

Definition 3.3.3. A **cone** c over diagram $\mathcal{D}$ with vertex $X \in \mathscr{C}$ is a natural transformation from $ct_X \Rightarrow \mathcal{D}$. More explicitly a cone consists in an object and a family of morphisms $(X, \{\lambda_j\}_{j\in J}), \lambda_j : X \to X_j$ (where the maps are usually called the legs of the cone) such that for any arrow $f : j \to k$ in J, we have the following commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ X_j \arrow[rr, "\mathcal{D}f"'] & & X_k \end{tikzcd}$$

Example. *We can picture a case like this. We consider a category with objects arranged like the left diagram, and a cone would be the object hovering above all of them with arrows going down to all of the objects (right diagram).*

$$\begin{tikzcd} & D_2 \arrow[dr, "h"] & & \\ D_1 \arrow[ur, "f"] \arrow[rr, "g"'] & & D_3 \arrow[r, shift left, "j"] & D_4 \arrow[l, shift left, "k"] \end{tikzcd} \qquad \begin{tikzcd} & & C \arrow[ddll, "\lambda_1"'] \arrow[d, "\lambda_2"] \arrow[dd, bend left, "\lambda_3"] \arrow[ddr, "\lambda_4"] & \\ & & D_2 \arrow[d, "h"'] & \\ D_1 \arrow[urr, "f"] \arrow[rr, "g"'] & & D_3 \arrow[r, shift left, "j"] & D_4 \arrow[l, shift left, "k"] \end{tikzcd}$$

Each arrow going down would commute with one of the arrows below and the correspondent down arrow to the codomain object.

Example. *If we consider category* $(\mathbb{R}, \leq)$ *and for* $S \subseteq \mathbb{R}$, *subcategory* $J = (S, \leq)$ *and the diagram given by the inclusion, then to give a cone over* $\mathcal{D}$ *with vertex* X *is equivalent to saying that* X *is a lower bound of* S.

On the other hand, if we consider two cones over diagram $\mathcal{D}, ct_X \Rightarrow \mathcal{D}, cte_Y \Rightarrow \mathcal{D}$, to give a natural transformation $ct_X \Rightarrow ct_Y$ from one cone to the other that makes the following diagram commute

$$\begin{tikzcd} ct_X \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & ct_Y \arrow[dl, Rightarrow] \\ & \mathcal{D} & \end{tikzcd}$$

is equivalent to giving a morphism between $f : X \to Y$ such that for any j, $\lambda_j = \mu_j \circ f$. We then say that cone X factorizes over cone Y.

A limit of a diagram $\mathcal{D}$ will be a universal cone i.e. a cone over which any other cone factorizes in a unique way. The definition is the following:

Definition 3.3.4. A cone $(L, \{\lambda_j\}_{j\in J})$ is a limit of diagram $\mathcal{D}$ if for any other cone $(C, \{\gamma_j\}_{j\in J})$ over $\mathcal{D}$ it uniquely factors through it, meaning that there exists a unique natural transformation $ct_C \Rightarrow ct_L$ such that the following diagram commutes

$$\begin{tikzcd} ct_C \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & ct_L \arrow[dl, Rightarrow] \\ & \mathcal{D} & \end{tikzcd}$$

As it was said before, this is equivalent to having a unique arrow $f : C \to L$ such that for any index j, $\gamma_j = \lambda_j \circ f$. This is the universal property of the limit of $\mathcal{D}$. We will say $\mathcal{D}$ has a limit if there exists a limit cone over it.

Example (Infimum)**.** *Given $S \subseteq \mathbb{R}$, let's consider diagram $\mathcal{D}$ mentioned in the example before. To give a limit over $\mathcal{D}$ is the same to be the infimum of set S, since it's the biggest lower bound.*

Example (Terminal objects)**.** *Given the empty category J, all diagram $\mathcal{D}$ from this category is empty. A cone over $\mathcal{D}$ is just an object $X \in \mathscr{C}$ without any arrows (the vertex). By the limit's definition, an object is terminal if and only if it's the limit of the empty diagram.*

Example (Products)**.** *A diagram of form $J = \bullet \quad \bullet$ consists in two objects $X_1, X_2 \in \mathscr{C}$. A limit of $\mathcal{D}$ consists in a cone*

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 \end{tikzcd}$$

such that for any other cone over $\mathcal{D}$ with vertex Z, there exists a unique morphism so that the following diagram commutes:

$$\begin{tikzcd}
 & Z \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & X \arrow[dl] \arrow[dr] & \\
X_1 & & X_2
\end{tikzcd}$$

By definition, we would say that the limit of $\mathcal{D}$ is the product of X_1, X_2.

Exercises

Exercise 3.3.1. *If we consider the following category J*

$$\begin{tikzcd}
 & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

and the correspondent diagram $\mathcal{D}$ with form J. What is the limit of this diagram?