Seminario de teoría de categorías

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October 14, 2024

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Chapter 1

Categories: objects and arrows

Definition 1.0.1 (Category)**.** A category $\mathscr{C}$ consists of the following data:

- A class of *objects*,
$$Ob(\mathscr{C})$$
(we usually write $X \in \mathscr{C}$ instead of $X \in Ob(\mathscr{C})$).

- For every two objects $X, Y \in \mathscr{C}$, a class of *morphisms* (or *arrows*) *from* X *to* Y,
$$\mathrm{Hom}_{\mathscr{C}}(X, Y)$$
(we sometimes write $\mathrm{Hom}(X, Y)$ or $\mathscr{C}(X, Y)$ instead of $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, and $f\colon X \to Y$ instead of $f \in \mathrm{Hom}(X, Y)$.

- A *composition law*, associating to morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, a *composition morphism*
$$g \circ f\colon X \to Z.$$
Arrows and the composition law are subject to the following conditions:
  - *Associativity*: given composable morphisms f, g, h,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
  - Existence of *identity arrows*: for any X, there exists a morphism $id_X\colon X \to X$, such that, for all $f\colon X \to Y$ and $g\colon Z \to X$,
$$f \circ id_X = f, \qquad id_X \circ g = g.$$

Example. *Some basic examples:*

1. *The category* **Set** *of all sets:*
   - *Ob*(**Set**) *is the class of all sets.*
   - $\mathrm{Hom}(X, Y)$ *is the set of all mappings* $f \colon X \to Y$
   - $g \circ f$ *is the usual composition of mappings.*
2. *The category* **Top** *of topological spaces has*
   - *topological spaces* $(X, \mathcal{T})$ *as objects (in other words, Ob*(**Set**) *is the class of all topological spaces).*
   - *continuous mappings* $f \colon (X, \mathcal{T}) \to (Y, \mathcal{T})$ *as morphisms, with the usual composition.*

   Notice that this requires two results from topology: (1) composition of continuous mappings is continuous and (2) the identity mapping is continuous.
3. *The category* **Group** *of groups has*
   - *groups as objects.*
   - *group homomorphisms as arrows, with the usual composition.*

   This requires the result that compositions of group homomorphisms are group homomorphisms.

All of the above examples consist of certain sets (with extra structure in some cases) and mappings between them, but this is not always the case:

1. *Posets give rise to categories:*
   For example, we may regard $(\mathbb{N}, \leq)$ *as a category having*
   - *natural numbers as objects.*
   - *for any* $n, m \in \mathbb{N}$,

     $$\mathrm{Hom}(n, m) = \begin{cases} \{n \to m\}, & \text{if } n \leq m \\ \emptyset, & \text{if } n \nleq m \end{cases}$$

     Here $\{n \to m\}$ *is just any set with a single element, which we regard as the arrow* $n \to m$*. These sets must be chosen so that* $n \to m$ *equals* $n' \to m'$ *if and only if* $n = n'$ *and* $m = m'$*.*

Since there is at most one arrow between objects, composition is defined in the only posible way.

Observe that the reflexivity and transitivity properties of partial orders are needed in order for this to form a category.

Similarly, any set X gives rise to a category

$$(\mathcal{P}(X), \subseteq),$$

and any topological space $(X, \mathcal{T})$ gives rise to a category

$$(\mathcal{T}, \subseteq).$$

2. *Groups as categories:*
   Given a group G, with binary operation "$\cdot$", the category BG has

   – *a single object, written $\bullet$.*
   – *elements $g \in G$ as arrows $\bullet \xrightarrow{g} \bullet$.*
   – *The composition of two arrows $g, g' \in G$ is the arrow $g \circ g' = g \cdot g'$.*

3. *The category associated to a unitary ring:*
   Given a unitary ring R, with binary operations "$+$" and "$\cdot$", the category $\mathscr{C}_R$ has

   – *a single object, written $\bullet$.*
   – *elements $r \in R$ as arrows $\bullet \xrightarrow{r} \bullet$.*
   – *The composition of two arrows $r, r' \in R$ is the arrow $r \circ r' = r \cdot r'$.*

   Observe that the condition that R is unitary is required by the axiom that categories must have identity arrows.

4. *Given a field k, the matrix category $\mathbf{M}at_k$ has*

   - *natural numbers as objects.*
   - *For any $n, m \in \mathbb{N} \setminus \{0\}$, a morphism $n \to m$ is an $m \times n$ matrix with entries in k. Composition of such morphisms corresponds to product of matrices. There is a single arrow $0 \to m$ and $m \to 0$, for every $m \in \mathbb{N}$, and composition with these arrows is determined by lack of choice.*

Definition 1.0.2. Given a category $\mathscr{C}$, the *opposite category* $\mathscr{C}^{op}$ has

- same objects as $\mathscr{C}$, that is

$$Ob(\mathscr{C}^{op}) = Ob(\mathscr{C}).$$

- Given $X, Y \in \mathscr{C}^{op}$, each morphism $X \to Y$ in $\mathscr{C}$ is an arrow $Y \to X$ in $\mathscr{C}^{op}$, that is,

$$\mathrm{Hom}_{\mathscr{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

Example. *This is an example of a simple category and its opposite.*

$$\mathscr{C} \qquad\qquad \begin{tikzcd} & \overset{2}{\bullet} \\ 1\ \bullet \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & \overset{3}{\bullet} \end{tikzcd}$$

$$\mathscr{C}^{\ op} \qquad\qquad \begin{tikzcd} & \overset{2}{\bullet} \arrow[dl, "f^{op}"'] \\ 1\ \bullet & \\ & \overset{3}{\bullet} \arrow[ul, "g^{op}"] \end{tikzcd}$$

$$\begin{aligned} Hom(1, 2) &= f \\ Hom(1, 3) &= g \\ Hom(2, 3) = Hom(2, 1) &= \\ = Hom(3, 1) = Hom(3, 2) &= \emptyset \end{aligned} \qquad \begin{aligned} Hom(2, 1) &= f^{op} \\ Hom(3, 1) &= g^{op} \\ Hom(2, 3) = Hom(1, 2) &= \\ = Hom(1, 3) = Hom(3, 2) &= \emptyset \end{aligned}$$

Definition 1.0.3 (Small and Locally small categories)**.**

A category $\mathscr{C}$ is **locally small** if, for all $X, Y \in \mathscr{C}$, $\mathrm{Hom}(X, Y)$ is a set (as opposed to a proper class).

A category is **small** if it is locally small and $Ob(\mathscr{C})$ is also a set.

Example. *All categories introduced above are locally small. An example of non-locally small category is the one with*

– *a single object* $\bullet$.

– *Any set* X *is an arrow* $\bullet \xrightarrow{X} \bullet$. *The composition of* $\bullet \xrightarrow{X} \bullet$ *and* $\bullet \xrightarrow{Y} \bullet$ *is* $\bullet \xrightarrow{X \cup Y} \bullet$.

Remark. *From now on, we will always consider locally small categories.*

1.1 Subcategories

Definition 1.1.1. Given a category $\mathscr{C}$, we say that a subset $\mathscr{D} \subseteq \mathscr{C}$ is a **subcategory** of $\mathscr{C}$ if it is a subcollection of objects and morphisms, satisfying that

- All domains and codomains of morphisims in $\mathscr{D}$ are contained.
- The identity morphism of every object in $\mathscr{D}$ is contained.
- Every posible composition between morphisms in $\mathscr{D}$ is contained.

Example. *Some basic examples of subcategories are*

1. ***Haus*** *is the subcategory of* ***Top*** *consisting of Hausdorff topological spaces.*
2. ***Ab*** *is the subcategory of* ***Group*** *consisting of abelian groups.*

Definition 1.1.2 (Full and Wide subcategories)**.**

Let $\mathscr{D}$ be a subcategory of $\mathscr{C}$.

- $\mathscr{D}$ ia a **full subcategory** of $\mathscr{C}$ if $\forall\, X, Y \in \mathscr{D}$, it is satisfied that
$$Hom_{\mathscr{D}}(X,Y) = Hom_{\mathscr{C}}(X,Y)$$
- $\mathscr{D}$ ia a **wide subcategory** of $\mathscr{C}$ if
$$Ob(\mathscr{D}) = Ob(\mathscr{C})$$

Example. *The subcategory $\mathscr{C}$ of* ***Group*** *consisting of*

- $Ob(\mathscr{C}) = Ob($***Group***$)$.
- *The arrows in $\mathscr{C}$ are the group isomorphisms.*

Example. *We consider the following categories:*

1. *The category* ***Rng*** *of all rings:*
   - $Ob(\mathbf{Rng})$ *is conformed by all rings.*
   - *Ring homomorphisms are arrows, with the usual composition.*
2. *The category* ***Ring*** *of all unitary rings:*

- *$Ob(\mathbf{Ring})$ is conformed by all unitary rings.*
- *Ring homomorphisms $\mathscr{R} \longrightarrow \mathscr{S}$ satisfying $1_{\mathscr{R}} \mapsto 1_{\mathscr{S}}$ are arrows with the usual composition.*

*It is clear that **Ring** is a subcategory of **Rng** but it is not full, as morphisms in **Ring** need to take the multiplicative identity of the first ring into the multiplicative identity of the second one.*

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Proposition 1.1.1. *Let $\mathscr{D}$ be a subcategory of $\mathscr{C}$. Then, $\mathscr{D}^{op}$ is a subcategory of $\mathscr{C}^{op}$*

1.2 Special Objects

Definition 1.2.1. An object $I \in \mathscr{C}$ is *initial* if, for every object $X \in \mathscr{C}$, there exists a unique morphism

$$I \to X.$$

Definition 1.2.2. An object $T \in \mathscr{C}$ is *terminal* (or *final*) if, for every object $X \in \mathscr{C}$, there exists a unique morphism

$$X \to T.$$

Terminal and initial are dual notions.

Definition 1.2.3. An object is a *zero object* if it is initial and terminal.

Example. *Some basic examples:*

1. *Special objects in category* $(\mathbb{N}, \leq)$*:*
   - *The initial object in this category is 0, since* $0 \leq n \ \forall n \in \mathbb{N}$ *and so there exists a unique arrow from 0 to any other natural number. The uniqueness comes from the fact that there exists only one arrow that goes from one object to another (greater) one.*
   - *There is no final object in this category, since there isn't any natural number greater that all of them.*
   - *Since there's no final object, there's no zero object in this category.*

   Note: if instead of $\mathbb{N}$ *we considered* $\mathbb{Z}$*, we wouldn't have an initial object either.*

2. *In **Set** and **Top**:*
   - *We have that* $\emptyset$ *is initial since we can always consider the inclusion as the only arrow from* $\emptyset$ *to any set or topological space.*
   - *We have that any set with only an element is terminal, since there's only one way to define a map from a set to the singleton.*
   - *There are no zero objects since* $\emptyset$ *is the only initial object in these categories.*

3. *In the **Ring** category,* $\mathbb{Z}$ *is an initial object since all arrows must send 0 and 1 to the new ring's neutral elements, and* $\{0\}$ *is terminal. However, in **Rng** we have that* $\{0\}$ *is a zero object.*

Exercises

Exercise 1.2.1. *If we have two initial objects from the same category, then they're isomorphic.*

Exercise 1.2.2. *Let* X_1 *be an initial object, and let* $X_2 \cong X_1$*. Then* X_2 *is initial.*

Analogous results are valid for terminal objects.

Exercise 1.2.3. *If* $|X| \neq 1$ *then* X *is not terminal.*

Exercise 1.2.4. *Let's consider category* $\mathscr{B}$ *that has objects* A, B, C, D, F *and arrows* $\alpha, \beta, \gamma, \delta, \sigma, \beta\alpha = \sigma\delta$ *and* $\sigma\tau$*. What are the initial and terminal objects?*

$$\begin{tikzcd}
& B \arrow[dr, "\beta"] & \\
A \arrow[ur, "\alpha"] \arrow[dr, "\delta"'] & & D \\
& C \arrow[ur, "\sigma"'] & \\
& & F \arrow[ul, "\tau"]
\end{tikzcd}$$

1.3 Monomorphisms, epimorphisms and isomorphisms

Monos & epis

Definition 1.3.1 (Monomorphism)**.** Let $\mathscr{C}$ be a category, and $X, Y, Z \in \mathscr{C}$.

We say that an arrow $f : X \longrightarrow Y$ is a **monomorphism** if for any pair of arrows $g, g' : Z \longrightarrow X$ holding the equality $f \circ g = f \circ g'$, they satisfy $g = g'$.

Definition 1.3.2 (Epimorphism)**.** Let $\mathscr{C}$ be a category, and $X, Y, Z \in \mathscr{C}$.

We say that an arrow $f : X \longrightarrow Y$ is a **epimorphism** if for any pair of arrows $h, h' : Y \longrightarrow Z$ holding the equality $h \circ f = h' \circ f$, they satisfy $h = h'$.

Example. *Some examples of monomorphisms and epimorphisms are:*

- *The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism in **Ring**.*
- *A morphism $f \colon X \to Y$ in **Haus** is an epimorphism if and only if $f(X)$ is dense in Y.*

Definition 1.3.3 (Split monomorphism and epimorphism)**.**

Let $\mathscr{C}$ be a category, $X, Y \in \mathscr{C}$ and $f \in Hom_{\mathscr{C}}(X, Y)$, $g \in Hom_{\mathscr{C}}(Y, X)$ such that

$$g \cdot f = id_X$$

Then f is called a **split monomorphism** and g is called **split epimorphism**.

Proposition 1.3.1. *Split monomorphisms and epimorphisms are respectively monomorphisms and epimorphisms.*

Proposition 1.3.2. *Let $f : X \longrightarrow Y$ be a monomorphism (or epimorphisim) in $\mathscr{C}$. Then, f is an epimorphism (or monomorphisim) in $\mathscr{C}^{op}$.*

Proposition 1.3.3 (Operations with monomorphism)**.** *(epimorphism by duality):*

- *Composition of monomorphisms is monomorphisms.*
- *If $g \circ f$ is a monomorphism, then f is a monomorphism.*

Proposition 1.3.4 (Subcategories reflect monomorphism)**.**
Let $\mathscr{A} \hookrightarrow \mathscr{B}$ be a subcategory, and let f be an arrow in $\mathscr{A}$. If f is a monomorphism in $\mathscr{B}$, then f is a monomorphism in $\mathscr{A}$.

Proposition 1.3.5 (Special arrows from special objects)**.**
If I is initial, any arrow $X \to I$ is an epimorphism

Remark. *It's important to separate the usual definitions for monomorphism and epimorphisms used in representable categories from the more general, categorical one:*

- *Being mono doesn't always imply being injective, just like being epi doesn't always imply being surjective. The first case is more difficult to see since in general in categories made from algebraic structures we do have that mono implies injectivity, but a counterexample for epimorphisms is the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in* ***Ring****.*
- *Being a split monomorphism implies being an injection in* ***Sets*** *but the other way around is not true.*

Exercises

Exercise 1.3.1. *Prove proposition 1.3.1*

Exercise 1.3.2. *The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in* ***Ring****.*

Exercise 1.3.3. *Prove proposition 1.3.5*

Exercise 1.3.4. *Split monomorphism implies injectivity in* ***Sets***

Isomorphisms

Definition 1.3.4 (Isomorphism)**.** Let $\mathscr{C}$ be a category, and $X, Y \in \mathscr{C}$.

An arrow $f : X \longrightarrow Y$ is said to be an **isomorphisim** if there exists another arrow $f^{-1} : Y \longrightarrow X$ satisfying that $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$.

Example. *Some examples of isomorphisms are:*

1. *A mapping $f\colon X \to Y$ is an isomorphism in* ***Set*** *if and only if it is a bijection.*
2. *A group homomorphism ϕ is an isomorphism in* ***Group*** *if and only if it is bijective.*
3. *The isomorphisms in* ***Top*** *are the homeomorphisms.*
4. *The only isomorphisms in $(\mathbb{N}, \leq)$ are the identities.*
5. *A morphism $X \to Y$ in $\mathscr{C}$ is the same as an isomorphism $Y \to X$ in $\mathscr{C}^{op}$.*
6. *Given an object $X \in \mathscr{C}$, an* ***automorphism*** *of X is just an isomorphism $X \to X$. This gives rise to an obvious automorphism subcategory*
$$Aut(X) \hookrightarrow End(X) \hookrightarrow \mathscr{C}.$$

Proposition 1.3.6 (Basic properties of isomorphisms)**.**

1. *Inverses of isomorphisms are unique.*
2. *Any composition of isomorphisms is an isomorphism.*
3. *If the composition of two arrows is an isomorphism and one of them is an isomorphism, then so is the other.*

Proposition 1.3.7 (Isomorphisms and subcategories)**.**
Let $\mathscr{A} \hookrightarrow \mathscr{B}$ be a subcategory, and let f be an arrow in $\mathscr{A}$.

1. *If f is an isomorphism in $\mathscr{A}$, then f is an isomorphism in $\mathscr{B}$*
2. *Full subcategories reflect isomorphisms:*
If $\mathscr{A}$ is a full subcategory and f is an isomorphism in $\mathscr{B}$, then f is an isomorphism in $\mathscr{A}$.

(Obs: the fullness condition above cannot be dropped).

Proposition 1.3.8 (Characterization of isomorphisms)**.**
A morphism is an isomorphism if and only if it is a split monomorphism and a split epimorphism.

Remark. *The condition of being split is necessary. We'll give some examples.*

1. *Considerate a category* $\mathscr{C}$ *given by*

   - $Ob(C) = \{a, b\}$
   - *Appart from the identitty arrows, there only exists an arror* $f : a \longrightarrow b$.

$$\begin{tikzcd} a \arrow[r, mapsto, "f"] \arrow[loop below, "id_a"] & b \arrow[loop below, "id_b"] \end{tikzcd}$$

   It is clear that f *is a monomorphism and an epimorphisim, but can't be an isomorphism because it doesn't exist an arrow* $g : a \longrightarrow b$.

2. *In* ***Top*** *category, we have that*

   - *An arrow is a monomorphism if and only if it is injective.*
   - *An arrow is an epimorphism if and only if it is surjective.*

   In conclusion, every bijective arrow in ***Top*** *is a mono and an epimorphism but it is only an isomorphisms if its inverse is contonuous.*

Chapter 2

Functors

Definition 2.0.1 (Functor)**.** A functor F from category $\mathscr{C}$ to $\mathscr{D}$, written $F : \mathscr{C} \to \mathscr{D}$ is

- An arrow between the objects of each category

$$F : Obj_{\mathscr{C}} \to Obj_{\mathscr{D}}$$

- And, for each $X, Y \in \mathscr{C}$ another arrow

$$F : Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$$

  such that:

  1. $F(id_X) = id_{F(X)}, \forall\, X \in \mathscr{C}$
  2. $F(g \circ f) = F(g) \cdot F(f)$ for any pair of composable arrows f, g.

Basically, a functor consists of a mapping on objects and a mapping on arrows that preserves all of the structure of a category, namely domains and codomains, composition, and identities. It can be represented by the next diagram:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] & \mathscr{D} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & FX \arrow[d, "Ff"] \\
Y \arrow[r, mapsto] & FY
\end{tikzcd}$$

Example.

1. *The classical, most simple examples is the* ***identity functor*** $\mathscr{C} \to \mathscr{C}$ *defined in the obvious way.*

2. *The other classical functor is the* ***inclusion functor****.*

   Let $\mathscr{C}$ *be a category and* $\mathscr{D} \subseteq \mathscr{C}$ *a subcategory. We define the inclusion functor*

   $$F : \mathscr{D} \hookrightarrow \mathscr{C}$$

   such that

   $$FX = X \qquad \forall\, X \in \mathscr{D}$$
   $$Ff = f \qquad \forall\, f \in Hom_{\mathscr{D}}(X, Y)$$

   It's trivial to see that it satisfies the axioms of functors.

   $$\begin{tikzcd} \mathscr{D} \arrow[r, phantom, "\overset{F}{\hookrightarrow}"] & \mathscr{C} \\ X \arrow[r, mapsto] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, mapsto] & Y \end{tikzcd}$$

3. *The* ***forgetful functor*** U *can be defined for the "classical" categories defined with a set and some underlying structure like* $\mathbf{Top}, \mathbf{Group}, \mathbf{Rng}$ *or* $\mathbf{Ring}$*. The functor makes the category "forget" it's underlying structure and stick with the original set and its arrows as normal maps. For example, for the category of topological spaces we can consider* $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$*.*

4. *We can define an endofunctor in the* ***Set*** *category*

   $$\mathcal{P} : \boldsymbol{Set} \longrightarrow \mathbf{Set}$$

   which sends a set A *to its power set* $\mathcal{P}(A)$*, and given a function* $f : A \longrightarrow B$*, it sends* f *to a function between theirs respectives powers sets,*
   $\mathcal{P}f : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$ *satisfying*

   $$\mathcal{P}(f)(A') = f(A') \in \mathcal{P}(B) \qquad \forall\, A' \subseteq A$$

5. *We'll introduce the dotted topology category,* $\mathbf{Top}_*$*, wich consists on*
   - $Ob(\mathbf{Top}_*) = \{(X, x) : X \text{ is a topological space}, x \in X\}$
   - $Hom_{\mathbf{Top}_*}((X, x), (Y, y)) = \{f : X \longrightarrow Y :\ f \text{ is continuous},\ f(x) = y\}$

   Then, the fundamental group defines a functor

   $$\pi_1 : Top_* \longrightarrow Group$$

   which carries continuous functions $f : (X, x) \longrightarrow (Y, y)$ *to group homomorphims* $f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, y)$.

Definition 2.0.2. Let $\mathscr{C}, \mathscr{D}, \mathscr{E}$ categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$, $G : \mathscr{D} \longrightarrow E$. We define the functor composition of G and F as

$$G \circ F : \mathscr{C} \longrightarrow \mathscr{E}$$

which acts like we show in the following diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[rr, bend left, "G \circ F"] & \mathscr{D} \arrow[r, "G"] & \mathscr{E} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & FX \arrow[r, mapsto] \arrow[d, "Ff"] & GFX \arrow[d, "GFf"] \\
Y \arrow[r, mapsto] & FY \arrow[r, mapsto] & GFY
\end{tikzcd}$$

Definition 2.0.3 (Isomorphic categories)**.**

Let $\mathscr{C}, \mathscr{D}$ be categories. We say that $\mathscr{C}, \mathscr{D}$ are isomorphic categories, $\mathscr{C} \cong \mathscr{D}$ if there exists functors

$$F : \mathscr{C} \longrightarrow \mathscr{D}$$

$$G : \mathscr{D} \longrightarrow \mathscr{C}$$

mutually inverse to each other. Then, we say that F and G are ***isomorphisms of categories***.

Exercise 2.0.1. *Let* G, H *be groups. The group isomorphisms between* G *and* H *induce categorical isomorphisms between* BG *and* BH.

Lemma 2.0.1. *Functors preserve isomorphisms.*

Proof. Let $\mathscr{C}, \mathscr{D}$ categories, $F : \mathscr{C} \longrightarrow \mathscr{D}$ a functor, $X \in \mathscr{C}, Y \in \mathscr{D}$ and $f : X \longrightarrow Y$ an isomorphism with inverse $g : Y \longrightarrow X$.

Applying the axioms which define functors and that $g \cdot f = id_X$ we have that

$$Fg \cdot Ff = F(g \cdot f) = F(id_X) = id_X$$

which means that Ff is the left inverse of Fg.

Arguing by duality, we prove that Ff is the right inverse of Fg, and, as a result, Ff and Fg are inverse isomorphisms. $\square$

Proposition 2.0.2. *Let $F : \mathscr{A} \to \mathscr{B}$ be a category isomorphism. Then*

1. *$f \in \mathscr{A}$ is a monomorphism (and respectively and epimorphism) if and only if $Ff \in \mathscr{B}$ is an monomorphism (and respectively and epimorphism).*

2. *An object $I \in \mathscr{A}$ is initial if and only if $FI \in \mathscr{B}$ is initial. The same holds for terminal and zero objects.*

Also, a functor $F : \mathscr{A} \to \mathscr{C}$ is an isomorphism if and only if $F^{op} : \mathscr{A}^{op} \to \mathscr{C}^{op}$ is ana isomorphism.

Proof. We will only prove the first part of the proposition.
Assume $f \in \mathscr{A}$ is a monomorphism. Assume there exists $g, g' \in \mathscr{B}$ such that $Ff \circ g = Ff \circ g'$. Then since F is a category isomorphism we have that

$$F^{-1} \circ (Ff \circ g) = (F^{-1}Ff) \circ (F^{-1}g) = (F^{-1} \circ F)f \circ F^{-1}(g) = f \circ F^{-1}(g)$$

and also

$$F^{-1} \circ (Ff \circ g') = (F^{-1}Ff) \circ (F^{-1}g') = (F^{-1} \circ F)f \circ F^{-1}(g') = f \circ F^{-1}(g')$$

Since f monomorphism we can deduce $F^{-1}(g) = F^{-1}(g')$ and from F being a category isomorphism we get that $g = g'$.
In the other direction, if we assume that Ff is a monomorphism and we assume that $\exists g, g' \in \mathscr{A}$ such that $f \circ g = f \circ g'$ then

$$F(f \circ g) = Ff \circ Fg = Ff \circ Fg' = F(f \circ g')$$

and so we can deduce $Fg = Fg'$ from Ff being mono and from F being a category isomorphism we have that $g = F^{-1}Fg = F^{-1}Fg' = g'$. $\square$

2.1 Contravariant Functors

Definition 2.1.1 (Contravariant Functor)**.**

Given two categories $\mathscr{C}, \mathscr{D}$, a **contravariant functor** between them is a functor $F : \mathscr{C}^{op} \longrightarrow \mathscr{D}$. By definition, this functor:

- Associates every object $X \in \mathscr{C}$ to an object $FX \in \mathscr{D}$.
- Associates any arrow $f^{op} : X \longrightarrow Y$ to an arrow $Ff^{op} : FX \longrightarrow FY$, where $X, Y \in \mathscr{C}$ in such a way that:
  - For every composable pair of arrows f^{op}, g^{op}
  $$Ff^{op} \circ Fg^{op} = F(f^{op} \circ g^{op})$$
  - $\forall\ X \in \mathscr{C}$
  $$Fid_X = 1_{FX}$$

However, this functors can be thought of in terms of the original category. If we write $\bar{F}f$ for Ff^{op}, then we can see the functor F as a functor $\bar{F}$ such that:

- Associates every object $X \in \mathscr{C}$ to an object $\bar{F}X \in \mathscr{D}$.
- Associates any arrow $f : X \longrightarrow Y$ to an arrow $\bar{F}f : \bar{F}Y \longrightarrow \bar{F}X$, where $X, Y \in \mathscr{C}$ in such a way that:
  - For every composable pair of arrows f, g
  $$\bar{F}f \circ \bar{F}g = \bar{F}(g \circ f)$$
  - $\forall\ X \in \mathscr{C}$
  $$\bar{F}id_X = 1_{\bar{F}X}$$

This is why contravariant functor are usually drawn with a diagram like the following one:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] & \mathscr{D} \\
X \arrow[r, mapsto] \arrow[d, "f"'] & FX \\
Y \arrow[r, mapsto] & FY \arrow[u, "Ff"']
\end{tikzcd}$$

Example. *This are a few of the most typical examples:*

1. *A typical example is the* ***contravariant power functor*** P : ***Set*** $\to$ ***Set*** *(or more correctly the* $\textbf{\textit{Set}}^{op} \to$ ***Set*** *power functor) that sends any set to it's power set* $P(X)$ *and any map between sets* $f : X \to Y$ *to the arrow* Ff *that sends any subset* $U \subset Y$ *to it's inverse image* $f^{-1}(U) \subset X$.

2. *If* ***Vect*** *are the finite dimension vector spaces over field* $\mathbb{K}$*, then the* ***dual functor*** *is a contravariant functor that associates every vector space* V *to its dual space* $V* = Hom(V, \mathbb{K})$ *and every linear map* $f : V \to W$ *to the map* Ff *such that for every* $\alpha \in W* = Hom(W, \mathbb{K})$ *then it pre-composes f to have a map from* V *to* $\mathbb{K}$ *(it associates* α *to* $\alpha \circ f$*).*

Remark. *It's important to be careful when we talk about contravariant functors. In general, we can think of a contravariant functor as a functor from a category* $\mathscr{A}$ *to a category* $\mathscr{B}$ *that just changes the arrows' directions. That's an easier way to visualise how the functor is working, but we need to take into account that all the theory that we've previously proved for "normal" (covariant) functors doesn't work for contravariant functors defined this way, so to apply all this theory we need to think of the contravariant functors just as a normal functor from* $\mathscr{A}^{op}$ *to* $\mathscr{B}$*.*

The Hom Functor

There's a very important example of covariant and contravariant functors provided by the **Hom** sets.

Definition 2.1.2 (Covariant Hom Functor)**.** Let $\mathscr{C}$ be a locally small category. Then, for each $X \in \mathscr{C}$ we can define a functor $Hom(X, \_)$ that sends

- Each object $Y \in \mathscr{C}$ to the object $Hom(X, Y) \in$ **Set**
- Each arrow $Y \to Z$ in $\mathscr{C}$ to an arrow $Hom(X, f)$ from $Hom(X, Y)$ to $Hom(X, Z)$ that sends each $\alpha \in Hom(X, Y)$ to $f \circ \alpha$

$$
\begin{tikzcd}
Hom(X,\_) : \mathscr{C} \arrow[r] & Set \\
Y \arrow[r, mapsto] \arrow[d, "f"'] & Hom(X,Y) \arrow[d, "f \circ \_"] \\
{} \arrow[r, mapsto] & {} \\
Z \arrow[r, mapsto] & Hom(X,Z)
\end{tikzcd}
$$

Similarly we can define the contravariant version of this functor:

Definition 2.1.3 (Contravariant Hom Functor)**.** Let $\mathscr{C}$ be a locally small category. Then, for each $X \in \mathscr{C}$ we can define a functor $Hom(\_, X)$ that sends

- Each object $Y \in \mathscr{C}$ to the object $Hom(Y, X) \in \mathbf{Set}$
- Each arrow $Y \to Z$ in $\mathscr{C}$ to an arrow $Hom(f, X)$ from $Hom(Z, X)$ to $Hom(Y, X)$ that sends each $\alpha \in Hom(Z, X)$ to $\alpha \circ f \in Hom(Y, X)$

$$
\begin{tikzcd}
Hom(\_,X) : \mathscr{C} \arrow[r] & Set \\
Y \arrow[r, mapsto] \arrow[dd, "f"'] & Hom(Y,X) \\
 \arrow[r, mapsto] & \\
Z \arrow[r, mapsto] & Hom(Z,X) \arrow[uu, "\_ \circ f"']
\end{tikzcd}
$$

We will see the importance of these functors in later chapters.

Exercises

Exercise 2.1.1. *If $F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{A}$ functors, prove that $G \circ F$ is a functor.*

Exercise 2.1.2. *If X, Y are isomorphic in $\mathscr{C}$, then given a functor $F : \mathscr{C} \to \mathscr{D}$ prove that F(X), F(Y) are isomorphic in $\mathscr{D}$.*

2.2 Natural transformations

Definition 2.2.1. Given two functors $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ a **natural transformation** between them (typically written $\alpha : F \longrightarrow G$) consisits of a family of morphisms in $\mathscr{D}$

$$(\alpha_X)_{X \in \mathscr{C}} \text{ where each } \alpha_X : FX \to GX$$

such that for any morphism $f : X \to Y$ in $\mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \\
FY \arrow[r, "\alpha_Y"'] & GY \arrow[u, "Gf"']
\end{tikzcd}$$

Each α_X is called a component of the natural transformation.
When the definition holds, it's usual to say that the functors are related in a natural manner, just like if a family of arrows define the components of a natural transformation we say that they are natural. Sometimes it's typical to display natural transformation with a diagram like the following one:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathscr{D}
\arrow[from=U, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

A **natural isomorphism** is a natural transformation in which every component α_X is an isomorphism.

Example. *Here are a few examples of natural transformations:*

1. *("Trivial" example)*
   Let's assume we have $\mathscr{C}$ the trivial category (one object, one arrow) and a category $\mathscr{D}$ with two objects a, b and one arrow between them. We consider the following functors:

$$\begin{array}{rcl}
F: \mathscr{C} & \longrightarrow & \mathscr{D} \\
\{\cdot\} & \longmapsto & a \\
1. & \longmapsto & 1_a
\end{array}
\qquad
\begin{array}{rcl}
G: \mathscr{C} & \longrightarrow & \mathscr{D} \\
\{\cdot\} & \longmapsto & b \\
1. & \longmapsto & 1_b
\end{array}$$

   Since there's only one possible arrow in $\mathscr{D}$ we fave that the only natural transformation that we can define is the unique arrow between a and b.

2. *Let's consider the following two functors:*

$$\begin{tikzcd} F: Set \arrow[rr] & & Set \\ X \arrow[d] \arrow[rr, mapsto] & & ((X \times X) \times X) \arrow[d] \\ & f \arrow[r, mapsto] & Ff \\ Y \arrow[rr, mapsto] & & ((Y \times Y) \times Y) \end{tikzcd} \qquad \begin{tikzcd} G: Set \arrow[rr] & & Set \\ X \arrow[d] \arrow[rr, mapsto] & & X^3 \arrow[d] \\ & f \arrow[r, mapsto] & Gf \\ Y \arrow[rr, mapsto] & & Y^3 \end{tikzcd}$$

where $Ff((x_1, x_2), x_3) = ((f(x_1), f(x_2)), f(x_3)$ *and* $Gf(x_1, x_2, x_3) = (f(x_1), f(x_2), f(x_3))$. *For each set we define*

$$\alpha_X(((x_1, x_2), x_3)) = (x_1, x_2, x_3)$$

Exercise 2.2.1. $(\alpha_X)_{X \in Sets}$ *is a natural transformation*

3. *The determinant is a natural transformation. Let's consider the following functors:*

$$\begin{tikzcd} F: CRing \arrow[r] & Grp \\ A \arrow[d, "f"'] \arrow[r, mapsto] & GL_n(A) \arrow[d, "GL_n(f)"] \\ R \arrow[r, mapsto] & GL_n(R) \end{tikzcd} \qquad \begin{tikzcd} G: CRing \arrow[r] & Grp \\ A \arrow[d, "f"'] \arrow[r, mapsto] & A^* \arrow[d, "f^*"] \\ R \arrow[r, mapsto] & R^* \end{tikzcd}$$

Here $GL_n(f)$ *is obtained by applying* f *to each matrix entry, and* f^* *is the restriction of* f *to the units.*
Note that that if M *is an* $n \times n$ *matrix with entries in the commutative ring* K*, the determinant of the matrix will be a unit if* M *is non-singular. Therefore* $f : GL_n(K) \to K^*$ *will be a group morphism, and since it's defined by the same formula for all rings we have that* $f^* \circ det_A = det_R \circ GL_n(f)$.

4. *The pair of functors* $Hom(A, \_), Hom(\_, B)$ *are related in a natural manner.*

Exercises

Exercise 2.2.2. *Suppose* $\alpha : F \Rightarrow G$ *is a natural isomorphism. Show that the inverses of the components define the components of a natural isomorphism* $\alpha^{-1} : G \Rightarrow F$

Exercise 2.2.3. *Let $T \in \mathscr{D}$ be a terminal object and $F : \mathscr{C} \to \mathscr{D}$ a functor. Let's define a functor $\boldsymbol{Const_T} : \mathscr{C} \to \mathscr{D}$ that sends every object in $\mathscr{C}$ to T and every map to the identity in T. Give a natural transformation between the functors.*

Exercise 2.2.4. *What is a natural transformation between two groups seen as one-object categories? (Between BG, BH for G, H groups)*

2.3 Equivalence of categories

Natural isomorphisms allow us to define the notion of *equivalence of categories* which generalizes the idea of homotopy equivalence.

Definition 2.3.1. An **equivalence of categories** is a pair of functors $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ together with natural isomorphisms $\eta : 1_{\mathscr{C}} \cong GF$ and $\xi : FG \cong 1_{\mathscr{D}}$. We will say that $\mathscr{C}, \mathscr{D}$ are equivalent and write $\mathscr{C} \simeq \mathscr{D}$ if there exists an equivalence of categories between them.