Seminario de teoría de categorías

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Chapter 1

Categories: objects and arrows

Definition 1.0.1 (Category)**.** A category $\mathscr{C}$ consists of the following data:

- A class of *objects*,
$$Ob(\mathscr{C})$$
(we usually write $X \in \mathscr{C}$ instead of $X \in Ob(\mathscr{C})$).

- For every two objects $X, Y \in \mathscr{C}$, a class of *morphisms* (or *arrows*) *from* X *to* Y,
$$\mathrm{Hom}_{\mathscr{C}}(X, Y)$$
(we sometimes write $\mathrm{Hom}(X, Y)$ or $\mathscr{C}(X, Y)$ instead of $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, and $f \colon X \to Y$ instead of $f \in \mathrm{Hom}(X, Y)$.

- A *composition law*, associating to morphisms $f \colon X \to Y$ and $g \colon Y \to Z$, a *composition morphism*
$$g \circ f \colon X \to Z.$$
Arrows and the composition law are subject to the following conditions:

  – *Associativity*: given composable morphisms f, g, h,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

  – Existence of *identity arrows*: for any X, there exists a morphism $id_X \colon X \to X$, such that, for all $f \colon X \to Y$ and $g \colon Z \to X$,
$$f \circ id_X = f, \qquad id_X \circ g = g.$$

Example. *Some basic examples:*

1. *The category* **Set** *of all sets:*
   - *Ob*(**Set**) *is the class of all sets.*
   - $\mathrm{Hom}(X, Y)$ *is the set of all mappings* $f\colon X \to Y$
   - $g \circ f$ *is the usual composition of mappings.*
2. *The category* **Top** *of topological spaces has*
   - *topological spaces* $(X, \mathcal{T})$ *as objects (in other words, Ob*(**Set**) *is the class of all topological spaces).*
   - *continuous mappings* $f\colon (X, \mathcal{T}) \to (Y, \mathcal{T})$ *as morphisms, with the usual composition.*

   Notice that this requires two results from topology: (1) composition of continuous mappings is continuous and (2) the identity mapping is continuous.
3. *The category* **Group** *of groups has*
   - *groups as objects.*
   - *group homomorphisms as arrows, with the usual composition.*

   This requires the result that compositions of group homomorphisms are group homomorphisms.

All of the above examples consist of certain sets (with extra structure in some cases) and mappings between them, but this is not always the case:

1. *Posets give rise to categories:*
   For example, we may regard $(\mathbb{N}, \leq)$ *as a category having*
   - *natural numbers as objects.*
   - *for any* $n, m \in \mathbb{N}$,

     $$\mathrm{Hom}(n, m) = \begin{cases} \{n \to m\}, & \text{if } n \leq m \\ \emptyset, & \text{if } n \not\leq m \end{cases}$$

     Here $\{n \to m\}$ *is just any set with a single element, which we regard as the arrow* $n \to m$. *These sets must be chosen so that* $n \to m$ *equals* $n' \to m'$ *if and only if* $n = n'$ *and* $m = m'$.

Since there is at most one arrow between objects, composition is defined in the only posible way.

Observe that the reflexivity and transitivity properties of partial orders are needed in order for this to form a category.

Similarly, any set X gives rise to a category

$$(\mathcal{P}(X), \subseteq),$$

and any topological space $(X, \mathcal{T})$ gives rise to a category

$$(\mathcal{T}, \subseteq).$$

2. *Groups as categories:*
   Given a group G, with binary operation "$\cdot$", the category BG has
   - *a single object, written $\bullet$.*
   - *elements $g \in G$ as arrows $\bullet \xrightarrow{g} \bullet$.*
   - *The composition of two arrows $g, g' \in G$ is the arrow $g \circ g' = g \cdot g'$.*
3. *The category associated to a unitary ring:*
   Given a unitary ring R, with binary operations "$+$" and "$\cdot$", the category $\mathscr{C}_R$ has
   - *a single object, written $\bullet$.*
   - *elements $r \in R$ as arrows $\bullet \xrightarrow{r} \bullet$.*
   - *The composition of two arrows $r, r' \in R$ is the arrow $r \circ r' = r \cdot r'$.*

   Observe that the condition that R is unitary is required by the axiom that categories must have identity arrows.
4. *Given a field k, the matrix category $\mathbf{M}at_k$ has*
   - *natural numbers as objects.*
   - *For any $n, m \in \mathbb{N} \setminus \{0\}$, a morphism $n \to m$ is an $m \times n$ matrix with entries in k. Composition of such morphisms corresponds to product of matrices. There is a single arrow $0 \to m$ and $m \to 0$, for every $m \in \mathbb{N}$, and composition with these arrows is determined by lack of choice.*

Definition 1.0.2. Given a category $\mathscr{C}$, the *opposite category* $\mathscr{C}^{op}$ has

- same objects as $\mathscr{C}$, that is

$$Ob(\mathscr{C}^{op}) = Ob(\mathscr{C}).$$

- Given $X, Y \in \mathscr{C}^{op}$, each morphism $X \to Y$ in $\mathscr{C}$ is an arrow $Y \to X$ in $\mathscr{C}^{op}$, that is,

$$\mathrm{Hom}_{\mathscr{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

Example. *This is an example of a simple category and its opposite.*

$$\mathscr{C} \qquad \begin{tikzcd} & \overset{2}{\bullet} \\ 1 \bullet \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & \overset{3}{\bullet} \end{tikzcd} \qquad\qquad \mathscr{C}^{op} \qquad \begin{tikzcd} & \overset{2}{\bullet} \arrow[dl, "f^{op}"'] \\ 1 \bullet & \\ & \overset{3}{\bullet} \arrow[ul, "g^{op}"] \end{tikzcd}$$

$$\begin{aligned} Hom(1,2) &= f \\ Hom(1,3) &= g \\ Hom(2,3) = Hom(2,1) &= \\ = Hom(3,1) = Hom(3,2) &= \emptyset \end{aligned} \qquad\qquad \begin{aligned} Hom(2,1) &= f^{op} \\ Hom(3,1) &= g^{op} \\ Hom(2,3) = Hom(1,2) &= \\ = Hom(1,3) = Hom(3,2) &= \emptyset \end{aligned}$$

Definition 1.0.3 (Small and Locally small categories)**.**

A category $\mathscr{C}$ is **locally small** if, for all $X, Y \in \mathscr{C}$, $\mathrm{Hom}(X, Y)$ is a set (as opposed to a proper class).

A category is **small** if it is locally small and $Ob(\mathscr{C})$ is also a set.

Example. *All categories introduced above are locally small. An example of non-locally small category is the one with*

– *a single object* $\bullet$.

– *Any set* X *is an arrow* $\bullet \xrightarrow{X} \bullet$. *The composition of* $\bullet \xrightarrow{X} \bullet$ *and* $\bullet \xrightarrow{Y} \bullet$ *is* $\bullet \xrightarrow{X \cup Y} \bullet$.

Remark. *From now on, we will always consider locally small categories.*

1.1 Subcategories

Definition 1.1.1. Given a category $\mathscr{C}$, we say that a subset $\mathscr{D} \subseteq \mathscr{C}$ is a **subcategory** of $\mathscr{C}$ if it is a subcollection of objects and morphisms, satisfying that

- All domains and codomains of morphisims in $\mathscr{D}$ are contained.
- The identity morphism of every object in $\mathscr{D}$ is contained.
- Every posible composition between morphisms in $\mathscr{D}$ is contained.

Example. *Some basic examples of subcategories are*

1. ***Haus*** *is the subcategory of* ***Top*** *consisting of Hausdorff topological spaces.*
2. ***Ab*** *is the subcategory of* ***Group*** *consisting of abelian groups.*

Definition 1.1.2 (Full and Wide subcategories)**.**

Let $\mathscr{D}$ be a subcategory of $\mathscr{C}$.

- $\mathscr{D}$ ia a **full subcategory** of $\mathscr{C}$ if $\forall\, X, Y \in \mathscr{D}$, it is satisfied that
$$Hom_{\mathscr{D}}(X, Y) = Hom_{\mathscr{C}}(X, Y)$$
- $\mathscr{D}$ ia a **wide subcategory** of $\mathscr{C}$ if
$$Ob(\mathscr{D}) = Ob(\mathscr{C})$$

Example. *The subcategory $\mathscr{C}$ of* ***Group*** *consisting of*

- $Ob(\mathscr{C}) = Ob($***Group***$)$.
- *The arrows in $\mathscr{C}$ are the group isomorphisms.*

Example. *We consider the following categories:*

1. *The category* ***Rng*** *of all rings:*
   - $Ob(\mathbf{Rng})$ *is conformed by all rings.*
   - *Ring homomorphisms are arrows, with the usual composition.*
2. *The category* ***Ring*** *of all unitary rings:*

- *Ob*(**Ring**) *is conformed by all unitary rings.*
- *Ring homomorphisms* $\mathscr{R} \longrightarrow \mathscr{S}$ *satisfying* $1_{\mathscr{R}} \mapsto 1_{\mathscr{S}}$ *are arrows with the usual composition.*

*It is clear that **Ring** is a subcategory of **Rng** but it is not full, as morphisms in **Ring** need to take the multiplicative identity of the first ring into the multiplicative identity of the second one.*

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Proposition 1.1.1. *Let* $\mathscr{D}$ *be a subcategory of* $\mathscr{C}$*. Then,* $\mathscr{D}^{op}$ *is a subcategory of* $\mathscr{C}^{op}$

1.2 Special Objects

Definition 1.2.1. An object $I \in \mathscr{C}$ is *initial* if, for every object $X \in \mathscr{C}$, there exists a unique morphism

$$I \to X.$$

Definition 1.2.2. An object $T \in \mathscr{C}$ is *terminal* (or *final*) if, for every object $X \in \mathscr{C}$, there exists a unique morphism

$$X \to T.$$

Terminal and initial are dual notions.

Definition 1.2.3. An object is a *zero object* if it is initial and terminal.