# Seminario de teoría de categorías

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# Contents

# Chapter 1

# Categories: objects and arrows

**Definition 1.0.1** (Category)**.** A category $\mathscr{C}$ consists of the following data:

- A class of *objects*,
$$Ob(\mathscr{C})$$
(we usually write $X \in \mathscr{C}$ instead of $X \in Ob(\mathscr{C})$).

- For every two objects $X, Y \in \mathscr{C}$, a class of *morphisms* (or *arrows*) *from* $X$ *to* $Y$,
$$\mathrm{Hom}_{\mathscr{C}}(X, Y)$$
(we sometimes write $\mathrm{Hom}(X, Y)$ or $\mathscr{C}(X, Y)$ instead of $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, and $f \colon X \to Y$ instead of $f \in \mathrm{Hom}(X, Y)$.

- A *composition law*, associating to morphisms $f \colon X \to Y$ and $g \colon Y \to Z$, a *composition morphism*
$$g \circ f \colon X \to Z.$$
Arrows and the composition law are subject to the following conditions:

  - *Associativity*: given composable morphisms $f, g, h$,
  $$h \circ (g \circ f) = (h \circ g) \circ f.$$

  - Existence of *identity arrows*: for any $X$, there exists a morphism $id_X \colon X \to X$, such that, for all $f \colon X \to Y$ and $g \colon Z \to X$,
  $$f \circ id_X = f, \qquad id_X \circ g = g.$$

**Example.** *Some basic examples:*

1. *The category* **Set** *of all sets:*
   - *Ob*(**Set**) *is the class of all sets.*
   - $\mathrm{Hom}(X, Y)$ *is the set of all mappings* $f: X \to Y$
   - $g \circ f$ *is the usual composition of mappings.*
2. *The category* **Top** *of topological spaces has*
   - *topological spaces* $(X, \mathcal{T})$ *as objects (in other words, Ob*(**Set**) *is the class of all topological spaces).*
   - *continuous mappings* $f: (X, \mathcal{T}) \to (Y, \mathcal{T})$ *as morphisms, with the usual composition.*

   *Notice that this requires two results from topology: (1) composition of continuous mappings is continuous and (2) the identity mapping is continuous.*
3. *The category* **Group** *of groups has*
   - *groups as objects.*
   - *group homomorphisms as arrows, with the usual composition.*

   *This requires the result that compositions of group homomorphisms are group homomorphisms.*

*All of the above examples consist of certain sets (with extra structure in some cases) and mappings between them, but this is not always the case:*

1. *Posets give rise to categories:*
   *For example, we may regard* $(\mathbb{N}, \leq)$ *as a category having*
   - *natural numbers as objects.*
   - *for any* $n, m \in \mathbb{N}$,

     $$\mathrm{Hom}(n, m) = \begin{cases} \{n \to m\}, & \text{if } n \leq m \\ \emptyset, & \text{if } n \not\leq m \end{cases}$$

     *Here* $\{n \to m\}$ *is just any set with a single element, which we regard as the arrow* $n \to m$*. These sets must be chosen so that* $n \to m$ *equals* $n' \to m'$ *if and only if* $n = n'$ *and* $m = m'$.

*Since there is at most one arrow between objects, composition is defined in the only posible way.*

*Observe that the reflexivity and transitivity properties of partial orders are needed in order for this to form a category.*

*Similarly, any set $X$ gives rise to a category*

$$(\mathcal{P}(X), \subseteq),$$

*and any topological space $(X, \mathcal{T})$ gives rise to a category*

$$(\mathcal{T}, \subseteq).$$

2. *Groups as categories:*
   *Given a group $G$, with binary operation "$\cdot$", the category $BG$ has*
   - *a single object, written $\bullet$.*
   - *elements $g \in G$ as arrows $\bullet \xrightarrow{g} \bullet$.*
   - *The composition of two arrows $g, g' \in G$ is the arrow $g \circ g' = g \cdot g'$.*
3. *The category associated to a unitary ring:*
   *Given a unitary ring $R$, with binary operations "$+$" and "$\cdot$", the category $\mathscr{C}_R$ has*
   - *a single object, written $\bullet$.*
   - *elements $r \in R$ as arrows $\bullet \xrightarrow{r} \bullet$.*
   - *The composition of two arrows $r, r' \in R$ is the arrow $r \circ r' = r \cdot r'$.*

   *Observe that the condition that $R$ is unitary is required by the axiom that categories must have identity arrows.*
4. *Given a field $k$, the matrix category $\mathbf{M}at_k$ has*
   - *natural numbers as objects.*
   - *For any $n, m \in \mathbb{N} \setminus \{0\}$, a morphism $n \to m$ is an $m \times n$ matrix with entries in $k$. Composition of such morphisms corresponds to product of matrices. There is a single arrow $0 \to m$ and $m \to 0$, for every $m \in \mathbb{N}$, and composition with these arrows is determined by lack of choice.*

**Definition 1.0.2.** Given a category $\mathscr{C}$, the *opposite category* $\mathscr{C}^{op}$ has

- same objects as $\mathscr{C}$, that is
$$Ob(\mathscr{C}^{op}) = Ob(\mathscr{C}).$$
- Given $X, Y \in \mathscr{C}^{op}$, each morphism $X \to Y$ in $\mathscr{C}$ is an arrow $Y \to X$ in $\mathscr{C}^{op}$, that is,
$$\mathrm{Hom}_{\mathscr{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

TO DO LIST: Clases de objetos, si hay una única clase de hom y entonces hay asignaciones de dominio y codominio o si hay clases disjuntas X,Y;
NOTACIÓN: Hom

**Definition 1.0.3** (Small and Locally small categories)**.**

A category $\mathscr{C}$ is **locally small** if, for all $X, Y \in \mathscr{C}$, $\mathrm{Hom}(X, Y)$ is a set (as opposed to a proper class).

A category is **small** if it is locally small and $Ob(\mathscr{C})$ is also a set.

**Example.** *All categories introduced above are locally small. An example of non-locally small category is the one with*

- *a single object* $\bullet$.
- *Any set* $X$ *is an arrow* $\bullet \xrightarrow{X} \bullet$. *The composition of* $\bullet \xrightarrow{X} \bullet$ *and* $\bullet \xrightarrow{Y} \bullet$ *is* $\bullet \xrightarrow{X \cup Y} \bullet$.

**Remark.** *From now on, we will always consider locally small categories.*

## 1.1 Subcategories

**Definition 1.1.1.** Given a category $\mathscr{C}$, we say that a subset $\mathscr{D} \subseteq \mathscr{C}$ is a **subcategory** of $\mathscr{C}$ if it is a subcollection of objects and morphisms, satisfying that

- All domains and codomains of morphisims in $\mathscr{D}$ are contained.
- The identity morphism of every object in $\mathscr{D}$ is contained.
- Every posible composition between morphisms in $\mathscr{D}$ is contained.

**Example.** *Some basic examples of subcategories are*

1. ***Haus*** *is the subcategory of* ***Top*** *consisting of Hausdorff topological spaces.*

2. ***Ab*** *is the subcategory of* ***Group*** *consisting of abelian groups.*

3.

**Definition 1.1.2** (Full and Wide subcategories)**.**

Let $\mathscr{D}$ be a subcategory of $\mathscr{C}$.

- $\mathscr{D}$ ia a **full subcategory** of $\mathscr{C}$ if $\forall\ X, Y \in \mathscr{D}$, it is satisfied that

$$Hom_{\mathscr{D}}(X, Y) = Hom_{\mathscr{C}}(X, Y)$$

- $\mathscr{D}$ ia a **wide subcategory** of $\mathscr{C}$ if

$$Ob(\mathscr{D}) = Ob(\mathscr{C})$$

**Example.** *Ejemplo de wide subcategory*

**Example.** *We consider the following categories:*

1. *The category* ***Rng*** *of all rings:*
   - $Ob(\mathbf{Rng})$ *is conformed by all rings.*
   - *Ring homomorphisms are arrows, with the usual composition.*
2. *The category* ***Ring*** *of all unitary rings:*
   - $Ob(\mathbf{Ring})$ *is conformed by all unitary rings.*
   - *Ring homomorphisms* $\mathscr{R} \longrightarrow \mathscr{S}$ *satisfying* $1_{\mathscr{R}} \mapsto 1_{\mathscr{S}}$ *are arrows with the usual composition.*

*It is clear that* ***Ring*** *is a subcategory of* ***Rng*** *but it is not full, as morphisms in* ***Ring*** *need to take the multiplicative identity of the first ring into the multiplicative identity of the second one.*

**Example.** *We consider the following categories:*

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   - *$Ob(\mathbf{Ring})$ is conformed by all unitary rings.*
   - *Ring homomorphisms $\mathscr{R} \longrightarrow \mathscr{S}$ satisfying $1_{\mathscr{R}} \mapsto 1_{\mathscr{S}}$ are arrows with the usual composition.*

*It is clear that* ***Ring*** *is a subcategory of* ***Rng*** *but it is not full, as morphisms in* ***Ring*** *need to take the multiplicative identity of the first ring into the multiplicative identity of the second one.*

**Proposition 1.1.1.** *Let $\mathscr{D}$ be a subcategory of $\mathscr{C}$. Then, $\mathscr{D}^{op}$ is a subcategory of $\mathscr{C}^{op}$*

## 1.2 Special Objects

acabada TO DO LIST:

1. Ejemplo 2: RNG, objeto 0, objeto inicial y objeto final PREGUNTAR SI PONGO ESTO O YA SON MUCHOS EJEMPLOS XD bueno se ve que lo he puesto igual
2. flecha 0 QUE ERA ESTO
3. Ejercicios de Clara LOS ENUNCIO PERO QUEDA PENDIENTE VER DONDE PONER LA RESOLUCIÓN
4. salvo isomorfismo único: Consecuencia: Set, Top y Ring no tienen objeto 0 QUÉ TIENE Q VER LO DEL ISOMORFISMO ÚNICO EN ESTE CASO

**Definition 1.2.1.** An object $I \in \mathscr{C}$ is *initial* if, for every object $X \in \mathscr{C}$, there exists a unique morphism
$$I \to X.$$

**Definition 1.2.2.** An object $T \in \mathscr{C}$ is *terminal* (or *final*) if, for every object $X \in \mathscr{C}$, there exists a unique morphism
$$X \to T.$$

Terminal and initial are dual notions.

**Definition 1.2.3.** An object is a *zero object* if it is initial and terminal.

**Example.** *Some basic examples:*

1. *Special objects in category* $(\mathbb{N}, \leq)$*:*

    - *The initial object in this category is 0, since 0* $\leq n \ \forall n \in \mathbb{N}$ *and so there exists a unique arrow from 0 to any other natural number. The uniqueness comes from the fact that there exists only one arrow that goes from one object to another (greater) one.*
    - *There is no final object in this category, since there isn't any natural number greater that all of them.*
    - *Since there's no final object, there's no zero object in this category.*

    *Note: if instead of* $\mathbb{N}$ *we considered* $\mathbb{Z}$*, we wouldn't have an initial object either.*

2. *In **Set** and **Top**:*

    - *We have that* $\emptyset$ *is initial since we can always consider the inclusion as the only arrow from* $\emptyset$ *to any set or topological space.*
    - *We have that any set with only an element is terminal, since there's only one way to define a map from a set to the singleton.*
    - *There are no zero objects since* $\emptyset$ *is the only initial object in these categories.*

3. *In the **Ring** category,* $\mathbb{Z}$ *is an initial object since all arrows must send 0 and 1 to the new ring's neutral elements, and* $\{0\}$ *is terminal. However, in **Rng** we have that* $\{0\}$ *is a zero object.*

## Exercises

puedo poner los enunciados en letra no itálica por favor?? also checkear que la enumeración está bien

**Exercise 1.2.1.** *If we have two initial objects from the same category, then they're isomorphic.*

**Exercise 1.2.2.** *Let* $X_1$ *be an initial object, and let* $X_2 \cong X_1$*. Then* $X_2$ *is initial.*

Analogous results are valid for terminal objects.

**Exercise 1.2.3.** *If* $|X| \neq 1$ *then* $X$ *is not terminal.*

**Exercise 1.2.4.** *Let's consider category* $\mathscr{B}$ *that has objects* $A, B, C, D, F$ *and arrows* $\alpha, \beta, \gamma, \delta, \sigma, \beta\alpha = \sigma\delta$ *and* $\sigma\tau$*. What are the initial and terminal objects?*

$$\begin{tikzcd}
 & B \arrow[dr, "\beta"] & \\
A \arrow[ur, "\alpha"] \arrow[r, "\delta"'] & C \arrow[r, "\sigma"'] & D \\
 & & F \arrow[ul, "\tau"]
\end{tikzcd}$$

## 1.3 Monomorphisms, epimorphisms and isomorphisms

TO DO LIST: Sacado de miniseminario y apuntes del master

1. Anillos conmutativos: mono=epi=no divisor de 0

**Definition 1.3.1** (Monomorphism)**.** Let $\mathscr{C}$ be a category, and $X, Y, Z \in \mathscr{C}$.

We say that an arrow $f : X \longrightarrow Y$ is a **monomorphism** if for any pair of arrows $g, g' : Z \longrightarrow X$ holding the equality $f \circ g = f \circ g'$, they satisfy $g = g'$.

**Definition 1.3.2** (Epimorphism)**.** Let $\mathscr{C}$ be a category, and $X, Y, Z \in \mathscr{C}$.

We say that an arrow $f : X \longrightarrow Y$ is a **epimorphism** if for any pair of arrows $h, h' : Y \longrightarrow Z$ holding the equality $h \circ f = h' \circ f$, they satisfy $h = h'$.

**Example.** *Some examples of mono and epimorphisms are:*

- *The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in **Ring**.*
- *A morphism $f \colon X \to Y$ in **Haus** is an epimorphism if and only if $f(X)$ is dense in $Y$.*

**Definition 1.3.3** (Split monomorphism and epimorphism)**.**

Let $\mathscr{C}$ be a category, $X, Y \in \mathscr{C}$ and $f \in Hom_{\mathscr{C}}(X, Y)$, $g \in Hom_{\mathscr{C}}(Y, X)$ such that

$$g \cdot f = id_X$$

Then $f$ is a monomorphisms and is said to be a **split monomorphism** and $g$ is an epimorphism called **split epimorphism**.

**Proposition 1.3.1.** *Split monomorphisms and epimorphisms are monomorphisms and epimorphisms.*

**Proposition 1.3.2.** *Let $f : X \longrightarrow Y$ be a monomorphism or an epimorphisim in $\mathscr{C}$. Then, $f$ is respectively a monomorphism or an epimorphisim in $\mathscr{C}^{op}$.*

**Proposition 1.3.3** (Operations with monomorphism)**.** *(epimorphism by duality):*

- *Composition of monomorphism is monomorphism.*
- *If $g \circ f$ is monomorphism, then $f$ is monomorphism.*

**Proposition 1.3.4** (Subcategories reflect monomorphism)**.**
*Let $\mathcal{A} \hookrightarrow \mathcal{B}$ be a subcategory, and let $f$ be an arrow in $\mathcal{A}$. If $f$ is monomorphism in $\mathcal{B}$, then $f$ is monomorphism in $\mathcal{A}$.*

**Proposition 1.3.5** (Special arrows from special objects)**.**
*If $I$ is initial, any arrow $X \to I$ is an epimorphism*

**Proposition 1.3.6.** *Every split monomorphism is a monomorphism. Every split epimorphism is an epimorphism.*

pregunta al aire esto no estaría bien demostrar algo a modo ejercicio? Y luego, estaría bien mencionar lo de q split mono implica inyeccion q implica mono (o algo as) para remarcar la diferencia entre las definiciones q tenemos asimiladas

## Isomorphisms

**Definition 1.3.4** (Isomorphism)**.** Let $\mathscr{C}$ be a category, and $X, Y \in \mathscr{C}$.

An arrow $f : X \longrightarrow Y$ is said to be an **isomorphisim** if there exists another arrow $f^{-1} : Y \longrightarrow X$ satisfying that $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$.

**Example.** *Some examples of isomorphisms are:*

1. *A mapping $f\colon X \to Y$ is an isomorphism in **Set** if and only if it is a bijection.*
2. *A group homomorphism $\phi$ is an isomorphism in **Group** if and only if it is bijective.*
3. *The isomorphisms in **Top** are the homeomorphisms.*
4. *The only isomorphisms in $(\mathbb{N}, \leq)$ are the identities.*
5. *A morphism $X \to Y$ in $\mathscr{C}$ is the same as an isomorphism $Y \to X$ in $\mathscr{C}^{op}$.*

6. *Given an object $X \in \mathscr{C}$, an* ***automorphism*** *of $X$ is just an isomorphism $X \to X$. This gives rise to an obvious automorphism subcategory*

$$Aut(X) \hookrightarrow End(X) \hookrightarrow \mathscr{C}.$$

algun ejemplo estaría bien explicarlo si hacemos clase de ejercicios(?

**Proposition 1.3.7** (Basic properties of isomorphisms)**.**

1. *Inverses of isomorphisms are unique.*
2. *Any composition of isomorphisms is an isomorphism.*
3. *If the composition of two arrows is an isomorphism and one of them is an isomorphism, then so is the other.*

**Proposition 1.3.8** (Isomorphisms and subcategories)**.**
*Let $\mathcal{A} \hookrightarrow \mathcal{B}$ be a subcategory, and let $f$ be an arrow in $\mathcal{A}$.*

1. *If $f$ is an isomorphism in $\mathcal{A}$, then $f$ is an isomorphism in $\mathcal{B}$*
2. *Full subcategories reflect isomorphisms:*
   *If $\mathcal{A}$ is a full subcategory and $f$ is an isomorphism in $\mathcal{B}$, then $f$ is an isomorphism in $\mathcal{A}$.*

*(Obs: the fullness condition above cannot be dropped).*

**Proposition 1.3.9** (Characterization of isomorphisms)**.**
*A morphism is an isomorphism if and only if it is a split monomorphism and a split epimorphism.*

**Remark.** *The condition of being split is necessary. We'll give some examples.*

1. *Considerate a category $\mathscr{C}$ given by*
   - $Ob(C) = \{a, b\}$
   - *Appart from the identitty arrows, there only exists an arror $f : a \longrightarrow b$.*

   *It is clear that $f$ is a monomorphism and an epimorphisim, but can't be an isomorphism because it doesn't exist an arrow $g : a \longrightarrow b$.*
2. *In* ***Top*** *category, we have that*

$$\begin{tikzcd} a \arrow[r, mapsto, "f"] \arrow[loop below, "id_a"] & b \arrow[loop below, "id_b"] \end{tikzcd}$$

- *An arrow is a monomorphism if and only if it is injective.*
- *An arrow is an epimorphism if and only if it is surjective.*

*In conclusion, every bijective arrow in* ***Top*** *is a mono and an epimorphism but it is only an isomorphisms if its inverse is contonuous.*

# Chapter 2

# Functors

**Definition 2.0.1** (Functor)**.** A functor $F$ from category $\mathscr{C}$ to $\mathscr{D}$, written $F : \mathscr{C} \to \mathscr{D}$ is

- An arrow between the objects of each category

$$FX : Obj_{\mathscr{C}} \to Obj_{\mathscr{D}}$$

- And, for each $X, Y \in \mathscr{C}$ another arrow

$$Ff : Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$$

  such that:

  1. $F(id_X) = id_{F(X)}, \forall\ X \in \mathscr{C}$
  2. $F(g \circ f) = F(g) \cdot F(f)$ for any pair of composable arrows $f, g$.

Basically, a functor consists of a mapping on objects and a mapping on arrows that preserves all of the structure of a category, namely domains and codomains, composition, and identities. It can be represented by the next diagram:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] & \mathscr{D} \\
X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\longrightarrow"] & FX \arrow[d, "Ff"] \\
Y \arrow[r] & FY
\end{tikzcd}$$

**Notation.** *From now on, we'll use $F$ to refer to both $FX$ and $Ff$.*

Siempre que tienes una subcategoría tienes un funtor

**Example.** 1. *The classical, most simple examples are the **identity functor** $\mathscr{C} \to \mathscr{C}$ defined in the obvious way, and the **inclusion functor** defined for any subcategory $\mathscr{D} \hookrightarrow \mathscr{C}$.*

2. *The **forgetful functor** $U$ can be defined for the "classical" categories defined with a set and some underlying structure like* **Top**, **Group**, **Rng** *or* **Ring**. *The functor makes the category "forget" it's underlying structure and stick with the original set and its arrows as normal maps. For example, for the category of topological spaces we can consider* $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$.

**Lemma 2.0.1.** *Functors preserve isomorphisms.*

*Proof.* Let $\mathscr{C}, \mathscr{D}$ categories, $F : \mathscr{C} \longrightarrow \mathscr{D}$ a functor, $X \in \mathscr{C}, Y \in \mathscr{D}$ and $f : X \longrightarrow Y$ an isomorphism with inverse $g : Y \longrightarrow X$.

Applying the axioms which define functors and that $g \cdot f = id_X$ we have that

$$Fg \cdot Ff = F(g \cdot f) = F(id_X) = id_X$$

which means that $Ff$ is the left inverse of $Fg$.

Arguing by duality, we prove that $Ff$ is the right inverse of $Fg$, and, as a result, $Ff$ and $Fg$ are inverse isomorphisms. □

## 2.1 Contravariant Functors

**Definition 2.1.1** (Contravariant Functor)**.**

Given two categories $\mathscr{C}, \mathscr{D}$, a **contravariant functor** between them is a functor $F : \mathscr{C}^{op} \longrightarrow \mathscr{D}$. Explicitly, this consists of

- An object $FX \in \mathscr{D}\ \forall\ X \in \mathscr{C}$.
- An arrow $Ff : FY \longrightarrow FX$ for any arrow $f : X \longrightarrow Y$, where $X, Y \in \mathscr{C}$.

satisfying the following conditions

- For every composable pair of arrows $f, g$

$$Ff \circ Fg = F(g \circ f)$$

- $\forall\ X \in \mathscr{C}$

$$Fid_X = 1_{FX}$$

$$\begin{tikzcd}
\mathscr{C}^{op} \arrow[r, "F"] & \mathscr{D} \\
X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\longrightarrow"] & FX \\
Y \arrow[r] & FY \arrow[u, "Ff"']
\end{tikzcd}$$

**Example.**

# Chapter 3

# Adjoints

# Chapter 4

# Limits

# Chapter 5

# Yoneda's Lemma

List of categories

- The category of sets

# Bibliography