TME - Part 1

Guillermo Abad López

1)

Law of total probability: $P(A) = \sum_{i=1}^{n} P(A|E_i)\, P(E_i)$ if $\begin{cases} E_i \cap E_j = \emptyset \text{ for } i \neq j \quad (*) \\ S = \bigcup_{i=1}^{n} E_i \quad (**) \end{cases}$

To prove this, let's start from the righ side:

$$\boxed{\sum_{i=1}^{n} P(A|E_i)\, P(E_i) = \sum_{i=1}^{n} P(A \cap E_i) \overset{(*)}{=} P\left(A \cap \bigcup_{i=1}^{n} E_i\right) \overset{(**)}{=} P(A \cap S) = P(A)}$$

2)

a)

It's a binomial distribution, where either you have an accident or you don't:

$$\boxed{P_n(K) = \binom{n}{K} p^K (1-p)^{n-K} = \binom{n}{K} \left(\frac{5}{n}\right)^K \left(1 - 5/n\right)^{n-K}}$$

$(5 = np,\ p = \frac{5}{n})$

In the limit case $p \to 0,\ n \to \infty$ but $p \cdot n = \mu = 5$, we go to a Poisson distribution:

$$\boxed{P(K) = \frac{\mu^K e^{-\mu}}{K!} = \frac{5^K e^{-5}}{K!}}$$

b)

- $\boxed{p(0) \cup p(1)} \overset{\text{(independent!)}}{=} p(0) + p(1) = \frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!} = (1+5)\, e^{-5} = \boxed{6 e^{-5} \approx 0{,}04 = 4\%}$
- $\boxed{p(5)} = \frac{5^5 e^{-5}}{5!} = \frac{5^4 e^{-5}}{4!} = \boxed{\frac{625}{24} e^{-5} \simeq 0{,}1752 = 17{,}5\%}$

c)

If that day wasn't a special day without cars, and we asume a more or less homogenous density of traffic during the day, we could say that with 5σ level of confidence, the mean is smaller than 16 death/day, because:

$$\boxed{p(0)_{\mu=16} = e^{-16} = 1{,}12 \cdot 10^{-7} < 5\sigma \text{ level}}$$

If the mean was 16, what we saw would happen only once every 14.000 years.

We could also say with confidence $\approx 4\sigma$ level that the mean is smaller than 10 death/days, because:

$$\boxed{p(0)_{\mu=10} = e^{-10} = 4{,}5 \cdot 10^{-5} \approx 4\sigma \text{ level}}$$

In this case the event would happen once every 100 years.

So probably the mean < 10 death/day, but it could also be that one day isn't sufficient time to make a good estimation, because special events could happen the whole day, such as a covid-19 lockdown, or a free pollution day, etc...

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3)

$$M_n = E[(x-\mu)^n] = \sum_{x=0}^{\infty} (x-\mu)^n p(x) = \sum_{x=0}^{\infty} (x-\mu)^n \frac{\mu^x e^{-\mu}}{x!}$$

$$\hookrightarrow \frac{dM_n}{d\mu} = \sum_{x=0}^{\infty} n(x-\mu)^{n-1}(-1)p(x) + \sum_{x=0}^{\infty} (x-\mu)^n \times \frac{x\mu^{x-1}e^{-\mu}}{x!} + \sum_{x=0}^{\infty} (x-\mu)^n \frac{\mu^x(-1)e^{-\mu}}{x!} =$$

$$= -n\sum_{x=0}^{\infty} (x-\mu)^{n-1} p(x) + \sum_{x=0}^{\infty} (x-\mu)^n p(x)\left(\frac{x}{\mu}-1\right) =$$

$$= -n M_{n-1} + \frac{1}{\mu}\sum_{x=0}^{\infty} (x-\mu)^n p(x)(x-\mu) =$$

$$= -n M_{n-1} + \frac{1}{\mu} M_{n+1}$$

so finally we see that:

$$\boxed{M_{n+1} = \mu\left(\frac{dM_n}{d\mu} + n M_{n-1}\right)}$$

4)

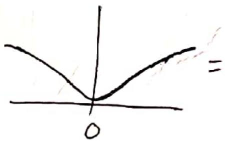
You can't calculate the mean and the variance, they are not defined, "there is not enough time to converge to the mean, between drastic jumps of $|\underline{x}|$, so it never converges".

But let's try nevertheless:

(it's the simple case $x_0=0$, $\gamma=1$, but it doesn't converge for any case.)

$$\boxed{E[\underline{x}]} = \int_{-\infty}^{\infty} x \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{2\pi}\log(x^2+1)\Big|_{-\infty}^{\infty} = \infty - \infty \quad \text{Does not converge!}$$

But if we plot the function:



$= \frac{1}{2\pi}\log(x^2+1)$

We can see it is centered at 0, and it is symmetric, so we deduce that both ∞ and $-\infty$, in the result when we substitute, behave the same giving a kind of "0" result. But at the end it's not convergent nevertheless.

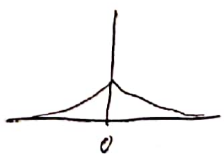
It happens the same for the variance:

$E[(\underline{x}-\mu)^2]$, because μ is not defined, we can't do it. Even if we wanna take $\mu=0$, which would be wrong, it also doesn't converge:

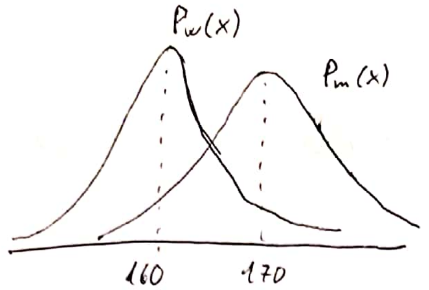
$$\boxed{E[\underline{x}^2]} = \int_{-\infty}^{\infty} x^2 \frac{1}{\pi(1+x^2)}\,dx = x - \frac{1}{\tan(x)}\Big|_{-\infty}^{\infty} = 2\left(\infty - \frac{1}{\tan(\infty)}\right) \quad \text{Does not converge!}$$

It's characteristic function in the other hand, is well defined:

$$\boxed{\varphi_{\underline{x}}(t) = E[e^{it\underline{x}}]} = \int_{-\infty}^{\infty} e^{itx}\frac{1}{\pi(1+x^2)}\,dx = \boxed{e^{-|t|} = \frac{1}{e^{|t|}}}$$



5)



(❀ Because I had problems with this excercice, I did numerical simulations, to check my result, where I got the result 86,095 % of the times the men will be taller, I'll add them after the excercice)

- First I tried: $$\int_{-\infty}^{\infty} \frac{e^{\frac{-(x-170)^2}{2\cdot 7^2}}}{7\sqrt{2\pi}} \cdot \underbrace{\left(\int_{-\infty}^{x} \frac{e^{\frac{-(y-160)^2}{2\cdot 6^2}}}{6\sqrt{2\pi}}\, dy\right)} dx$$

Which would get the probability that $y < x$: $\begin{cases} y = \text{Women height rand. var.} \\ x = \text{Men height rand. var.} \end{cases}$ and the integrate the product $p(x)\, p(y<x\,|\,x)$, to take into account all possible x.

But the integral does not converge, and I couldn't find any good serie to approximate it, using Mathematica, so I moved on.

(Mathematica results:
$$\int_{-\infty}^{x} \frac{e^{\frac{-(y-160)^2}{2\cdot 6^2}}}{6\sqrt{2\pi}}\, dy = \frac{1}{2} + \frac{1}{2}\,\mathrm{Erf}\left(\frac{-160+x}{6\sqrt{2}}\right) \simeq \frac{1}{2} + \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{-160+x}{6\sqrt{2}}\right)^{2n+1}}{n!\,(2n+1)}$$
which didn't converge, neither it's integral, it gave very different results if you picked 5 terms, 6 or 7...)

- So then I tried defining $\underline{D = X - Y}$, and because X and Y are independent, D follows a gaussian with:

$$\hat{D} = \hat{X} - \hat{Y} = 10 \quad \text{and} \quad \sigma_D^2 = \sigma_x^2 + \sigma_y^2 = 7^2 + 6^2$$

And the only thing left was integrating the new gaussian over:

$$\int_{0}^{\infty} \frac{e^{\frac{-(D-10)^2}{(6^2+7^2)\cdot 2}}}{\sqrt{6^2+7^2}\,\sqrt{2\pi}}\, dD = \boxed{0,860962}$$

$\left(\frac{\text{Men taller}}{\begin{matrix} X > Y \\ D > 0 \end{matrix}}\right)$

Answer: 86,1% of the times the men will be taller

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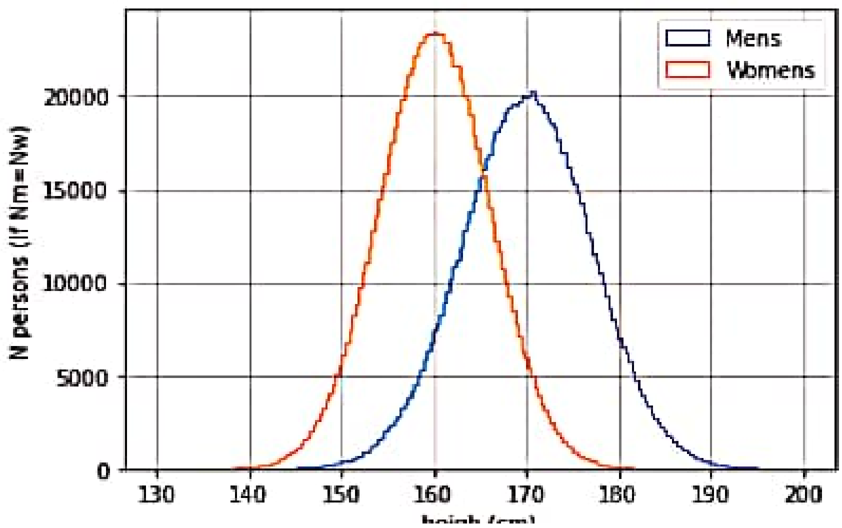
In [42]:

```python
%matplotlib inline
import scipy.stats as scp
import matplotlib.pyplot as plt
import numpy as np

N=10000000

x_dist= scp.norm(loc=170,scale=7)
y_dist= scp.norm(loc=160,scale=6)

counter=0
for i in range(N):
    x= x_dist.rvs(1)
    y= y_dist.rvs(1)
    d=x-y
    if d > 0:
        counter+=1
print(counter/N*100,"%")
```

```
86.09790000000001 %
```

In [40]:

```python
%matplotlib inline
import scipy.stats as scp
import matplotlib.pyplot as plt
import numpy as np


N=1000000
Nbins=200
bins= np.linspace(130,200,Nbins)

x_dist= scp.norm(loc=170,scale=7)
y_dist= scp.norm(loc=160,scale=6)

x_points= x_dist.rvs(N)
y_points= y_dist.rvs(N)

hx= plt.hist(x_points,bins, histtype=u'step', label='Mens')
hy= plt.hist(y_points,bins, histtype=u'step', label='Womens')

plt.grid(True)
plt.xlabel('heigh (cm)')
plt.ylabel('N persons (if Nm=Nw)')
plt.legend(loc=1)
plt.show()
```



6)

We have a multinomial distribution which fullfills:

$$\sum_{i=1}^{l} p_i = 1 \quad , \quad \sum_{i=1}^{l} \underline{k_i} = n \quad , \quad p_n(\vec{k}) = \frac{n!}{\prod_i k_i!} \prod_{i=1}^{l} p_i^{k_i}$$

First we are going to do it the hard way, to then show that it can be done way quicker, considering n independent experiments:

$$\boxed{\hat{k}_i} = E(\underline{k_i}) = \sum_{k_i=0}^{n} k_i\, P_n(k_i) = \sum_{k_i=1}^{n} k_i\, P_n(k_i) \overset{⊛}{=}$$

$$= \sum_{k_i=1}^{n} k_i \sum_{\substack{k_{j\neq i}=0 \\ \text{with } \sum_s k_s = n}}^{n} P_n(\vec{k}\,|\,k_i) = \sum_{k_i=1}^{n} k_i \sum_{\substack{k_{j\neq i}=0 \\ \text{with } \sum_s k_s = n}}^{n} \frac{n!}{\prod_\alpha k_\alpha!} \prod_\alpha^{l} p_\alpha^{k_\alpha} =$$

$$= \sum_{k_i=1}^{n} \sum_{\substack{k_{j\neq i}=0 \\ \text{with } \sum_s k_s = n}}^{n} \frac{n!}{\prod_{\alpha\neq i} k_\alpha!\,(k_i-1)!}\, p_i \prod_{\alpha\neq i}^{l} p_\alpha^{k_\alpha}\, p_i^{k_i-1} \overset{⊛⊛}{=}$$

$$= \sum_{k_i'=0}^{n-1} \sum_{\substack{k_{j\neq i}=0 \\ \text{with } \sum_s k_s' = n-1}}^{n} \frac{(n-1)!}{\prod_\alpha k_\alpha'!} (n p_i) \prod_{\alpha=1}^{l} p_\alpha^{k_\alpha'} \overset{⊛⊛⊛}{=}$$

$$= (n p_i) \sum_{\substack{k_j'=0 \\ \text{with } \sum_s k_s' = n'}}^{n'} \frac{n'!}{\prod_\alpha k_\alpha'!} \prod_{\alpha=1}^{l} p_\alpha^{k_\alpha'} \overset{⊛⊛⊛⊛}{=}$$

$$= (n p_i)\left(\sum_{\alpha=1}^{l} p_\alpha\right)^{n'} = (n p_i)\cdot 1^{n'} = \boxed{n p_i}$$

Where we have used:

$$⊛: \; P_n(k_i) = \sum_{\vec{k}\text{ with } k_i} P_n(\vec{k}\,|\,k_i) = \sum_{\substack{k_{j\neq i}=0 \\ \text{with } \sum_s k_s = n}}^{n} P_n(\vec{k}\,|\,k_i)$$

$$⊛⊛: \; k_i' \equiv k_i - 1 \quad \text{and} \quad k_{j\neq i}' = k_j$$

$⊛⊛⊛$: $n' = n-1$ and $\sum_{k_{j\neq i}=0}^{n} = \sum_{k_{j\neq i}=0}^{n-1}$ because the case n means that $k_i = 0$ and so $k_i P_n(k_i) = 0$

$⊛⊛⊛⊛$: <u>Multinomial theorem:</u>

$$(p_1+p_2+\dots+p_l)^n = \sum_{l=0} \binom{n}{l} \prod p^l$$

Now let's compute it again, taking the E_n experiment as n E independent experiments, so:

$$\boxed{\hat{K}_i} = E(\underline{K_i}) \overset{\circledast}{=} E\left(\sum_{\beta=1}^{n} \underline{K_i}^{(\beta)}\right) \overset{\circledast\circledast}{=} \sum_{\beta=1}^{n} E(\underline{K_i}^{(\beta)}) = \sum_{\beta=1}^{n}\left(\sum_{K_i=0,1} K_i\, p_1(K_i)\right) \overset{\circledast\circledast\circledast}{=}$$

$$= \sum_{\beta=1}^{n}\left(1\cdot \underbrace{p_1(K_i=1)}_{p_i}\right) = \sum_{\beta=1}^{n} p_i = \boxed{n\, p_i}$$

where:

$\circledast$: βth experiment outcome for K_i (ϕ or 1) $= K_i^{(\beta)}$

$\circledast\circledast$: Independent experiments: $E(A+B) = E(A) + E(B)$

$\circledast\circledast\circledast$: $K_i = 0$ makes $K_i\, p(K_i) = 0$ and $p_1(\vec{K}) = \dfrac{\prod_{j=1}^{} p_j^{K_j}}{\prod_i K_i!} = \dfrac{p_i^{1}}{1} = p_i$ (remains $p_i \neq 0$)

And because this is much easier, we are going to compute the covariance matrix in this way:

(introducing α'th and β'th experiments)

$$\boxed{C_{ij}} = E(\underline{K_i}\underline{K_j}) - E(\underline{K_i})\,E(\underline{K_j}) = E\left(\sum_{\alpha,\beta=1}^{n} \underline{K_i}^{(\alpha)} \underline{K_j}^{(\beta)}\right) - (np_i)(np_j) =$$

(if $\alpha=\beta$ only one $K_i \neq 0$ so i must $= j$)

$$= E\left(\sum_{\alpha=1}^{n} (\underline{K_i}^{(\alpha)})^2 \delta_{ij} + \sum_{\alpha\neq\beta}^{n} \underline{K_i}^{(\alpha)} \underline{K_j}^{(\beta)}\right) - n^2 p_i p_j =$$

(if $\alpha\neq\beta$ they are independent experiments: $E(A\cdot B) = E(A)\cdot E(B)$)

$$= \delta_{ij} \sum_{\alpha=1}^{n} E\left[(\underline{K_i}^{(\alpha)})^2\right] + \sum_{\alpha\neq\beta}^{n} E(\underline{K_i}^{(\alpha)})\,E(\underline{K_j}^{(\beta)}) - n^2 p_i p_j =$$

$$= \delta_{ij} \sum_{\alpha=1}^{n} p_i + \sum_{\alpha\neq\beta}^{n} p_i p_j - n^2 p_i p_j =$$

$$\left(\sum_{K_i=0,1} K_i^2\, p_1(K_i) = 1^2\cdot \underbrace{p_1(K_i=1)}_{p_i} = p_i\right)$$

$$= \delta_{ij}\, n p_i + n(n-1) p_i p_j - \cancel{n^2 p_i p_j} =$$

$$\left(\sum_{\alpha=1}^{n} \sum_{(\beta\neq\alpha)=1}^{n} = n\cdot(n-1)\right)$$

(n possibilities for α; 1 less for β each time ($\beta\neq\alpha$))

$$= \delta_{ij}\, n p_i - n p_i p_j = \boxed{n p_i (\delta_{ij} - p_j)}$$

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7)

$$P(k) = \frac{\mu^k e^{-\mu}}{k!}$$

First, using Mathematica, we can easily see:

$$\boxed{\hat{k} = E(\underline{k}) = \sum_{k=0}^{\infty} k\, p(k) = \sum_{k=0}^{\infty} \frac{k \mu^k e^{-\mu}}{k!} = \mu}$$

$$\boxed{\sigma^2(\underline{k}) = E[(\underline{k}-\mu)^2] = \sum_{k=0}^{\infty} (k-\mu)^2 p(k) = \sum_{k=0}^{\infty} \frac{(k-\mu)^2 \mu^k e^{-\mu}}{k!} = \mu}$$

Now let's compute them by hand:

$$\boxed{\hat{k} = E(\underline{k}) = \sum_{k=0}^{\infty} \frac{k \mu^k e^{-\mu}}{k!} = \sum_{k=1}^{\infty} \frac{k \mu^k e^{-\mu}}{k!} = \mu \sum_{k=1}^{\infty} \frac{\mu^{k-1} e^{-\mu}}{(k-1)!} \overset{(k'=k-1)}{=}}$$

$$= \mu \sum_{k'=0}^{\infty} \frac{\mu^{k'} e^{-\mu}}{k'!} = \mu \underbrace{\sum_{k'=0}^{\infty} p(k')}_{1} = \boxed{\mu}$$

$$\boxed{\sigma^2(k) = E[(\underline{k}-\mu)^2]} = \sum_{k=0}^{\infty} \frac{(k-\mu)^2 \mu^k e^{-\mu}}{k!} = \sum_{k=0}^{\infty} \frac{(k^2-2k\mu+\mu^2)\ \mu^k e^{-\mu}}{k!} =$$

$$= \mu \sum_{k=1}^{\infty} \frac{k \mu^{k-1} e^{-\mu}}{(k-1)!} - 2\mu^2 \sum_{k=1}^{\infty} \frac{\mu^{k-1} e^{-\mu}}{(k-1)!} + \mu^2 \sum_{k=0}^{\infty} \frac{\mu^k e^{-\mu}}{k!} \overset{(k'=k-1)}{=}$$

$$= \mu \sum_{k'=0}^{\infty} \frac{(k'+1) \mu^{k'} e^{-\mu}}{k'!} - 2\mu^2 \sum_{k'=0}^{\infty} \frac{\mu^{k'} e^{-\mu}}{k'!} + \mu^2 \underbrace{\sum_{k=0}^{\infty} p(k)}_{1} =$$

$$= \mu \Big[\sum_{k'=1}^{\infty} \frac{k' \mu^{k'} e^{-\mu}}{k'!} + \underbrace{\sum_{k'=0}^{\infty} p(k')}_{1} \Big] - 2\mu^2 \underbrace{\sum_{k'=0}^{\infty} p(k')}_{1} + \mu^2 \overset{(k''=k'-1=k-2)}{=}$$

$$= \mu \Big[\mu \sum_{k''=0}^{\infty} \frac{\mu^{k''} e^{-\mu}}{k''!} + 1 \Big] - 2\mu^2 + \mu^2 =$$

$$= \mu \Big[\mu \underbrace{\sum_{k''=0}^{\infty} p(k'')}_{1} + 1 \Big] - \mu^2 = \mu(\cancel{\mu}+1) - \cancel{\mu^2} = \boxed{\mu}$$

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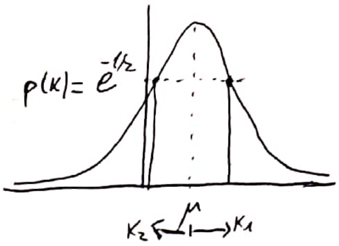
8)

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



$K = \mu \pm \text{width} \longrightarrow \text{width} = K_1 - \mu$

so comparing $p(K_1)$ with $e^{-1/2}$ we will obtain:

$$p(K_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(K_1-\mu)^2}{2\sigma^2}} = e^{-1/2} \; ;$$

$$e^{-\frac{(K_1-\mu)^2}{2\sigma^2}} = \sigma\sqrt{2\pi}\, e^{-1/2} \; ;$$

$$-\frac{(K_1-\mu)^2}{2\sigma^2} = \ln(\sigma\sqrt{2\pi}) - \frac{1}{2} \; ;$$

$$-(\text{width})^2 = 2\sigma^2 \ln(\sigma\sqrt{2\pi}) - \sigma^2 \; ;$$

$$\boxed{|\text{width}| = \sqrt{\sigma^2 - 2\sigma^2 \ln(\sigma\sqrt{2\pi})} = \sigma\sqrt{1 + 2\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)}}$$

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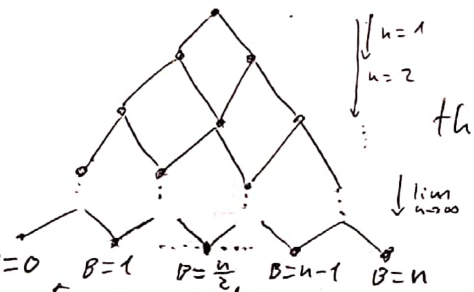
9)

In a galton board:



the last positions will be obtained only when a concrete nº of right bounces is achived, which we are going to call $B^{(i)}$ (each individual bounce), with return values $\begin{cases} B^{(i)} = 1 & p(B^{(i)}=1) = \frac{1}{2} \text{ "right"} \\ B^{(i)} = 0 & p(B^{(i)}=0) = \frac{1}{2} \text{ "left".} \end{cases}$

So then our final position for the ball will be the sum of those $B^{(i)}$, so we define:

$$\underline{B} = \sum_{i=1}^{n} \underline{B^{(i)}} \quad , \text{ where } B^{(i)} \text{ follows a binomial with } \mu = \frac{1}{2} \text{ and } \sigma^2 = \frac{1}{4}.$$

If we then normalize this distribution $\underline{B}$ of the balls in the final position, as:

$$\underline{B'} = \frac{1}{n}\sum_{i=1}^{n} B^{(i)} \xrightarrow{\lim_{n\to\infty}} \left(\rho(B') = \text{Gaussian with } \mu = \frac{1}{2} \text{ and } \sigma^2 = \frac{1}{4n} \right)$$

And finally from the <u>central limit theorem</u> we know that $\underline{B'}$ will follow a Gaussian in the $\lim_{n\to\infty}$. So we know that $\underline{B}$ will follow a gaussian with $\mu = \frac{n}{2}$ and $\sigma^2 = \frac{1}{4}$. ✓✓

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10)

We are going to consider we have n independent Binomial distributions, which can give either 1 for "positive" or 0 for "negative", so:

$$p(k^{(i)}=1) = \frac{1}{200}^{1} \cdot \frac{199}{200}^{1-1} = \frac{1}{200} \equiv p$$

$$p(k^{(i)}=0) = \frac{1}{200}^{0} \frac{199}{200}^{1-0} = \frac{199}{200} \equiv q$$

So, as we know if K is the total of positives:

$$\hat{K} = np = \frac{n}{200} \quad \text{and} \quad \sigma^2(\underline{K}) = npq = \frac{n \cdot 199}{200^2}$$

If we want to determine this $\hat{K}$ with an error of 1%, let's define the "frequency":

$$\underline{X} = K/n \qquad \text{which has} \qquad \hat{x} = \frac{1}{200} = R \quad \text{and} \quad \sigma^2(x) = \frac{pq}{n} = \frac{199}{200^2 \cdot n}$$

If we consider 1% of accuracy that $3\sigma \approx \frac{R}{100}$, which means that 99,7% of the times this experiment would happen the obtained $\underline{R}$ would land in a 1% error margin of the actual $R = \frac{1}{200}$, we get:

$$\left.\begin{aligned} 3\sigma &= \frac{3 \cdot \sqrt{199}}{200\sqrt{n}} \\ \frac{R}{100} &= \frac{1}{100 \cdot 200} \end{aligned}\right\} \quad \sqrt{n} = 3 \cdot 100\sqrt{199} \implies \boxed{n \approx 9 \cdot 199 \cdot 10^4 \approx 2 \cdot 10^7}$$

Doing the same for $1\sigma, 2\sigma, \dots a\sigma$, we get that:

$$\boxed{n \approx 2a^2 \cdot 10^6 \quad \text{for a confidence level of } a\sigma \text{ that } \underline{R} \text{ would land on the actual } R = \frac{1}{200} \pm 1\%.}$$

11)

Let's check normalization:

$$\int_0^\infty e^{-t/\tau}\,dt = -\tau\, e^{-t/\tau}\Big|_0^\infty = -(-\tau) = \tau \quad ✓$$

So now let's find the likelyhood function:

$$\mathcal{L}(\tau) = \prod_{i=1}^{n} f(t_i, \tau) = \prod_{i=1}^{n} \tau^{-1} e^{-t_i/\tau}$$

And the log-likelihood is then:

$$\log(\mathcal{L}(\tau)) = \sum_{i=1}^{n} \left(-\log(\tau) - t_i/\tau\right)$$

So maximizing this, gives:

$$\left(\hat{t} = \frac{1}{n}\sum_{i=1}^{n} t_i\right)$$

$$0 = \frac{d\log(\mathcal{L}(\tau))}{d\tau} = \sum_{i=1}^{n}\left(-\frac{1}{\tau} + \frac{t_i}{\tau^2}\right) = -\frac{\cancel{n}}{\cancel{\tau}} + \frac{\cancel{n}\,\hat{t}}{\cancel{\tau^2}} \longrightarrow \boxed{\tau = \hat{t}}$$

As we would expect!

Checking it's a maximum:

$$\frac{d^2\log(\mathcal{L}(t))}{dt} = \frac{n}{\tau^2} - 2\frac{n\hat{t}}{\tau^3}$$

$$\frac{d^2\log(\mathcal{L}(t))}{dt}\Big|_{\tau=\hat{t}} = \frac{n}{\hat{t}^2} - 2\frac{n}{\hat{t}^2} < 0$$

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12)

Let's first calculate how many protons (p^+) are in the 500 tons of water:

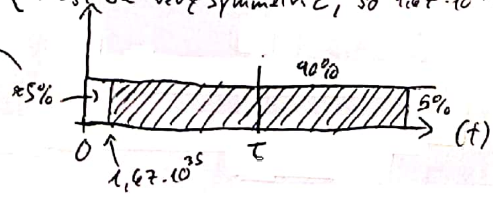
$$500 \text{ tons } H_2O \cdot \frac{10^6 g}{1 \text{ ton}} \cdot \frac{1 \text{ mol } H_2O}{18 \text{ g } H_2O} \cdot \frac{6{,}022 \cdot 10^{23}\, H_2O}{1 \text{ mol } H_2O} \cdot \frac{10\, p^+}{1\, H_2O} = \boxed{1{,}67 \cdot 10^{32}\, p^+}$$

Because we are in particle decays, where the probability gets is the same always, we can actually count all the protons as a single one that has survived $1000 \frac{\text{days}}{p} \cdot 1{,}67 \cdot 10^{32}\, p^+ = \boxed{1{,}67 \cdot 10^{35} \text{ days}}$

So, if we have a single decay that has survived this time, to have a 90% confidence level. We need that:

(Because it's so plain the function, it must be very symmetric, so $1{,}67 \cdot 10^{35} \approx 5\%$)

$$\int_0^{1{,}67\cdot 10^{35}} \frac{e^{-t/\tau}}{\tau}\,dt \approx 0{,}05$$



$$\downarrow$$

$$1 - e^{-\frac{1{,}67\cdot 10^{35}}{\tau}} \approx 0{,}05 \longrightarrow \tau = \frac{1{,}67\cdot 10^{35}}{-\ln(0{,}95)} = 3{,}26 \cdot 10^{36} \text{ days} =$$

$$= \boxed{8{,}92 \cdot 10^{33} \text{ years}}$$

13)

$$\hat{S}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\underline{x_i} - \hat{\underline{x}})^2 \quad \text{where} \quad \hat{\underline{x}} = \frac{1}{n} \sum_{i=1}^{n} \underline{x_i}$$

with:

- $E(\hat{\underline{x}}) = \hat{x} \equiv \mu_1$
- $\sigma^2(\hat{\underline{x}}) = \frac{\sigma^2(\underline{x_i})}{n}$
- $E(\hat{S}^2) = \sigma^2(\underline{x_i})$

Let's now compute $\sigma^2(\hat{S}^2)$, starting by:

- $$\sigma^2(\hat{S}^2) = E[(\hat{S}^2 - \underbrace{E(\hat{S}^2)}_{\sigma^2(x_i)})^2] = E[(\hat{S}^2 - \sigma^2(x_i))^2] = E[(\hat{S}^2)^2] - 2\underbrace{E[\hat{S}^2]}_{\sigma^2(x_i)}\sigma^2(\underline{x_i}) + (\sigma^2(\underline{x_i}))^2 =$$

$$= E[(\hat{S}^2)^2] - \sigma^2(\underline{x_i})^2$$

And because $\hat{S}^2$ only depends on differences between $\underline{x_i} - \hat{\underline{x}}$, the result will be the same if we make achange of variable: $x' = x - \mu$. This is like changing from our distribution to another one identical but centered at 0, they both share the same value for $\hat{S}^2$, so let's do this other case which will be easier.

For this new distribution, we will have:

- $\mu_i = 0$ for i odd ($\mu_1 = \hat{x} = 0$, $\mu_3 = 0$, $\mu_5 = 0, \dots$)
- $\mu_2 = E[\underline{x_i}^2] = \frac{1}{n}\sum_{i=1}^{n} E[\underline{x_i}^2] = \frac{1}{n} E[\sum_{i=1}^{n} \underline{x_i}^2] = \frac{1}{n} E[\sum_{i=1}^{n} (\underline{x_i} - \hat{\underline{x}})^2] = \sigma^2(x_i)$

And with all this clear let's compute the unkown term: $E[(\hat{S}^2)^2]$

$$E[(\hat{S}^2)^2] = E[(\frac{1}{n-1}\sum_{i=1}^{n}(\underline{x_i} - \hat{\underline{x}})^2)^2] = \frac{1}{(n-1)^2} E[(\sum_{i=1}^{n}(\underline{x_i}^2 - 2\underline{x_i}\hat{\underline{x}} + \hat{\underline{x}}^2))^2] =$$

$$= \frac{1}{(n-1)^2} E[(\sum_{i=1}^{n} \underline{x_i}^2 - 2n\hat{\underline{x}}^2 + n\hat{\underline{x}}^2)^2] = \frac{1}{(n-1)^2} E[(\sum_{i=1}^{n} \underline{x_i}^2 - n\hat{\underline{x}}^2)^2] =$$

$$= \frac{1}{(n-1)^2} E[(\sum_{i=1}^{n} \underline{x_i}^2)^2 - 2n\hat{\underline{x}}^2(\sum_{i=1}^{n} \underline{x_i}^2) + n^2(\hat{\underline{x}}^2)^2] =$$

$$= \frac{1}{(n-1)^2} E[(\sum_{i=1}^{n} \underline{x_i}^2)^2 - 2n(\frac{1}{n}\sum_{j=1}^{n} \underline{x_j})^2(\sum_{i=1}^{n} \underline{x_i}^2) + n^2(\frac{1}{n}\sum_{j=1}^{n} \underline{x_j})^4] =$$

$$= \frac{1}{(n-1)^2} E[\underbrace{(\sum_{i=1}^{n} \underline{x_i}^2)^2}_{(1)} - \underbrace{\frac{2}{n}(\sum_{j=1}^{n} \underline{x_j})^2(\sum_{i=1}^{n} \underline{x_i}^2)}_{(2)} + \underbrace{\frac{1}{n^2}(\sum_{j=1}^{n} \underline{x_j})^4}_{(3)}]$$

where:

$$(1) = E\Big[\big(\sum_{i=1}^{n} x_i^2\big)^2\Big] = E\Big[\sum_{i=1}^{n} x_i^4 + \sum_{i\neq j} x_i^2 x_j^2\Big] = \sum_{i=1}^{n} \overbrace{E(x_i^4)}^{\mu_4} + \sum_{i\neq j} \overbrace{E(x_i^2)}^{\mu_2}\, \overbrace{E(x_j^2)}^{\mu_2} =$$

$$= \underline{n\mu_4 + n\cdot(n-1)(\mu_2)^2}$$

$$(2) = -\frac{2}{n} E\Big[\big(\sum_{j=1}^{n} x_j\big)^2 \big(\sum_{i=1}^{n} x_i^2\big)\Big] = -\frac{2}{n} E\Big[\big(\sum_{j=1}^{n} x_j^2 + \sum_{j\neq k} x_j x_k\big)\big(\sum_{i=1}^{n} x_i^2\big)\Big] =$$

$$= -\frac{2}{n} E\Big[\sum_{i=1}^{n} x_i^4 + \sum_{i\neq j} x_i^2 x_j^2 + \sum_{k\neq j=i} x_k x_i^3 + \sum_{j\neq k=i} x_j x_i^3 + \sum_{k\neq j\neq i} x_i^2 x_j x_k\Big] =$$

$$= -\frac{2}{n}\Big(\sum_{i=1}^{n} E(x_i^4) + \sum_{i\neq j} E(x_i^2)E(x_j^2) + \sum_{k\neq j=i} E(x_k)E(x_i^3) + \sum_{j\neq k=i} E(x_j)E(x_i^3) + \sum_{k\neq j\neq i} E(x_i^2)E(x_j)E(x_k)\Big) =$$

$$= -\frac{2}{n}\Big(n\mu_4 + n\cdot(n-1)\,\mu_2\cdot\mu_2 + n\cdot(n-1)\cancelto{0}{\mu_1}\cancelto{0}{\mu_3} + n\cdot(n-1)\cancelto{0}{\mu_1}\cancelto{0}{\mu_3} + n\cdot(n-1)(n-2)\,\mu_2\cancelto{0}{\mu_1}\cancelto{0}{\mu_1}\Big) =$$

$$= \underline{-\frac{2}{n}\big(n\mu_4 + n(n-1)(\mu_2)^2\big)}$$

$$(3) = \frac{1}{n^2} E\Big[\big(\sum_{i=1}^{n} x_i\big)^4\Big] = \frac{1}{n^2} E\Big[\sum_{i=1}^{n} x_i^4 + 4\sum_{i\neq j} x_j^3 x_i + 3\sum_{i\neq j} x_i^2 x_j^2 + 6\sum_{i\neq j\neq k} x_i^2 x_j x_k + \sum_{i\neq j\neq k\neq l} x_i x_j x_k x_l\Big] =$$

$$= \frac{1}{n^2}\Big[n\mu_4 + 4\,n\cdot(n-1)\,\mu_3\cancelto{0}{\mu_1} + 3n(n-1)\,\mu_2\mu_2 + 6\,n\cdot(n-1)(n-2)\,\mu_2\cancelto{0}{\mu_1\mu_1} +$$

$$+ n\cdot(n-1)\cdot(n-2)\cdot(n-3)\,\cancelto{0}{\mu_1\cdot\mu_1\cdot\mu_1\cdot\mu_1}\Big] = \underline{\frac{1}{n^2}\big[n\mu_4 + 3n(n-1)(\mu_2)^2\big]}$$

And finally putting all these terms together, one gets:

$$E[(\hat{s}^2)^2] = \frac{1}{(n-1)^2}\Big(\big(1-\frac{2}{n}+\frac{1}{n^2}\big)n\mu_4 + \big(1-\frac{2}{n}+\frac{3}{n^2}\big)n(n-1)(\mu_2)^2\Big) =$$

$$= \frac{1}{(n-1)^2}\frac{1}{n}\Big(\big(\overbrace{n^2-2n+1}^{(n-1)^2}\big)\mu_4 + (n^2-2n+3)(n-1)(\mu_2)^2\Big) =$$

$$= \frac{1}{n}\Big(\mu_4 + \frac{n^2-2n+3}{n-1}(\mu_2)^2\Big)$$

And adding the $\sigma^2(x_i)$ term we get:

$$\sigma^2(\hat{s}^2) = \frac{1}{n}\Big(\mu_4 + \frac{n^2-2n+3}{n-1}(\mu_2)^2\Big) - (\mu_2)^2 = \frac{1}{n}\Big(\mu_4 + \frac{\cancel{n^2}-2n+3-\cancel{n^2}+n}{n-1}(\mu_2)^2\Big) =$$

$$= \frac{1}{n}\Big(\mu_4 + \frac{-n+3}{n-1}(\mu_2)^2\Big) = \frac{1}{n}\Big(\mu_4 - \frac{n-3}{n-1}(\mu_2)^2\Big)$$

Which undoind the change of coordinates $\underline{x' = x - \mu}$, finally is:

$$\boxed{\sigma^2(\hat{s}^2) = \frac{1}{n}\Big(m_4 - \frac{n-3}{n-1}\sigma^4\Big)}$$

14)

To get random samples generated in any distribution, we only need:

1) generate a random uniform distribution from $(0,1)$ $\equiv \boxed{u}$

2) Know the cdf of the distribution, which should be uniformly distributed in the y-axis. $\equiv \boxed{G(x)}$

3) Invert the cdf and apply it to u. $\equiv \boxed{G^{-1}(\underline{u}) = \underline{K}}$

And the variable $\underline{K} = G^{-1}(\underline{u})$ will be distributed according to the initial desired distribution $g(K)$.

In our case we already have u, we only need the cdf of the Breit-Wigner distribution. But in the Wiki I only found the pdf, so we need to integrate that:

$$f(x) = \frac{K}{(x^2-M^2)^2+M^2\Gamma^2} \longrightarrow F(y) = \int_0^y f(x)\,dx =$$

$$=$$

And invert it: $F^{-1}(K) =$

So finally: $\underline{K} = F^{-1}(\underline{u}) =$

will be distributed following a Breit-Wigner distribution.

15)

We want a pdf $\propto (M_v^2+t)^{-2}$, so $g(x) = \dfrac{N}{(M_v^2+t)^2}$ where N is the Normalization cte.

The process we need to follow is the one described in the previous exercice:

1) Having u distributed uniformly $(0,1)$

2) $$G(x) = \int_0^x \frac{N}{(M_v^2+t)^2}\,dt = \frac{xN}{M_v^2(M_v^2+x)}$$

3) Invert it: $$G^{-1}(K) = -\frac{M_v^4 K}{M_v^2 K - N}$$

4) Apply it to u: $$\boxed{\underline{t} = G^{-1}(\underline{u}) = -\frac{M_v^4 \underline{u}}{M_v^2 \underline{u} - N}}$$

$\underline{t}$ is now distributed $\propto (M_v^2+t)^{-2}$