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Homework 5

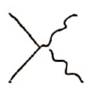
Consider the theory will Lagrangian density:

- $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* (D^\mu \phi) - m_0^2 \phi^* \phi$

$\begin{cases} A_\mu \text{ Abelian gauge} \\ \phi \text{ Complex scalar} \end{cases}$

If we now substitute $D_\mu \phi = (\partial_\mu + i e_0 A_\mu)\phi$, we get:

$$\begin{cases} \mathcal{L}_0 = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{free } A_\mu} + \underbrace{(\partial_\mu \phi)^* (\partial^\mu \phi) - m_0^2 \phi^* \phi}_{\text{free } \phi} \\ \mathcal{L}_i = \underbrace{-i e_0 A^\mu [\phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi]}_{\text{interaction}} + \underbrace{e_0^2 A_\mu A^\mu \phi^* \phi}_{\text{interaction}} \end{cases} \Longrightarrow \underline{\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_i}$$

 interaction  interaction

1)

Let's first introduce the renormalized fields, mass and charge:

- $A_{r\mu} = \frac{A_\mu}{\sqrt{Z_A}} \longleftrightarrow$ the previous propagator $\frac{-i Z_A g^{\mu\nu}}{q^2}$ becomes $\frac{-i g^{\mu\nu}}{q^2}$ ✓
- $\phi_r = \frac{\phi}{\sqrt{Z_\phi}} \longleftrightarrow$ the previous propagator $\frac{i Z_\phi}{p^2 - m^2}$ becomes $\frac{i}{p^2 - m^2}$ ✓
- $e = \frac{e_0}{Z_e}$
- $m = \frac{m_0}{\sqrt{Z_m}}$

Which makes our lagrangian:

$$\begin{cases} \mathcal{L}_0 = -\frac{Z_A}{4} F_{r\mu\nu} F_r^{\mu\nu} + Z_\phi [(\partial_\mu \phi_r)^* (\partial^\mu \phi_r) - Z_m m^2 \phi^* \phi] \\ \mathcal{L}_i = -i Z_e \sqrt{Z_A} Z_\phi \, e A_r^\mu [\phi_r^* (\partial_\mu \phi_r) - (\partial_\mu \phi_r^*) \phi_r] + Z_e^2 Z_A Z_\phi e^2 A_r^\mu A_{r\mu} \phi_r^* \phi_r \end{cases}$$

Which, if we write the renormalized fields without suffix $\phi_r, A_r^\mu \to \phi, A^\mu$, and we split it into the physical Lagrangian and their counterterm ct:

$$\boxed{\mathcal{L} = \mathcal{L}_0' + \mathcal{L}_i' + \mathcal{L}_{0\,ct} + \mathcal{L}_{i\,ct}}$$

$$\left.\begin{aligned} \mathcal{L}_0' &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi \\ \mathcal{L}_i' &= -ie A^\mu [\phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi] + e^2 A_\mu A^\mu \phi^* \phi \end{aligned}\right\} \text{physical } \mathcal{L}'$$

$$\left.\begin{aligned} \mathcal{L}_{0\,ct} &= -\frac{\delta_1}{4} F_{\mu\nu} F^{\mu\nu} + \delta_2 (\partial_\mu \phi)^* (\partial^\mu \phi) - \delta_3 m^2 \phi^* \phi \\ \mathcal{L}_{i\,ct} &= -\delta_4 ie A^\mu [\phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi] + \delta_5 e^2 A_\mu A^\mu \phi^* \phi \end{aligned}\right\} \text{counterterms } \mathcal{L}_{ct}$$

where:

$$\left.\begin{aligned} \delta_1 &= Z_A - 1 \equiv \delta_A \\ \delta_2 &= Z_\phi - 1 \equiv \delta_\phi \\ \delta_3 &= Z_\phi Z_m - 1 \equiv \delta_m \\ \delta_4 &= Z_e \sqrt{Z_A}\, Z_\phi - 1 \equiv \delta_Y \\ \delta_5 &= Z_e^2 Z_A Z_\phi - 1 \equiv \delta_X \end{aligned}\right\} \left(4 \text{ "d.o.f"}: \quad Z_A = \frac{A_{\mu 0}^2}{A_{\mu r}^2}, \quad Z_\phi = \frac{\phi_0^2}{\phi_r^2}, \quad Z_e = \frac{e_0}{e}, \quad Z_m = \frac{m_0^2}{m^2} \right)$$

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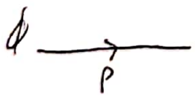
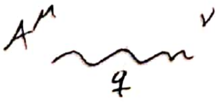
2)

Physical part

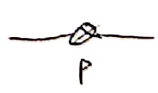

$= \dfrac{i}{p^2 - m^2}$

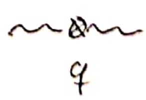

$= \dfrac{-i g^{\mu\nu}}{q^2}$


$= -ie (p_\mu^1 + p_\mu^2)$

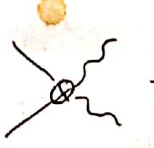

$= 2ie^2 g^{\mu\nu}$

Counterterm part


$= i(\delta_\phi p^2 - \delta_m m^2)$


$= -i\delta_A (g^{\mu\nu} q^2 - q^\mu q^\nu)$


$= -i\delta_Y (p_\mu^1 + p_\mu^2)$


$= 2i\delta_X (p_\mu^1 + p_\mu^2)$

But from gauge invariance we see that the covariant derivatives must remain the same, for the new parameters/fields, so:

$$e_0 A_{0\mu} = e A_{r\mu} \quad ; \quad Z_e \cancel{e} \sqrt{Z_A} \cancel{A_{r\mu}} = \cancel{e} \cancel{A_{r\mu}} \longrightarrow \boxed{Z_e\sqrt{Z_A}=1}$$

Which looking back makes everything much easier, because:

$$\begin{cases} \delta_1 = Z_A - 1 \equiv \underline{\delta_A} \\ \delta_2 = Z_\phi - 1 = \underline{\delta_\phi} \\ \delta_3 = Z_\phi Z_m - 1 = \underline{\delta_m} \\ \delta_4 = \cancel{Z_e\sqrt{Z_A}} Z_\phi - 1 = Z_\phi - 1 = \delta_x = \underline{\delta_\phi} \\ \delta_5 = \cancel{Z_e^2}\cancel{Z_A} Z_\phi - 1 = Z_\phi - 1 = \delta_x = \underline{\delta_\phi} \end{cases}$$

so the Lagrangian ends with 3 "dof": $\underline{\delta_A, \delta_\phi, \delta_m}$ or $\underline{Z_A, Z_\phi, Z_m}$.

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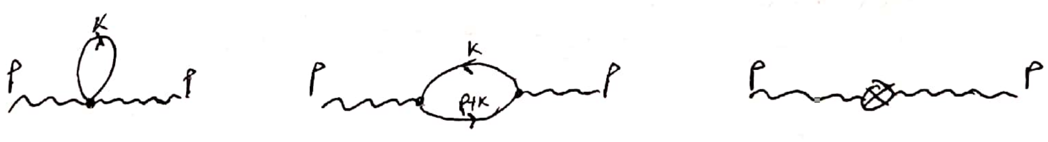
3) The relevant diagrams will be:

(At higher order this propagators would be multiplied by $Z_i = 1 + \cancel{\delta_i}^{\,0}$ but $\delta_i \propto e^2$)



① ② ③

$$① = \int \frac{d^dK}{(2\pi)^d}\, 2ie^2 g^{\mu\nu} \frac{i}{K^2-m^2}\Bigg|_{d=4}$$

$$② = \int \frac{d^dK}{(2\pi)^d} \left(-ie(p+K+K)^\mu\right) \frac{i}{K^2-m^2}\,\frac{i}{(K+p)^2-m^2}\left(-ie(p+K+K)^\nu\right)\Bigg|_{d=4}$$

$$③ = -i\,\delta_A \left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)$$

$$\boxed{i\Pi_2^{\mu\nu}(q) = ① + ② + ③}$$

Let's start computing them then:

$$\left(\begin{array}{l} d=4-\epsilon \\ \Gamma(-1)\ \text{divergence} \end{array}\right)$$

$$① = -2e^2 g^{\mu\nu}\int \frac{d^dK}{(2\pi)^d}\frac{1}{K^2-m^2}\Big|_{d=4} = \frac{i2e^2 g^{\mu\nu}}{(4\pi)^{d/2}}\left(\frac{1}{m^2}\right)^{1-\frac{d}{2}}\Gamma(1-d/2)\Big|_{d=4} \approx \boxed{\frac{i2e^2g^{\mu\nu}m^2}{16\pi^2}\,\frac{1}{\epsilon}}$$

$$② = e^2\int\frac{d^dK}{(2\pi)^d}\frac{(p+2K)^\mu(p+2K)^\nu}{(K^2-m^2)\,((p+K)^2-m^2)}\Big|_{d=4} = \boxed{\left(\frac{ie^2p^\mu p^\nu}{24\pi^2}-\frac{ie^2g^{\mu\nu}p^2}{6\pi^2}\right)\frac{1}{\epsilon}}$$

So then the full one-loop contribution will be (on-shell):

$$\boxed{\Pi_2^{\mu\nu}(p^2=m^2)} = \left(\overbrace{\frac{2e^2g^{\mu\nu}m^2}{16\pi^2}}^{①} - \overbrace{\frac{e^2g^{\mu\nu}m^2}{6\pi^2}+\frac{e^2p^\mu p^\nu}{24\pi^2}}^{②}\right)\frac{1}{\epsilon}\ \overbrace{-\delta_A\left(g^{\mu\nu}m^2-p^\mu p^\nu\right)}^{③} =$$

$$-\frac{e^2g^{\mu\nu}m^2}{24\pi^2}$$

$$= -\frac{e^2}{24\pi^2\epsilon}\left(g^{\mu\nu}m^2-p^\mu p^\nu\right) - \delta_A\left(g^{\mu\nu}m^2-p^\mu p^\nu\right) =$$

$$= \boxed{-\underbrace{\left(\frac{e^2}{24\pi^2\epsilon}+\delta_A\right)}_{\Pi(p^2)}\left(g^{\mu\nu}m^2-p^\mu p^\nu\right)}$$

Which with the renormalization condition $\Pi_2^{\mu\nu}(p^2=m^2)=0$, gives: $\delta_A = -\frac{e^2}{24\pi^2\epsilon}$

Lastly, we are asked to check the Wards identity:

$$\boxed{q_\mu\Pi_2^{\mu\nu}(q)} = \Pi(q^2)\,q_\mu\left(g^{\mu\nu}m^2-q^\mu q^\nu\right) = \Pi(q^2)\left(q^\nu m^2-\overset{m^2}{\cancel{q^2}}q^\nu\right) =$$

$$= \Pi(q^2)\left(\cancel{q^\nu m^2}-\cancel{q^\nu m^2}\right) = \boxed{0}\quad ✓$$

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