Classical Field Theory

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Lagrangian classical field theory

The action in classical mechanics

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, \mathrm{d}t$$

The solution is an extreme of the action.

Local field theory

$$q_i \to \phi(x) \quad ; \quad L = \int \mathrm{d}^3x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$\mathcal{L} \equiv$lagrangian density.

$$S = \int \mathrm{d}t \int \mathrm{d}^3x\, \mathcal{L}(\phi, \partial_\mu \phi) = \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

Equations of motion: (e.o.m.)

- Extreme of the action
- by given contour conditions on the border:

$$S = \int_\Omega \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu \phi) \;\; ; \;\; \Sigma = \partial\Omega \;\; ; \;\; \phi|_\Sigma = \text{constant}$$

$$\phi \to \phi + \delta\phi \Rightarrow \frac{\delta S}{\delta \phi} = 0$$

$$\begin{aligned} 0 = \delta S &= \int_\Omega \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\partial_\mu \partial_\mu \delta\phi \right) \quad , \quad [\delta\partial_\mu \phi \equiv \partial_\mu \delta\phi] \\ &= \int_\Omega \mathrm{d}^4x \Bigg(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi \right)}_{\text{Gauss-Ostrogradsky th.}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \delta\phi \Bigg) \end{aligned}$$

$$\int_\Omega \mathrm{d}^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi \right) = \int_\Sigma \mathrm{d}^3x\, n_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi = 0 \;\; , \;\; [\delta\phi|_\Sigma = 0]$$

$$\Rightarrow \delta S = \int_\Omega \mathrm{d}^4x\, \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) = 0 \quad \forall \delta\phi$$

Euler-Lagrange equations for a field

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0$$

- $\mathcal{L}$ is not unique:

$$\begin{aligned} \mathcal{L}' &= \mathcal{L} + \partial_\mu \mathcal{K}^\mu(\phi) \\ S' &= S + \int_\Omega \mathrm{d}^4x\, \partial_\mu \mathcal{K}^\mu(\phi) = S + \underbrace{\int_\Sigma \mathrm{d}^3x\, n_\mu \mathcal{K}^\mu(\phi)}_{\text{constant}} \; , \; [\phi|_\Sigma = \text{cnt}] \end{aligned}$$

$$\Rightarrow \delta S' = \delta S$$

S' and S give the same equations of motion.

Definition: Conjugate momentum (canonical momentum):

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

Definition: hamiltonian density:

$$\mathcal{H} = \Pi \partial_0 \phi - \mathcal{L}$$

Definition: hamiltonian:

$$H = \int \mathrm{d}^3x \, \mathcal{H} = \int \mathrm{d}^3x \, (\Pi \partial_0 \phi - \mathcal{L})$$

Example 1: Real Klein-Gordon field: $\phi \in \mathbb{R}$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2\right) = \frac{1}{2}\left(\left(\frac{\partial \phi}{\partial t}\right)^2 - \left(\frac{\partial \phi}{\partial x^i}\right)^2 - m^2\phi^2\right)$$

e.o.m.

$$\begin{aligned} -\partial_t \frac{\partial \mathcal{L}}{\partial \partial_t \phi} - \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \phi} + \frac{\partial \mathcal{L}}{\partial \phi} &= 0 \\ -\partial_t^2 \phi + \partial_i^2 \phi - m^2 \phi &= 0 \\ -\partial_\mu \partial^\mu \phi - m^2 \phi &= 0 \quad \text{Klein-Gordon eq.} \end{aligned}$$

Momentum: $\Pi_\phi = \dfrac{\partial \mathcal{L}}{\partial \partial_t \phi} = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \partial^t \phi = \partial^0 \phi = \dfrac{\partial \phi}{\partial t} = \dot{\phi}$

Careful with indices!: $g^{00} = +1$: $\partial^0 = \partial_0 = \frac{\partial}{\partial t}$ but $g^{ii} = -1$: $\partial^i = -\partial_i = -\frac{\partial}{\partial x^i}$.
Hamiltonian density:

$$\begin{aligned} \mathcal{H} &= \Pi_\phi \partial_0 \phi - \mathcal{L} = \partial^0 \phi \partial_0 \phi - \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \\ &= \frac{1}{2}(\partial^0 \phi \partial_0 \phi + (\partial_i \phi)^2 + m^2 \phi^2) = \frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\partial_i \phi)^2 + \frac{1}{2} m^2 \phi^2 \end{aligned}$$

Example 2: Complex Klein-Gordon field: $\phi \in \mathbb{C}$

$\phi = \phi_R + i\phi_I \Rightarrow$ 2 independent degrees of freedom

$\Rightarrow$ take ϕ, ϕ^* as independent

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*$$

$$\begin{aligned}
\text{e.o.m. } \phi^* &: \quad -\partial_\mu \partial^\mu \phi - m^2 \phi = 0 \\
\text{e.o.m. } \phi &: \quad -\partial_\mu \partial^\mu \phi^* - m^2 \phi^* = 0 \\
\Pi_\phi &= \partial^0 \phi^* = \partial_0 \phi^* \\
\Pi_{\phi^*} &= \partial^0 \phi = \partial_0 \phi \\
\mathcal{H} &= \Pi_\phi \partial_0 \phi + \Pi_{\phi^*} \partial_0 \phi^* - \mathcal{L} \\
&= \Pi_\phi \Pi_{\phi^*} + \Pi_{\phi^*} \Pi_\phi - \Pi_{\phi^*} \Pi_\phi - \partial_i \phi \partial^i \phi + m^2 \phi \phi^* \\
&= \Pi_\phi \Pi_{\phi^*} + \partial_i \phi \partial_i \phi^* + m^2 \phi \phi^* \\
&= \Pi_\phi \Pi_{\phi^*} + \nabla \phi \cdot \nabla \phi^* + m^2 \phi \phi^*
\end{aligned}$$

Noether's Theorem

For each global symmetry there is a conserved current

- Symmetry: leaves e.o.m. invariant
- Global: independent of point

if: $\mathcal{L} \longrightarrow \mathcal{L} + \partial_\mu \mathcal{K}^\mu \Rightarrow$ There is a conserved current

Internal symmetries

$$\mathcal{L}(\phi_i, \partial_\mu \phi_i) \quad , \quad i = 1, \cdots, N$$

transformation of the fields

$$\begin{aligned} x^\mu &\to x'^\mu = x^\mu && (1)\\ \phi_i(x) &\to \phi'_i(x') = G(\phi(x)) \simeq \phi_i(x) + \delta\phi_i(x) = \phi_i(x) + \varepsilon^a F_{ia}(\phi, \partial_\mu \phi) \end{aligned}$$

If this transformation is a symmetry:

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \partial_\mu \mathcal{K}^\mu \tag{2}$$

On the other hand:

$$\begin{aligned} S' &= \int \mathrm{d}^4x\, \mathcal{L}' = \int \mathrm{d}^4x\, \mathcal{L}(\phi', \partial_\mu \phi') \\ &= \int \mathrm{d}^4x \left\{ \mathcal{L}(\phi, \partial_\mu\phi) + \frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\partial_\mu\phi_i \right\} + \cdots \\ \delta S = S' - S &= \int \mathrm{d}^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\partial_\mu\phi_i \right\} \end{aligned}$$

$\delta\phi_i$ is independent of space-time:

$$\begin{aligned} \delta\phi_i &= \varepsilon^a F_{i,a}(\phi, \partial_\mu\phi) \\ \delta\partial_\mu\phi_i &= \partial_\mu \delta\phi_i \\ \delta S = S' - S &= \int \mathrm{d}^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\partial_\mu\delta\phi_i}_{\text{integrate by parts}} \right\} \end{aligned}$$

$$\delta S = \int \mathrm{d}^4x \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \right)}_{\text{e.o.m.}=0} \delta\phi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\phi_i \right)$$

If ϕ_i are solutions to the e.o.m.:

$$\delta S = \int \mathrm{d}^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\phi_i \right) \quad \text{for } \phi_i \text{ solutions e.o.m.} \tag{3}$$

this change (3) must be equal to the change induced by $\mathcal{K}^\mu$ (2)

$$\delta S = S' - S = \int \mathrm{d}^4x\, \mathcal{L}' - \int \mathrm{d}^4x\, \mathcal{L} = \int \mathrm{d}^4x\, \partial_\mu \mathcal{K}^\mu \tag{4}$$

δS in (3) must be equal to δS in (4) for any spacte-time volume

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i - \mathcal{K}^\mu \right) = 0 \text{ for } \phi_i \text{ solutions e.o.m.}$$

Conserved current

$$\begin{aligned} \partial_\mu J^\mu &= 0 \\ J^\mu &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i - \mathcal{K}^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \varepsilon^a F_{i,a}(\phi, \partial_\mu \phi) - \mathcal{K}^\mu \end{aligned} \tag{5}$$

$\mathcal{K}^\mu \propto \varepsilon^a \Rightarrow J_a^\mu \Rightarrow$ one conserved current for each parameter.

Define: $Q = \int \mathrm{d}^3 x \, J^0$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \partial_0 Q = \int \mathrm{d}^3 x \, \partial_0 J^0 = -\int \mathrm{d}^3 x \, \partial_i J^i = -\int_\infty \mathrm{d}^2 x \, \boldsymbol{n} \cdot \boldsymbol{J}$$

If the fields $\phi_i \to 0$ at $x_i \to \infty$

Conserved charge Q

$$Q = \int \mathrm{d}^3 x \, J^0 \; ; \; \frac{\mathrm{d}Q}{\mathrm{d}t} = 0 \tag{6}$$

Example: complex Klein-Gordon field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi \phi^*$$

$$\begin{aligned} \phi' &= e^{-i\alpha} \phi \simeq (1 - i\alpha)\phi \; , \; \delta\phi = -i\alpha\phi \\ \phi^{*\prime} &= e^{i\alpha} \phi^* \simeq (1 + i\alpha)\phi^* \; , \; \delta\phi^* = +i\alpha\phi^* \end{aligned}$$

$\mathcal{L}$ is invariant: $\mathcal{L}' = \mathcal{L} \Rightarrow \mathcal{K}^\mu = 0$

$$\begin{aligned} J^\mu &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} \delta\phi^* = \partial^\mu \phi^* (-i\alpha\phi) + \partial^\mu \phi (i\alpha\phi^*) \\ &= +i\alpha(-\phi \partial^\mu \phi^* + \phi^* \partial^\mu \phi) \end{aligned}$$

Conserved current (electromagnetic current)

$$J^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

Conserved charge (electric charge)

$$Q = \int \mathrm{d}^3 x \, J^0 = i \int \mathrm{d}^3 x \, (\phi^* \partial^0 \phi - \phi \partial^0 \phi^*)$$

Example: N complex Klein-Gordon fields, same mass

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \quad , \quad \Phi^\dagger = \left(\phi_1^* \cdots \phi_N^*\right)$$

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi$$

$\mathcal{L}$ invariant ($\mathcal{K}^\mu = 0$) general linear unitary transformations of Φ:

$$U \in SU(N) : \{U^\dagger = U^{-1} \quad , \quad \det(U) = 1\}$$

$$\begin{aligned}
\Phi' &= U\Phi \simeq (1 - i\alpha^a T_a)\Phi \\
T_a &\equiv \text{generators, hermitic matrices: } T_a^\dagger = T_a \\
\phi_i' &= \phi_i - i\alpha^a T_a^{ij}\phi_j \\
\phi_i^{*\prime} &= \phi_i^* + i\alpha^a T_a^{ij*}\phi_j^* = \phi_i^* + i\alpha^a T_a^{ji}\phi_j^* \\
J^\mu &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i^*}\delta\phi_i^* = \partial^\mu \phi_i^*(-i\alpha^a T_a^{ij}\phi_j) + \partial^\mu \phi_i (i\alpha^a T_a^{ji})\phi_j^* \\
&= i\alpha^a(-(\partial^\mu \Phi^\dagger) T_a \Phi + \Phi^\dagger T_a \partial^\mu \Phi)
\end{aligned}$$

as many conserved currents as $T_a \equiv SU(N)$ generators

Conserved currents

$$J_a^\mu = i(\Phi^\dagger T_a \partial^\mu \Phi - (\partial^\mu \Phi^\dagger) T_a \Phi)$$

Explicit example: $SU(2)$

(proton, neutron), (electron, ν_e),:

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad , \quad U = e^{-i\vec{\alpha}\cdot\vec{\sigma}/2}$$

$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$
$\vec{\sigma} \equiv$ Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow$ 3 conserved currents

$$J_i^\mu = i \left(\Phi^\dagger \frac{\sigma_i}{2} \partial^\mu \Phi - (\partial^\mu \Phi^\dagger) \frac{\sigma_i}{2} \Phi \right)$$

- To analyze field theory, and the conserved currents, we need to use group theory, more specifically, continuous groups, also known as
  $\Rightarrow$ Lie Groups
- To analyze the space-time symmetries, we need to know the symmetry group structure of special relativity space-time:
  $\Rightarrow$ Poincaré Group