Quantum Field Theory, 2021/2022
Exercise sheet 5: QED, LSZ
Hand-in: November 10, 2020

5.1. Consider Compton scattering:

$$e^-(p_1)\gamma(k_1) \to e^-(p_2)\gamma(k_2)$$

(a) Draw all possible Feynman diagrams at leading order in perturbation theory

(b) Compute (using the Feynman rules) the corresponding invariant transition matrix element $\mathcal{M}$ for each of them

(c) A gauge transformation in position space $A^\mu(x) \to A^\mu(x) + \partial^\mu\Lambda(x)$ can be translated to momentum space as:

$$\varepsilon^\mu(k) \to \varepsilon^\mu(k) + \lambda k^\mu$$

show that the amplitudes corresponding to each individual diagram are **not** gauge invariant, but the sum of all diagrams gives a gauge-invariant total amplitude.

5.2. The Optical Theorem for the transition matrix states that:

$$2\mathrm{Im}(\mathcal{T}) = \mathcal{T}^\dagger\mathcal{T}$$

(a) By inserting the appropriate external and intermediate particle states, show that for a two particle state k_1, k_2:

$$\mathrm{Im}\left(\mathcal{M}(k_1k_2 \to k_1k_2)\right) = 2E_{CM}p_{CM}\sigma^{tot}(k_1, k_2 \to \text{anything})$$

where the total cross-section is defined as:

$$\sigma^{tot}(k_1, k_2 \to \text{anything}) = \frac{1}{4E_{CM}p_{CM}}\sum_f \int \mathrm{dLIPS}_q|\mathcal{M}(k_1k_2 \to f)|^2$$

and the Lorentz-Invariant-Phase-Space (LIPS) factor for a final state f of n particles with momentum q_i is:

$$\mathrm{dLIPS}_q = (2\pi)^4\delta^4\left(k_1 + k_2 - \sum_{i=1}^n q_i\right)\prod_{i=1}^n \frac{\mathrm{d}^3q_i}{(2\pi)^3 2E_i}$$

(b) Now, instead of a two-particle state in the external line, we can use the theorem for a one-particle state to compute

$$\langle p|\mathcal{T}|p\rangle \quad ;$$

work with a complex Klein-Gordon field and:

i. Use the LSZ reduction formula to relate the transition matrix element $\mathcal{M}(p \to p)$ to the $\Sigma^{1PI}(p^2)$ defined at class.

ii. Now, if $\Sigma^{1PI}(p^2)$ has an imaginary part, the pole of the full propagator has an imaginary part (show it!) and no longer corresponds to the physical mass, we define the mass M, instead as the real part of the pole:

$$p^2 - m^2 - \mathrm{Re}(\Sigma^{1PI}(p^2))\big|_{p^2=M^2} = 0$$

show that, if $\Sigma^{1PI}(p^2)$ is small, and we can approximate $\mathrm{Im}(\Sigma^{1PI}(p^2)) \simeq \mathrm{Im}(\Sigma^{1PI}(M^2))$ in the vicinity of $p^2 \sim M^2$, then the propagator has the Breit-Wigner resonance shape:

$$|\Delta_F^{full}(p^2)|^2 \sim \left| \frac{1}{p^2 - M^2 + im\Gamma} \right|^2$$

identify the full decay width Γ as a function of Σ^{1PI}, and use the optical theorem to give the expression of the full decay width as a sum over final states.