HW2

sábado, 14 de mayo de 2022 13:00

# Homework 2

Due date: 21/04/2022

**Problem 1:**

Consider QCD with only one flavour $n_f = 1$.

a) Use Grassmann integration identities to express the partition function as

$$\mathscr{Z} = \mathscr{C} \int \mathscr{D}A \det\left\{1 - \frac{i}{i\partial\!\!\!/ - aM_R}(-ig_R)\, b A\!\!\!/^c t^c\right\} e^{i\int d^4x \mathscr{L}_R(A_R, g_R, \mu)} = \int DA\, Dc\, e^{i\int d^4x\, \mathcal{L}(n=1)} \tag{1}$$

with $\mathscr{C}$, $a$ and $b$ constants and $\mathscr{L}_R$ the renormalised pure glue ($n_f = 0$) Lagrangian. Express $a$ and $b$ in terms of the field strength constants of the theory. Moments of this partition function generates all correlators with no external fermions.

b) The fermionic determinant in the partition function leads to correction to the pure glue correlation functions, as determined from the pure gauge Lagrangian. Using the matrix identity

$$\log(\det \mathscr{M}) = \mathrm{Tr}\{\log \mathscr{M}\} \tag{2}$$

exponentiate the determinant and interpret it as the modification of the pure glue Lagrangian induced by the heavy quark, $\Delta S$. By expanding in powers of $g_R$, determine the leading order correction to the pure glue Lagrangian which is quadratic in the gauge field and express it in the form

$$\Delta S = g_R^2 \int \frac{d^4q}{(2\pi)^4} A_\mu^a(-q) A_\nu^b(q) \mathscr{V}_{ab}^{\mu\nu}(q, M_R, \mu) \tag{3}$$

Define first $\mathscr{V}(q, M_R, \mu)$ by an integral. Discuss the diagrammatic interpretation of the corresponding integral.

c) Perform the integral $\mathscr{V}(q, M_R, \mu)$ . Regularise any divergence in the $MS$-scheme. Annalise the Lorentz structure of the divergent contribution. Interpret its effect when this modified action is combined with the counter-terms of the $n_f = 1$ theory.

d) Annalise the finite part of the integral. Assume that you are only interested in correlation functions with momentum $q << M_R$. Expand the finite piece to leading order in $q/M_R$. Annalise the Lorentz structure of this leading order contribution and interpret its form. Identify the leading order contribution to the decoupling parameter $\xi_A$.

a)

Let's start showing the full renormalized Lagrangian for QCD $\mathcal{L}_R(g_R, M_R, \mu^\epsilon)$ which if $\psi_{i\alpha}$ only has a Dirac indice ($\alpha = 1, \dots, 4$) and a color one ($i = r, g, b$) [no flavour indice], is $\equiv \mathcal{L}(n_f = 1)$, and if we set $\psi = 0$ we will have the Lagrangian for pure glue theory $\equiv \mathcal{L}(n_f = 0)$:

$$\begin{aligned}
\mathcal{L}_R(g_R, m_R, \mu) = {} & \mathcal{L}_{\text{t.l.}}(g_R\mu^\epsilon, m_R) \\
& + \delta_\psi^{1\,\text{Loop}} \bar{\psi}_R i\partial\!\!\!/ \psi_R - \left(\delta_m^{1\,\text{Loop}} + \delta_\psi^{1\,\text{Loop}}\right) m_R \bar{\psi}_R \psi_R - \delta_1^{1\,\text{Loop}} g_R \mu^\epsilon \bar{\psi} A\!\!\!/_R{}^A t^A \psi \\
& - \delta_A^{1\,\text{Loop}} \frac{1}{4}\left(\partial_\mu A_{R\nu} - \partial_\nu A_{R\mu}\right)\left(\partial^\mu A_R{}^\nu - \partial^\nu A_R{}^\mu\right) \\
& + \delta_1^{\prime\,1\,\text{Loop}} \frac{1}{2} g_R \mu^\epsilon f^{ABC} A_R{}^{\mu B} A_R{}^{\nu C}\left(\partial_\mu A_{R\nu}^A - \partial_\nu A_{R\mu}^A\right) \\
& - \delta_4^{1\,\text{Loop}} \frac{1}{4} g_R{}^2 \mu^{2\epsilon} f^{ABC} f^{AB'C'} A_{R\mu}^B A_{R\nu}^C A_R{}^{B'\mu} A_R{}^{C'\nu} \\
& - \delta_c^{\prime\,1\,\text{Loop}} \bar{c}_R^A \partial^2 c_R{}^A + \delta_1^{\prime\prime\,1\,\text{Loop}} g_R \mu^\epsilon f^{ABC} \bar{c}_R^A \partial_\mu A_R{}^{\mu B} c_{Rc}
\end{aligned} \tag{1.36}$$

$$\mathcal{L}_{\mathrm{t.l.}}(g_{\mathrm{R}}\mu^{\epsilon}, m_{\mathrm{R}}) = \bar{\psi}_{\mathrm{R}}\left(i\partial\!\!\!/ - g_{\mathrm{R}}\mu^{\epsilon} A\!\!\!/_{\mathrm{R}}{}^{A} t^{A} - m_{\mathrm{R}}\right)\psi_{\mathrm{R}} - \frac{1}{4}F^{A}_{\mu\nu}(A_{\mathrm{R}}, g_{\mathrm{R}}\mu^{\epsilon})F^{\mu\nu A}(A_{\mathrm{R}}, g_{\mathrm{R}}\mu^{\epsilon})$$
$$-\frac{1}{2}\lambda_{R}(\partial\cdot A_{\mathrm{R}})^{2} + \bar{c}^{A}_{\mathrm{R}}\left[\partial^{2}\delta^{AB} - g_{\mathrm{R}}\mu^{\epsilon} f^{ABC}\partial_{\mu}A_{\mathrm{R}}{}^{\mu B}\right]c_{\mathrm{R}}{}^{C}. \qquad (1.34)$$

We will start from the expression for the partition function:

$$Z = \int DA\, D\bar{\Psi}\, D\Psi\, e^{i\int d^4x\, \mathcal{L}(n_f=1)} = C\int DA\, \det\left\{1 - \frac{i}{i\partial\!\!\!/ - aM_R}(-ig_R)\, b\, A\!\!\!/^a_R T^a\right\} e^{i\int d^4x\, \mathcal{L}(n_f=0)} \qquad (1)$$

$$\Updownarrow$$

$$\int d\bar{\Psi}\, d\Psi\, e^{i\int d^4x\, \mathcal{L}(n_f=1)} \xrightarrow{\text{(Integrate out the terms with } \bar{\Psi} \text{ and } \Psi)} e^{i\int d^4x\, \mathcal{L}(n_f=0)} \cdot X$$

So we only need to check that the factor $X$ is in fact the given by the statement of the excercice. For this let's start with the $\mathcal{L}$ terms that have $\bar{\Psi}$ or $\Psi$:

$$\mathcal{L}_{\Psi} = \bar{\Psi}_R\left(i\partial\!\!\!/ - g_R\mu^{\epsilon} A\!\!\!/^a_R T^a - M_R\right)\Psi_R + \delta^{(1)}_{\Psi}\bar{\Psi}_R i\partial\!\!\!/\Psi_R -$$
$$-(\delta^{(1)}_{\Psi} + \delta^{(1)}_{m})M_R\bar{\Psi}_R\Psi_R - \delta^{(1)}_{1} g_R\mu^{\epsilon}\bar{\Psi}_R A\!\!\!/^a_R T^a\Psi_R =$$
$$= \bar{\Psi}_R\underbrace{\left\{(1+\delta^{(1)}_{\Psi})\, i\partial\!\!\!/ - (1+\delta^{(1)}_{1})\, g_R\mu^{\epsilon} A\!\!\!/^a_R T^a - (1+\delta^{(1)}_{\Psi} + \delta^{(1)}_{m})M_R\right\}}_{\equiv \Delta(\delta^{(1)}_{\Psi}, \delta^{(1)}_{1}, \delta^{(1)}_{m}, g_R, \mu^{\epsilon}, A_R, M_R, x)}\Psi_R$$

Now let's see what this terms give when we integrate $\bar{\Psi}$ and $\Psi$ out:

$$Z = \int DA\, D\bar{\Psi}\, D\Psi\, e^{i\int d^4x\, \mathcal{L}(n_f=1)} =$$
$$= \int DA\, e^{i\int d^4x\, \mathcal{L}(n_f=0)}\left(\int D\bar{\Psi}\, D\Psi\, e^{i\overbrace{\int d^4x\, \bar{\Psi}(x)\,\Delta(x)\,\Psi(x)}^{\bar{\Psi}\Delta\Psi \text{ (grassman variables)}}}\right) =$$

Going to discrite picture

$$= \int DA\, e^{i\int d^4x\, \mathcal{L}(n_f=0)}\left(\prod_i\int D\bar{\Psi}_i\, D\Psi_i\, e^{i\bar{\Psi}_i\Delta_{ij}\Psi_j}\right) =$$

Going into diagonal base of $\Delta = \begin{pmatrix}\delta_1 & & \\ & \ddots & \\ & & \delta_n\end{pmatrix}$

$$= \int DA\, e^{i\int d^4x\, \mathcal{L}(n_f=0)}\left(\prod_i\int D\bar{\Psi}'_i\, D\Psi'_i\, e^{i\delta_i\bar{\Psi}'_i\Psi'_i}\right) =$$

$e^x = 1 + \frac{x}{1!} + \cancel{\frac{x^2}{2!}} + \cancel{\dots}$
($\Psi^2 = 0$ for Grassman variables)

$$= \int DA\, e^{i\int d^4x\, \mathcal{L}(n_f=0)}\left(\prod_i\int D\bar{\Psi}'_i\, D\Psi'_i\,(1 + i\delta_i\bar{\Psi}'_i\Psi'_i)\right) =$$
$$= \int DA\, e^{i\int d^4x\, \mathcal{L}(n_f=0)}\left\{\underbrace{\prod_i\delta_i}_{\det(\Delta)}\left(\overset{0}{\cancel{\int D\bar{\Psi}'_i D\Psi'_i}} + i\overset{-1}{\cancel{\int D\bar{\Psi}'_i D\Psi'_i\,\bar{\Psi}'_i\Psi'_i}}\right)\right\} =$$
$$= \int DA\, e^{i\int d^4x\, \mathcal{L}(n_f=0)}\left(-i\det(\Delta)\right)$$

And then the determinant gives:

$$\det(\Delta) = \det\Big\{\underbrace{(1+\delta_4^{(1)})\, i\not{\partial} - (1+\delta_4^{(1)}+\delta_m^{(1)}) M_R}_{K} - (1+\delta_1^{(1)})\, g_R \gamma^\epsilon A_R^a T^a\Big\}$$

$$= \det(K)\,\det\left(1 - \left[(1+\delta_1^{(1)})\, g_R \gamma^\epsilon A_R^a T^a\right] K^{-1}\right) =$$

$$= \det(K)\,\det\left(1 - \frac{(1+\delta_1^{(1)})\, g_R \gamma^\epsilon A_R^a T^a}{(1+\delta_4^{(1)})\, i\not{\partial} - (1+\delta_4^{(1)}+\delta_m^{(1)}) M_R}\right) =$$

$$= \det(K)\,\det\left(1 - \frac{\frac{1+\delta_1^{(1)}}{1+\delta_4^{(1)}}\, g_R \gamma^\epsilon A_R^a T^a}{i\not{\partial} - \frac{1+\delta_4^{(1)}+\delta_m^{(1)}}{1+\delta_4^{(1)}} M_R}\right)$$

So the partition function is:

$$Z = \int DA\, e^{i\int d^4x\, \mathcal{L}(u_f=0)} \left(-i \det(\Delta^{-1})\right) =$$

$$= \underbrace{-i\det(K)}_{\mathcal{C}} \int DA\, \det\left(1 - \frac{\overbrace{\frac{1+\delta_1^{(1)}}{1+\delta_4^{(1)}}}^{b}\, g_R \gamma^\epsilon A_R^a T^a}{i\not{\partial} - \underbrace{\frac{1+\delta_4^{(1)}+\delta_m^{(1)}}{1+\delta_4^{(1)}}}_{a} M_R}\right) e^{i\int d^4x\, \mathcal{L}(u_f=0)}$$

Which comparing with (1), we get:

$$\boxed{a = \frac{1+\delta_4^{(1)}+\delta_m^{(1)}}{1+\delta_4^{(1)}} \;;\quad b = \frac{1+\delta_1^{(1)}}{1+\delta_4^{(1)}}} \;;\quad \mathcal{C} = -i\det(K)$$

---

b)

Let's start from the determinant and work with it a bit:

$$\det\left(1 + \frac{b\, g_R \gamma^\epsilon A_R^a T^a}{i\not{\partial} - a M_R}\right) = \exp\left\{\log\left[\det\left(1 + \frac{b\, g_R \gamma^\epsilon A_R^a T^a}{i\not{\partial} - a M_R}\right)\right]\right\} =$$

$\log(\det(M)) = \mathrm{tr}(\log(M))$

$$= \exp\left\{\mathrm{Tr}\left[\log\left(1 + \frac{b\, g_R \gamma^\epsilon A_R^a T^a}{i\not{\partial} - a M_R}\right)\right]\right\} \simeq$$

$\log(1+x) \simeq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$

$$\simeq \exp\left\{\mathrm{tr}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{b\, g_R \gamma^\epsilon A_R^a T^a}{i\not{\partial} - a M_R}\right)^n}{n}\right]\right\}$$

Now let's add this to the exponential as corrections to the pure glue balls theory:

$$e^{i\Delta S} = e^{tr(\ )} \Longrightarrow \Delta S = -i\,tr\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{b\, g_R \mu^{\epsilon} A_R^a T^a}{i\not\partial - aM_R}\right)^n\right)$$

And because $T^a$ are traceless matrices, the first order term vanishes:

$$\Delta S = -i\,Tr\left(-\frac{1}{2}\left(\frac{b\, g_R\mu^{\epsilon} A_R^a T^a}{i\not\partial - aM_R}\right)^2\right) + O(\cancelto{0}{g_R^4}) \simeq$$

This is the real order of the terms, I'll write it in compact form: $\frac{A}{i\partial - aM}$, but we have to keep in mind the real order for the computations.

$$\simeq \frac{i}{2} b^2 g_R^2 \mu^{2\epsilon}\, Tr\left(\frac{1}{i\partial_\alpha\gamma^\alpha - aM_R} A^a_\mu \gamma^\mu T^a \frac{1}{i\partial_\beta\gamma^\beta - aM_R} A^b_\nu\gamma^\nu T^b\right) =$$

(Trace in the 3 spaces)

$$= \frac{i}{2} b^2 g_R^2\mu^{2\epsilon}\left(\sum_{i,color}\ \sum_{\gamma,Dirac}\int \langle x,i,\gamma| \frac{A^a_\mu\gamma^\mu T^a}{i\partial_\alpha\gamma^\alpha - aM_R}\overset{(\mathbb{1})}{\downarrow}\frac{A^b_\nu\gamma^\nu T^b}{i\partial_\beta\gamma^\beta - aM_R}|x,i,\gamma\rangle\, d^4x\right) =$$

(Take out color part)

$$= \frac{i}{2}b^2g_R^2\mu^{2\epsilon}\left(\cancelto{C(R)\delta^{ab}}{\sum_{i,color}\langle i|T^aT^b|i\rangle}\right)\left(\sum_{\gamma,Dirac}\int\langle x,\gamma|\frac{A^a_\mu\gamma^\mu}{i\partial_\alpha\gamma^\alpha-aM_R}\overbrace{\int|p\rangle\langle p|\,d^4p}^{\mathbb{1}}\ \frac{A^b_\nu\gamma^\nu}{i\partial_\beta\gamma^\beta-aM_R}|x,\gamma\rangle\, d^4x\right) =$$

(*), (**)

$$= \frac{i}{2}b^2g_R^2\mu^{2\epsilon}C(R)\delta^{ab}\left(\sum_{\gamma,Dirac}\iiiint \overbrace{\langle x,\gamma|q\rangle}^{\langle\gamma|e^{ixq}} \frac{A^a_\mu(q-p)\gamma^\mu}{q_\alpha\gamma^\alpha - aM_R}\ \frac{A^b_\nu(p-k)\gamma^\nu}{p_\beta\gamma^\beta-aM_R}\overbrace{\langle k|x,\gamma\rangle}^{(e^{-ixk}|\gamma\rangle)}\, d^4x\, d^4p\, d^4q\, d^4k\right) =$$

$$\left(\int e^{ix(q-p)}dx = \delta(q-p)\right)$$

$$= \frac{i}{2}b^2g_R^2\mu^{2\epsilon}C(R)\delta^{ab}\left(\sum_{\gamma,Dirac}\iiint \delta(q-k)\langle\gamma|\frac{A^a_\mu(q-p)\gamma^\mu}{q_\alpha\gamma^\alpha-aM_R}\ \frac{A^b_\nu(p-k)\gamma^\nu}{p_\beta\gamma^\beta-aM_R}|\gamma\rangle\, d^4p\, d^4q\, d^4k\right) =$$

$(k\to q)$

$$= \frac{i}{2}b^2g_R^2\mu^{2\epsilon}C(R)\delta^{ab}\left(\sum_{\gamma,Dirac}\iint \langle\gamma|\frac{A^a_\mu(q-p)\gamma^\mu}{q_\alpha\gamma^\alpha-aM_R}\ \frac{A^b_\nu(p-q)\gamma^\nu}{p_\beta\gamma^\beta-aM_R}|\gamma\rangle\, d^4p\, d^4q\right) =$$

$$\left(\ell\equiv p-q;\quad p=\ell+q;\ dp=d\ell\right)$$

$$= \frac{i}{2}b^2g_R^2\mu^{2\epsilon}C(R)\delta^{ab}\left(\sum_{\gamma,Dirac}\iint \langle\gamma|\frac{A^a_\mu(-\ell)\gamma^\mu}{q_\alpha\gamma^\alpha-aM_R}\ \frac{A^b_\nu(\ell)\gamma^\nu}{(q+\ell)_\beta\gamma^\beta-aM_R}|\gamma\rangle\, d^4\ell\, d^4q\right) =$$

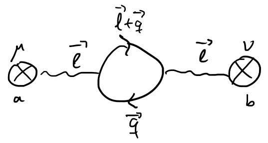
$$= \frac{i}{2}b^2g_R^2\mu^{2\epsilon}C(R)\delta^{ab}\int A^a_\mu(-\ell)A^b_\nu(\ell)\left(\sum_{\gamma,Dirac}\int\langle\gamma|\frac{1}{q_\alpha\gamma^\alpha-aM_R}\gamma^\mu\frac{1}{(q+\ell)_\beta\gamma^\beta-aM_R}\gamma^\nu|\gamma\rangle\, d^4q\right)d^4\ell =$$

$$= g_R^2\int d^4\ell\ A^a_\mu(-\ell)A^b_\nu(\ell)\left\{\frac{i}{2}b^2\mu^{2\epsilon}C(R)\delta^{ab}\int Tr_\gamma\left(\frac{1}{\not q - aM_R}\gamma^\mu\frac{1}{\not q+\not\ell - aM_R}\gamma^\nu\right)d^4q\right\}$$

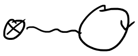
So now to get $\Delta S = g_R^2\int\frac{d^4q}{(2\pi)^4}A^a_\mu(-q)A^b_\nu(q)\gamma^{\mu\nu}_{ab}(q,M_R,\mu)$, we need:

$$\boxed{V^{\mu\nu}_{ab}(l, M_R, \mu) = \frac{i}{2} b^2 \mu^{2\epsilon} C(R)\,\delta_{ab} \int tr_\gamma\left(\frac{1}{\not q - aM_R}\gamma^\mu \frac{1}{\not q + \not l - aM_R}\gamma^\nu\right)\frac{d^4q}{(2\pi)^4}}$$

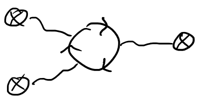
The diagramatic explanation of this integral is we have an internal fermion propagator given by two gluons sources:

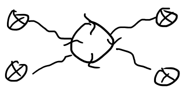
$V^{\mu\nu}_{ab} \propto$ 


And we can even generalize this idea for all the terms of $\Delta S$:

$\Delta S^{(1)} \propto$  $= 0$

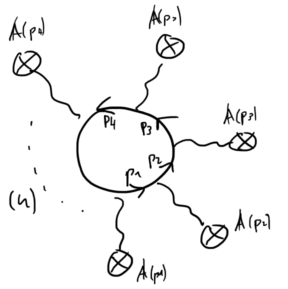
$\Delta S^{(2)} \propto$ 

$\Delta S^{(3)} \propto$  $= 0$

$\Delta S^{(4)} \propto$ 

$\vdots$

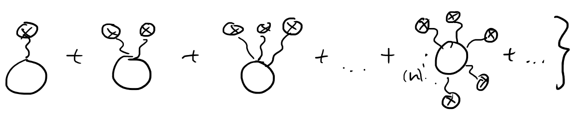
$(e = g_R b \mu^\epsilon)$

$\Delta S^{(n)} \propto$ 


$$= -\frac{1}{n}\int d^4p_1 \dots d^4p_n \; tr_\gamma\left\{(-ieA(p_1))S_F(p_1)\dots(-ieA(p_n))S_F(p_n)\right\} =$$

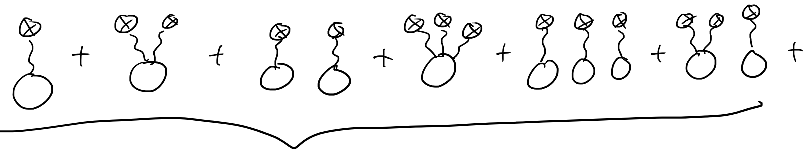
$$= -\frac{1}{n} tr\left\{\left(\frac{i}{i\not\partial - aM_R}(-ieA)\right)^n\right\}$$

Which finally means, that:

$$\det\left(1 + \frac{b\, g_R \mu^\epsilon A^a_R T^a}{i\not\partial - aM_R}\right) = \exp\{i\Delta S\} = \exp\Big\{$$ 


$\Big\} =$

(The $\frac{1}{n}$ factor of the exponential gets absorbed with the symmetry factors!)

$= 1 +$ 

corrections to pure glue theory with no external fermion !!!

Which is exactly what the statement told us to expect. It also makes total sense that $\Delta S$ only includes the connected corrections, since it's the log( All corrections ), same as when in QFT subject we computed $\log(Z(\lambda, J))$!

$$\left( \begin{array}{l} \circledast \\ \langle q | A | k \rangle = A(q-k) \end{array} \longleftrightarrow \begin{cases} A|p\rangle = \int A(q-p)\, |q\rangle\, d^4q \\ \langle p| A = \int A(p-q)\, \langle q|\, d^4 q \end{cases} \right)$$

($\circledast\circledast$ We will skip the $(2\pi)^4$ factors in the excercice and consider they are inside of $d^4p$)

---

c)

Let's now compute the integral in $V^{\mu\nu}_{ab}$:

$$V^{\mu\nu}_{ab}(\ell, M_R, \mu) = \frac{i}{2} b^2 \mu^{2\epsilon} C(R)\, \delta_{ab} \int Tr_\gamma \left( \frac{1}{\not q - aM_R} \gamma^\mu \frac{1}{\not q + \not \ell - aM_R} \gamma^\nu \right) \frac{d^4 q}{(2\pi)^4} =$$

For that, first we have to work out the trace inside it:

$$Tr_\gamma \left( \frac{1}{\not q - aM_R} \gamma^\mu \frac{1}{\not q + \not \ell - aM_R} \gamma^\nu \right) = Tr_\gamma \left( \frac{\not q + aM_R}{q^2 - (aM_R)^2} \gamma^\mu \frac{\not q + \not \ell + aM_R}{(q+\ell)^2 - (aM_R)^2} \gamma^\nu \right) =$$

$\Big\downarrow$ $\left( tr(\text{odd n} \ \gamma) = 0 \right)$

$$= \frac{q_\alpha (q+\ell)_\beta\, tr(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu) + (aM_R)^2\, tr(\gamma^\mu \gamma^\nu)}{[q^2 - (aM_R)^2]\,[(q+\ell)^2 - (aM_R)^2]} =$$

$\Big\downarrow$ $\left( \begin{array}{l} tr(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu} \\ tr(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu) = 4(g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu} + g^{\alpha\nu} g^{\mu\beta}) \end{array} \right)$

$$= \frac{4\{ q^\mu (q+\ell)^\nu + q^\nu (q+\ell)^\mu + [(aM_R)^2 - q(q+\ell)]\, g^{\mu\nu} \}}{[q^2 - (aM_R)^2]\,[(q+\ell)^2 - (aM_R)^2]} =$$

Now let's put this inside the integral:

$$V^{\mu\nu}_{ab}(\ell, M_R, \mu) = \frac{i}{2} b^2 \mu^{2\epsilon} C(R)\, \delta_{ab} \int \frac{4\{ q^\mu (q+\ell)^\nu + q^\nu (q+\ell)^\mu + [(aM_R)^2 - q(q+\ell)]\, g^{\mu\nu} \}}{[q^2 - (aM_R)^2]\,[(q+\ell)^2 - (aM_R)^2]} \frac{d^4 q}{(2\pi)^4} =$$

$\Big\downarrow$ $\left( \begin{array}{l} \text{Introduce } \int_0^1 \ldots dx \\ \text{Feynman parameters} \end{array} \right)$

$$= 2i\, b^2 \mu^{2\epsilon} C(R)\, \delta_{ab} \iint_0^1 \frac{q^\mu (q+\ell)^\nu + q^\nu (q+\ell)^\mu + [(aM_R)^2 - q(q+\ell)]\, g^{\mu\nu}}{(q^2 + 2x\, q\, \ell + x \ell^2 - (aM_R)^2)^2}\, dx\, \frac{d^4 q}{(2\pi)^4} =$$

$\Big\downarrow$ $(p = q + x\ell)$

$$= 2i\, b^2 \mu^{2\epsilon} C(R)\, \delta_{ab} \iint_0^1 \frac{2 p^\mu p^\nu - g^{\mu\nu} p^2 - 2x(1-x)\ell^\mu \ell^\nu + g^{\mu\nu}[(aM_R)^2 + x(1-x)\ell^2]}{(p^2 + x(1-x)\ell^2 - (aM_R)^2)^2}\, dx\, \frac{d^4 p}{(2\pi)^4} =$$

$\Big\downarrow$ $\left( \text{Wick rotation} \ \ p^0 = i p^0_E \right)$

$$= -2\, b^2 \mu^{2\epsilon} C(R)\, \delta_{ab} \iint_0^1 \frac{(-\frac{2}{d} + 1)\, g^{\mu\nu} p_E^2 - 2x(1-x)\ell^\mu \ell^\nu + g^{\mu\nu}[(aM_R)^2 + x(1-x)\ell^2]}{(p_E^2 \underbrace{- x(1-x)\ell^2 + (aM_R)^2}_{\equiv +\Delta})^2}\, dx\, \frac{d^4 p_E}{(2\pi)^4} =$$

And using the formulas:

- $$\int \frac{d^d p_E}{(2\pi)^d} \frac{1}{(p_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \xrightarrow{n=2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\cancel{\Gamma(2)}} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

- $$\int \frac{d^d p_E}{(2\pi)^d} \frac{p_E^2}{(p_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \xrightarrow{n=2} \frac{d}{2} \frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} = \frac{-\Delta}{(1-\frac{2}{d})} \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

We then get:

$$V^{\mu\nu}_{ab}(l, M_R, \mu) = \frac{-2}{(4\pi)^{d/2}} b^2 \mu^{2\epsilon} C(R)\, \delta_{ab} \int_0^1 \Big\{ -g^{\mu\nu} \overbrace{\Delta}^{(\cancel{aM_R})^2 - x(1-x)l^2} - 2x(1-x)\, l^\mu l^\nu + g^{\mu\nu}\left[(\cancel{aM_R})^2 + x(1-x)l^2\right] \Big\} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\, dx =$$

$$= \frac{-2}{(4\pi)^{d/2}} b^2 \mu^{2\epsilon} C(R)\, \delta_{ab} \int_0^1 \left\{ 2x(1-x)l^2 g^{\mu\nu} - 2x(1-x)\, l^\mu l^\nu \right\} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\, dx =$$

$$= \frac{-4}{(4\pi)^{d/2}} b^2 \mu^{2\epsilon} C(R)\, \delta_{ab} \left(l^2 g^{\mu\nu} - l^\mu l^\nu\right) \int_0^1 x(1-x) \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\, dx$$

Which when we take the limit $d \to 4$ and using the expansions:

- $\Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + O(\epsilon)$
- $\frac{1}{\Delta^{\epsilon/2}} = 1 - \frac{\epsilon}{2}\log(\Delta) + O(\epsilon^2)$
- $\frac{1}{(4\pi)^{-\epsilon/2}} = 1 + \frac{\epsilon}{2}\log(4\pi) + O(\epsilon^2)$
- $\frac{1}{\mu^{2\epsilon}} = 1 + 2\epsilon\log(\mu) + O(\epsilon^2)$

$(x^{d/2} = x^2 x^{-\epsilon/2})$

$$\frac{\mu^{2\epsilon}\Gamma(\frac{\epsilon}{2})}{(4\pi)^{d/2}\Delta^{\epsilon/2}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log(\Delta) - \gamma + \log(4\pi) + 4\log(\mu) + O(\epsilon)\right) =$$

$$= \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\mu^4}{\Delta}\right) + O(\epsilon)\right)$$

We finally get:

$$\boxed{V^{\mu\nu}_{ab}(l, M_R, \mu)} = \frac{-4}{(4\pi)^2} b^2 C(R)\, \delta_{ab} \left(l^2 g^{\mu\nu} - l^\mu l^\nu\right) \int_0^1 x(1-x)\left(\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\mu^4}{\Delta}\right)\right) dx = \left(\int_0^1 x(1-x)\,dx = \frac{1}{6}\right)$$

$$\boxed{= \frac{-4b^2 C(R)}{(4\pi)^2}\, \delta_{ab} \left(l^2 g^{\mu\nu} - l^\mu l^\nu\right) \left\{ \frac{1}{3\epsilon} - \frac{\gamma}{6} + \int_0^1 x(1-x)\log\left(\frac{4\pi\mu^4}{\Delta}\right) dx \right\}}$$

Where we see that the divergent contribution has a Transerse Lorentz structure that fullfills the Ward Identity. Which is the same Lorentz structure as that of the gauge field propagator, corroborating the diagrams we found previusly: .

All this could be easily seen also from the start, where: $\underline{V} \propto \int Tr_\gamma\left(\frac{1}{\not{q} - aM_R}\gamma^\mu \frac{1}{\not{q}+\not{l} - aM_R}\gamma^\nu\right)\frac{d^4q}{(2\pi)^4} \propto$ gauge propagator, but now we can see this more explicitly:

$$\boxed{g_R^2 V^{\mu\nu}_{ab}(\ell, M_R, \mu)_{div}} = -\frac{g_R^2}{4\pi}\frac{b^2 C(R)}{3\pi}\delta_{ab}(\ell^2 g^{\mu\nu} - \ell^\mu \ell^\nu)\frac{1}{\epsilon} =$$

$(C(R) = T_F$ in class notes$)$

$$= -(\ell^2 g^{\mu\nu} - \ell^\mu\ell^\nu) T_F \delta_{ab}\frac{\alpha}{3\pi} b^2 \frac{1}{\epsilon} =$$

$$= -b^2 \Pi^{\mu\nu}_{ab}(\ell)^{(1\,loop)} \underset{(\text{Order } \epsilon^1\ (b=1))}{\simeq} \boxed{-\Pi^{\mu\nu}_{ab}(\ell)^{(1\,loop)}}$$

All this tells us that this divergence will go away whenever the 1 fermion-loop divergence of the gauge propagator goes away. So introducing the counter terms of the $n_f = 1$ theory will cancel this divergence:

$$\left.\begin{aligned} &-\delta_A^1 \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &\text{with } \delta_A^{(1)} = C_1\left(\frac{1}{\epsilon} + C_M\right) \underset{(MS)}{=} \frac{C_1}{\epsilon}\end{aligned}\right\} \xrightarrow{\text{cancels}} -i\Pi^{(1)}_{ab\mu\nu} = i(q^2 g_{\mu\nu} - q_\mu q_\nu)\delta_{AB}\frac{C_1}{\epsilon}$$

---

## d)

The finite part of the integral will be (again) the corrections to the renormalized pure glue theory, which also have the Lorentz structure of the gauge propagator, being transverse and fullfilling the Ward Identity. Which is necessary for it to be the fermion interactions correction to the pure glue propagator.

Now if we force $\ell \ll M_R^2$, we then get:

$$\log\left(\frac{4\pi\mu^4}{\Delta}\right) = \log(4\pi\mu^4) - \log\left[(aM_R)^2 - x(1-x)\ell^2\right] = \log\left(\frac{4\pi\mu^4}{(aM_R)^2}\right) - \log\left[1 - x(1-x)\left(\frac{\ell}{aM_R}\right)^2\right] =$$

$$= \log\left(\frac{4\pi\mu^4}{(aM_R)^2}\right) + \sum_{n=1}^{\infty}\frac{1}{n}\left(x(1-x)\left(\frac{\ell}{aM_R}\right)^2\right)^n \simeq \log\left(\frac{4\pi\mu^4}{(aM_R)^2}\right) + x(1-x)\left(\frac{\ell}{aM_R}\right)^2 + O\left(\left(\frac{\ell}{aM_R}\right)^4\right)$$

$$\left(\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3}\dots\right)$$

So we then have:

$$V^{\mu\nu}_{ab}(\ell, M_R, \mu) = \frac{-4b^2 C(R)}{(4\pi)^2}\delta_{ab}(\ell^2 g^{\mu\nu} - \ell^\mu\ell^\nu)\left\{\frac{1}{3\epsilon} - \frac{\gamma}{6} + \int_0^1 x(1-x)\log\left(\frac{4\pi\mu^4}{\Delta}\right)dx\right\} \simeq$$

$$\simeq \frac{-4b^2 C(R)}{(4\pi)^2}\delta_{ab}(\ell^2 g^{\mu\nu} - \ell^\mu\ell^\nu)\left\{\frac{1}{3\epsilon} - \frac{\gamma}{6} + \frac{\log\left(\frac{4\pi\mu^4}{(aM_R)^2}\right)}{6} + \left(\frac{\ell}{aM_R}\right)^2\int_0^1 x^2(1-x)^2 dx\right\} =$$

$$\left(\int_0^1 x^2(1-x)^2 dx = \frac{1}{30}\right)$$

$$= \frac{-4b^2 C(R)}{3(4\pi)^2}\delta_{ab}(\ell^2 g^{\mu\nu} - \ell^\mu\ell^\nu)\left\{\underbrace{\frac{1}{\epsilon}}_{div} + \underbrace{\frac{\log\left(\frac{4\pi\mu^4}{(aM_R)^2}\right) - \gamma}{2} + \frac{1}{10}\left(\frac{\ell}{aM_R}\right)^2}_{finite}\right\}$$

Now having the leading term, we see that obviously we have the same Lorentz structure as before, but now we also see that the amplitude goes with $\ell^2$ for very massive fermions.

The decoupling parameter $\xi_A = \frac{\lambda'_A}{\lambda_A} = \frac{1 + \Pi(k^2=0, \mu, M)}{1 + \Pi'(k^2=0, \mu)} = \frac{1 + C'\left(\log\left(\frac{4\pi\mu^4}{(aM_R)^2}\right) - \gamma\right)}{1 + C'(1-\gamma)} \simeq \boxed{1 + C'' \log\left(\frac{\mu^4}{(aM_R)^2}\right)}$