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Exercise sheet 2: Real K.G. field

1)

$$\begin{cases} [\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y}) \\ [\phi(t,\vec{x}), \phi(t,\vec{y})] = 0 \\ [\pi(t,\vec{x}), \pi(t,\vec{y})] = 0 \end{cases}$$

$$\begin{cases} \phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^{\dagger} e^{ipx}\right) \\ \pi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} (-iE_p)\left(a_{\vec{p}} e^{-ipx} - a_{\vec{p}}^{\dagger} e^{ipx}\right) \end{cases}$$

Let's start proving $a_{\vec{p}} = \frac{1}{\sqrt{2E_p}} \int d^3x\, e^{ipx}\left(i\dot{\phi}(x) + E_p\phi(x)\right)$:

- $$\frac{1}{\sqrt{2E_p}}\int d^3x\, e^{ipx}\int\frac{d^3p'}{(2\pi)^3\sqrt{2E_{p'}}}\Big[\overbrace{E_{p'}\left(a_{\vec{p}'}e^{-ip'x} - a_{\vec{p}'}^{\dagger}e^{ip'x}\right)}^{i\dot{\phi}(x)} + \overbrace{E_p\left(a_{\vec{p}'}e^{-ip'x} + a_{\vec{p}'}^{\dagger}e^{ip'x}\right)}^{E_p\phi(x)}\Big] =$$

$$= \frac{1}{2\sqrt{E_p}}\int d^3x\, e^{ipx}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{E_{p'}}}\left[(E_{p'}+E_p)\,a_{\vec{p}'}e^{-ip'x} + (E_p - E_{p'})\,a_{\vec{p}'}^{\dagger}e^{ip'x}\right] =$$

$$= \frac{1}{2\sqrt{E_p}}\iint\frac{d^3x\,d^3p'}{(2\pi)^3\sqrt{E_{p'}}}\left[(E_{p'}+E_p)\,a_{\vec{p}'}e^{-i(p'-p)x} + (E_p - E_{p'})\,a_{\vec{p}'}^{\dagger}e^{i(p'+p)x}\right] =$$

$$\left(\int\frac{d^3x}{(2\pi)^3}e^{i(p-p')x} = \delta(p-p')\right)$$

$$= \frac{1}{2\sqrt{E_p}}\int\frac{d^3p'}{\sqrt{E_{p'}}}\left[(E_{p'}+E_p)\,a_{\vec{p}'}\,\delta(p'-p) + (E_p - E_{p'})\,a_{\vec{p}'}^{\dagger}\,\delta(p'+p)\right] =$$

$$(E_{-p} = E_p)$$

$$= \frac{1}{2E_p}\left[(E_p + E_p)\,a_{\vec{p}} + \overbrace{(E_p - E_p)}^{0}\,a_{\vec{p}}^{\dagger}\right] = \frac{\cancel{2E_p}}{\cancel{2E_p}}\,a_{\vec{p}} = \boxed{a_{\vec{p}}}$$

So now we can compute our commutators:

- $$\boxed{[a_{\vec{p}}, a_{\vec{q}}^{\dagger}]} = \frac{1}{2\sqrt{E_pE_q}}\iint d^3x\,d^3y\, e^{i(px-qy)}\,[i\dot{\phi}(x) + E_p\phi(x), -i\dot{\phi}(y) + E_q\phi(y)] =$$

$$(-i\delta^3(\vec{x}-\vec{y})) \qquad (i\delta^3(\vec{x}-\vec{y}))$$

$$= \frac{1}{2\sqrt{E_pE_q}}\iint d^3x\,d^3y\, e^{i(px-qy)}\left(iE_q[\dot{\phi}(x), \phi(y)] - iE_p[\phi(x), \dot{\phi}(y)]\right) =$$

$$= \frac{1}{2\sqrt{E_pE_q}}\underbrace{\int d^3x\, e^{i(p-q)x}}_{(2\pi)^3\delta(p-q)}(E_q + E_p) = \frac{E_q + E_p}{2\sqrt{E_pE_q}}(2\pi)^3\delta(p-q)$$

(Effectively every time we have this in an integral p will be equal to p = q, so:)

$$= \frac{\cancel{2E_p}}{\cancel{2E_p}}(2\pi)^3\delta(p-q) = \boxed{(2\pi)^3\delta(p-q)}$$

$$\bullet\ \boxed{[a_{\vec p}, a_{\vec q}]} = \frac{1}{2\sqrt{E_p E_q}} \iint d^3x\, d^3y\; e^{i(px+qy)} \left[i\dot\phi(x) + E_p \phi(x),\ i\dot\phi(y) + E_q \phi(y) \right] =$$

$$= \frac{1}{2\sqrt{E_p E_q}} \iint d^3x\, d^3y\; e^{i(px+qy)} \Big(i E_q \underbrace{\cancel{[\dot\phi(x), \phi(y)]}}_{(-i\delta^3(\vec x - \vec y))} + i E_p \underbrace{\cancel{[\phi(x), \dot\phi(y)]}}_{(i\delta^3(\vec x - \vec y))} \Big) =$$

$$= \frac{1}{2\sqrt{E_p E_q}} \underbrace{\int d^3x\, e^{i(p+q)x}}_{((2\pi)^3 \delta^3(p+q))} \left(E_q - E_p \right) = (2\pi)^3 \delta(p+q)\, \frac{E_q - E_p}{2\sqrt{E_p E_q}} =$$

($E_p = E_{-p}$
And effectively we will have that $q = -p$, so $E_q = E_{-p} = E_p$)

$$= (2\pi)^3 \delta(p+q)\, \frac{\cancelto{0}{E_p - E_{-p}}}{2\sqrt{E_p E_p}} = \boxed{0}$$

$$\bullet\ \boxed{[a^\dagger_{\vec p}, a^\dagger_{\vec q}]} = \frac{1}{2\sqrt{E_p E_q}} \iint d^3x\, d^3y\; e^{-i(px+qy)} \left[-i\dot\phi(x) + E_p \phi(x),\ -i\dot\phi(y) + E_q \phi(y) \right] =$$

$$= \frac{1}{2\sqrt{E_p E_q}} \iint d^3x\, d^3y\; e^{-i(px+qy)} \Big(-i E_q \underbrace{\cancel{[\dot\phi(x), \phi(x)]}}_{(-i\delta^3(\vec x - \vec y))} - i E_p \underbrace{\cancel{[\phi(x), \dot\phi(y)]}}_{(i\delta^3(\vec x - \vec y))} \Big) =$$

$$= \frac{1}{2\sqrt{E_p E_q}} \underbrace{\int d^3x\, e^{-i(p+q)x}}_{((2\pi)^3 \delta^3(p+q))} \left(-E_q + E_p \right) = (2\pi)^3 \delta^3(p+q)\, \frac{E_p - E_q}{2\sqrt{E_p E_q}} = \boxed{0}$$

(same as before)

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2)

$$\begin{cases} (t, \vec x) \xrightarrow{\text{Parity}} (t, -\vec x) \equiv (\tilde x) \\ \phi(t, \vec x) \longrightarrow P\phi(t, \vec x) P^{-1} = \eta_p\, \phi(t, -\vec x) \equiv \eta_p \phi(\tilde x) \end{cases} \qquad \begin{cases} P|0\rangle = |0\rangle \\ \eta_p = \pm 1 \longrightarrow \eta_p^2 = 1 \end{cases}$$

a)

$(P^{-1}P = \mathbb{1})$ $(P^{-1}P = \mathbb{1})$

$$\bullet\ \boxed{P\mathcal{L}P^{-1}} = P\, \frac{1}{2} \left(\partial_\mu \phi(x) \partial^\mu \phi(x) - m^2 \phi^2(x) \right) P^{-1} =$$

$$= \frac{1}{2} \left[\partial_\mu (P\phi(x)P^{-1})\, \partial^\mu (P\phi(x)P^{-1}) - m^2 (P\phi(x)P^{-1})(P\phi(x)P^{-1}) \right] =$$

$$= \frac{1}{2} \left[\cancelto{1}{\eta_p^2}\, \partial_\mu \phi(\tilde x) \partial^\mu \phi(\tilde x) - \cancelto{1}{\eta_p^2}\, m^2 \phi^2(\tilde x) \right] = \frac{1}{2} \left[(\cancel{-}\partial_{\tilde\mu}) \phi(\tilde x) (\cancel{-}\partial^{\tilde\mu}) \phi(\tilde x) - m^2 \phi^2(\tilde x) \right] =$$

$$= \frac{1}{2} \left[\partial_{\tilde\mu} \phi(\tilde x) \partial^{\tilde\mu} \phi(\tilde x) - m^2 \phi(\tilde x) \right] = \boxed{\mathcal{L}(\tilde x)}$$

(it's the same because integrating this over all space give the same Lagrangian.)

$$\boxed{PLP^{-1}} = \int_{-\infty}^{\infty} d^3x\, \mathcal{L}(\tilde x) = \int_{\infty}^{-\infty} -d^3(-x)\, \mathcal{L}(\tilde x) = \int_{-\infty}^{\infty} d^3(\tilde x)\, \mathcal{L}(\tilde x) = \boxed{L}$$

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b)

- $\boxed{P|\vec{k}_1,\vec{k}_2,\dots,\vec{k}_n\rangle} = P\left(2^{n/2}\sqrt{E_1E_2\cdots E_n}\; a_{\vec{k}_1}^{\dagger} a_{\vec{k}_2}^{\dagger}\dots a_{\vec{k}_n}^{\dagger}|0\rangle\right) =$

$$= 2^{n/2}\sqrt{E_1E_2\cdots E_n}\; P\, a_{\vec{k}_1}^{\dagger} a_{\vec{k}_2}^{\dagger}\dots a_{\vec{k}_n}^{\dagger}|0\rangle =$$

$$= 2^{n/2}\sqrt{E_1E_2\cdots E_n}\; a_{-\vec{k}_1}^{\dagger} a_{-\vec{k}_2}^{\dagger}\dots a_{-\vec{k}_n}^{\dagger}(\eta_p)^n \overbrace{P|0\rangle}^{|0\rangle} = \circledast$$

$$= \boxed{(\eta_p)^n|-\vec{k}_1,-\vec{k}_2,\dots,-\vec{k}_n\rangle}$$

$\circledast$

$$P a_{\vec{k}}^{\dagger} P^{-1} = \frac{1}{\sqrt{2E_k}}\int_{-\infty}^{\infty} d^3x\; e^{ikx}\Big(i\underbrace{(P\dot{\phi}(x)P^{-1})}_{\eta_p\dot{\phi}(\tilde{x})} + E_k\underbrace{(P\phi(x)P^{-1})}_{\eta_p\phi(\tilde{x})}\Big) =$$

$$= \frac{\eta_p}{\sqrt{2E_k}}\int_{-x=\infty}^{-x=-\infty} -d^3(\tilde{x})\; e^{i(-k)\tilde{x}}\left(i\dot{\phi}(\tilde{x}) + E_k\phi(\tilde{x})\right) =$$

$$= \eta_p\int_{\tilde{x}=-\infty}^{\tilde{x}=\infty} d^3(\tilde{x})\; e^{i(-k)\tilde{x}}\left(i\dot{\phi}(\tilde{x}) + E_k\phi(\tilde{x})\right) = \eta_p\, a_{-\vec{k}}^{\dagger}$$

Which means that $P a_{\vec{k}}^{\dagger} = \eta_p\, a_{-\vec{k}}^{\dagger} P$, so $P a_{\vec{k}_1}^{\dagger} a_{\vec{k}_2}^{\dagger} = \eta_p^2\, a_{-\vec{k}_1}^{\dagger} a_{-\vec{k}_2}^{\dagger} P \dots$ etc

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3)

$:P_K: = \int d^3x\, :\pi(x)\,\partial_K\phi(x):$ in terms of a and $a^{\dagger}$

First let's compute P_K in a's, $a^{\dagger}$'s from: $P_K = \int d^3x\,(\pi(x)\,\partial_K\phi(x))$:

- $$P_K = \int d^3x\int\frac{d^3p\, d^3q}{(2\pi)^6\, 2\sqrt{E_pE_q}}(-E_p q_K)\left[a_{\vec{p}}a_{\vec{q}}e^{-i(p+q)x} + a_{\vec{p}}^{\dagger}a_{\vec{q}}^{\dagger}e^{i(p+q)x} - a_{\vec{p}}a_{\vec{q}}^{\dagger}e^{-i(p-q)x} - a_{\vec{p}}^{\dagger}a_{\vec{q}}e^{i(p-q)x}\right] =$$

$$= \int\frac{d^3p\, d^3q}{(2\pi)^3\, 2\sqrt{E_pE_q}}(-E_p q_K)\left[a_{\vec{p}}a_{\vec{q}}\delta(p+q) + a_{\vec{p}}^{\dagger}a_{\vec{q}}^{\dagger}\delta(p+q) - a_{\vec{p}}a_{\vec{q}}^{\dagger}\delta(p-q) - a_{\vec{p}}^{\dagger}a_{\vec{q}}\delta(p-q)\right] =$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{-\cancel{E_p}}{2\cancel{E_p}}\left[(-P_K)a_{\vec{p}}a_{-\vec{p}}e^{-i2E_pt} + (-P_K)a_{\vec{p}}^{\dagger}a_{-\vec{p}}^{\dagger}e^{i2E_pt} - (P_K)a_{\vec{p}}a_{\vec{p}}^{\dagger}\cancel{e^{i0}}^{\,1} - (P_K)a_{\vec{p}}^{\dagger}a_{\vec{p}}\cancel{e^{i0}}^{\,1}\right] =$$

$$= \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}P_K\Big[\underbrace{\cancel{a_{\vec{p}}a_{-\vec{p}}e^{-i2E_pt}}^{\,0} + \cancel{a_{\vec{p}}^{\dagger}a_{-\vec{p}}^{\dagger}e^{i2E_pt}}^{\,0}}_{\text{It's an antisymmetric term integrated from }-\infty\text{ to }\infty} + a_{\vec{p}}a_{\vec{p}}^{\dagger} + a_{\vec{p}}^{\dagger}a_{\vec{p}}\Big] = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}P_K\left[a_{\vec{p}}a_{\vec{p}}^{\dagger} + a_{\vec{p}}^{\dagger}a_{\vec{p}}\right]$$

$$\begin{cases} P_K \to -P_K & \text{antisimetric} \\ E_p \to E_{-p} = E_p & \text{simetric} \\ a_p a_{-p} \to a_{-p}a_p = a_p a_{-p} & \end{cases}$$

so finally:

$$\boxed{:P_K: = \int\frac{d^3p}{(2\pi)^3}P_K\, a_{\vec{p}}^{\dagger}a_{\vec{p}}}$$

4)

$$D_R(x-y) = \theta(x^0-y^0)[\phi(x),\phi(y)]$$

Prove that $(\partial_\mu\partial^\mu + m^2)\, D_R(x-y) = -i\delta^4(x-y)$, so first let's compute each term:

- $\partial^\mu D_R(x-y) = \dfrac{\partial\theta(x^0-y^0)}{\partial x_0}[\phi(x),\phi(y)] + \theta(x^0-y^0)\dfrac{\partial[\phi(x),\phi(y)]}{\partial x_\mu} =$ ($\theta(x^0-y^0)$ only depends in x^0, $\phi(y)$ doesn't depend on x^μ)

$$= \underbrace{\delta(x^0-y^0)[\phi(x),\phi(y)]}_{(\text{only has } \mu=0,\ \partial_0\partial^0)} + \underbrace{\theta(x^0-y^0)[\partial^\mu\phi(x),\phi(y)]}_{(\text{has all the } \mu,\ \partial_\mu\partial^\mu)}$$

- $\partial_\mu\partial^\mu D_R(x-y) = \underbrace{\partial_0(\delta(x^0-y^0))[\phi(x),\phi(y)] + \delta(x^0-y^0)[\partial_0\phi(x),\phi(y)]}_{(\text{only has } \mu=0,\ \partial_0\partial^0)} +$

$$+ \underbrace{\partial_\mu(\theta(x^0-y^0))[\partial^\mu\phi(x),\phi(y)]}_{(\text{only survive } \mu=0,\ \partial_0\partial^0)} + \underbrace{\theta(x^0-y^0)[\partial_\mu\partial^\mu\phi(x),\phi(y)]}_{(\text{has all the } \mu\text{'s},\ \partial_\mu\partial^\mu)}$$

But we know that $f(x)\,\delta'(x) = -f'(x)\,\delta(x)$, so we get:

- $\partial_\mu\partial^\mu D_R(x-y) = \cancel{-\delta(x^0-y^0)[\partial_0\phi(x),\phi(y)]} + \cancel{\delta(x^0-y^0)[\partial_0\phi(x),\phi(y)]} +$ $[(\partial_\mu\partial^\mu+m^2)\phi=0$ K.G. eq$]$

$$+ \delta(x^0-y^0)[\partial^0\phi(x),\phi(y)] + \theta(x^0-y^0)[-m^2\phi(x),\phi(y)]$$

And the other term

- $m^2 D_R(x-y) = m^2\,\theta(x^0+y^0)[\phi(x),\phi(y)]$

So everything together becomes:

- $(\partial_\mu\partial^\mu + m^2)\, D_R(x-y) = \delta(x^0-y^0)[\dot\phi(x),\phi(y)] + \cancel{(m^2-m^2)}^{\,0}\,\theta(x^0-y^0)[\phi(x),\phi(y)] =$

$$= \delta(x^0-y^0)[\dot\phi(x^0,\vec{x}),\phi(y^0,\vec{y})]$$

And finally, effectively the delta makes that x^0 and y^0 will be equal in any integration, so:

$$\boxed{(\partial_\mu\partial^\mu+m^2)\, D_R(x-y)} = \delta(x^0-y^0)\,\underbrace{[\dot\phi(x^0,\vec{x}),\phi(x^0,\vec{y})]}_{-i\delta^3(\vec{x}-\vec{y})} = \boxed{-i\,\delta^4(x-y)}$$

Comprovation 4)

We know that $D_R(x-y) = \theta(x^0-y^0)\,[\phi(x),\phi(y)] = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2}$

Proof:

- $$[\phi(x),\phi(y)] = \int \frac{d^3p\, d^3q}{(2\pi)^6\, 2\sqrt{E_p E_q}}\left[a_p e^{-ipx} + a_p^\dagger e^{ipx},\; a_q e^{-iqy} + a_q^\dagger e^{iqy}\right] =$$
$$= \int \frac{d^3p\, d^3q}{(2\pi)^6\, 2\sqrt{E_p E_q}}\left(e^{-i(px-qy)}\underbrace{[a_p, a_q^\dagger]}_{(2\pi)^3\delta^3(p-q)} + e^{i(px-qy)}\underbrace{[a_p^\dagger, a_q]}_{(-(2\pi)^3\delta^3(p-q))}\right) =$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right)$$

- $$i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2} = \theta(x^0-y^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(e^{-ip(x-y)} - e^{-ip(y-x)}\right)$$

(done in class)

So let's check if it's the Green function aswell:

$$\boxed{(\partial_\mu\partial^\mu + m^2)}\, i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2} = i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\frac{(-ip^\mu)(-ip_\mu)+m^2}{p^2-m^2} =$$

$$= i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\underbrace{\frac{-p^2+m^2}{p^2-m^2}}_{-1} = -i\int \frac{dp^0}{2\pi}\,\delta^3(\vec{x}-\vec{y})\, e^{-ip^0(x_0-y_0)} =$$

$$= -i\,\delta^3(\vec{x}-\vec{y})\,\delta(x^0-y^0) = \boxed{-i\delta^4(x-y)}$$