Advanced General Relativity

Action principle and field equations in low-energy string theory

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February 28, 2022

We consider the action:

$$I = \int d^4x\sqrt{-g}\Big(R - \frac{1}{2}(\nabla\phi)^2 - e^{-\phi}F^2 - 2\Lambda e^{\phi}\Big) \tag{1}$$

where ϕ is a scalar field, F is the electromagnetic field, and Λ is a number.

Field equations

We first vary I with respect to the scalar field ϕ:

$$\begin{aligned} \phi &\to \phi + \delta\phi & e^{-\phi} &\to e^{-\phi}(1 - \delta\phi) \\ \nabla_\mu\phi\nabla^\mu\phi &\to \nabla_\mu\phi\nabla^\mu\phi + 2\nabla_\mu\phi\nabla^\mu(\delta\phi) & e^{\phi} &\to e^{\phi}(1 + \delta\phi) \end{aligned}$$

Then:

$$\begin{aligned} \delta I &= \int dV\Big(-\nabla_\mu\phi\nabla^\mu(\delta\phi) + e^{-\phi}F^2\delta\phi - 2\Lambda e^{\phi}\delta\phi\Big) \\ &= \int dV\Big(\nabla^2\phi\,\delta\phi + e^{-\phi}F^2\delta\phi - 2\Lambda e^{\phi}\delta\phi\Big) + \text{boundary terms} \\ &= \int dV\Big(\nabla^2\phi + e^{-\phi}F^2 - 2\Lambda e^{\phi}\Big)\,\delta\phi \end{aligned}$$

and setting $\delta I = 0$ we obtain:

$$\nabla^2\phi + e^{-\phi}F^2 - 2\Lambda e^{\phi} = 0 \tag{2}$$

Next, we vary with respect to A_μ:

$$\begin{aligned} A_\mu &\to A_\mu + \delta A_\mu & \nabla_\mu A_\nu\nabla^\mu A^\nu &\to \nabla_\mu A_\nu\nabla^\mu A^\nu + 2\nabla^\mu A^\nu\nabla_\mu(\delta A_\nu) \\ & & \nabla_\mu A_\nu\nabla^\nu A^\mu &\to \nabla_\mu A_\nu\nabla^\nu A^\mu + 2\nabla^\mu A^\nu\nabla_\nu(\delta A_\mu) \end{aligned}$$

and so

$$\begin{aligned} F^2 = \nabla_\mu A_\nu\nabla^\mu A^\nu - \nabla_\mu A_\nu\nabla^\nu A^\mu &\to F^2 + (2\nabla^\mu A^\nu\nabla_\mu(\delta A_\nu) - 2\nabla^\mu A^\nu\nabla_\nu(\delta A_\mu)) \\ &= F^2 + 2(\nabla^\mu A^\nu\nabla_\mu(\delta A_\nu) - \nabla^\nu A^\mu\nabla_\mu(\delta A_\nu)) \\ &= F^2 + 2F^{\mu\nu}\nabla_\mu(\delta A_\nu) \end{aligned}$$

Then:

$$\begin{aligned} \delta I &= \int dV\Big(-e^{-\phi}2F^{\mu\nu}\nabla_\mu(\delta A_\nu)\Big) \\ &= -2\int dV\, e^{-\phi}F^{\mu\nu}\nabla_\mu(\delta A_\nu) \\ &= 2\int dV\,\nabla_\mu(e^{-\phi}F^{\mu\nu})\,\delta A_\nu + \text{boundary terms} \end{aligned}$$

and so, again setting $\delta I = 0$ we obtain:

$$\nabla_\mu(e^{-\phi}F^{\mu\nu}) = 0 \tag{3}$$

Finally, we must vary I with respect to the metric:

$$
\begin{aligned}
I &= \int d^4x\sqrt{-g}\Big(R - \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - e^{-\phi}g^{\alpha\rho}g^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma} - 2\Lambda e^{\phi}\Big) \\
&\to \int d^4x\sqrt{-g}\Big(1 - \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\Big) \\
&\quad \Big(R + R_{\alpha\beta}\delta g^{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta} - \frac{1}{2}(g^{\alpha\beta} + \delta g^{\alpha\beta})\nabla_\alpha\phi\nabla_\beta\phi - e^{-\phi}(g^{\alpha\rho} + \delta g^{\alpha\rho})(g^{\beta\sigma} + \delta g^{\beta\sigma})F_{\alpha\beta}F_{\rho\sigma} - 2\Lambda e^{\phi}\Big)
\end{aligned}
$$

and so:

$$
\begin{aligned}
\delta I = \int d^4x\sqrt{-g}\Big(&R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{4}g_{\mu\nu}(\nabla\phi)^2 - 2e^{-\phi}g^{\rho\sigma}F_{\rho\mu}F_{\sigma\nu} + \frac{1}{2}g_{\mu\nu}e^{-\phi}F^2 + \Lambda e^{\phi}g_{\mu\nu}\Big)\delta g^{\mu\nu} \\
&+ \text{boundary terms}
\end{aligned}
$$

And so the equation of motion is:

$$
G_{\mu\nu} - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{4}g_{\mu\nu}(\nabla\phi)^2 - 2e^{-\phi}F_{\rho\mu}F^{\rho}{}_{\nu} + \frac{1}{2}e^{-\phi}g_{\mu\nu}F^2 + \Lambda e^{\phi}g_{\mu\nu} = 0
$$

We may take the trace of this equation to obtain:

$$
\begin{aligned}
0 &= -R - \frac{1}{2}(\nabla\phi)^2 + (\nabla\phi)^2 - 2e^{-\phi}F^2 + 2e^{-\phi}F^2 + 4\Lambda e^{\phi} \\
0 &= \frac{1}{2}R - \frac{1}{4}(\nabla\phi)^2 - 2\Lambda e^{\phi}
\end{aligned}
$$

And so:

$$
R_{\mu\nu} - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi - 2e^{-\phi}\Big(F_{\rho\mu}F^{\rho}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F^2\Big) - \Lambda e^{\phi}g_{\mu\nu} = 0
$$

as required.

Stress-energy tensor

The stress-energy tensor is given by:

$$
\begin{aligned}
T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta I_{\text{matter}}}{\delta g^{\mu\nu}} \\
&= -2\Big(-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{4}g_{\mu\nu}(\nabla\phi)^2 - 2e^{-\phi}F_{\rho\mu}F^{\rho}{}_{\nu} + \frac{1}{2}g_{\mu\nu}e^{-\phi}F^2 + \Lambda e^{\phi}g_{\mu\nu}\Big) \\
&= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 + 4e^{-\phi}\Big(F_{\rho\mu}F^{\rho}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F^2\Big) - 2\Lambda e^{\phi}g_{\mu\nu}
\end{aligned}
$$

and here we pause to observe that the parenthesized term is the standard stress-energy tensor for the electromagnetic field, a fact that will be useful in a moment.

We now compute the divergence of the terms of the stress-energy tensor in two steps.

First:

$$
\begin{aligned}
\nabla_\mu\Big(\nabla^\mu\phi\nabla_\nu\phi - \frac{1}{2}\delta^\mu_\nu\nabla^\alpha\phi\nabla_\alpha\phi - 2\Lambda e^{\phi}\delta^\mu_\nu\Big) &= \nabla^2\phi\nabla_\nu\phi + \nabla^\mu\phi\nabla_\mu\nabla_\nu\phi - \nabla^\alpha\phi\nabla_\nu\nabla_\alpha\phi - 2\Lambda e^{\phi}\nabla_\nu\phi \\
&= \nabla^2\phi\nabla_\nu\phi - 2\Lambda e^{\phi}\nabla_\nu\phi \\
&= -e^{-\phi}F^2\nabla_\nu\phi
\end{aligned}
$$

where at the last line we assumed that the equation of motion (2) for ϕ is satisfied.

Second:

$$
\begin{aligned}
\nabla_\mu\left(4e^{-\phi}F_\rho{}^\mu F^\rho{}_\nu - e^{-\phi}\delta^\mu_\nu F^2\right) &= 4\nabla_\mu\left(e^{-\phi}F_\rho{}^\mu\right)F^\rho{}_\nu + 4e^{-\phi}F_\rho{}^\mu\nabla_\mu F^\rho{}_\nu - e^{-\phi}\nabla_\nu F^2 + e^{-\phi}F^2\nabla_\nu\phi \\
&= 4e^{-\phi}F^{\rho\mu}\nabla_\mu F_{\rho\nu} - e^{-\phi}\nabla_\nu F^2 + e^{-\phi}F^2\nabla_\nu\phi \\
&= 4e^{-\phi}\nabla_\mu\left(F^{\rho\mu}F_{\rho\nu}\right) - 4e^{-\phi}F_{\rho\nu}\nabla_\mu F^{\rho\mu} - e^{-\phi}\nabla_\nu F^2 + e^{-\phi}F^2\nabla_\nu\phi \\
&= 4e^{-\phi}\nabla_\mu\left(F^{\rho\mu}F_{\rho\nu} - \frac{1}{4}\delta^\mu_\nu F^2\right) - 4e^{-\phi}F_{\rho\nu}\nabla_\mu F^{\rho\mu} + e^{-\phi}F^2\nabla_\nu\phi \\
&= e^{-\phi}F^2\nabla_\nu\phi
\end{aligned}
$$

where at the second line we used the equation of motion (3) for F and at the last line two terms vanished because:

- the stress-energy tensor for the electromagnetic field is always divergence-free, and
- F itself is divergence-free when there is no source term for the electromagnetic field (which is the case for the given action I).

And so, finally, the two divergences cancel, and:

$$\nabla_\mu T^\mu{}_\nu = 0$$

Dilaton gravity in string frame

We now turn our attention to the action:

$$I = \int d^4x\sqrt{-g}\,e^{-2\Phi}\left(R + 4(\nabla\Phi)^2\right) \tag{4}$$

We first vary with respect to Φ to obtain:

$$
\begin{aligned}
\delta I &= \int dV\left(-2e^{-2\Phi}R\delta\Phi - 8e^{-2\Phi}(\nabla\Phi)^2\delta\Phi + 8e^{-2\Phi}\nabla_\mu\Phi\nabla^\mu(\delta\Phi)\right) \\
&= \int dV\left(-2e^{-2\Phi}R\delta\Phi - 8e^{-2\Phi}(\nabla\Phi)^2\delta\Phi - 8\nabla^\mu\left(e^{-2\Phi}\nabla_\mu\Phi\right)\delta\Phi\right) + \text{boundary terms} \\
&= \int dV\left(-2e^{-2\Phi}R - 8e^{-2\Phi}(\nabla\Phi)^2 + 16e^{-2\Phi}(\nabla\Phi)^2 - 8e^{-2\Phi}\nabla^2\Phi\right)\delta\Phi \\
&= \int dV 2e^{-2\Phi}\left(-R + 4(\nabla\Phi)^2 - 4\nabla^2\Phi\right)\delta\Phi
\end{aligned}
$$

and so the equation of motion is:

$$R - 4(\nabla\Phi)^2 + 4\nabla^2\Phi = 0$$

Next, we vary with respect to the metric:

$$
\begin{aligned}
\delta I &= \int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}e^{-2\Phi}\left(R + 4(\nabla\Phi)^2\right) + e^{-2\Phi}R_{\mu\nu}\delta g^{\mu\nu} + 4e^{-2\Phi}(\nabla_\mu\Phi\nabla_\nu\Phi)\delta g^{\mu\nu}\right) \\
&= \int d^4x\sqrt{-g}\,e^{-2\Phi}\left(\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - 2g_{\mu\nu}(\nabla\Phi)^2 + 4\nabla_\mu\Phi\nabla_\nu\Phi\right)\delta g^{\mu\nu}
\end{aligned}
$$

and so the equation of motion is:

$$G_{\mu\nu} - 2g_{\mu\nu}(\nabla\Phi)^2 + 4\nabla_\mu\Phi\nabla_\nu\Phi = 0$$

Now, taking the trace, we obtain:

$$
\begin{aligned}
0 &= -R - 8(\nabla\Phi)^2 + 4(\nabla\Phi)^2 \\
0 &= R + 4(\nabla\Phi)^2
\end{aligned}
$$

and so the equations of motion simplify to:

$$\nabla^2\Phi - 2(\nabla\Phi)^2 = 0 \qquad\qquad R_{\mu\nu} + 4\nabla_\mu\Phi\nabla_\nu\Phi = 0 \tag{5}$$

On the other hand, we observe the following curious fact:

$$\begin{aligned}
&\int d^4x\sqrt{-g}\,2e^{-2\Phi}(\nabla_\mu\Phi\nabla_\nu\Phi)\delta g^{\mu\nu} \\
&= \int d^4x\sqrt{-g}\,\left(e^{-2\Phi}\nabla_\mu\nabla_\nu\Phi - \nabla_\mu\left(e^{-2\Phi}\nabla_\nu\Phi\right)\right)\delta g^{\mu\nu} \\
&= \int d^4x\sqrt{-g}\,e^{-2\Phi}\nabla_\mu\nabla_\nu\Phi + \text{boundary terms}
\end{aligned}$$

and this invariance of the action establishes the relation $2\nabla_\mu\Phi\nabla_\nu\Phi = \nabla_\mu\nabla_\nu\Phi$, along with the following alternate way to write the equations of motion (5):

$$\nabla^2\Phi - 2(\nabla\Phi)^2 = 0 \qquad\qquad R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi = 0 \tag{6}$$

We now consider a conformal transformation under which the metric and volume element change according to:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{-2\Phi}g_{\mu\nu} \qquad\qquad d^4x\sqrt{-g} \to d^4x\sqrt{-\tilde{g}}\,e^{4\Phi}$$

Under this mapping, the Ricci scalar transforms according to the following well-known[1] formula:

$$R \to \tilde{R} = e^{2\Phi}\left(R + 6\nabla^2\Phi - 6(\nabla\Phi)^2\right) = e^{2\Phi}\left(R + 6(\nabla\Phi)^2\right)$$

where we made use of the first of the equations of motion (6).
Now the action (4) may be written as:

$$\begin{aligned}
I &= \int d^4x\sqrt{-\tilde{g}}\,e^{4\Phi}e^{-2\Phi}\left(e^{-2\Phi}\tilde{R} - 6(\nabla\Phi)^2 + 4(\nabla\Phi)^2\right) \\
&= \int d^4x\sqrt{-\tilde{g}}\left(\tilde{R} - 2e^{2\Phi}(\nabla\Phi)^2\right) \\
&= \int d^4x\sqrt{-\tilde{g}}\left(\tilde{R} - 2e^{2\Phi}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\right) \\
&= \int d^4x\sqrt{-\tilde{g}}\left(\tilde{R} - 2\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\right)
\end{aligned}$$

and if we now set $\phi = 2\Phi$, we obtain:

$$I = \int d^4x\sqrt{-\tilde{g}}\left(\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

which agrees with definition (1) of the action in the Einstein frame, at least for the case $\Lambda = 0, A = 0$

[1] Well-known to Wikipedia contributors, at least.