Guillermo Abad, HW 2

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Sheet 2: Grand Unified Theories and Cosmological Applications

Exercise 1.

Let us move now to Grand Unified Theories, such as $SU(5)$, where the proton is unstable and can decay e.g. as $p \to e^+ \gamma$. We wish to estimate the corresponding decay rate, Γ_p.

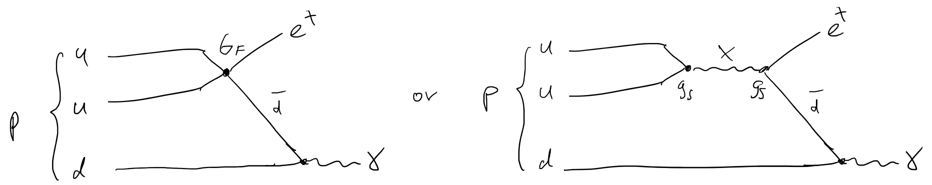
i) Justify why $\Gamma_p \propto G_X^2\, m_p^5$, with $G_X \sim g_5^2/M_X^2$. Why this structure?

ii) Let N_0 be the number of protons in a "tank of water experiment" for proton decay. Using the standard law of radioactive decay, show that if after a time t there has been no decay, then the lifetime of the proton is $\tau_p > N_0 t$.

iii) Current experiments at the Super-Kamiokande neutrino laboratory in Japan (consisting of a cylindrical stainless steel tank 41.4 m tall and 39.3 m in diameter enclosing 50,000 tons of ultra-purified water) indicate that $\tau_p > 10^{36}$ yr. Use this experimental information and the theoretical result of **i)** to estimate a lower bound on M_X. Assume also that g_5 is of order of the $SU(2)$ gauge coupling of the Standard Model.

i)

The decay of the proton, given the SU(5) theory is given by processes like the following ones:



then we see that the M matrix of this procesos $M \propto G_F = g_5^2/M_X^2$, so:

$$\Gamma \propto \int d^4p\, |M|^2 \propto M^2(\not p) \propto G_F^2(\not p) \propto \frac{g_5^4}{M_X^4}(\not p)$$ (p integrated out, so no p dependence.)

And by dimensional analisis, because $[\Gamma] = \frac{1}{[t]} = [E]$ and $\left[\frac{g_5^4}{M_X^4}\right] = \frac{1}{[E^4]}$ we see that we need another factor of $[E^5]$, which in a decay rate can only come from m_p^5, so we conclude that:

$$\boxed{\Gamma_p \propto m_p^5\, G_F^2 \propto \frac{m_p^5\, g_5^4}{M_X^4}}$$

ii)

The probability that from 1 original proton, we still have 1 proton after a time t, is:

$$P_{1|1}(t) = e^{-\Gamma t}$$

The mean time is defined as the time where this probability becomes $\frac{1}{2}$, which means there is $\frac{1}{2}$ chance it has decayed, and $\frac{1}{2}$ chance it has not. This time is:

$$\left.\begin{array}{l} P_{1|1}(\tau) = e^{-\Gamma\tau} \\ P_{1|1}(\tau) = \frac{1}{2} \end{array}\right\} \quad \tau = \frac{-\ln(1/2)}{\Gamma} \quad ; \quad \boxed{\tau = \frac{\ln(2)}{\Gamma}}$$

If instead of 1 proton, we now have N_0 protons, considering their decays as independent succeses, the probability of still having N after a time t will be the product of the N individual cases:

$$P_{N_0|N_0}(t) = e^{-\Gamma t} \cdot \overset{(N_0)}{\cdots} \cdot e^{-\Gamma t} = \left(e^{-\Gamma t}\right)^{N_0} = e^{-\Gamma N_0 t}$$

So comparing with the time for which the probability of no-decayment is $\frac{1}{2}$ again, we then have:

$$\left.\begin{array}{l} P_{N_0|N_0}(\tau') = e^{-\Gamma N_0 \tau'} \\ P_{N_0|N_0}(\tau') = \frac{1}{2} \end{array}\right\} \quad \tau' = \frac{-\ln(1/2)}{N_0 \Gamma} \quad ; \quad \boxed{\tau' = \frac{\ln(2)}{N_0 \Gamma}}$$

And now comparing both, we see that the lifetime of a single proton (τ) is N times that of the time where non of the N_0 protons have decayed (τ'):

$$\boxed{\tau = N_0 \cdot \tau'}$$

iii)

Using $\tau_p = \frac{\ln(2)}{\Gamma_p}$ from ii), and that $\Gamma_p = G_X^2 m_p^5 = \frac{g_5^4 m_p^5}{M_X^4}$, we get:

$$\boxed{\tau_p = \frac{\ln(2)}{\Gamma_p} = \frac{\ln(2)}{G_X^2 m_p^5} = \frac{\ln(2) M_X^4}{g_5^4 m_p^5}}$$

And from the experiments we have $\tau_p > 10^{36}$ years, which gives us:

$$\frac{\ln(2) M_X^4}{g_5^4 m_p^5} > 10^{36} \text{years} \cdot \frac{3 \cdot 10^7 s}{1 \text{year}} \cdot \frac{1/eV}{7 \cdot 10^{-16} s} = 0{,}5 \cdot 10^{68}\ GeV^{-1}$$

Finally using the value of the SU(2) standard model coupling for $g_5 \approx 0{,}5$, and the mass of the proton $m_p \approx 1\,\text{GeV}$, we get:

$$M_X{}^4 > \frac{10^{68} \cdot g_5{}^4 \cdot m_p{}^5}{2\ln(2)} \; ; \quad \boxed{M_X >} \frac{10^{17} \cdot \cancel{g_5}^{\approx 1/2} \cdot \cancel{m_p}^{\approx 1} \cdot \cancel{\sqrt[4]{m_p}}^{\approx 1}}{\cancel{\sqrt[4]{2\ln(2)}}^{\approx 1}} \approx \boxed{5 \cdot 10^{16}\,\text{GeV}}$$

Exercise 2. i) Let us consider the Gell-Mann/Nishijima relation $Q = T_3 + \frac{Y}{2} = T_3 + c\frac{\lambda_0}{2}$ applied to the λ_0 generator of $SU(5)$, which we gave in class. Take the 5-plet and evaluate both sides. Find that the matching implies $c = -\sqrt{5/3}$.

ii) Using the 5-plet of fermions of $SU(5)$ and the structure of the covariant derivative, write down as explicit as possible the interaction Lagrangian of gauge bosons with fermions, making a clear distinction between the SM interactions (QCD and Electroweak) and the extra gauge boson interactions associated to the X and Y gauge fields.

iii) In the previous question, you've made use of the gauge boson matrix given in class. Find out in literature the explicit structure of all the $SU(5)$ generators and confirm (by explicit computation) that the gauge boson matrix is the one we employed. (Please quote your bibliographical source).

iv) How do you think one could construct the interactions between the $SU(5)$ gauge bosons with the 10-plet of fermions? Check (and quote) the literature, if necessary.

i)

From the Gell-Mann/Nishijima relation we arrive to:

$$Q = T_3 + \frac{Y}{2} = T_3 + c\frac{\lambda_0}{2} \; ; \qquad 0 = Q - T_3 - \frac{c}{2}\lambda_0$$

And applying $|\psi\rangle = (\bar{d}_1, \bar{d}_2, \bar{d}_3, e, -\nu_e)$ to this operators we have:

$$\left\{\begin{aligned}
Q|\psi\rangle &= Q\begin{pmatrix}\bar{d}_1\\ \bar{d}_2\\ \bar{d}_3\\ e\\ -\nu_e\end{pmatrix} = \begin{pmatrix}1/3 & \bar{d}_1\\ 1/3 & \bar{d}_2\\ 1/3 & \bar{d}_3\\ -1 & e\\ 0 & -\nu_e\end{pmatrix}\\
T_3|\psi\rangle &= T_3\begin{pmatrix}\bar{d}_1\\ \bar{d}_2\\ \bar{d}_3\\ e\\ -\nu_e\end{pmatrix} = \begin{pmatrix}0 & \bar{d}_1\\ 0 & \bar{d}_2\\ 0 & \bar{d}_3\\ -1/2 & e\\ 1/2 & -\nu_e\end{pmatrix}\\
\lambda_0|\psi\rangle &= \frac{1}{\sqrt{15}}\begin{pmatrix}-2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3\end{pmatrix}\begin{pmatrix}\bar{d}_1\\ \bar{d}_2\\ \bar{d}_3\\ e\\ -\nu_e\end{pmatrix} = \begin{pmatrix}-2/\sqrt{15} & \bar{d}_1\\ -2/\sqrt{15} & \bar{d}_2\\ -2/\sqrt{15} & \bar{d}_3\\ \sqrt{3/5} & e\\ \sqrt{3/5} & -\nu_e\end{pmatrix} = \frac{1}{-\sqrt{5/3}}\begin{pmatrix}2/3 & \bar{d}_1\\ 2/3 & \bar{d}_2\\ 2/3 & \bar{d}_3\\ -1 & e\\ -1 & -\nu_e\end{pmatrix}
\end{aligned}\right.$$

So from the previous relations, we have to fullfill

$$0 = \left[\begin{pmatrix}1/3\\1/3\\1/3\\-1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\-1/2\\1/2\end{pmatrix} - \frac{c}{2}\,\frac{-1}{\sqrt{5/3}}\begin{pmatrix}2/3\\2/3\\2/3\\-1\\-1\end{pmatrix}\right]\begin{pmatrix}\bar d_1\\\bar d_2\\\bar d_3\\e\\-\nu_e\end{pmatrix} = \left[\begin{pmatrix}1/3\\1/3\\1/3\\-1/2\\-1/2\end{pmatrix} + \frac{c}{\sqrt{5/3}}\begin{pmatrix}1/3\\1/3\\1/3\\-1/2\\-1/2\end{pmatrix}\right]\begin{pmatrix}\bar d_1\\\bar d_2\\\bar d_3\\e\\-\nu_e\end{pmatrix}$$

From where we finally find: $\boxed{c = -\sqrt{5/3}}$

ii)

The interaction Lagrangian is:

$$\mathcal{L}_{int} = -i^2 g_5\, \bar\Psi A \Psi = g_5\, \bar\Psi \vec A \vec T \Psi$$

where:

- $$\Psi = \begin{pmatrix}\bar d_1\\\bar d_2\\\bar d_3\\e\\-\nu_e\end{pmatrix} \equiv \begin{pmatrix}\bar d\\ \ell\end{pmatrix}\ \begin{matrix}\text{(down quarks)}\\ \text{(leptons)}\end{matrix}$$

- $$A = \begin{pmatrix} G^1_1 - \frac{2}{\sqrt{30}}B & G^1_2 & G^1_3 & \bar X^1 & \bar Y^1 \\ G^2_1 & G^2_2 - \frac{2}{\sqrt{30}}B & G^2_3 & \bar X^2 & \bar Y^2 \\ G^3_1 & G^3_2 & G^3_3 - \frac{2}{\sqrt{30}}B & \bar X^3 & \bar Y^3 \\ X^1 & X^2 & X^3 & \frac{1}{\sqrt2}W^3 + \frac{3}{\sqrt{30}}B & W^+ \\ Y^1 & Y^2 & Y^3 & W^- & -\frac{1}{\sqrt2}W^3 + \frac{3}{\sqrt{30}}B \end{pmatrix} \equiv \begin{pmatrix} G & \bar X \\ X & W \end{pmatrix}$$

If $A_\mu = A_\mu^a T^a$ $\begin{cases} A_\mu^a = G_\mu^a & \text{for } a = 1,\dots,8 \\ A_\mu^{b+8} = W_\mu^b & \text{for } b = 1,\dots,3 \\ A_\mu^{c+11} = X^c & \text{for } c = 1,\dots,12 \\ A_\mu^{24} = B_\mu \end{cases}$, $\vec T$ is going to be the su(5) base.

su(5) base
$$\begin{cases} T^0 = \lambda_0 = \frac{1}{\sqrt{15}}\begin{pmatrix}-2&&&&\\&-2&&&\\&&-2&&\\&&&3&\\&&&&3\end{pmatrix} & \text{[Global } U(1) \text{ sector} \longleftrightarrow B] \\ T^a = \begin{pmatrix} T^a_{SU(3)} & 0 \\ 0 & 0\end{pmatrix} \text{ for } a = 1,\dots,8 & \text{[QCD sector} \longleftrightarrow G] \\ T^{b+8} = \begin{pmatrix} 0 & 0 \\ 0 & T^b_{SU(2)}\end{pmatrix} \text{ for } b = 1,\dots,3 & \text{[Electroweak sector} \longleftrightarrow W] \end{cases}$$

$$T^{c+11} = \frac{1}{\sqrt2}\left\{ \begin{pmatrix} 0_{3\times3} & \begin{smallmatrix}1/i&\\&\\&\end{smallmatrix} \\ \begin{smallmatrix}1/-i&&\\&&\end{smallmatrix} & 0_{2\times2}\end{pmatrix}, \begin{pmatrix} 0_{3\times3} & \begin{smallmatrix}&1/i\\&\\&\end{smallmatrix} \\ \begin{smallmatrix}&&\\1/-i&&\end{smallmatrix} & 0_{2\times2}\end{pmatrix}, \begin{pmatrix} 0_{3\times3} & \begin{smallmatrix}&\\1/i&\\&\end{smallmatrix} \\ \begin{smallmatrix}&1/-i&\\&&\end{smallmatrix} & 0_{2\times2}\end{pmatrix}, \begin{pmatrix} 0_{3\times3} & \begin{smallmatrix}&\\&1/i\\&\end{smallmatrix} \\ \begin{smallmatrix}&&\\&1/-i&\end{smallmatrix} & 0_{2\times2}\end{pmatrix}, \begin{pmatrix} 0_{3\times3} & \begin{smallmatrix}&\\&\\1/i&\end{smallmatrix} \\ \begin{smallmatrix}&&1/-i\\&&\end{smallmatrix} & 0_{2\times2}\end{pmatrix}, \begin{pmatrix} 0_{3\times3} & \begin{smallmatrix}&\\&\\&1/i\end{smallmatrix} \\ \begin{smallmatrix}&&\\&&1/-i\end{smallmatrix} & 0_{2\times2}\end{pmatrix} \right\}$$

for $c = 1,\dots,12$ [Mi… $\longleftrightarrow X$]

$$T^{c} = \frac{1}{\sqrt{2}}\left\{ \begin{pmatrix} \begin{pmatrix} 0&0&0 \\ 1/{-i} & & \end{pmatrix} & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} 0&0&0 \\ & 1/{-i} & \end{pmatrix} & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} 0&0&0 \\ & & 1/{-i} \end{pmatrix} & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{pmatrix}, \dots \right\}$$

for $c = 1, \dots, 12$ [Mixing sector $\longleftrightarrow$ X]

then
$$\boxed{A_\mu = \left(\vec{G}_\mu \cdot \frac{\vec{\lambda}_{3\times3}}{2} \oplus \mathbb{1}_{2\times2}\right) + \left(\mathbb{1}_{3\times3} \oplus \vec{W}_\mu \frac{\vec{\sigma}_{2\times2}}{2}\right) + B_\mu \overset{\lambda_0}{\overset{\shortparallel}{T^0}} + \vec{X}_\mu \vec{T}_X} =$$

$$= \left(\vec{G}_\mu \vec{T}_{SU(3)} \oplus \mathbb{1}_{2\times2}\right) + \left(\mathbb{1}_{3\times3} \oplus \vec{W}_\mu \vec{T}_{SU(2)}\right) + B_\mu T^0 + \vec{X}_\mu \vec{T}_X =$$

$$= \left(\vec{G}_\mu \vec{T}_{SU(3)} \oplus \vec{W}_\mu \vec{T}_{SU(2)}\right) + B_\mu T^0 + \vec{X}_\mu \vec{T}_X = \begin{pmatrix} G^1_1 - \frac{2}{\sqrt{30}}B & G^1_2 & G^1_3 & \bar{X}^1 & \bar{Y}^1 \\ G^2_1 & G^2_2 - \frac{2}{\sqrt{30}}B & G^2_3 & \bar{X}^2 & \bar{Y}^2 \\ G^3_1 & G^3_2 & G^3_3 - \frac{2}{\sqrt{30}}B & \bar{X}^3 & \bar{Y}^3 \\ X^1 & X^2 & X^3 & \frac{1}{\sqrt{2}}W^3 + \frac{3}{\sqrt{30}}B & W^+ \\ Y^1 & Y^2 & Y^3 & W^- & -\frac{1}{\sqrt{2}}W^3 + \frac{3}{\sqrt{30}}B \end{pmatrix}$$

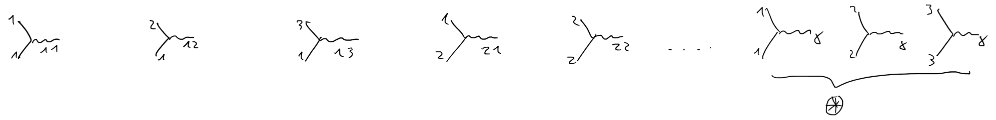
So then the interaction Lagrangian is:

$$\mathcal{L}_{int} = (d\ \bar{\ell}) \begin{pmatrix} G & \bar{X} \\ X & W \end{pmatrix} \begin{pmatrix} \bar{d} \\ \ell \end{pmatrix} = (d\ \bar{\ell}) \begin{pmatrix} G\bar{d} + \bar{X}\ell \\ X\bar{d} + W\ell \end{pmatrix} = dG\bar{d} + d\bar{X}\ell + \bar{\ell}X\bar{d} + \bar{\ell}W\ell$$

where we see that:

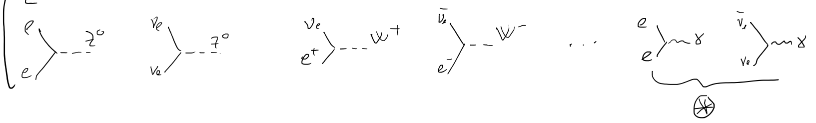
- the quark part is $dG\bar{d}$, where a quark emits a gluon, or quark-antiquark anihilate to G.

$$\left(d_1 G_{11} \bar{d}_1 + d_1 G_{12} \bar{d}_2 + d_1 G_{13} \bar{d}_3 + d_2 G_{21} \bar{d}_1 + d_2 G_{22} \bar{d}_2 + \dots + d_1 B \bar{d}_1 + d_2 B \bar{d}_2 + d_3 B \bar{d}_3 \right)$$



- the lepton part is $\bar{\ell}W\ell$, where leptons emit W bosons, or lepton-antilepton anihilate to W.

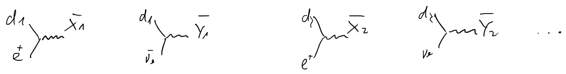
$$\left(\bar{e} W^0 e - \nu_e W^0 \bar{\nu}_e + \bar{e} W^+ \nu_e + \bar{\nu}_e W^- e + \dots + \bar{e} B e + \bar{\nu}_e B \nu_e \right)$$



- the mixing part is $d\bar{X}\ell + \bar{\ell}X\bar{d}$, where quarks transforms into lepton through X emission.

$$(d_1\ d_2\ d_3) \begin{pmatrix} \bar{X}_1 & \bar{Y}_1 \\ \bar{X}_2 & \bar{Y}_2 \\ \bar{X}_3 & \bar{Y}_3 \end{pmatrix} \begin{pmatrix} e \\ \nu_e \end{pmatrix} = (d_1 d_2 d_3) \begin{pmatrix} \bar{X}_1 e + \bar{Y}_1 \nu_e \\ \bar{X}_2 e + \bar{Y}_2 \nu_e \\ \bar{X}_3 e + \bar{Y}_3 \nu_e \end{pmatrix} =$$

$$= d_1 \bar{X}_1 e + d_1 \bar{Y}_1 \nu_e + d_2 \bar{X}_2 e + d_2 \bar{Y}_2 \nu_e + d_3 \bar{X}_3 e + d_3 \bar{Y}_3 \nu_e$$



- the photon part is inside $dG\bar{d}$ and $\bar{\ell}W\ell$, as a global U(1). ⊛

iii)

Now we have to repeat the "quick" computation of A, we did in ii), but this time, being more rigorous and showing all the redefinitions of our fields, etc...

To do so, let's start by saying that we need a base of traceless matrix, that are normalized as $\delta(T^a T^b) = \frac{\delta^{ab}}{2}$, It's obvious that:

- $T^a = \begin{pmatrix} T^a_{SU(3)} & 0 \\ 0 & 0 \end{pmatrix}$ for $a = 1, \dots, 8$ [QCD sector $\longleftrightarrow$ G]

- $T^{b+8} = \begin{pmatrix} 0 & 0 \\ 0 & T^b_{SU(2)} \end{pmatrix}$ for $b = 1, \dots, 3$ [Electroweak sector $\longleftrightarrow$ W]

fullfill those criteria, and also for the off-diagonal terms it's obvious that this natural extension of the Gell-mann matrices will also work:

- $T^{c+11} = \frac{1}{\sqrt{2}} \left\{ \left(\begin{array}{ccc|cc} 0&0&0&1/i& \\ 0&0&0&& \\ 0&0&0&& \\ \hline 1/-i&&&0&0 \\ &&&0&0 \end{array}\right), \left(\begin{array}{ccc|cc} 0&0&0&&1/i \\ 0&0&0&& \\ 0&0&0&& \\ \hline &&&0&0 \\ 1/-i&&&0&0 \end{array}\right), \left(\begin{array}{ccc|cc} 0&0&0&& \\ 0&0&0&1/i& \\ 0&0&0&& \\ \hline &1/-i&&0&0 \\ &&&0&0 \end{array}\right), \left(\begin{array}{ccc|cc} 0&0&0&& \\ 0&0&0&&1/i \\ 0&0&0&& \\ \hline &&&0&0 \\ &1/-i&&0&0 \end{array}\right), \left(\begin{array}{ccc|cc} 0&0&0&& \\ 0&0&0&& \\ 0&0&0&1/i& \\ \hline &&1/-i&0&0 \\ &&&0&0 \end{array}\right), \left(\begin{array}{ccc|cc} 0&0&0&& \\ 0&0&0&& \\ 0&0&0&&1/i \\ \hline &&&0&0 \\ &&1/-i&0&0 \end{array}\right) \right\}$
  for $c = 1, \dots, 12$ [Mixing sector $\longleftrightarrow$ X]

And finally we will need the last diagonal operator that mixes SU(3) and SU(2), traceless:

- $T^0 = \lambda_0 = \frac{1}{\sqrt{15}} \begin{pmatrix} -2 &&&& \\ & -2 &&& \\ && -2 && \\ &&& 3 & \\ &&&& 3 \end{pmatrix}$ [Global U(1) sector $\longleftrightarrow$ B]

Now we have to multiply each of those by the coefficients $A_\mu = A_\mu^a T^a$ and add all of them:

$$A = \frac{1}{2} \left(\begin{array}{ccc|cc} \overbrace{A^3 + \frac{A^8}{\sqrt{3}}}^{G_1^1 \cdot \sqrt{2}} + \overbrace{\frac{2A^0}{\sqrt{15}}}^{\frac{2B}{\sqrt{30}} \sqrt{2}} & \overbrace{A^1 - iA^2}^{G_2^1 \cdot \sqrt{2}} & \overbrace{A^4 - iA^5}^{G_3^1 \cdot \sqrt{2}} & \overbrace{A^9 - iA^{10}}^{X_1 \sqrt{2}} & \overbrace{A^{15} - iA^{16}}^{Y_1 \sqrt{2}} \\ A^1 + iA^2 = G_1^2 \cdot \sqrt{2} & \overbrace{-A^3 + \frac{A^8}{\sqrt{3}}}^{G_2^2 \cdot \sqrt{2}} + \frac{2A^0}{\sqrt{15}} & \overbrace{A^6 - iA^7}^{G_3^2 \sqrt{2}} & \overbrace{A^{11} - iA^{12}}^{X_2 \sqrt{2}} & \overbrace{A^{17} - iA^{18}}^{Y_2 \sqrt{2}} \\ A^4 + iA^5 = G_1^3 \cdot \sqrt{2} & A^6 + iA^7 = G_2^3 \cdot \sqrt{2} & \frac{2A^8}{\sqrt{3}} + \frac{2A^0}{\sqrt{15}} = G_3^3 \sqrt{2} & \overbrace{A^{13} - iA^{14}}^{X_3 \sqrt{2}} & \overbrace{A^{19} - iA^{20}}^{Y_3 \sqrt{2}} \\ \hline A^9 + iA^{10} & A^{11} + iA^{12} & A^{13} + iA^{14} & Z^0 \; A^{23} - \overbrace{\frac{3A^0}{\sqrt{15}}}^{\frac{2B}{\sqrt{30}} \sqrt{2}} & \overbrace{A^{21} - iA^{22}}^{W^+ \sqrt{2}} \\ \underbrace{\overbrace{A^{15} + iA^{16}}^{X_1 \sqrt{2}}}_{Y_1 \cdot \sqrt{2}} & \underbrace{\overbrace{A^{17} + iA^{18}}^{X_2 \sqrt{2}}}_{Y_2 \cdot \sqrt{2}} & \underbrace{\overbrace{A^{19} + iA^{20}}^{X_3 \sqrt{2}}}_{Y_3 \cdot \sqrt{2}} & \underbrace{A^{21} + iA^{22}}_{W^- \cdot \sqrt{2}} & \underbrace{-A^{23}}_{-Z^0} - \frac{3A^0}{\sqrt{15}} \end{array}\right)$$

Mixing
gluons
photon
W^+, W^-
Z^0

So finally doing the colors redefinitions on the fields, we are left with:

$$A_\mu = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|cc} G_{1\mu}^1 + \frac{2B_\mu}{\sqrt{30}} & G_{2\mu}^1 & G_{3\mu}^1 & X_\mu^{1c} & Y_\mu^{1c} \\ G_{1\mu}^2 & G_{2\mu}^2 + \frac{2B_\mu}{\sqrt{30}} & G_{3\mu}^2 & X_\mu^{2c} & Y_\mu^{2c} \\ G_{1\mu}^3 & G_{2\mu}^3 & G_{3\mu}^3 + \frac{2B_\mu}{\sqrt{30}} & X_\mu^{3} & Y_\mu^{3c} \\ \hline X_\mu^1 & X_\mu^2 & X_\mu^3 & \frac{Z_\mu}{\sqrt{2}} - \sqrt{\frac{3}{10}} B_\mu & W_\mu{}^+ \\ Y_\mu^1 & Y_\mu^2 & Y_\mu^3 & W_\mu{}^- & -\frac{Z_\mu}{\sqrt{2}} - \sqrt{\frac{3}{10}} B_\mu \end{array}\right)$$

iv)

In the previous cases everything worked out:

$$\text{5-rep } \psi \Longrightarrow \langle \psi_5 | A | \psi_5 \rangle = X \text{ (scalar)} = \mathcal{L}_{int}$$

But now the product of $\psi A \psi$ doesn't give a scalar, so we need something else:

$$\text{10-rep } \psi \Longrightarrow \chi_{10} A \chi_{10} = M \text{ (matrix)} \neq \mathcal{L}_{int} \Longrightarrow tr(M) = \mathcal{L}_{int}$$

✓

so the answer is then:

$$\boxed{\mathcal{L}_{int} = tr\left[\chi_{10} A \chi_{10}\right]}$$

$$\mathcal{L}^F = i\frac{1}{2}\mathrm{Tr}\left\{\overline{\Psi}_{10} \not{D} \Psi_{10}\right\} + i\overline{\Psi}_{\overline{5}} \not{D} \Psi_{\overline{5}} = \frac{1}{2} i \mathrm{Tr}\left\{\overline{\Psi}_{10} \not{\partial} \Psi_{10}\right\} + i\overline{\Psi}_{\overline{5}} \not{\partial} \Psi_{\overline{5}} + \mathcal{L}^F_{\mathrm{int}} \,,$$

$$\mathcal{L}^F_{\mathrm{int}} = -g_5 \mathrm{Tr}\left\{\overline{\Psi}_{10} \gamma^\mu A_\mu \Psi_{10}\right\} + g_5 \overline{\Psi}_{\overline{5}} \gamma^\mu A^T_\mu \Psi_{\overline{5}} \,.$$

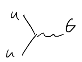
, from Bibliography ⊛

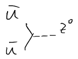
If we do a very quick computation with $\chi_{10} \equiv \begin{pmatrix} \bar{u} & u/d \\ -\overline{u/d}^{+} & \bar{e} \end{pmatrix}$

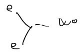
$$tr(\chi_{10} A \chi_{10}) = tr\left[\begin{pmatrix} u & \overline{u/d} \\ -\overline{u/d}^{+} & e \end{pmatrix} \begin{pmatrix} G & \bar{X} \\ X & W \end{pmatrix} \begin{pmatrix} \bar{u} & u/d \\ -u/d^{+} & \bar{e} \end{pmatrix}\right] =$$

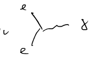
$$= tr\left[\begin{pmatrix} u & \overline{u/d} \\ -\overline{u/d}^{+} & e \end{pmatrix} \begin{pmatrix} G\bar{u} - \bar{X} u/d^{+} & G u/d + \bar{X}\bar{e} \\ X\bar{u} - W u/d^{+} & X u/d + W\bar{e} \end{pmatrix}\right] =$$

$$= tr\left[u G \bar{u} - u\bar{X} u/d^{+} + \overline{u/d} X u - \overline{u/d}\, W u/d^{+} - \overline{u/d}^{+} G\, u/d - \overline{u/d}^{+} \bar{X}\bar{e} + eX u/d + eW\bar{e}\right]$$

From where we quickly see terms resembling: 

or 

or 

or 

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Bibliographical references for the whole ex. 2

As bibliographical resources I can quote some paper I read to make memory, but all the derivations from the SU(5) algebra are self contained, from what we learn with

Javier Virto last month. Anyway here there are some references:

- The Algebra of Grand Unified Theories
  John Baez and John Huerta

- An overview of SU(5) grand unification
  Nicola Canzano

- **The gauge group SU(5) as a simple GUT**
  Marina von Steinkirch

- Seminar
  SU(5) and SO(10) Unification
  Ivan Nišandžić
  Advisor: dr. Jure Zupan

- **The SU(5) Grand Unification Theory Revisited** ✳
  Miguel Crispim Romão

- Wikipedia:
  - Georgi-Glashow Model
  - Weak hypercharge
  - Weak isospin
  - Neutrino / Antineutrino
  - Down quark
  - Gell-Mann Matrices

Exercise 6. Let us estimate the freeze-out temperature of neutrinos, T_f, by considering the order of magnitude of the corresponding cross-section in Fermi's theory:

$$\sigma_\nu \sim G_F^2\, s\,,$$

where s is the squared center-of-mass energy.

i) Justify this formula using plausible physical arguments and dimensional analysis . Then use it to estimate the value of T_f.

ii) Verify that neutrinos contribute the following to the cosmological mass parameter:

$$\Omega_\nu\, h^2 = \frac{90\,\zeta(3)}{11\,\pi^4}\,\frac{\sum_\nu m_\nu}{T_0^\gamma}\,\Omega_\gamma^0 \simeq \frac{\sum_\nu m_\nu}{94\,eV}\,.$$

At what cosmic time (in seconds) this contribution was fixed once and forever?

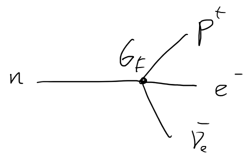
iii) Why is not possible to apply this result to a non-relativistic WIMP like e.g. a neutralino χ?

iv) What should be the value of m_ν (assuming that each family contributes the same) if these particles were to be the only ones responsible for the missing mass in the Universe.

v) What is the experimentally allowed range of the neutrino masses from cosmology and in Particle Physics? What is then their maximal contribution to Ω_m^0? How it compares the neutrino contribution to Ω_m^0 with the luminous contribution from shining stars?

i)

In fermi theory, we will have:



$$\Longrightarrow \sigma \sim |M|^2 \sim G_F^2$$

And then from dimensional analisis:

$$\left.\begin{array}{l}[\sigma] = GeV^{-2} \\ [G_F] = GeV^{-2}\end{array}\right\} \quad [\sigma] \sim (GeV^2)\cdot[G_F^2]$$

So we will need a term of units of Energy^2, which in this case can only be the squared center of mass energy "s", so:

$$\boxed{\sigma \sim G_F^2 \cdot s}$$

Going now to the computation of the temperature T_{fr} from the PDG:

$$\langle\sigma v\rangle \sim \frac{T_f^2}{m_W^4} \longrightarrow T_f \sim m_W^2\sqrt{\sigma v} = m_W^2 G_F \sqrt{s\cdot v}$$

And to estimate the value, we are going to consider:

- $G_F = 1.17 \cdot 10^{-5}\,GeV^{-2}$
- $m_W = 80{,}4\ GeV$
- $\sqrt{s} \simeq 939\ MeV$
- $\sqrt{v} \sim 1$ (relativistic)

$$\boxed{T_f \sim 80{,}4^2 \cdot 1{,}17 \cdot 10^{-5} \cdot 939 \sim 71\ MeV}$$

Which should be 1 MeV, as we know ~ CBB, the reason of the discrepancy might be that this was an approximate computation.

ii)

Because the neutrinos are hot relics, it's abundance before decoupling Y_{Eq}, was constant. Using Kolb Turner book, we know that:

$$Y_\infty = Y_{Eq} \simeq 0{,}278\ \frac{g_{eff}}{g_{*,s}} \longrightarrow n_4 = s_0 Y_\infty = 824\ \frac{g_{eff}}{g_*}$$

(Present entropy)

Then, because the present relic mass density is:

$$\rho_4 = s_0 Y_\infty m = n_4 \cdot m \longrightarrow \Omega_4 h^2 = 7{,}82 \cdot 10^{-2}\ \frac{g_{eff}}{g_{*,s}}\ m$$

and because the neutrinos decouple at $T \sim 1\ MeV$:

$$\left.\begin{aligned} g_{*,s} &= g_* = 10{,}75 \\ g_{eff} &= 2 \cdot \frac{3}{4} = \frac{3}{2} = 1{,}5 \end{aligned}\right\} \quad \boxed{\Omega_\nu h^2 = 7{,}82 \cdot 10^{-2}\ \frac{g_{eff}}{g_{*,s}}\ \Sigma_\nu m_\nu}$$

Considering the different types.

And now we need to use:

$$Y_\infty = Y_{eq} = \frac{\zeta(3)/_{\pi^2}\ g_{eff}\ \cancel{T^3}}{\frac{2\pi^2}{45}\ g_{*,s}\ \cancel{T^3}} = \frac{90\ \zeta(3) \cdot 11}{11\pi^4 \cdot 4}\ \frac{g_{eff}}{g_{*,s}}$$

with:

$$g_{*,s} = \sum_b g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_f g_i \left(\frac{T_i}{T}\right)^3$$

and because:

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}$$

we finally obtain:

$$\Omega_\nu = \frac{\sum_\nu m_\nu n_0}{3 H_0^2 M_{Pl}^2} \quad \text{where} \quad n_\nu = s_0 Y_\infty \Longrightarrow \boxed{\Omega_\nu h^2 = \frac{a0\ \zeta(3)}{11\ \pi^4}\ \frac{\sum_\nu m_\nu}{T_0{}^{\gamma}}\ \Omega_\gamma^0}$$

And now the exercise also asks to compute the cosmic time, for that we consider that the recombination epoch time: $z \sim 1000$ corresponds to the freeze-out temperature, we previously computed $T_f \sim 1\,MeV$:

$$H_0 t = \int_z^\infty \frac{dz}{(1+z)\,H(z)} = \int_z^\infty \frac{dz}{(1+z)^3} = -\frac{1}{2}\frac{1}{(1+z)^2}\Big|_{1000}^{\infty} \simeq 5\cdot 10^{-7}$$

so finally

$$\boxed{t = \frac{5\cdot 10^{-7}}{67{,}4} = 7{,}5\cdot 10^{-9}\,\frac{Mp}{Km}\,s \simeq 2{,}3\cdot 10^{11}\,s}$$

iii)

Because for the computation we assumed we had a hot relic, but a non-relativistic WIMP would be a cold relic.

In this case the abundance changes significantly depending on the freeze-out temperature.

$$\left(\text{In class we obtained:}\quad \Omega_\chi h^2 \sim 0{,}1\left(\frac{4\,pb}{\langle\sigma v\rangle}\right)\right)$$

iv)

Well, it's easy to see, that if all families contribute equally we will have:

$$\sum_\nu m_\nu = 3m_\nu \implies \Omega_\nu h^2 \simeq \frac{3\,m_\nu}{94}$$

And if they are responsible for all the dark matter, then:

$$\Omega_\nu = \Omega_{DM} \simeq 0{,}25 \implies \frac{h^2}{4} \simeq \frac{3\,m_\nu}{94} \implies \boxed{m_\nu = \frac{94}{4\cdot 2\cdot 3} = 3{,}91\,eV}$$

$h \sim 0{,}7^2 = 0{,}5$

v)

From the current age of the universe, we know that

$$\Omega_0 h^2 \lesssim 1 \implies \boxed{m_\nu \lesssim 94\,eV}$$

And from PDG we can check that $m_\nu \lesssim 1{,}1\,eV$, so:

$$\boxed{\Omega_\nu} \simeq \frac{3m_\nu}{94h^2} \underset{(0,7^2=0,5)}{\leq} \frac{6,6}{94} = \boxed{0,0702}$$

which would be their maximal posible contribution to Ω^0_m, not enough to explain DM.

And for the last question, we know that the luminous contribution is:

$$\boxed{\Omega_{visible} \simeq 0,02 - 0,03}$$

So it's obvious that the maximal neutrino contribution will be the double or more than that from the luminous contribution.

$$\boxed{\frac{\Omega_\nu}{\Omega_m^0} \leq \frac{0,07}{0,3} \simeq 0,24} \quad \text{and} \quad \boxed{\frac{\Omega_{vis}}{\Omega_m^0} = \frac{0,03}{0,3} \simeq 10}$$

But it is the maximal, which means that actually both could be comparable!

Exercise 8. In Supersymmetry (SUSY) the vacuum energy is exactly zero. The divergent integral of the zero point energy of any boson field is compensated in SUSY by the corresponding fermionic partner contribution, which is equal but opposite in sign. Why? Is this consistent with the structure of the SUSY potential that we saw in class?

i) The above feature of SUSY is encoded in the innermost part of its "genetic code", namely in its anticommuting algebra. Indeed, using the SUSY graded Lie algebra, and in particular the basic ingredient of it

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\,\sigma^\mu_{\alpha\dot\beta}\,P_\mu\,,$$

check explicitly that the SUSY vacuum energy is zero:

$$\langle 0|H|0\rangle = 0\,,$$

where $H = P^0$ is the Hamiltonian.

ii) Even more, check that the SUSY vacuum satisfies

$$\langle 0|P_\mu|0\rangle = 0\,.$$

To complete the proof that the zero vacuum energy is indeed the minimal one you need to show that any other state satisfies

$$\langle\psi|H|\psi\rangle \geq 0\,.$$

In SUSY theories for any bosonic field we have a superpartner fermionic field related by what we call a supertransformation, with algebras $Q, \bar{Q}$:

$Q|boson\rangle = |fermion\rangle$ (super partner) $\qquad \bar{Q}|fermion\rangle = |boson\rangle$ (super partner)

Because this generators commute with translation generators, the superpartners must have the same mass:

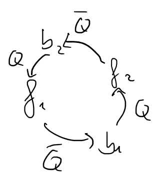
$$M_b^2|boson\rangle = P^2|boson\rangle = P^2 Q|fermion\rangle \overset{[P_\mu,Q]=0}{=} Q P^2|fermion\rangle =$$

$$= Q M_f^2|fermion\rangle = M_f^2 \, |boson\rangle$$

$$\Downarrow$$

$$M_b = M_f$$

So, all the states in a supermultiplot have the same mass, and the same quantity of bosons than fermions:



there will always be the same nº of b's and f's!

Therefore, because fermion loops carry a minus sign respect the bosonic ones, the divergent contributions to the vaccum energy will cancel!

Now to relate this with the superpotential, we have:

$(W_p = -pW)$

$$\begin{cases} Q_1 \equiv \frac{1}{2}(p\sigma_1 + W\sigma_2) \\ Q_2 \equiv \frac{1}{2}(p\sigma_2 - W\sigma_1) \end{cases} \Longrightarrow \left.\begin{array}{l} Q_1^2 = \frac{1}{4}\left(p^2 + W^2 + \sigma_1\sigma_2\, pW + \sigma_2\sigma_1\, W_p\right) \\ Q_2^2 = \frac{1}{4}\left(p^2 + W^2 - \sigma_2\sigma_1\, pW - \sigma_1\sigma_2\, W_p\right) \end{array}\right\} Q_1^2 = Q_2^2$$

it's the Hamiltonian!

$$\left.\begin{array}{l} \{Q_1,Q_1\} = 2Q_1^2 \\ \{Q_2,Q_2\} = 2Q_2^2 \end{array}\right\} = \frac{1}{2}\left(p^2 + W^2 + 2i\sigma_3 \frac{-i\hbar}{2}\frac{dW}{dx}\right) = H \Longrightarrow H = \frac{1}{2}\{Q_\alpha, \bar{Q}_\alpha\}$$

And then $\langle 0|H|0\rangle = \langle 0|Q_\alpha \overset{0}{\bar{Q}_\alpha|0\rangle} = 0$ $\quad$ ($Q|0\rangle = 0$ and $\bar{Q}|0\rangle = 0$)

Also finally comment that the vaccuum should be invariant under Lorentz and SUSY transformations, and because the actual transformations look like:

$$\left.\begin{matrix}\delta\phi \sim \psi \\ \delta\psi \sim \partial_\mu \phi\end{matrix}\right\}$$ vacuum can have only a constant scalar part ϕ_0.

And so the <u>SUSY potential will be consistent</u>!

i)

For checking explicitly that the vacuum energy is 0, from:

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu$$

We are going to multiply both sides by $\sigma^{\mu'}_{\dot\beta\alpha}$, getting:

$$\sigma^{\mu'}_{\dot\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\underbrace{\sigma^{\mu'}_{\dot\beta\alpha}\sigma^\mu_{\alpha\dot\beta}}_{tr(\sigma^{\mu'}\sigma^\mu)=2\delta^{\mu\mu'}} P_\mu \implies P^\mu = \frac{1}{4}\sigma^\mu_{\dot\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\}$$

and so:

$$\boxed{\langle 0|H|0\rangle} = \langle 0|P^0|0\rangle = \langle 0|\frac{1}{4}\overset{\delta_{\dot\beta\alpha}}{\cancel{\sigma^0_{\dot\beta\alpha}}}\{Q_\alpha, \bar{Q}_{\dot\beta}\}|0\rangle =$$

$$= \frac{1}{4}\langle 0|\{Q_\alpha, \bar{Q}_\alpha\}|0\rangle = \frac{1}{4}\{\langle 0|Q_\alpha \overset{0}{\cancel{\bar{Q}_\alpha|0\rangle}} + \langle 0|\bar{Q}_\alpha \overset{0}{\cancel{Q_\alpha|0\rangle}}\} = \boxed{0}$$

(Unbroken SUSY $Q|0\rangle = 0$ and $\bar{Q}|0\rangle = 0$!!!)

ii)

Now to check $\langle 0|P^\mu|0\rangle$ instead than $\langle 0|P^0|0\rangle$, we do the same, but keeping the index μ, which gives:

$$\boxed{\langle 0|P^\mu|0\rangle} = \langle 0|\frac{1}{4}\sigma^\mu_{\dot\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\}|0\rangle = \frac{\sigma^\mu_{\dot\beta\alpha}}{4}\langle 0|\{Q_\alpha, \bar{Q}_{\dot\beta}\}|0\rangle =$$

$$= \frac{\sigma^\mu_{\dot\beta\alpha}}{4}\{\langle 0|Q_\alpha \overset{0}{\cancel{\bar{Q}_{\dot\beta}|0\rangle}} + \langle 0|\bar{Q}_{\dot\beta} \overset{0}{\cancel{Q_\alpha|0\rangle}}\} = \sigma^\mu_{\dot\beta\alpha}\cdot 0 = \boxed{0}$$

And finally we need to proof the energy is positive defined for any state, that is:

$$\langle\psi|H|\psi\rangle \geq 0 \quad \forall\psi$$

Which can be easily seen writting H in terms of Q's again, which gives:

$$\langle \psi | H | \psi \rangle = \langle \psi | \{ Q_\alpha , \bar{Q}_\alpha \} | \psi \rangle = \langle \psi | Q_\alpha \bar{Q}_\alpha | \psi \rangle + \langle \psi | \bar{Q}_\alpha Q_\alpha | \psi \rangle =$$

$$= \langle \bar{Q} \psi | \bar{Q} \psi \rangle + \langle Q \psi | Q \psi \rangle = \underbrace{| \bar{Q} \psi |^2}_{\geq 0} + \underbrace{| Q \psi |^2}_{\geq 0} \geq 0$$