8.821/8.871 Holographic duality

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2014

Lecture 6

1.3: HOLOGRAPHIC PRINCIPLE

If we do treat black hole as an "ordinary" QM object, an important implication would be the holographic principle.

Consider an isolated system mass E and entropy S_0 in an asymptotic flat spacetime. Let A be the area of the smallest sphere hat encompasses the system, and M_A to be the mass of a black hole with the same horizon area, we must have $E < M_A$, otherwise the system would be already a black hole.

Now add $M_A - E$ energy to the system (keeping A fixed), we shall obtain a black hole with mass M_A, since

$$S_{BH} \geqslant S_0 + S'$$

where S' is the entropy of added energy, we conclude that

$$S_0 \leqslant S_{BH} = \frac{A}{4\hbar G_N}$$

i.e. the maximal entropy inside a region bounded by area A is

$$S_{max} = \frac{A}{4\hbar G_N}$$

Recall the definition of entropy in quantum statistical physics:

$$S = -\operatorname{Tr} \rho \log \rho$$

where ρ is the density matrix for the state of a system. For a system with N-dimensional Hilbert space

$$S_{max} = \log N$$

Hence the "effective" dimension of the Hilbert space for a system inside a region of area A is bounded by

$$\log N \leqslant \frac{A}{4\hbar G_N} = \frac{A}{4l_p^2}$$

Remarks:

1. For a system of n spins, the Hilbert space dimension $N = 2^n$.
2. The dimension of $\mathcal{H}$ for a single harmonic oscillator is infinite. But for a quantum system with finite number of degrees of freedom (d. o. f.), the dimension of $\mathcal{H}$ below some finite energy scale is always finite, that's why we have "effective" in the above description.
3. In typical physical systems,
$$\text{number of d. o. f.} \sim \log N$$
thus we can write
$$\text{number of d. o. f. of any quantum gravity system} \leqslant \frac{A}{4l_p^2}$$
4. The bound is violated in non-gravitational systems whose number of d. o. f. (or $\log N$) is proportional to the volume rather than area of the system. *e.g.* for a lattice of spins with lattice spacing a, total number of spins is $\frac{V}{a^3} = \frac{A}{a^2}\frac{L}{a} \gg \frac{A}{l_p^2}$ for large enough L. Also $N = 2^{V/a^3}$, we have $S_{max} = \frac{V}{a^3}\log 2 \geqslant S_{BH}$ for large enough volume. In other words, quantum gravity leads to a huge reduction of d. o. f.

Holographic principle: In quantum gravity, a regime of boundary area A can be fully described by no more than $\frac{A}{4\hbar G_N} = \frac{A}{4l_p^2}$ degrees of freedom, *i.e.* degree of freedom per Planck area. Black hole brings quantum gravity to a macroscopic level.

1.3: LARGE N EXPANSION OF GAUGE THEORIES

We now look at clues to holographic duality from field theory side.

Consider QCD which can be described as $SU(3)$ gauge theory with fundamental quarks. The Lagrangian reads

$$\mathcal{L} = \frac{1}{g_{YM}^2}\left[-\frac{1}{4}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} - i\overline{\Psi}(\not{D} - m)\Psi\right]$$

where $D_\mu = \partial_{mu} - iA_\mu$, A_{mu} are 3×3 Hermitian matrices and can be expressed as $A_\mu = A_\mu^a T^a$, with $T^a \in SU(3)$. In such a theory, coupling becomes strong in IR ($\Lambda_{QCD} \sim 250$ MeV), there is no small parameter to expand. It is still an open problem to derive IR properties of QCD from the first principle.

t' Hooft in 1974 suggested take number of color $N = 3$ as a parameter, *i.e.* promote A_μ to $N \times N$ hermitian matrices and consider $N \to \infty$ limit and do a $\frac{1}{N}$ expansion. It is an ingenious idea, unfortunately, QCD still cannot be solved to leading order in the large N limit. Surprisingly, there is an correspondence between the large N gauge theory and the string theory. The key is the fields are matrices. As an illustration, we will consider a scalar theory:

$$\mathcal{L} = -\frac{1}{g_2}\operatorname{Tr}\left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{4}\Phi^4\right]$$

where g is the cooling constant, $\Phi(x) = \Phi^a_b(x)$ is a $N\times N$ hermitian matrix, *i.e.* $\Phi^a{}_b{}^* = \Phi^b{}_a$. In terms of components

$$\mathcal{L} = -\frac{1}{g^2}\left[\frac{1}{2}(\partial_\mu\Phi^a{}_b)(\partial^\mu\Phi^b{}_a) + \frac{1}{4}\Phi^a{}_b\Phi^b{}_c\Phi^c{}_d\Phi^d{}_a\right]$$

$\mathcal{L}$ is invariant under $U(N)$ "global" symmetry:

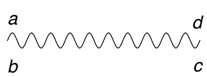
$$\Phi(x) \to U\Phi(x)U^\dagger$$

where U is any constant $U(N)$ matrices.

Remarks

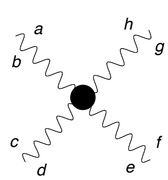
1. It is a theory of N^2 scalar fields.
2. One can also consider other types of matrices, *e.g.* $N\times N$ real symmetric matrix, the corresponding symmetry will be $SO(N)$.
3. One could also introduce gauge fields to make the $U(N)$ symmetry local (this point is important and we will discuss it later).

Here we list the Feynman rules for this theory: The propagator:

$$\langle\Phi^a{}_b(x)\Phi^c{}_d(y)\rangle =$$ 

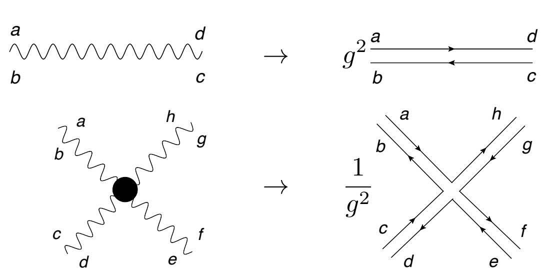
$$= g^2\delta^a{}_d\delta^c{}_b G(x-y)$$

The fermion vertex:



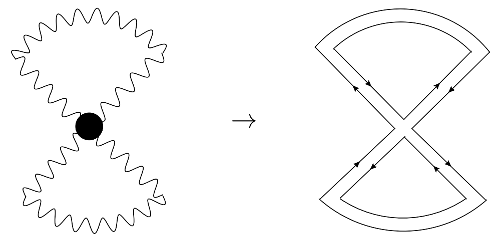
$$= \frac{1}{g^2}\delta^a{}_h\delta^c{}_b\delta^e{}_d\delta^f{}_g$$

So here we can adapt the double line notation:

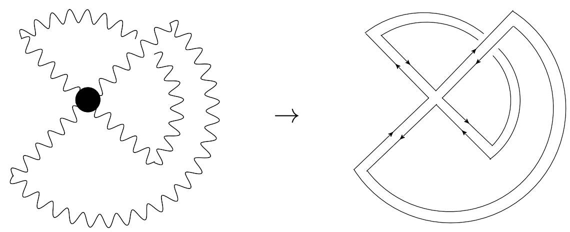


Vacuum energy

We consider vacuum bubbles, *i.e.* diagrams with no external legs. The lowest order diagrams will be



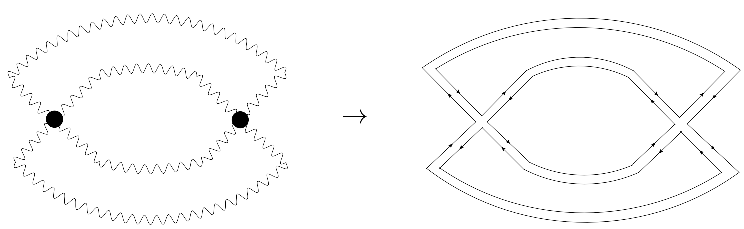
(1)



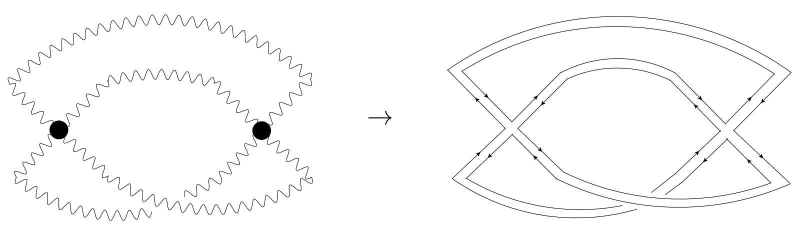
(2)

In the case of diagram 1, each contracted index line gives N, so the total contribution will be of the order $N^3 \frac{(g^2)^2}{g^2} = N^3 g^2$. In the case of diagram 2, there is only one contracted line, the total contribution will be of the order Ng^2. The difference comes from the fact that the matrices do not commute. In the first case, the diagram can be drawn on a plane without crossing lines, we call it a planner digram; while in the second case, the diagram cannot be drawn on a plane without crossing lines, we call it a non-planar diagram.

If we consider next order in the perturbation theory



(3)

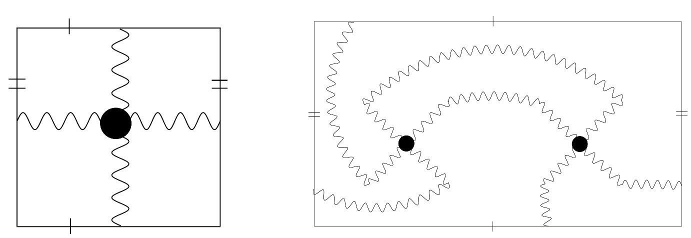


(4)

The first diagram gives the order of $N^4 g^4$, the second diagram gives the order of $N^2 g^4$. We can further consider higher order diagrams, but how can we obtain general N-counting? And how to classify all the non-planar diagrams?

To answer the above questions, we make 2 observations

- Diagrams 2 and 4 can be drawn on a torus without crossing lines.



(5)

- The power of N for each diagram equal to the number of faces in the diagram after we straighten it out.

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Lecture 7

In fact, any orientable two dimensional surface is classified topologically by an integer h, called the genus. The genus is equal to the number of "holes" that the surface has (Fig. 1).

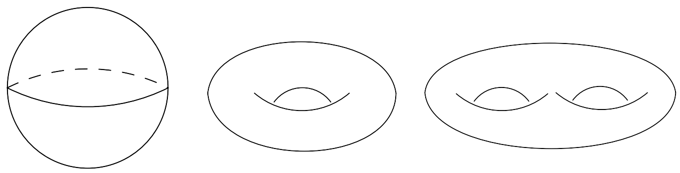

Figure 1: sphere (genus-0), torus (genus-1) and double torus(genus-2).

An topological invariant of the manifold is the Euler character:

$$\chi = 2 - 2h$$

Here we make some claims:

1. For any non-planar diagram, there exists an integer h, such that the diagram can be straightened out (*i.e.* non-crossing) on a genus-h surface, but not on a surface with a smaller genus.
2. For any non-planar diagram, the power of N that comes from contracting propagators is given by the number of faces on such a genus-h surface, *i.e.* the number of disconnected regions separated by the diagram.

Both claims are self-evident after a bit practices.

In general, a vacuum diagram has the following dependence on g^2 and N:

$$A \sim (g^2)^E (g^2)^{-V} N^F$$

where E is the number of propagators, V is the number of vertices, F is the number of faces. This does not give a sensible $N\to\infty$ limit or $1/N$ expansion, since there is no upper limit on F. However, 't Hooft suggests that we can take the limit $N\to\infty$ and $g^2\to 0$ but keep $\lambda = g^2 N$ fixed. Then

$$A \sim (g^2 N)^{E-V} N^{F+V-E} = \lambda^{L-1} N^{\chi} = \lambda^{L-1} N^{2-2h}$$

where L is the number of loops. The relation $\chi = F + V - E$ is guaranteed by the following theorem.

Theorem: Given a surface composed of polygons with F faces, E edges and V vertices, the Euler character satisfy

$$\chi = F + V - E = 2 - 2h$$

Since each Feynman diagram can be considered as a partition of the surface separating it into polygons, then the above theorem also works for our counting in N.

Thus in this limit, to the leading order in N is the planar diagrams

$$N^2(c_0 + c_1\lambda + c_2\lambda^2 + \cdots) = N^2 f_0(\lambda)$$

Because $\log Z$ evaluates the sum of all vacuum diagrams, we can conclude, including higher order $1/N^2$ corrections:

$$\log Z = \sum_{h=0}^{\infty} f_h(\lambda) = N^2 f_0(\lambda) + f_1(\lambda) + \frac{1}{N^2} f_2(\lambda) + \cdots$$

The first term comes from the planar diagrams, second term from the genus-1 diagrams, etc.

There is a heuristic way to understand $\log Z = O(N^2) + \cdots$. Since $Z = \int D\Phi e^{iS[\Phi]}$ and we can rewrite the Lagrangian as

$$\mathcal{L} = \frac{N}{\lambda}\,\mathrm{Tr}\left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{4}\Phi^4\right]$$

The trace also gives a factor of N, thus $\mathcal{L} \sim O(N^2)$, we have $\log Z \sim O(N^2)$.

Clearly our discussion only depends on the matrix nature of the fields. So for any Lagrangian of matrix valued fields of the form

$$\mathcal{L} = \frac{N}{\lambda}\,\mathrm{Tr}\,(\cdots)$$

we would have

$$\log Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda)$$

To summarize, in the 't Hooft limit, $1/N$ expansion is the same as topological expansion in terms of topology of Feynman diagrams.

General observables

Now we have introduced two theories:

$$\begin{aligned}
&(a) \qquad \mathcal{L} = -\frac{1}{g^2}\,\mathrm{Tr}\left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{4}\Phi^4\right] \\
&(b) \qquad \mathcal{L} = \frac{1}{g_{YM}^2}\left[-\frac{1}{4}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} - i\overline{\Psi}(\not{D} - m)\Psi\right]
\end{aligned}$$

(a) is invariant under the global $U(N)$ transformation: $\Phi \to U\Phi U^\dagger$ with U constant $U(N)$ matrix, *i.e.* the theory has a global $U(N)$ symmetry. (b) is invariant under local $U(N)$ transformation:
$A_\mu \to U(x)A_\mu U^\dagger(x) - i\partial_\mu U(x)U^\dagger(x)$ with $U(x)$ any $U(N)$ matrix, the theory has a $U(N)$ gauge symmetry.

On the other hand, consider allowed operators in the two theories. In (a), operators like ${\Phi^a}_b$ are allowed, although it is not invariant under global $U(N)$ symmetry. But in (b), allowed operators must be gauge invariant, so ${\Phi^a}_b$ is not allowed. So if we consider gauge theories: $\mathcal{L} = \mathcal{L}(A_\mu, \Phi, \cdots)$, the allowed operators will be

$$\begin{aligned}
&\text{Single-trace operators} : \mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}), \mathrm{Tr}(\Phi^n), \cdots \\
&\text{Multiple-trace operators} : \mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu})\,\mathrm{Tr}\,(\Phi^2), \mathrm{Tr}\,(\Phi^2)\,\mathrm{Tr}\,(\Phi^n)\,\mathrm{Tr}\,(\Phi^n), \cdots
\end{aligned}$$

We denote single-trace operators as $\mathcal{O}_k$, $k = 1, \cdots$ represents different operators. Then multiple-trace ones will be like $\mathcal{O}_m\mathcal{O}_n(x)$, $\mathcal{O}_{m_1}\mathcal{O}_{m_2}\mathcal{O}_{m_3}(x), \cdots$

So general observables will be correlation functions of gauge invariant operators, here we focus on local operators:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c \tag{1}$$

Note that it is enough to focus on single-trace operators since multiple-trace ones are products of them. Since we are working in the t' hooft limit, we want to know how correlation (Eq. 1) scales in the large N limit. There is a trick, consider

$$Z\,[J_1, \cdots, J_n] = \int DA_\mu D\Phi \cdots \exp\,(iS_{eff}) = \int DA_\mu D\Phi \cdots \exp\left[iS_0 + iN\sum_j \int J_i(x)\mathcal{O}_i(x)\right]$$

Then the correlation (Eq. 1) can be expressed as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c = \frac{\delta^n \log Z}{\delta J_1(x_1)\cdots\delta J_n(x_n)}|_{J_1=\cdots=J_n=0}\frac{1}{(iN)^n} \tag{2}$$

With $\mathcal{O}_i$ single-trace operators, S_{eff} has the form $N\,\mathrm{Tr}\,(\cdots)$. So we have

$$\log Z\left[J_1,\cdots,J_n\right]=\sum_{h=0}^{\infty}N^{2-2h}f_h(\lambda,\cdots)$$

Applying Eq. (2),

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c\sim N^{2-n}\left[1+O(\frac{1}{N^2})+\cdots\right]$$

e.g.

$$\begin{aligned}\langle\mathbb{1}\rangle &\sim O(N^2)+O(N^0)+\cdots\\ \langle\mathcal{O}\rangle &\sim O(N)+O(N^{-1})+\cdots\\ \langle\mathcal{O}_1\mathcal{O}_2\rangle_c &\sim O(N^0)+O(N^{-2})+\cdots\\ \langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_c &\sim O(N^{-1})+O(N^{-3})+\cdots\end{aligned}$$

All leading order contributions come from planar diagrams.

Physical implications:

1. In the large N limit, $\mathcal{O}(x)|0\rangle$ can be interpreted as creating a single-particle state ("glue ball"). Similarly $:\mathcal{O}_1\cdots\mathcal{O}_n(x):|0\rangle$ represents n-particle state.
   - since $\langle\mathcal{O}_i\mathcal{O}_j\rangle\sim O(N^0)$, we can diagonalize them such that $\langle\mathcal{O}_i\mathcal{O}_j\rangle\propto\delta^i{}_j$.
   - $\langle\mathcal{O}_i(x)\mathcal{O}_j^2(y)\rangle\sim O(N^{-1})\to 0$ as $N\to\infty$, *i.e.* there is no mixing between single-trace and multiple-trace operators in the large N limit.
   - $\langle\mathcal{O}_1\mathcal{O}_2(x)\mathcal{O}_1\mathcal{O}_2(y)\rangle=\langle\mathcal{O}_1(x)\mathcal{O}_1(y)\rangle\langle\mathcal{O}_2(x)\mathcal{O}_2(y)\rangle+\langle\mathcal{O}_1\mathcal{O}_2(x)\mathcal{O}_1\mathcal{O}_2(y)\rangle_c$, the first term is the multiple of independent propagators of $\mathcal{O}_1$ and $\mathcal{O}_2$ states, the second term scales like $O(N^{-2})$.

   Note that it is not necessary there exists a stable on-shell particle associated with $\mathcal{O}_i(x)|0\rangle$.

2. The fluctuations of "glue balls" are suppressed:
   suppose $\langle\mathcal{O}\rangle\neq 0\sim O(N)$, the variance of $\langle\mathcal{O}\rangle$ is $\langle\mathcal{O}^2\rangle-\langle\mathcal{O}\rangle^2=\langle\mathcal{O}^2\rangle_c\sim O(1)$, *i.e.* $\frac{\sqrt{\langle\mathcal{O}^2\rangle_c}}{\langle\mathcal{O}\rangle}\sim N^{-1}\to 0$. Also $\langle\mathcal{O}_1\mathcal{O}_2\rangle=\langle O_1\rangle\langle O_2\rangle+\langle\mathcal{O}_1\mathcal{O}_2\rangle_c$, the first term scales as $O(N^2)$ while the second term scales as $O(1)$. Thus in the large N limit, it is more like a classical theory.

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Lecture 8

Reminder from last lecture

We consider correlation functions of gauge invariant (local) operators, including single-trace operators $\mathcal{O}_k$ and multiple-trace operators like $\mathcal{O}_m\mathcal{O}_n(x)$. Without loss of generality, we can focus on single-trace operators since multiple-trace ones are products of them.

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c = \sum_{n=0}^{\infty} N^{2-n-2h} F_n^{(h)}(x_1,\cdots,x_n;\lambda) = O(N^{2-n}) + O(N^{-n}) + O(N^{-n-2}) + \cdots$$

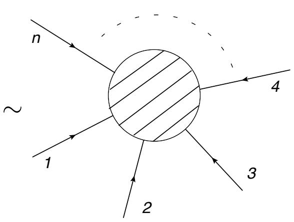
The first term comes from planar diagrams, second one from torus diagrams, third one from double torus diagrams etc.

Physical implications:

1. In the large N limit, $\mathcal{O}(x)|0\rangle$ can be interpreted as creating a single-particle state ("glue ball"). Similarly $:\mathcal{O}_1\cdots\mathcal{O}_n(x):|0\rangle$ represents n-particle state.
2. Fluctuations of "glue balls" are suppressed, *i.e.* $\frac{\sqrt{\langle\mathcal{O}^2\rangle_c}}{\langle\mathcal{O}\rangle} \sim N^{-1}\to 0$, if $\langle\mathcal{O}\rangle \neq 0$.

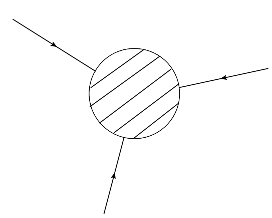
In this lecture we first continue to give physical implications:

3. If we interpret

$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c \sim$  $\sim O(N^{2-n}) + \cdots$

as "scattering amplitude" of n "glue balls", then to the leading order in $N\to\infty$, the scattering only involve tree-level interactions (only classical), among the glue ball states.

- Consider

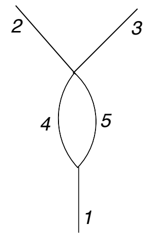
 $\sim \frac{1}{N} \sim \tilde{g}$

suppose we treat it as a basic vertex with coupling $\tilde{g}$, then the tree-level amplitude for n-particle scatterings scales as $\tilde{g}^{n-2} \sim N^{2-n}$.

- We can also include higher order vertices, but they should satisfy:

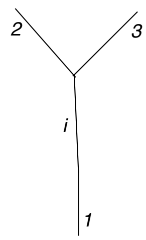
 $\sim \tilde{g}^2,$  $\sim \tilde{g}^3, \quad \cdots$

- There are no more than one-particle intermediate states. Consider *e.g.* $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle \sim O(\frac{1}{N})$. If we insert a complete set of states at all possible places, due to large N counting, all states other than-single particle ones are suppressed:



$$= \langle \mathcal{O}_1 : \mathcal{O}_4 \mathcal{O}_5 : \rangle \langle : \mathcal{O}_4 \mathcal{O}_5 : \mathcal{O}_2 \mathcal{O}_3 \rangle \sim O(N^{-3})$$

Compared to



$$= \langle \mathcal{O}_1 \mathcal{O}_i \rangle \langle \mathcal{O}_i \mathcal{O}_2 \mathcal{O}_3 \rangle \sim O(N^{-1})$$

i.e. all "loops" of glue balls are suppressed.

In summary, at leading order in $1/N$ expansion, we obtain a classical theory of glue balls, with interaction among glue balls given by $\tilde{g} \sim \frac{1}{N}$.

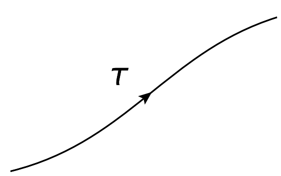
More explicitly

$$\begin{aligned} \text{Gauge theory with finite } \hbar \text{ in the } N \to \infty \text{ limit} \quad &= \quad \text{Glue ball theory with } \hbar \to 0 \\ \text{Perturbative expansion in } \frac{1}{N} \quad &= \quad \text{Loops of glue balls perturbative in } \hbar \end{aligned}$$

We will now show that these resemble a string theory.

1.5: LARGE N EXPANSION AS A STRING THEORY

QFT can be considered as a theory of "particles". The standard quantization approach is second quantization. In the first quantization approach, we directly quantize the motion of a particle in spacetime.



$$\to X^\mu(\tau)$$

We have

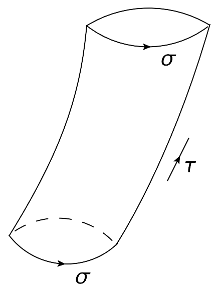
$$Z = \int DX^\mu(\tau) e^{iS_{particle}}$$

where

$$S_{particle} = m \int dl = m \int d\tau \frac{dl}{d\tau} = m \int d\tau \sqrt{g_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{\tau}}$$

If we want to include interactions like $\lambda \phi^3$, we need to add them by hand.

In string theory, similarly, we need to quantize the motions of strings in spacetime.



$\rightarrow$ worldsheet $\Sigma : X^\mu(\tau, \sigma)$

Take a similar quantization approach

$$Z = \int DX^\mu(\tau) e^{iS_{string}} \tag{1}$$

The simplest form of S_{string} is the Nambu-Goto action

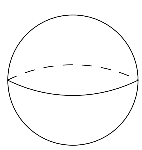
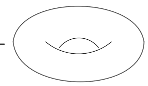
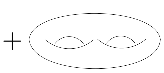
$$S_{NG} = T \int_\Sigma dA$$

here $T = \frac{1}{2\pi\alpha'}$ is the string tension (mass per unit length). $dA = \sqrt{-\det h_{ab}} d\sigma d\tau$ is the infinitesimal area of the world sheet with the induced matrix $h_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$.

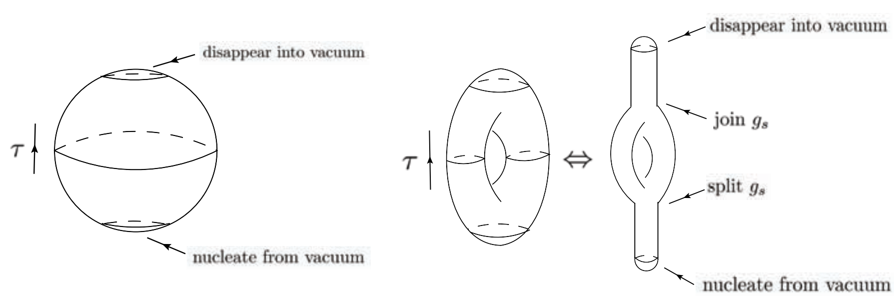
To "define" and evaluate Eq. (1), the most convenient way is to go to Euclidean signature. For vacuum processes:

$$Z_{string} = \sum_{\text{all closed surfaces}} e^{-S_{NG}} = \sum_{h=0}^{\infty} e^{-\lambda\chi} \sum_{\text{surface with given topology}} e^{-S_{NG}}$$

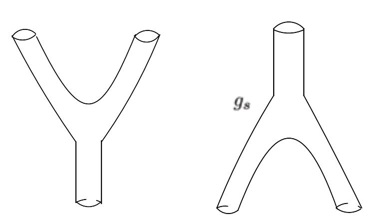
here $\chi = 2 - 2h$ denotes the weight for different topologies, λ can be thought as the "chemical potential" for topology. If we define $g_s = e^\lambda$, the vacuum includes diagrams like

 $g_s^2 = e^{-2\lambda} +$  $g_s^0 = e^{-2\lambda} +$  $g_s^2 = e^{2\lambda} + \cdots$

There is a remarkable fact about string theory: summing over topology of all surfaces automatically includes interactions of strings. In fact this fully specifies string interactions with no freedom of making arbitrary choices. To see this



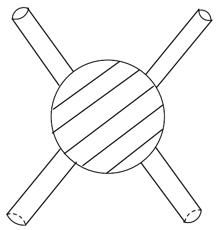
The surface can be thought as the vacuum bubble, at the south pole the string nucleates from vacuum and at the north pole, the string disappears into the vacuum. The torus can be thought as the one loop diagram, the string split into two strings and then join together again with interaction strength g_s on each vertex. Thus the basic interaction vertices are the splitting and rejoining of the strings, the coupling strength is $g_s = e^\lambda$:



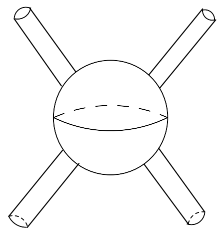
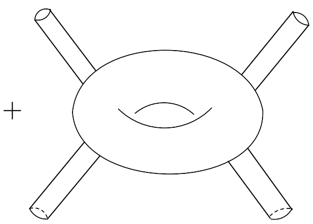
Now we include external strings, *e.g.*

$$\text{string} + \text{string} \quad \rightarrow \quad \text{string} + \text{string}$$

In the diagrammatic language:



$=$ sum of all surfaces with four boundaries

$=$  $+$  $+\cdots$

$$= \sum_{h=0}^{\infty} e^{-\lambda\chi} \sum_{\text{surfaces of given topology}}$$

where $\chi = 2 - 2h - n$, where n is the number of boundaries (number of external strings).

Thus for n-string scattering process (including vacuum processes, *i.e.* n=0)

$$A_n = \sum_{h=0}^{\infty} g_s^{n-2+2h} F_n^{(h)} = g_s^{n-2} F_n^{(0)} + g_s^{n} F_n^{(1)} + g_s^{n+2} F_n^{(2)} + \cdots$$

The first term comes from tree-level diagrams (sphere topology), second term comes from 1-loop diagrams (torus topology), third term comes from 2-loop ones (double-torus topology) etc.

Now comparing with the large N expansion of a gauge theory as we discussed earlier (including n=0)

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_c = \sum_{h=0}^{\infty} N^{2-n-2h} f_n^{(h)} = N^{2-n} f_n^{(0)} + N^{-n} f_n^{(1)} + N^{-n-2} f_n^{(2)}$$

The first term comes from planar diagrams (sphere topology), second one comes from torus diagrams, third one comes from double-torus diagrams, etc.

We see an identical mathematical structure of the two theories with the identification:

$$\begin{aligned} e^{\lambda} = g_s \quad &\leftrightarrow \quad \frac{1}{N} \\ \text{external strings} \quad &\leftrightarrow \quad \text{"glue balls" (single-trace operator) } \mathcal{O}_i(x)|0\rangle \\ \text{sum over string world sheet of given topology} \quad &\leftrightarrow \quad \text{sum over Feynman diagrams of given topology} \\ \text{topology of the worldsheet} \quad &\leftrightarrow \quad \text{topology of Feynman diagrams} \end{aligned}$$

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8.821/8.871 Holographic duality

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Lecture 9

Reminder from last lecture

Recall that each Feynman diagram can be considered as a partition of a genus-h surface. The scattering amplitude of n particles on genus-h surface can be written as

$$f_n^{(h)} = \sum_{\text{all Feynman diagrams of genus h}} G = \sum_{\text{all possible triangulations of genus h surface}} G$$

Here G represents the expression for each diagram. Similarly in string theory, we have n-string scattering process

$$F_n^{(h)} = \int_{\text{genus h surfaces with n boundaries}} DXe^{-S_{string}} = \sum_{\text{all possible triangulations of genus-h surfaces with n boundaries}} e^{-S_{string}}$$

If we could identify G with some $e^{-S_{string}}$, we will then have:

$$\begin{aligned}
\text{a large N gauge theory} &= \text{a string theory} \\
\frac{1}{N}\text{ expansion} &= \text{perturbative expansion in } g_s \\
\text{large N limit (classical theory of glue balls)} &= \text{classical string theory} \\
\text{single-trace operators (glue balls)} &= \text{string states}
\end{aligned}$$

In fact this identification is difficult

1. G is expressed as products of field theory propagators integrated over spacetime, there is no obvious connection to $e^{-S_{string}}$. Note that the action S_{string} gives a map from the world sheet Σ to the target space $\mathcal{M}$ (spacetime manifold)
$$(\sigma,\tau)\rightarrow X^\mu(\sigma,\tau)$$
In such a map, we can make choices of spacetime manifold $\mathcal{M}$, the specific forms of the action S_{string}, we can also have "internal" d. o. f. living on the world sheet with no immediate spacetime. For example, it can be superstrings, including fermions on the worldsheet.

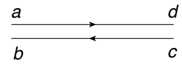
2. String theory is formulated in the continuum, while the Feynman diagrams at best has a discrete version (triangulation of the manifold). Even if there exists a connection, we expect that the geometric picture of G to emerge only in the strong coupling limit, *i.e.* when the Feynman diagrams with many (infinite) vertices dominate.

3. For simple theories, like matrix integrals or matrix quantum mechanics, one could go pretty far in relating them to some low-dimensional string theory, see *e.g.* Sec. II in Ref. [1]. But in general it is not possible for higher dimensions.

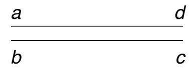
Generalizations:

1. We have so far been restricted to matrix-valued fields, *i.e.* fields in the adjoint representation of $U(N)$ gauge group. One could also include fields in the fundamental representation (quarks)
$$q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$$
e.g. , vacuum diagrams now include loops of quarks, which can be classified topologically by 2d surfaces with boundaries, then it corresponds to a string theory with both open and closed strings.

2. So far we considered $U(N)$ gauge group,

$$\langle \Phi^a{}_b(x) \Phi^c{}_d(y) \rangle =$$ 

If instead, we consider $SO(N)$ or $SP(N)$, then there is no difference between the two indices of the fields

$$\langle \Phi_{ab} \Phi_{cd} \rangle =$$ 

The corresponding Feynman diagrams will live on non-orientable string theories, and it corresponds to non-orientable string theories.

Now take *e.g.* large N generalization of QCD in (3+1)d Minkowski spacetime. Suppose $\frac{1}{N}$ expansion can be described by a string theory, what can we say about it?

The simplest guess would be a string theory in (3+1)d Minkowski space

$$ds^2 = -dt^2 + d\vec{x}^2 = \eta_{\mu\nu} dX^\mu dX^\nu$$

We can consider Nambu-Goto action

$$S_{NG} = \frac{1}{2\pi\alpha'} \int_\Sigma dA$$

or the Polyakov action which is equivalent to S_{NG} classically. But this does not work:

1. Such a string theory is inconsistent for $D \neq 10, 26$, where D is the spacetime dimension.
2. Take a string theory in 10d with $\mathcal{M}_4 \times \mathcal{N}$, where $\mathcal{N}$ is some compact manifold. Such a theory contains a massless spin-2 particle (graviton) in $\mathcal{M}_4$, which is not present in Yang-Mills theory.

To solve the problem, we can either think about more exotic string actions or consider other target space. Actually there are hints for considering a 5d string theory:

1. Holographic principle
   String theory necessarily contains gravity, to be consistent with holographic principle, such a gravity theory should be in 5d.
2. The consistency of string theory itself
   It needs to include a Liouville mode which behaves as on extra dimensions.
3. Geometrization of renormalization group flows (pure hindsight).

Now consider a string Y in 5d spacetime. It should at least have all the symmetries of 4d YM theories, *e.g.* translations, Lorentz symmetries etc. *i.e.* consider

$$ds^2 = a^2(z) \left[dz^2 + \eta_{\mu\nu} dX^\mu dX^\nu \right] \tag{1}$$

which is the most general metric consistent with 4d Poincare symmetries. For a general gauge theory, not more can be said. But if a theory is conformal, or simply scale invariant, Eq. 1 should be the AdS metric. This is simple to see. If Eq. 1 is invariant under scaling transformation

$$X^\mu \to \lambda X^\mu$$

Then we must have $z \to \lambda z$ and $a(\lambda z) = \frac{1}{\lambda} a(z)$, which means $a(z) = \frac{R}{z}$ with R constant.

At last, to close this chapter, we make a list of the history, of the discovery of the holographic duality.

1974 (continued)	lattice QCD (Wilson), confining strings
1993-1994	holographic principle (t' Hooft, Susskind)
1995	D-branes (Polchinski)
1997 June	need 5d string theory to describe QCD (Polyakov)
1997 Nov	AdS/CFT(Maldacena)
1998 Feb	connection between holographic principle and large N gauge theory/string theory duality (Witten)

References

[1] Igor R. Klebanov, arXiv:hep-th/9108019

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Fall 2014

Chapter 2: Deriving AdS/CFT

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2014

Lecture 16

Important equations for this lecture from the previous ones:

1. The spacetime metric from N D3-branes in IIB SUGRA , equation (13) and (14) in lecture 15:

$$ds^2 = f(r)\Big(-dt^2 + d\vec{x}^2\Big) + h(r)\Big(dr^2 + r^2 d\Omega_e^2\Big) \quad ; \tag{1}$$

$$f(r) = \frac{1}{h(r)} = H^{-1/2}(r) \quad , \quad H(r) = 1 + \frac{R^4}{r}^4 \quad , \quad R^4 = N\frac{4}{\pi^2}G_N T_3 = N4\pi g_s \alpha' \tag{2}$$

2. The relation between the gravitational constant G_N and string theory's g_s and α', equation (15) in lecture 12:

$$G_N = 8\pi^6 g_s^2 \alpha'^2 \tag{3}$$

2.2: D-BRANES AS SPACETIME GEOMETRY (cont.)

From the spacetime metric given in equation (1) and (2), the physical interpretation of R can be seen:

1. For $r \to \infty$, $f(r) = h(r) = 1$, as the spacetime geometry is asymptotically flat.
2. For $r \gg R$, then one arrives at the long-range Coulomb potential $\sim \frac{1}{r^4}$ in $D = 10$ due to a 3D object:

$$f(r) = 1 + \mathcal{O}\Big(\frac{R^4}{r^4}\Big) \quad , \quad h(r) = 1 + \mathcal{O}\Big(\frac{R^4}{r^4}\Big) \tag{4}$$

3. For $r \sim R$, the deformation of spacetime metric from D3-branes become significant, with the curvature $\sim R^{-2}$. In order for $\alpha' R^{-2} \ll 1$ (so that SUGRA is valid), one need $g_s N \gg 1$ and $g_s \ll 1$.
4. For $r \to 0$ as one approaches the D3-branes, then $H(r) \approx \frac{R^4}{r^4}$:

$$ds^2 = \frac{r^2}{R^2}\Big(-dt^2 + d\vec{x}^2\Big) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 \tag{5}$$

The spacetime is now factorized into $AdS_5 \times S^5$, with the S^5 has a constant radius R. Another interesting feature of this metric is that $r = 0$ is now sits at an infinite proper distance away, as the branes seems to be essentially disappeared (no source) and there're only the deformed geometry and F_5 flux in spacetime.

Now, we has 2 descriptions of N D-branes:

1. Description A: D-branes in flat spacetime where open strings can end.

2. Description B: Deformed spacetime metric given in equation (4) with F_5 fluxes on S^5 where only closed strings can propagate.

These 2 descriptions are expected to be equivalent. In priciple, both of them can be extended to be valid for all α' and g_s. This is a surprising statement, but no much can be done about it, since both sides are complicated and not very well known. In 1997, J. Maldacena considered a special limit of this equivalent, the low energy limit (fixed the energy scale E and take $\alpha' \to 0$, or fixed α' and take $E \to 0$), and it is known nowadays as the AdS/CFT correspondence:

1. Description A: Open strings give $\mathcal{N} = 4$ SYM theory with the gauge group $U(N)$ and the Yang-Mills coupling $g_{YM}^2 = 4\pi g_s$, closed strings give graviton and other massless fields, and note that the coupling between massless open and closed strings:

$$G_N \sim g_s^2 \alpha'^4 \tag{6}$$

As $E \to 0$, the $\mathcal{N} = 4$ SYM decouples from gravitons and other closed string modes. Effectively, the theory is that of $\mathcal{N} = 4$ SYM and free gravitons.

2. Description B: From the spacetime metric of N D3-branes, one should be careful with which time to use and define the energy. The energy of D3-branes in description A is defined with t given in equation (1), which is the time at $r = \infty$. At a general value of r, the local proper time $d\tau = H^{-1/4}(r)dt$ so then the local energy $E_\tau = H^{-1/4}E$. For $r \gg R$, $H(r) \approx 1$ and $E^2\alpha' \to 0$, hence all massive string modes decouple. For $r \ll R$, $H(r) \approx \frac{R^4}{r^4}$, and the low energy limit $E^2\alpha' \to 0$ means:

$$E_\tau^2 \frac{r^2}{R^2}\alpha' \to 0 \quad \Rightarrow \quad E_\tau^2 \frac{r^2}{\sqrt{4\pi g_s N}} \to 0 \tag{7}$$

This means, for any E_τ, the low energy limit means $r \to 0$. Which means, for sufficiently small r (close to the D3-branes), any massive stringy modes are allowed. The $r \to 0$ region has $AdS_5 \times S^5$ geometry with full stringy description, so the low energy limit is that of the free gravitons at $r = \infty$ and full string theory (with D-branes, which translational dynamics is actually playing an important role) in $AdS_5 \times S_5$ – these 2 sectors decouple.

Equating description A and B at low energy, one has $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$ (characterized by g_{YM}^2 and N) is equivalent to the full IIB superstring theory in $AdS_5 \times S^5$ (characterized by g_s and $\frac{R^2}{\alpha'}$) with D-branes. With the help from equation (2), one gets the relations:

$$g_{YM}^2 = 4\pi g_s \quad , \quad g_{YM}^2 N = \frac{R^4}{\alpha'^2} \quad , \quad \frac{G_N}{R^8} = \frac{\pi^4}{2N^2} \tag{8}$$

2.3: AdS/CFT DUALITY

2.3.1: AdS SPACETIME

From equation (5), the AdS spacetime metric:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 \tag{9}$$

If $\vec{x}$ is d-dimensional then this metric described AdS_{d+1} spacetime. R is the AdS curvature radius, and r runs from 0 to the boundary ∞. From the general relativity Einstein's field equation point of view, AdS is a spacetime of constant curvature with negative cosmological constant:

$$\mathcal{R}_{MN} - \frac{1}{2}g_{MN}(\mathcal{R} - 2\Lambda) = 0 \;\; ; \;\; \Lambda < 0 \tag{10}$$

The solution of the given tensor equation:

$$\mathcal{R} = \frac{2(d+1)}{d-1}\Lambda \quad , \quad \Lambda = -\frac{1}{2}d(d-1)\frac{1}{R^2} \rightarrow \mathcal{R} = -d(d+1)R^2 \quad , \quad \mathcal{R}_{MNPQ} = -R^2(g_{MP}g_{NQ} - g_{MQ}g_{NP}) \tag{11}$$

Another convenient choice for coordinates in AdS space is $z = \frac{R^2}{r^2}$, runs from the boundary 0 to ∞:

$$ds^2 = \frac{R^2}{z^2}\Big(- dt^2 + d\vec{x}^2 + dz^2\Big) \tag{12}$$

It should be noted that equation (9) and (12) only cover 1 part of the full AdS spacetime, called the Poincare patch. Indeed, to cover the whole AdS spacetime one needs an infinite number of copies of the Poincare patch. The global AdS_{d+1} spacetime can be described as a hyperboloid in a flat Lorentz spacetime of signature $(2, d)$:

$$X_{-1}^2 + X_0^2 - \vec{X}^2 = R^2 \quad , \quad ds^2 = -dX_{-1}^2 - dX_0^2 + d\vec{X}^2 \tag{13}$$

Let's look more closely to the geometrical structure of AdS space:

1. The Poincare coordinates:

$$r = X_{-1} + X_d \quad , \quad x^\mu = \frac{R}{r}X^\mu \tag{14}$$

   Therefore, the coordinates described by equation (9) and (12) only corresponds to the $r > 0$ branch.

2. The global coordinates:

$$X_0 = R\sqrt{1+r^2}\cos\tau \quad , \quad X_{-1} = R\sqrt{1+r^2}\sin\tau \quad , \quad X_0^2 + X_{-1}^2 = R^2(1+r^2) \quad , \quad \vec{X}^2 = R^2r^2 \tag{15}$$

   Let τ runs from $-\infty$ to $+\infty$, then:

$$ds^2 = R^2\left(- (1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1}^2\right) \tag{16}$$

   For $r = \tan\rho$ with $\rho \in \left[0, \frac{\pi}{2}\right]$:

$$ds^2 = \frac{R^2}{\cos^2\rho}\Big(- d\tau^2 + d\rho^2 + \sin^2\rho d\Omega_{d-1}^2\Big) \tag{17}$$

   This choice of coordinates has the AdS center at $\rho = 0$ and the AdS boundary at $\rho = \frac{\pi}{2}$, and the geometry of the boundary is $S^{d-1} \times \mathbb{R}$.

   The spacetime interval in the boundary can be calculated with:

$$ds^2_{boundary} \sim -d\tau^2 + d\Omega_{d-1}^2 \tag{18}$$

   It takes a light ray $\tau = \frac{\pi}{2}$ to reach the boundary, but a massive particle can never reach the boundary since at some point it will turned back by gravitational pull. The AdS spacetime is like a confining box of size $\sim R$.

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Chapter 2: Deriving AdS/CFT

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Lecture 17

Important equations for this lecture from the previous ones:

1. The global AdS_{d+1} can be described as a hyperboloid in a flat Lorentzian spacetime of signature $(2, d)$, equation (13) in lecture 16:
$$X_{-1}^2 + X_0^2 - \vec{X}^2 = R^2 \tag{1}$$

2. The Poincare patch, equation (12) in lecture 16:
$$ds^2 = \frac{R^2}{z^2}\Big(- dt^2 + dz^2 + d\vec{x}^2\Big) \tag{2}$$

3. The relation between the gravitational constant G_N and string theory's g_s and α', equation (15) in lecture 12:
$$G_N = 8\pi^6 g_s^2 \alpha'^2 \tag{3}$$

2.3.1: AdS SPACETIME (cont.)

Since the global AdS_{d+1} can be defined as in equation (1), it is clear that the spacetime has an $SO(2, d)$ isometry, which is the same as the conformal group in flat d-dimensional Minkowski space. This can be seen from the Poincare patch, which is given in equation (2):

1. P^μ: d generators, translations along $x^\mu = (t, \vec{x})$.

2. $M^{\mu 0}$: $\frac{1}{2}d(d-1)$ generators, Lorentz transformation for x^μ.

3. Scaling: 1 generator, take z to λz and x^μ to λx^μ.

4. Special conformal transformation (SCT): d generators
$$z \to z' = \frac{z}{1 + 2bx + b^2 A} \quad , \quad x^\mu \to x'^\mu = \frac{x^\mu + b^\mu A}{1 + 2bx + b^2 A} \quad , \quad A = z^2 + x^2 \tag{4}$$
One can try to understand the SCT as the combination of inversion $z \to \frac{z}{A}$ and $x^\mu \to \frac{x^\mu}{A}$, translation by b, then the very same inversion again.

Altogether, there are $\frac{1}{2}(d+1)(d+2)$ generators.

2.3.2: STRING THEORY IN AdS$_5\times$ S^5

A perturbative string theory can be characterized by g_s and α'. Since $AdS_5\times S^5$ is a homogeneous spacetime of constant curvature everywhere, string theory in this spacetime background has only 2 dimensional parameters, g_s and $\frac{\alpha'}{R^2}$. As G_N is related to g_s, given in equation (3), another choice for parameters (which is more convenient in the classical gravity limit) is with $\frac{G_N}{R^8}$ and $\frac{\alpha'}{R^2}$.

The classical gravity limit gives IIB SUGRA:

$$g_s\to 0 \quad , \quad \frac{\alpha'}{R}\to 0 \quad , \quad \frac{G_N}{R^8}\to 0 \tag{5}$$

The classical string limit:

$$g_s\to 0 \quad , \quad \frac{\alpha'}{R^2}\to \text{finite} \tag{6}$$

The compactness of S^5 make it convenient to express a 10D field in terms of a tower of AdS_5 fields by expending it in terms of 5-spherical harmonic function, for example, with a scalar:

$$\Phi(x^\mu,z,\Omega)=\sum_l \Phi_l(x^\mu,z)Y_l(\Omega) \tag{7}$$

As the gravity is essentially 5D, the graviton zero-mode on S^5 with V_5 is the volume of S^5:

$$\frac{1}{16\pi G_N}\int d^5x d^5\Omega\sqrt{-G_{10}}\mathcal{R}_{10}=\frac{V_5}{16\pi G_N}\int d^5x\sqrt{-G_5}\mathcal{R}_5 \tag{8}$$

Hence, the effective 5D gravitational constant:

$$G_{N\,5}=\frac{G_N}{V_5}=\frac{G_N}{\pi^3R^5} \tag{9}$$

After dimensional reduction on S^5, then the action can be written as:

$$S=\frac{1}{16\pi G_{N\,5}}\int d^5x\Big(\mathcal{L}_{gravity}+\mathcal{L}_{matter}\Big) \tag{10}$$

2.3.3: $\mathcal{N}=4$ SYM theory

The fields content of interests are A_μ, Φ^i (with $i=1,2,...,6$) and the Weyl spinor χ^A_α ($A=1,2,3,4$), they are all in the adjoint representation of $U(N)$

. Altogether, the number of on-shell degrees of freedom:

$$\Bigg(8(\text{bosonic})+8(\text{fermionic})\Bigg)\times N^2=16N^2 \tag{11}$$

The interacting part are actually $SU(N)$ while the $U(1)$ part decouples and free:

$$\Phi^i=\Phi^i_aT^a_{SU(N)}+\phi^iT_{U(1)} \quad , \quad A_\mu=A^a_\mu T^a_{SU(N)}+B_\mu T_{U(1)} \quad ; \quad T_{U(1)}\sim 1_{N\times N} \quad , \quad \Big[T_{U(1)},T^a_{SU(N)}\Big]=0 \tag{12}$$

Specifically, he ϕ^i and B_μ fields (decouple from $SU(N)$ degrees of freedom) governed by a free theory.

There's a subtlety in why the AdS bulk space is equivalent to $SU(N)$ but not the whole $U(N)$. Indeed, $U(1)$ actually corresponds to the translational degree of freedom of the whole N D3-branes, so by taking that group on one side and the D-branes out of the other, one left with the correspondence between the CFT $\mathcal{N}=4$ SU(N) SYM and IIB stringy physics in $AdS_5 \times S^5$ background. And there are many other ways to see that, such as in the dual string theory in $AdS_5 \times S^5$ all modes couple to gravity, therefore the noninteracting part $U(1)$ shouldn't be included.
Also, note that at tree level, correlation functions of $U(N)$ and $SU(N)$ are the same, but in general it's not true at higher loop orders. However, as N goes to infinity, then even at loop orders these 2 gauge groups are in agreement.

The piece of string theory of interests is about the centor of mass motion of the N D3-branes:

$$\mathcal{L} = -\frac{1}{g_{YM}^2} \operatorname{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(D^\mu\Phi^i\right)\left(D_\mu\Phi_i\right) + \left[\Phi^i,\Phi^j\right]^2\right) + \text{fermionic} \tag{13}$$

The properties of the theory:

1. It has $\mathcal{N}=4$ SUSY: in (1+3)D, with $\mathcal{N}=1$ the supercharge Q_α is a Weyl spinor, while $\mathcal{N}=4$ has 4 of such conserved Q_α^A supercharge ($A=1,2,3,4$), maximally allowed SUSY for a renormalizable field theory.

2. The SYM coupling g_{YM} is dimensionless classically, and in the quantum theory one also get the β-function of the coupling to be zero, therefore it is a genuine dimensionless parameter.

3. The theory is conformally invariant, which is a conformal field theory (CFT), because the vanishing of β-function indicates that the theory doesn't have a scale.

   The full bosonic symmetries are $SO(2,d)\otimes SO(6)$, and by including SUSY, one arrives at the superconformal symmetry (SCFT) $PSU(2,2|4)$. Since $\mathcal{N}=4$ is the most symmetry (thus likely simplest) among all interacting theories in (1+3)D.

In a CFT, the basic objects are local operators with a definite scaling dimension Δ (conformal dimension):

$$O(x) \to O'(x) = \lambda^\Delta O(\lambda x) \tag{14}$$

The typical observables are correlation functions of local operators. The conformal symmetries determine 2-point and 3-point correlation functions up to a constant:

$$\langle O(x_1)O(x_2)\rangle = \frac{C\delta_{\Delta_1,\Delta_2}}{|x_1-x_2|^{2\Delta_1}} \quad , \quad \langle O(x_1)O(x_2)O(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}} \tag{15}$$

Some comment about the equivalent between $\mathcal{N}=4$ SYM with gauge group $SU(N)$ in $\mathbb{R}^{1,3}$ and type IIB string in $AdS_5\times S^5$ (Poincare patch):

1. $\mathcal{R}^{1,3}$ is the boundary of the Poincare patch of AdS_5

2. IIB string theory in such background gives a 5D gravity theory.

The equivalence can be considered as a realization of the holographic principle, and this lead to a nontrivial prediction that $\mathcal{N}=4$ on $S^3\times\mathbb{R}$ is equivalent to type IIB string theory in the global $AdS_5\times S^5$.

Why we can (and certainly should) identify the CFT $\mathcal{N}=4$ $SU(N)$ SYM on Minkowski space of (1+3)D (D3-branes worldvolume) with the boundary of AdS? The reason is that it's the only possible $SO(2,4)$-invariant between these 2 spaces. Also, the AdS/CFT mapping is naturally formulated with buld/boundary [1] (indeed, this can be guessed from that the AdS boundary condition should be controlled by the CFT of interests). Another evidence is that the number of degrees of freedom seems to be matched. Hence, it is expected that the relation is holographic.

For a quantum gravitational theory, spacetime fluctuates, so what do we really mean by $AdS_5 \times S^5$? Well, indeed, for finite g_s, the quantum gravitational fluctuation can be large, one should interpret $AdS_5 \times S^5$ as specifying the asymptotic structure of the bulk spacetime (the boundary conditions for the bulk quantum gravity).

Reference

[1] Witten, E. (1998). *Anti de Sitter space and holography.* arXiv preprint hep-th/9802150

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Fall 2014

Chapter 3: Duality Toolbox

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2014

Lecture 18

3.1: GENERAL ASPECTS

3.1.1: IR/UV CONNECTION

As seen before, equipped with holographic principle, we can deduce $\mathcal{N} = 4$ super Yang-Mills theory in (3+1) dimension from AdS gravity. However, from field theory perspective, where does the extra dimension come from? The answer lies already in the way we take the low energy limit. As we see before, as we approach the center of D3 brane, namely $r \to 0$, we get the low energy limit of boundary theory. In other words, the extra dimension can be considered as representing the energy scale of the boundary theory!

Since this is a very important point, let us go over it again using AdS metric:

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) \qquad t, \vec{x}: \text{ defined in boundary units} \tag{1}$$

Local proper time and proper length at same z are

$$d\tau = \frac{R}{z}dt \qquad dl = \frac{R}{z}dx \tag{2}$$

which implies the relations between the energies and distances in boundary and bulk:

$$E_{YM} = \frac{R}{z}E_{loc} \qquad d_{YM} = \frac{z}{R}d_{loc} \tag{3}$$

We can interpret these relations as follows: for the same bulk processes at different z, i.e. with the same E_{loc} and d_{loc}, the corresponding processes in the field on the boundary are

$$E_{YM} = \frac{1}{z}(E_{loc}R) \propto \frac{1}{z} \qquad d_{YM} = z\frac{d_{loc}}{R} \propto z \tag{4}$$

In particular, we have

$$z \to 0 \implies E_{YM} \to \infty,\ d_{YM} \to 0 \text{ (UV processes of SYM)} \tag{5}$$

$$z \to \infty \implies E_{YM} \to 0,\ d_{YM} \to \infty \text{ (IR processes of SYM)} \tag{6}$$

Generally, a typical bulk process has an energy scale $E_{loc} \sim \frac{1}{R}$, which implies the energy scale of corresponding boundary process $E_{YM} \sim \frac{1}{z}$. This relation connecting IR and UV in boundary theory by the "depth" of corresponding process in bulk is called IR/UV connection.

Remarks:

1. Putting an IR cuttoff in AdS at $z = \epsilon \iff$ In the boundary introducing a short-distance (UV) cutoff at $\Delta x \sim \epsilon$ or energy cutoff at $E \propto \frac{1}{\epsilon}$.

2. In a conformal theory on $\mathbb{R}^{3,1}$, there exists arbitrarily low energy excitations, corresponding to $z \to \infty$ region in the bulk (Poincare patch). If a theory has an energy gap, bulk spacetime has to "end" at a finite proper distance. An example of CFT on $S^3 \times \mathbb{R} \iff$ gravity in global AdS is given in problem set.

3.1.2: MATCHING OF SYMMETRIES

We have the following table showing the symmetries of $\mathcal{N}=4$ SYM and IIB string theory in $\mathrm{AdS}_5 \times S_5$:

$\mathcal{N}=4$ SYM		IIB string theory in $\mathrm{AdS}_5 \times S_5$
conformal $SO(4,2)$	$\Longleftrightarrow$	isometry of AdS_5: $SO(4,2)$
global (internal) $SO(6)$	$\Longleftrightarrow$	isometry of S^5: $SO(6)$
SUSY (global): 4 ($\mathcal{N}$)+4 (CFT SUSY) Weyl spinors: 32 components of real supercharges	$\Longleftrightarrow$	same amount of local SUSY

Table 1: Symmetries of $\mathcal{N}=4$ SYM and IIB string theory in $\mathrm{AdS}_5 \times S_5$

Remarks:

1. Isometry of $\mathrm{AdS}_5 \times S_5$ is a subgroup of diffeomorphisms (coordinate transformations), which are *local* symmetries on gravity side. We thus have

$$\text{Global Symmetries} \Longleftrightarrow \text{Local Symmetries (gauge)}$$

2. What is special about isometry?
This is subgroup which leaves the asymptotic form of the metric invariant, which can be regarded as those gauge transformations (diffeomorphism) falling off sufficiently fast at infinity. In other words, isometries are *large gauge transformations* , which can be considered as global part of the diffeomorphisms.

3. The story works in general:

CFT in Mink_d		AdS_{d+1} gravity
conformal $SO(d,2)$	$\Longleftrightarrow$	isometry $SO(d,2)$
global $U(1)$	$\Longleftrightarrow$	local $U(1)$
global SUSY	$\Longleftrightarrow$	local SUSY

Table 2: General symmetries matching

3.1.3: MATCHING OF PARAMETERS

According to our discussion in Chapter 2, we summarize in the following table to show the relations between parameters of SYM and gravity:

$\mathcal{N}=4$ SYM		IIB in $\mathrm{AdS}_5 \times S_5$
g_{YM}^2	$=$	$4\pi g_s$
$\lambda \equiv g_{YM}^2 N$	$=$	$\frac{R^4}{\alpha'^2}$
$\frac{\pi^4}{2N^2}$	$=$	$\frac{G_N}{R^8}$

Table 3: Parameters of $\mathcal{N}=4$ SYM and IIB string theory in $\mathrm{AdS}_5 \times S_5$

We often consider dimensional reduction on S^5 to get the effective gravitational constant in AdS_5. This can be done by simply integrating out the volume of S^5 in our action:

$$\frac{1}{G_5} = \frac{V_5}{G_N} \implies \frac{G_5}{R^3} = \frac{\pi}{2N^2} \tag{7}$$

where we used the volume of S^5 is $V_5 = \pi^3 R^5$. We are specially interested in following two limits:

1. Semi-classical gravity limit. In this limit we set $\hbar = 1$ and treat gravity as classical background field and restrict length scales much larger than string length. Hence, we equivalently have quantum field theory in curved spacetime. Our parameters take the limit of $G_N/R^8 \to 0$ and $\alpha'/R^2 \to 0$, which corresponds

to $N \to \infty$ and $\lambda \to \infty$ respectively by table 3. This result has a remarkable physical interpretation: *strong coupling limit is described by classical gravity!* Furthermore, we may find

$$\frac{1}{N^2} \text{ corrections in CFT} \iff \text{quantum gravity corrections}$$
$$\frac{1}{\sqrt{\lambda}} \text{ corrections in CFT} \iff \frac{\alpha'}{R^2} \text{ corrections (string has a finite size)}$$

2. Classical string limit. In this limit we release the restriction of α'/R^2 above, which can be considered as classical string theory in curved spacetime. Similarly, in this limit, we have

$$N \to \infty \iff \frac{G_N}{R^8} \to 0 \ (g_s \to 0)$$
$$\lambda \text{ arbitrary} \iff \frac{\alpha'}{R^2} \text{ arbitrary}$$

which is exactly 't Hooft limit.

3.1.4: MATCHING OF THE SPECTRUM

From now on, we will consider semi-classical gravity regime in the bulk (*i.e.* QFT in a curved spacetime). The following table shows the matching of Hilbert space in the general correspondence, not necessary to be $\mathcal{N} = 4$ SYM:

boundary theory		bulk theory
representations of conformal group $SO(2,4)$	$\iff$	representations of isometry $SO(2,4)$
conformal local operators	$\iff$	bulk fields
scalar operators $\mathcal{O}$	$\iff$	scalar fields ϕ
vector operators J_μ	$\iff$	vector fields A_M
tensor operators $T_{\mu\nu}$	$\iff$	tensor fields h_{MN}

Table 4: Correspondence of Hilbert space

If there are other symmetries, quantum numbers and representations under them, they should also match. Actually, spectrum of IIB supergravity on S^5 has been worked out long ago and for all of them counter parts have been found in $\mathcal{N} = 4$ SYM. The most important correspondence in their spectra are as follows:

$\mathcal{N} = 4$ SYM		IIB supergravity
Lagrangian: $\mathcal{L}_{\mathcal{N}=4}$	$\iff$	dilaton: Φ
$SO(6)$: J^a_μ	$\iff$	gauge field reduced from S^5: A^a_μ
stress tensor: $T_{\mu\nu}$	$\iff$	metric perturbations: h_{MN}

Table 5: Correspondence of Hilbert space

Given an operator $\mathcal{O}(x)$ in a filed theory, a natural thing to is to deform the theory by adding a source term for $\mathcal{O}(x)$:

$$\int d^d x \phi_0(x)\mathcal{O}(x) \tag{8}$$

where if $\phi_0(x)$ is a constant and generally $\mathcal{O} \sim \mathrm{Tr}(\phi_1\phi_2\cdots))$, this is equivalent to changing the coupling in $\mathcal{O}(x)$. Now the question raises: what does this operation mean in the bulk? Let us consider it in a specific way. We know in IIB supergravity, $g_s = e^{\langle\Phi\rangle}$, where Φ is dilaton. For a spacetime with boundary like AdS, we can identify

$$\langle\Phi\rangle = \Phi_\infty \qquad \text{value of } \Phi \text{ at the boundary} \tag{9}$$

because we may naturally assume there is no fluctuation of Φ approaching boundary. Since in the correspondence of parameters, $g^2_{YM} = 4\pi g_s$, we will get a relation $g^2_{YM} = 4\pi e^{\Phi_\infty}$. Note the coupling constant $1/g^2_{YM}$ can be regarded as the (uniform) source C for $\mathcal{L}_{\mathcal{N}=4}$, and the deformation of form of (8) of the boundary Lagrangian is $\int \delta C \mathcal{L}_{\mathcal{N}=4}$, we have

$$\int \delta C \mathcal{L}_{\mathcal{N}=4} = \frac{1}{4\pi} e^{-\Phi_\infty}(-\delta\Phi_\infty) \int \mathcal{L}_{\mathcal{N}=4} \tag{10}$$

We conclude that this deformation corresponds to changing the boundary value Φ_∞ of dilaton Φ, or in this way we say the bulk field Φ corresponds to boundary operator $\mathcal{L}_{\mathcal{N}=4}$. Indeed, one can extend this duality to more general cases:

$$\int d^d x \phi_0(x) \mathcal{O}(x) \text{ in boundary theory } \Longleftrightarrow \text{ bulk field } \phi(x) \text{ corresponding to } \mathcal{O}(x) \text{ with boundary value } \phi_0(x) \tag{11}$$

up to some (re)normalization factor because of mass-dimension relation that we will explain in the next lecture.

One can use this identification to argue: (a) any conserved current J^μ in boundary is dual to a bulk gauge field, *i.e.* global boundary symmetry $\Longleftrightarrow$ bulk gauge symmetry. (b) stress tensor is always dual to metric perturbations. To see (a), consider deforming the boundary theory by

$$\int d^d x a_\mu(x) J^\mu(x) \tag{12}$$

where we work in Poincare patch and based on our duality: $a_\mu(x) = A_\mu(x,z)|_{z=0}$, where A_μ is the bulk dual vector field. Since (12) is invariant under

$$a_\mu \to a_\mu + \partial_\mu \Lambda(x) \tag{13}$$

for any $\Lambda(x)$ as long as a_μ vanishes fast enough in boundary and J^μ is conserved by boundary global symmetry, we expect bulk dynamics to be invariant under some gauge symmetry of which (13) is a subset. This is exactly the case that we endow A_μ $U(1)$ gauge symmetry, which has transformation of

$$A_M(x,z) \to A_M(x,z) + \partial_M \tilde{\Lambda}(x,z) \tag{14}$$

and approaching boundary the transverse components will be

$$A_\mu(x,z)|_{z=0} \to A_\mu(x,z)|_{z=0} + \partial_\mu \tilde{\Lambda}(x,z)|_{z=0} \qquad \Lambda(x) \equiv \tilde{\Lambda}(x,z)|_{z=0} \tag{15}$$

which coincides with (13). To see (b), note adding $\int d^d x h_{\mu\nu} T^{\mu\nu}$ to boundary theory can be considered as deforming the boundary metric as

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu} \equiv g^{(b)}_{\mu\nu} \tag{16}$$

From AdS metric

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) = \frac{R^2}{z^2} dz^2 + g_{\mu\nu} dx^\mu dx^\nu \tag{17}$$

where

$$g_{\mu\nu}(z, x^\mu)|_{z\to 0} = \frac{R^2}{z^2} \eta_{\mu\nu} \tag{18}$$

we may expect that

$$g_{\mu\nu}(z, x^\mu)|_{z\to 0} = \frac{R^2}{z^2} g^{(b)}_{\mu\nu} \tag{19}$$

which implies $T_{\mu\nu}$ should correspond to metric perturbations, *i.e.* the bulk theory must involve gravity.

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