1. GAUGE "SYMMETRY" AND SPIN-1 PARTICLES

Consider a set of fermions ψ^i, $i = 1 \cdots N$.
Lorentz invariance requires:

$$\mathcal{L}_\psi = \bar{\psi}^i i \not{\partial} \psi^i - m_i \bar{\psi}^i \psi^i + \text{more derivatives}$$
$$+ \text{more } \psi\text{'s}.$$

Let's write this in a slightly different form.
Define:

$$P_L \equiv \frac{1-\gamma_5}{2} \quad ; \quad P_R \equiv \frac{1+\gamma_5}{2}$$

and $\psi_L \equiv P_L \psi$; $\psi_R \equiv P_R \psi$ $\quad$ (so $\psi = \psi_L + \psi_R$)

Then (Exercise) $\quad$ ($\bar{\psi}_L \equiv \bar{\psi} P_R$, etc.)

$$\mathcal{L}_\psi = \bar{\psi}_L^i i \not{\partial} \psi_L^i + \bar{\psi}_R^i i \not{\partial} \psi_R^i - m_i \left(\bar{\psi}_L^i \psi_R^i + \bar{\psi}_R^i \psi_L^i \right)$$

This Lagrangian is invariant under a GLOBAL $U(N)$ unitary transformation (for $m_i = m$)

$$\psi_L^i \to U^{ij} \psi_L^j \quad ; \quad \psi_R^i \to U^{ij} \psi_R^j$$

When $N=1$ this is the familiar phase $U(1)$ symmetry

$$\psi \to e^{i\alpha}\psi$$

(which is extended to $U(1)^N \subset U(N)$, $\psi_i \to e^{i\alpha_i}\psi_i$).
$[U(N) = U(1) \times SU(N)]$

Also, if $m_i = 0$ then the global symmetry is larger, $U(N)_L \times U(N)_R$:

$$\psi_L^i \to U_L^{ij}\,\psi_L^j \quad ; \quad \psi_R^i \to U_R^{ij}\,\psi_R^j$$

In the $N=1$ this corresponds to the two symmetries:

$$\psi \to e^{i\alpha}\psi \quad \text{and} \quad \psi \to e^{i\alpha\gamma_5}\psi$$

The second one is called a CHIRAL SYMMETRY.

If we allow the transformation to be different at each space-time point, it is no longer a symmetry because of the derivative term, i.e:

$$\psi \to e^{i\alpha(x)}\psi \;\Rightarrow\; \mathcal{L} \to \mathcal{L} \underbrace{- j_\mu(x)\,\partial^\mu\alpha(x)}_{\delta\mathcal{L}}$$

with $j_\mu(x) = \bar{\psi}\gamma_\mu\psi$.

We can, however, impose the local symmetry, by introducing a vector field, a "connection" $A_\mu(x)$ that redefines what we mean by 'derivative' in the case of local transformations:

$$i\partial \longrightarrow iD \equiv i\partial - gA$$

Then, the above local transformation becomes:

$$\delta\mathcal{L} = -j^\mu(x)\left(\partial_\mu\alpha(x) + \delta A_\mu(x)\right)$$

and thus we recover the symmetry when

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x)$$

and so the modified Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi$$ is "GAUGE INVARIANT".

We can now complete the Lagrangian with new allowed terms, e.g.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\Psi}(i\not{D} - m)\Psi + \cdots$$

with $F^{\mu\nu} \equiv [D^\mu, D^\nu]$ (gauge-invariant).

Up to dimension = 4 there are no other possible terms. In particular, a mass term

$$m^2 A_\mu A^\mu$$

is not gauge invariant. Thus imposing the local symmetry introduces a new massless spin-1 particle. In this case, the photon. the resulting theory is QED.

The case with $N>1$ fermions and local $SU(N)$ symmetry is leads to NON-ABELIAN GAUGE THEORY.

"Gauging" a global symmetry is the source of very interesting theories:

Local Poincaré invariance $\Longrightarrow$ GR
Local Supersymmetry $\Longrightarrow$ SUGRA
Local $SU(N)$ $\Longrightarrow$ YM theory.

Thus many times gauge symmetry is invoked as a Principle (the "Gauge Principle"). The truth is, any theory of massless spin-1 particles is, at low-energy, a gauge theory.

Massive and massless vector particles

Let's start with a basic discussion on the ~~ka~~ one-particle states in relativistic QM. The source is Ch. 2 of Weinberg's QFT Vol 1.

[Notes A]

Thus, massive spin 1 particles have 3 degrees of freedom, while massless ~~spin 1~~ particles only have helicity. There are particles with helicity + and particles with helicity −, and both are "different" particles from the point of view of ~~Lore~~ Poincaré symmetry, since helicity is Lorentz-invariant.

- In Parity-conserving theories we can decide to give both states the same name.

  E.g: Photons and Gluons have 2-degrees of freedom: helicity + and −.

  (Also gravitons)

- In Parity-violating theories we will distinguish both helicities.

E.g in the SM we have:

$+1/2$ helicity neutrinos $\rightarrow \nu$

$-1/2$ helicity anti-neutrinos $\rightarrow \bar{\nu}$

$+1/2$ helicity electrons e_L

$-1/2$ helicity electrons e_R

Let's consider now massive spin 1 particles.

In order to describe them in a Lorentz covariant way we use a 4-vector field A_μ.

Each component must satisfy the KG equation:

$$\left[(i\partial)^2 - m^2 \right] A_\mu = 0 \qquad (1)$$

in order for its momentum to be on the mass shell.

But A_μ has 4 degrees of freedom, while as we have seen, a massive spin 1 particle has 3 d.o.f.

We can go to the CM frame and describe the vector by its spin: $\nwarrow p_\mu = (m, \vec{0})$

$$A_\mu^{CM} = (0, \vec{A})$$

that is:

$$p_\mu \tilde{A}^\mu = 0 \qquad \text{(in momentum space)}$$

In position space this means imposing

$$\partial_\mu A^\mu = 0. \qquad (2)$$

Exercise: (a) Show that (1) + (2) is equivalent to

$$\partial^\mu \underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}} + m^2 A_\nu = 0$$

~~Exercise~~: (b) Show that the above eq. EOM follows from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^2$$

with $\cancel{F_{\mu\nu} F^{\mu}}$ $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$

Now let's couple the field A_μ to a current J_μ (could be a matter current):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^2 + A_\mu J^\mu$$

this modifies the E.O.M to

$$\partial^\mu F_{\mu\nu} + m^2 A_\nu = -J_\nu$$

Acting ~~on~~ on it with ∂^ν we find:

$$m^2 \partial^\mu A_\mu = -\partial_\mu J^\mu$$

$\Rightarrow$ Since $\partial_\mu A^\mu = 0$ we find that $\partial_\mu J^\mu = 0$

<u>thus</u>: We can only couple the vector to a <u>CONSERVED CURRENT</u>.

(<u>or</u>: the source of the spin 1 particle is a conserved current)

Now let's put the mass $m \to 0$.

then the Lagrangian looks like ~~QED~~:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu$$

We see that this Lagrangian is invariant under $A_\mu \to A_\mu - \partial_\mu \Lambda(x)$ (Exercise) (3)

i.e. it is "GAUGE INVARIANT"

Note: $\partial_\mu \Lambda(x) = -e^{-i\Lambda(x)} (i\partial_\mu) e^{i\Lambda(x)}$

This is not a symmetry, but a redundancy. It tells us that the fields A_μ and $A_\mu - \partial_\mu \Lambda(x)$ describe the same physical state. It reflects the fact that massless ~~particle have~~ A_μ only has 2 d.o.f. (not 3) and then from the 3 independent components of A_μ we need to get rid of one by imposing an additional condition (a gauge condition).

In momentum space, Eq. (2) has the form:

$$k_\mu \cdot \varepsilon^\mu(k) = 0$$

$\uparrow$ polarization vector ($\sim \tilde{A}^\mu$)

Since $k^2=0$, we can always change the polarization vector by:

$$\varepsilon^\mu(k) \longrightarrow \varepsilon^\mu(k) + \lambda k^\mu$$

This is the gauge transformation in mom space.
(the gauge condition is that ε and $\varepsilon+\lambda k$ are to be identified).

Let's try to understand ~~the~~ more the $m \to 0$ limit.

We start computing the propagator for the massive A_μ. We write: (Exercise)

$$\cancel{\mathcal{L} = \frac{1}{2} A_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right.}$$

$$\mathcal{L} = \frac{1}{2} A_\mu \underbrace{\left[g^{\mu\nu}(\partial^2+m^2) - \partial^\mu\partial^\nu \right]}_{\mathcal{O}_A^{\mu\nu}} A_\nu + A_\mu J^\mu$$

The Feynman propagator is given by

$$\tilde{\mathcal{O}}_A^{\mu\nu} \cancel{\text{Prop}} (\tilde{D}_F)_{\nu\rho} = i\,\delta^\mu{}_\rho$$

that is:

$$\left[-(k^2-m^2)\, g_{\mu\nu} - k_\mu k_\nu \right] \tilde{D}_F^{\nu\rho}(k) = i\delta_\mu{}^\rho$$

~~with~~ which leads to: (Exercise)

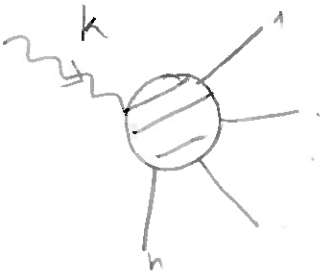
$$\boxed{\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2-m^2+i\varepsilon}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2} \right)}$$

How can we do the limit $m \to 0$? It looks like the term $k^\mu k^\nu / m^2$ explodes.

The key observation is that this term <u>does not</u> contribute to physical processes.

This follows from the <u>Ward-Takahashi</u> identity:

Let's ~~At~~ consider an on-shell amplitude with an external ~~on-shell~~ photon:

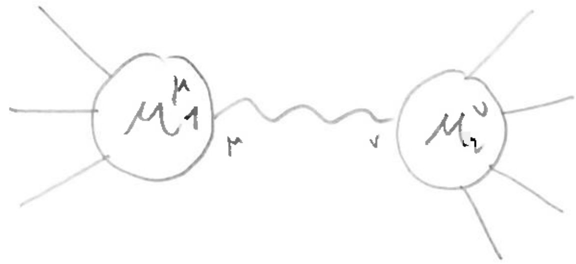


$$\mathcal{M}_\mu(k, p_1, \dots, p_n)$$

$$= i\mathcal{M}_\mu\, \varepsilon^\mu(k) \overset{!}{=} i\mathcal{M}_\mu\left(\varepsilon^\mu(k) + \lambda k^\mu \right)$$

then $\boxed{k^\mu \mathcal{M}_\mu = 0}$ $\left\{ \begin{array}{l} \text{Zee } 11.7 \\ \text{Peskin } 7.4 \end{array} \right\}$

Consider now an internal photon propagator:



The term $\frac{k_\mu k_\nu}{m^2}$ will not contribute:

$$\mathcal{M}_1^\mu \mathcal{M}_2^\nu \, k_\mu k_\nu = 0$$

and it is thus irrelevant. In fact we can add a $k_\mu k_\nu / k^2$ term with an arbitrary coefficient:

$$\boxed{\tilde{D}_F^{\mu\nu} = \frac{-i}{k^2 + i\varepsilon} \left(g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right)}$$

Gauge parameter

$\xi = 1 \equiv$ Feynman gauge

$\xi = 0 \equiv$ Landau gauge

~~Note:~~

~~Also we see (again) that~~

$\varepsilon_\mu \to \varepsilon_\mu + \lambda k_\mu$ ~~does not change $\mathcal{M}_\mu \varepsilon^\mu$~~.

What's happening?

Consider the three basis polarization vectors of a massive vector with $k_\mu = (\omega, 0, 0, k)$

$$\varepsilon_\mu^{(1)} = (0, 1, 0, 0)$$
$$\varepsilon_\mu^{(2)} = (0, 0, 1, 0)$$
(transverse)

$$\varepsilon_\mu^{(3)} = (k, 0, 0, \omega)/m$$
(longitudinal)

one can check that

$$\varepsilon_\mu^{(a)} \varepsilon^{(a)\mu} = -1$$
$$k \cdot \varepsilon^{(a)} = 0$$
$$k \cdot k = \omega^2 - k^2 \equiv m^2$$

the amplitude to emit a longitudinal meson is:

$$\varepsilon_\mu^{(3)} \cdot \mathcal{M}^\mu = (k\mathcal{M}^0 + \omega \mathcal{M}^3)/m$$
$$= \Big(k\mathcal{M}^0 + \underbrace{\sqrt{k^2+m^2}}_{k + \frac{m^2}{2k} + \dots}\mathcal{M}^3\Big)/m$$
$$= \underbrace{\frac{k \cdot \mathcal{M}}{m}}_{(k,0,0,k)} + \frac{m}{2k}\mathcal{M}^3 + \dots$$

But $k \cdot \mathcal{M} = 0 \Rightarrow \varepsilon_\mu^{(3)} \cdot \mathcal{M} = \mathcal{O}(m) \longrightarrow 0$

$\Rightarrow$ $\boxed{\text{the longitudinal photon decouples from physical processes}}$

Exercise: Consider a process with a final-state photon, and the sum over polarizations

$$\overline{|\mathcal{M}|^2} \equiv \sum_a \varepsilon_\mu^{(a)*} \varepsilon_\nu^{(a)} \mathcal{M}^\mu(k) \mathcal{M}^{\nu *}(k)$$

Demonstrate that the Ward Identity ensures one can do

$$\sum_a \varepsilon_\mu^{(a)*} \varepsilon_\nu^{(a)} \longrightarrow -g_{\mu\nu}$$

effectively including all 4 polarizations in the sum.

(Peskin pg. 160)

Can we compute the propagator for the massless photon directly as we did for the massive one?

Problem: $\mathcal{O}_{A_0}^{\mu\nu} = [g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu]$

is a singular operator. $\Rightarrow$ Need to fix the gauge in the functional integral.

(we'll review this) $\leftarrow$

Summary: Massless spin 1 particles can only be described by a gauge theory.

or:

$$\boxed{\text{YM is the \underline{unique} low-energy theory describing massless spin 1 particles}}$$

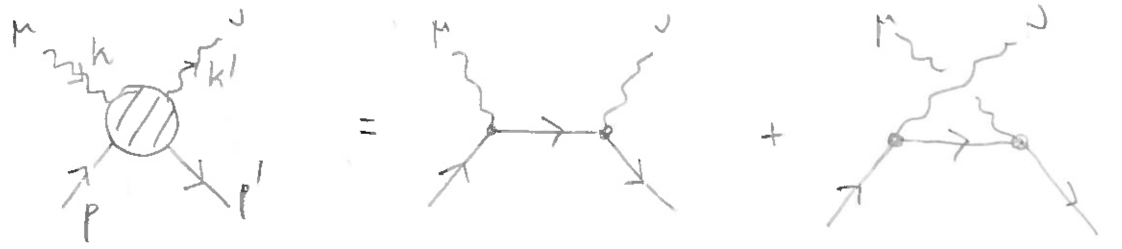
{ See <u>first 2 lectures</u> on

"Robustness of GR. Attempts to Modify Gravity"

by N. Arkani-Hamed. (Youtube) }

Let's now consider one final exercise:

Consider Compton scattering:



$$+ \mathcal{O}(\alpha^2)$$

$$= i\left(\mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu}\right) \varepsilon_\mu(k)\, \varepsilon_\nu^*(k') + \mathcal{O}(\alpha^2)$$

We have:

$$\mathcal{M}_1^{\mu\nu} = -i\, \bar{u}(p')(ie\gamma^\nu) \cdot \frac{i(\not{p}+\not{k}+m)}{(p+k)^2 - m^2}(ie\gamma^\mu)\, u(p)$$

$$= -e^2 \frac{\bar{u}(p')\gamma^\nu(\not{p}+\not{k}+m)\gamma^\mu u(p)}{(p+k)^2 - m^2}$$

$$\mathcal{M}_2^{\mu\nu} = -e^2 \frac{\bar{u}(p')\gamma^\mu(\not{p}'-\not{k}+m)\gamma^\nu u(p)}{(p'-k)^2 - m^2}$$

Let's check Ward's identity. Note that:

- $$\bar{u}(p')\gamma^\nu(\not{p}+\not{k}+m)\not{k}\, u(p) = \bar{u}(p')\gamma^\nu \overbrace{\not{p}\not{k}}^{2p\cdot k - \not{k}\not{p}} u(p) + m\, \bar{u}(p')\gamma^\nu \not{k}\, u(p)$$

$$= 2p\cdot k\ \bar{u}(p')\gamma^\nu u(p)$$

- $$\bar{u}(p')\not{k}(\not{p}'-\not{k}+m)\gamma^\nu u(p) = \bar{u}(p')\overbrace{\not{k}\not{p}'}^{2p'\cdot k - \not{p}'\not{k}}\gamma^\nu u(p) + m\, \bar{u}(p')\not{k}\gamma^\nu u(p)$$

$$= 2p'\cdot k\ \bar{u}(p')\gamma^\nu u(p)$$

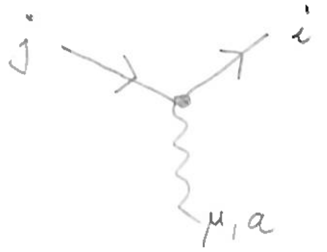
- $$\frac{2p\cdot k}{(p+k)^2 - m^2} = 1 \quad \text{and} \quad \frac{2p'\cdot k}{(p'-k)^2 - m^2} = -1$$

Then: $$k_\mu \mathcal{M}_1^{\mu\nu} = -e^2\, \bar{u}(p')\gamma^\nu u(p) = -k_\mu \mathcal{M}_2^{\mu\nu}$$

So: $k_\mu \mathcal{M}_i^{\mu\nu}$ are both non-zero but they cancel: $k_\mu(\mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu}) = 0$. ✓

Now let's imagine that we have several massless spin-1 particles: A_μ^a $a = 1, \dots, n$, as well as several "species" of fermions ψ_i $i = 1 \dots N$.

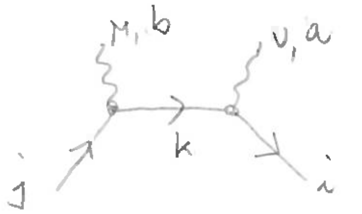
In general we will have a vertex like:



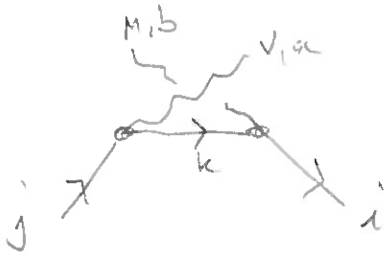
$= i g \gamma_\mu \cdot T^a_{ij}$

some coefficients

Then:



$= i\mathcal{M}_1 \cdot T^a_{ik} T^b_{kj}$



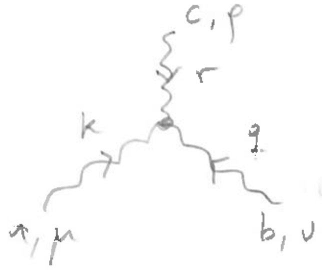
$= i\mathcal{M}_2 \, T^b_{ik} T^a_{kj}$

and so $k_\mu \mathcal{M}^{\mu\nu} = -g^2 \bar{u}(p') \gamma^\nu u(p) \left[T^a, T^b\right]_{ij}$

only zero if $[T^a, T^b] = 0 \quad \forall\, a, b$.

This case is called "abelian gauge theory" for obvious reasons, and corresponds to n different "photons" independent of each other.

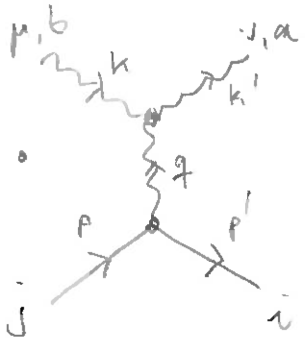
The more interesting "non-abelian" case seems to violate the Ward Identity. However there is one way to fix this: by considering the possibility that the A^a_μ have self-interactions:



$$= g f^{abc} \underbrace{\left[g_{\mu\nu}(k-q)_\rho + g_{\nu\rho}(q-r)_\mu + g_{\rho\mu}(r-k)_\nu \right]}_{\text{by dim. analysis and symmetry.}}$$

Then there is an additional contribution to $\mathcal{M}$:

$k_\mu \cdot$



Exercise

$$= i g^2 f^{abc} T^c_{ij} \, \bar{u}(p') \gamma_\rho u(p) \frac{-i}{q^2} \times$$

$$\times \left(k^\nu \underbrace{(-k'-k)^\rho}_{-2k-q} + k^\rho \underbrace{(k-q)^\nu}_{2k-k'} + g^{\rho\nu} \underbrace{(q+k')}_{2k'-k} \cdot k \right)$$

needed → $\boxed{k' \cdot \varepsilon(k') = 0 \;!!}$

$$= -g^2 f^{abc} T^c_{ij} \cdot \bar{u}(p') \gamma^\nu u(p)$$

$\Rightarrow$ The Ward Identity is satisfied if: $\boxed{[T^a, T^b] = i f^{abc} T^c}$

[Notes A]

A proper Lorentz transformation Λ is implemented on physical quantum states by a unitary operator $\mathcal{U}(\Lambda)$ [Wigner's Symmetry Rep. theorem]:

$$|\Psi\rangle \longrightarrow \mathcal{U}(\Lambda)\,|\Psi\rangle$$

The generators are the operators P^μ, J_i, K_i, with $P^0 = H$ the Hamiltonian. P_i and J_i are conserved, but K_i are not ($[H, K_i] \neq 0$).

It seems a good idea to express a state in terms of eigenvectors of P^μ:

$$P^\mu\,|p,\sigma\rangle = p^\mu\,|p,\sigma\rangle$$

where σ labels the rest of the quantum numbers.

One-particle states are defined as those for which the label σ is discrete.

We have that $\mathcal{U}(\Lambda)\, P^\mu\, \mathcal{U}^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\, P^\nu$

$$P^\mu\, \mathcal{U}(\Lambda)\, |p,\sigma\rangle = \Lambda^\mu{}_\nu\, \mathcal{U}(\Lambda)\, P^\nu\, |p,\sigma\rangle = (\Lambda p)^\mu\, \mathcal{U}(\Lambda)\, |p,\sigma\rangle$$

and thus:

$$\mathcal{U}(\Lambda)\, |p,\sigma\rangle = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)\, |\Lambda p, \sigma'\rangle$$

Expressing the matrix $C_{\sigma'\sigma}$ in block-diagonal form, each irreducible block will correspond to a particle type. These irreducible blocks are associated to IRREDUCIBLE REPS OF THE LORENTZ GROUP. Thus we need to study such representations.

Now let's consider the states with $p^2 \geq 0$ and $p^0 > 0$. For each p^2 we will choose a "standard momentum":

- $k_\mu = (1,0,0,1)$ eV for $p^2 = 0$
- $k_\mu = (M,0,0,0)$ for $p^2 = M^2$

Any other p_μ can be obtained by a so-called "standard Lorentz transformation" $L(p)$:

$$p^\mu = L^\mu{}_\nu(p)\, k^\nu$$

We now <u>define</u> :

$$|p,\sigma\rangle \equiv \sqrt{\frac{k^0}{p^0}}\, \mathcal{U}(L(p))\, |k,\sigma\rangle$$

this means that the labels σ are specified for k and then this translates to p.

We then have:

$$\mathcal{U}(\Lambda)\, |p,\sigma\rangle = \sqrt{\frac{k^0}{p^0}}\, \mathcal{U}(\Lambda L(p))\, |k,\sigma\rangle$$

$$= \sqrt{\frac{k^0}{p^0}}\, \mathcal{U}(L(\Lambda p)) \cdot \mathcal{U}(\underbrace{W(\Lambda,p)}_{= L^{-1}(\Lambda p)\, \Lambda\, L(p)})\, |k,\sigma\rangle$$

Note that $W^\mu{}_\nu k^\nu = k^\mu$. The set of W define what is called the LITTLE GROUP : the group of Lorentz trans. that leave k_μ invariant.

$$LG(k) = \{ W \in \mathcal{L} \mid W^\mu{}_\nu k^\nu = k^\mu \}$$

For $W \in LG$,

$$\mathcal{U}(W)\, |k,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W)\, |k,\sigma'\rangle$$

and the $D_{\sigma'\sigma}$ furnish a unitary representation of LG.

The general transformation on $|p,\sigma\rangle$ is then

$$\mathcal{U}(\Lambda)\,|p,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p)) \sqrt{\frac{k^0}{p^0}}\; \underbrace{\mathcal{U}(L(\Lambda p))\,|k,\sigma'\rangle}_{\sqrt{\frac{(\Lambda p)^0}{k^0}}\,|\Lambda p,\sigma'\rangle}$$

$$= \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\,|\Lambda p,\sigma'\rangle$$

We thus only need to find the reps. of the L.G.
This is called the "Method of Induced Representations".

In the two cases of interest, the L.G. is:

- $p^2 = M^2 > 0 \longrightarrow LG = SO(3)$
- $p^2 = 0 \longrightarrow LG = ISO(2)$

In the first case ($p^2 = M^2$) we already know the representations from angular momentum in QM:
$(2j+1)$-dim representations of spin $j = 0, \frac{1}{2}, 1, \cdots$
So the label σ means (j,m), and Lorentz transformations only change m ($\in \{-j, \cdots, 0, \cdots, j\}$).

In the second case $(p^2 = 0)$ we have:

$$D_{\sigma'\sigma}(W) = e^{i\theta\sigma}\,\delta_{\sigma'\sigma}$$

where θ is the angle of rotation around the 3-axis. (See Weinberg Sect. 2.5).

These are all 1-dimensional representations. It turns out that the only possible values for σ are

$$\sigma = 0, \pm 1/2, \pm 1, \ldots$$