Gauge Theories of the Standard Model — Part I

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1. Generators of $SU(5)$ and Weinberg's angle in $SU(5)$ GUT's

The simplest model for a unified theory containing the standard model is based on $SU(5)$.

(a) How many generators does $SU(5)$ have?

(b) An easy choice is to start writing the following 11 independent generators:

$$T^a = \begin{pmatrix} T^a_{SU(3)} & 0 \\ 0 & 0 \end{pmatrix} , \qquad T^{b+8} = \begin{pmatrix} 0 & 0 \\ 0 & T^b_{SU(2)} \end{pmatrix} ,$$

with $a = 1, .., 8$, $b = 1, .., 3$. Convince yourself that this is correct, and a good idea. Build 12 of the remaining generators of $SU(N)$

$$T^{c+11} = \begin{pmatrix} 0_{3\times 3} & \begin{matrix} * & * \\ * & * \\ * & * \end{matrix} \\ \begin{matrix} * & * & * \\ * & * & * \end{matrix} & 0_{2\times 2} \end{pmatrix} \qquad c = 1, \dots, 12 \ ,$$

explicitly out of Pauli matrices, normalized such that $\mathrm{Tr}\big[T^a T^b\big] = \frac{1}{2}\,\delta^{ab}$. Do they commute with the other generators T^a and T^{b+8}? Should they commute?

(c) The last generator of $SU(5)$ is diagonal. Construct a diagonal generator T^{24} which commutes with T^a and T^{b+8} (but not with T^{c+11}), is traceless, hermitian and is normalized according to $\mathrm{Tr}\big[T^i T^{24}\big] = \frac{1}{2}\delta^{i\,24}$, for $i = 1, .., 24$.

(d) The $SU(5)$ gauge field is given by the 5×5 matrix $A_\mu = A^a_\mu T^a$. Write down explicitly A_μ in matrix form using the following very very very convenient notation: $A^a_\mu \equiv G^a_\mu$ for $a = 1, .., 8$, $A^{b+8}_\mu \equiv W^b_\mu$ for $b = 1, .., 3$ and $A^{24}_\mu \equiv B_\mu$. (Why is this convenient?)

(e) Couple the gauge field A_μ calculated in part (d) to a fermion Ψ in the fundamental representation of $SU(5)$:

$$\mathcal{L}_\Psi = g_5 \bar{\Psi} \not{A} \Psi$$

where

$$\Psi_k = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \end{pmatrix} .$$

Find the couplings of Ψ_4 and Ψ_5 to B_μ and W^3_μ in terms of the $SU(5)$ coupling g_5.

(f) The couplings of B_μ and W^3_μ to Ψ should be identified with g' and g of the SM. Calculate the Weinberg angle

$$\sin^2\theta_{\mathrm{w}} = \frac{g'^2}{g'^2 + g^2} \ ,$$

and compare the value with the experimental result $\sin^2\theta_{\mathrm{w}} \simeq 0.23 \pm 0.01$. What could be the reason for the discrepancy?

2. Spontaneous breaking of $SU(5)$ by fields in the Adjoint

Consider a gauge theory with the gauge group $SU(5)$, coupled to a scalar field Φ in the adjoint representation.

(a) The adjoint representation of $SU(N)$ is the real representation of dimension $N^2 - 1$. The generators are given by the structure constants of the group. A field in the adjoint representation is a $(N^2 - 1)$-vector Φ^a. However it is very convenient to arrange the $N^2 - 1$ components of this vector into a $N \times N$ matrix Φ defined as:

$$\Phi \equiv \Phi^a T^a \ ,$$

where T^a are the $(N^2 - 1)$ generators in the **fundamental** representation. We know that under a gauge transformation $\Phi^a \to (U_{\text{adj.}})^{ab} \Phi^b$, and that the covariant derivative is $D_\mu \Phi^a = [\delta^{ab}\partial_\mu - ig(A_\mu^{\text{adj.}})^{ab}]\Phi^b$, where $(A_\mu^{\text{adj.}})^{ab} = A_\mu^c (t^c_{\text{adj.}})^{ab} = -if^{abc}A_\mu^c$. Show that:

(i) The matrix Φ transforms as $\Phi \to U\Phi U^\dagger$, with U in the fundamental.

(ii) The covariant derivative of Φ is given by $D_\mu \Phi = \partial_\mu \Phi - ig[A_\mu, \Phi]$.

(iii) The covariant derivative is covariant, that is, transforms exactly like Φ.

(iv) The only allowed kinetic term for the adjoint scalar Φ^a is $\mathcal{L}_\Phi^{\text{kin}} = \frac{1}{2}\text{Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)]$.

(b) Assume that the potential for this scalar field forces it to acquire a nonzero vacuum expectation value. Two possible choices for this expectation values are

$$\langle\Phi\rangle = A\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix} \qquad \langle\Phi\rangle = B\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

where A and B are arbitrary constants. For each case, work out the spectrum of gauge bosons and the unbroken symmetry group. For this you should identify the relevant terms in $\mathcal{L}_\Phi^{\text{kin}}$, and notice that in the matrix notation an unbroken generator is defined by $[T^a, \langle\Phi\rangle] = 0$. Start by proving this statement.

3. Baryon-Number violating operators in the SM and in $SU(5)$ GUT's

(a) Write a couple (or more) Lorentz-invariant dimension 6 local operators built out of SM fields, invariant under the SM gauge group and which break Baryon Number.

(b) Now consider an $SU(5)$ gauge theory coupled to a fermion Ψ in the $\bar{\mathbf{5}}$ representation of $SU(5)$ and a fermion Φ in the $\mathbf{10}$. The $\bar{\mathbf{5}}$-field Ψ can be represented by a column 5-vector Ψ_i and the $\mathbf{10}$-field can be represented by an antisymmetric 10×10 matrix Φ_{ij}, transforming as: $\Psi_i \to U^{\dagger}_{ij}\Psi_j$ and $\Phi_{ij} \to U_{ik}\Phi_{kj}$, where U is the gauge transformation matrix in the fundamental.

Write all possible 4-fermion $SU(5)$- (and Lorentz-) invariant dimension-6 operators.

(c) We arrange all known fermions in $SU(5)$ representations in the following way:

$$\Psi = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ -\nu \end{pmatrix}_L , \quad \Phi = \begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ u_1^c & u_2^c & u_3^c & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{pmatrix}_L .$$

Expand the operators in part (b) in terms of u, d, e, ν fields. Do you recover the SM operators of part (a)?

(d) Draw a tree-level Feynman diagram for a Baryon-number-violating process mediated by an $SU(5)$ gauge boson. Compute the corresponding amplitude in the limit where the CM energy is much smaller than the mass of the gauge boson M_X. In this limit the propagator can be written as $\mathcal{P} = -i/M_X^2$. This amplitude is equal to the matrix element of a dimension-six operator times some coefficient. Find the operator and the coefficient. What is needed to suppress the rate of such Baryon-number-violating processes?