Lecture 2:
Cosmic time. Doppler effect and scale factor. Robertson-Walker metric.

Review: Metric of flat space-time (Minkowski space).

Proper time: For any particle or object moving along a trajectory, the proper time is the time that would be measured by a watch riding with the object.

L: $c^2\, d\tau^2 \equiv c^2\, dt^2 - dx^2 - dy^2 - dz^2$. Note the notation $dx^2 = (dx)^2$.
The proper time is a Lorentz invariant, or a scalar. Any observer agrees on its value. We can also talk about proper distance. You can check this as an exercise with Lorentz transformations.
Minkowski space: in Newtonian physics, time is separated into past and future relative to an event. In relativity, relative to a given event, space-time is separated into past, future, and elsewhere.

Diagram of moving frame: light ray emitted from origin, reflecting at $x = L/2$, coming back to origin. Draw x', y' axes for moving observer, and same light rays for moving observer.

B: Doppler effect:

L: Now, we consider light being emitted by a moving source of velocity v, with wavelength λ, frequency $\nu = c/\lambda$, period $P_s = 1/\nu = \lambda/c$. Let us imagine we could see a watch ticking with period P in the source.

Q: What is the period according to the observer? $P = \gamma P_s$.

L: But the observed period is not this, because the light-travel time is changing as the source moves. During the time P, the source moves by vP, and it gets closer to the observer by $vP\cos\theta$. So, $P_{obs} = \gamma P_s(1 - v/c\cos\theta)$.

Examples: blueshift and redshift.

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L: Definition of cosmic time: First, we must define a time coordinate. In special relativity, time is relative and depends on the observer's motion, so what do we mean by age of the Universe? In an expanding universe filled with matter or some other form of energy, there is a special frame at every location: the frame where matter is at rest. The cosmological time t is the proper time measured by an observer at rest relative to the matter, since the time of the Big Bang.

Q: Remember the concept of comoving coordinates. Let us call x, y, z the comoving coordinates near an observer. Locally, the metric should be like Minkowski space. How do we write it in terms of the comoving coordinates?

$$c^2 d\tau^2 = c^2 dt^2 - a^2(t)[dx^2 + dy^2 + dz^2]\ . \tag{1}$$

L: This is valid only near the observer, we'll see later how this is generalized. For now, we can see the relation to redshift. Generally, we can normalize the scale factor so that it is unity at some present time: $a(t_0) = 1$.

Q: What is the rate of change with cosmic time of the proper distance $d(t)$ to a galaxy? $\dot{d}(t) = \dot{a}/a d$.

L: However, we do not observe galaxies at their present distance, but in the past light-cone.

Q: How do we find the cosmic time when a galaxy at position r emitted the light we receive now? $cdt/a(t) = dr$.

Important: Comoving distance to a source that emitted at time t_e, and we see at time t_0:

$$r = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} \,. \tag{2}$$

Q: Now, we relate the redshift to the scale factor: a galaxy at fixed position r emits two signals, at t_e and at $t_e + P_e$. The first signal is received at t_0 by us. When is the second signal received?

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+P_e}^{t_0+P_0} \frac{dt}{a(t)} \,; \qquad \int_{t_e}^{t_e+P_e} \frac{dt}{a(t)} = \int_{t_0}^{t_0+P_0} \frac{dt}{a(t)} \,. \tag{3}$$

This implies $P_0 = P_e/a(t_e)$. The same relation holds for wavelength, and the inverse one holds for frequency. So

$$1 + z = \frac{1}{a(t)} \,. \tag{4}$$

L: Another way to see that: consider a series of observers, they send signals along a direction on which there are many observers, such that the spacing between signals is the same as the spacing between observers. There is obviously the same density of observers and signals. So the wavelength must be stretched just like the scale factor.

L: Another way to see that: we can define "comoving" or conformal time: $d\eta = a(t)dt$. Then:

$$c^2 d\tau^2 = a^2(t)[c^2 d\eta^2 - dx^2 - dy^2 - dz^2] \,. \tag{5}$$

In the diagram $\eta - x$, it is clear that the conformal duration of emission and reception is the same. But real time gets stretched so that $dt = a(t)d\eta$.

L: Concept of geodesic and parallel transport in general relativity. In detail, the equation for geodesic is

$$\frac{d^2 x^\alpha}{ds^2} = \Gamma^\alpha_{\mu\lambda}\, x^\mu x^\lambda \,. \tag{6}$$

The Christoffel symbols are related to the metric, but they are not a tensor:

$$\Gamma^\alpha_{\mu\lambda} = \frac{1}{2} g^{\alpha\nu} \left(\frac{\partial g_{\nu\mu}}{\partial x^\lambda} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right) \,. \tag{7}$$

Intuitive understanding of geodesic, parallel transport: analogy to major circles in sphere, and carrying a gyroscope indicating a fixed direction.

L: Cosmological principle: the Universe is homogeneous and isotropic on large scales.

The Friedmann-Robertson-Walker metric is the most general metric for a universe that is homogeneous and isotropic. The cosmological principle combined with general relativity leads to the conclusion that there are only three possibilities for the geometry of the universe: closed, flat, or open. These correspond to positive, zero or negative curvature *for space alone*.

L: In general, the space part could be curved, but homogeneity then demands constant curvature. In general, the metric has to be spherically symmetric (because of isotropy) and could be of the form:

$$c^2 d\tau^2 = c^2 dt^2 - a^2(t)[f^2(r)dr^2 + \xi^2(r)(d\theta^2 + \sin^2\theta d\phi^2)] \,. \tag{8}$$

But we can always change to a new radial coordinate $dr_{new} = f(r)dr_{old}$, so let us set $f(r) = 1$.

Example: if space has the geometry of a sphere, then the space part of the metric is

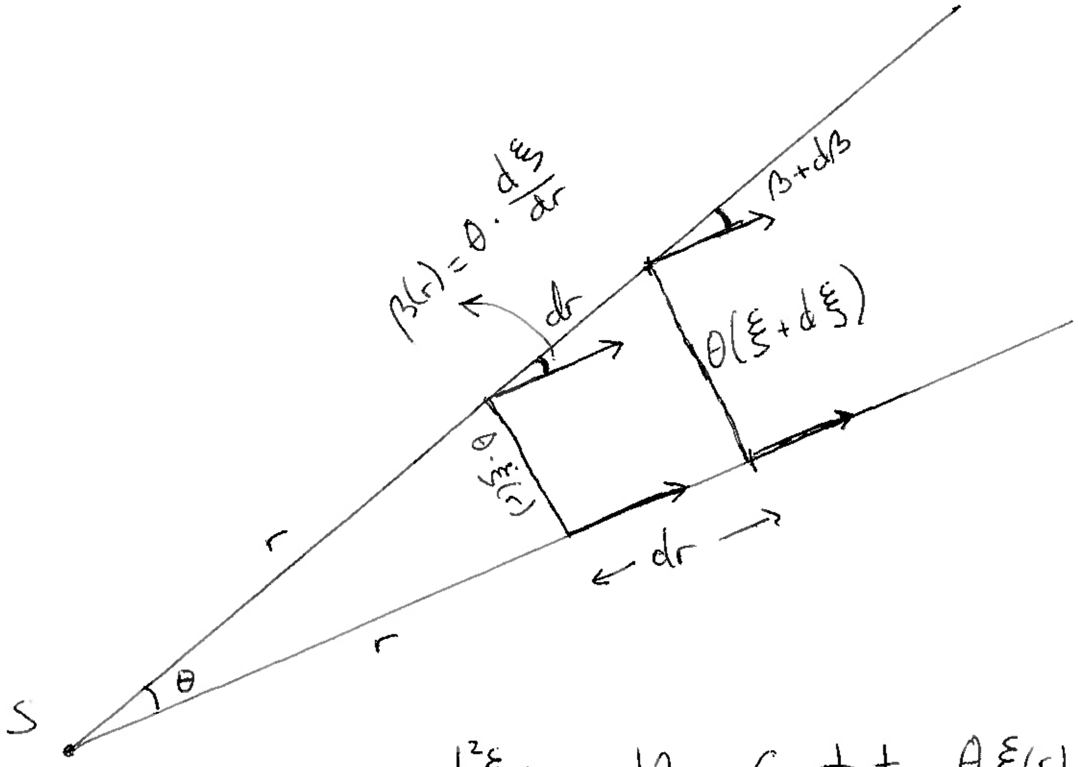
$$ds^2 = R^2(dr^2 + \sin^2 r d\Omega) \ . \tag{9}$$

and if space is flat then

$$ds^2 = (dr^2 + r^2 d\Omega) \ . \tag{10}$$

We now want to find what the most general possible metric is.

L: For that, let us consider two geodesics corresponding to two light rays coming out of a source, separated by a small angle θ. At the comoving distance r the separation between the two light-rays is $\theta\,\xi(r)$, and if we take a vector parallel to a light-ray and we parallel-transport it across to the other light-ray (maintaining the angle with a geodesic constant), they form an angle $\beta = \theta\,\partial\xi/\partial r$. In a homogeneous space of constant curvature, the change in β along the light-ray can only be proportional to the area as can be seen when we do a circuit around two areas.



$$\theta \cdot \frac{d^2\xi}{dr^2} dr = d\beta = \underbrace{\text{Constant}}_{= (\pm A^2)} \cdot \theta\,\xi(r) \cdot dr$$

So:

B-Q:

$$\frac{d^2\xi}{dr^2} dr = \pm A^2 \xi \, dr \ . \tag{11}$$

Q: What is the solution to this equation?

$$\text{minus sign} : \xi \propto \sin(Ar) ; \qquad A = 0 : \ \xi \propto r ; \qquad \text{plus sign} : \xi \propto \sinh(Ar) \ . \tag{12}$$

L: So the final solution is that the space can have positive or zero or negative constant curvature. We can more easily write this as:

$$c^2 d\tau^2 = c^2 dt^2 - a^2(t)[dr^2 + S_k^2(r)(d\theta^2 + \sin^2\theta d\phi^2)] \ , \tag{13}$$

where $S_k(r) = r$ for $k = 0$ (flat case), $S_k(r) = R\sin(r/R)$ for $k = 1$ (closed case), and $S_k(r) = R\sinh(r/R)$ for $k = -1$ (open case). We can also express this in terms of a new variable $x = S_k(r)$, and then $dx = S_k'(r)\, dr$, which gives $dr = dx/\sqrt{1 - kx^2/R^2}$, and:

$$c^2 d\tau^2 = c^2 dt^2 - a^2(t)[\frac{dx^2}{1 - kx^2/R^2} + x^2(d\theta^2 + \sin^2\theta d\phi^2)] \ , \tag{14}$$

Examples: Perimeter and radius of a circle, sum of angles of triangles.

Summary: The cosmological principle implies that the metric of the universe on a large scale can have only one of three solutions: open, flat, or closed geometry. The cosmic time t is the time measured by an observer at rest with the matter. Such a free-falling observer will have constant comoving coordinates. All the freedom that is left for our cosmological model (and all we need to find out from observations) is $a(t)$, k, and R.