Excercices Mescia

martes, 14 de junio de 2022 18:14

1)

1. *Earlier alternative to the $SU(2)\times U(1)$ gauge group.*

To succefully explain weak and electromagnetic interactions, we introduced the Intermediate Vector Boson Theory, $\mathcal{L}^{IVB}$. This theory involves 3 gauge bosons with the corresponding 3 Noether currents: 2 charged ones, $J^\mu_{weak} = \frac{1}{2}\bar{\nu}\gamma^\mu(1-\gamma_5)e$ and $(J^\mu_{weak})^\dagger$, and 1 neutral, $J^\mu_{em} = \bar{e}\gamma^\mu e$. Now the associated generators (charges), T_+, T_- and Q, do not define a closed algebra, namely

$$[T_+(t), T_-(t)] \neq Q,$$

and the first consequence of it is that the theory is not renormalizabe (even for $M_W = 0$): at one-loop new operators can come out, for example terms proportional to $[T_+(t), T_-(t)] = 2T_3$ are absent in the tree-level theory.

At class we solved this type of problem by introducing new Noether charges (so new gauge bosons). At this point we have been able to close the algebra but we ended up with an extended symetry group, $SU(2)\times U(1)$, namely 4 gauge bosons. Although nowdays this choice, namely the Standard Model, is well suported by data, at earlier times several alternatives were suggested. These alternatives are still useful exercises when you want to extend the SM above the M_Z scale, TeV scale LHC physics.

For example, we can make a closed algebra with the T_+, T_- and Q, namely

$$[T_+(t), T_-(t)] = aQ, \quad \text{amb } a \text{ constant}$$

by including new fermions, namely E^+ and N respectively new charged and neutral heavy lepton. In this case, we don't need new gauge bosons and the group is $SU(2)$ but we need new fermions.

Let's consider the following Lagrangian

$$\mathcal{L} = \bar{T}_L i\partial\!\!\!/ T_L + \bar{T}_R i\partial\!\!\!/ T_R + \bar{s}_R i\partial\!\!\!/ s_R, \qquad \text{with}$$

$$T_L = \begin{pmatrix} E_L^+ \\ \nu_L\cos\alpha + N_L\sin\alpha \\ e_L \end{pmatrix}, \quad T_R = \begin{pmatrix} E_R^+ \\ N_R \\ e_R \end{pmatrix}, \quad s_R = N_R\cos\alpha - \nu_R\sin\alpha$$

Here $\psi_L = \frac{1-\gamma_5}{2}\psi$ and $\psi_R = \frac{1+\gamma_5}{2}\psi$. Moreover e, ν, E and N are Dirac fields with $q_e = -q_E = -1$ and $q_\nu = q_N = 0$ electric charges respectively. Assuming $T_{L,R}$ and s_R respectively triplet and singlet of $SU(2)$, namely

$$T'_{L,R} = e^{i\alpha_i t^i} T_{L,R}, \quad \text{and} \quad s'_{L,R} = s_R$$

$$t^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad t^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad t^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

the theory is invariant under $SU(2)$ and $U(1)_{em}$.

- Please check that t^i define a (triplet) representation of $SU(2)$, namely they have the algebra of $SU(2)$ even though they are 3×3 matrix!
- Please write the $SU(2)$ and electric Noether currents of this theory and check that $[T_+(t), T_-(t)] \propto Q$, where $T_\pm = T_1 \pm iT_2$.
- Please comment why this choice is now fucked up!, which experimental does disfavour this option?. Please don't think much exotic!

a)

Let's check the commutators of the generators, to see if they fullfill the SU(2) algebra:

$$[t^1, t^2] = \frac{1}{2}\left[\begin{pmatrix} 0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix} 0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} - \begin{pmatrix} 0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\begin{pmatrix} 0&1&0\\1&0&1\\0&1&0\end{pmatrix}\right] = \frac{1}{2}\left[\begin{pmatrix} i&0&-i\\0&0&0\\i&0&-i\end{pmatrix} - \begin{pmatrix} -i&0&-i\\0&0&0\\i&0&i\end{pmatrix}\right] = \begin{pmatrix} i&0&0\\0&0&0\\0&0&-i\end{pmatrix} = i t^3$$

$$[t^3, t^1] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix} 0&1&0\\1&0&1\\0&1&0\end{pmatrix} - \begin{pmatrix} 0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix} 1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0&1&0\\0&0&0\\0&-1&0\end{pmatrix} - \begin{pmatrix} 0&0&0\\1&0&-1\\0&0&0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&1&0\\-1&0&1\\0&-1&0\end{pmatrix} = i t^2$$

$$[t^2, t^3] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\begin{pmatrix} 1&0&0\\0&0&0\\0&0&-1\end{pmatrix} - \begin{pmatrix} 1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix} 0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0&0&0\\i&0&i\\0&0&0\end{pmatrix} - \begin{pmatrix} 0&-i&0\\0&0&0\\0&-i&0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&i&0\\i&0&i\\0&i&0\end{pmatrix} = i t^1$$

so we have seen that:

$$\boxed{[t^i, t^j] = i\epsilon^{ijk} t^k \qquad \underline{\text{SU(2) algebra!}}} \quad ✓✓$$

b)

Applying the gauge transformations to the Lagrangian we get:

(T, T_R, T_L are particles
$T_{1,2,3}$ or $T_{+,-}$ are charges
t_i are SU(2) generators)

$$\mathcal{L} = \bar{T}_L i \not{\partial} T_L + \bar{T}_R i \not{\partial} T_R + \bar{S}_R i \not{\partial} S_R$$

$\underline{\text{SU(2) transformation}}$

$$\mathcal{L}' = \mathcal{L} - \bar{T}_L (\not{\partial}\alpha_i) t^i T_L - \bar{T}_R (\not{\partial}\alpha_i) t^i T_R = \mathcal{L} + \alpha_i \partial_\mu \left[\bar{T}_L \gamma^\mu t^i T_L + \bar{T}_R \gamma^\mu t^i T_R\right] - \text{total derivative}$$

(⊛ $\underline{\text{For posterior comparison, here would be a U(1) transformation:}}$

$$\mathcal{L}' = \mathcal{L} - \sum_{\text{charged } \psi} \bar{\psi}_i (\not{\partial}\beta) q_\psi \psi_i = \mathcal{L} + \beta\, \partial_\mu \left(\sum_{\text{charged } \psi} q_\psi \bar{\psi}_i \gamma^\mu \psi_i\right) - \text{total derivative}$$)

from the SU(2) transformations we see that the Noether currents are:

$$J_w^{\mu i} = \bar{T}_L \gamma^\mu t^i T_L + \bar{T}_R \gamma^\mu t^i T_R$$

$$\hookrightarrow J_w^{0i} = T_L^\dagger t^i T_L + T_R^\dagger t^i T_R = T^\dagger t^i T \Rightarrow \begin{cases} T_1 \equiv \int J_w^{01} d^3x \\ T_2 \equiv \int J_w^{02} d^3x \\ T_3 \equiv \int J_w^{03} d^3x \equiv \boxed{Q} \end{cases}$$

$$\boxed{T_+ = T_1 + iT_2} \qquad \left(t_1 + it_2 = \sqrt{2}\begin{pmatrix} 0&1&0\\0&0&1\\0&0&0\end{pmatrix}\right)$$

$$\boxed{T_- = T_1 - iT_2} \qquad \left(t_1 - it_2 = \sqrt{2}\begin{pmatrix} 0&0&0\\1&0&0\\0&1&0\end{pmatrix}\right)$$

(⊛ We will see that this third component actually corresponds to the typical U(1) charge!)

so we will then have: $\left[e_L^\dagger (\nu_e \cos\alpha + N\sin\alpha)_L + (\nu_e^\dagger \cos\alpha + N^\dagger \sin\alpha)_L E_L^+ + e_R^\dagger N_R + N_R^\dagger E_R^+\right] d^3x$

$$T_+ = \frac{\sqrt{2}}{2}\int \left(T_L^\dagger \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0\end{pmatrix} T_L + T_R^\dagger \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0\end{pmatrix} T_R\right) d^3x = \frac{1}{\sqrt{2}}\int \left(T_L^{\uparrow\dagger} T_L^0 + T_L^{0\dagger} T_L^{\downarrow} + T_R^{\uparrow\dagger} T_R^0 + T_R^{0\dagger} T_R^{\downarrow}\right) d^3x$$

$$= \frac{1}{\sqrt{2}}\int \left(E_L^{+\dagger}(\nu_{e_L} \cos\alpha + N_L \sin\alpha) + (\nu_{e_L}^\dagger \cos\alpha + N_L^\dagger \sin\alpha) e_L + E_R^{+\dagger} N_R + N_R^\dagger e_R\right) d^3x$$

$$T_- = \frac{\sqrt{2}}{2}\int \left(T_L^\dagger \begin{pmatrix} 0&0&0\\1&0&0\\0&1&0\end{pmatrix} T_L + T_R^\dagger \begin{pmatrix} 0&0&0\\1&0&0\\0&1&0\end{pmatrix} T_R\right) d^3x = \frac{1}{\sqrt{2}}\int \left(T_L^{0\dagger} T_L^{\uparrow} + T_L^{\downarrow\dagger} T_L^0 + T_R^{0\dagger} T_R^{\uparrow} + T_R^{\downarrow\dagger} T_R^0\right) d^3x$$

$$= \frac{1}{\sqrt{2}}\int \left(e_L^\dagger (\nu_{e_L} \cos\alpha + N_L \sin\alpha) + (\nu_{e_L}^\dagger \cos\alpha + N_L^\dagger \sin\alpha) E_L^+ + e_R^\dagger N_R + N_R^\dagger E_R^+\right) d^3x \quad \left(= T_+^\dagger \ \underline{✓✓ \text{ checks}}\right)$$

$$Q = \int \left(E_L^{+\dagger} E_L^+ - e_L^\dagger e_L + E_R^{+\dagger} E_R^+ - e_R^\dagger e_R\right) d^3x = \int \left(E^{+\dagger} E^+ - e^\dagger e\right) d^3x$$

(⊛ Where by comparison with the U(1) transf. we see that E^+, e^- would be the charged particles under the U(1) symmetry!)

Finally we have to show that $[T_+(t), T_-(t)] \propto Q(t)$:

$$\boxed{[T_+, T_-]} = [T_1 + iT_2, T_1 - iT_2] = -i[T_1, T_2] + i[T_2, T_1] = -i2\overbrace{[T_1, T_2]}^{iT_3 \text{ (from a))}} = 2T_3 = \boxed{2Q}$$

or

$$\boxed{[T_+, T_-]} = \frac{1}{2}\int T^\dagger [t_+, t_-]\, T\, d^3x = \frac{1}{2}\int T^\dagger \sqrt{2}^2 \left[\begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\1&0&0\\0&1&0 \end{pmatrix} - \begin{pmatrix} 0&0&0\\1&0&0\\0&1&0 \end{pmatrix}\begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}\right] T\, d^3x =$$

$$= \int T^\dagger \left[\begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0\\0&1&0\\0&0&1 \end{pmatrix}\right] T\, d^3x = \int T^\dagger \begin{pmatrix} 1&0&0\\0&0&0\\0&0&-1 \end{pmatrix} T\, d^3x = \boxed{2Q}$$

c)

In the previous sections we have seen that in such a model the only neutral current is the electromagnetic current, so the discovery of weak neutral currents ruled this model out!

The experiment that discovered this was presented in a seminar at CERN on 1973, where from the results they concluded that the Z^0 boson must exist!

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2)

2. Unitarity Violation in Gauge Theories.

Consider the IVB Lagragian

$$\mathcal{L}_{IVB} = -\frac{1}{2}W^{\mu\nu}W^\dagger_{\mu\nu} + m^2 W^\mu W^+_\mu + \bar{\ell}_L \not{D} \ell_L$$

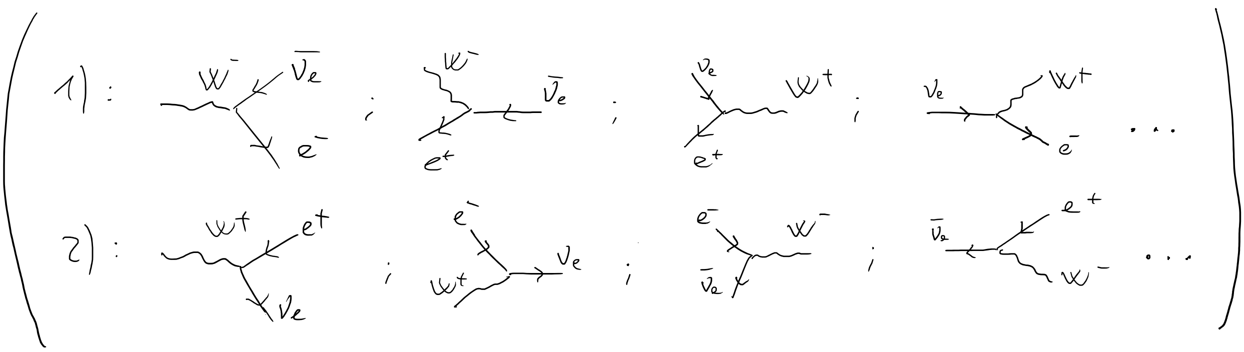
where $D_\mu = \partial_\mu + ig_w W^+_\mu \tau^- + ig_w W^-_\mu \tau^+$, ℓ is the letpon doublet. Consider the amplitude of $\sigma(\nu + \bar{\nu} \to W^+_L + W^-_L)$ where W_L indicates a longitudinally-polarized W.

- Check if the theory has unitarity violation, namely $\lim_{s\to\infty} s\sigma(s) \to \infty$.
- Please work out a new interaction to cancel the possible unitarity violation. First, write down the effective operator you need to preserve unitarity and finally work out a renormalized theory including the possible gauge interaction which can generate this effective operator.

a)

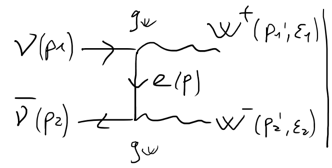
$$\mathcal{L}_{int} = \bar{\ell}_L(\not{D} - \not{\partial})\ell_L \underset{\uparrow}{=} ig_w(\bar{\nu}_e, e^+)_L[\not{W}^+ t^- + \not{W}^- t^+]\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \underset{\uparrow}{=} ig_w[\overset{1)}{\not{W}^+ e^-_L \bar{\nu}_e} + \overset{2)}{\not{W}^- \nu_e e^+_L}]$$

$$\left(\ell_L = \begin{pmatrix}\nu_{e_L}\\ e^-_L\end{pmatrix};\ \bar{\ell}_L = \begin{pmatrix}\bar{\nu}_{eL}\\ e^+_L\end{pmatrix}\right) \qquad \left(t^+ = \begin{pmatrix}0&0\\1&0\end{pmatrix};\ t^- = \begin{pmatrix}0&1\\0&0\end{pmatrix}\right)$$



Consider the reaction $\nu\bar{\nu} \to W^+W^-$, with both W's being longitudinally polarized:

$$\sigma\left(\nu+\bar{\nu} \longrightarrow W_L^+ + W_L^-\right) \propto \left|\mathcal{M}_e(\nu+\bar{\nu} \to W_L^+ + W_L^-)\right|^2 = \left|\quad\right|^2 \sim \left|\frac{g_W^2}{p - m_e}\right|^2$$



More exactly:

$$\mathcal{M}_e(\nu+\bar{\nu} \longrightarrow W_L^+ + W_L^-) = -i\,\bar{\nu}_L(p_2)\,(-ig_W)\,\not{\varepsilon}_2\,\frac{i4}{\not{p} - m_e}\,\not{\varepsilon}_1\,(-ig_W)\,\nu_L(p_1) =$$

$$= -g_W^2\,\bar{\nu}(p_2)\,(1-\gamma^5)\,\frac{\not{\varepsilon}_2(\not{p}-m_e)\not{\varepsilon}_1}{p^2 - m_e^2}\,(1-\gamma^5)\,\nu(p_1) =$$

$$= -2g_W^2\,\bar{\nu}(p_2)\,\frac{\not{\varepsilon}_2(\not{p}-m_e)\not{\varepsilon}_1}{p^2-m_e^2}\,(1-\gamma^5)\,\nu(p_1)$$

And because, $W^{+/-}$ are longitudinals, in the high energy limit we have:

$$\varepsilon_{1or2} = \varepsilon_{1or2}^{(3)} = \frac{(p'_{1or2}, 0, 0, E'_{1or2})}{M_W} \xrightarrow[s\to\infty]{} p'_{1or2} \approx E'_{1or2} - \frac{M_W^2}{2E'_{1or2}} + \dots \longrightarrow \varepsilon_{1or2\,\mu} \approx \frac{p'_{1or2\,\mu}}{M_W} + O(M_W/E_{1or2})$$

So in the high energy limit, writting $p = p_1 - p_1' = p_2' - p_2$, we will have:

$$(\not{p}_1 - \not{p}_1')\not{p}_1'\,\nu(p_1) = (2p_1\cdot p_1' - p_1'^2)\,\nu(p_1)$$

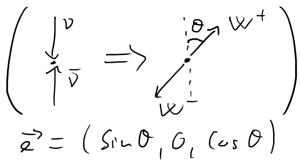
$$\mathcal{M}_e(\nu+\bar{\nu} \longrightarrow W_L^+ + W_L^-) \approx -\frac{2g_W^2}{M_W^2}\,\bar{\nu}(p_2)\,\frac{\not{p}_2'(\not{p}_1 - \not{p}_1' - m_e)\not{p}_1'}{(p_1-p_1')^2 - m_e^2}\,(1-\gamma^5)\,\nu(p_1) =$$

$$= -\frac{2g_W^2}{M_W^2}\,\bar{\nu}(p_2)\,\frac{\not{p}_2'(2p_1p_1' - p_1'^2)}{\cancel{p_1^2} + p_1'^2 - 2p_1p_1' - \cancel{m_e^2}}\,(1-\gamma^5)\,\nu(p_1) =$$

$$= \frac{2g_W^2}{M_W^2}\,\bar{\nu}(p_2)\,\not{p}_2'\,(1-\gamma^5)\,\nu(p_1)$$

Going to to the CM frame of reference $\left(E_1 = E_2 \equiv E \text{ with } \vec{p}_1 = -\vec{p}_2\right)$:

$$\begin{cases} p_1 = (E, 0, 0, E) \\ p_2 = (E, 0, 0, -E) \end{cases} \quad\text{and}\quad \begin{cases} p_1' = (E, 0, 0, p'\vec{e}) \\ p_2' = (E, 0, 0, -p'\vec{e}) \end{cases} \quad\text{where}$$



$\vec{e} = (\sin\theta, 0, \cos\theta)$

And because ν and $\bar{\nu}$ have opposite helicities $\left(\chi_{1/2} = \binom{1}{0} \text{ and } \chi_{-1/2} = \binom{0}{1}\right)$:

$$\bar{\nu}(p_2)\,\not{p}_2'\,(1-\gamma^5)\,\nu(p_1) = \underbrace{\sqrt{E}\,\chi_{1/2}^\dagger\left(\overbrace{\frac{\vec{\sigma}\cdot\vec{p}_2}{E}}^{-\sigma_z}, -1\right)}_{\bar{\nu}(p_2)}\underbrace{\begin{pmatrix} E & p'\vec{\sigma}\cdot\vec{e} \\ -p'\vec{\sigma}\cdot\vec{e} & -E \end{pmatrix}}_{\not{p}_2'}\underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{(1-\gamma^5)}\underbrace{\sqrt{E}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}_1}{E} \end{pmatrix}\chi_{-1/2}}_{\nu(p_1)} =$$

$$= -E\,\chi_{1/2}^\dagger\,4\,(E - p'\vec{\sigma}\cdot\vec{e})\,\chi_{-1/2} = 4Ep'\sin\theta$$

$$\left(-4E\left(E\,\cancel{\chi_{1/2}^\dagger\chi_{-1/2}}^{\,0} - p'\left[\sin\theta\,\overbrace{\chi_{1/2}^\dagger\sigma_x\chi_{-1/2}}^{1} + \cos\theta\,\overbrace{\chi_{1/2}^\dagger\sigma_z\chi_{-1/2}}^{0}\right]\right)\right)$$

So finally we have obtained: $\left(\frac{g_W^2}{M_W^2} = \frac{G_F}{\sqrt{2}} \text{, fermi coupling} \right)$

$$\boxed{\mathcal{M}_e(\nu+\bar\nu \to W_L^+ + W_L^-) = \frac{2 g_W^2}{M_W^2} 4 E p' \sin\theta \xrightarrow[s\to\infty]{} 8 g_W^2 E^2/M_W^2 \sin\theta = \frac{8}{\sqrt{2}} G_F E^2 \sin\theta \sim s}$$

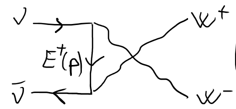
$$\boxed{|\mathcal{M}(\nu\bar\nu \to W^+W^-)|^2 \xrightarrow[s\to\infty]{} 64 \frac{g_W^4}{M_W^4} E^4 \sin^2\theta = 32 G_F^2 E^4 \sin\theta \sim s^2}$$

So obviously we have a UV problem because:

$$\boxed{\lim_{s\to\infty} s\cdot\sigma(s) \propto s\cdot|\mathcal{M}(s)|^2 \propto s^3 \longrightarrow \infty\ !!!}$$

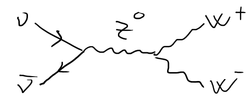
b)

u-channel:



(heavy-lepton alternative)

s-channel:



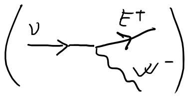
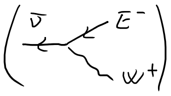
(neutral vector boson altern.)

To cancel this bad high-energy behaviour we need other diagrams at the same orders for this interaction. For this there are two possibilities:
(Another t channel diagram won't help since it will have the same sign)

- u-channel:

The heavy-lepton alternative would work with the effective operators:

$\bar\nu E^+ W^-$ and $\nu E^- W^+$

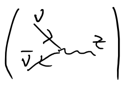
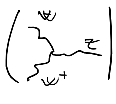
 

with the same coupling $g_{E^+} = g_W$ (It's the same diagram)

- s-channel:

The neutral vector boson alternative need of the term:

$\nu\bar\nu Z$ and $W^+W^- Z$

but in this case, we need to check how strong the coupling has to be:

$$\triangleright\ -\mathcal{M}_e(\nu+\bar\nu \to W_L^- + W_L^+) = -\frac{2 g_W^2}{M_W^2} \bar v(p_2)\, \not{p}_2' (1-\gamma^5)\, \nu(p_1)$$

$$\triangleright\ \mathcal{M}_Z(\nu+\bar\nu \to W_L^- + W_L^+) = -i\, \bar v_L(p_2)(-i g_{Z\nu})\, \nu_L(p_1)\, L_{\alpha\mu\nu}\, \varepsilon^\mu(p_1')\, \varepsilon^\nu(p_2') \frac{\quad}{(p_1+p_2'')^2 - M_Z^2}$$

$$= -\frac{g_{Z\nu}\, g_{ZW}}{M_Z^2} \bar v(p_2)\, \not{p}_2' (1-\gamma^5)\, \nu(p_1)$$

$$\boxed{g_{Z\nu}\, g_{ZW} = 2 g_W^2}$$

$$= - \frac{\cdots}{M_w^2}\, \bar{v}(p_2)\, \not{p}_2 \,(1-\gamma^5)\, v(p_1)$$

(And chosing the W^+W^-Z term to have Yang Mills structure, the $L_{\alpha\mu\nu}$ has to be:

$$L_{\alpha\mu\nu} = -i g_{zw}\left((p_2'-p_1')_\alpha g_{\mu\nu} - (2p_2'+p_1')_\nu g_{\alpha\mu} + (2p_1'+p_2')_\mu g_{\alpha\nu} \right)$$
)

So finally using the neutral vector boson, we arrive to the Lagrangian:

$$\mathcal{L}_{int} = i g_w \left[\not{W}^+ e_L^- \bar{\nu}_e + \not{W}^- \nu_e e^+ \right] + i g_{Z\nu} \nu\bar{\nu} \not{Z} + i g_{Ze} e\bar{e} \not{Z} + g_{ZW} W^+ W^- \not{Z}$$

Which we can identify with the interaction Lagrangian of the $SU(2)\times U(1)$ gauge theory, as we now are going to show to end this excercice.

First, the interaction Lagrangian given by the covariant derivative of the gauge manifold is:

$$\mathcal{L}_{SU(2)\times U(1)_{int}} = \bar{\ell}_L \left(\frac{g'}{2} \not{A} Y + \frac{g}{2} \vec{\not{B}} \cdot \vec{\tau} \right) \ell_L \quad \text{where} \quad \begin{cases} W_\mu^\pm = \dfrac{B_\mu^1 \mp i B_\mu^2}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} (\overbrace{1, \mp i, 0}^{B_\mu}, \overbrace{0}^{A_\mu})_{\vec{B}, A} \\ Z_\mu = \dfrac{-g' A_\mu + g B_\mu^3}{\sqrt{g^2+g'^2}} = \dfrac{1}{\sqrt{g^2+g'^2}} (0, 0, g, -g')_{\vec{B}, A} \\ \gamma_\mu = \dfrac{g A_\mu + g' B_\mu^3}{\sqrt{g^2+g'^2}} = \dfrac{1}{\sqrt{g^2+g'^2}} (0, 0, g', g)_{\vec{B}, A} \end{cases}$$

massive; massless

(where $Y, \vec{\tau}$ are the generators of $U(1)$ and $SU(2)$) (which obtain their masses with the Higgs mechanism)

Which checks outs because:

$$\vec{\not{B}} \cdot \vec{\tau} = \begin{pmatrix} 0 & i\not{B}_1 \\ i\not{B}_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\not{B}_2 \\ \not{B}_2 & 0 \end{pmatrix} + \begin{pmatrix} iB_3 & 0 \\ 0 & -iB_3 \end{pmatrix} = i \begin{pmatrix} \not{B}_3 & \not{B}_1 + i\not{B}_2 \\ \not{B}_1 - i\not{B}_2 & -\not{B}_3 \end{pmatrix} \sim \begin{pmatrix} (Z,\gamma)_1 & W^- \\ W^+ & (Z,\gamma)_2 \end{pmatrix}$$

[where the diagonal terms (which are some combinations of Z and γ) combined with the $U(1)$ generator gives the correct int. also.]

$$\bar{\ell}_L \vec{\not{B}}\vec{\tau}\, \ell \sim (\bar{e}\ \bar{\nu}_e)_L \begin{pmatrix} (Z,\gamma)_1 & W^- \\ W^+ & (Z,\gamma)_2 \end{pmatrix} \begin{pmatrix} e \\ \nu_e \end{pmatrix} \sim \bar{e}_L W^- \nu_{e_L} + \bar{\nu}_{e_L} W^+ e_L + \bar{e}_L (Z,\gamma)_1 e_L + \bar{\nu}_e (Z,\gamma)_2 \nu_e \quad \checkmark\checkmark$$

And the last term that we need (ZW^+W^-), is also included in this $SU(2)\times U(1)$ gauge theory, but in the gauge Kinetic term because of it's non-abelian nature:

$$\mathcal{L}_{SU(2)\times U(1)_{Kin}} = -\frac{1}{4} F_{\mu\nu}^{\ell} F^{\ell\,\mu\nu} - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} \quad \text{where} \quad \begin{matrix} F_{\mu\nu}^{\ell} = \partial_\nu B_\mu^\ell - \partial_\mu B_\nu^\ell + g\, \varepsilon_{ij\ell} B_\mu^i B_\nu^j & (SU(2) \text{ part}) \\ f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu & (U(1) \text{ part}) \end{matrix}$$

$\longrightarrow$ which gives Interactions

$$\begin{cases} \partial B^\ell \cdot \varepsilon_{ij\ell}\, g (B^i \cdot B^j) \sim ZW^+W^- \text{ and } \gamma W^+ W^- \quad \checkmark\checkmark \\ \varepsilon_{ij\ell}\varepsilon_{i'j'\ell}\, g (B^i \cdot B^j)\, g (B^{i'} \cdot B^{j'}) \sim W^+W^+W^-W^-,\ W^+W^-ZZ,\ W^+W^-\gamma\gamma \text{ and } W^+W^-\gamma Z \end{cases}$$

So finally we see that <u>the $SU(2)\times U(1)$ gauge theory contains all the necessary interactions, and it also predicts even more, which has been later checked !!!</u>

3)

3. $SU(2)$ **gauge group and Alternative universe.**

Once Upon a Time there was a world invariant under the non-abelian $SU(2)$ gauge group. The $SU(2)$ group is spontaneously broken as $SU(2) \to U(1)_{em}$ and the two gauge bosons $W^{1,2}$ get mass (whereas the $W^3 = \gamma$ keeps massless).

Please, introduce a Higgs field to generate the correct mass pattern of the gauge fields.

The initial unbroken gauge group will contribute as:

$$\vec{W}\vec{\tau} = W_1 \cdot \tau_1 + W_2 \tau_2 + W_3 \tau_3 = i \begin{pmatrix} W_3 & W_1 + i W_2 \\ W_1 - i W_2 & -W_3 \end{pmatrix}$$

Let's introduce now the Higgs field, and check it's contribution and see how the vaccum expectation value (VEV, $\hat{\phi} = \phi - \phi_0$) changes the coupling terms, depending in which representation this Higgs fields transforms:

a) Scalar field in the fundamental of SU(2) [spinor representation]:

If ϕ is in the fundamental the covariant derivative will be:

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha - ig A_\mu^a T^a_{fund\,\alpha\kappa} \phi_\kappa = \partial_\mu \phi_\alpha - i \frac{g}{2} A_\mu^a \sigma^a_{\alpha\kappa} \phi_\kappa$$

When we add the VEV ($\hat{\phi} = \phi - \phi_0$), the coupling terms becomes:

$$|D_\mu \phi|^2 = |D_\mu \hat{\phi}|^2 + (D_\mu \phi_0)^\dagger (D^\mu \hat{\phi}) + (D_\mu \hat{\phi})^\dagger (D^\mu \phi_0) + \underline{|D_\mu \phi_0|^2}$$

Where the only term that contains only 2 gauge bosons (future mass term) is the last one:

$$|D_\mu \phi_0|^2 = g^2 \phi_0^\dagger T^a T^b \phi_0 A_\mu^a A^{\mu b} = g^2 \phi_0^\dagger \frac{T^a T^b + T^b T^a}{2} \phi_0 A_\mu^a A^{\mu b} =$$

$$= \frac{1}{4} g^2 \phi_0^\dagger \delta^{ab} \phi_0 A_\mu^a A^{\mu b} = \frac{g^2}{4} |\phi_0|^2 (A_\mu^a)^2 \quad \text{(Symetrizing)} \quad \left(\{T^a, T^b\} = \tfrac{1}{2}\delta^{ab}\right)$$

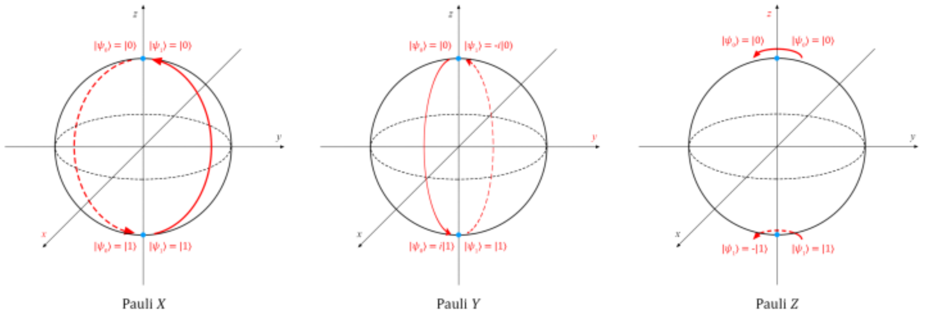
Which compared with a typical mass term $\left(\frac{1}{2} m_{ab}^2 A_\mu^a A^{\mu b}\right)$:

$$m_a^2 = 2\left(\frac{g^2}{4} |\phi_0|^2\right) \longrightarrow \boxed{m_a = \frac{g|\phi_0|}{\sqrt{2}} \quad \forall a}$$

doesn't work for us !!!

So all Gauge boson adquire a mass independently of the chosen VEV.

Which can be seen more easily, with the actions of the generators (algebra) and with the action of the group basis:



[Effects of the generators of the fundamental SU(2)]

Algebra action:

$$\left.\begin{array}{l} i\frac{\sigma_x}{2}\begin{pmatrix}a\\b\end{pmatrix}=\frac{i}{2}\begin{pmatrix}b\\a\end{pmatrix} \\ i\frac{\sigma_y}{2}\begin{pmatrix}a\\b\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-b\\a\end{pmatrix} \\ i\frac{\sigma_z}{2}\begin{pmatrix}a\\b\end{pmatrix}=\frac{i}{2}\begin{pmatrix}a\\-b\end{pmatrix} \end{array}\right\} \quad \boxed{\begin{array}{c}T_i\,\Phi_0\neq 0\\ (\forall\,\Phi_0\neq 0)\end{array}} \Longrightarrow$$

Group basis action:

$$\left\{\begin{array}{l} SU(2)_{g_1}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}b\\-a\end{pmatrix}\longrightarrow \left.\begin{array}{l}a=b\\b=-a\end{array}\right\}\ \Phi_0=0 \\ SU(2)_{g_2}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0&i\\i&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}ib\\ia\end{pmatrix}\longrightarrow \left.\begin{array}{l}a=ib\\b=ia\end{array}\right\}\ \Phi_0=0 \\ SU(2)_{g_3}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}i&0\\0&-i\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}ia\\-ib\end{pmatrix}\longrightarrow \left.\begin{array}{l}a=ia\\b=-ib\end{array}\right\}\ \Phi_0=0 \end{array}\right.$$

No element of SU(2) (group), in the fundamental has eigen value 1: $\boxed{\begin{array}{c}SU(2)\cdot\Phi_0\neq\Phi_0\\(\forall\Phi)\end{array}}$

<u>All 3 generators are SSB in the fundamental!</u>

b) <u>Scalar field in the adjoint [vector representation]:</u>

If ϕ is in the adjoint the covariant derivative will be: $\left(T^a_{adj\,\alpha\kappa}=-if^{a\alpha\kappa}=-i\,\epsilon_{a\alpha\kappa}\right)$

$$D_\mu\Phi_\alpha=\partial_\mu\Phi_\alpha-ig\,A^a_\mu\,T^a_{adj\,\alpha\kappa}\,\Phi_\kappa=\partial_\mu\Phi_\alpha-g\,\epsilon_{a\alpha\kappa}\,A^a_\mu\,\Phi_\kappa$$

When we add the VEV $(\hat\Phi\equiv\phi-\Phi_0)$, the coupling terms becomes:

$$|D_\mu\Phi|^2=|D_\mu\hat\Phi|^2+(D_\mu\Phi_0)^\dagger(D^\mu\hat\Phi)+(D_\mu\hat\Phi)^\dagger(D^\mu\Phi_0)+\underline{|D_\mu\Phi_0|^2}$$

Where the only term that contains only 2 gauge bosons (future mass term) is the last one:

$$|D_\mu\Phi_0|^2=g^2\,\Phi_0^\dagger\,T^a\,T^b\,\Phi_0\,A^a_\mu A^{\mu b}=g^2\,\Phi^\dagger_{0\kappa}\,\overbrace{\epsilon_{a\alpha\kappa}\,\epsilon_{b\alpha\ell}}^{\delta_{ab}\delta_{\kappa\ell}-\delta_{a\ell}\delta_{\kappa b}}\,\Phi_{0\ell}\,A^a_\mu A^{\mu b}=$$

$$=g^2\left(\delta_{ab}|\Phi_0|^2-\Phi^\dagger_{0b}\,\Phi_{0a}\right)A^a_\mu A^{\mu b}\ \xrightarrow[\text{the } i \text{ direction}]{\text{if we set } \Phi_0 \text{ in}}\ =g^2\,|\Phi_0|^2\,A^a_\mu A^{\mu a}\quad\text{for } a\neq i$$

Which compared with a typical mass term $\left(\frac{1}{2}m_{ab}^2\,A^a_\mu A^{\mu b}\right)$:

$$m_{a\neq i}^2=2g^2|\Phi_0|^2\longrightarrow\boxed{\left\{\begin{array}{l}m_{a\neq i}=\sqrt{2}\,g\,|\Phi_0|\\ m_i=0\end{array}\right.}$$

Works for us!!!

<u>So all Gauge boson except the direction of the VEV gained mass in the adjoint (vector repre.)!</u>

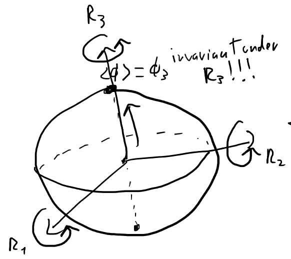
Which can be seen more easily, with the actions of the generators (algebra) and with the action of the group basis:

<u>Algebra action:</u>

$$\begin{cases} L_1\phi_0 = \begin{pmatrix} 0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\-c\\b\end{pmatrix} \longrightarrow L_1\phi_0 = 0 \text{ if } \phi_0 \text{ points in } \vec{1} \text{ direction } \phi_0 = \begin{pmatrix}v\\0\\0\end{pmatrix} \\ L_2\phi_0 = \begin{pmatrix} 0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}c\\0\\-a\end{pmatrix} \longrightarrow L_2\phi_0 = 0 \text{ if } \phi_0 \text{ points in } \vec{2} \text{ direction } \phi_0 = \begin{pmatrix}0\\v\\0\end{pmatrix} \\ L_3\phi_0 = \begin{pmatrix} 0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-b\\a\\0\end{pmatrix} \longrightarrow L_3\phi_0 = 0 \text{ if } \phi_0 \text{ points in } \vec{3} \text{ direction } \phi_0 = \begin{pmatrix}0\\0\\v\end{pmatrix} \end{cases}$$

$$\Downarrow$$

<u>Group basis action:</u>



$$\begin{cases} R_1\begin{pmatrix}0\\0\\v\end{pmatrix} \neq \begin{pmatrix}0\\0\\v\end{pmatrix} \\ R_2\begin{pmatrix}0\\0\\v\end{pmatrix} \neq \begin{pmatrix}0\\0\\v\end{pmatrix} \\ R_3\begin{pmatrix}0\\0\\v\end{pmatrix} = \begin{pmatrix}0\\0\\v\end{pmatrix} \end{cases} ; \quad \begin{cases} R_1\begin{pmatrix}0\\v\\0\end{pmatrix} \neq \begin{pmatrix}0\\v\\0\end{pmatrix} \\ R_2\begin{pmatrix}0\\v\\0\end{pmatrix} = \begin{pmatrix}0\\v\\0\end{pmatrix} \\ R_3\begin{pmatrix}0\\v\\0\end{pmatrix} \neq \begin{pmatrix}0\\v\\0\end{pmatrix} \end{cases} ; \quad \begin{cases} R_1\begin{pmatrix}v\\0\\0\end{pmatrix} = \begin{pmatrix}v\\0\\0\end{pmatrix} \\ R_2\begin{pmatrix}v\\0\\0\end{pmatrix} \neq \begin{pmatrix}v\\0\\0\end{pmatrix} \\ R_3\begin{pmatrix}v\\0\\0\end{pmatrix} \neq \begin{pmatrix}v\\0\\0\end{pmatrix} \end{cases}$$

Only the rotation in the same axis than $\langle\phi\rangle$ leave it invariant!

<u>All the generators will SSB, except the one in the direction we move the vacua!</u>

So we see that <u>for obtaining the correct mass pattern we need the Higgs to be in the adjoint/vector representation (3 components), and then that the vacuum is in the 3rd component!</u>

Which in most of the literature is written as: $\boxed{\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\v\end{pmatrix}_{adj}}$

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4)

4. Higgs as SU(2) septuplet? or as even higher SU(2) representation?

Let us consider a $SU(2)\otimes U(1)_Y$ model with an Higgs multiplet of isospin I and hypercharge $Y = Q - I_3$, where $I_3 = -I, \cdots, +I$ and Q the electric charge.

From the $SU(2)\otimes U(1)_Y$ covariant derivative, obtain the W and Z masses with I and Y general. ~~Determine I and Y to satisfy the~~ experimental constraint

$$\rho = \frac{M_W^2}{M_Z^2 c_W^2} = 1 \qquad \text{with} \quad c_W^2 = \frac{g^2}{g^2 + (g')^2}$$

with g and g' are the $SU(2)_L$ and $U(1)_Y$ couplings, respectively.

Initial data:

Higgs multiplet: $|I, Y\rangle_R$; $\rho = \dfrac{M_W^2}{M_Z^2 \cos\theta_W^2} = \dfrac{M_W^2}{M_Z^2}\dfrac{g^2+g'^2}{g^2}$ $\begin{cases} g \sim SU(2)_L \\ g' \sim U(1)_Y \end{cases}$; $\begin{cases} W_\mu^\pm = \dfrac{B_\mu^1 \mp i B_\mu^2}{\sqrt{2}} \\ Z_\mu = \dfrac{-g' A_\mu + g B_\mu^3}{\sqrt{g^2+g'^2}} = -S_W A_\mu + C_W B_\mu \\ \gamma_\mu = \dfrac{g A_\mu + g' B_\mu^3}{\sqrt{g^2+g'^2}} = C_W A_\mu + S_W B_\mu \end{cases}$

If ϕ is a complex scalar the covariant derivative will be:

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha - i g B_\mu^a T^a_{R\,\alpha\kappa}\phi_\kappa - i g' A_\mu \frac{Y}{2}\phi_\alpha$$ (where T_R is the SU(2) generator in the R representation)

When we add the VEV ($\hat{\phi} \equiv \phi - \phi_0$), the coupling terms becomes:

$$|D_\mu \phi|^2 = |D_\mu\hat{\phi}|^2 + (D_\mu\phi_0)^\dagger(D^\mu\hat{\phi}) + (D_\mu\hat{\phi})^\dagger(D^\mu\phi_0) + \underline{|D_\mu\phi_0|^2}$$

Where the only term that contains only 2 gauge bosons (future mass term) is the last one:

$$|D_\mu\phi_0|^2 = g^2\phi_0^\dagger T_R^a T_R^b \phi_0 B_\mu^a B^{\mu b} + g'^2\frac{Y^2}{4}A_\mu^2|\phi_0|^2 + gg' B_\mu^a A_\mu \frac{Y}{2}[T_R^a,\phi]\phi$$

$$\left(gg' B_\mu^a \underset{\overset{\|}{(-T^a_{R\,\alpha\kappa})}}{T^{a\dagger}_{R\,\alpha\kappa}}\phi_\kappa^\dagger A_\mu \frac{Y}{2}\phi_\alpha + gg' A_\mu\frac{Y}{2}\phi_\alpha^\dagger B_\mu T^a_{R\,\alpha\kappa}\phi_\kappa\right)$$

And because:

$$|W^\pm|^2 = \frac{|B_1|^2+|B_2|^2}{2} \quad ; \quad |Z|^2 = \left(\frac{gB_3+g'A}{\sqrt{g^2+g'^2}}\right)^2 = \frac{g'^2A^2+g^2B_3^2+2gg'AB_3}{g^2+g'^2}$$

For the $W^\pm$ mass we only need the diagonal terms of B_1 and B_2, so it will be like:

$$g^2\phi_0^\dagger \frac{T_R^1T_R^1+T_R^2T_R^2}{2}\phi_0 W^{\pm 2} = g^2\phi_0^\dagger\frac{|T|^2-T_R^{3\,2}}{2}\phi_0 W^{\pm 2}$$

And because the Higgs is not charged:

$$0 = Q\phi = \left(T_3+\frac{1}{2} Y\right)\phi \longrightarrow T_3\phi = -\frac{Y}{2}\phi \longrightarrow T_3^2\phi = \frac{Y^2}{4}\phi$$

And finally because the Higgs multiplet has I, I_3 and Y ($|T|^2\phi = I_3(I_3+1)\phi$), the mass term for $W^\pm$ will be:

$$\frac{g^2}{2}\phi_0^\dagger(|T|^2-T_R^{3\,2})\phi_0 W^{\pm 2} = \frac{g^2}{2}\phi_0^\dagger\phi_0\left[I_3(I_3+1)-\frac{Y^2}{4}\right]W^{\pm 2}$$

so comparing with $\frac{1}{2} m_W^2 W^{\pm 2}$ the mass will be:

$$\boxed{m_W^2 = g^2|\phi_0|^2\left[I_3(I_3+1)-\frac{Y^2}{4}\right]} \quad ; \quad m_W = g|\phi_0|\sqrt{I_3(I_3+1)-\frac{Y^2}{4}}$$

And now for the Z case, because the photon is massless all the remaining mass terms of the Lagrangian correspond to m_Z, which also works with our previous definition of Z:

$$g^2\phi_0^\dagger T_R^{3\,2}\phi_0 B^{3\,2} + g'^2\phi_0^\dagger\frac{Y^2}{4}\phi_0A^2 = gg'B_\mu^aA_\mu\frac{Y}{2}[T_R^a,\phi]\phi = \frac{Y^2}{4}|\phi_0|^2(gB_3+g'A)^2 =$$

$$= \frac{Y^2}{4}|\phi_0|^2\frac{\overbrace{g^2+g'^2}^{1}}{g^2+g'^2}(gB_3+g'A)^2 = \frac{Y^2}{4}|\phi_0|^2(g^2+g'^2)\left(\frac{gB_3+g'A}{\sqrt{g^2+g'}}\right)^2 = \frac{Y^2}{4}|\phi_0|^2(g^2+g'^2)Z^2$$

so finally comparing with $\frac{1}{2} m_Z^2 Z^2$ the Z mass is:

$$\boxed{m_Z^2 = (g^2+g'^2)\, \frac{Y^2}{2}\, |\phi_0|^2 = \frac{g^2}{c_W^2} \frac{Y^2}{2} |\phi_0|^2} \quad ; \quad m_Z = \sqrt{g^2+g'^2}\, \frac{Y}{\sqrt{2}} |\phi_0| = \frac{g}{c_W} \frac{Y}{\sqrt{2}} |\phi_0|^2$$

Which means the experimental constrain becomes:

$$1 = \rho = \frac{m_W^2}{m_Z^2} \frac{g^2+g'^2}{g^2} = \frac{m_W^2}{m_Z^2 c_W^2} = \frac{\cancel{g^2} \cancel{|\phi_0|^2} \left[I_3(I_3+1) - \frac{Y^2}{4} \right]}{\cancel{\frac{g^2}{c_W^2}} \frac{Y^2}{2} \cancel{|\phi_0|^2} \cancel{c_W^2}} = \frac{I_3(I_3+1) - \frac{Y^2}{4}}{Y^2/2}$$

so, from the experimental constrain we see that the Higgs particle has to fullfill:

$$I_3(I_3+1) - \frac{Y^2}{4} = \frac{Y^2}{2} \quad ; \quad I_3(I_3+1) - 3\frac{Y^2}{4} = 0 \quad ; \quad \boxed{4 I_3(I_3+1) - 3Y^2 = 0}$$

which by adding +1 in each side can also be written as:

$$4I_3^2 + 4I_3 + 1 - 3Y^2 = 1 \quad ; \quad \boxed{(2I_3+1)^2 - 3Y^2 = 1}$$

Finally to answer the question we need to see which values of I_3 and Y fullfill this:

- $\boxed{I_3 = \frac{1}{2},\ Y = 1}$ (the usual SM Higgs doublet)
- $\boxed{I_3 = 3,\ Y = 4}$ (a 7-plet containing a maximally charged state)
- $\boxed{I_3 = \frac{25}{2},\ Y = 15}$ (a 26-plet containing a maximally charged state)