New symmetries?

$$\mathcal{L} = i\overline{\Psi}\not{D}\Psi - \frac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu}$$

QED with massless Dirac fermion, charge 1

U(1) gauge symmetry:

$$\Psi(x) \to e^{-ig\Gamma(x)}\Psi(x) \ ,$$

$$\overline{\Psi}(x) \to e^{+ig\Gamma(x)}\overline{\Psi}(x) \ ,$$

$$A^{\mu}(x) \to A^{\mu}(x) - \partial^{\mu}\Gamma(x)$$

Axial U(1) symmetry:

$$\Psi(x) \to e^{-i\alpha\gamma_5}\Psi(x)$$

$$\overline{\Psi}(x) \to \overline{\Psi}(x)e^{-i\alpha\gamma_5}$$

$$\{\gamma_5,\gamma_\mu\} = 0, \quad \gamma_5^2 = 1$$

Noether current:

$$j^\mu = \frac{\delta L}{\delta(\partial_\mu \phi_a)}\delta\phi_a$$

$$j^{\mu}_{\mathrm{A}}(x) \equiv \overline{\Psi}(x)\gamma^{\mu}\gamma_5\Psi(x)$$

$$\partial^\mu j_\mu = 0$$

Local axial symmetry?

$$S(A) \equiv \int d^4x \, \overline{\Psi} i \not{D} \Psi \qquad\qquad D_\mu = \partial_\mu - igA_\mu$$

$$Z(A) \equiv \int \mathcal{D}\Psi \, \mathcal{D}\overline{\Psi} \, e^{iS(A)}$$

$$\Psi(x) \to J(x,y)\Psi(y) = e^{-i\alpha(x)\gamma_5} \delta^4(x-y)\Psi(y)$$

$$\overline{\Psi}(x) \to \overline{\Psi}(y) J(x,y) = \overline{\Psi}(y) e^{-i\alpha(x)\gamma_5} \delta^4(x-y)$$

$$A_\mu \to A_\mu$$

This corresponds to a change of path integral integration variables

$$\mathcal{D}\Psi \, \mathcal{D}\overline{\Psi} \to (\det J)^{-2} \mathcal{D}\Psi \, \mathcal{D}\overline{\Psi} \qquad\qquad \psi_i = J_{ij} \psi'_j \ ,$$

fermionic

$\gamma_5^2 = 1$....naively...

$$\mathrm{Det}\left(e^{i\beta(x)\gamma^5}\right) = \mathrm{Det}\begin{pmatrix} e^{-i\beta(x)} & 0 & 0 & 0 \\ 0 & e^{-i\beta(x)} & 0 & 0 \\ 0 & 0 & e^{i\beta(x)} & 0 \\ 0 & 0 & 0 & e^{i\beta(x)} \end{pmatrix} = 1$$

Local axial symmetry?

$$S(A) \to S(A) + \int d^4x\, j^\mu_{\mathrm{A}}(x)\partial_\mu \alpha(x)$$

$$\to S(A) - \int d^4x\, \alpha(x)\partial_\mu j^\mu_A(x)$$

$$Z(A) \to \int \mathcal{D}\Psi\, \mathcal{D}\overline{\Psi}\, e^{iS(A)} e^{-i\int d^4x\, \alpha(x)\partial_\mu j^\mu_{\mathrm{A}}(x)}$$

If this is a symmetry $\partial_\mu j^\mu_A = 0$

True classically in massless limit:

$$\left(i\not{D} - m\right)\psi = 0$$

$$\partial^\mu\left(\overline{\psi}\gamma_\mu\gamma_5\psi\right) = \overline{\psi}\,\overleftarrow{\not{\partial}}\gamma_5\psi - \overline{\psi}\gamma_5\overrightarrow{\not{\partial}}\psi = -i\overline{\psi}(e\not{A} - m)\gamma_5\psi - i\overline{\psi}\gamma_5(e\not{A} - m)\psi$$

$$= 2im\overline{\psi}\gamma_5\psi$$

Axial gauge symmetry

$$S(A) \equiv \int d^4x \overline{\psi} i \not{D} \gamma_5 \psi$$

$$\Psi(x) \to J(x,y)\Psi(y) = e^{-i\alpha(x)\gamma_5}\delta^4(x-y)\Psi(y)$$

$$\overline{\Psi}(x) \to \overline{\Psi}(y)J(x,y) = \overline{\Psi}(y)e^{-i\alpha(x)\gamma_5}\delta^4(x-y)$$

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

$$S(A) \to S(A)$$

Anomalies

In all cases the axial symmetry is anomalous

… breaks down at the quantum level

Classical v/s quantum symmetry

Classical symmetry: $\phi \rightarrow \phi + \delta\phi, \quad S(\phi) \rightarrow S(\phi)$

Quantum symmetry $\int d\phi\, e^{iS(\phi)} \xrightarrow{?} \int d\phi\, e^{iS(\phi)}$

i.e. measure must be invariant too

Path integral treatment (Fujikawa method)

$$\mathcal{D}\Psi\,\mathcal{D}\overline{\Psi} \to (\det J)^{-2}\,\mathcal{D}\Psi\,\mathcal{D}\overline{\Psi}$$

$$J(x,y) = \delta^4(x-y)e^{-i\alpha(x)\gamma_5}$$

$$\boxed{(\det J)^{-2} = \exp\left[2i\int d^4x\,\alpha(x)\,\mathrm{Tr}\,\delta^4(x-x)\gamma_5\right]}$$

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Definition of fermion path integral measure $D\bar{q}Dq$

$$q(x) = \sum_n a_n \phi_n(x), \qquad \bar{q}(x) = \sum_n \phi_n^\dagger(x) \bar{b}_n$$

anticommuting

$$\not{\partial}\phi_n(x) = \lambda_n \phi_n(x)$$

$$\phi_n(x) = u_r e^{ik.x}, \qquad u_r = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$D\bar{q}Dq \rightarrow \Pi_n d\bar{b}_n \Pi_m da_m$$

$$q(x) = \sum_n a_n \phi_n(x), \qquad \bar{q}(x) = \sum_n \phi_n^\dagger(x) \bar{b}_n$$

$$\mathrm{D}\bar{q} Dq \rightarrow \Pi_n d\bar{b}_n \Pi_m da_m$$

Chiral transformation:

$$q(x) \rightarrow q'(x) = (1 + i\alpha\gamma_5) q(x) \;=\; \sum_n a'_n \phi_n(x)$$

$$a'_n = \sum_m \int \phi_n^* (1 + i\alpha\gamma_5) \phi_m a_m d^4x \;\; \equiv \sum_m C_{nm} a_m$$

$$\Pi_n da_n \rightarrow \Pi_n da'_n = \frac{1}{\det C_{nm}} \Pi_n da_n$$

$$\log(\det C) = Tr(\log C)$$

$$\frac{1}{\det C_{mn}} \simeq exp\left(-i \sum_n \int \alpha \phi_n^* \gamma_5 \phi_n d^4x \right)$$

Summation ill defined …need to regulate

$$\frac{1}{\det C_{mn}} \simeq exp\left(-i\sum_n \int \alpha\phi_n^* \gamma_5 \phi_n d^4x\right)$$

$$\sum_n \phi_n{}^*(x)\gamma_5\phi_n(x) \rightarrow \lim_{M\rightarrow\infty} \sum_n \phi_n{}^*(x)\gamma_5 e^{-\lambda_n^2/M^2}\phi_n(x)$$

$$= \lim_{M\rightarrow\infty, y\rightarrow x} Tr\gamma_5 e^{-(\not{D}/M)^2}\delta(x-y)$$

$$\boxed{(Det\ C)^{-2} = \lim_{M\rightarrow\infty,\ y\rightarrow x} \exp\left(-2i\int d^4x\,\alpha(x) Tr\,\gamma_5 e^{-(\not{D}/M)^2}\delta(x-y)\right)}$$

c.f.

$$\boxed{(\det J)^{-2} = \exp\left[2i\int d^4x\,\alpha(x)\,\mathrm{Tr}\,\delta^4(x-x)\gamma_5\right]}$$

Path integral treatment (Fujikawa method)

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$$J(x,y) = \delta^4(x-y)e^{-i\alpha(x)\gamma_5}$$

$$\boxed{(\det J)^{-2} = \exp\left[2i\int d^4x\,\alpha(x)\,\mathrm{Tr}\,\delta^4(x-x)\gamma_5\right]}$$

To define this need to regulate the integral .. e.g.

$$\delta^4(x-y) = Lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}e^{\partial_x^2/M^2}e^{ik(x-y)} \propto Lim_{M\to\infty}e^{-(x-y)^2M^2}$$

Here use scheme consistent with gauge invariance and with regularisation of $Z(A) = \det(i\not{D})$

$$\delta^4(x-y) = \int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)} \to Lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}e^{(i\not{D}_x)^2/M^2}e^{ik(x-y)}$$

c.f.

$$\boxed{(Det\ C)^{-2} = \lim_{M\to\infty,\ y\to x}\exp\left(-2i\int d^4x\,\alpha(x)Tr\,\gamma_5 e^{-(\not{D}/M)^2}\delta(x-y)\right)}$$

$$\delta^4(x-y)=\int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)}\rightarrow Lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}e^{(i\not{D}_x)^2/M^2}e^{ik(x-y)}$$

$$=Lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)}e^{(i\not{D}_x-\not{k})^2/M^2}$$

$$(i\not{D}-\not{k})^2=\not{k}^2-i\{\not{k},\not{D}\}-\not{D}^2$$

$$=-k^2-i\{\gamma^\mu,\gamma^\nu\}k_\mu D_\nu-\gamma^\mu\gamma^\nu D_\mu D_\nu$$

$$=-k^2+2ik\cdot D+D^2+2iS^{\mu\nu}D_\mu D_\nu \qquad \left(\gamma^\mu\gamma^\nu=\tfrac{1}{2}\left(\{\gamma^\mu,\gamma^\nu\}+[\gamma^\mu,\gamma^\nu]\right)=-g^{\mu\nu}-2iS^{\mu\nu}\right)$$

$$=-k^2+2ik\cdot D+D^2+gS^{\mu\nu}F_{\mu\nu} \qquad \frac{1}{2}[D_\mu,D_\nu]=-\frac{1}{2}igF_{\mu\nu}$$

$$k\rightarrow k'/M$$

$$\delta^4(x-y)\rightarrow M^4\int\frac{d^4k}{(2\pi)^4}e^{iMk(x-y)}\,e^{-k^2}e^{2ik\cdot D/M+D^2/M^2+gS^{\mu\nu}F_{\mu\nu}/M^2}$$

$$\mathrm{Tr}\,\delta^4(x-x)\gamma_5\rightarrow M^4\int\frac{d^4k}{(2\pi)^4}e^{-k^2}\,\mathrm{Tr}\,e^{2ik\cdot D/M+D^2/M^2+gS^{\mu\nu}F_{\mu\nu}/M^2}\gamma_5$$

Expand to M^{-4}

$$\boxed{Lim_{M\to\infty}Tr\delta^4(x-x)\gamma_5\rightarrow\tfrac{1}{2}g^2\int\frac{d^4k}{(2\pi)^4}e^{-k^2}(\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma})(\mathrm{Tr}\,S^{\mu\nu}S^{\rho\sigma}\gamma_5)}$$

$$i\varepsilon^{\mu\nu\rho\sigma}$$

$$\boxed{(\det J)^{-2} = \exp\left[2i\int d^4x\,\alpha(x)\,\mathrm{Tr}\,\delta^4(x-x)\gamma_5\right]}$$

$$= \exp\left[-\frac{ig^2}{16\pi^2}\int d^4x\,\alpha(x)\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}(x)F_{\rho\sigma}(x)\right]$$

$$Z(A) \to \int \mathcal{D}\Psi\,\mathcal{D}\overline{\Psi}\;e^{iS(A)}e^{-i\int d^4x\,\alpha(x)[(g^2/16\pi^2)\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,F_{\mu\nu}(x)F_{\rho\sigma}(x)+\partial_\mu j^\mu_{\mathrm{A}}(x)]}$$

$$\Rightarrow \quad \boxed{\partial_\mu j^\mu_{\mathrm{A}} = -\frac{g^2}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma}}$$

Exact result!

$$Note: \quad \partial_\mu j^\mu_V = 0 \text{ since } Tr(S^{\mu\nu}S^{\rho\sigma}) = 0$$

$$Z(A)=\int D\psi D\overline{\psi}\exp\left[iS(A)+\beta(x)\left\{\partial_\mu j^{\mu 5}+\frac{g^2}{16\pi^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}(x)\right\}\right]$$

$$c.f.\;\; Z(A)\equiv\int\mathcal{D}\Psi\,\mathcal{D}\overline{\Psi}\;e^{iS(A)}$$

$$\partial_\mu j^\mu_{\mathrm{A}}=-\frac{g^2}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

Exact result

Non Abelian case

$$A_\mu\equiv T^a_{\mathrm{R}}A^a_\mu \qquad\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ig[A_\mu,A_\nu]$$

Analysis proceeds as before except trace includes gauge indices

$$\begin{aligned}\partial_\mu j^\mu_{\mathrm{A}}&=-\frac{g^2}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma}\\&=-\frac{g^2}{16\pi^2}T(\mathrm{R})\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}\end{aligned}$$

$$\boxed{(\det J)^{-2} = \exp\left[2i\int d^4x\,\alpha(x)\,\mathrm{Tr}\,\delta^4(x-x)\gamma_5\right]}$$

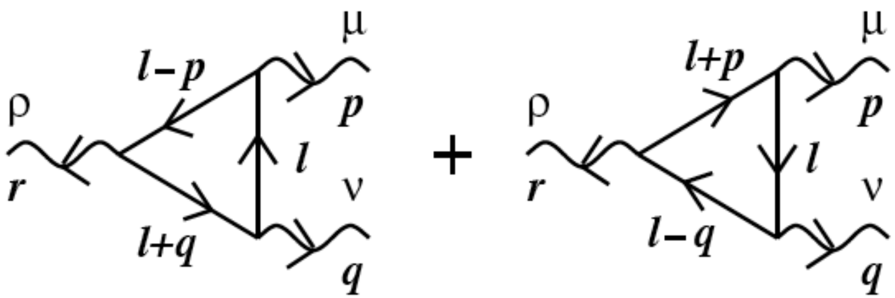
$$= \exp\left[-\frac{ig^2}{16\pi^2}\int d^4x\,\alpha(x)\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}(x)F_{\rho\sigma}(x)\right]$$

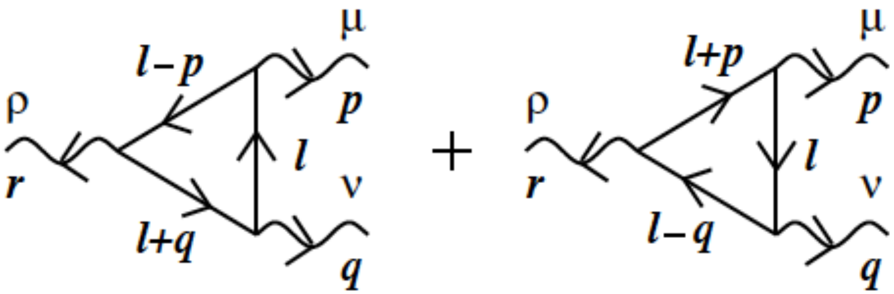
$$Z(A) \to \int \mathcal{D}\Psi\,\mathcal{D}\overline{\Psi}\;e^{iS(A)}e^{-i\int d^4x\,\alpha(x)\left[(g^2/16\pi^2)\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,F_{\mu\nu}(x)F_{\rho\sigma}(x)+\partial_\mu j^\mu_{\mathrm{A}}(x)\right]}$$

$$\Rightarrow \quad \boxed{\partial_\mu j^\mu_{\mathrm{A}} = -\frac{g^2}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma}}$$

Exact result!

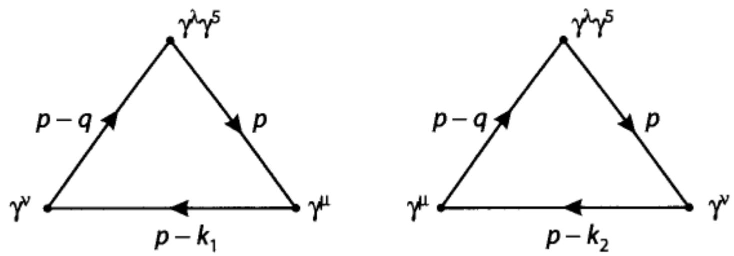
Result previously obtained by perturbative calculation





Furry's theorem: In QED any scattering amplitude with no external fermions and odd number of photons vanishes (charge conjugation)

What happens with γ_5 coupling –(charge conjugation broken)?



$q_\mu = (k_1 + k_2)_\mu$

$\not{k}_1 = \not{p} - (\not{p} - \not{k}_1)$ $\quad$ $(\not{p} - \not{k}_2) - (\not{p} - \not{q})$

$$k_{1\mu}\,\Delta^{\lambda\mu\nu}(k_1, k_2) = (-1)i^3 \int \frac{d^4 p}{(2\pi)^4}$$

$$k_{1\mu}\,\mathrm{tr}\left(\gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{q}}\gamma^\nu \frac{1}{\not{p}-\not{k}_1}\gamma^\mu \frac{1}{\not{p}} + \gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{q}}\gamma^\mu \frac{1}{\not{p}-\not{k}_2}\gamma^\nu \frac{1}{\not{p}}\right)$$

$\not{p} \to \not{p} + \not{k}_1$

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1, k_2)$$
$$= i\int \frac{d^4 p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{q}}\gamma^\nu \frac{1}{\not{p}-\not{k}_1} - \gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{k}_2}\gamma^\nu \frac{1}{\not{p}}\right) = 0?$$

But integrals divergent … not justified to shift integration variable

$$\int_{-\infty}^{\infty} dp\,(f(p+a) - f(p)) = \int_{-\infty}^{+\infty} dp\left(a\frac{d}{dp}f(p) + \cdots\right) = a(f(+\infty) - f(-\infty)) + \cdots$$

Regulating divergent integrals with ϒ$_5$?

$$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4i\varepsilon^{\mu\nu\rho\sigma}$$

… continue to d dimensions??

Pauli Villars regularisation?

$$P_{\mathrm{L}}\frac{\not p}{p^2} \to P_{\mathrm{L}}\left(\frac{-\not p}{p^2} - \frac{-\not p+\Lambda}{p^2+\Lambda^2}\right)$$

…not gauge invariant??

Regularisation choice: Euclidean space

$$\int d_E^4 p\Big[f(p+a)-f(p)\Big]=\int d_E^4 p\Big[a^\mu \partial_\mu f(p)\ldots\Big]$$

Gauss' theorem

$$=\lim_{P\to\infty} a^\mu\left(\frac{P_\mu}{P}\right)f(p)S_3(P)$$

Rotate back to Minkowski space

$$\int d^4 p\Big[f(p+a)-f(p)\Big]=\lim_{P\to\infty} ia^\mu\left(\frac{P_\mu}{P}\right)f(p)(2\pi^2 P^3)$$

$$\int d^4 p\left[f(p+a)-f(p)\right]=\lim_{P\to\infty} ia^{\mu}\left(\frac{P_{\mu}}{P}\right)f(p)(2\pi^2 P^3)$$

$$f(p)=\mathrm{tr}\left(\gamma^{\lambda}\gamma^{5}\frac{1}{\not p-\not k_2}\gamma^{\nu}\frac{1}{\not p}\right)=\frac{\mathrm{tr}[\gamma^5(\not p-\not k_2)\gamma^{\nu}\not p\gamma^{\lambda}]}{(p-k_2)^2p^2}=\frac{4i\varepsilon^{\tau\nu\sigma\lambda}k_{2\tau}p_{\sigma}}{(p-k_2)^2p^2}$$

$$k_{1\mu}\Delta^{\lambda\mu\nu}=\frac{i}{(2\pi)^4}\lim_{P\to\infty} i(-k_1)^{\mu}\frac{P_{\mu}}{P}\frac{4i\varepsilon^{\tau\nu\sigma\lambda}k_{2\tau}P_{\sigma}}{P^4}2\pi^2P^3=\frac{i}{8\pi^2}\varepsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}$$

Here $a_{\mu}=k_{1\mu}$

Non-zero!

….. regularisation does not preserve gauge invariance

Underlying problem is $\Delta^{\lambda\mu\nu}(k_1,k_2)$ undefined until regulated

Compute $\Delta^{\lambda\mu\nu}(a,k_1,k_2)-\Delta^{\lambda\mu\nu}(k_1,k_2)$

$$\Delta^{\lambda\mu\nu}(k_1,k_2)=(-1)i^3\int\frac{d^4p}{(2\pi)^4}$$
$$\mathrm{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p-\not a}\gamma^\nu\frac{1}{\not p-\not k_1}\gamma^\mu\frac{1}{\not p}+\gamma^\lambda\gamma^5\frac{1}{\not p-\not a}\gamma^\mu\frac{1}{\not p-\not k_2}\gamma^\nu\frac{1}{\not p}\right)$$

Here $f(p)= tr(\gamma^\lambda\gamma^5\frac{1}{\not p-\not a}\gamma^{\hat\nu}\frac{1}{\not p-\not k_1}\gamma^\mu\frac{1}{\not p})$

$$\lim_{P\to\infty}\frac{\mathrm{tr}(\gamma^\lambda\gamma^5\not P\gamma^\nu\not P\gamma^\mu\not P)}{P^6}=\frac{2P^\mu tr(\gamma^\lambda\gamma^5\not P\gamma^\nu\not P)-P^2\,\mathrm{tr}(\gamma^\lambda\gamma^5\not P\gamma^\nu\gamma^\mu)}{P^6}=\frac{-4iP^2P_\sigma\varepsilon^{\sigma\nu\mu\lambda}}{P^6}$$

$$\boxed{\begin{aligned}\Delta^{\lambda\mu\nu}(a,k_1,k_2)-\Delta^{\lambda\mu\nu}(k_1,k_2)&=\frac{4i}{8\pi^2}\lim_{P\to\infty}a^\omega\frac{P_\omega P_\sigma}{P^2}\varepsilon^{\sigma\nu\mu\lambda}+\{\mu,k_1\leftrightarrow\nu,k_2\}\\&=\frac{i}{8\pi^2}\varepsilon^{\sigma\nu\mu\lambda}a_\sigma+\{\mu,k_1\leftrightarrow\nu,k_2\}\end{aligned}}$$

$$\boxed{\begin{aligned}\Delta^{\lambda\mu\nu}(a,k_1,k_2)-\Delta^{\lambda\mu\nu}(k_1,k_2)&=\frac{4i}{8\pi^2}\lim_{P\to\infty}a^{\omega}\frac{P_\omega P_\sigma}{P^2}\varepsilon^{\sigma\nu\mu\lambda}+\{\mu,k_1\leftrightarrow\nu,k_2\}\\&=\frac{i}{8\pi^2}\varepsilon^{\sigma\nu\mu\lambda}a_\sigma+\{\mu,k_1\leftrightarrow\nu,k_2\}\end{aligned}}$$

$$a=\alpha(k_1+k_2)+\beta(k_1-k_2)$$

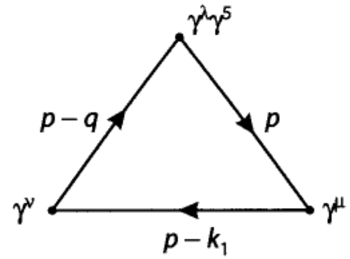
$$\Delta^{\lambda\mu\nu}(a,k_1,k_2)=\Delta^{\lambda\mu\nu}(k_1,k_2)+\frac{i\beta}{4\pi^2}\varepsilon^{\lambda\mu\nu\sigma}(k_1-k_2)_\sigma$$

But $$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2)=\frac{i}{8\pi^2}\varepsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}$$

Vector current conserved if $\beta=\dfrac{1}{2}$

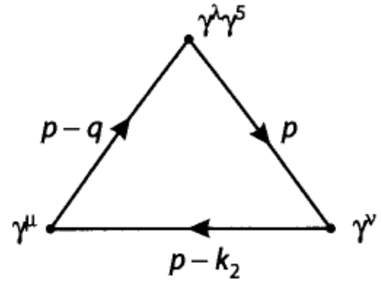
But what about the axial current?

$$q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2) = q_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) + \frac{i}{4\pi^2}\varepsilon^{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma}$$



$$q_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) = i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^5 \frac{1}{\not p - \not q}\gamma^\nu \frac{1}{\not p - \not k_1}\gamma^\mu - \gamma^5 \frac{1}{\not p - \not k_2}\gamma^\nu \frac{1}{\not p}\gamma^\mu\right) + \{\mu, k_1 \leftrightarrow \nu, k_2\}$$

$$= \frac{i}{4\pi^2}\varepsilon^{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma}$$



$$\boxed{q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2) = \frac{i}{2\pi^2}\varepsilon^{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma}}$$

i.e. $\partial_\mu j^\mu_{\mathrm{A}} = -\frac{g^2}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ as before

Chiral gauge theories

$$\Psi \equiv \begin{pmatrix} \chi \\ \xi^{\dagger} \end{pmatrix} \text{ in representation R } \equiv \text{ 2 Weyl fields } \chi \in \mathrm{R}, \xi \in \overline{R}$$

$$\Psi_{M} = \begin{pmatrix} \omega \\ \omega^{\dagger} \end{pmatrix} \text{ Majorana field in real representation R}', \ \omega, \omega^{\dagger} \in \mathrm{R}'$$

i.e.Majorana field in real rep R is equivalent to single LH Weyl field in R

Chiralgauge theories

- Consider single (LH) Weyl field ψ in complex representation R

  … parity violating…``chiral” ($\psi^\dagger$ in inequivalent $\overline{R}$)

$$\mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu D_\mu \psi - \frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} \ , \qquad D_\mu = \partial_\mu - igA^a_\mu T^a_{\mathrm{R}}$$

$\left(\text{No mass term } \psi\psi \quad (\mathrm{I} \notin \mathrm{R} \otimes \mathrm{R},\ \mathrm{R} \text{ complex})\right)$

- Can write this in terms of Dirac spinor

$$P_L \Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$$

$$P_L = \frac{1}{2}(1-\gamma_5)$$

$$P_L^2 = P_L$$

e.g. U(1) gauge invariant theory

$$\mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu (\partial_\mu - igA_\mu)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$Q_\psi = +1$$

$$\Rightarrow \quad \mathcal{L} = i\overline{\Psi}\gamma^\mu(\partial_\mu - igA_\mu)P_{\mathrm{L}}\Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$P_{\mathrm{L}}\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$$

$$P_L = \frac{1}{2}(1-\gamma_5)$$

$$P_L^2 = P_L$$

Feynman rules:

$$\tilde{S}(p) = -\frac{P_L \not{p}}{p^2}$$

$$ig\gamma^\mu P_L$$

Chiral symmetry

$$\psi_{L(R)} = \tfrac{1}{2}(1 \mp \gamma_5)\psi = P_{L(R)}\psi$$

$$-\gamma_5 = P_L - P_R$$

$$e^{-i\alpha\gamma_5}\psi_L = e^{i\alpha(P_L - P_R)}\psi_L = e^{i\alpha P_L}\psi_L = e^{i\alpha}\psi_L$$

$$e^{-i\alpha\gamma_5}\psi_R = e^{i\alpha(P_L - P_R)}\psi_R = e^{-i\alpha P_R}\psi_R = e^{-i\alpha}\psi_R$$

$$\psi_{L(R)} \rightarrow e^{i\alpha_{L(R)}}\psi_{L(R)}$$

$\alpha_L = \alpha_R$ vectorlike

$\alpha_L \neq \alpha_R$ chiral $(\alpha_L = -\alpha_R$ axial$)$

Chiral - no mass term $\psi\psi$

Chiral gauge symmetry anomaly

ψ_L in representation R

$$j^{a\mu} = \overline{\Psi} T_{\mathrm{R}}^{a} \gamma^{\mu} P_{\mathrm{L}} \Psi$$

Chiral anomaly

$$D_{\mu}^{ab} j^{b\mu} = \frac{g^2}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} \partial_{\mu} \mathrm{Tr}\Big[T_{\mathrm{R}}^{a} (A_{\nu} \partial_{\rho} A_{\sigma} - \tfrac{1}{2} i g A_{\nu} A_{\rho} A_{\sigma})\Big]$$

RHS not gauge invariant…gauge theory inconsistent!

Chiral gauge symmetry

$$\left\{T_N^a,T_N^b\right\}=\frac{1}{N}\delta^{ab}+d^{abc}T_N^c \qquad SU(N)$$

$$\boxed{\begin{aligned}\mathrm{Tr}(T^aT^bT^c) &= \tfrac{1}{2}\,\mathrm{Tr}(T^a[T^b,T^c])+\tfrac{1}{2}\,\mathrm{Tr}(T^a\{T^b,T^c\})\\ &=\tfrac{1}{2}iT(R)f^{abc}+\tfrac{1}{4}A(R)d^{abc}\end{aligned}}$$

ψ_L in representation R

$$A(N)=1$$

$$j^{a\mu} = \overline{\Psi}T^a_{\mathrm{R}}\gamma^\mu P_{\mathrm{L}}\Psi$$

Chiral anomaly

$$D^{ab}_\mu j^{b\mu} = \frac{g^2}{24\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\partial_\mu \mathrm{Tr}\left[T^a_{\mathrm{R}}(A_\nu\partial_\rho A_\sigma - \tfrac{1}{2}igA_\nu A_\rho A_\sigma)\right]$$

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu\partial_\rho A_\sigma) = \varepsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu\partial_\rho A_\sigma$$

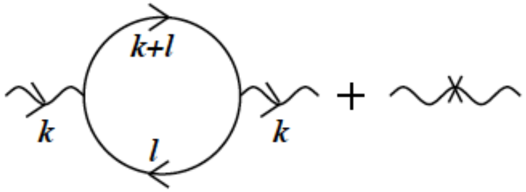
$$\varepsilon^{\mu\nu\rho\sigma}\partial_\mu A^b_\nu\partial_\rho A^c_\sigma\,\mathrm{Tr}(T^aT^bT^c)\propto A(R) \qquad (\mu,\nu\leftrightarrow\rho,\sigma \quad b,c \quad \text{symmetric})$$

$$\mathrm{Tr}(T^aT^b[T^c,T^d]) = -\tfrac{1}{2}T(\mathrm{R})f^{cde}f^{abe}+iA(\mathrm{R})f^{cde}d^{abe} \qquad (c,d \text{ antisymmetric})$$

$$\tfrac{1}{3}(f^{cde}f^{abe}+f^{dbe}f^{ace}+f^{bce}f^{ade}) = 0(Jacobi) \qquad (b,c,d \text{ antisymmetric})$$

$$\varepsilon^{\mu\nu\rho\sigma}A^b_\nu A^c_\rho A^d_\sigma\,\mathrm{Tr}(T^aT^bT^cT^d)\propto A(R)$$

QED



$$i\Pi^{\mu\nu}(k) = (-1)(ig)^2\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{N^{\mu\nu}}{(\ell+k)^2\ell^2} - i(Z_3-1)(k^2g^{\mu\nu}-k^\mu k^\nu) + O(g^4)$$

$$N^{\mu\nu} = \mathrm{Tr}[P_\mathrm{L}(\not\ell+\not k)\gamma^\mu P_\mathrm{L}P_\mathrm{L}\not\ell\gamma^\nu P_\mathrm{L}] = \mathrm{Tr}[(\not\ell+\not k)\gamma^\mu\not\ell\gamma^\nu P_\mathrm{L}]$$

$$\left(N^{\mu\nu}\right)_{\gamma_5\text{ term}} \to -\tfrac{1}{2}\mathrm{Tr}[(\not\ell+\not k)\gamma^\mu\not\ell\gamma^\nu\gamma_5]$$

$$= 2i\varepsilon^{\alpha\mu\beta\nu}(\ell+k)_\alpha\ell_\beta$$

$$= 2i\varepsilon^{\alpha\mu\beta\nu}k_\alpha\ell_\beta\,. \quad \to 0$$

(No γ_5 divergence)

Single charged Weyl fermion contributes ½ Dirac fermion