Quantum Field Theory, 2021/2022
Exercise sheet 1: Classical Field Theory & Symmetries
Hand-in: September 29, 2021

1.1. Consider a massless, non-interacting, real, Klein-Gordon field ϕ:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$$

one can define a dilatation operation acting on space-time:

$$x'^\mu = e^\alpha x^\mu \quad ; \quad \phi'(x') = e^{-d\alpha}\phi(x) \tag{1}$$

with d a constant parameter.

(a) Prove that, choosing appropriately the parameter d, the the dilatation transformation of eq. (1) is a symmetry of the theory.
Hint: Work directly with the action, and show that the action is invariant.

(b) Find the conserved current associated with the symmetry (1).

(c) Now, suppose that we add a mass-term, or a 4-particle interaction term:

$$\mathcal{L}_m = \mathcal{L} - \frac{1}{2}m^2\phi^2 \;\; ; \;\; \mathcal{L}_\lambda = \mathcal{L} - \lambda\phi^4$$

Is the dilatation operation still a symmetry of the lagrangians $\mathcal{L}_m$, $\mathcal{L}_\lambda$?

1.2. Consider the electromagnetic 4-vector potential, A_μ, and define the electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We can write a free-field lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2}$$

(a) Find the equations of motion. You can write them as a function of $F_{\mu\nu}$.

(b) Find the conjugate momentum of the fields A_μ, Π_μ. What is the value of Π_0?

(c) The Maxwell equations in the presence of matter can be written as:

$$\partial_\mu F^{\mu\nu} = j^\nu(x)$$

How can we modify the lagrangian (2) to obtain these equations of motion?

(d) The free-field lagrangian of eq. (2) is obviously invariant under the gauge transformations:

$$A_\mu \to A_\mu + \partial_\mu\Lambda(x)$$

so that the theory has a gauge symmetry. This is not the case for the lagrangian found in the previous question. Which conditions must fulfill $j^\mu(x)$ such that the gauge transformations are a symmetry of the theory?