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Charge Shielding and Quark Confinement in the Massive Schwinger Model*

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The Schwinger model is quantum electrodynamics with massless fermions in two dimensions. It is known that the asymptotic states of the theory contain no states corresponding to free fermions ("quark trapping") and that local charge conservation is spontaneously broken ("Higgs phenomenon"). We investigate to what extent these phenomena persist when the fermion is given a bare mass. We find quark trapping but no Higgs phenomenon. The second of these results is dependent on mass perturbation theory; the first is not.

1. INTRODUCTION AND CONCLUSIONS

The Schwinger model [1] is quantum electrodynamics with a massless fermion in two-dimensional space-time. In a covariant gauge, the model is defined by the Lagrange density,

$$\mathscr{L} = \bar{\psi} i \partial_\mu \gamma^\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 - e \bar{\psi} \gamma^\mu \psi A_\mu , \tag{1.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \tag{1.2}$$

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and α is the usual gauge-fixing parameter. The theory is super-renormalizable; it requires no infinite renormalizations other than a trivial redefinition of the zero of energy density.

The model is exactly solvable and is known to possess the following interesting properties [1, 2]:

(1) The Higgs phenomenon occurs. Local electric charge conservation is spontaneously broken, but no Goldstone boson appears because the Goldstone mode may be gauged away.

(2) The Nambu–Goldstone phenomenon occurs. Global chiral symmetry is spontaneously broken, and the vacuum is infinitely degenerate. Here also no Goldstone boson appears, but for a different reason: The local chiral current is afflicted with an anomaly, and a conserved local current is needed for the Goldstone theorem [3].

(3) Charge shielding occurs. If we couple an external c-number current, J_μ, to the theory,

$$\mathscr{L} \to \mathscr{L} - J_\mu A^\mu, \tag{1.3}$$

then, unlike free electrodynamics, there is no long-range force between widely separated external charges. This is connected with the Higgs phenomenon. If we were in four dimensions, we would say the photon acquires a mass. However, in two dimensions, there is no photon, even for free electrodynamics, because there are no transverse directions, and all we can say is that the long-range Coulomb force disappears.

(4) Quark trapping occurs. Here, "quark," with some nomenclatural presumption, refers to the fundamental fermion of the theory. If we consider the space of gauge-invariant states, those states that are obtained by applying gauge-invariant operators to the vacua, then this space contains no states that correspond to widely separated quarks. Indeed, the only particle in the theory is a free meson of mass $e/\pi^{1/2}$, which can be thought of as a quark–antiquark bound state. This is connected with charge shielding. If we attempt to separate a quark–antiquark pair, when the separation is sufficiently great, it is energetically favorable for a new pair to materialize from the vacuum. The new quark is attracted to the original antiquark and the new antiquark is attracted to the original quark. This both shields the long-range force and insures that what we are separating is not a quark and an antiquark but two quark–antiquark bound states.

In this paper, we investigate to what extent these properties persist in a generalization of the model in which the fermions are given a mass,

$$\mathscr{L} \to \mathscr{L} - m_0 \bar{\psi}\psi. \tag{1.4}$$

The massive model is not exactly solvable; however, it is possible to do perturbation theory in the mass parameter [4].

Of course, the mass term explicitly breaks chiral invariance, so Property (2) is no longer at issue. However, Properties (1), (3), and (4) are still of interest. We focus on Property (3), charge shielding, since, in the massless theory, this is connected with both the Higgs phenomenon and quark trapping. Our main results are:

(i) In mass perturbation theory, the long-range force between external charges of arbitrary magnitude does not disappear. From this we infer that the Higgs phenomenon does not occur.

(ii) However, if the external charges are integral multiples of the fundamental charge e then the long-range force *does* disappear. This is an exact result, independent of mass perturbation theory. (Of course, it is also true in mass perturbation theory.) From this, we infer that quark trapping *does* take place.

These conclusions, quark trapping but no Higgs phenomenon, are the same as those that have recently been found for the strong-coupling approximation to lattice gauge theories [5].

Additional insight into the contrast between the massless and the massive models can be obtained by expressing our results diagrammatically. Fig. 1 shows all the Feynman diagrams that contribute to the interaction between external charges. The direct exchange of an electromagnetic propagator, Fig. 1a, gives a long-range force in both the massless and massive models. In the massless model, this force is cancelled by the first graph in Fig. 1b; the vacuum polarization function has a pole at zero momentum that cancels the pole in Fig. 1a. The

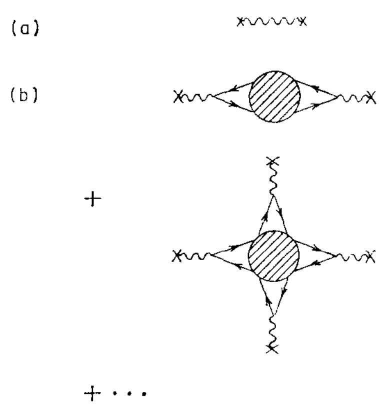

FIG. 1. All the diagrams that contribute to the interaction energy of an external c-number current distribution. The crosses denote the current distribution, the wavy lines denote electromagnetic propagators, and the directed lines denote fermion propagators.

remaining diagrams in Fig. 1b make no contribution to the long-range force. (Indeed, they vanish identically.) The situation is very different for the massive model. Here, Fig. 1a is cancelled not just by the first graph in Fig. 1b, but by the whole sum, and then only for special values of the external charges. In particular, this implies that the vacuum polarization function has no pole at zero momentum, in agreement with the results of ordinary perturbation theory (in e).

Section 2 reviews the massless Schwinger model and mass perturbation theory; Section 3 gives our computations.

2. The Schwinger Model Revisited

The (massless) Schwinger model is exactly solvable; in any gauge, all the Green's functions of the theory may be computed in closed form. From these Green's functions, the Hilbert space of states can be constructed in the standard way. As for any gauge theory, the space one constructs in this way depends on one's choice of gauge. For our purposes, it will suffice to restrict ourselves to the gauge-invariant Hilbert space, that which is constructed from the Green's functions for strings of gauge-invariant operators. Of course, this is only a subspace of the space one would obtain in any given gauge. Nevertheless, it is sufficient to describe the result of any conceivable experiment if one accepts the dogma that only gauge-invariant operators are observables.

It turns out [6] that this space contains an infinite family of orthonormal vacuum states. The vacua can be labeled by an angle θ, which runs from 0 to 2π,

$$\langle \theta' \mid \theta \rangle = \delta(\theta - \theta'), \tag{2.1}$$

and

$$P_\mu \mid \theta \rangle = 0. \tag{2.2}$$

To distinguish the vacua, it is convenient to define a unitary operator, U, by

$$U \mid \theta \rangle = e^{i\theta} \mid \theta \rangle. \tag{2.3}$$

Of course, there are also states of nonzero energy. These are most conveniently constructed with the aid of a free field φ:

$$\mathscr{H} = \frac{1}{2} :\pi^2 + (\nabla\varphi)^2 + \frac{e^2}{\pi} \varphi^2:, \tag{2.4}$$

where π is the canonical momentum density and the colons denote conventional normal ordering. φ commutes with U, and all the states of the theory are obtained by applying smeared polynomials in φ to the vacua.

All gauge-invariant observables can be expressed in terms of U and φ. For our computations, we will need the expression for the electromagnetic current:

$$j^\mu \equiv :\bar{\psi}\gamma^\mu\psi: = \frac{1}{\pi^{1/2}}\,\epsilon^{\mu\nu}\partial_\nu\varphi. \tag{2.5}$$

We will also need the expressions for the scalar and pseudoscalar chiral eigendensities:

$$\sigma_+ \equiv :\bar{\psi}(1+\gamma_5)\,\psi: = ce\,:\exp(2i\pi^{1/2}\,\varphi):\,U \tag{2.6a}$$

and

$$\sigma_- \equiv :\bar{\psi}(1-\gamma_5)\,\psi: = ce\,:\exp(-2i\pi^{1/2}\,\varphi):\,U^+ \tag{2.6b}$$

where c is a numerical constant, related to Euler's constant. (We shall not need the explicit value of c for our computations.)

The connection between this structure and the qualitative features of the model described in Section 1 should be clear. Because we are restricting ourselves to gauge-invariant quantities, we do not see the spontaneous breakdown of gauge invariance. However, the spontaneous breakdown of chirality is announced by the appearance of the infinitely degenerate vacua. In the vacuum labeled by θ, the expectation values of $\sigma_\pm$ are $\exp(\pm i\theta)$. Global chiral transformations rotate one vacuum into another. There are no fermions in the theory, nor are there Goldstone bosons; the only particle is a free scalar meson of mass $e/\pi^{1/2}$, etc.

We now turn to the massive model. Eqs. (3.6a) and (3.6b) tell us how to write the mass term in terms of U and φ:

$$\mathscr{H} = \frac{1}{2}\,:\pi^2 + (\nabla\varphi)^2 + \frac{e^2}{\pi}\,\varphi^2:$$
$$+\,\frac{1}{2}\,cem_0\,:\exp(2i(\pi)^{1/2}\,\varphi):\,U + \frac{1}{2}\,cem_0\,:\exp(-2i\pi^{1/2}\,\varphi):\,U^+. \tag{2.7}$$

Since this expression commutes with U, we may partially diagonalize the Hamiltonian by restricting ourselves to an eigenspace of U with eigenvalue $e^{i\theta}$. On such a subspace

$$\mathscr{H} \equiv \mathscr{H}_\theta = \frac{1}{2}\,:\pi^2 + (\nabla\varphi)^2 + \frac{e^2}{\pi}\,\varphi^2: + cem_0\,:\cos(2(\pi)^{1/2}\,\varphi + \theta):. \tag{2.8}$$

This form of the Hamiltonian density for the massive Schwinger model explicitly displays the structure of the massless model. From it, a perturbation series in m_0 (mass perturbation theory) can be constructed by standard methods. It will also be the starting point for our investigation of charge shielding. Some comments should be made:

(1) The massive Schwinger model is not an exactly solvable theory, and it is not known whether Eq. (2.8) defines a physically sensible theory for any values of m_0 and θ [7]. We will assume that it does define a sensible theory, but the reader should be warned that this is only an assumption.

(2) One of the most startling features of Eq. (2.8) is that it contains a parameter, the angle θ, which is independent of m_0 and e, the only parameters in the original Lagrangian, and which, at least apparently, is physically significant. (For example, in mass perturbation theory, the mass of the meson depends non-trivially on θ.) This is in striking contrast to what usually happens when we add a symmetry-breaking term to the Hamiltonian of a theory that displays spontaneous symmetry breakdown. In the usual case, the symmetry-breaking term removes the degeneracy among the vacua of the original theory, and all of the vacua other than the one of lowest energy become unstable, decaying through the emission of Goldstone bosons. For the massive Schwinger model, the mass term removes the degeneracy, just as in the usual case. Indeed, to lowest order in mass perturbation theory, the energy per unit length is given by

$$\mathscr{E}(\theta) = cem_0 \cos \theta. \tag{2.9}$$

Nevertheless, unlike the usual case, all the vacua remain stable, [8] because there are no Goldstone bosons. (More precisely, because in the unperturbed theory it is not possible to change the expectation value of φ over a large region with no cost in energy except at the boundary of the region.)

(3) Eq. (2.8) is written in terms of canonical Bose fields alone. One might naively think that this immediately implies that the theory contains no fermions. This is false (at least in two dimensions). Both weakly-coupled φ^4 theory and the sine–Gordon equation are counterexamples [9]. Of course, no fermions appear in mass perturbation theory, but neither do any appear in ordinary perturbation theory for either φ^4 or sine–Gordon theory. To find whether the fermions really disappear (that is to say, whether quark trapping occurs), we must construct more careful arguments. We now turn to this task.

3. External Currents

We wish to consider the massive Schwinger model coupled to an external current distribution, J^μ,

$$\mathscr{L} = \bar{\psi} i \partial_\mu \gamma^\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 - m_0 \bar{\psi}\psi - eA_\mu \bar{\psi}\gamma^\mu\psi - A_\mu J^\mu. \tag{3.1}$$

We will restrict ourselves to the case in which J^μ is conserved and the total charge vanishes. In this case, J^μ may be written as the curl of a scalar c-number field, which we denote by D,

$$J^\mu = \epsilon^{\mu\nu}\partial_\nu D. \tag{3.2}$$

D is defined only up to an additive constant; we will fix this constant by demanding that D vanish at spatial infinity. Fig. 2 is a sketch of the form of D for the case on which we shall ultimately focus: Two opposite charges of magnitude Q separated by a distance L.

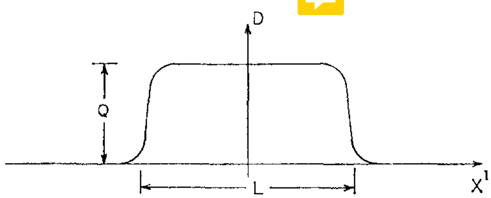

FIG. 2. The form of the D-function for a static charge distribution consisting of two opposite charges of magnitude Q separated by a distance L.

Figure 1 shows the contribution to the vacuum energy of the external current in terms of the fermion/photon form of the Lagrangian, Eq. (3.1). We would like to rewrite this in terms of the boson description of Section 2. This is easy to do: Fig. 1a makes a contribution to the action given by

$$\begin{aligned} I_a &= -\tfrac{1}{2}\int d^2x\, J^\mu \Box^{-2} J_\mu \\ &= -\tfrac{1}{2}\int d^2x\, \epsilon^{\mu\nu}\partial_\nu D \Box^{-2}\epsilon_{\mu\lambda}\partial^\lambda D \\ &= -\tfrac{1}{2}\int d^2x\, D^2. \end{aligned} \tag{3.3}$$

Likewise, all the graphs of Fig. 1b are equivalent to an interaction between J^μ and the fermion current of the form

$$\begin{aligned} I_b &= -e\int d^2x\, J^\mu \Box^{-2} j_\mu \\ &= -\frac{e}{\pi^{1/2}}\int d^2x\, \epsilon^{\mu\nu}\partial_\nu D \Box^{-2}\epsilon_{\mu\lambda}\partial^\lambda \varphi \\ &= -\frac{e}{\pi^{1/2}}\int d^2x\, D\varphi. \end{aligned} \tag{3.4}$$

Here we have used Eq. (2.5) at the second line.

Adding all this to Eq. (2.8), we obtain the pleasantly simple expression,

$$\mathscr{H} = \frac{1}{2} :\pi^2 + (\nabla\varphi)^2 + \frac{e^2}{\pi}\left(\varphi + \frac{\pi^{1/2}}{e} D\right): + m_0 ec :\cos(2\pi^{1/2}\varphi + \theta):. \tag{3.5}$$

Note that all traces of the nonlocal photon propagator have disappeared; this expression involves only local interactions between φ and D.

This observation enables us to extract the long-range part of the interaction between external charges. Let us consider the D-field sketched in Fig. 2 for large L. As we increase L, all that happens is that the region in which D equals Q increases in size. Thus, the term in the interaction energy that grows with L is simply L times the vacuum energy per unit length for a *constant* D-field.

The computation of this quantity is simplified by defining

$$\varphi' = \varphi + (\pi^{1/2}/e)\, D. \tag{3.6}$$

The Hamiltonian density for $D = Q$ then becomes

$$\mathscr{H} = \frac{1}{2} :\pi^2 + (\nabla\varphi')^2 + \frac{e^2}{\pi}\varphi'^2: + m_0 ce :\cos\left(2\pi^{1/2}\varphi' + \theta - \frac{2\pi Q}{e}\right):. \tag{3.7}$$

Thus, if we define $\mathscr{E}(\theta)$ to be the vacuum energy per unit length in the absence of external charges, as in Section 2, we find for the interaction energy between two widely separated external charges,

$$E = \left[\mathscr{E}\left(\theta - \frac{2\pi Q}{e}\right) - \mathscr{E}(\theta)\right] L + \cdots. \tag{3.8}$$

where the ... indicate terms that become constant as L goes to infinity.

The argument is now complete. For the Hamiltonian (3.7) is a periodic function of Q with period e; a fortiori, so is the vacuum energy. Thus, there is no long-range force if Q is an integral multiple of e. (Note that this result does not depend on mass perturbation theory.) On the other hand, for arbitrary Q, the long-range force is present, at least in mass perturbation theory, as can be seen from Eq. (2.9). These are the results stated in Section 1.

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6. Most of what follows can be found in the papers of Ref. [2].
7. J. Frohlich has been able to establish the validity of all the usual axioms for some closely related models (private communication).
8. We differ here from the conclusions in Ref. [4].
9. J. Goldstone and R. Jackiw, *Phys. Rev.*, **D11** (1975), 1486. S. Coleman, *Phys. Rev.*, **D11** (1975), 2088. "Weak coupling" above means small dimensionless coupling constant when the Lagrangian is written in a form in which spontaneous symmetry breakdown is manifest; this is strong coupling in the usual sense of constructive field theory.