

Deconfinement in $d=1$: Asymptotic and half-asymptotic particles

R. Shankar¹ and Ganpathy Murthy²¹*Department of Physics, Yale University, New Haven, Connecticut 06520, USA*²*Department of Physics and Astronomy, University of Kentucky, Lexington Kentucky 40506-0055, USA*

(Received 11 August 2005; published 9 December 2005)

The notion of deconfinement in two $d=1$ models, the Schwinger model and the Heisenberg chain, is reexamined. Both have half-asymptotic excitations (where particles and antiparticles must alternate) and also truly asymptotic particles which are half as many in number. The two kinds of particles are related by a complicated transformation. The main purpose of this paper is to highlight the relationship between asymptotic and half-asymptotic particles. The relevance of our findings to higher dimensions is briefly discussed.

DOI: [10.1103/PhysRevB.72.224414](https://doi.org/10.1103/PhysRevB.72.224414)

PACS number(s): 75.70.Cn, 75.47.Pq, 72.25.Mk, 79.60.Jv

I. INTRODUCTION

We take a closer look at deconfinement in $d=1$ and point out that the situation is more complicated than is generally appreciated. We consider two examples, the massive Schwinger model and the Heisenberg spin chain, wherein the degrees of freedom one often considered as deconfined are really *half asymptotic*. That is, particles and antiparticles can move arbitrarily far from each other, provided they alternate along the line. In both cases however, there exist truly asymptotic, unrestricted particles, but these are fewer in number and related to the half-asymptotic ones by complicated transformations. Many results presented here are scattered in the literature, and we claim credit only for filling in some gaps and highlighting the distinction between asymptotic and half-asymptotic particles.

II. EXAMPLE 1: THE MASSIVE SCHWINGER MODEL

Schwinger posed and solved the eponymous model of the electrodynamics of massless Dirac fermions of charge e and showed that the final spectrum consisted only of a single boson of mass $\frac{e}{\sqrt{\pi}}$. This result, readily established using bosonization, is understood as the confinement of charged particles by the linearly rising Coulomb potential (which is what electrodynamics reduces to in one space dimension). The case with nonzero fermion mass m was studied by numerous authors, most notably Coleman and co-workers.¹ They pointed out that electrodynamics in $d=1$ admits an extra parameter θ that corresponds to a background electric field $E_\theta = e\frac{\theta}{2\pi}$ produced by charges $\pm e\frac{\theta}{2\pi}$ at $x = \mp\infty$. In the absence of matter fields, the field lines due to these boundary charges would run undiminished in density from $-\infty$ to $+\infty$. In the massless case any such field would be screened immediately and at no cost by the fermions via what is essentially Debye screening. Thus θ would be effectively zero. (In the bosonized version, θ can be shifted away.) In the massive case, a suitable integer part of the background charges would be screened by the fermions, leaving a background field obeying $-\frac{e}{2} < E_\theta \leq \frac{e}{2}$, that is with $\theta = \frac{2\pi E_\theta}{e}$ limited to $-\pi < \theta \leq \pi$. Note that even though massive fermions cost a finite amount of energy, they are worth producing if bulk energy density can be reduced.

Let us consider the phase diagram of this model as a function of θ , m , and e . We study the weak-coupling limit in the fermionic version and then pass on to the bosonized version for strong coupling. We will begin by simply reviewing Coleman's work. Our contributions, pertaining to the interpretation and clarification of the phase transitions (and its relation to our main theme) will then follow.

The Hamiltonian is

$$H = \int \psi^\dagger (\alpha p + \beta m) \psi dx - \frac{e^2}{4} \iint \rho(x) |x-y| \rho(y) dx dy - \frac{e^2 \theta}{2\pi} \int x \rho(x) dx, \quad (1)$$

where $e\rho$ is the charge density and α and β are Dirac (Pauli) matrices. The first term is the fermion kinetic energy, the second is the electrostatic energy of dynamic charges, and the third is the coupling of the matter fields to the background field. (The energy to set up the background field is not shown.)

Gauss' law relates the electric field E due the dynamic charges as per

$$\frac{dE}{dx} = e\rho. \quad (2)$$

Note that this E is the field due to the fermions only and does not include the background field. However, the divergence of the total field E_T is the same as that of E .

Using Gauss' law can rewrite the last term in Eq. (1) as follows:

$$- \frac{e\theta}{2\pi} \int x \frac{dE(x)}{dx} dx = E_\theta \int E(x) dx, \quad (3)$$

where $E_\theta = e\theta/2\pi$. Indeed, the total electrical energy density $\frac{1}{2}E_T^2 = \frac{1}{2}(E+E_\theta)^2$, generates a term $E_\theta^2/2$ that is suppressed, a term $E^2/2$ which is the second term in Eq. (1), and a cross term which gives the the last term $E_\theta E$. It will be useful to remember for later use that the θ term is the integral of $E_\theta E$.

Let us start with empty space permeated by the background field E_θ and θ chosen to be positive. (Negative values lead to the same physics upon making suitable sign changes.)

Suppose we introduce a fermion-antifermion pair $F\bar{F}$ with the antifermion to the left. We see that E_T starts at the value $\frac{e\theta}{2\pi}$ at $x=-\infty$ and drops down to $e(\frac{\theta}{2\pi}-1)$ between the charges and goes back to $e\frac{\theta}{2\pi}$ to the right of F . The reader can verify easily that in the region between charges there is an extra energy per unit length $\delta U = 1 - \frac{\theta}{\pi} > 0$. Thus charges are linearly confined in a generic case. (Had the F occurred to the left, the field between F and \bar{F} and confining potential would have been even larger.)

However, at $\theta=\pi$, the charges can be separated with no extra cost (ignoring residual short range forces) *provided \bar{F} lies to the left of F* . (A similar thing happens at $\theta=-\pi$, where the external field is reversed, provided we reverse the ordering of F and \bar{F} .) We can introduce any number of such pairs at finite cost provided F and \bar{F} alternate.

The particles at $\theta=\pi$ were described as half-asymptotic by Coleman. These are the naive “spinons” of this problem. They do not constitute genuine deconfined degrees of freedom because of the restriction that fermions and antifermions alternate. For example, if we are sitting next to a fermion, millions of miles away from any other matter, we know that on either side of it must lie antifermions. What we need are objects that can be created with no such restriction. We will find them, but they will be related to F and \bar{F} in a very complicated way. There will also be half as many of them.

To proceed we must bosonize. For our purposes, bosonization is a set of rules²⁻⁶ given by

$$\psi^\dagger \alpha p \psi = \frac{\pi^2 + (\nabla \phi)^2}{2}, \quad (4)$$

$$\psi^\dagger \beta \psi = -c\Lambda \cos(\sqrt{4\pi}\phi), \quad (5)$$

$$\psi^\dagger \psi = \rho = \frac{1}{\sqrt{\pi}} \frac{\partial \phi}{\partial x}, \quad (6)$$

where Λ is the cutoff, and c is a constant. Comparing the last equation to Gauss’s law we see that ϕ is essentially the electric field due to the dynamic charges

$$E = \frac{e}{\sqrt{\pi}} \phi \quad (7)$$

and that the net fermion charge is proportional to $\phi(\infty) - \phi(-\infty)$. We will limit ourselves to the neutral sector so that we may choose $\phi(\infty) = \phi(-\infty) = 0$.

The bosonized Hamiltonian becomes, upon doing some integrations by parts (recalling that ϕ vanishes at infinity), defining $\phi + \frac{\theta}{2\sqrt{\pi}}$ as the new ϕ , and using $\nabla^2|x-y| = 2\delta(x-y)$,

$$H = \int dx \left[\frac{\pi^2 + (\nabla \phi)^2}{2} + \frac{e^2}{2\pi} \phi^2 - mc\Lambda \cos(\sqrt{4\pi}\phi - \theta) \right]. \quad (8)$$

Note that due to the shift, ϕ now is proportional to the *total electric field* E_T due to external and dynamic charges and that in the massless case $m=0$, θ drops out.

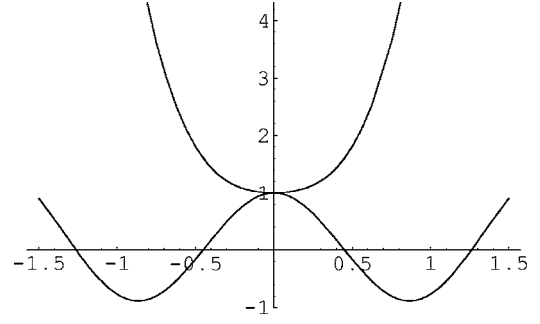


FIG. 1. The scalar potential $v(\phi)$ for $\gamma=\theta-\pi=0$ for two cases, with single and double minima. When $\gamma \neq 0$, there is a single asymmetric minimum.

Let us examine the scaled potential energy [Fig. 1]

$$v(\phi) = \frac{V(\phi)}{cm\Lambda} = \lambda \phi^2 + \cos(\sqrt{4\pi}\phi + \gamma), \quad (9)$$

where

$$\gamma = \pi - \theta \quad (10)$$

and

$$\lambda = \frac{e^2}{2\pi m\Lambda c}. \quad (11)$$

First consider weak coupling $\lambda \ll 1$. Start with $\gamma=0$. The cosine has an infinite number of minima. Focus on the ones nearest to the origin, near $\phi = \pm \frac{\sqrt{\pi}}{2}$, since the other minima will be lifted to higher energies when we turn on the $\lambda \phi^2$ term. These two values of ϕ correspond to total electric fields $\pm e/2$. How do we get two values for E_T ? The value $e/2$ corresponds to the background field at $\theta=\pi$ while the one at $-e/2$ corresponds to the case when an \bar{F} and F have been produced and sent to $\mp\infty$ (at zero energy cost per unit volume) to produce a degenerate configuration. For a small $\gamma > 0$, there is just one minimum near $\phi = \sqrt{\pi}/2$. As the potential $v(\phi)$ has a unique minimum, there are no solitons and hence no fermions. This is the effect of linear confinement at $\theta \neq \pi$. As γ passes through zero, to negative values, $\langle \phi \rangle$ jumps to roughly $-\sqrt{\pi}/2$. The transition at $\gamma=0$ is thus first order. (When γ becomes negative, θ exceeds π , at which point an $F\bar{F}$ pair is sent to spatial infinity to bring θ to the allowed interval $|\theta| \leq \pi$.)

Let us now focus on the critical line $\gamma=0$ (that is $\theta=\pi$) and slowly raise λ . At small λ , we have two degenerate minima in ϕ near $\pm\sqrt{\pi}/2$. Solitons connecting these two vacua can be identified with the old F and \bar{F} . However, since we have only two minima, an increase in $\langle \phi \rangle$ must be followed by a decrease in $\langle \phi \rangle$ and vice versa, i.e., particles must alternate with antiparticles. These solitons and antisolitons are the half-asymptotic particles.

At some very large value of λ (or e^2), the curvature of the bare potential $v(\phi)$ at the origin is positive and there is just one ground state at $\phi=0$. There are only neutral bosons beyond this value of λ , and no half-asymptotic fermions. Due to quantum fluctuations the phase transition will occur at a

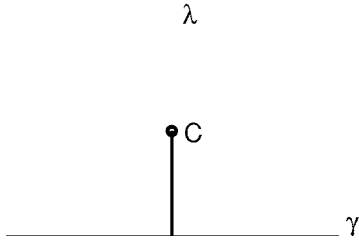


FIG. 2. Phase diagram in the $\lambda = e^2/2\pi m\Lambda c$ and $\gamma = \pi - \theta$ plane. The dark line denotes a first order transition terminating at C , a second order critical point. In the Ising model mapping, $\lambda - \lambda_C$ plays the role of t , the temperature, and $\gamma = \pi - \theta$ plays the role of the magnetic field h .

smaller value, λ_C . The phase diagram is then as shown in Fig. 2. This concludes the review of past work.

We now ask about the nature of C , the end point of the line of first order transitions. As the only symmetry in the problem is $\phi \rightarrow -\phi$, the transition is generically in the Ising class. This has been noticed before during the course of an analysis of confinement and deconfinement in a certain limit of the multifrequency Sine-Gordon model.⁷ Clearly $\lambda - \lambda_C$ plays the role of temperature t and $\gamma = \pi - \theta$ plays the role of the symmetry breaking magnetic field h . Let us discuss this in more detail. For small λ , i.e., deep in the ordered side, the low energy configurations are a series of kinks as the ground state jumps from one value of $\langle \phi \rangle$ to its negative. We can then describe the kinks by domain wall fermion operators c^\dagger , c . They are fermionic because there can only be zero or one wall at a site. (They could also be hard core bosons. In one dimension these are equivalent, but it is a lot easier to deal with noninteracting fermions than hardcore bosons.) Bear in mind that these fermions do not care if we are going from an up to a down vacuum or vice versa, unlike in the Schwinger model where fermions and antifermions were used to describe the two kinds of transitions. This point will be elaborated on shortly. These domain wall fermions can move and also annihilate in pairs. It is well known that this wall dynamics describes the Ising model in operator form (using the transfer matrix). Equivalently, we can associate an Ising spin σ_3 with the two vacua (call them “up” and “down” vacua) and describe the low energy physics with the Ising Hamiltonian

$$H_I = - \sum_n \sigma_3(n) \sigma_3(n+1) - (1+t) \sum_n \sigma_1(n), \quad (12)$$

where the first term assigns a cost to domain walls and the second term allows spin flip, i.e., vacuum fluctuations and $t=0$ is the critical point. Let us define two Majorana fermions ψ_1 and ψ_2 [related to the Hermitian operators $c + c^\dagger$ and $i(c - c^\dagger)$] by the strings

$$\psi_{1,2}(n) = \frac{1}{\sqrt{2}} \left(\prod_{k=-\infty}^{n-1} \sigma_1(k) \right) [\sigma_{2,3}(n)] \quad (13)$$

and which obey the anticommutation rules

$$[\psi_\alpha(n), \psi_\beta(m)]_+ = \delta_{\alpha\beta} \delta_{mn}. \quad (14)$$

In terms of these

$$H_I = -2i \left[\sum_n \psi_1(n) \psi_2(n+1) + (1+t) \sum_n \psi_1(n) \psi_2(n) \right]. \quad (15)$$

It is these Majorana fermions that are truly liberated at the first order transition $\gamma=0$. They can be created at will. Unlike the Schwinger fermions or antifermions that *shift* ϕ in one direction or another in a half-space, the Majorana fermions *reflect* ϕ to minus the value in a half space. While the half-asymptotic fermion creation operator of the Schwinger model only act on the “down” vacuum sending it to “up” in half of space (and the antifermion does the opposite), the Majorana fermion can act on either vacuum and flip it to the other. We may schematically represent these two kinds of fermions by the strings

$$\Psi_{\text{Schwinger}}^\dagger \simeq \prod_{k=-\infty}^{n-1} \sigma_+(k), \quad (16)$$

$$\psi_{\text{Ising-Majorana}} \simeq \prod_{k=-\infty}^{n-1} \sigma_1(k). \quad (17)$$

It is clear that there is no simple relation between the two. To summarize: (i) the end point of the first-order transitions at $\gamma=0$ or $\theta=\pi$ is an Ising transition, the point C in Fig. 1. (ii) The charged fermions of the Schwinger model are deconfined at the first-order transition at $\theta=\pi$ but are half asymptotic, while the Majorana fermions are truly deconfined objects and correctly describe the second order end point. The Majorana fermions, being domain walls, are also confined away from the line $\gamma=0$ since the spins in one side or other of a domain wall are pointing opposite to the applied field. (iii) The number of truly deconfined objects is half that of the half-asymptotic particles. (iv) There is no simple relation between the two kinds of particles.

Note that the fact that Majorana fermions, related to domain walls of the ordered side, also describe the disordered side (where $\langle \phi \rangle = 0$). This is due to the self-duality of the Ising model: if we formed Majorana kinks out of the disorder variables

$$\mu_1(n) = \sigma_3(n) \sigma_3(n+1), \quad \mu_3(n) = \prod_{k=-\infty}^n \sigma_1(k) \quad (18)$$

we would get essentially the same operators as in Eq. (13) except for the exchange between ψ_1 and ψ_2 . Of course, these fermions would cease to exist deep in the disordered phase when λ exceeds the value beyond which the bare potential $v(\phi)$ has only a single minimum at $\phi=0$.

III. EXAMPLE 2: THE SPIN CHAIN AND SPINONS

One place where spinons arise is in the study of the following Hamiltonian of spin- $\frac{1}{2}$ degrees of freedom studied in depth by Haldane:⁸

$$H = J_1 \sum_n \mathbf{S}(n) \cdot \mathbf{S}(n+1) + J_2 \sum_n \mathbf{S}(n) \cdot \mathbf{S}(n+2) + \gamma \sum_n (-1)^n \mathbf{S}(n) \cdot \mathbf{S}(n+1). \quad (19)$$

Let us rescale J_1 , the coefficient of the Heisenberg interaction, to unity. With only this term, the model is exactly solvable and known to have power law decay of all correlations. We are interested in the case⁸ wherein $J_2 > J_{2c}$, when the system exhibits spontaneous dimerization, that is

$$\langle (-1)^n \mathbf{S}(n) \cdot \mathbf{S}(n+1) \rangle = \pm \Omega_{\text{SP}}, \quad (20)$$

where “SP” denotes the degenerate Spin-Peierls ground states. Let us call them “up” and “down” vacua, respectively. In such a state the bond energy between neighboring spins alternates in strength as we move across the lattice. In the last term, the coefficient γ is an external field that couples to this order parameter and chooses one of the two degenerate states. It will be turned off in what follows unless otherwise stated. It is known that the case $J_2 = 0.5J_1$, the Majumdar-Ghosh Hamiltonian,⁹ lies in this dimerized phase. At this point H is a sum of operators that project any three spins into a triplet state. It follows that when any two of the three form a singlet it is annihilated by this projector. The two ground states then are of the form

$$|\text{down}\rangle = \cdots B_{12} B_{34} B_{56} \cdots, \quad (21)$$

$$|\text{up}\rangle = \cdots B_{23} B_{45} B_{67} \cdots, \quad (22)$$

where B_{ij} is a singlet formed out of the spins at sites i and j . In the up (down) state the strong bond has an even (odd) numbered site at its left end.

A spinon at $n=0$ is imagined as follows. From $-\infty$ to $n=-1$ we have the up vacuum, at $n=0$ there is a free (unpaired) spin, followed by the down vacuum whose strong bond begins at $n=1$. Further down the chain we can insert an antispinon, but it has to be at an odd site so we can switch back to the up vacuum. It appears that there are four states open to a spinon: it can have its spin up or down and it can occur as we go from the up to down vacuum or the other way around. Note however that an antispinon that switches from up vacuum to down must be followed by a spinon that switches from down to up. This alternation is reminiscent of the $F\bar{F}$ pairs in the Schwinger model. We shall see once again that compared to these degrees of freedom that are half-asymptotic, the number of really independent deconfined degrees of freedom are half as many. Note that the third term in Eq. (19) (proportional to γ) if turned on, will lead to a confinement of spinons just like a magnetic field confines domain walls in the Ising model. The half-asymptotic spinons are thus deconfined at a first order transition when γ vanishes.

We shall exhibit the half-asymptotic nature of these spinons in another language which parallels the Schwinger model more closely. First we begin with Haldane's mapping¹⁰ of the spin chain to the nonlinear sigma model

with the following Lagrangian density: The Lagrangian density in terms of the unit three-vector representing Néel order \mathbf{n} is

$$\mathcal{L} = \frac{1}{2g^2} (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n}) + i\lambda(|\mathbf{n}|^2 - 1) + \frac{i\theta}{4\pi} \mathbf{n} \cdot \frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial \tau}. \quad (23)$$

The vector \mathbf{n} is normalized to unity by the Lagrange multiplier field λ and g is the coupling constant related to the microscopic spin hamiltonian in some manner that does not concern us. The parameter θ equals π when $\gamma=0$.

This model in turn can be mapped into the CP^1 model¹¹ with the Euclidean Lagrangian density

$$\mathcal{L} = \int d^2x \left[\frac{2}{g^2} |D_\mu z|^2 + i\lambda(\bar{z}z - 1) + \frac{i\theta}{2\pi} \varepsilon_{\mu\nu} \partial_\mu a_\nu \right], \quad (24)$$

$$D_\mu = \partial_\mu - i a_\mu, \quad a_\mu = -i \bar{z} \partial_\mu z, \quad (25)$$

where z is a two-component spinor, related to \mathbf{n} via

$$\mathbf{n} = \bar{z} \sigma z. \quad (26)$$

Note that the gauge field a_μ does not have any dynamics at this stage.

The equality of the two Lagrangians can be shown by using Fierz identities such as

$$\sigma_{ab} \cdot \sigma_{cd} = -\delta_{ab} \delta_{cd} + 2\delta_{ad} \delta_{bc}. \quad (27)$$

The relation $\mathbf{n} = \bar{z} \sigma z$ implies that z may be rotated in phase from point to point, which is why we get a gauge theory. The reason such gauge theories are attractive is that spinor excitations appear naturally in them. On the other hand, spinor excitations are difficult, if not impossible, to construct if one works directly with the vector \mathbf{n} .

If z had N components instead of 2, with $N \rightarrow \infty$, we would proceed as follows:¹² (i) Integrate over the z quanta of which there are N species. (ii) Find the saddle point of the effective action with an N in front of the trace-log coming from the integration. (iii) Note that λ has a mean value (which gives a mass M to the z) and negligible fluctuations. (iv) Note that the field a_μ acquires dynamics, i.e., has an action proportional to $F_{\mu\nu}^2$ which mediates a linear potential between the z quanta. (v) Observe that the θ or topological term corresponds to a background field with charges $\theta/2\pi$ times the z -quanta charge. One way to see this is to write the term as a line integral around the boundary of space-time using Stokes' theorem. At any given time slice we intercept two Wilson lines of charges $\pm\theta/2\pi$ at spatial infinity. From our discussion of this term in the Schwinger model we see that $\varepsilon_{\mu\nu} \partial_\mu a_\nu$ is the electric field of the matter charges.

While the large N picture is incorrect for $N=1$ if J_2 is small, it correctly describes the dimerized phase, $J_2 > J_{2c}$ because here the z quanta are massive, as assumed in the derivation of the effective action for a_μ . The effective theory at low energies is thus given by the Lagrangian density

$$\mathcal{L} = |(\partial_\mu - ie a_\mu)z|^2 - M^2 \bar{z}z - \frac{1}{4} F_{\mu\nu}^2 + \frac{ie\theta}{2\pi} \varepsilon_{\mu\nu} \partial_\mu a_\nu. \quad (28)$$

The a_μ field has been rescaled to get to this form.¹² Note that in the effective theory, a_μ is an independent field, not slaved to z , and e is a parameter arising during z integration.

We see that this is just the Schwinger model with the trivial modification that the particles and antiparticle carry spin and are bosons and the global symmetry group is $SU(2)$. The θ term can be viewed as the product of the external field E_θ due to boundary charges $\pm e\theta/2\pi$ and the electric field due to dynamic charges, $E = \varepsilon_{\mu\nu} \partial_\mu a_\nu$. At $\theta = \pi$ (or $\gamma = 0$) the total E_T field reverses sign whenever one crosses a particle or antiparticle since the particle charge is double the external charges at $\pm\infty$.

There is thus very strong circumstantial evidence that the z and \bar{z} quanta are the half-asymptotic spinons previously caricatured as the dangling spins that separated the two dimerized states with opposite values of Ω_{SP} . To firm up the connection we need to show that the electric field $\varepsilon_{\mu\nu} \partial_\mu a_\nu$ which goes up every time we cross a z or goes down when we cross a \bar{z} is none other than the order parameter density $(-1)^k \mathbf{S}(k) \cdot \mathbf{S}(k+1)$ that does the same when we cross a spinon or antispinon.

We offer two proofs. First, we can argue that if we change the Lagrangian by adding a term proportional to the integral of $\varepsilon_{\mu\nu} \partial_\mu a_\nu$ this changes θ away from π and causes linear confinement, which is exactly what happens if we add a term proportional to the Spin-Peierls order parameter, i.e., turn on γ in Eq. (19).

A more microscopic answer can be given to readers more familiar with the sigma model description of the spin chain in terms of the unit three vector $\mathbf{n} = \bar{z}\sigma z$ and the associated angular momentum $\mathbf{L} = \mathbf{n} \times \partial \mathbf{n} / \partial t$. Starting with Haldane's decomposition¹⁰ of \mathbf{S} , in terms of \mathbf{a} , the lattice spacing \mathbf{L} and \mathbf{n} ,

$$\begin{aligned} \Omega_{SP} &= (-1)^k \mathbf{S}(k) \cdot \mathbf{S}(k+1) \\ &= (-1)^k [\mathbf{aL}(k) + (-1)^k \mathbf{n}(k)] \cdot (k \rightarrow k+1) \\ &= \mathbf{aL}(k) \cdot [\mathbf{n}(k) - \mathbf{n}(k+1)] + \text{oscillating terms} = \\ &\quad -a^2 \mathbf{L} \cdot \nabla \mathbf{n} \rightarrow a^2 \mathbf{n} \cdot \frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial t}, \end{aligned}$$

which is proportional to the topological density in the \mathbf{n} language, which in turn is given in the CP^1 language by $\varepsilon_{\mu\nu} \partial_\mu a_\nu$. Thus we have related Ω_{SP} to the electric field of the CP^1 model.

If the z quanta are half-asymptotic what are the really deconfined particles and how many of them are there? Once again bosonization provides the answer, now applied to the Jordan Wigner fermions formed from spin operators. The details will not be furnished here since they are copiously described in the published literature (see, for example, Refs. 5, 6, and 8).

The bosonized Lagrangian density is

$$\mathcal{L} = \alpha [(\partial_\tau \phi)^2 + (\nabla \phi)^2] - v \cos(4\phi), \quad (29)$$

where α and v are functions of J_2 (and J_1 which can be set to unity by overall rescaling). The normalization of the field ϕ is as in Sachdev's book⁵ and but not the same as the one used in the Schwinger model. Space-time anisotropy has also been removed by a suitable rescaling of coordinates. The only important points are the following. (i) The coupling v becomes negative for $J_1 > J_{2c}$, when we enter the Spin-Peierls state. This is the regime of interest to us. (ii) The boson field is related to the underlying spin variables as follows:

$$\sin(2\phi) = (-1)^n \mathbf{S}(n) \cdot \mathbf{S}(n+1) \equiv \Omega_{SP}, \quad (30)$$

$$\cos(2\phi) = (-1)^n S_z(n) \equiv \Omega_N, \quad (31)$$

$$\sum_k S_z(k) = \frac{\phi(\infty) - \phi(-\infty)}{2\pi}, \quad (32)$$

where the labels SP and N on the order parameters stand for Spin-Peierls and Néel, respectively. (iii) The underlying spin variables are invariant under $\phi \rightarrow \phi + \pi$. Thus for example, even though there may seem to be an infinite number of vacua in ϕ they describe a finite number of physically distinct situations.

Let us examine the vacuum and soliton structure of the potential energy

$$v(\phi) = |v| \cos(4\phi) \quad (33)$$

whose minima are at

$$\phi^* = (2m+1) \frac{\pi}{4}. \quad (34)$$

At these minima Ω_N vanishes while

$$\Omega_{SP} = \pm 1 \text{ according as } m \text{ is even/odd.} \quad (35)$$

Let us again refer to these vacua where $\Omega_{SP} = \pm 1$ as up and down vacua. Note that there are just two kinds of vacua here, depending on whether m is even or odd. In other words even though m has an infinite range of values, physics depends on m modulo 2.

Suppose we start out with $m=0$ (up vacuum) at $x=-\infty$. We can go to the down vacuum either by going to $m=-1$ or $m=+1$. These two values of m differ by 2, the corresponding ϕ 's differ by π , which makes these vacua physically equal as far as the spins go. However, these solitons carry spins $S_z = \mp \frac{1}{2}$ depending on which way we move in ϕ . These are the two spin states of the spinon that takes an up vacuum to a down vacuum. However, no distinction is made between this and a spinon that connects a down vacuum to an up vacuum, one that starts out at say $m=1$ and ends up at $m=0$ or $m=2$ corresponding to $S_z = \mp \frac{1}{2}$. That is, these domain walls are once again neutral. There is just one kind of domain wall here, it carries spin and separates the two vacua, and the same domain wall implements an up to down or down to up transition.

On considering the partition function of the lattice CP^1 model invented by Seiberg,¹³ at $\theta = \pi$, Affleck¹⁴ has noticed that since particles and antiparticles have to alternate (if we

want zero coulomb energy), the charge label is not needed and the partition function, which is a sum over loops reduces to that of a system of neutral particles, i.e., with no orientation. He comments that (for general n in the CP^{n-1} model) the $SU(n)$ model seems to have reduced to an $O(n)$ model. We emphasize that despite this reduction upon the restriction of the full model to the zero coulomb energy sector, the symmetry of the CP^1 model is firmly anchored at $SU(2)$, with the spinons with $S_z = \pm \frac{1}{2}$ forming an isospinor. A different but related perspective on the ideas of this section has been formulated by Mudry and Fradkin,¹⁵ who show that the naive spinons in a strongly coupled gauge theory can never truly deconfine in 1D, and that the truly free objects are solitons.

IV. CONCLUSIONS AND OUTLOOK

We re-examined deconfinement in $d=1$, which is after all the source of many of the most fruitful ideas in the physics of strongly correlated electrons. We found that the situation was more complex than is generally appreciated. In the two cases we studied, the massive Schwinger model and dimerized spin chain, the particles often viewed as deconfined were only half-asymptotic (had to be created with particles and antiparticle alternating). However, in both cases there were also truly deconfined asymptotic particles, but these were half-as many in number and related in a complicated way to the more obvious but half-asymptotic ones. In both the Schwinger model and the spontaneously dimerized spin-1/2 chains there were two vacua, called up and down, and the half-asymptotic particles and antiparticles were associated with the passage from down to up or up to down vacua, respectively. The truly asymptotic ones were associated with domain walls that simply lay between any two vacua, that is say, they seemed “neutral.” In the Schwinger model the original fermions and antifermions were half-asymptotic on critical line $\theta=\pi$ while the Majorana fermion was truly asymptotic and described the Ising transition. In the magnetized problem, there were two kinds of domain walls going from down to up and up to down vacua, each with two values of spin. In the CP^1 version we had particles and antiparticles, namely, z and \bar{z} quanta, each with two values of spin. Both were half-asymptotic. The truly asymptotic particles, however, only carried spin and could connect any one vacuum to the other.

In other words, we could either work with half-asymptotic particles that came in two varieties (associated with up to down and down to up walls or particle and antiparticles) but occurred with the alternating constraint, or with fully asymptotic particles (that were neutral) but could occur freely. If one works with half-asymptotic particles but forgets the constraint that particles and antiparticles alternate, one will be off by a factor of two in things such as specific heat.

In these examples the doubling occurred only in the gauge description of the problem. Why not just limit ourselves to the non gauge descriptions that did not have this redundancy? The answer is that the very notion of deconfinement exists only in the gauge version and gauge models are the focus of a lot of attention, and rightly so, since they allow

one to introduce fractionalized objects into the theory.

It is interesting to ask how the Ising model with its domain walls could be mapped into a gauge theory. We first attach an extra label to the domain walls calling the ones interpolating from down spin to up spin vacua as particles those that do the reverse as antiparticles. However, we must now demand that particles and antiparticles alternate. One way to enforce this is to couple them to a gauge potential that produces linear confinement, introduce a background field corresponding to $\theta=\pi$, and let them deconfine as half-asymptotic particles forced to alternate in the low-energy sector, i.e., end up with the massive Schwinger model.

One might think that the factor of two redundancy in the gauge version is due to the usual redundancy of gauge descriptions, i.e., because the Schwinger fermions and the z quanta are not gauge invariant. To see that this is not so one need only consider QED in the Coulomb phase, say in three dimensions: The electron and positron operators are not gauge invariant, but these particles are very much part of the spectrum. The reduction in true degrees of freedom arises because we want to limit ourselves to the low energy sector, with no Coulomb energy. In the Schwinger model for example, we can have fermions and antifermions in any order if we are willing to go to states of high electrostatic energy. A better analogy is to consider the restriction to the lowest Landau level in the FQHE problem. This restriction changes a lot of commutation relations, x and y become conjugate, the x - y plane becomes the phase space and the problem becomes one dimensional.

To better understand the impact on physics in $d>1$, we need to recall the history. This has to be necessarily brief given the rather long time separating today from Anderson’s proposal¹⁶ that spinons that appear in $d=1$ could also appear in $d=2$. For this to happen, the spin system must be in a featureless liquid state. Rokhsar and Kivelson¹⁸ introduced a model of hard core quantum dimers (representing spin singlet nearest-neighbor bonds) which they solved exactly at a particular point (known now as the RK point). At the RK point spinons are indeed deconfined. Read and Sachdev,¹⁷ using large N generalizations of $SU(2)$ spins, found that the systems like to either have Néel order or valence bond solid (VBS) order. They also pointed out that the Rokhsar-Kivelson model had order on both sides of the RK point, implying that it was an isolated point of deconfinement. This fact was made explicit by Moessner, Sondhi, and Fradkin,¹⁹ who appealed to the quantum Lifshitz theory introduced by Henley.²⁰ It turns out that the RK point is a result of fine tuning (in terms of underlying spins) and the deconfinement at criticality it exhibits is not generic.²¹ In the meantime, an entire deconfined phase has been uncovered²² by Moessner and Sondhi in the triangular lattice hardcore dimer model. Fractionalized deconfined phases have also been proposed in other contexts as well.²³ Most recently, the idea that deconfinement at criticality could be generic in a class of spin models that also exhibited non-Landau phase transitions (where two different order parameters vanished from either side) has been vigorously pursued by Senthil *et al.*²⁴

Can our study here shed any light on these works in $d>1$. While it is true that in $d>1$, as in $d=1$, spinons are confined by gauge fields, it does not appear that there is any

double counting either in the caricatures in terms of spins or gauge theory descriptions. The mechanism for deconfinement in $d=1$ (an external field getting reversed by the field due to the particles without bulk energy cost) does not seem applicable in $d=2$. In other words, it seems as if particles will be either confined or librated in $d=2$ with no room for half-asymptotic particles, the very notion being limited to $d=1$ where particle ordering makes sense. Furthermore, the deconfining transitions we considered were generically first order, while the focus in $d>1$ has been in continuous second-order transitions. The only common feature may be that the relation between deconfined particles and their con-

fined versions on either side of the transition is hopelessly complicated.

ACKNOWLEDGMENTS

We are grateful for many illuminating conversations with I. Affleck, Vadim Oganesyan, Subir Sachdev, T. Senthil, Shivaji Sondhi, Ashwin Vishwanath, and Xiao-Gan Wen. We acknowledge the National Science Foundation for Grant Nos. DMR 0354517 (R.S.) and DMR Grant No. 0371611 (GM) and the Aspen Center for Physics where this work was completed.

-
- ¹S. Coleman, R. Jackiw, and L. Susskind, *Ann. Phys. (N.Y.)* **93**, 267 (1975) and of more relevance to us S. Coleman, *ibid.* **101**, 239 (1976).
- ²S. Coleman, *Phys. Rev. D* **11**, 2088 (1975).
- ³R. Shankar (unpublished).
- ⁴E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison Wesley, Redwood City, CA, 1991).
- ⁵S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999). See in particular Chap. 14.
- ⁶T. Gimarchi, *Quantum Physics in One Dimension*, (Oxford University Press, Oxford, 2004), Chap. 7.
- ⁷G. Delfino and G. Mussardo, *Nucl. Phys. B* **516**, 675 (1998).
- ⁸The earliest and exhaustive discussion of the spin chain with a dimerized phase is by F. D. M. Haldane, *Phys. Rev. B* **25**, R4925 (1982).
- ⁹C. K. Majumdar and D. K. Ghosh, *J. Phys. C* **3**, 911 (1970).
- ¹⁰F. D. M. Haldane, *J. Appl. Phys.* **57**, 3359 (1985), *Phys. Rev. Lett.* **57**, 1488 (1986). See also the elaboration by I. Affleck, *Nucl. Phys. B* **265** 409 (1985).
- ¹¹For an introduction to the CP^N model see A. d'Adda, M. Luscher, P. Di Vecchia, *Nucl. Phys. B* **143**, 63 (1978).
- ¹²E. Witten, *Nucl. Phys. B* **149**, 285 (1979).
- ¹³N. Seiberg, *Phys. Rev. Lett.* **53**, 637 (1994).
- ¹⁴I. Affleck, *Phys. Rev. Lett.* **66**, 2429 (1991).
- ¹⁵C. Mudry and F. Fradkin, *Phys. Rev. B* **50**, 11409 (1994).
- ¹⁶P. W. Anderson, *Mater. Res. Bull.* **8**, 153 (1973), P. Fazekas and P. W. Anderson, *Philos. Mag.* **30**, 423 (1974).
- ¹⁷N. Read and S. Sachdev, *Phys. Rev. Lett.* **62**, 1694 (1989); *Phys. Rev. B* **42**, 4568 (1990).
- ¹⁸D. S. Rokhsar and S. A. Kivelson, *Phys. Rev. Lett.* **61**, 2376 (1988); See also E. Fradkin and S. Kivelson, *Mod. Phys. Lett. B* **4**, 225 (1990) who mapped it on to a gauge theory.
- ¹⁹R. Moessner, S. L. Sondhi, and E. Fradkin, *Phys. Rev. B* **65**, 024504 (2001).
- ²⁰C. L. Henley, *J. Stat. Phys.* **89**, 483 (1997); E. Ardonne, P. Fendley, and E. Fradkin, *Ann. Phys. (N.Y.)* **310**, 493 (2004).
- ²¹E. Fradkin, D. A. Huse, R. Moessner, V. Oganesyan, and S. L. Sondhi, *Phys. Rev. B* **69**, 224415 (2004); A. Vishwanath, L. Balents, and T. Senthil, *ibid.* **69**, 224416 (2004).
- ²²R. Moessner and S. L. Sondhi, *Phys. Rev. Lett.* **86**, 1881 (2001); *Phys. Rev. B* **68**, 054405 (2003).
- ²³T. Senthil, and M. P. A. Fisher, *Phys. Rev. B* **62**, 7850 (2000); R. Moessner, S. L. Sondhi, and E. Fradkin, *Phys. Rev. B* **65**, 024504 (2002); G. Misguich, D. Serban, and V. Pasquier, *Phys. Rev. Lett.* **89**, 137202 (2002).
- ²⁴T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, *Science* **303**, 1490 (2004); T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, *Phys. Rev. B* **70**, 144407 (2004).