Lecture 3:
Dynamics of the expansion: the evolution of the Universe.

L: We have seen that Einstein's equations, once combined with the cosmological principle, restrict the metric of the universe to be of the form of Robertson-Walker. This has a radius of curvature and the function $a(t)$ as free parameters. But Einstein equations also tell us how the function $a(t)$ is related to the mass-energy content of the Universe. The space curvature is determined by the radius of curvature, but the space-time curvature depends also on $a(t)$ and is the quantity that relates to the mass-energy density and pressure. This relation is what is called the Friedmann equation, and is derived from Einstein's equations.

We can't do this derivation here without going deeper into the mathematics of General Relativity, but we can see the Newtonian analogy by considering not a uniform universe, but a finite sphere that we cut off the Universe, of uniform density with radius $r_s(t)$, and mass $M_s = (4\pi/3)\rho(t) r_s(t)^3$.

Q: Consider a particle at the surface of the sphere. What is the acceleration of its motion due to the gravity of the sphere?

$$\text{Acceleration} = -\frac{GM_s}{r_s(t)^2} = \frac{d^2 r_s}{dt^2} \ . \tag{1}$$

L: Multiplying by dr_s/dt and integrating, we find

$$\frac{1}{2}\left(\frac{dr_s}{dt}\right)^2 = \frac{GM_s}{r_s(t)} + U \ . \tag{2}$$

This is just a statement of conservation of energy: the kinetic plus potential energy of the sphere must be conserved. Now, replacing $r_s(t) = a(t) R_s$, we find

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{R_s^2}\frac{1}{a(t)^2} \ . \tag{3}$$

Q: The constant U here is the total energy per unit mass. How will the sphere evolve if $U < 0$? How about if $U > 0$? And if $U = 0$?

L: Newton's theory gives us an analogous solution to the evolution of the Universe when we consider a small sphere, for which the expansion is non-relativistic. But in Newton's theory, an infinite, homogeneous Universe leads to contradictions: why should some points accelerate, and why should there be a center that does not accelerate? Which frame is inertial and which is not? The calculation of the force is inconsistent: it diverges in absolute value, so the result depends on the ordering of the contributions from distant mass elements. For example, if we cut out the Universe in ellipsoids we can get an anisotropic potential and collapse, a different result than if we cut it out in spherical shells.

Q: Consider the sum $1 + x - x^2/2 + x^3/3 - x^4/4 + ...$ For $x = 1$, what is the result? We obtain $\ln(2)$, but by reordering the sum we can get any result we want. Something similar happens when we compute the force from the homogeneous Universe in Newton's theory.

Q: How is the problem that no point should be accelerating in an absolute way resolved in General Relativity? In General Relativity there are no inertial frames valid over all space any more. There are instead free-falling frames, which are local. Every point in a homogeneous, expanding universe is in free-fall, or following a geodesic. They are all equivalent. Free-falling frames in different locations accelerate relative to each other owing to tides, but you cannot say which one is accelerating in an absolute sense. Acceleration of free-falling frames in different locations is also relative.

L: Relativity solution: the solution of Einstein's equations when we plug in the Robertson-Walker metric is Friedmann's equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{kc^2}{R_0^2}\frac{1}{a(t)^2} \ . \tag{4}$$

Important changes in the correct relativistic form: it is the total energy (including kinetic energy of particles, radiation and anything else) that matters. And the Newtonian total energy U is related to the space curvature.

Q: Let's apply Friedmann's equation at the present time. What is the left-hand side? H_0^2. So

$$H_0^2 = \frac{8\pi G}{3c^2}\epsilon_0 - \frac{kc^2}{R_0^2} \ . \tag{5}$$

L: We define the critical density as

$$\rho_c(t) = \frac{\epsilon_c(t)}{c^2} = \frac{3H^2(t)}{8\pi G} \ . \tag{6}$$

Q: If the total density is (greater, equal, less) than the critical density, what is the space curvature?

Exercise: For $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, check that the critical density at the present age of the Universe is $9.20 \times 10^{-30}\mathrm{g\,cm^{-3}} = 1.36 \times 10^{11}\,M_\odot\,\mathrm{Mpc^{-3}}$.

The total energy density parameter is: $\Omega(t) = \epsilon(t)/\epsilon_c(t)$. Therefore, once we measure the total energy density in the universe, we know the space curvature: closed for $\Omega > 1$, flat for $\Omega = 1$, open for $\Omega < 1$.

We can define this for every component of the Universe: for matter, $\Omega_m(t) = \epsilon_m(t)/\epsilon_c(t)$, for radiation $\Omega_r(t) = \epsilon_r(t)/\epsilon_c(t)$, etc. Present values are $\Omega_{m0} = \epsilon_{m0}/\epsilon_{c0}$, etc.

We can rewrite Friedman's equation, using the present day total cosmological energy density, $\Omega_0 = \epsilon_0/\epsilon_{c0} = \epsilon_0\, 8\pi G/(3c^2 H_0^2)$. From Friedman's equation at the present time, we derive $kc^2/R_0^2 = H_0^2(\Omega_0 - 1)$, so the radius of curvature of the Universe is $R_0 = cH_0^{-1}(k/(\Omega_0 - 1)^{1/2}$. We use also a present normalization of the scale factor $a_0 = 1$, so that $a = 1/(1+z)$. Then,

$$H^2(z) = H_0^2\Omega_0\epsilon(z)/\epsilon_0 + H_0^2(1-\Omega_0)(1+z)^2 \ . \tag{7}$$

If we know how the energy density varies with redshift, $\epsilon(z)$, we can compute the cosmic time t at redshift z. For example, if the universe has only matter, $\epsilon(z) = \epsilon_0(1+z)^3$, so

$$H^2(z) = H_0^2\Omega_0(1+z)^3 + H_0^2(1-\Omega_0)(1+z)^2 \ . \tag{8}$$

L: In general, we say that a galaxy is "at redshift z" to designate its distance. This means when the galaxy emitted the light we now receive, the universe was smaller by a factor $1+z$. We are interested in computing the age of the universe at redshift z,

$$dt = \frac{dt}{da}da = \frac{da}{H(z)a} = -\frac{dz}{H(z)(1+z)} \ ; \qquad t(z) = \int_z^\infty \frac{dz}{H(z)(1+z)} = \int_0^a \frac{da}{Ha} \ , \tag{9}$$

and the comoving distance: from $dr = c\,dt/a(t)$, we derive

$$r = c\int_0^z \frac{dz}{H(z)} \ . \tag{10}$$

L: These last two relations are generally valid: once we know $H(z)$ from the Friedmann equation, we can calculate $r(z)$ and $t(z)$ from the redshift, the quantity we observe. We can do this for any model, once we know R_0 and how ϵ varies with a.

Summary: Once we know all the components contributing to $\epsilon(z)$, and how they vary with redshift, we can compute the time and comoving distance as a function of redshift. This tells us also the evolution of the universe: how the scale factor varies with time. The total content of energy is related to the curvature of space.

L: Example: a flat universe containing only matter. In this case,

$$H(z)^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^3} = H_0^2(1+z)^3 \ . \tag{11}$$

So now we can find the time:

$$\int_{t_0}^{t_e} dt = -\int_0^z \frac{dz'}{H_0(1+z')^{5/2}} \ ; \qquad t_0 - t_e = \frac{2}{3H_0}\left(1 - \frac{1}{(1+z)^{3/2}}\right) \ . \tag{12}$$

The age of the universe at redshift z is

$$t(z) = -\int_\infty^z \frac{dz'}{H_0(1+z')^{5/2}} = \frac{2}{3H_0}\frac{1}{(1+z)^{3/2}} \ . \tag{13}$$

The comoving distance is

$$r = \int_0^z \frac{c\,dz}{H(z)} = \frac{2c}{H_0}\left[1 - \left(\frac{1}{1+z}\right)^{1/2}\right] \ . \tag{14}$$

L: Next example: curvature only (empty universe). Then we have:

$$(\dot{a})^2 = -\frac{kc^2}{R_0^2} \ . \tag{15}$$

R_0 is the present radius of curvature, and also the comoving radius if we choose the normalization $a_0 = 1$ at present (since $R_0 = a_0 R$).

Q: What is the solution? For $k = 0$ we must have a static universe, that is Minkowski space. For $k = -1$, we have an open, empty universe, $\dot{a} = \pm(c/R_0)$. The expanding universe has $a(t) = t/t_0$, with $t_0 = R_0/c$.

Q: What is the age of the universe in terms of the Hubble constant in this case? $H_0 = \dot{a}(t_0) = c/R_0$, so $t_0 = 1/H_0$.

Q: What is the Hubble constant at other times, $H(t)$? $H(t) = \dot{a}/a = 1/t = 1/(at_0) = (1+z)/t_0$.

Q: What is the comoving distance to a galaxy at redshift z?

$$r = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_0^z \frac{c\,dz}{H(z)} = ct_0 \int_0^z \frac{dz}{1+z} = ct_0 \ln(1+z) \ . \tag{16}$$

L: Another example (you can do it as exercise): we can have matter and curvature together.