Supplemental Lecture 21

Bosonization in 1+1 Dimensions and Solving the Schwinger Model

Introduction to the Foundations of Quantum Field Theory
For Physics Students
Part VIII

Abstract

We derive the duality map between fermions and bosons in 1+1 dimensions and apply this method to solve the Schwinger model, quantum electrodynamics in 1+1 dimensions. For massless interacting fermions, the solution is a free but massive neutral boson. The physics of the model is discussed from the perspective of quantum choromodynamics and other non-abelian gauge theories. The Schwinger model illustrates the physics of flux tubes, confinement, the Higgs mechanism, the chiral anomaly, chiral symmetry breaking, mass generation and theta-vacua in a simple setting.

Prerequisite: This lecture continues topics started in Supplementary Lectures 19 and 20.

This lecture supplements material in the textbook: Special Relativity, Electrodynamics and General Relativity: From Newton to Einstein (ISBN: 978-0-12-813720-8) by John B. Kogut. The term "textbook" in these Supplemental Lectures will refer to that work.

Keywords: Bosonization, 1+1 Dimensions, Correlation Functions, Operator Product Expansions, Confinement, Higgs Mechanism, The Chiral Anomaly, Chiral Symmetry Breaking, Mass Generation, The Schwinger Model. Quantum Electrodynamics in 1+1 Dimensions, Theta Vacua.

Contents

1. Introduction. Goals of This Lecture.

The Schwinger model, electrodynamics in 1+1 dimensions, was introduced and solved by J. Schwinger [1]. The model was popularized as a simple model of confinement in the early days of quantum chromodynamics [2]. In fact, the model stimulated a great deal of productive thinking about non-perturbative phenomena in field theory in an era previously dominated by perturbation theory. The model provided fundamental insights into the physical origin of the chiral anomaly, mass generation, confinement and the Higgs mechanics.

The problem of confinement in 3+1 dimensional quantum chromodynamics consists of the following challenges: can the theory's colored quarks be weakly coupled at short distances, but be strongly coupled at large distances so that they cannot be isolated? From the perspective of an experimentalist, can a theory exist in which its fundamental constituents can be observed and have their quantum numbers measured in high resolution experiments, like deep inelastic scattering, and yet those fundamental constituents cannot be produced individually in the final states of those experiments? Early in the quantum chromodynamics era, these ideas were highly debated, although now they are accepted as orthodoxy, even though they have not been derived from analysis of quantum chromodynamics. This is why model field theories were very important in the early days of quantum chromodynamics and the demonstration of confinement in the Schwinger model was an important step in the development of the field. In 3+1 dimensions one imagines that electric flux tubes form at large distances due to the running of the theory's coupling constant to large values at large distances, a fact supported by numerical simulations of SU(3) Yang Mills theory in 3+1 dimensions. In 1+1 dimensions, electric flux cannot spread out and the classical model confines charge with a linear force law so confinement in the Schwinger model with massive fermions is not surprising. However, the fact that the quantized theory is a free massive scalar field even in the limit where the fermion mass goes to zero is quite striking and inspirational. The quantum nature of the Schwinger model and its non-trivial vacuum are essential here. We have already seen how the Dirac sea plays a critical role in the theory's violation of chiral symmetry and its chiral anomaly. Here we will see that the chiral anomaly leads to the mass generation of the scalar field.

The solution of the Schwinger model will be done through bosonization in order to illustrate a non-perturbative method. Actually, the physics of the model can be extracted via perturbative methods because of the simplicity of kinematics in 1+1 dimensions. The bosonization method grew out of methods originally developed in statistical mechanics to analyze and solve two dimensional spin systems [3]. The cross fertilization between statistical field theory methods and high energy physics field theory methods was essential here and has led to many joint triumphs more recently.

2. Fermions in 1+1 Dimensions.

We have discussed fermions in 1+1 dimensions in the previous two lectures. In the chiral representation in 1+1 dimensions used in the previous lecture, the action reads,

$$S = \int d^2x \ \left(i\psi_+^\dagger(\partial_0 - \partial_1)\psi_+ + i\psi_-^\dagger(\partial_0 + \partial_1)\psi_-\right) \qquad 2.1$$

Here $\psi_\pm$ are chiral eigenstates as discussed in Sec. 4 of lecture 20. In that lecture we wrote out and studied the vector current $j^\mu = \bar{\psi}\gamma^\mu\psi$ and the axial vector current $j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$.

Let's write out these fields $\psi_\pm$ in terms of fermion creation and annihilation operators [4]. It follows from Eq. 2.1 that the chiral fermion ψ_- is right-moving. We interpret ψ_- as a quantized field from here forward, following the formalism for complex scalar fields introduced in lecture 14, except for the changes needed to execute bosons → fermions that will be discussed below. We use the Schrodinger picture where states have time dependence but operators do not. So, ψ_- will be expanded in terms of creation and annihilation operators for $p > 0$. Introduce a creation operator $c_p^{(-)\dagger}$ for anti-particles and an annihilation operator $b_p^{(-)}$ for particles,

$$\psi_-(x) = \int_0^\infty \frac{dp}{2\pi}\left(b_p^{(-)}e^{ipx} + c_p^{(-)\dagger}e^{-ipx}\right) \qquad 2.2a$$

And similarly for ψ_+ which is left-moving,

$$\psi_+(x) = \int_{-\infty}^0 \frac{dp}{2\pi}\left(b_p^{(+)}e^{ipx} + c_p^{(+)\dagger}e^{-ipx}\right) \qquad 2.2b$$

The creation and annihilation operators are postulated to obey anti-commutation relations,

$$\left\{b_p^{(\pm)}, b_q^{(\pm)\dagger}\right\} = \left\{c_p^{(\pm)}, c_q^{(\pm)\dagger}\right\} = 2\pi\delta(p-q) \qquad 2.3$$

with other anti-commutators vanishing. For example,

$$\left\{b_p^{(\pm)\dagger}, b_q^{(\pm)\dagger}\right\} = 0 \qquad 2.4$$

So $b_p^{(\pm)\dagger}b_p^{(\pm)\dagger} = 0$ which insures the Pauli exclusion principle, two fermions cannot occupy the same state, and $b_q^{(\pm)\dagger}b_p^{(\pm)\dagger} = -b_p^{(\pm)\dagger}b_q^{(\pm)\dagger}$ which enforces the fact that when fermions are interchanged, the wave function changes sign. The vacuum is defined by the conditions that it is annihilated by $b_p^{(\pm)}$ and $c_p^{(\pm)}$, $b_p^{(\pm)}|0>=$ $c_p^{(\pm)}|0>$=0. The $b_p^{(\pm)\dagger}$ creates particles and the $c_p^{(\pm)\dagger}$ creates anti-particles.

The anti-commutation relations Eq. 2.3 and 2.4 imply the basic anti-commutation relations for the field operators. For example, using Eq. 2.2a and 2.2b one computes,

$$\{\psi_\pm(x)\,, \psi_\pm^\dagger(y)\,\} = \delta(x-y) \qquad 2.5$$

with the other anti-commutators vanishing.

Field theories need careful analysis at very short distances and very high energies. To avoid ambiguous expressions we modify Eq. 2.2 to read,

$$\psi_-(x) = \int_0^\infty \frac{dp}{2\pi}\left(b_p^{(-)}e^{ipx} + c_p^{(-)\dagger}e^{-ipx}\right)e^{-p/2\Lambda} \qquad 2.6a$$

and

$$\psi_+(x) = \int_{-\infty}^{0} \frac{dp}{2\pi}\left(b_p^{(+)} e^{ipx} + c_p^{(+)\dagger} e^{-ipx}\right) e^{-|p|/2\Lambda} \qquad 2.6\text{b}$$

where Λ is a momentum cutoff which must be taken to ∞ at the end of all calculations. $1/\Lambda$ appears in many calculations so we define $\epsilon \equiv 1/\Lambda$.

We will need the vacuum expectation values of products of field operators [4]. For example,

$$< \psi_-(x)\psi_-^\dagger(y) > = \int_0^\infty \frac{dp\, dq}{(2\pi)^2} < b_q^{(-)}\, b_p^{(-)\dagger} > e^{iqx-ipy-(p+q)/2\Lambda}$$

$$= \int_0^\infty \frac{dp}{2\pi} e^{ip(x-y+i\epsilon)} = \frac{i}{2\pi}\frac{1}{(x-y)+i\epsilon} \qquad 2.7$$

Let's understand some aspects of this result. First, note that it depends on the difference $x - y$ as a consequence of translation invariance. In addition, it varies as $(x-y)^{-1}$ by dimensional analysis: each ψ_- carries the dimensions $\left[L^{-1/2}\right]$, so $\psi_-(x)\psi_-^\dagger(y)$ should vary as an inverse length and the only candidate is $x - y$. We also see that the high momentum cutoff Λ was essential to obtain a well-defined result when $x \to y$. This is reasonable because when $x \to y$, the high momentum fluctuations in the fields become especially important. If the theory were interacting, all these issues would come up again, but the dynamics would make analogous calculations more challenging! In such theories, the product of two operators at x and y and their singular behavior as $x \to y$ contains essential information concerning the theory's high energy dynamics.

Can we retrieve the basic anti-commutator Eq. 2.5 from this analysis? First, a short calculation like the one just done, shows that $< \psi_-^\dagger(x)\psi_-(y) >$ equals Eq. 2.7. Therefore, the vacuum expectation value of the commutator is,

$$< \{\psi_-(x), \psi_-^\dagger(y)\} > = \frac{i}{2\pi}\left(\frac{1}{(x-y)+i\epsilon} + \frac{1}{-(x-y)+i\epsilon}\right) = \frac{1}{\pi}\frac{\epsilon}{(x-y)^2+\epsilon^2} \qquad 2.8$$

and the right hand side becomes $\delta(x-y)$ as $\epsilon \to 0$, recalling a basic formula from one's introductory course in quantum mechanics,

$$\frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{(x-y)^2+\epsilon^2} = \delta(x-y)$$

Other bits of algebra we will need later are : $< \psi_+(x)\psi_+^\dagger(y) > = \ < \psi_+^\dagger(x)\psi_+(y) >$, and

$$< \psi_+(x)\psi_+^\dagger(y) > = -\frac{i}{2\pi}\frac{1}{(x-y)-i\epsilon} \qquad 2.9$$

2. Bosonization in 1+1 Dimensions.

It was known since the 1960s that there are intimate relations between fermion fields and the exponentials of boson fields in 1+1 dimensions [4]. They were part of the bag of tricks specialists in model field theories had developed. Parallel developments had occurred in statistical physics. For example, the Jordan-Wigner relation had been used to rewrite versions of the two dimensional Ising model in terms of Dirac and/or Majorana fermions. The Jordan-Wigner formula constructs fermion operators out of non-local products of Pauli matrices. This magic will be reviewed in a later lecture on two dimensional spin models. These analyses inspired the field theoretic bosonization methods we will discuss in this section. It leads to the solution of several 1+1 dimensional fermion field theories, like the Schwinger model, quantum electrodynamics in 1+1 dimensions, which we will discuss in the next section.

Building fermions out of bosons faces several hurdles. First, the construction must respect the fact that fermion fields satisfy basic anti-commutation rules while bosons satisfy commutation rules! And the mapping will only be useful if it takes local fermion actions to local boson actions. Since exponentials will be involved, let’s focus on boson fields that range in magnitude from zero to 2π. We start with the action,

$$S = \int d^2x \ \frac{1}{8\pi}\left(\partial_\mu\phi\right)^2 \qquad 3.1$$

where the coefficient is chosen with hindsight – to make the fermion ↔ boson correspondence work.

Let’s write down the conserved currents implied by the action Eq. 3.1. The Euler-Lagrange equations imply that ϕ satisfies the Klein-Gordon equation, $\partial^\mu\partial_\mu\phi = 0$. This implies that there is a conserved current

$$j^\mu = \frac{1}{4\pi} \ \partial^\mu\phi \qquad 3.2$$

The associated conserved charge is, $Q = \int dx\, j^0 = \frac{1}{4\pi}\int dx\, \dot{\phi}$, and the “phase” operator $e^{i\phi}$ carries $Q = 1$, as the reader should verify. In 1+1 dimensions, Eq. 3.1 gives rise to another conserved current (almost) trivially,

$$j_T^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\rho}\partial_\rho\phi \qquad 3.3a$$

Clearly, $\partial_\mu j_T^{\mu} = 0$ because $\epsilon^{\mu\rho}$ is anti-symmetric. The associated charge has a topological character: suppose the theory is defined on a circle of radius R instead of the linear x-axis. Then,

$$Q_T = \frac{1}{2\pi}\int_0^{2\pi R} dx\, \partial_1\phi \qquad 3.3b$$

and it counts the number of times ϕ goes over its range 2π as x winds around the circle.

We have seen a structure like the two currents j^{μ} and j_T^{μ} before. From Eq. 3.2 and 3.3, we have $j_T^{\mu} = 2\,\epsilon^{\mu\rho}\, j_\rho$. Compare this to j^{μ} and j_A^{μ} for free massless fermions. Reproducing Eq. 3.11 and 3.12 from lecture 20,

$$j^0 = \chi_+^{\dagger}\chi_+ + \chi_-^{\dagger}\chi_- \qquad j^1 = -\chi_+^{\dagger}\chi_+ + \chi_-^{\dagger}\chi_-$$

$$j_A^0 = \chi_+^{\dagger}\chi_+ - \chi_-^{\dagger}\chi_- \qquad j_A^1 = -\chi_+^{\dagger}\chi_+ - \chi_-^{\dagger}\chi_-$$

We read off,

$$j_A^{\mu} = \epsilon^{\mu\rho}\, j_\rho \qquad 3.4$$

for free fermions in 1+1 dimensions.

Returning to bosons, these points suggest that we introduce a second scalar field $\tilde{\phi}$ which satisfies,

$$\partial^{\mu}\phi = 2\,\epsilon^{\mu\rho}\,\partial_\rho\tilde{\phi} \qquad 3.5$$

so that $j_T^{\mu} = -\frac{1}{\pi}\partial^{\mu}\tilde{\phi}$ which pairs with Eq. 3.2. Eq. 3.5 can be written out as $\partial_0\phi = -2\partial_1\tilde{\phi}$ and $\partial_1\phi = -2\partial_0\tilde{\phi}$, relations we will need below.

In addition, the analogy with fermions suggest that we introduce "chiral bosons". The Klein-Gordon equation $\partial^\mu \partial_\mu \phi = 0$ can be written in light cone coordinates $(x^\pm = x^0 \pm x^1 = t \pm x)$, $\partial_+ \partial_- \phi = 0$, which is solved with chiral bosons,

$$\phi = \phi_-(t-x) + \phi_+(t+x) \qquad 3.6$$

Eq. 3.5 and 3.6 implies that $\tilde{\phi}$ can also be written in terms of chiral bosons,

$$\tilde{\phi} = \tfrac{1}{2}\big(\phi_-(t-x) - \phi_+(t+x)\big) \qquad 3.7$$

To see this, begin with Eq. 3.5, $\partial_0 \phi = -2\partial_1 \tilde{\phi}$ and $\partial_1 \phi = -2\partial_0 \tilde{\phi}$. Next, take linear combinations and find $\partial_+ \phi = -2\partial_+ \tilde{\phi}$ and $\partial_- \phi = -2\partial_- \tilde{\phi}$. If we substitute Eq. 3.6 and 3.7 here, we find consistency.

We can also solve Eq. 3.6 and 3.7 for $\phi_\pm$,

$$\phi_\pm = \tfrac{1}{2}\big(\phi \mp 2\,\tilde{\phi}\big) \qquad 3.8$$

We can verify these tricky relations. For example, using $\partial_0 \phi = -2\partial_1 \tilde{\phi}$ and $\partial_1 \phi = -2\partial_0 \tilde{\phi}$, we can compute

$$\partial_1 \phi_- = \frac{1}{2}\big(\partial_1 \phi + 2\partial_1\,\tilde{\phi}\big) = \frac{1}{2}\big(-2\partial_0 \tilde{\phi} - \partial_0\,\phi\big) = -\partial_0 \left(\frac{\phi + 2\,\tilde{\phi}}{2}\right) = -\partial_0 \phi_-$$

and learn that $(\partial_1 + \partial_0)\phi_- = \partial_+ \phi_- = 0$ which checks.

Now let's explicitly quantize the boson ϕ. To begin, we expand the operator $\phi(x)$ in plane waves,

$$\phi(x) = \sqrt{4\pi} \int \tfrac{dp}{2\pi\sqrt{2E}} \big(a_p e^{ipx} + a_p^\dagger e^{-ipx}\big) e^{-E/2\Lambda} \qquad 3.9a$$

where we have followed the canonical quantization procedures of lecture 14, using the action Eq. 3.1, and $E = |p|$. Again, we are using the Schrodinger picture here (states have time dependence but operators do not). Note the convergence factor $e^{-\epsilon E}$ where $\epsilon = 1/2\Lambda$. We can form the conjugate momentum field π accompanying ϕ from Eq. 3.1, $\mathcal{L} = \frac{1}{8\pi}\big(\partial_\mu \phi\big)^2$, $\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \frac{1}{4\pi}\dot{\phi}$, and write out this operator in plane waves using the canonical methods presented in Appendix A of lecture 15,

$$\pi(x) = -\frac{i}{\sqrt{4\pi}} \int \frac{dp}{2\pi} \sqrt{\frac{E}{2}} \left(a_p e^{ipx} - a_p^\dagger e^{-ipx}\right) e^{-E/2\Lambda} \qquad 3.9b$$

The $a_p^\dagger$ and a_p are creation and annihilation operators for bosons in momentum state p. If we choose the normalization for the basic boson commutator to be,

$$\left[a_p, a_q^\dagger\right] = 2\pi\, \delta(p-q) \qquad 3.10a$$

then we have the correct canonical commutator,

$$[\phi(x), \pi(y)] = i\, \delta(x-y) \qquad 3.10b$$

according to canonical quantization of scalar fields presented in lecture 14 and in the Appendices of lecture 15.

Now we turn to the special features of 1+1 dimensions. First, let's form the dual scalar field $\tilde{\phi}$ and then the chiral bosons $\phi_\pm$. The dual scalar field obeys $\partial_1 \tilde{\phi} = \frac{-1}{2} \dot{\phi} = -2\pi\, \pi(x)$, so we have to integrate to construct $\tilde{\phi}(x)$ from $\pi(x)$,

$$\tilde{\phi}(x) = -2\pi \int_{-\infty}^{x} dx'\ \pi(x') \qquad 3.11$$

and we see explicitly the non-local character of the dual scalar field $\tilde{\phi}(x)$. From Eq. 3.8 this implies that the chiral bosons are also non-local,

$$\phi_\pm = \frac{1}{2}\left(\phi \pm 4\pi \int_{-\infty}^{x} dx'\ \pi(x')\right) \qquad 3.12$$

Let's evaluate these expressions for $\phi_\pm$ by using the plane wave expansions for ϕ and π. Doing the integral over the plane waves in Eq. 3.12, we find,

$$\phi_-(x) = \sqrt{\pi} \int \frac{dp}{2\pi\sqrt{2E}} \left(1 + \frac{|p|}{p}\right) \left(a_p e^{ipx} + a_p^\dagger e^{-ipx}\right) e^{-\epsilon E}$$

$$= \sqrt{2\pi} \int_0^\infty \frac{dp}{2\pi\sqrt{E}} \left(a_p e^{ipx} + a_p^\dagger e^{-ipx}\right) e^{-\epsilon E} \qquad 3.13a$$

and we see that only modes with $p > 0$ contribute to ϕ_-. Similarly,

$$\phi_-(x) = \sqrt{2\pi} \int_{-\infty}^{0} \frac{dp}{2\pi\sqrt{E}} \left(a_p e^{ipx} + a_p^\dagger e^{-ipx}\right) e^{-\epsilon E} \qquad 3.13b$$

where $E = |p|$ and $\epsilon = 1/2\Lambda$.

Now we can calculate various correlation functions of $\phi_\pm(x)$ just as we did for chiral fermions. Begin with the commutator,

$$[\phi_\pm(x), \phi_\pm(y)] = \pm\pi \int_{-\infty}^{y} dy' \, [\phi(x), \pi(y')] \pm \pi \int_{-\infty}^{x} dx' \, [\pi(x'), \phi(y)] = \mp\pi i \; sgn(x-y) \quad 3.14$$

The non-locality of the fields is again apparent. In addition, the mixed commutator is,

$$[\phi_+(x), \phi_-(y)] = -i\pi \quad 3.15$$

which is a perplexing result: we would have expected zero in Eq. 3.15 but the $p = 0$ mode has made an unexpectedly finite contribution! Clearly it is easy to miss this contribution: the terms in Eq. 3.13 contain factors of $p/|p|$ which are ambiguous at $p = 0$. The real space calculations Eq. 3.14 and 3.15 are somewhat more reliable.

Now let's turn to connected correlation functions in real space and compare them to our earlier discussion of free chiral fermions. We need,

$$G_\pm(x,y) = < \phi_\pm(x)\phi_\pm(y) > - < \phi_\pm(0)\phi_\pm(0) > \quad 3.16$$

Substituting in the plane wave expansions Eq. 3.13,

$$G_-(x,y) = \pi \int_0^\infty \frac{dp \, dq}{(2\pi)^2} \frac{2}{\sqrt{pq}} < a_p a_q^\dagger > \left(e^{ipx - iqy} - 1\right) e^{-(p+q)/2\Lambda}$$

$$= \pi \int_0^\infty \frac{dp}{\pi p} \left(e^{ip(x-y)} - 1\right) e^{-p/\Lambda} = \ln\left(\frac{\epsilon}{\epsilon - i(x-y)}\right) \quad 3.17a$$

where $\epsilon = 1/\Lambda$, which was engineered to agree with the fermion calculation obtained earlier. The result is familiar: at large distances G_- behaves as the logarithm of the distance. This is characteristic of a massless boson in 1+1 dimensions.

A similar calculation also gives,

$$G_+(x,y) = \ln\left(\frac{\epsilon}{\epsilon + i(x-y)}\right) \quad 3.17b$$

Compare these results to the fermion calculation Eq. 2.7. They are (essentially) related by exponentiation! Could we make a precise correspondence between the products of chiral fermions and the products of exponentials of chiral boson fields? This is precisely where we are headed!

We want to consider operators like $e^{i\phi(x)}$. We can also consider derivatives such as $\partial_\mu\phi$. Other familiar operators like ϕ^2 are not appropriate here because they are not periodic. We should also be careful to normal order the operators so that in a product of several operators all the annihilation operators lie to the right of the creation operators (This is the basic definition of normal ordering). The normal ordering process is traditionally denoted by a pair of colons: $:\theta_1\theta_2 \ldots :$ Let's begin the discussion with the product of "vertex" operators, $: e^{i\phi(x)} :$. We have to manipulate the exponential of linear combinations of harmonic oscillator operators. We have a lot(!) of experience in this after studying coherent states in past lectures, such as lecture 14 and 15. We will need the Baker-Hausdorf Theorem for two operators, *A* and *B*, whose commutator, $[A, B]$, is a regular commuting number,

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{3.18a}$$

and,

$$e^A e^B = e^B e^A e^{[A,B]} \tag{3.18b}$$

To apply this theorem to $e^{i\phi(x)}$, we need only take the case $A = \alpha a + \beta a^\dagger$ and $B = \gamma a + \delta a^\dagger$, where a and $a^\dagger$ are harmonic oscillator annihilation and creation operators and $\alpha, \beta, \gamma, \delta$ are numbers. Then,

$$: e^A :: e^B : = e^{\beta a^\dagger} e^{\alpha a} e^{\delta a^\dagger} e^{\gamma a} = e^{\beta a^\dagger} e^{\delta a^\dagger} e^{\alpha a} e^{\gamma a} e^{\alpha\delta} =: e^{A+B} : e^{<AB>} \tag{3.19}$$

where we observed that $\alpha\delta =< \alpha\delta\, a a^\dagger > = < (\alpha a + \beta a^\dagger)(\gamma a + \delta a^\dagger) > = < AB >$. Now we can apply this simplification to the product of vertex operators $: e^{i\phi(x)} :$ because ϕ is just many terms of the form of operator *A*. The first application we want is the vacuum expectation value,

$$<: e^{i\phi_-(x)} :: e^{-i\phi_-(y)} : > = <: e^{i\phi_-(x) - i\phi_-(y)} : > \; e^{G_-(x,y)} = \frac{\epsilon}{\epsilon - i(x-y)} \tag{3.20a}$$

and similarly,

$$<: e^{i\phi_+(x)} :: e^{-i\phi_+(y)} : > = \frac{\epsilon}{\epsilon + i(x-y)} \tag{3.20b}$$

where we observed that $<: e^{i\phi_-(x)-i\phi_-(y)}: > = <: e^{i\phi_+(x)-i\phi_+(y)}: > = 1$ because the normal ordering makes the vacuum value of all the terms vanish except for the leading term 1. (Here the reader sees the real motivation for normal ordering!) We see here that our hunch was right! Exponentiating boson fields produces fermion correlators!

3. The Fermion-Boson Dictionary.

Now compare Eq. 3.20a and 3.20b with Eq. 2.7 and 2.9,

$$\psi_-(x) \leftrightarrow \sqrt{\tfrac{1}{2\pi\epsilon}} : e^{i\phi_-(x)}: \tag{4.1a}$$

and,

$$\psi_+(x) \leftrightarrow \sqrt{\tfrac{1}{2\pi\epsilon}} : e^{-i\phi_+(x)}: \tag{4.1b}$$

Note the presence of the ultraviolet cutoff in these formulas. In applications we will have to absorb these factors into renormalized parameters.

Now we turn to fermion bilinears, the observables of the fermion theory. The mass term is easy,

$$\bar{\psi}\,\psi = \psi_-^\dagger\psi_+ + \psi_+^\dagger\psi_- \leftrightarrow \tfrac{1}{2\pi\epsilon}\left(: e^{i\phi_-(x)}: : e^{i\phi_+(x)}: + h.c.\right) \tag{4.2}$$

But we can apply Eq. 3.19 here,

$$\bar{\psi}\,\psi \leftrightarrow -\tfrac{1}{2\pi\epsilon}\left(: e^{-i\phi(x)} + e^{+i\phi(x)}:\right) = -\tfrac{1}{\pi\epsilon} : \cos\phi(x): \tag{4.3a}$$

Similarly, the pseudo-scalar bilinear is,

$$\overline{\imath\psi}\,\gamma^5\psi \leftrightarrow -\tfrac{1}{\pi\epsilon} : \sin\phi(x): \tag{4.3b}$$

The boson forms of the fermionic vector and axial vector currents involve additional subtleties. Recall the currents written in terms of chiral fermion fields from the previous lecture,

$$j^0 = \psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_- \qquad\qquad j^1 = -\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_- \tag{4.4}$$

To find the boson expressions here, begin with $\psi_-^\dagger(x)\psi_-(y)$ and take $x \to y$. One general lesson we are learning in this lecture is that the product of operators can have simgularities which require careful analysis. This is called "Operator Product Expansion" methodology and is fundamental in field theory and its applications to high energy and statistical physics.

Consider first the product,

$$\psi_-^\dagger(x)\psi_-(x) = \lim_{y\to x} \psi_-^\dagger(x)\psi_-(y)$$

$$\leftrightarrow \frac{1}{2\pi\epsilon} \lim_{y\to x} :e^{-i(\phi_- x)}::e^{+i\phi_-(y)}: \; = \; \frac{1}{2\pi\epsilon} \lim_{y\to x} :e^{-i(\phi_-(x)-\phi_-(y))}: e^{G_-(x,y)}$$

Expanding,

$$\psi_-^\dagger(x)\psi_-(x) \leftrightarrow \frac{1}{2\pi\epsilon} \lim_{y\to x} \left(1 - i(x-y)\frac{\partial\phi_-(x)}{\partial x} + ..\right)\frac{\epsilon}{\epsilon - i(x-y)} = \frac{1}{2\pi}\frac{\partial\phi_-(x)}{\partial x} + \lim_{y\to x}\left(\frac{1}{2\pi(x-y)}\right) \qquad 4.5a$$

We can do a similar analysis for $\psi_+^\dagger(x)\psi_+(x)$ and summarize,

$$:\psi_\pm^\dagger(x)\psi_\pm(x): \leftrightarrow \frac{1}{2\pi}\frac{\partial\phi_\pm(x)}{\partial x} \qquad 4.5b$$

where we have normal ordered the fermion operators to eliminate the (divergent) constant on the right hand side of Eq. 4.5a. Now we can return to the fermion vector and axial vector currents,

$$j^0 \leftrightarrow -\frac{1}{2\pi}\frac{\partial(\phi_+ + \phi_-)}{\partial x} = -\frac{1}{2\pi}\frac{\partial\phi}{\partial x} \qquad 4.6a$$

And

$$j^1 \leftrightarrow -\frac{1}{2\pi}\frac{\partial(\phi_- - \phi_+)}{\partial x} = -\frac{1}{\pi}\frac{\partial\tilde{\phi}}{\partial x} = \frac{1}{2\pi}\frac{\partial\phi}{\partial t} \qquad 4.6b$$

where we recalled $\tilde{\phi} = \frac{1}{2}(\phi_- - \phi_+)$ and $-2\partial_1\tilde{\phi} = \partial_0\phi$ in the last step. We learn that the fermion vector current is related to the bosonic topological current! In particular,

$$j^\mu \leftrightarrow -j_T^\mu = -\frac{1}{2\pi}\epsilon^{\mu\rho}\partial_\rho\phi \qquad 4.7a$$

The same analysis relates the fermion axial vector current to the bosonic current,

$$j_A^\mu \leftrightarrow -j^\mu = -\frac{1}{4\pi}\partial^\mu\phi \qquad 4.7b$$

And collecting all these tricks together, some additional calculations using these methods show that one can also associate the kinetic energies of the dual theories,

$$\bar{\psi}\, i\gamma^{\mu}\partial_{\mu}\, \psi \leftrightarrow \frac{1}{8\pi}\left(\partial_{\mu}\phi\right)^{2} \tag{4.8}$$

What lessons have we learned from these constructions, Eq. 4.1-4.8? In the midst of the mapping, we had $\psi_{\pm}$ in the fermion description and ϕ, $\tilde{\phi}$ and $\phi_{\pm}$ in the boson description. However, we found a great simplification when we focused on the boson description of just the fermion action and fermion observables (bilinears): only the local, causal, canonical field ϕ appeared! This means that the mapping ends with a local, causal bosonic description of the fermionic Schwinger model! The reader should consult the Dictionary to see where the non-local fields $\tilde{\phi}$ and $\phi_{\pm}$ were replaced with ϕ in the fermion $\leftrightarrow$ boson correspondences for physical observables. So, $\tilde{\phi}$ and $\phi_{\pm}$ just served as scaffolding to accommodate the non-local character of the correspondence. But they disappear at the end!

With these results in hand, it is finally(!) time to have some fun! Let's solve the Schwinger model!

4. The Solution of the Schwinger Model.

Why should we care about 1+1 dimensional quantum electrodynamics? This model actually played an extraordinary role in the early days of Quantum Chromodynamics since it illustrates quark confinement, chiral symmetry breaking, mass generation by symmetry breaking and strong dynamics, and the importance of the quantum vacuum and Theta-vacua. Some of these points were reviewed in Section 1 above. Another point which is illustrated very nicely by the Schwinger model is the connection between quark confinement and chiral symmetry breaking [5]. In quantum chromodynamics a vector color current couples to the colored gauge fields. One models the field theoretic confinement mechanism using electric flux tubes. But this brings up a challenge. Note that the vector current of the fermionic Schwinger model Eq. 4.4 preserves chirality. But then it appears that the forces of confinement cannot turn the trajectories of massless quarks around and keep them confined into colorless states! In order for fermions to turn around under the influence of a vector force which preserves chirality, it must be that the fermion that turns around picks up chirality from the vacuum! In other words, the confinement mechanism is only successful if confinement implies chiral symmetry breaking and the vacuum consists of a chiral condensate, an unbounded source of chirality! So, confinement really has a multi-body aspect to be successful for massless constituent quarks! This picture of confinement has good support from computer simulations.

After all this heady talk and motivation, let's return to the modest environs of 1+1 dimensional quantum electrodynamics. The action of the model for massless fermions reads,

$$S = \int d^2x \, \left(\tfrac{1}{2}E^2 + \bar{\psi}\, i\gamma^\mu\left(\partial_\mu - eA_\mu\right)\psi\right) \qquad 5.1$$

Because of the model's dimensionality, it has some simple properties which are different from its more familiar 3+1 dimensional version. In particular, the electric charge has dimensions of [mass], so there is a natural distinction in the Schwinger model between high and low energies. The model is called "super-renormalizable" which means that its high energy behavior ($|p| \gg e$) is given by free fields while its low energy physics ($|p| \le e$) is strongly coupled. If we write out the covariant derivative, $\partial_\mu - eA_\mu$, in the action, we see that the vector fermion current couples to the gauge field, $eA_\mu j^\mu$. The conservation of the vector current is critical to the sensible properties of the theory. For example, under a gauge transformation, $A_\mu \to A_\mu + \partial_\mu G$, where G is an invariant gauge function. The action picks up an extra term $e\partial_\mu G j^\mu$. Integrating by parts, this becomes $-eG\partial_\mu j^\mu$, which only vanishes if the current is conserved locally. We will check that bosonization of the 1+1 model is consistent with current conservation.

Now bosonize Eq. 5.1 using the Dictionary of the previous Section,

$$S = \int d^2x \, \left(\tfrac{1}{2}E^2 + \tfrac{1}{8\pi}\left(\partial_\mu\phi\right)^2 + \tfrac{1}{2\pi} eA_\mu \epsilon^{\mu\rho}\partial_\rho\phi\right) \qquad 5.2$$

Doing an integration by parts on the third term and identifying the electric field $E = F_{01}$, we have

$$S = \int d^2x \, \left(\tfrac{1}{2}E^2 + \tfrac{e}{2\pi}\phi E + \tfrac{1}{8\pi}\left(\partial_\mu\phi\right)^2\right) \qquad 5.3$$

Now, the Eulerr-Lagrange equation for the scalar field ϕ produces its wave equation,

$$\partial^\mu\partial_\mu\phi = -2eE \qquad 5.4$$

But if we recall that the axial current is given by $j_A^\mu = -\frac{1}{2\pi}\partial^\mu \phi$, we see that Eq. 5.4 simply expresses the chiral anomaly that we struggled through in the previous lecture! In particular, substituting $j_A^\mu = -\frac{1}{2\pi}\partial^\mu \phi$ into the equation of motion Eq. 5.4, we have,

$$\partial_\mu j_A^\mu = \frac{e}{\pi} E \qquad 5.5$$

which is precisely correct. Now we can completely solve the Schwinger model by reading off the equation of motion for the electric field from the action Eq. 5.3,

$$E = -\frac{e}{2\pi}\,\phi \qquad 5.6$$

Substituting back into the action, we see that ϕ is a free scalar field of mass, $m^2 = e^2/\pi$, and the periodicity of ϕ has been lost,

$$S = \frac{1}{8\pi}\int d^2x\,\left(\left(\partial_\mu \phi\right)^2 - \frac{e^2}{\pi}\phi^2\right) \qquad 5.7$$

Note how many basic principles of quantum field theory this result illustrates! We started with massless fermions interacting over long distances with a strong electromagnetic field and we end with free, non-interacting, massive scalars. The long range linear potential provided by the electric field E confines the fermions, the strong force and the theory's quantum vacuum conspire to break chiral symmetry through the chiral anomaly derived in Eq. 5.5 here and studied extensively in the previous lecture, and the theory becomes a neutral free massive boson.

Now let's consider two modifications of the simplest Schwinger model in Eq. 5.1. First let's change the boundary conditions and put the model into a fixed external electric field [6]. We parametrize this term with a variable "theta" and write,

$$S = \int d^2x\,\left(\frac{1}{2}E^2 + \frac{\theta}{2\pi}E + \bar{\psi}\,i\gamma^\mu\left(\partial_\mu - eA_\mu\right)\psi\right) \qquad 5.8$$

It is interesting to see how the quantum field theory responds to this boundary condition. We can redo the analysis in Eq. 5.4-5.7. In particular, the relation between E and ϕ in Eq. 5.6 becomes,

$$E = -\frac{e}{2\pi}(\theta + \phi) \qquad 5.9$$

and the action becomes,

$$S = \frac{1}{8\pi}\int d^2x\ \left(\left(\partial_\mu\phi\right)^2 - \frac{e^2}{\pi}(\theta+\phi)^2\right) \qquad 5.10$$

But now we can make a shift $\phi \to \theta + \phi$ and return to Eq. 5.7! So, for massless fermions, the θ-term drops out of the local dynamics of the model! What does this mean? It shows that the vacuum adjusts to any external electric field and screens it completely! This is the Higgs mechanism!

There is more to discuss here but those topics will be deferred to a later lecture. The external electric field and the θ-term led to the idea of θ-vacua which is important in 3+1 dimensional non-abelian gauge theories.

It is interesting that the physics is qualitatively different if the mass of the fermion *m* is different from zero. Then a term $m\bar{\psi}\psi$ must be added to the action Eq. 5.8. Using the boson correspondence Eq. 4.3a, the action Eq. 5.8 becomes,

$$S = \int d^2x\ \left(\frac{1}{2}E^2 + \frac{\theta}{2\pi}E + \bar{\psi}\ i\gamma^\mu\left(\partial_\mu - eA_\mu\right)\psi + \frac{m}{\pi\epsilon}\cos\phi\right) \qquad 5.11$$

Now the steps Eq. 5.4-5.7 produce the bosonized action,

$$S = \frac{1}{8\pi}\int d^2x\ \left(\left(\partial_\mu\phi\right)^2 - \frac{e^2}{\pi}(\theta+\phi)^2 + \frac{8m}{\epsilon}\cos\phi\right) \qquad 5.12$$

Now when we shift the field $\phi \to \theta + \phi$, the action retains its dependence on θ through the fermion mass term. The minimum of the potential is now at,

$$\sin\phi = -\frac{e^2\epsilon}{4\pi m}(\theta+\phi) \qquad 5.13$$

By plotting both sides of this equation we see that for large *m* there are many solutions. However, if $\theta \neq \pi$, there is clearly a unique ground state. Therefore, the model with a non-zero fermion mass illustrates confinement as we discussed it in the introduction Section 1. The external field cannot be screened completely by the fermions of the theory. So, long range flux tubes of E fields persist and confine charged impurities that one can place in the theory. The strength of the confining long range force is proportional to m and has been studied in the literature [7]. The physics at $\theta = \pi$ is rather special and interesting and will be discussed elsewhere.

This exercise in bosonization is the simplest example of a family of duality maps that take models of strongly interacting degrees of freedom and maps them onto other models of weakly interacting fields. Other examples where the mapping relates interesting models with local variables and local interactions will be discussed in later lectures.

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