Guillermo Abad López

QFT Sheet 4

4.1)

a)

$$\boxed{[\hat{a}_{3\vec{k}} - \hat{a}_{0\vec{k}}, \hat{a}^{\dagger}_{3\vec{q}} - \hat{a}^{\dagger}_{0\vec{q}}]} = \underbrace{[\hat{a}_{3\vec{k}}, \hat{a}^{\dagger}_{3\vec{q}}]}_{-g_{33}(2\pi)^3\delta^3(\vec{k}-\vec{q})} - \underbrace{[\hat{a}_{3\vec{k}}, \hat{a}^{\dagger}_{0\vec{q}}]}_{0} - \underbrace{[\hat{a}_{0\vec{k}}, \hat{a}^{\dagger}_{3\vec{q}}]}_{0} + \underbrace{[\hat{a}_{0\vec{k}}, \hat{a}^{\dagger}_{0\vec{q}}]}_{-g_{00}(2\pi)^3\delta^3(\vec{k}-\vec{q})} =$$

$$= (2\pi)^3\delta^3(\vec{k}-\vec{q}) \, \underbrace{[-g_{33} - g_{00}]}_{0} \boxed{= 0}$$

b)

From the Gupta-Bleuler condition, for physical states we obtain:

$$(a_{3\vec{q}} - a_{0\vec{p}})|\Psi\rangle = 0 \longrightarrow a_{3\vec{p}}|\Psi\rangle = a_{0\vec{p}}|\Psi\rangle$$

Which means, that destructing a longitudinal or a scalar photon has to give the same state. This gives us the intuition that our states will have to be formed by:

$$|\Psi\rangle = \underbrace{|\Psi_T\rangle}_{\text{transverse part}} \underbrace{|\Psi_{SL}\rangle}_{\text{scalar-longitudinal part}} \Big\} \text{ product state (independent!)} \quad \text{with } |\Psi_{SL}\rangle \text{ formed by combinations of } (\hat{a}^{\dagger}_{3\vec{q}} - \hat{a}^{\dagger}_{0\vec{q}})|0\rangle \; \circledast$$

$$\left(\circledast \; (a_{3\vec{p}} - a_{0\vec{p}})|\Psi_{SL}\rangle = (a_{3\vec{p}} - a_{0\vec{p}})\underbrace{(a^{\dagger}_{3\vec{q}} - a^{\dagger}_{0\vec{q}})|0\rangle}_{|\Psi_{SL}\rangle} \underset{\text{from a)}}{=} [a_{3\vec{p}} - a_{0\vec{p}}, a^{\dagger}_{3\vec{q}} - a^{\dagger}_{0\vec{q}}]\,|0\rangle = 0 \checkmark \; \text{physical state!} \right)$$

So it's obvious that for different momentums we can have also our state be:

$$|\Psi_{SL}\rangle = \prod_i^{\infty} (\hat{a}^{\dagger}_{3\vec{q}_i} - \hat{a}^{\dagger}_{0\vec{q}_i})|0\rangle \qquad \text{for different } q_i\text{'s (they commute)}$$

(It's not so obvious, but the mathematics also tell us that this will work for any power of this states:

$$(a_{3\vec{p}} - a_{0\vec{p}})|\Psi_{SL}\rangle = (a_{3\vec{p}} - a_{0\vec{p}})\underbrace{(a^{\dagger}_{3\vec{q}} - a^{\dagger}_{0\vec{q}})^n|0\rangle}_{|\Psi_{SL}\rangle} =$$

$$= \Big[\underbrace{[a_{3\vec{p}} - a_{0\vec{p}}, a^{\dagger}_{3\vec{q}} - a^{\dagger}_{0\vec{q}}]}_{0} + (a^{\dagger}_{3\vec{q}} - a^{\dagger}_{0\vec{q}})(a_{3\vec{p}} - a_{0\vec{p}})\Big](a^{\dagger}_{3\vec{q}} - a^{\dagger}_{0\vec{q}})^{n-1}|0\rangle$$

$$\vdots$$

$$= (a^{\dagger}_{3\vec{q}} - a^{\dagger}_{0\vec{q}})^n (a_{3\vec{q}} - a_{0\vec{q}})|0\rangle = \boxed{0}$$

So our state can have the form: $|\Psi_{SL}\rangle = \prod_i^{\infty} (\hat{a}^{\dagger}_{3\vec{q}_i} - \hat{a}^{\dagger}_{0\vec{q}_i})^{n_i}|0\rangle \quad \forall n_i$

This finally says us that we can have a linear combinations of all the possible n_i's, to get the most general state:

$$\boxed{|\psi_{SL}\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots c(n_1,n_2,\dots) \prod_{i=1}^{\infty} (\hat{a}^{+}_{3\vec{q}_i} - \hat{a}^{+}_{0\vec{q}_i})^{n_i} |0\rangle}$$

Let's check that it fullfills the Gupta-Bouler condition, to confirm our supositions:

$$\boxed{\alpha_j |\psi_{SL}\rangle} = (\alpha_j) \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots c(n_1,n_2,\dots) \prod_{i=1}^{\infty} (\alpha_i^{+})^{n_i} |0\rangle =$$

(α_j commutes with all α_i^+)

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots c(n_1,n_2,\dots) \prod_{i=1}^{\infty} (\alpha_i^{+})^{n_i} \cancel{(\alpha_j)|0\rangle} = \boxed{0}$$

$(a|0\rangle = 0)$

physical! ✓✓

c)

$$\langle\psi_{SL}|\psi_{SL}\rangle = \left(\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots\right)\left(\sum_{n_1'=0}^{\infty} \sum_{n_2'=0}^{\infty} \dots\right) c^*(n_1,n_2,\dots)\, c(n_1',n_2',\dots) \cdot$$

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \underbrace{\langle 0| (\hat{a}_{3\vec{k}_j} - \hat{a}_{0\vec{k}_j})^{n_j} (\hat{a}^{+}_{3\vec{q}_i} - \hat{a}^{+}_{0\vec{q}_i})^{n_i} |0\rangle}_{\text{but } \alpha_j \text{ commutes with any } \alpha_i^+} =$$

$$= \left((\Sigma\Sigma..)(\Sigma\Sigma..)\, c^* c \prod\prod\right) \langle 0| (\hat{a}^{+}_{3\vec{q}_i} - \hat{a}^{+}_{0\vec{q}_i})^{n_i} (\hat{a}_{3\vec{k}_j} - \hat{a}_{0\vec{k}_j})^{n_j} |0\rangle$$

which gives 0 except all n_j and $n_i = 0$, which gives:

$$\boxed{\langle\psi_{SL}|\psi_{SL}\rangle = c^*(0,0,\dots)\, c(0,0,\dots) \overbrace{\langle 0|0\rangle}^{1} = |c(0,0,\dots)|^2}$$

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4.2)

$$H_{int} = \frac{\lambda}{3!}\phi^3$$

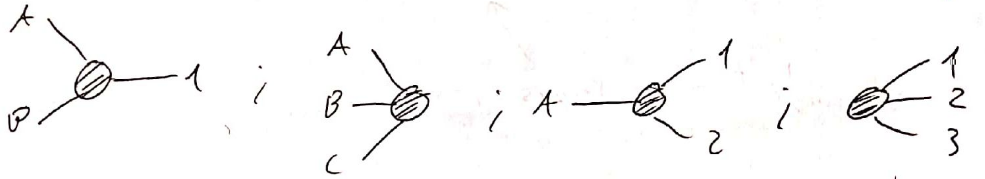
a)

$$\langle f|S|i\rangle = \langle f|\,\overset{S^{(0)}}{\mathbb{1}} \overset{S^{(1)}}{- i\int d^4x_1\, T\{H_{int}(x_1)\}} + \overset{S^{(2)}}{\frac{(-i)^2}{2!}\int d^4x_1\, d^4x_2\, T\{H_{int}(x_1)\,H_{int}(x_2)\}} + \dots$$

One vertex interactions means keeping the second term $S^{(1)}$:

$$\langle f|S^{(1)}|i\rangle = -i\int d^4x_1\, \langle f|\,T\{H_{int}(x_1)\}\,|i\rangle = -i\int d^4x_1\, \frac{\lambda}{3!}\langle f|\,T\{\phi^3_{x_1}\}\,|i\rangle$$

3 particle interaction means we have this possibles initial and final states:

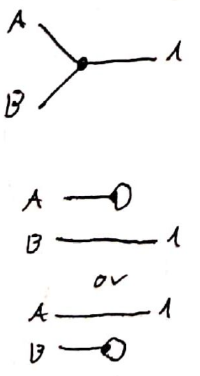


① ; ② ; ③ ; ④

▷ Let's compute each one and check they give the same except for the $\vec{p}$ delta:

$$① = \frac{-i\lambda}{3!}\int d^4x_1\, \langle 1|\,T\{\phi_{x_1}\phi_{x_1}\phi_{x_1}\}\,|A,B\rangle$$

$$= \frac{-i\lambda}{3!}\int d^4x_1 \left(3!\,\langle 1|\phi_{x_1}\phi_{x_1}\phi_{x_1}|AB\rangle + \underbrace{3\,\langle 1|\phi_{x_1}\phi_{x_1}\phi_{x_1}|AB\rangle + 3\,\langle 1|\phi_{x_1}\phi_{x_1}\phi_{x_1}|AB\rangle}\right)$$



First term is the fully contracted, which the first field has 3 states to chose, the second one can chose between the remaing 2 states, and the last field can't chose, giving 3! possibilities.

Second and third terms are the partially connected diagrams, which each has 3 possibilities, but we are not going to take into account, because, we actually reabsorb this partially connected diagrams in the normalization!

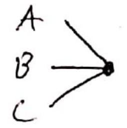
so ① renormalized gives:

$$\boxed{①} = -i\lambda\,\frac{\cancel{3!}}{\cancel{3!}}\int d^4x_1\, \langle 0|\,e^{ip_1x_1}\,e^{-ip_Ax_1}\,e^{-ip_Bx_1}\,|0\rangle = -i\lambda\int d^4x_1\, e^{(p_1-p_A-p_B)x_1}\,\langle 0|0\rangle =$$

$$= \boxed{-i\lambda\,(2\pi)^4\,\delta(p_1-p_A-p_B)}$$

▷ Let's go for ② then:

$$② = \frac{-i\lambda}{3!}\int d^4x \,\langle 0|T\{\phi_x\phi_x\phi_x\}|ABC\rangle = \frac{-i\lambda}{3!}\int d^4x \,\left(3!\,\langle 0|\phi_x\phi_x\phi_x|ABC\rangle\right) =$$



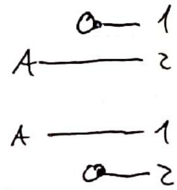
Here we only have fully contracted terms, if not $\langle 0|\psi\rangle = \langle\psi|0\rangle = 0$, and we easly see we have 3! possible combinations again

$$\boxed{②} = -i\lambda\int d^4x \,\langle 0|e^{-ip_A x}\,e^{-ip_B x}\,e^{-ip_C x}|0\rangle = \boxed{-i\lambda\,(2\pi)^4\,\delta(p_A+p_B+p_C)}$$

▷ For ③ now, this time skipping partially connected diagrams:

$$③ = \frac{-i\lambda}{3!}\int d^4x \,\langle 1,2|T\{\phi_x\phi_x\phi_x\}|A\rangle = \frac{-i\lambda}{3!}\int d^4x \,\left(3!\,\langle 1,2|\phi_x\phi_x\phi_x|A\rangle\right)$$

We have the same as in ① but mirrored, so it's the same



do not contribute

$$\boxed{③} = -i\lambda\int d^4x \,\langle 0|e^{ip_1 x}\,e^{ip_2 x}\,e^{-ip_A x}|0\rangle = \boxed{-i\lambda\,(2\pi)^4\,\delta(p_1+p_2-p_A)}$$

▷ And finally ④ which is ② mirrored:

$$\boxed{④} = \frac{-i\lambda}{3!}\int d^4x \,\langle 1,2,3|T\{\phi_x\phi_x\phi_x\}|0\rangle = \frac{-i\lambda}{3!}\int d^4x \,\left(3!\,\langle 1,2,3|\phi_x\phi_x\phi_x|0\rangle\right) =$$

$$= \boxed{-i\lambda\,(2\pi)^4\,\delta(p_1+p_2+p_3)}$$

We only have this term

So finally we see that the interaction Feynman rule for a 3particle vertex interaction is:

$$\boxed{-i\lambda\,(2\pi)^4\,\delta\left(\sum_i p_i - \sum_f p_f\right)}$$

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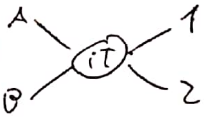
b)

$\langle k_1,k_2|iT|p_A,p_B\rangle =$  $= \dfrac{(-i)^2}{2!}\displaystyle\int d^4x\, d^4y\, \langle k_1,k_2|T\{H_{int}(x)\,H_{int}(y)\}|p_A,p_B\rangle =$

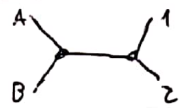
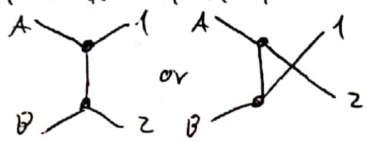
$$= \frac{-\lambda^2}{3!^2\, 2!}\int d^4x\, d^4y\, \langle k_1 k_2|T\{\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y\}|p_A,p_B\rangle =$$

Let's show all the possible contractions:

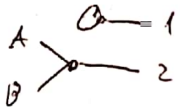
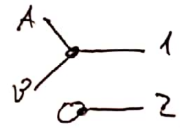
VV

- $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$ or $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$

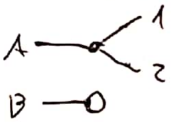
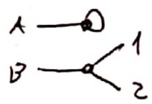
(3!² possibilities and $x \leftrightarrow y$ symmetry $\downarrow$ 3!²·2 possibilities)

 or 

- $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$ or $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$

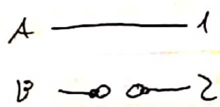
- $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$ or $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$

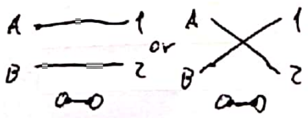
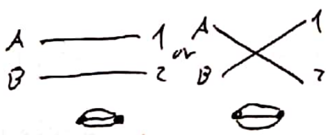
- $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$ or $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$

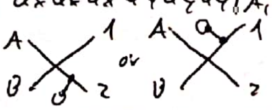
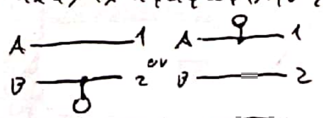
- $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$ or $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$

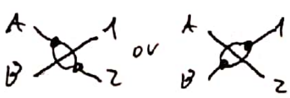
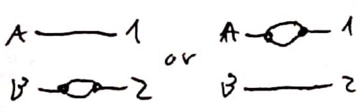
- $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$ or $\langle 1,2|\phi_x\phi_y\phi_x\phi_y\phi_y\phi_y|A,B\rangle$

- $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$ or $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$

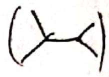
 

- $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$ or $\langle 1,2|\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y|A,B\rangle$

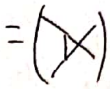
 

XX Do not contribute! (Not fully connected)

So, taking only the fully connected terms:

$$\boxed{\langle k_1 k_2 | iT | p_A p_B \rangle} = -\lambda^2 \frac{\cancel{3!^2}\,\cancel{2}}{\cancel{3!^2}\,\cancel{2!}} \int d^4x\, d^4y \Big[\langle 0 | e^{ip_1 x} e^{ip_2 x} \Delta_F(x-y)\, e^{-ip_A y} e^{-ip_B y} | 0 \rangle +$$ 

$$+ \langle 0 | e^{ip_1 x} e^{ip_2 y} \Delta_F(x-y)\, e^{-ip_A x} e^{-ip_B y} | 0 \rangle +$$ 

$$+ \langle 0 | e^{ip_1 x} e^{ip_2 y} \Delta_F(x-y)\, e^{-ip_A y} e^{-ip_B x} | 0 \rangle \Big] =$$ 

$$\left(\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip(x-y)} \right)$$

$$= -\lambda^2 \int d^4x\, d^4y\, \Delta_F(x-y)\, e^{i[(p_1+p_2)x-(p_A+p_B)y]} + e^{i[(p_1-p_A)x+(p_2-p_B)y]} + e^{i[(p_1-p_B)x+(p_2-p_A)y]} =$$

$$= -i\lambda^2 \int \frac{d^4x\, d^4y\, d^4p}{(2\pi)^4\, p^2-m^2+i\epsilon} e^{-ip(x-y)} \left(e^{(\;)} + e^{(\;)} + e^{(\;)} \right) =$$

$$= -i\lambda^2 (2\pi)^4 \int \frac{d^4p}{p^2-m^2+i\epsilon} \Big[\delta(p_1+p_2-p)\,\delta(p_A+p_B-p) +$$

$$+ \delta(p_A+p-p_1)\,\delta(p_2+p-p_B) +$$

$$+ \delta(p_A+p-p_2)\,\delta(p_B+p-p_1) \Big] =$$

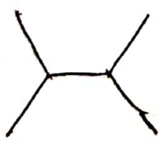
$$= -i\lambda^2 (2\pi)^4 \left[\frac{\delta(p_A+p_B-p_1-p_2)}{(p_1+p_2)^2-m^2+i\epsilon} + \frac{\delta(p_1+p_2-p_A-p_B)}{(p_1-p_A)^2-m^2+i\epsilon} + \frac{\delta(p_1+p_2-p_A-p_B)}{(p_2-p_A)^2-m^2+i\epsilon} \right] =$$

$$= \boxed{i\mathcal{M}\, (2\pi)^4\, \delta(p_A+p_B-p_1-p_2)}$$

with $\boxed{\mathcal{M}} = -\lambda^2 \left[\frac{1}{\underbrace{(p_1+p_2)^2}_{s}-m^2} + \frac{1}{\underbrace{(p_1-p_A)^2}_{t}-m^2} + \frac{1}{\underbrace{(p_2-p_A)^2}_{u}-m^2} \right] =$

$$= \boxed{-\lambda^2 \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2} \right]}$$

That is why we call the corresponding diagrams:



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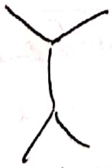

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