Lorentz Group

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Lorentz Group

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1}$$

Leave invariant the 4-product: $x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu = t^2 - x^2 - y^2 - z^2$

$$\begin{aligned}
g_{\mu\nu} x'^\mu x'^\nu &= g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma = g_{\rho\sigma} x^\rho x^\sigma \quad \forall x \\
g_{\rho\sigma} &= g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma & (2) \\
g &= \Lambda^T g \Lambda & (3)
\end{aligned}$$

- Determinant

$$(\det(\Lambda))^2 = 1 \quad \Rightarrow \quad \det(\Lambda) = \pm 1 \tag{4}$$

- 00 component

$$1 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \quad \Rightarrow \quad (\Lambda^0{}_0)^2 \geq 1 \quad \Rightarrow \quad \begin{cases} \Lambda^0{}_0 \geq 1 \\ \Lambda^0{}_0 \leq -1 \end{cases} \tag{5}$$

	Orthochronus $\Lambda^0{}_0 \geq 1$	non-Orthochronus $\Lambda^0{}_0 \leq -1$
Proper $\quad \det(\Lambda) = 1$	$\mathcal{L}_\uparrow^+$	$\mathcal{L}_\downarrow^+$
Improper $\quad \det(\Lambda) = -1$	$\mathcal{L}_\uparrow^-$	$\mathcal{L}_\downarrow^-$

- $\mathcal{L}_\uparrow^+$: Subgroup. Rotations & boosts. Lie group. The only one that forms a group.

  $$\Lambda_P$$

  This part is connected to the identity.
- $\mathcal{L}_\downarrow^+$: Change the sign of time, and an odd number of space coordinates. Includes total inversion:
  $\Lambda_P \times \{\mathrm{diag}(-,-,+,+); \mathrm{diag}(-,+,-,+); \mathrm{diag}(-,+,+,-); \mathrm{diag}(-,-,-,-)\}$
- $\mathcal{L}_\uparrow^-$: Change an odd number of space coordinates. Includes parity (all space inversions):
  $\Lambda_P \times \{\mathrm{diag}(+,-,+,+); \mathrm{diag}(+,+,-,+); \mathrm{diag}(+,+,+,-); \mathrm{diag}(+,-,-,-)\}$
- $\mathcal{L}_\downarrow^-$: Change the sign of time,
  and an even number of space coordinates. Includes time-inversion:
  $\Lambda_P \times \{\mathrm{diag}(-,+,+,+); \mathrm{diag}(-,-,-,+); \mathrm{diag}(-,-,+,-); \mathrm{diag}(-,+,-,-)\}$

Proper orthochronus Lorentz Group: $\mathcal{L}_{\uparrow}^{+}$

Infinitesimal Lorentz transformation:

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}$$

$$g_{\rho\sigma} = g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\mu\nu}(\delta^{\mu}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}_{\sigma} + \omega^{\nu}{}_{\sigma}) = g_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + \mathcal{O}(\omega)^2$$

$$\Rightarrow \boxed{\omega_{\rho\sigma} = -\omega_{\sigma\rho}} \Rightarrow \text{6 parameters} \begin{cases} \text{3 rotations} & (R) \\ \text{3 boosts} & (L) \end{cases}$$

$$R_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_\theta & -s_\theta \\ 0 & 0 & s_\theta & c_\theta \end{pmatrix} \; ; \; R_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta & 0 & s_\theta \\ 0 & 0 & 1 & 0 \\ 0 & -s_\theta & 0 & c_\theta \end{pmatrix} \; ; \; R_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta & -s_\theta & 0 \\ 0 & s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; ;$$

$$L_x = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; ; \; L_y = \begin{pmatrix} \cosh\eta & 0 & \sinh\eta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\eta & 0 & \cosh\eta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; ; \; L_z = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} \; ;$$

$\eta =$rapidity: $\gamma = \cosh\eta$, $\gamma\beta = \sinh\eta$, $\beta = \tanh\eta$, $\eta = \frac{1}{2}\ln\frac{1+\beta}{1-\beta}$, additive.

Generators: $\delta\theta \ll 1$, $\delta\eta \ll 1$

Rotation generators

$$R_x(\delta\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\delta\theta \\ 0 & 0 & \delta\theta & 1 \end{pmatrix} = \mathbb{1} - i\delta\theta J^1 \quad ; \quad J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$R_y(\delta\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\delta\theta & 0 & 1 \end{pmatrix} = \mathbb{1} - i\delta\theta J^2 \quad ; \quad J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$R_z(\delta\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta\theta & 0 \\ 0 & \delta\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - i\delta\theta J^3 \quad ; \quad J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Boost generators

$$L_x(\delta\eta) = \begin{pmatrix} 1 & \delta\eta & 0 & 0 \\ \delta\eta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - i\delta\eta K^1 \quad ; \quad K^1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L_y(\delta\eta) = \begin{pmatrix} 1 & 0 & \delta\eta & 0 \\ 0 & 1 & 0 & 0 \\ \delta\eta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - i\delta\eta K^2 \quad ; \quad K^2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L_z(\delta\eta) = \begin{pmatrix} 1 & 0 & 0 & \delta\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \delta\eta & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - i\delta\eta K^3 \quad ; \quad K^3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

Not hermitian[a]:

$$K^{l\dagger} = -K^l$$

[a] because the the boosts are *non-compact*.

Lie algebra

$$[J^k, J^l] = i\epsilon^{klm} J^m \;\; ; \;\; [K^k, K^l] = -i\epsilon^{klm} J^m \;\; ; \;\; [J^k, K^l] = i\epsilon^{klm} K^m \;\; ; \;\; (k, l, m \in \{1, 2, 3\})$$

- Rotations: closed algebra. Rotation group subgroup of $\mathcal{L}_\uparrow^+$
- Boosts do not close algebra. Not a subgroup.
- Define:

$$A^m = \frac{1}{2}(J^m + iK^m) \;\; ; \;\; B^m = \frac{1}{2}(J^m - iK^m) \tag{6}$$

  - A^m and B^m are hermitic
  - verify the $SU(2)$ Lie algebra:

$$[A^k, A^l] = i\epsilon^{klm} A^m \;\; ; \;\; [B^k, B^l] = i\epsilon^{klm} B^m \;\; ; \;\; [A^k, B^l] = 0$$

$\Rightarrow$ Lorentz group is locally isomorph to $SU(2) \times SU(2)$:

$$\mathcal{L}_\uparrow^+ \simeq SU(2) \times SU(2) \text{ locally}$$

  - $SU(2)$ irreducible representations (irreps) are those of spin.

Irreducibe representations of $\mathcal{L}_\uparrow^+$

(j_1, j_2) of dimension $(2j_1 + 1)(2j_2 + 1)$.

- The representations are not unitary:

$$\Lambda = \exp\{-i(\theta^m J^m + \eta^m K^m)\} \equiv \exp\{-i(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\eta}\cdot\boldsymbol{K})\}$$

$$\Lambda^{-1} = \exp\{i(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\eta}\cdot\boldsymbol{K})\} \neq \exp\{i(\boldsymbol{\theta}\cdot\boldsymbol{J} - \boldsymbol{\eta}\cdot\boldsymbol{K})\} = \Lambda^\dagger$$

- Under parity ($\in \mathcal{L}_\uparrow^-$)

$$(t,\boldsymbol{x}) \to (t,-\boldsymbol{x}) \Rightarrow \boldsymbol{\beta} \to -\boldsymbol{\beta} \Rightarrow \boldsymbol{J} \to \boldsymbol{J} \ , \ \ \boldsymbol{K} \to -\boldsymbol{K} \Rightarrow \boldsymbol{A} \leftrightarrow \boldsymbol{B}$$

  Representations:

$$(j_1, j_2) \leftrightarrow (j_2, j_1)$$

  $\Rightarrow$ not invariant under parity ($\mathcal{L}_\uparrow^-$) unless $j_1 = j_2$
  QED and QCD are theories with parity conservation.

Alternative representation

6 infinitesimal parameters $\omega_{\mu\nu}$:

$$\Lambda = \exp\left\{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right\}$$

$$\begin{aligned} J^k &= \frac{1}{2}\epsilon^{klm}J^{lm} \quad ; \quad \begin{cases} J^1 = J^{23} = -J^{32} \\ J^2 = J^{31} = -J^{13} \\ J^3 = J^{12} = -J^{21} \end{cases} \\ K^k &= J^{0k} = -J^{k0} \end{aligned}$$

$$\begin{aligned} \theta^k &= \frac{1}{2}\epsilon^{klm}\omega^{lm} \quad ; \quad \begin{cases} \theta^1 = \omega^{23} = -\omega^{32} = \omega_{23} = -\omega_{32} \\ \theta^2 = \omega^{31} = -\omega^{13} = \omega_{31} = -\omega_{13} \\ \theta^3 = \omega^{12} = -\omega^{21} = \omega_{12} = -\omega_{21} \end{cases} \\ \eta^k &= \omega^{0k} = -\omega^{k0} = -\omega_{0k} = \omega_{k0} \end{aligned}$$

Generators:

$$(J^{\mu\nu})^{\rho}{}_{\sigma} = i(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma}) \tag{7}$$

Lie algebra:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$$

Vector & Tensor representations

Vector representation

Vector rep.: defining representation. Dim=4:

$$\mathbf{4} : \Lambda^{\mu}{}_{\nu} = [\exp\{-i(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\eta}\cdot\boldsymbol{K})\}]^{\mu}{}_{\nu} = \left[\exp\left\{-\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right\}\right]^{\mu}{}_{\nu}$$

$J^{\rho\sigma}$ defined in eq. (7).
Acts over a covariant vector V^{μ}.

$$V^{\mu} \to V'^{\mu} = \Lambda^{\mu}{}_{\nu}V^{\nu}$$

A contra-variant vector V_{μ} equivalent rep.:

$$V_{\mu} \to V'_{\mu} = \Lambda_{\mu}{}^{\nu}V_{\nu} \;\; ; \;\; \Lambda_{\mu}{}^{\nu} = g_{\mu\rho}\Lambda^{\rho}{}_{\sigma}g^{\sigma\nu}$$

These representations are irreducible.

Tensor representations

We can build tensors:

$$\mathbf{4} \otimes \mathbf{4} : T^{\mu\nu} \to T^{\mu'\nu'} = \Lambda^{\mu'}{}_{\rho}\Lambda^{\nu'}{}_{\sigma} T^{\rho\sigma}$$

Dim=16. It is reducible.
Decompose each tensor as:

$$T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} T + A^{\mu\nu} + S^{\mu\nu}$$

- $T = T^{\mu\nu} g_{\mu\nu}$: Trace, it is invariant, Dim=1
- $A^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})$: Anti-symmetric, Dim=6
- $S^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) - \frac{1}{4} g^{\mu\nu} T$: Symmetric, Traceless, Dim=9

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9}$$

Higher rank tensors will also be reducible representations.

Spinor representations

- Build the irreps of $\mathcal{L}_\uparrow^+$ from the ones of $SU(2)$.

Remember from QM rotation group

Start from spin $1/2$:

$$J^k = \frac{1}{2}\sigma^k \;\; ; \;\; \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\; ; \;\; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;\; ; \;\; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;\; ;$$

A spinor field $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ is transformed by: $\psi' = \exp\{-i\boldsymbol{\alpha} \cdot \frac{\boldsymbol{\sigma}}{2}\}\psi$

Higher dimension irreps:

- spinor tensor product + Clebsch-Gordan reduction, e.g.:

$$\frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{0} \oplus \mathbf{1}$$

$\Rightarrow$ irreps of $\mathcal{L}_\uparrow^+$: (j_1, j_2): Tensorial products of $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, Dim= $(2j_1 + 1)(2j_2 + 1) = 2$
2-component Weyl spinors

2-component Weyl spinors: $\begin{cases} \text{left-handed} & \psi_L \in (\frac{1}{2}, 0) \quad ; \\ \text{right-handed} & \psi_R \in (0, \frac{1}{2}) \end{cases}$

Transform under A and B as 2-component spinors, and singlets, and:

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{J} + i\boldsymbol{K}) \;\;,\;\; \boldsymbol{B} = \frac{1}{2}(\boldsymbol{J} - i\boldsymbol{K}) \;\; \Rightarrow \boldsymbol{J} = \boldsymbol{A} + \boldsymbol{B} \;\;,\;\; \boldsymbol{K} = -i(\boldsymbol{A} - \boldsymbol{B})$$

Transformations under rotations and boosts:

$$\psi_L \quad : \quad \boldsymbol{A} = \frac{\boldsymbol{\sigma}}{2}, \boldsymbol{B} = \boldsymbol{0} \Rightarrow \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2} \;\;;\;\; \boldsymbol{K} = -i\frac{\boldsymbol{\sigma}}{2}$$

$$\Lambda_L = \exp\left\{(-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right\}$$

$$\psi_R \quad : \quad \boldsymbol{A} = \boldsymbol{0}, \boldsymbol{B} = \frac{\boldsymbol{\sigma}}{2} \Rightarrow \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2} \;\;;\;\; \boldsymbol{K} = i\frac{\boldsymbol{\sigma}}{2}$$

$$\Lambda_R = \exp\left\{(-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right\}$$

$\sigma^2\sigma^i\sigma^2 = -\sigma^{i*} \Rightarrow \sigma^2\Lambda_L^*\sigma^2 = \Lambda_R, \Rightarrow$ define the conjugate spinors:

$$\psi_L^c \equiv i\sigma^2\psi_L^* \in (0, \frac{1}{2}) \;\;;\;\; \psi_R^c \equiv -i\sigma^2\psi_R^* \in (\frac{1}{2}, 0)$$

Weyl spinors are complex: $\psi \in \mathbb{R} \xrightarrow{\text{boost}} \psi' \in \mathbb{C}$

Fields representation

The fields will be in irreducible representations of the Lorentz Group.

$$\phi'(x') = G(\phi(x)) \simeq \phi(x) + \frac{i}{2}\omega_{\mu\nu} S^{\mu\nu} \phi(x)$$

$S^{\mu\nu}$: irrep of Lorentz group generators over ϕ.
Noether's theorem:
$\delta\phi = \phi'(x) - \phi(x) = \frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\phi(x) + \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\rho}{}_{\sigma} x^{\sigma}\partial_{\rho}\phi(x) \equiv -\frac{i}{2}\omega_{\mu\nu}\mathcal{J}_{\phi}^{\mu\nu}\phi(x)$
$(J^{\mu\nu})^{\rho}{}_{\sigma}$: coordinate transformations generators – eq. (7)

$$\delta x^{\rho} = -\frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\rho}{}_{\sigma} x^{\sigma}$$

Scalar fields

Seen before: $S^{\mu\nu} = 0$.

$$\delta\phi(x) = \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\rho}{}_{\sigma}x^{\sigma}\partial_{\rho}\phi(x) \equiv -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi(x)$$

$$L^{\mu\nu} = -(J^{\mu\nu})^{\rho}{}_{\sigma}x^{\sigma}\partial_{\rho} = -i(x^{\nu}\partial^{\mu} - x^{\mu}\partial^{\nu}) = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) \tag{8}$$

Orbital angular momentum

- Left handed fields: $\mathcal{J}_L^{\mu\nu} = S_L^{\mu\nu} + L^{\mu\nu}$
  Orbital part defined in eq. (8), and:

$$-\frac{i}{2}\omega_{\mu\nu}S_L^{\mu\nu} = \Lambda_L - \mathbb{1} = -i(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\eta}\cdot\boldsymbol{K}) = \frac{-i}{2}(\boldsymbol{\theta} - i\boldsymbol{\eta})\cdot\boldsymbol{\sigma}$$

  Generators for ψ_L:

$$\text{Rotations}: \mathcal{J}^i = L^i + S^i = L^i + \frac{\sigma^i}{2} \quad ; \quad \text{boosts}: \mathcal{K}^k = K^k - \frac{i}{2}\sigma^k$$

- Right-handed fields: $\mathcal{J}_R^{\mu\nu} = S_R^{\mu\nu} + L^{\mu\nu}$
  Orbital part defined in eq. (8), and:

$$-\frac{i}{2}\omega_{\mu\nu}S_R^{\mu\nu} = \Lambda_R - \mathbb{1} = -i(\boldsymbol{\theta}\cdot\boldsymbol{J} + \boldsymbol{\eta}\cdot\boldsymbol{K}) = \frac{-i}{2}(\boldsymbol{\theta} + i\boldsymbol{\eta})\cdot\boldsymbol{\sigma}$$

  Generators for ψ_R:

$$\text{Rotations}: \mathcal{J}^i = L^i + S^i = L^i + \frac{\sigma^i}{2} \quad ; \quad \text{boosts}: \mathcal{K}^k = K^k + \frac{i}{2}\sigma^k$$

Under parity: $\psi_L \leftrightarrow \psi_R$

$\Rightarrow$ Not convenient if the theory conserves parity (QED, QCD)

Definition: **Dirac** 4-component spinor

$$\psi_D(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}$$

transforms as

$$\psi(x) \to \psi'(x') = \Lambda_D \psi(x) \quad ; \quad \Lambda_D = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}$$

under parity $x^\mu \to \tilde{x}^\mu = (t, -\boldsymbol{x})$

$$\psi(x) \to \psi'(\tilde{x}) = \begin{pmatrix} \psi_R(\tilde{x}) \\ \psi_L(\tilde{x}) \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \psi(\tilde{x})$$

The charge conjugate:

$$\psi^c = \begin{pmatrix} \psi_R^c \\ \psi_L^c \end{pmatrix} = \begin{pmatrix} -i\sigma^2 \psi_R^* \\ i\sigma^2 \psi_L^* \end{pmatrix} = -i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \psi^*$$

Majorana spinor

self-conjugate 4-component spinor (up to a phase)

$$\psi_R = \xi i\sigma^2 \psi_L^* \Rightarrow \psi_M = \begin{pmatrix} \psi_L \\ \xi i\sigma^2 \psi_L^* \end{pmatrix} \quad ; \quad |\xi|^2 = 1$$

has two degrees of freedom and:

$$\psi_M^c = \begin{pmatrix} \xi^* \psi_L \\ i\sigma^2 \psi_L^* \end{pmatrix} = \xi^* \psi_M$$

Vector Fields

Vectors transform as x^μ:

$$V^\mu \to V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x)$$

$$\mathcal{J}^{\mu\nu} = S_V^{\mu\nu} + L^{\mu\nu}$$

$S_V^{\mu\nu}$: same form as the x^μ transformation $(J^{\mu\nu})^\rho{}_\sigma$ from eq. (7):

$$(S_V^{\mu\nu})^\rho{}_\sigma = (J^{\mu\nu})^\rho{}_\sigma = i(g^{\mu\rho}\delta^\nu_\sigma - g^{\nu\rho}\delta^\mu_\sigma)$$

Poincaré Group

Lorentz + Translations:

$$x^\mu \to x'^\mu = x^\mu + a^\mu$$

Generators: components of 4-momenta, $a^\mu = \varepsilon^\mu$:

$$\begin{aligned} x'^\mu = (\mathbb{1} - i\varepsilon_\rho P^\rho) x^\mu \quad &\Rightarrow \quad \delta x^\mu = \varepsilon^\mu = -i\varepsilon_\rho P^\rho x^\mu \\ &\Rightarrow \quad P^\rho = i\partial^\rho \end{aligned}$$

Poincaré algebra:

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [P^\mu, J^{\rho\sigma}] &= i(g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho) \\ [J^{\mu\nu}, J^{\rho\sigma}] &= i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}) \end{aligned}$$

Specifying for rotation/boosts:

$$\begin{array}{lcl} [P^0, J^k] = 0 & ; & [P^0, K^k] = iP^k \\ [P^k, J^l] = i\epsilon^{klm} P^m & ; & [P^k, K^l] = iP^0 \delta^{kl} \end{array}$$

Particle state representations

Casimir operators of the Poincaré group:

$$m^2 = P^\mu P_\mu \quad ; \quad W_\mu W^\mu$$

Pauli-Lubanski operator: $W^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma$
Particle states: labeled by the irreps of these operators

$$\begin{aligned}
[W^\mu, P^\alpha] &= -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}[J_{\nu\rho}P_\sigma, P^\alpha] \\
&= -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(J_{\nu\rho}[P_\sigma, P^\alpha] + [J_{\nu\rho}, P^\alpha]P_\sigma) \\
&= \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}(g^\alpha{}_\nu P_\rho - g^\alpha{}_\rho P_\nu)P_\sigma \\
&= \frac{i}{2}(\varepsilon^{\nu\alpha\rho\sigma}P_\rho P_\sigma + \varepsilon^{\mu\nu\alpha\sigma}P_\nu P_\sigma) = 0
\end{aligned}$$

- $\boldsymbol{m^2 > 0}$: Go to the proper reference frame of the particle $p^\mu = (m, 0, 0, 0)$, then:

$$\left.\begin{array}{l} W^0 = 0 \\ W^i = -\frac{m}{2}\varepsilon^{ijk0}J^{jk} = mJ^i \end{array}\right\} \Rightarrow W^\mu W_\mu = -m^2 j(j+1)$$

  $\Rightarrow$ Particle states are labeled by: mass, total spin.

  $\Rightarrow$ Each particle has $(2j+1)$ states.

- $\boldsymbol{m^2 = 0}$: $W^\rho W_\rho = 0$, and $P_\rho W^\rho = 0 \Rightarrow$

$$W^\rho = hP^\rho$$

  h: helicity = $\pm s$ with $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

  $\Rightarrow$ Only two states for each particle

- Other representations: not realized in nature ($m^2 = 0$, continuous spin; ...)