Week 9

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1)

a)

$$n \longrightarrow p\, e^-\, \bar\nu_e$$
$$(A) \quad (1)\ (2)\ (3)$$

$$\mathcal{L} = 2\sqrt{2}\, G\left(\bar n \gamma^\mu \frac{1-g_A\gamma^5}{2} p\, \bar\nu_e \gamma_\mu P_L e + \bar e P_L \gamma_\mu \nu_e\, \bar p \frac{1-g_A\gamma^5}{2}\gamma^\mu n\right)$$

- $\langle 1 | \bar p \frac{1-g_A\gamma^5}{2}\gamma^\mu n | A\rangle = \bar u(1) \frac{1-g_A\gamma^5}{2}\gamma^\mu u(A)$
- $\langle 2,3 | \bar e P_L \gamma_\mu \nu_e | 0\rangle = \bar u(2) P_L \gamma_\mu v(3)$

(because:

$p \approx u a_1 + v b_1^\dagger$; $n \approx u a_A + v b_A^\dagger$; $e = u a_2 + v b_2^\dagger$; $\nu = u a_3 + v b_3^\dagger$; and: $\langle 1| \approx \langle 0| a_1$
$\bar p \approx \bar u a_1^\dagger + \bar v b_1^\dagger$; $\bar n \approx \bar u a_A^\dagger + \bar v b_A$; $\bar e = \bar u a_2^\dagger + \bar v b_2$; $\bar\nu = \bar u a_3^\dagger + \bar v b_3$; $|A\rangle \approx a_A^\dagger |0\rangle$
$\langle 2,3| \approx \langle 0| a_2 b_3$)

so our amplitude element is:

$$\mathcal{M} = \sqrt{2}\, G\left(\bar u(2) P_L \gamma_\mu v(3)\ \bar u(1)\,(1-g_A\gamma^5)\gamma^\mu u(A)\right)$$

And squaring it we obtain:

$$|\mathcal{M}|^2 = G^2 \sum_{\lambda's=\pm}\Big(\bar u(2) P_L \gamma_\mu v(3)\, \bar u(1)\gamma^\mu u(A)\, \underbrace{u^\dagger(A)\gamma^{\nu\dagger}\gamma^0 u(1)}_{\bar u(A)\gamma^0\overbrace{\gamma^{\nu\dagger}\gamma^0}^{\gamma^\nu} u(1)}\, \underbrace{v(3)^\dagger \gamma_\nu^\dagger P_L \gamma^0 u(2)}_{\bar v(3)\overbrace{\gamma^0\gamma_\nu^\dagger\gamma^0}^{\gamma_\nu} P_R u(2)} -$$
$$- g_A \bar u(2) P_L \gamma_\mu v(3)\, \bar u(1)\gamma^\mu u(A)\, u^\dagger(A)\gamma^{\nu\dagger}\gamma^5\gamma^0 u(1)\, v(3)^\dagger\gamma_\nu^\dagger P_L\gamma^0 u(2) -$$
$$- g_A \bar u(2) P_L \gamma_\mu v(3)\, \bar u(1)\gamma^5\gamma^\mu u(A)\, u^\dagger(A)\gamma^{\nu\dagger}\gamma^0 u(1)\, v(3)^\dagger\gamma_\nu^\dagger P_L\gamma^0 u(2) +$$
$$+ g_A^2 \bar u(2) P_L \gamma_\mu v(3)\, \bar u(1)\gamma^5\gamma^\mu u(A)\, u^\dagger(A)\gamma^{\nu\dagger}\gamma^5\gamma^0 u(1)\, v(3)^\dagger\gamma_\nu^\dagger P_L\gamma^0 u(2)\Big) =$$
$$= G^2 \sum_{\lambda's=\pm}\Big[\mathrm{tr}\left(\bar u(1)\gamma^\mu u(A)\,\bar u(A)\gamma^\nu u(1)\right) - g_A\,\mathrm{tr}\left(\bar u(1)\gamma^\mu u(A)\,\bar u(A)\gamma^\nu(-\gamma^5)u(1)\right) -$$
$$- g_A\,\mathrm{tr}\left(\bar u(1)(-\gamma^5)\gamma^\mu u(A)\,\bar u(A)\gamma^\nu u(1)\right) + g_A^2\,\mathrm{tr}\left(\bar u(1)(-\gamma^5)\gamma^\mu u(A)\,\bar u(A)\gamma^\nu(-\gamma^5)u(1)\right)\Big]\cdot$$

- $\mathrm{tr}\left(\bar u(2) P_L \gamma_\mu v(3)\, \bar v(3)\gamma_\nu P_R u(2)\right)$ (factor comú a tots!)

And using that $\sum_{s=\pm} u(p)\,\bar{u}(p) = \not{p} + m$ and $\sum_{s=\pm} v(p)\,\bar{v}(p) = \not{p} - m$, at the same time as $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$, we get:

$$|M|^2 = G^2\Big[\mathrm{tr}\big((\not{p}_1+m_1)\gamma^\mu(\not{p}_A+m_A)\gamma^\nu\big) + g_A\,\mathrm{tr}\big((\not{p}_1+m_1)\gamma^\mu(\not{p}_A+m_A)\gamma^\nu\gamma^5\big) +$$

$$+ g_A\,\mathrm{tr}\big((\not{p}_1+m_1)\gamma^5\gamma^\mu(\not{p}_A+m_A)\gamma^\nu\big) + g_A^2\,\mathrm{tr}\big((\not{p}_1+m_1)\gamma^5\gamma^\mu(\not{p}_A+m_A)\gamma^\nu\gamma^5\big)\Big]\cdot$$

$$\cdot\,\mathrm{tr}\big((\not{p}_2+m_2)P_L\gamma_\mu(\not{p}_3-\overset{0}{\cancel{m_3}})\underbrace{P_L\gamma_\nu}_{\gamma_\nu P_R}\big)$$

And using that $\mathrm{tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_p}) = \mathrm{tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_p}\gamma^5) = 0$ if p is odd and that $\{\gamma^5,\gamma^\mu\}=0$, and that $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5)=0$, we get

$$|M|^2 = G^2\Big[\mathrm{tr}(\not{p}_1\gamma^\mu\not{p}_A\gamma^\nu) + m_1 m_A\,\mathrm{tr}(\gamma^\mu\gamma^\nu) +$$

$$+ g_A\,\mathrm{tr}(\not{p}_1\gamma^\mu\not{p}_A\gamma^\nu\gamma^5) + g_A m_1 m_A\,\mathrm{tr}(\overset{0}{\cancel{\gamma^\mu\gamma^\nu\gamma^5}}) +$$

$$+ g_A\,\mathrm{tr}(\overbrace{\not{p}_1\gamma^5}^{-\gamma^5\not{p}_1}\gamma^\mu\not{p}_A\gamma^\nu) + g_A m_1 m_A\,\mathrm{tr}(\overset{0}{\cancel{\gamma^5\gamma^\mu\gamma^\nu}}) +$$

$$+ g_A^2\,\mathrm{tr}(\underbrace{\not{p}_1\gamma^5}_{-\gamma^5\not{p}_1}\gamma^\mu\not{p}_A\gamma^\nu\gamma^5) + g_A^2 m_1 m_A\,\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^5)\Big]\cdot$$

$$\cdot\,\frac{1}{4}\,\mathrm{tr}\big(\not{p}_2(1-\gamma^5)\gamma_\mu\not{p}_3(1-\gamma^5)\gamma_\nu\big) =$$

$$= G^2\Big[\mathrm{tr}(\not{p}_1\gamma^\mu\not{p}_A\gamma^\nu) + m_1 m_A\,\mathrm{tr}(\gamma^\mu\gamma^\nu) + g_A\big(\cancel{\mathrm{tr}(\not{p}_1\gamma^\mu\not{p}_A\gamma^\nu\gamma^5)} - \cancel{\mathrm{tr}(\not{p}_1\gamma^\mu\not{p}_A\gamma^\nu\gamma^5)}\big) +$$

$$- g_A^2\,\mathrm{tr}(\not{p}_1\gamma^\mu\not{p}_A\gamma^\nu) + g_A^2 m_1 m_A\,\mathrm{tr}(\gamma^\mu\gamma^\nu)\Big]\cdot$$

$$\cdot\,\frac{1}{4}\,\mathrm{tr}\big(\not{p}_2(1-\gamma^5)\gamma_\mu\not{p}_3(1-\gamma^5)\gamma_\nu\big)$$

Using now that, $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$, $\mathrm{tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) = \mathrm{tr}(\gamma^{\mu_n}\ldots\gamma^{\mu_1})$, $\mathrm{tr}(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2) = 4\,[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1p_2)g^{\mu\nu}]$ and finally that $\mathrm{tr}(\gamma^\mu(1-\gamma^5)\not{p}_1\gamma^\nu(1-\gamma^5)\not{p}_2) = 2\,\mathrm{tr}(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2) + 8i\epsilon^{\mu\alpha\nu\beta}p_{1\alpha}p_{2\beta}$:

- $\mathrm{tr}(\not{p}_1\gamma^\mu\not{p}_A\gamma^\nu) = \mathrm{tr}(\gamma^\nu\not{p}_1\gamma^\mu\not{p}_A) = 4\,[p_1^\mu p_A^\nu + p_1^\nu p_A^\mu - (p_1p_A)\,g^{\mu\nu}]$
- $\mathrm{tr}(\not{p}_2(1-\gamma^5)\gamma_\mu\not{p}_3(1-\gamma^5)\gamma_\nu) = \mathrm{tr}(\gamma_\nu(1-\gamma^5)\not{p}_3\gamma_\mu(1-\gamma^5)\not{p}_2) = 2\,\mathrm{tr}(\gamma_\nu\not{p}_3\gamma_\mu\not{p}_2) + 8i\epsilon^{\nu\alpha\mu\beta}p_{3\alpha}p_{2\beta}$

So finally we can express our amplitude as:

$$|M|^2 = G^2\Big[(1-g_A^2)\,4\,[p_1^\mu p_A^\nu + p_1^\nu p_A^\mu - (p_1 p_A)\,g^{\mu\nu}] + (1+g_A^2)\,m_1 m_A\,4\,g^{\mu\nu}\Big]\cdot$$

$$\cdot\frac{1}{4}\Big(2\cdot 4\,[p_{3\mu} p_{2\nu} + p_{3\nu} p_{2\mu} - (p_3\cdot p_2)\,g_{\mu\nu}] + 8i\,\epsilon_{\nu\alpha\mu\beta}\,p_3^\alpha p_2^\beta\Big)$$

but we see that the levi-civitta simbol is contracted only with simmetric things, so it gives 0, and we only have to contract the first term.

$$(g^{\mu\nu}g_{\mu\nu} = g^\nu_\nu = 4)$$

$$|M|^2 = 8G^2\Big[(1-g_A^2)\big(2(p_1 p_2)(p_A p_3) + 2(p_1 p_3)(p_A p_2) \cancel{- 4(p_1 p_A)(p_3 p_2)}\big) + \cancel{(p_1 p_A)(p_3 p_2)\,g^{\mu\nu}g_{\mu\nu}} +$$

$$+ (1+g_A^2)\,m_1 m_A\,(2p_3 p_2 - p_3 p_2\,g^{\mu\nu}g_{\mu\nu})\Big]$$

$$= 16G^2\Big[(1-g_A^2)\big(m_1 E_2\, m_A E_3 + m_1 E_3\, m_A E_2\big) - 2(1+g_A^2)\,m_1 m_A\,p_2 p_3\Big] =$$

$$= 16\,G^2 m_1 m_A\Big[(E_2 E_3)\big[2(1-g_A^2) - 2(1+g_A^2)\big] + |\vec{p}_2|\,E_3\cos\theta\,\big[2(1+g_A^2)\big]\Big] =$$

$$= 16\,G^2 m_1 m_A\Big[-4g_A^2\,E_2 E_3 + (2+2g_A^2)\,|\vec{p}_2|\,E_3\cos\theta\Big]$$

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In order to obtain the decay width, we have to integrate over the 3-momenta:

$$\Gamma = \frac{1}{2m_A}\int\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\int\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\int\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3} |\overline{\mathcal{M}}|^2 (2\pi)^4 \delta(m_A - E_1 - E_2 - E_3)\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3)$$

Since we have that $E_1 = m_1$, we can carry out the integration over $\vec{p}_1$:

$$\Gamma = \frac{1}{4m_A m_1}\int\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\int\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3} |\overline{\mathcal{M}}|^2 (2\pi)\delta(m_A - m_1 - E_2 - E_3)$$

Since the $\bar{\nu}_e$ has no mass $\rightarrow E_3 = |\vec{p}_3|$, then the integral over $\vec{p}_3$ can also be carried out:

$$\Gamma = \frac{1}{4m_A m_1}\int\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\,\theta(E_3)\frac{E_3}{2\pi}|\overline{\mathcal{M}}|^2\Big|_{E_3 = m_A - m_1 \equiv E_0 - E_2} =$$

$$= \frac{1}{4m_A m_1}\int\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\,\theta(m_A - m_1 - E_2)\frac{E_3}{2\pi}16G^2 m_A m_1(-4g_A^2)E_2E_3\Big|_{E_3 = E_0 - E_2} =$$

$$\Gamma = 2G^2(-4g_A^2)\int\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\,\theta(m_A - m_1 - E_2)\frac{E_2(E_0-E_2)^2}{\pi}$$

If we consider that $m_2 \sim 0 \rightarrow E_2 \simeq |\vec{p}_2|$, and the electron energy spectrum will be:

$$\boxed{\frac{d\Gamma}{dE_2} = \frac{G^2}{2\pi^3}|\vec{p}_2|E_2(E_0-E_2)^2(-4g_A^2) \overset{E_2 \simeq |\vec{p}_2|}{\simeq} \frac{G^2}{2\pi^3}E_2^2(E_0-E_2)^2(-4g_A^2)}$$

Finally, the total decay width will be:

$$\boxed{\Gamma = \int_{m_2}^{E_0} dE_2\frac{d\Gamma}{dE_2}} \simeq \int_0^{E_0} dE_2\frac{G^2}{2\pi^3}E_2^2(E_0-E_2)^2(-4g_A^2) = \frac{G^2E_0^5}{60\pi^3}(-4g_A^2)$$

The electron mass can be neglected in this scenario. Since $E_0 = m_n - m_p$ and since the mass of the neutron and the proton is similar, $E_0 \sim 0$. The mass of the electron is not negligible for $E_0 \gtrsim m_e$, however, since we have that $E_0 \sim 0$, we can approximate $m_e \sim 0$.