Lecture 4:
Equation of state and acceleration of the expansion

Q: Friedmann's equation tells us a relation between Hubble's constant, the energy density, and the radius of curvature at any time. Remember that the energy density includes all forms of energy, *including the rest-mass of matter*. The evolution of the universe will depend on how the energy density varies as the universe expands. Let us take the derivative of Friedmann's equation, multiplied by a^2.

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon a^2 - \frac{kc^2}{R_0^2} \; ; \qquad 2\dot{a}\,\ddot{a} = \frac{8\pi G}{3c^2}(\dot{\epsilon}a^2 + 2\epsilon a\dot{a}) \; ; \qquad \frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\epsilon}\,\frac{a}{\dot{a}} + 2\epsilon\right) \; . \tag{1}$$

Q: So the evolution of the universe depends on how the energy density varies with the scale factor, as the universe expands. How do we determine that? What law do we use here?

L: First law of thermodynamics! $d(\epsilon V) = -p\, dV$.

Q: Take any comoving region, how does the volume vary with time? The volume of any region is $V(t) = a^3(t)V_c$, so $dV = 3a^2\dot{a}V_c dt = 3(\dot{a}/a)\, V\, dt$.

So,

$$\frac{d(\epsilon V)}{dt} = V\left[\dot{\epsilon} + 3\epsilon\frac{\dot{a}}{a}\right] = -3p\frac{\dot{a}}{a}\,V \; . \tag{2}$$

or

$$\dot{\epsilon} = -3(\epsilon + p)\frac{\dot{a}}{a} \; . \tag{3}$$

So, we find:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\,(\epsilon + 3p) \; . \tag{4}$$

L: So, to find out the evolution of the universe, all we need to know is the pressure, in addition to the energy density.

Q: What is the relation between energy density and pressure known as? It's an equation of state, that will generally depend on temperature.

Equations of state

Q: Now, let us look at some cases. What if the universe is filled with matter in the form of an ideal, non-relativistic gas? What is the relation between pressure and density? Ideal gas law:

$$p = nkT \simeq \frac{\epsilon}{\mu c^2}kT = \epsilon\,\frac{<v^2>}{3c^2} \; . \tag{5}$$

L: So, for matter that is *non-relativistic*, $p \ll \epsilon$, the pressure is negligible. In that case, $\epsilon \propto a^{-3}(t)$. The energy density is contributed mostly by the rest-mass of the gas, which is conserved in a comoving volume.

Q: What happens if the universe is made of radiation? What is the pressure? $p = \epsilon/3$, is generally true for isotropic radiation. Then, $\dot{\epsilon} = -4\epsilon(\dot{a}/a)$, so $\epsilon \propto a^{-4}$.

L: In general, the Universe may contain many components with different equations of state: matter, radiation, vacuum energy (as we shall see), and it can have curvature. The solution to the Friedmann equation for $a(t)$ can then be complicated.

L: Now, go back to the equation for the acceleration: $\ddot{a}/a = -4\pi G/(3c^2)(\epsilon + 3p)$. It seems like the universe should always decelerate the expansion.

Q: Or should it? Could the pressure be negative? And could the universe accelerate?

L: The Universe will accelerate if $p < -\epsilon/3$.

Q: The equation of state can in general be complicated. But the simple case $p = w\epsilon$ is often of interest ($w = 0$ for matter, $w = 1/3$ for radiation). How does the energy density change with the scale factor for this case?

$$\dot{\epsilon} = -3\frac{\dot{a}}{a}(1 + w)\epsilon \ . \tag{6}$$

So, $\epsilon \propto a^{-3(1+w)}$.

L: Another special case: the *vacuum energy* of a *cosmological constant*. Imagine that the vacuum has a fixed potential energy per unit volume, which gives a fixed energy density.

Q: If you have a box containing this stuff, and you want to make the box bigger, how much energy do you have to invest to pull on the walls of the box? $dE = \epsilon\, dV$. So how does the energy stored inside the box change? $dE = -p\, dV = \epsilon\, dV$. So $p = -\epsilon$.

L: This is called dark energy, or vacuum energy, of the cosmological constant case, the equation of state has $w = -1$.

L: For a model with only vacuum energy: $H(z) = H_0$, so $t_0 - t = \ln(1 + z)/H_0$, or $a = \exp[(t - t_0)/H_0]$, just like for the Steady-State universe. The comoving distance is $r = cz/H_0$.

Summary: To determine the evolution of the scale factor $a(t)$, we need to know the equation of state relating the pressure to the energy density. That tells us how the energy density will vary with the scale factor, and the Friedmann equation then determines $a(t)$. For the simple case $p = w\epsilon$, we have $\epsilon \propto a^{-3(1+w)}$. Non-relativistic matter has $w = 0$, radiation has $w = 1/3$. A component with $w < -1/3$ causes acceleration. The case of the cosmological constant is $w = -1$.

L: So, we have a universe that contains matter, radiation, and may contain vacuum energy with negative pressure, and may also have space curvature. So what is the equation for the evolution in this general case?

$$H^2(z) = H_0^2 \left[\Omega_{r,0}(1 + z)^4 + \Omega_{m,0}(1 + z)^3 + \Omega_{\Lambda,0} + (1 - \Omega_0)(1 + z)^2\right] \ . \tag{7}$$

As before, we can find the time and the comoving distance corresponding to any redshift by

$$t = \int_z^\infty \frac{dz}{H(z)(1 + z)} \ , \tag{8}$$

and

$$r = c\int_0^z \frac{dz}{H(z)} \ , \tag{9}$$

L: An important case: matter and cosmological constant. Defining the scale factor at which matter and cosmological constant have equal densities, $a_{m\Lambda} = [\Omega_{m0}/(1-\Omega_{m0})]^{1/3}$, the result of integrating the Friedmann equation can be written as

$$H_0 t = \frac{3}{3\sqrt{1-\Omega_{m.0}}} \ln \left[\left(\frac{a}{a_{m\Lambda}} \right)^{3/2} + \sqrt{1 + \left(\frac{a}{a_{m\Lambda}} \right)^3} \right] . \tag{10}$$

Notice that at large times, a goes like the exponential of time. At small times, $t \propto a^{3/2}$ (at very small times, radiation would become important and this equation for t as a function of a would become invalid).

L: *The Benchmark model*: all the present evidence, which we will describe later, indicates: $\Omega_{m,0} \sim 0.31$, $\Omega_0 = 1$, and so $\Omega_{\Lambda,0} \sim 0.69$ (radiation at present is a very small part of all the energy density). We have no space curvature, but we have some kind of dark energy with negative pressure!

L: More generally, instead of a cosmological constant there may be some form of dark energy where the pressure is some function of density that is unknown, and may even change with time. The bottom line is that we still don't know the function $a(t)$, and what will happen in the future, because we don't understand this dark energy!

L: For the case $p = w\epsilon$ for the dark energy, we have:

$$H^2(z) = H_0^2 \left[\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{de,0}(1+z)^{(1+w)/3} + (1-\Omega_0)(1+z)^2 \right] . \tag{11}$$

Observations (combining CMB, BAO, large-scale structure, supernovae, lensing) indicate not only that there is dark energy, but also that w is very close to -1, with $\sim 5\%$ accuracy at present, and the Universe is flat with the total Ω_0 being within 0.3% of 1.

Summary: We can solve Friedmann's equation and find $a(t)$, $H(z)$, the time as a function of redshift, the comoving distance as a function of redshift, for any model with any combination of components. The present observational evidence points to our benchmark model, which has at present matter and dark energy and has no curvature, and has also radiation which was dominant in the past.