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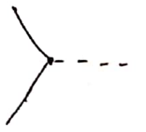
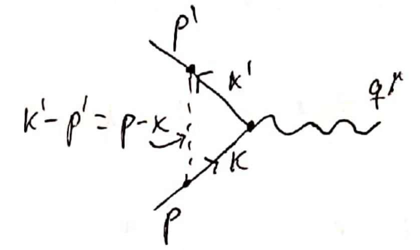
Homework 6:

Consider a real scalar ϕ with physical mass M coupled by $g\phi\bar{\psi}\psi$:

 $= -ig$ 

$= I$

1)

Let's then try to get an expression for I:

$$I = i\mathcal{M} = \int \frac{d^4k}{(2\pi)^4} (-ig)^2 \frac{i}{(p-k)^2 - m_\phi^2} \frac{i(\not{k}'+m_\psi)}{k'^2-m_\psi^2} (-ie\gamma^\mu) \frac{i(\not{k}+m_\psi)}{k^2-m_\psi^2} =$$

$$= eg^2 \int \frac{d^4k}{(2\pi)^4} \frac{(\not{k}'+m_\psi)\gamma^\mu(\not{k}+m_\psi)}{\left((p-k)^2-m_\phi^2\right)(k'^2-m_\psi^2)(k^2-m_\psi)^2} =$$

$$= eg^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{dx\,dy\,dz\,\delta^3(x+y+z-1)\,(\overbrace{\not{k}-\not{q}}^{k'}+m_\psi)\gamma^\mu(\not{k}+m_\psi)\cdot 2}{[xk^2+yk^2+zk^2+2yqk+yq^2+zp^2-2zpk-xm_\psi^2-ym_\psi^2-zm_\phi \ldots}$$

$$= eg^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{dx\,dy\,dz\,\delta^3(x+y+z-1)\,(\not{k}-\not{q}+m_\psi)\gamma^\mu(\not{k}+m_\psi)\cdot 2}{[k^2+2k(yq-zp)+yq^2+zp^2-(1-z)m_\psi^2-zm_\phi^2]^3} =$$

$$= eg^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 \frac{dx\,dy\,dz\,\delta^3(x+y+z-1)\,(\not{k}-\not{q}+m_\psi)\gamma^\mu(\not{k}+m_\psi)\cdot 2}{[\ell^2-\Delta]^3}$$

where $\ell = k+yq-zp$ and $\Delta = -xyq^2+(1-z^2)m_\psi^2+zm_\phi^2$

We can now switch the integrals, and $d^4k \to d^4\ell$:

$$I = eg^2 \int_0^1 dx\,dy\,dz\,\delta^3(x+y+z-1) \int \frac{d^4\ell}{(2\pi)^4} \frac{(\not{k}-\not{q}+m_\psi)\gamma^\mu(\not{k}+m_\psi)}{[\ell^2-\Delta]^3}$$

Everything looks nice, now we need to change the numerator:

$(\not{k}-\not{q}+m_\psi)\gamma^\mu(\not{k}+m_\psi)$ into $N_1\gamma^\mu + N_2\, i\sigma^{\mu\nu}q_\nu$, let's do it!

Let's then modify this numerator:

$$(\not k-\not q+m_\psi)\gamma^\mu(\not k+m_\psi) = \not k\gamma^\mu\not k + \not k\gamma^\mu m_\psi - \not q\gamma^\mu\not k - \not q\gamma^\mu m_\psi + m_\psi\gamma^\mu\not k + m_\psi^2\gamma^\mu =$$

$$= (\not l + z\not p - y\not q)\gamma^\mu(\not l + z\not p - y\not q) + (\not l+z\not p - y\not q)\gamma^\mu m_\psi - \not q\gamma^\mu(\not l + z\not p - y\not q) - \not q\gamma^\mu m_\psi$$

$$+ m_\psi\gamma^\mu(\not l + z\not p - y\not q) + m_\psi^2\gamma^\mu =$$

But we can substitute $\not p\to m_\psi$ at rightmost and $\not p' = \not q - \not p \to m_\psi$ at leftmost, also because the integral is symmetric, all terms linear in l will cancel, and finally also all the terms linear in γ^μ will also cancel:

$$= \not l\gamma^\mu\not l + z^2\not p\gamma^\mu\not p + y^2\not q\gamma^\mu\not q - zy(\not p\gamma^\mu\not q + \not q\gamma^\mu\not p) + m_\psi(z\not p - y\not q)\gamma^\mu -$$

$$- z\not q\gamma^\mu\not p + y\not q\gamma^\mu\not q - m\not q\gamma^\mu + zm_\psi\gamma^\mu\not p - ym_\psi\gamma^\mu\not q + m_\psi^2\gamma^\mu =$$

$$= \underbrace{\not l\gamma^\mu\not l}_{0} - y(1-y)\underbrace{\not q\gamma^\mu\not q}_{0} + z(1-y)\underbrace{\not q\gamma^\mu\not p}_{-2m_\psi p^\mu} - zy\underbrace{\not p\gamma^\mu\not q}_{2m_\psi p'^\mu} + z^2\underbrace{\not p\gamma^\mu\not p}_{2m_\psi p^\mu} + \cancelto{0}{m_\psi^2\gamma^\mu} +$$

$$+ m_\psi(1-y)\underbrace{\not q\gamma^\mu}_{-2p^\mu} + m_\psi z\underbrace{\not p\gamma^\mu}_{2p^\mu} - m_\psi y\underbrace{\gamma^\mu\not q}_{2p'^\mu} + m_\psi z\underbrace{\gamma^\mu\not p}_{0} =$$

$$= -2m_\psi z(1-y)p^\mu - 2m_\psi zyp'^\mu + 2m_\psi z^2p^\mu - 2m_\psi(1-y)p^\mu + 2m_\psi zp^\mu - 2m_\psi yp'^\mu =$$

$$= m_\psi p'^\mu(-2zy - 2y) + m_\psi p^\mu(2zy + 2y + 2z^2 - 2) =$$

$$= m_\psi(p'^\mu - p^\mu)(-2zy - 2y) + m_\psi p^\mu(2z^2 - 2) \leftarrow + m_\psi p'^\mu(z^2-1) - m_\psi p'^\mu(z^2-1)$$

$$= m_\psi\left[(p'^\mu + p^\mu)(z^2-1) + \underbrace{(p'^\mu - p^\mu)}_{q^\mu\simeq 0}(1 - z^2 - 2zy - 2y)\right] =$$

(if alone: $p'^\mu\gamma^\mu m + p^\mu\gamma^\mu m \to p'^\mu - p^\mu \propto m - m = 0$)

$$= m_\psi\left[\underbrace{(p'^\mu + p^\mu)(z^2-1)}\right] =$$

$$\left(-\cancel{2}m_\psi(z^2-1)\frac{i\sigma^{\mu\nu}q_\nu}{\cancel{2m_\psi}} + 2m_\psi\gamma^\mu(z^2-1)\right.$$

Gordon Identity, I haven't write them, but we have a $\bar u(p')[\dots]u(p)$ braket, in all the computation, due to the external legs (rightmost, leftmost) (same as)

$$= \boxed{m_\psi(1-z^2)i\sigma^{\mu\nu}q_\nu + 2m_\psi^2\gamma^\mu(z^2-1) = \text{Numerator}}$$

so: $\boxed{N_1 = 2m_\psi^2(z^2-1)}$ and $\boxed{N_2 = 2m_\psi(1-z^2)}$ they fullfill the dependances you gave ✓✓

2)

$$\left(\Gamma = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} \underline{F_2(q^2)}\right)$$

We will take $q \to 0$ for $\boxed{F_2(q^2)} = \int dx\,dy\,dz\,\delta^{(3)}(x+y+z-1)\int \frac{d^4 l}{(2\pi)^4}\, i g^2 \frac{2m_\psi^2(1-z^2)\cdot 2}{[l^2-\Delta]^3} =$

$$= \int dx\,dy\,dz\,\delta^{(3)}(x+y+z-1)\left[i g^2 \frac{-i}{(4\pi)^2}\frac{4m_\psi^2(1-z^2)}{2}\frac{1}{\Delta}\right] =$$

$$= \frac{g^2 m_\psi^2}{8\pi^2}\int dx\,dy\,dz\,\delta^3(x+y+z-1)\frac{(1-z^2)}{z m_\phi^2 + (1-z)^2 m_\psi^2} =$$

$$= \frac{g^2 m_\psi^2}{8\pi^2}\int_0^1 dz \frac{(1-z)(1-z^2)}{z m_\phi^2 + (1-z)^2 m_\psi^2} \simeq \frac{g^2 m_\psi^2}{8\pi^2}\left[\int_0^1 dz \frac{1}{z m_\phi^2 + (1-z)^2 m_\psi^2} - \frac{1}{m_\phi^2}\int_0^1 dz\,(1+z-z^2)\right] =$$

$$\boxed{= \frac{g^2 m_\psi^2}{8\pi^2}\left[\int_0^1 dz \frac{1}{z + (1-z)^2\frac{m_\psi^2}{m_\phi^2}} - \frac{7}{6}\right]}$$

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3)

Expanding for $m_\phi \gg m_\psi$, we get:

$$F_2(q^2) = \frac{g^2 m_\psi^2}{8\pi^2 m_\phi^2}\left[\frac{1}{1-\frac{m_\psi^2}{m_\phi^2}}\int_{\frac{m_\psi^2}{m_\phi^2}}^1 du\,\frac{1}{u} - \frac{7}{6}\right] =$$

$$\left(\delta = \frac{m_\psi}{m_\phi}\right)$$

$$= \frac{g^2\delta^2}{8\pi^2}\left[\frac{1}{1-\delta^2}\left(\ln(1) - \ln(\delta^2)\right) - \frac{7}{6}\right] \simeq$$

$$\simeq \frac{g^2\delta^2}{8\pi^2}\left[\ln(\delta^2) - \frac{7}{6}\right]$$

$$\boxed{a = \frac{7}{6\cdot 8\pi^2}}$$

$$\boxed{b = \frac{1}{8\pi^2}}$$

$$= g^2\frac{m_\psi^2}{m_\phi^2}\left[\frac{1}{8\pi^2}\ln\left(\frac{m_\psi^2}{m_\phi^2}\right) - \frac{7}{6\cdot 8\pi^2}\right]$$