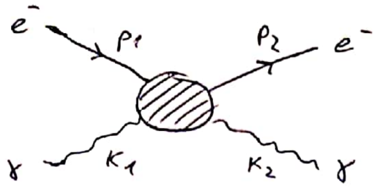
QFT, Sheet 5:

Guillermo Abad López

5.1)



with $\mathcal{L}_{int_{QED}} = e\bar{\Psi}\cancel{A}\Psi$

$e^-(p_1)\gamma(k_1) \longrightarrow e^-(p_2)\gamma(k_2)$

a)

- **1st order:** $\langle p_2 k_2 | e\bar{\Psi}\cancel{A}\Psi | p_1 k_1\rangle \longrightarrow$ not enough A's, we need 2 to contract k_1 and k_2, we can't leave one alone: $\left(\langle k_2 | A | k_1\rangle = \langle k_2 | 0\rangle = 0\right)$

- **2nd order:** $\langle p_2 k_2 | e^2 (\bar{\Psi}\cancel{A}\Psi)_x (\bar{\Psi}\cancel{A}\Psi)_y | p_1 k_1\rangle =$

$$= e^2 \langle p_2 | (\bar{\Psi}\gamma^\mu\Psi)_x (\bar{\Psi}\gamma^\nu\Psi)_y | p_1\rangle \langle k_2 | A_{\mu_x} A_{\nu_y} | k_1\rangle =$$

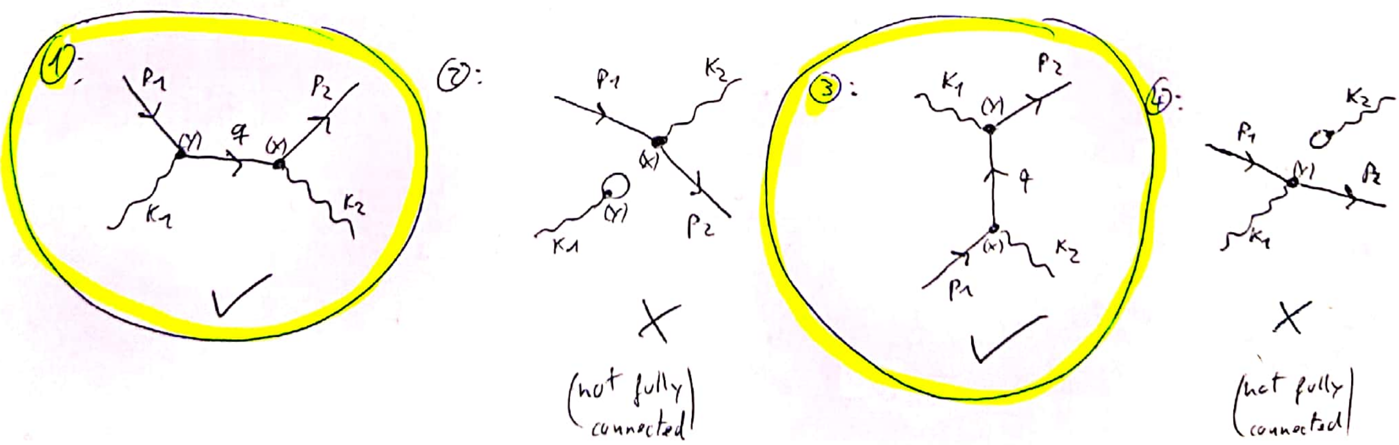
$$= e^2 \langle k_2 | A_{\mu x} A_{\nu y} | k_1\rangle \Big[\langle p_2 | \bar{\Psi}_x \gamma^\mu \Psi_x \bar{\Psi}_y \gamma^\nu \Psi_y | p_1\rangle + \quad (1)$$

$$+ \langle p_2 | \bar{\Psi}_x \gamma^\mu \Psi_x \bar{\Psi}_y \gamma^\nu \Psi_y | p_1\rangle + \quad (2)$$

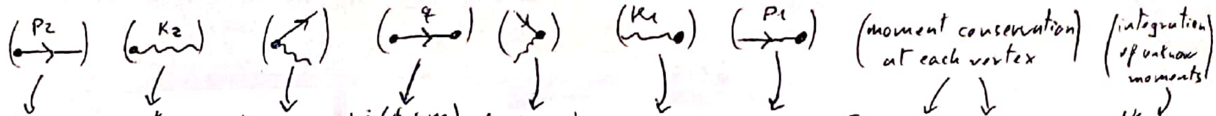
$$+ \langle p_2 | \bar{\Psi}_x \gamma^\mu \Psi_x \bar{\Psi}_y \gamma^\nu \Psi_y | p_1\rangle + \quad (3)$$

$$+ \langle p_2 | \bar{\Psi}_x \gamma^\mu \Psi_x \bar{\Psi}_y \gamma^\nu \Psi_y | p_1\rangle \Big] \quad (4)$$

(the only possible contractions except for $x \leftrightarrow y$)



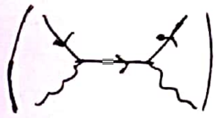
b)

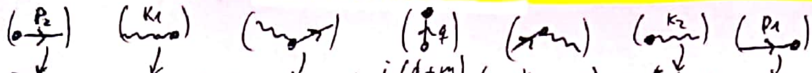


- $$iT_{①} = \sum_{\lambda,\lambda'} \int \bar{u}(p_2)\, \epsilon^*_{\mu\lambda}(k_2)\, \left((2\pi)^4 ie\gamma^\mu\right) \frac{i(\not{q}+m)}{q^2-m^2+i\varepsilon} \left((2\pi)^4 ie\gamma^\nu\right) \epsilon_{\nu\lambda'}(k_1)\, u(p_1)\, \delta(q-p_1-k_1)\, \delta(q-p_2-k_2) \frac{d^4q}{(2\pi)^4} =$$

$$= \sum_{\lambda,\lambda'} (-ie^2)\, \bar{u}(p_2)\, \epsilon^*_{\mu\lambda}(k_2)\, \gamma^\mu \frac{(\not{p}_1+\not{k}_1)+m}{(p_1+k_1)^2-m^2} \gamma^\nu \epsilon_{\nu\lambda'}(k_1)\, u(p_1)\, (2\pi)^4 \delta(p_1+k_1-p_2-k_2)$$

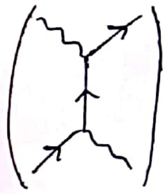
so $$\boxed{\mathcal{M}_{①}} = \frac{iT_{①}}{i(2\pi)^4\delta(p_1+k_1-p_2-k_2)} = -e^2 \sum_{\lambda,\lambda'} \bar{u}(p_2)\, \epsilon^*_{\mu\lambda}(k_2)\, \gamma^\mu \frac{(\not{p}_1+\not{k}_1)+m}{(p_1+k_1)^2-m^2} \gamma^\nu \epsilon_{\nu\lambda'}(k_1)\, u(p_1)$$





- $$iT_{②} = \sum_{\lambda,\lambda'} \int \bar{u}(p_2)\, \epsilon_{\mu\lambda}(k_1)\, \left((2\pi)^4 ie\gamma^\mu\right) \frac{i(\not{q}+m)}{q^2-m^2+i\varepsilon} \left((2\pi)^4 ie\gamma^\nu\right) \epsilon^*_{\nu\lambda'}(k_2)\, u(p_1)\, \delta(q-p_1+k_2)\, \delta(q-p_2+k_1) \frac{d^4q}{(2\pi)^4} =$$

$$= \sum_{\lambda,\lambda'} (-ie^2)\, \bar{u}(p_2)\, \epsilon_{\mu\lambda}(k_1)\, \gamma^\mu \frac{(\not{p}_1-\not{k}_2)+m}{(p_1-k_2)^2-m^2} \gamma^\nu \epsilon^*_{\nu\lambda'}(k_2)\, u(p_1)\, (2\pi)^4 \delta(p_1+k_1-p_2-k_2)$$



so $$\boxed{\mathcal{M}_{②}} = \frac{iT_{②}}{i(2\pi)^4\delta(p_1+k_1-p_2-k_2)} = -e^2 \sum_{\lambda,\lambda'} \bar{u}(p_2)\, \epsilon_{\mu\lambda}(k_1)\, \gamma^\mu \frac{(\not{p}_1-\not{k}_2)+m}{(p_1-k_2)^2-m^2} \gamma^\nu \epsilon^*_{\nu\lambda'}(k_2)\, u(p_1)$$

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c)

$$\boxed{\mathcal{M}_T} = \mathcal{M}_{①} + \mathcal{M}_{②} = -e^2 \sum_{\lambda,\lambda'} \epsilon^*_{\mu\lambda}(k_2)\, \epsilon_{\nu\lambda'}(k_1)\, \bar{u}(p_2) \Big(\gamma^\mu \overbrace{\frac{(\not{p}_1+\not{k}_1)+m}{\underbrace{(p_1+k_1)^2-m^2}_{\cancel{p_1^2}+\cancel{k_1^2}^{\,0}+2k_1p_1-\cancel{m^2}}}}^{①\text{ component}} \gamma^\nu + \gamma^\nu \overbrace{\frac{(\not{p}_1-\not{k}_2)+m}{\underbrace{(p_1-k_2)^2-m^2}_{\cancel{p_1^2}+\cancel{k_2^2}^{\,0}-2p_1k_2-\cancel{m^2}}}}^{②\text{ component}} \gamma^\mu \Big) u(p_1)$$

(different signs that will allow us to cancel the terms)

And if we apply the transformation $\epsilon^\mu(k) \longrightarrow \epsilon^\mu(k) + \lambda k^\mu$ we get that:

$$\epsilon^*_{\mu\lambda}(k_2)\, \epsilon_{\nu\lambda'}(k_1) \longrightarrow \left(\epsilon^*_{\mu\lambda}(k_2) + \lambda^* k_{2\mu}\right)\left(\epsilon_{\nu\lambda'}(k_1) + \lambda k_{1\nu}\right) =$$

$$= \epsilon^*_{\mu\lambda}(k_2)\, \epsilon_{\nu\lambda'}(k_1) + \lambda^* k_{2\mu}\, \epsilon_{\nu\lambda'}(k_1) + \lambda k_{1\nu}\, \epsilon^*_{\mu\lambda}(k_2) + |\lambda|^2 k_{2\mu} k_{1\nu}$$

so, contracting the μ's and ν's our $\mathcal{M}$ amplitude transforms as:

$$\mathcal{M}_T \longrightarrow \mathcal{M}_T - e^2 \sum_{\lambda,\lambda'} \bar{u}(p_2) \left(\Delta \right) u(p_1)$$

where $$\Delta = |\lambda|^2 \left(\frac{\not{k}_2[(\not{p}_1+\not{k}_1)+m]\not{k}_1}{2k_1p_1} - \frac{\not{k}_1[(\not{p}_1-\not{k}_2)+m]\not{k}_2}{2k_2p_1} \right) +$$
$$+ \lambda^* \left(\frac{\not{k}_2[(\not{p}_1+\not{k}_1)+m]\not{\epsilon}_{\lambda'}(k_1)}{2k_1p_1} - \frac{\not{\epsilon}_{\lambda'}(k_1)[(\not{p}_1-\not{k}_2)+m]\not{k}_2}{2k_2p_1} \right) +$$
$$+ \lambda \left(\underbrace{\frac{\not{\epsilon}^*_{\lambda}(k_2)[(\not{p}_1+\not{k}_1)+m]\not{k}_1}{2k_1p_1}}_{①\text{ components}} - \underbrace{\frac{\not{k}_1[(\not{p}_1-\not{k}_2)+m]\not{\epsilon}^*_{\lambda}(k_2)}{2k_2p_1}}_{②\text{ components}} \right)$$

From where we need proof that the ① components cancel the ② components and that $\Delta = 0$

To show that $\Delta = 0$, we will need to use that:

- $\not p \not k = p^\mu \gamma_\mu \gamma_\nu k^\nu \overset{(\{\gamma_\mu \gamma_\nu\} = 2g_{\mu\nu})}{=} p^\mu (2g_{\mu\nu} - \gamma_\nu \gamma_\mu) k^\nu = 2pk - \not k \not p \longrightarrow \boxed{\{\not p, \not k\} = 2pk}$ (*)

- $\not k \not k = 2\underbrace{\cancel{kk}}_{(0 = k \cdot k = m_\gamma)} - \not k \not k \longrightarrow \boxed{\not k \not k = 0}$ (**)

- $$\left.\begin{aligned} s &= (p_1 + k_1)^2 = p_1^2 + \cancel{k_1^2} + 2p_1k_1 = m^2 + 2p_1k_1 \\ s &= (p_2 + k_2)^2 = p_2^2 + \cancel{k_2^2} + 2p_2k_2 = m^2 + 2p_2k_2 \end{aligned}\right\} \boxed{p_1k_1 = p_2k_2}$$ (*****)

- $$\left.\begin{aligned} u &= (p_1 - k_2)^2 = p_1^2 + \cancel{k_2^2} - 2p_1k_2 = m^2 - 2p_1k_2 \\ u &= (p_2 - k_1)^2 = p_2^2 + \cancel{k_1^2} - 2p_2k_1 = m^2 - 2p_2k_1 \end{aligned}\right\} \boxed{p_1k_2 = p_2k_1}$$ (****)

- Dirac equation:

$$(\not p - m)\psi(p) = 0 \longrightarrow \boxed{(\not p - m)u(p) = 0}$$
$$(\not p + m)\bar\psi(p) = 0 \longrightarrow \boxed{\bar u(p)\,(\not p - m) = 0}$$ (***)

so let's start with the $|\lambda|^2$ term of Δ:

▷ $|\lambda|^2$ Term:

$$\frac{\not k_2[(\not p_1 + \not k_1) + m]\not k_1}{2k_1p_1} - \frac{\not k_1[(\overbrace{\not p_1 + \not k_2}^{\not k_2 - \not k_1}) + m]\not k_2}{2k_2p_1} \overset{(*),(**)}{=} \frac{\not k_2 \not k_1(-\not p_1 + m) + 2\not k_2\, p_1k_1}{2k_1p_1} - \frac{(-\not p_2 + m)\not k_1 \not k_2 + 2k_1p_2\not k_2}{2k_2p_1}$$

(comute $\not k_1$ till it reaches $\not k_2$)

which because we have $\bar u(p_2)\,\Delta\, u(p_1)$, $(-p+m)$ doesn't contribute (***):

$$\frac{2\not k_2 p_1k_1}{2k_1p_1} - \frac{2k_1p_2\not k_2}{2k_2p_1} \overset{(****)}{=} \frac{2\not k_2 p_1k_1}{2k_1p_1} - \frac{2k_2p_1\not k_2}{2k_2p_1} = \not k_2 - \not k_2 \boxed{= 0}$$

now let's do the same for the other 2 terms:

▷ λ term:

It's the exact same changing $\not k_2$ by $\not\epsilon_\lambda^*(k_2)$, so $\boxed{=0}$

▷ λ^* term:

It's the opposite, we gonna anticommute the $\not k_2$ instead of the $\not k_1$:

$$\frac{\not k_2[(\overbrace{\not p_1 + \not k_1}^{\not k_2 + \not k_2}) + m]\not\epsilon_{\lambda'}(k_1)}{2k_1p_1} - \frac{\not\epsilon_{\lambda'}(k_1)[(\not p_1 - \not k_2) + m]\not k_2}{2k_2p_1} \overset{(*)(**)}{=} \frac{\not k_2\not\epsilon_{\lambda'}(k_1)\cancel{(-\not k_2 + m)} + 2\not\epsilon_{\lambda'}(k_1)\,k_2p_2}{2k_1p_1} - \frac{\not\epsilon_{\lambda'}(k_1)\not k_2\cancel{(-\not p_1 + m)} + 2\not\epsilon_{\lambda'}(k_1)p_1k_2}{2k_2p_1}$$

$$\overset{(***)}{=} \frac{2\not\epsilon_{\lambda'}(k_1)k_2p_2}{2k_1p_1} - \frac{2\not\epsilon_{\lambda'}(k_1)p_1k_2}{2k_2p_1} \overset{(*****)}{=} \frac{\cancel{2}\not\epsilon_{\lambda'}(k_1)\cancel{k_1p_1}}{\cancel{2k_1p_1}} - \frac{2\not\epsilon_{\lambda'}(k_1)\cancel{p_1k_2}}{\cancel{2k_2p_1}} = \not\epsilon_{\lambda'}(k_1) - \not\epsilon_{\lambda'}(k_1) \boxed{= 0}$$

so we finally see that under gauge transformation the total $\mathcal{M}_T = \mathcal{M}_1 + \mathcal{M}_3$:

$$\boxed{\mathcal{M}_T \longrightarrow} \; \mathcal{M}_T - e^2 \sum_{\lambda\lambda'} \bar{u}(p_2) \cancelto{0}{\not{k}}\, u(p_1) \; \boxed{= \mathcal{M}_T}$$

invariant!!!

And because the terms of  cancelled with  , and not independently it's obvious that **$\mathcal{M}_1, \mathcal{M}_3$ are not invariant alone!**

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5.2)

a)

from the optical theorem we get:

$$\mathrm{Im}(T) = \frac{1}{2} T^\dagger T = \underbrace{2 E_{cm} P_{cm} \frac{T^\dagger T}{4 E_{cm} P_{cm}}}_{\frac{1}{2}}$$

and now adding $|k_1 k_2\rangle$ to left and righ:

$$\mathrm{Im}(\langle k_1 k_2 | T | k_1 k_2\rangle) = 2 E_{cm} P_{cm} \frac{\langle k_1 k_2 | T^\dagger T | k_1 k_2\rangle}{4 E_{cm} P_{cm}}$$

$$\mathrm{Im}\left[-i \mathcal{M}(k_1 k_2 \to k_1 k_2)(2\pi)^4 \delta(k_1 + k_2 - k_1 + k_2)\right] = 2 E_{cm} P_{cm} \frac{\langle k_1 k_2 | T^\dagger \overset{\left(\sum_f \int \prod_{i=1} \frac{d^3 q_i}{(2\pi)^3 2E_i} |f\rangle\langle f|\right)}{\overset{\|}{\mathbb{1}}} T | k_1 k_2\rangle}{4 E_{cm} P_{cm}}$$

$$\mathrm{Im}\left[\mathcal{M}(k_1 k_2 \to k_1 k_2)\right] = 2 E_{cm} P_{cm} \underbrace{\frac{i}{(2\pi)^4} \frac{\sum_f \int \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3 2E_i} \langle k_1 k_2 | T^\dagger | f\rangle\langle f | T | k_1 k_2\rangle}{4 E_{cm} P_{cm}}}_{①}$$

so we only need to proof that ① is our σ^{tot}:

$$① = \frac{i}{(2\pi)^4} \frac{1}{4 E_{cm} P_{cm}} \sum_f \int \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3 2E_i} \langle k_1 k_2 | \mathcal{M} | f\rangle\langle f | \mathcal{M} | k_1 k_2\rangle \left| i (2\pi)^4 \delta(k_1 + k_2 - \sum_f q_f)\right|^2 =$$

$$= i (2\pi)^4 \frac{1}{4 E_{cm} P_{cm}} \sum_f \int \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3 2E_i} \left|\langle k_1 k_2 | \mathcal{M} | f\rangle\right|^2 \delta(k_1 + k_2 - \sum_f q_f) =$$

$$= \frac{i}{4 E_{cm} P_{cm}} \sum_f \int d\mathrm{LIPS} \, |\mathcal{M}(k_1 k_2 \to f)|^2 = \sigma^{tot}(k_1 k_2 \to \text{anything}):$$

so, we finally see, that:

$$\boxed{\mathrm{Im}\left[\mathcal{M}(k_1 k_2 \to k_1 k_2)\right] = 2 E_{cm} P_{cm} \, \sigma^{tot}(k_1 k_2 \to \text{anything})}$$

b)

i) LSZ reduction formula:

$$\tilde{G}(p) = \left(\frac{i\sqrt{Z_\phi}}{p^2-M^2}\right)^2 \langle p|iT|p\rangle$$

so:

$$\frac{-i}{p^2-m^2-\Sigma^{1PI}(p^2)} = \frac{Z_\phi}{(p^2-m^2-\Sigma^{1PI}(p^2))^2} \langle p|iT|p\rangle$$

which relates to:

$$\boxed{\underset{\Delta(p\to p)}{\overset{\shortparallel}{\langle p|M|p\rangle}} = \frac{-i(p^2-m^2-\Sigma^{1PI}(p^2))}{Z_\phi\ (2\pi)^4}}$$

ii)

if $\Sigma^{1PI}(p^2)$ has imaginary part, so does in the denominator, so to have $p^2-m^2-\Sigma^{1PI}(p^2)=0$, we need to cancel it's imaginary part, saying, our pole will be imaginary.

$$\underbrace{\Sigma^{1PI}(p^2) \text{ small} \quad + \quad \mathrm{Im}(\Sigma^{1PI}(p^2)) \overset{\text{when } p^2\sim M^2}{\simeq} \mathrm{Im}(\Sigma^{1PI}(M^2))}$$

$$|\tilde{G}(p)|^2 = |\Delta_F^{full}(p^2)|^2 = \left|\frac{1}{p^2-m^2-\Sigma^{1PI}(p^2)}\right|^2 \simeq$$

$$\simeq \left|\frac{1-\left.\frac{d\Sigma^{1PI}(p^2)}{dp^2}\right|_{p^2=M^2}}{\underset{M^2\to p^2}{\lim}\ p^2-M^2}\right| \simeq \left|\frac{1}{p^2-M^2+(p^2-M^2)\left.\frac{d\Sigma^{1PI}(p^2)}{dp^2}\right|_{p^2=M^2}}\right|$$

so we see that

$$\boxed{\mathrm{im}\,\Gamma = (p^2-M^2)\left.\frac{d\Sigma^{1PI}(p^2)}{dp^2}\right|_{p^2=M^2}} \overset{\text{(from i))}}{=} p^2-M^2\left(-1+i Z_\phi (2\pi)^4 \frac{d\langle p|M|p\rangle}{dp^2}\right) =$$

$$= M^2-p^2+i(p^2-M^2) Z_\phi (2\pi)^4 \frac{d\sigma}{dp^2}$$

(from a))