Universitat de Barcelona

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DYNAMICS AND SPECTRUM OF THE SCHWINGER MODEL

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To my beloved ones, who I owe it all. Endure and transcend.

Abstract

The Schwinger model, named after Julian Schwinger [1], is the model describing 1+1 d (one spatial dimension + time) quantum electrodynamics. The massless model can be solved exactly and is known to possess some interesting properties [2]–[4] which make it an excellent toy model for other more complex theories (such as qcd).

The massless model exhibits confinement, a mass gap for the bound states, a periodic θ parameter independent from the couplings and the masses, an anomaly of a U(1) chiral symmetry and charge shielding. Also the massive model develops a topological discontinuity at a critical point of the ratio m/e, due to the periodicity of the independent θ term, which we relate to a phase transition of the system at $\theta = \pi$.

In this thesis, first we will present an extense review to gain intuition of the massless model at classical and quantum levels. And then to finish we will add a mass to the fermions which makes the model not exactly solvable, and we will work out the spectrum of the theory in the small coupling $e << m$ regime.

Contents

Chapter 0

Motivation and introduction

The Schwinger model, named after Julian Schwinger [1], is the model describing 1+1 d (one spatial dimension + time) quantum electrodynamics. The massless model can be solved exactly using operator methods [2] and is known to possess some interesting properties [3]–[5] such as:

- This model exhibits confinement for the charges similar to the quark confinement in QCD, because in two dimensions, classically, the Coloumb potential goes linearly as r, instead of 1/r as in the typical 4 dimensions (in 1+1 d the field lines that get out of a charge can't get separated in any other direction, so they keep together until they reach another charge).
- In this model there is also a mass gap, giving mass to the bound states of particle-antiparticles through the gauge field. The value of this mass gap will change with the ratio e/m for the massive case.
- The model has a periodic θ parameter independent from the couplings and the masses, that appears in the theory, behaving as an external electric field, that affects the bounding or unbounding of the particle-antiparticle pairs.
- There is also an anomaly in the model, where we will see that from the original two classical conserved currents massless QED has, only one can remain conserved at the quantum level. To preserve gauge invariance, the symmetry that will suffer the anomaly will be the chiral symmetry.
- In the model we will also have charge shielding, if we separate particle-antiparticle enough, the energy will be sufficient to create a new pair particle-antiparticle shielding the previous. Here the θ parameter acting as a background electric field will again play a significant role making it possible or not possible for pair creation.
- And finally the massive model will develop a phase transition for the periodic θ parameter at $\theta = \pi$, for a critical point of the ratio m/e.

This makes it an excellent toy model for other more complex theories such as qcd or other gauge theories of higher dimensions. Also due to its dimensional simplicity this model is a great way to approach all the previously mentioned subtleties of quantum field theories.

In the first chapter we will review this model at the classical level, studying its Lagrangian, the equations of motion, the U(1) global symmetries, the U(1) gauge symmetry, its conserved currents, all the way to the general solution of it, and to finish the chapter we will review the so called θ term which is related with the topology of the model.

Then in the second chapter we will do a quantisation of the model which we are going to follow by lot of interesting phenomena that happens at the quantum level, such as the ABJ anomaly and the mass gap of the theory, and then we will do a canonical quantization of the model in a circle and the spectrum that arises from it. Then we will relate the topology of the model with the θ parameter and the anomaly. And to end we will show that the fermion model can be described with bosons through bosonization.

Finally in the last chapter we will introduce the massive model which develops a phase transition at a critical point (at a concrete ratio of m/e) given by a topological discontinuity of the model, related with the periodicity of the θ parameter. Because this model is not exactly solvable you have to solve it in two different limit regimes [5] [6]:

- Weak coupling ($e << m$) where we can use perturbation theory in the coupling and non-relativistic approaches.
- Strong coupling ($e >> m$) where we can use perturbation theory in the bosonized version of the theory.

and to end we will work out the spectrum of the theory for the weak limit regime, where the θ parameter appears again with a phase transition.

Chapter 1

The classical Schwinger model, massless electrodynamics in 1+1 d

1.1 Introduction to massless classical electrodynamics

Since the Schwinger model is masless QED in 1+1 dimensions, first we are going to study the general QED Lagrangian ($\mathcal{L}_{QED}$) at the classical level for arbitrary dimensions. Concretely we will study its equations of motion, its behaviour under global and gauge symmetries and the conserved currents that arise from them.

1.1.1 Equations of motion in classical electrodynamics

The QED Lagrangian expressed in two equivalent ways takes the forms [7]:

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi = \frac{1}{2}A^{\mu}(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)A^{\nu} + \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi \quad (1.1)$$

where $D^{\mu} = \partial^{\mu} + ieA^{\mu}$ is the covariant derivative and $F^{\mu\nu} = \frac{-i}{e}[D^{\mu}, D^{\nu}] = D^{\mu}A^{\nu} - D^{\nu}A^{\mu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ is the field strength tensor (both of these terms are explained in equations eq.(A.2) and eq.(A.3) respectively in Appendix A). Also a slashed d-vector means: $\not{X} = X^{\mu}\gamma_{\mu}$ where γ^{μ} is a Dirac matrix that has to satisfy the Dirac Algebra $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$ and finally the bar over a field means $\bar{\psi} = \psi^{\dagger}\gamma^{0}$, which when we quantize the model will describe the antiparticle states.

The equations of motion for A^{ν}, ψ and $\bar{\psi}$ from this Lagrangian are:

$$\frac{\partial \mathcal{L}}{\partial A^{\mu}} - \partial^{\nu}\frac{\partial \mathcal{L}}{\partial(\partial^{\nu}A^{\mu})} = 0 \quad \longrightarrow \quad \partial_{\nu}F^{\nu\mu} = e\bar{\psi}\gamma^{\mu}\psi \quad (1.2)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial^{\mu}\frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\psi)} = 0 \quad \longrightarrow \quad \bar{\psi}(i\not{D} + m) = 0 \quad (1.3)$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial^{\mu}\frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\bar{\psi})} = 0 \quad \longrightarrow \quad (i\not{D} - m)\psi = 0 \quad (1.4)$$

where D_{μ} is acting on the left and on the right respectively. Also from these equations we see one conserved current $\partial_{\mu}(\bar{\psi}\gamma^{\mu}\psi) = \partial_{\mu}(\partial_{\nu}F^{\nu\mu}) = 0$ due to the anti-symmetry in the definition of the field strength tensor ($F_{\mu\nu} = -F_{\nu\mu}$).

So we are going to define the vector current as this conserved current, which later we will relate with a global U(1) symmetry.

$$j_V^{\mu} = \partial_{\nu}F^{\nu\mu} = \bar{\psi}\gamma^{\mu}\psi \quad (1.5)$$

1.1.2 $U(1)$ gauge symmetry for classical electrodynamics

Directly from the QED Lagrangian in eq.(1.1), we see that this theory has a $U(1)$ local gauge symmetry:

$$\begin{cases} \psi(x) \;\rightarrow\; e^{ie\alpha(x)}\psi(x) \\ A^\mu(x) \rightarrow\; A^\mu(x) - \partial^\mu\alpha(x) \end{cases} \tag{1.6}$$

which, due to Noether theorem, leads to the conserved current:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\delta A^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = F^{\mu\nu}\partial_\nu\alpha(x) - e\alpha(x)\bar{\psi}\gamma^\mu\psi \tag{1.7}$$

which if the fields fulfill the equations of motion of eq.(1.2), become:

$$j^\mu = -\partial_\nu(F^{\nu\mu}\alpha(x)) \quad\longrightarrow\quad \partial_\mu j^\mu = -\partial_\mu\partial_\nu(F^{\nu\mu}\alpha(x)) = 0 \tag{1.8}$$

where it's easy to see that the current is conserved, again due to the field strength tensor antisymmetry. And if we restrict ourselves to $\alpha(x) \rightarrow \alpha$, now α can be factorized out of the currents, and we see that current from eq.(1.8) becomes the vector current j^μ_V we already found directly from the equations of motion in eq.(1.2).

So we have seen, that the global vector current j^μ_V, which would be there even if there are no gauge fields, is mathematically equivalent to the global case of the gauge symmetry current.

1.1.3 Chiral symmetry for classical electrodynamics in even dimensions

For even-dimensional space-times, we can rewrite the Lagrangian in eq.(1.1) in its corresponding chiral states:

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\not{D}\psi_R - m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R) \tag{1.9}$$

with $\gamma^5 = i\gamma^0...\gamma^{d-1}$ ($\gamma^5\gamma^5 = 1$) where $\psi_L = P_L\psi = \frac{1-\gamma^5}{2}\psi$ are left-handed chirality fermions and $\psi_R = P_R\psi = \frac{1+\gamma^5}{2}\psi$ are right-handed chirality fermions, both of which are defined to be eigenstates of γ^5 with eigenvalues $\pm$:

$$\begin{cases} \gamma^5\psi_R = \gamma^5 P_R\psi = \gamma^5\frac{1+\gamma^5}{2}\psi = \frac{\gamma^5+1}{2}\psi = P_R\psi = \psi_R \\ \gamma^5\psi_L = \gamma^5 P_L\psi = \gamma^5\frac{1-\gamma^5}{2}\psi = \frac{\gamma^5-1}{2}\psi = -P_L\psi = -\psi_L \end{cases} \tag{1.10}$$

also it is easy to see that $\psi_L^\dagger\psi_R = \psi_R^\dagger\psi_L = 0$ since we obtain them from projecting to orthogonal subspaces of the Dirac indices: $P_L^2 = P_L$, $P_R^2 = P_R$ and $P_L P_R = 0$.

From this new Lagrangian in eq.(1.9), we see that only the mass term mixes the projections ψ_L and ψ_R, so if we had massless QED (m=0) we would have a theory with $U(1)_L \times U(1)_R$ symmetries, one independent transformation for each projection:

$$\begin{cases} \psi_R \;\rightarrow\; U_R\;\psi_R \\ \psi_L \;\rightarrow\; U_L\;\psi_L \end{cases} \quad\longrightarrow\quad \begin{cases} j^\mu_R = \bar{\psi}_R\gamma^\mu\psi_R \\ j^\mu_L = \bar{\psi}_L\gamma^\mu\psi_L \end{cases} \quad \text{with} \quad \begin{cases} \partial_\mu j^\mu_R = 0 \\ \partial_\mu j^\mu_L = 0 \end{cases} \tag{1.11}$$

but changing the basis, we can also express these transformations as a symmetry U_V that transforms both components in the same way, and then another symmetry U_A that would transform them in opposite ways:

$$\begin{cases} \psi \rightarrow U_V\psi = \begin{pmatrix} U\;\psi_R \\ U\;\psi_L \end{pmatrix} \\ \psi \rightarrow U_A\psi = \begin{pmatrix} U\;\psi_R \\ U^{-1}\psi_L \end{pmatrix} \end{cases} \quad\longrightarrow\quad \begin{cases} j^\mu_V = j^\mu_R + j^\mu_L \\ j^\mu_A = j^\mu_R - j^\mu_L \end{cases} \quad \text{with} \quad \begin{cases} \partial_\mu j^\mu_V = 0 \\ \partial_\mu j^\mu_A = 0 \end{cases} \tag{1.12}$$

which, since ψ_L and ψ_R are left/right-chiral states respectively and so they are eigenvectors of γ^5 with eigenvalues $\pm$, the currents can be rewritten, more simply, as:

$$\begin{cases} j_V^\mu = j_R^\mu + j_L^\mu = \bar{\psi}_R\gamma^\mu\psi_R + \bar{\psi}_L\gamma^\mu\psi_L = \bar{\psi}\gamma^\mu\psi \\ j_A^\mu = j_R^\mu - j_L^\mu = \bar{\psi}_R\gamma^\mu\psi_R - \bar{\psi}_L\gamma^\mu\psi_L = \bar{\psi}\gamma^\mu\gamma^5\psi \end{cases} \tag{1.13}$$

and if we compute the divergence of the axial current (using the equations of motion for the fermions fields, eq.(1.3) and eq.(1.4)) we get:

$$\partial_\mu j_A^\mu = (\not{\partial}\bar{\psi})\gamma^5\psi - \bar{\psi}\gamma^5(\not{\partial}\psi) = \; = i(m - e\not{A})\bar{\psi}\gamma^5\psi - \bar{\psi}\gamma^5 i(m + e\not{A})\psi = 2im\bar{\psi}\gamma^5\psi \tag{1.14}$$

which confirms that such current is conserved when m=0, and that massless QED will classically have chiral symmetry, meaning the projections of the fermion field, ψ_L and ψ_R are independent one from the other and conserved separately.

The lagrangian of such a massless QED theory ($\mathcal{L}_{QED_{m=0}}$) would be:

$$\mathcal{L}_{QED_{m=0}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{\partial}\psi - ej_V^\mu A_\mu \tag{1.15}$$

but as we are going to see in section 2.2, for even numbers of spacetime dimensions (1+1 massless QED for example) we will have an anomaly in the conservation of the axial current, telling us that at the quantum level such symmetry is not compatible with gauge invariance.

1.2 The classical Schwinger model: massless CED in 1+1 d

In this section we are going to show the interesting dynamics of the classical Schwinger model, so from now on, we will work with classical electrodynamics in a 1+1 d spacetime.

1.2.1 Gauge part of the Lagrangian

Focusing first on the gauge part of the Lagrangian of eq.(1.15), which is:

$$\mathcal{L}_G = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - ej_V^\mu A_\mu = \frac{1}{2}F_{01}^2 - e(j_V^0 A_0 + j_V^1 A_1) \;\text{ with }\; F_{\mu\nu} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \tag{1.16}$$

where j_V can be taken as an external source/current and the strength tensor ($F_{\mu\nu} = -F^{\mu\nu}$) only has one degree of freedom, the electric field $E = \partial_0 A_1 - \partial_1 A_0 = -(\partial_t \mathbf{A} + \vec{\boldsymbol{\nabla}} \cdot \phi)$.

In the absence of sources ($j_V = 0$), this would give the following equations of motion:

$$\partial_\nu F^{\nu\mu} = 0 \;\longrightarrow\; \partial_0 E = 0 \;\text{ and }\; \partial_1 E = 0. \tag{1.17}$$

which tells us that E is constant through spacetime, so in particular there won't be electromagnetic waves in d=1+1 dimensions.

In a general d-dimensional spacetime, the gauge field is A_μ with the index running over $\mu = 0, 1, ..., d-1$. However, not all of these components are physical. The standard way to isolate the physical degrees of freedom is to use the gauge symmetry $A_\mu \to A_\mu + \partial_\mu\omega$ to set $A_0 = 0$. This leaves us with only the spatial gauge fields $\mathbf{A}$. However, we still have to impose the equation of motion for $\mathbf{A}$ which is solved by insisting that $\vec{\boldsymbol{\nabla}} \cdot A = 0$. This projects out the longitudinal fluctuations of **A**, leaving us just with the transverse modes.

The upshot is that the gauge field in d dimensions carries $d-2$ physical degrees of freedom. When $d = 3 + 1$, these are the familiar two polarisation modes of the photon. However, in $d = 1 + 1$ dimensions, there are no transverse modes and the electromagnetic field has no propagating degrees of freedom [8].

Now adding the external current (j_V) from the Lagrangian in the right side (adding sources), we will have the following equations of motion:

$$\partial_\nu F^{\nu\mu} = ej_V^\mu \quad \longrightarrow \quad \begin{cases} \partial_1 E(t,x) = ej_V^0(t,x) \equiv e\rho_V(t,x) \\ \partial_0 E(t,x) = -ej_V^1(t,x) \equiv -e\mathbf{j}_V(t,x) \end{cases} \tag{1.18}$$

which are Maxwell equations for 1+1 d in covariant and components form respectively.

From here we can rudimentally find the Green's functions setting an eternal (occupying all time) point charge $\rho_V(x) = \rho_V\delta(x)$, and an instant current occupying all space $\mathbf{j}_V(t) = \mathbf{j}_V\delta(t)$ give respectively different contributions to the electric field [9], given by:

$$\begin{cases} E_\rho(t,x) = e\int_{-\infty}^{x} \rho_V\delta(x')dx' + F(t) = e\rho_V H(x) + F(t) \\ E_j(,x) = -e\int_{-\infty}^{t} \mathbf{j}_V\delta(t')dt' + G(x) = -e\mathbf{j}_V H(t) + G(x) \end{cases} \tag{1.19}$$

where H(r) is a step function, and $F(t)$ and $G(x)$ are the integration constants, which can be thought as contributions of each equation into the other as background fields at $-\infty$.

These solutions for the Green's functions tells us that the dynamics of the electric field, in the presence of matter, are given by:

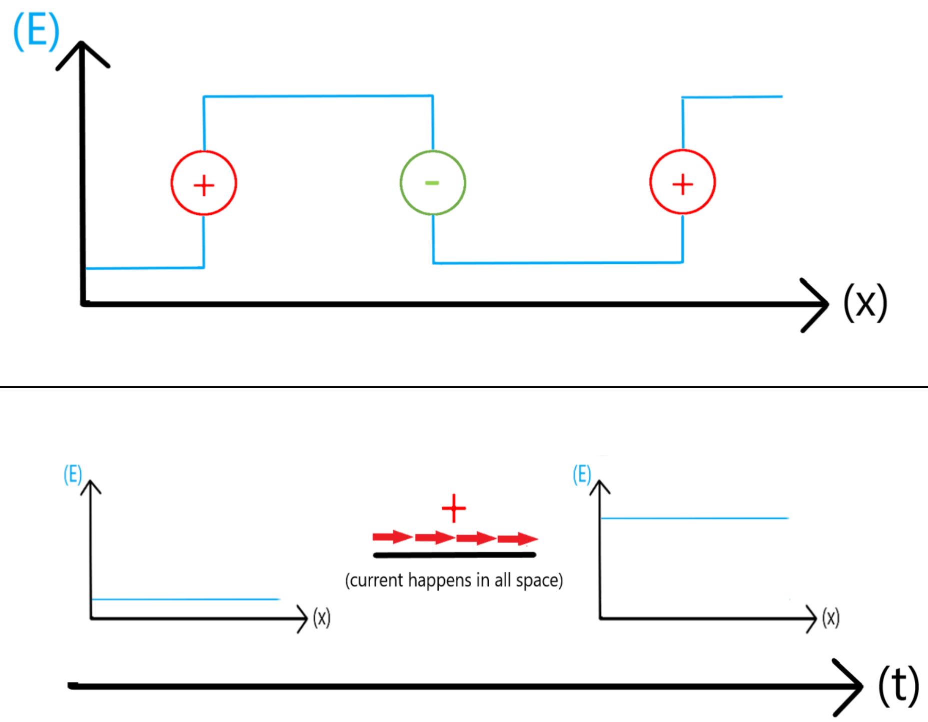


Figure 1.1: Graphical representation to the electric field contribution from point charges and from instant currents, Green's functions.

- The first contribution, tells us that a point charge ρ produces a change of the electric field in the charge position (is a step function in space, that precisely steps over the charge).
- The second contribution in the other hand, tells us that an instant of current $\mathbf{j}$ in all space adds a constant value for the electric field in all space from that moment on (is a step function in time, that increases precisely when the currents occurs).

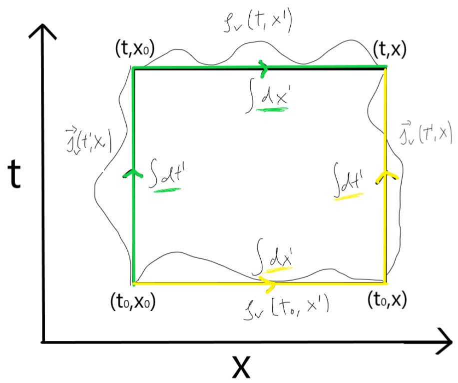
Finally say, that for realistic sources (not pointlike charges or instant currents), the contributions will just be their distribution times the Green's function result, meaning they will just increase or decrease the field more or less in function of their distribution (in space for charges ρ_V and in time for currents $\mathbf{j}_V$).

Concretely these contributions will translate into two types of electric field increase:

$$\begin{aligned} E(t,x) =& e\int_{-\infty}^{x} \rho_V(t,x')dx' + F(t) = E(t,x_0) + e\int_{x_0}^{x} \rho_V(t,x')dx' \\ E(t,x) =& -e\int_{-\infty}^{t} \mathbf{j}_V(t',x)dt' + G(x) = E(t_0,x) - e\int_{t_0}^{t} \mathbf{j}_V(t',x)dt' \end{aligned}$$

- where the first solution tells us that the electric field will be the same as that in some other left point plus the space integration of all the charge distribution (in space) until the original point.
- where the second solution tells you that the electric field will be the same as the one that was in that position some time ago minus the time integration over the current distribution (in time) until we reach that time.

which we can put together to obtain the space time propagator, in two ways, depending on the integration scheme:



$$E(t,x) = E(t_0,x_0) + e\int_{x_0}^{x} \rho_V(t,x')dx' - \int_{t_0}^{t} \vec{j}_V(t',x_0)dt'$$

$$E(t,x) = E(t_0,x_0) - e\int_{t_0}^{t} \vec{j}_V(t',x)dt' + e\int_{x_0}^{x} \rho_V(t_0,x')dx'$$

Figure 1.2: Schemes for the propagators, dependent on the integral path we chose.

But without knowing the dynamics of matter first, we can't find the final dynamics of the electric field: $E(t,x) = E(t_0,x_0) + \Delta E(t-t_0, x-x_0)$ which tells us the evolution of the field for any spacetime point, given only the charge distributions ρ_V. So in the next sections we are going to find the dynamics of matter and charges, to then find the general solution for the electric field in the classical Schwinger model.

But before that, we are going to see some interesting phenomena we can already understand, such as confinement and the effect of a background field in the model.

Infinite energy and confinement of charges in the classical Schwinger model

First, we are going to discuss the infinite energy of a point charge and then show the confinement of the classical Schwinger model. For that, we need to introduce the energy contained in the electric field [8]:

$$\mathcal{E} = \int dx \frac{1}{2} F_{01}^2 \tag{1.20}$$

And because a charge produces a constant electric field to infinity, this means that a classical point charge in d = 1+1 dimensions costs infinite energy, like in d=3+1, but for two very different reasons:

- In the d=3+1 case, the infinity comes from the divergence of the field near the charge, at small distances, which can be solved with other high-energy theories
- Instead in the case we are studying, in d=1+1, the infinity comes from a constant electric field that doesn't diverge anywhere, but that keeps going and going with non-decreasing behaviour until infinity. And this cannot be solved correct the behaviour in the small distance (high-energy) regimes.

So we conclude that the finite energy states in this model, must be neutrally charged.

So now let's consider the simplest neutrally charged state, which would be a charge q at position $x = -L/2$ and a charge $-q$ at position $x = +L/2$. From the equations of motion we get :

$$\partial_1 F^{01} = eq[\delta(-L/2) - \delta(L/2)] \longrightarrow F^{01} = \begin{cases} eq \text{ between the charges} \\ 0 \text{ outside} \end{cases} \tag{1.21}$$

where we have chosen the integration constant F to ensure vanishing electric field at $x = \pm\infty$ (no background field, we will do the other cases in the following subsections). The total energy stored in the electric field is then:

$$\mathcal{E} = \frac{e^2 q^2}{2} L \tag{1.22}$$

where we see that energy grows linearly with separation. In other words, (massive) **electric charges in $d = 1 + 1$ are classically confined!** (if there is no background field):

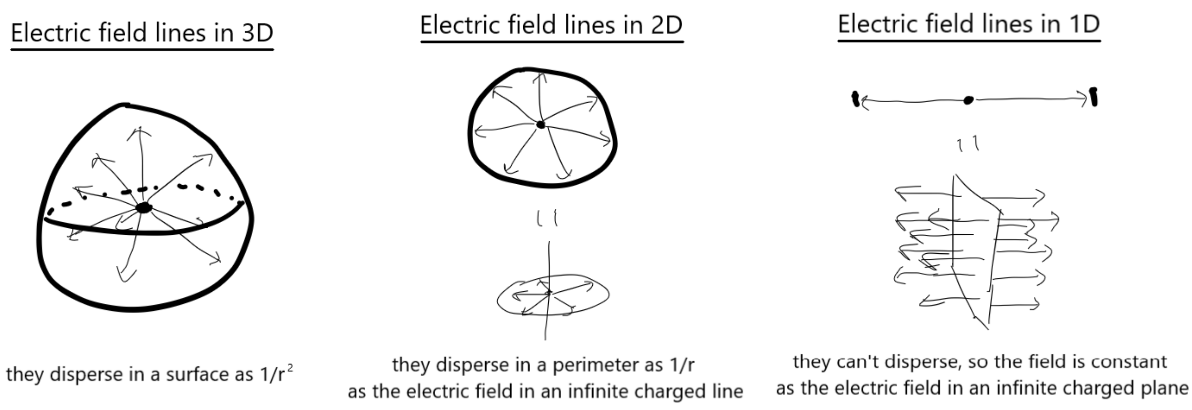


Figure 1.3: Drawings showing how electric lines can't escape in 1+1 d, so the electric field has to be constant.

the reason is that electric field is forced to form flux tubes, simply because it has nowhere else to go, as happens in an infinite uniformly charged plane in d=3+1. So, confinement appears rather naturally, indeed, in 1+1 dimensions, the Coulomb phase is the same thing as the confining phase.

The Theta parameter (θ) on the line, the classical background field

Now, let's introduce a property of the system that will let us work with non neutral states, a naturally arising background field of the model, a Theta term (θ).

As shown in the Annexes section C for four dimensional Maxwell theory, we can add another ingredient to pure Maxwell theory, in 1+1 d the analogous term will be:

$$\mathcal{L}_\theta = e\frac{\theta}{2\pi}F_{01} \quad \longrightarrow \quad \mathcal{L} = \frac{1}{2}F_{01}^2 + e\frac{\theta}{2\pi}F_{01}$$

which is a total derivative, and does not affect the classical equations of motion (but it will affect the quantum spectrum in posterior sections).

And using Green's theorem as in Annexes A, the action gets an extra:

$$S_\theta = e\frac{\theta}{2\pi}\int E\ d^2x = e\frac{\theta}{2\pi}\int F_{01}d^2x = e\frac{\theta}{2\pi}\int(\partial_0 A_1 - \partial_1 A_0)dx^0dx^1 = e\frac{\theta}{2\pi}\oint A_\mu d^\mu x$$

where its contribution is proportional to the integration of E in all space time, or what is the same, the total curvature of the gauge manifold in all space time. From this result we see that in order to keep the action stationary, θ can behave as a negative electric field compensating the action increments of the initial pure gauge action E^2.

What's more if we ignore the matter fields and treat F or E as a fundamental field, we obtain the following equation of motion:

$$E + e\frac{\theta}{2\pi} = 0 \ \longrightarrow \ E = -e\frac{\theta}{2\pi} \tag{1.23}$$

so θ may be interpreted as a constant and negative background electric field [4] [10] .

If we now compute the real gauge part of the equations of motion (respect A_μ), with this new term added, we get:

$$\partial_\nu\left(F^{\nu\mu} + e\frac{\theta}{2\pi}\right) = ej_V^\mu \ \longrightarrow \begin{cases}\partial_1 E = e\left(j_V^0 - \frac{\partial_1\theta}{2\pi}\right)\\ \partial_0 E = -e\left(j_V^1 + \frac{\partial_0\theta}{2\pi}\right)\end{cases} \tag{1.24}$$

where again we see that if θ changes we see that in both cases it increases/decreases exactly in an opposite way the electric field (with a coefficient in front).

So now, solving the first equation of motion (1+1 d Gauss law) we arrive to:

$$\partial_1\left(E(t,x) + e\frac{\theta}{2\pi}\right) = ej_V^0(t,xt) \ \longrightarrow \ E(t,x) = e\int_{-\infty}^{x} j_V^0(t,x')dx' - e\frac{\theta}{2\pi} \tag{1.25}$$

where we can consider the θ background field as being produced by a pair of positive-negative charges separated to both $\pm\infty$ respectively (as an infinitely separated capacitor).

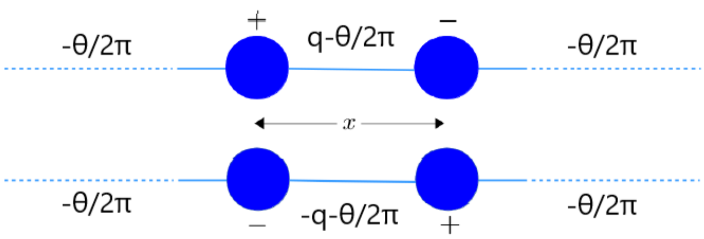


Figure 1.4: Two distinct configurations for a pair of positive-negative charges in a background field (we have set e=1 for clarity) [11]

This background field gives very interesting dynamics on the model. As Coleman first suggested [4], and following the last computation we did for the energy of the electric field produced by pair of positive-negative charges, if we consider one of these pairs of separated by a distance x, the energy difference between this configuration compared to a vacuum state, both in the same background field give:

$$\Delta\mathcal{E} = \frac{1}{2}\int\left[F_{01}^2 - \left(e\frac{\theta}{2\pi}\right)^2\right]dx = \frac{xe}{2}\left[\left(\frac{\theta}{2\pi}\pm q\right)^2 - \left(\frac{\theta}{2\pi}\right)^2\right] = \frac{xe}{2}\left[q^2 \pm q\frac{\theta}{\pi}\right]$$

which shows that when $|\theta| > \pi q$ having a pair of positive-negative charges will be energetically favorable!

We can interpret this as that if we had available pairs of positive-negative charge together in the center. we could screen the background field in all space (which contributes infinitely to the energy) moving more and more charges to their respective infinities, each time the θ background fields overcomes the value $q/2$, then we would "reset" the θ term to $-q/2$ which would be able increase again until it $q/2$ moving another pair of charges to its respective infinities, implying that the model with infinite charges available, would be periodic in $\theta/2\pi$ with period q.

And the energy density ε of the model with infinite available charges would behave as:

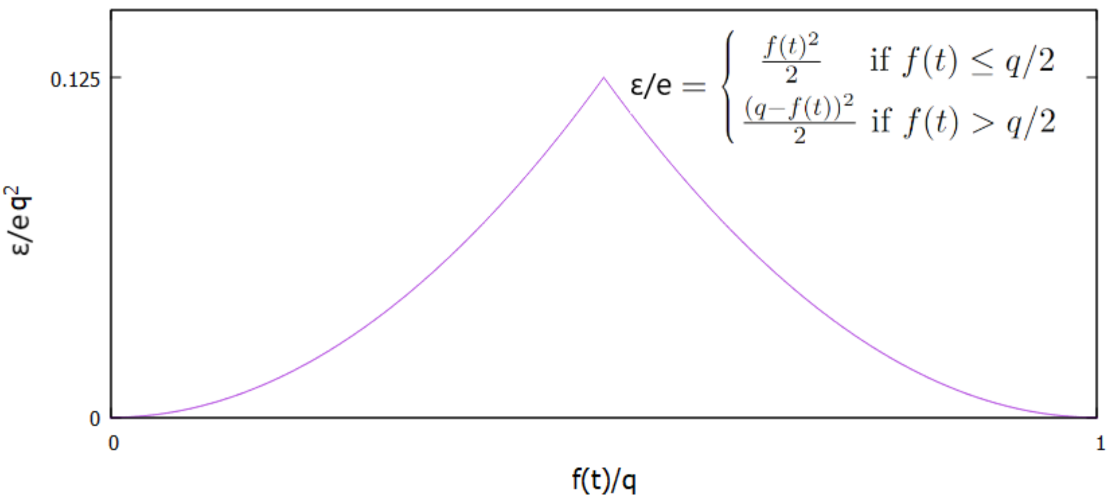


Figure 1.5: Plot of energy density divided by the coupling and the charge squared ε/eq^2, as a function of the background field (which we normalized to be between 1 and 0 as: $f(t) = \frac{\theta}{2\pi}$) divided by the charge $f(t)/q$ [11].

which has a discontinuity in the slope, with a sharp peak at $\theta/q = \pi$, showing the possibility of a phase transition on the model.

Which if we go back to our infinite available charges discussion, we again see the periodicity of $\theta/2\pi$ with period q, which if we consider unit charges ($q = 1$) means there is a discontinuity in the energy density behaviour at $\theta = \pi$ with θ having a period of 2π. So we keep going from one peak to another to another, while the available charges keep going to their respective $\pm\infty$ compensating the increasing θ field and resetting it from π to $-\pi$ in each peak.

But in the current classical theory, we don't have such infinite available charges, meaning the θ term will affect our physical fields, increasing them for example without any mechanism to counter this. We will have to wait for the quantum theory and its Dirac sea, until this becomes a reality.

Confinement of massive charges in function of the θ angle

To end the gauge part of the Lagrangian, we are going to discuss, the effect of a background field in the previously found confinement of charges (we will be talking about theoretical massive charges, since massless ones will always move at the speed of light).

Let's start our discussion with the already studied case, $\theta = 0$. A charge $q = 1$ emits to the right a constant electric field, $F_{01} = e^2$ and if a negative charge $q = -1$, sits at a distance L, then we are left with an energy in the electric field given by eq.(1.22):

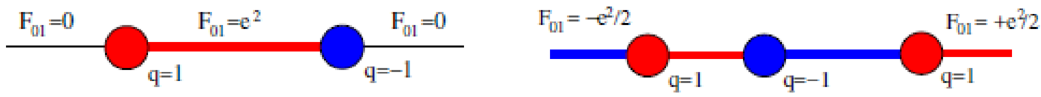
$$\mathcal{E} = \frac{1}{2}\int_{+q}^{-q} F_{01}^2 dx = \frac{e^2}{2} L$$

this linear growth in energy, as we already discussed implies confinement.

Let's now turn on θ and compute the energy density $\mathcal{E}$, to make the computations easier let's consider that both charges are found between $x = 0$ and $x = 1$, obtaining:

$$\begin{aligned}\mathcal{E} =& \frac{1}{2}\int F_{01}^2 = \frac{1}{2}\left(\int_0^{+q} F_{01}^2 dx + \int_{+q}^{-q} F_{01}^2 dx + \int_{-q}^{1} F_{01}^2 dx\right) = \\ =& \frac{1}{2}\left(\left(e\frac{\theta}{2\pi}\right)^2 (1-L) + \left(e - e\frac{\theta}{2\pi}\right)^2 L\right) = \frac{e^2}{2}\left(\left(\frac{\theta}{2\pi}\right)^2 + L\left(1-\frac{\theta}{\pi}\right)\right)\end{aligned}$$

where we see that we still have a dependence with the distance L, at least until we reach $\theta = \pi$ where the energy density $\mathcal{E}$ stops depending on L, implying there is no confinement!



a) When $\theta = 0$, there is a confining string between particles and anti-particles

b) When $\theta = \pi$, the string tensions cancel on either side and alternating particles/anti-particles feel no long-distance force.

Figure 1.6: Flux tubes between positive-negative charges, and their effect as θ varies [in a) the flux is confining, and in b) it is not, because the θ external field has equated half the flux tubes] [8].

When $\theta = \pi$ the electric field at $\pm\infty$ is $F_{01} = -e^2/2$, while the charges make the electric field jump $\Delta F_{01} = +e^2$ to reach $F_{01} = +e^2/2$ between them. And since the magnitude of the electric field doesn't change, the charges are free to roam along the line. We can follow this by a chain of alternating positive and negative particles, each of which is free to move at no extra cost of energy. In this case, the charges are no longer confined, at least when placed with a particular ordering along the line.

So summarizing, we will have the flux tubes confining the matter for almost all values of θ, but when the θ contribution is strong enough ($\theta = \pi$), these flux tubes are gonna get canceled by the θ background field, giving totally free positive-negative charges.

The fact that this result can be related with the external field becoming strong enough for pairs of positive-negative charges being energetically favorable, together with the fact that the θ parameter in the quantum theory becomes periodic with period 2π (check Annexes C), will give pretty interesting results when we quantize the model in the next chapter.

1.2.2 Fermionic part of the Lagrangian

Now we are going to see what these charges we talked about so much are made of, and how they behave, we are going to talk about the fermionic part of the Lagrangian in eq.(1.15), which is:

$$\mathcal{L}_f = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi = \bar{\psi}iD\!\!\!\!/\,\psi = \bar{\psi}iD_\mu\gamma^\mu\psi \tag{1.26}$$

in two dimensions the gamma matrices can be chosen to be:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.27}$$

and for this choice, the Dirac field ψ, which only has two spinor components, will be written as:

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \text{ and } \bar{\psi} = \begin{pmatrix} \psi_L^* & \psi_R^* \end{pmatrix} \tag{1.28}$$

so this time we can more explicitly see eq.(1.10) and see that ψ_R and ψ_L will be eigenfunctions of γ^5 having +1 and -1 eigenvalues respectively:

$$\gamma^5\psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} \psi_R \\ -\psi_L \end{pmatrix} \to \begin{cases} \gamma^5\psi_R = \psi_R \\ \gamma^5\psi_L = -\psi_L \end{cases} \tag{1.29}$$

Using this representation, we can rewrite the fermionic part of the Lagrangian (1.15), as:

$$\mathcal{L}_f = \bar{\psi}i(D_0\gamma^0 + D_1\gamma^1)\psi = \psi_R^* i(D_0 + D_1)\psi_R + \psi_L^* i(D_0 - D_1)\psi_L \tag{1.30}$$

which in the absence of gauge fields would give the following equations of motion:

$$\begin{cases} i(\partial_0 + \partial_1)\psi_R = 0 \\ i(\partial_0 - \partial_1)\psi_L = 0 \end{cases} \longrightarrow \begin{cases} \psi_R = \psi_R(t - x) \\ \psi_L = \psi_L(t + x) \end{cases} \longrightarrow \begin{cases} \psi_R \text{ moves to the right} \\ \psi_L \text{ moves to the left} \end{cases} \tag{1.31}$$

the solution to these equations are waves that move to the right and to the left in one dimensional space, at the speed of light [12] (Check this simulation).

Because the equations are real, we can easily see the behaviour of its complex conjugate part (when we add mass, there will be a relative i factor complicating things) [13]:

$$\begin{cases} -i(\partial_0 + \partial_1)\psi_R^* = 0 \\ -i(\partial_0 - \partial_1)\psi_L^* = 0 \end{cases} \longrightarrow \begin{cases} \psi_R^* = \psi_R^*(t - x) \\ \psi_L^* = \psi_L^*(t + x) \end{cases} \longrightarrow \begin{cases} \psi_R^* \text{ moves to the right} \\ \psi_L^* \text{ moves to the left} \end{cases} \tag{1.32}$$

where we see that the corresponding right and left complex conjugate fields also live up to their respective names, so we have seen that all the right-handed chirality fields will move to the right and all the left-handed chirality fields will move to the left!

It is also important to remark that 2D has the strange property that vector and axial currents are not independent of each other, due to the relation:

$$\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu \quad \longrightarrow \quad \gamma^0\gamma^5 = \gamma^1 \text{ and } \gamma^1\gamma^5 = \gamma^0 \tag{1.33}$$

which makes that the time and space components of both currents are related as:

$$j_A^\mu = -\epsilon^{\mu\nu} j_{V\nu} \quad \longrightarrow \quad \begin{cases} j_V \equiv (\rho_V\ ,\ \mathbf{j}_V) = \left(\bar{\psi}\gamma^0\psi\ ,\ \bar{\psi}\gamma^1\psi\right) \\ j_A \equiv (\rho_A\ ,\ \mathbf{j}_A) = \left(\bar{\psi}\gamma^1\psi\ ,\ \bar{\psi}\gamma^0\psi\right) \end{cases} \tag{1.34}$$

which means that the $\rho_V = \mathbf{j}_A$ and $\mathbf{j}_V = \rho_A$, so the charge density of one will be the 1d current of the other and viceversa, with:

$$\begin{cases} \rho_V = \mathbf{j}_A = \bar{\psi}\gamma^0\psi = \begin{pmatrix} \psi_L^* & \psi_R^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = |\psi_R|^2 + |\psi_L|^2 \\ \mathbf{j}_V = \rho_A = \bar{\psi}\gamma^1\psi = \begin{pmatrix} \psi_L^* & \psi_R^* \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = |\psi_R|^2 - |\psi_L|^2 \end{cases} \tag{1.35}$$

and if we now define the Noether charges as the typical $Q \equiv \int dx j^0$ with $\partial_t Q = 0$ for conserved currents ($\partial_\mu j^\mu = 0$). Then we will have:

$$\begin{cases} Q_R = \int \bar{\psi_R}\gamma^0\psi_R \; dx = \int \psi_R^*\psi_R \; dx = \int |\psi_R|^2 \; dx \\ Q_L = \int \bar{\psi_L}\gamma^0\psi_L \; dx = \int \psi_L^*\psi_L \; dx = \int |\psi_L|^2 \; dx \end{cases} \tag{1.36}$$

$$\begin{cases} Q_V = \int \bar{\psi}\gamma^0\psi \; dx = \int (|\psi_R|^2 + |\psi_L|^2) \; dx \\ Q_A = \int \bar{\psi}\gamma^1\psi \; dx = \int (|\psi_R|^2 - |\psi_L|^2) \; dx \end{cases} \tag{1.37}$$

Having both j_V and j_A currents of eq.(1.13) conserved, means that the total quantity of waves ($Q_V = Q_R + Q_L$) is conserved and that the difference between left-right moving ($Q_A = Q_R - Q_L$) is also conserved:

$$\begin{cases} \partial_t Q_V = 0 \quad \longrightarrow \quad \partial_t Q_R = -\partial_t Q_L \quad \longrightarrow \quad |\psi_{total}|^2_{\int dx} \; = \; \text{constant} \\ \partial_t Q_A = 0 \quad \longrightarrow \quad \partial_t Q_R = \partial_t Q_L \quad \longrightarrow \quad |\psi_R|^2_{\int dx}, \; |\psi_L|^2_{\int dx} \; \text{change equally} \end{cases} \tag{1.38}$$

and having both of these conditions is the same as having independently conservation of the quantity of left moving waves (Q_L) and of the quantity of right moving waves (Q_R):

$$\begin{cases} \partial_t Q_V = 0 \\ \partial_t Q_A = 0 \end{cases} \longleftrightarrow \begin{cases} \partial_t Q_R = 0 \\ \partial_t Q_L = 0 \end{cases} \longleftrightarrow \begin{cases} |\psi_R|^2_{\int dx} \; = \; \text{constant} \\ |\psi_L|^2_{\int dx} \; = \; \text{constant} \end{cases} \tag{1.39}$$

so classically we have conservation of the total of right and left moving fields independently.

But as we commented before, at the quantum level, Q_A is not going to be conserved, there will be an anomaly, which means that left waves will be able to pass to right waves and vice-versa, mixing left and right components. We will elaborate further on this in section 2.2, but for the moment just know that what really happens is:

$$\begin{array}{ccc} \underline{\text{Quantum theory}} & & \underline{\text{Classical theory}} \\ \begin{cases} |\psi_{total}|^2_{\int dx} \; = \; \text{constant} \\ |\psi_R|^2_{\int dx} \; \rightleftharpoons \; |\psi_L|^2_{\int dx} \end{cases} & \xrightarrow{\text{classical limit}} & \begin{cases} |\psi_R|^2_{\int dx} \; = \; \text{constant} \\ |\psi_L|^2_{\int dx} \; = \; \text{constant} \end{cases} \end{array}$$

1.2.3 The complete classical Schwinger model Lagrangian

If we now look at the full Schwinger model Lagrangian:

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}F_{01}^2 + \psi_R^* i(D_0 + D_1)\psi_R + \psi_L^* i(D_0 - D_1)\psi_L = \\ =& \frac{1}{2}F_{01}^2 + \psi_R^* i(\partial_0 + \partial_1)\psi_R + \psi_L^* i(\partial_0 - \partial_1)\psi_L - ej_V^0 A_0 - ej_V^1 A_1\end{aligned}$$

we can relate the previous charges/currents/sources for $\mathcal{L}_f$ and $\mathcal{L}_G$ and link them together, so that the complete equations of motion of the Schwinger model become:

$$\begin{cases}\partial_1 F^{10} = ej_V^0 = e\bar{\psi}\gamma^0\psi \;\rightarrow\; \partial_1 E = e(|\psi_R|^2 + |\psi_L|^2) \\ \partial_0 F^{01} = ej_V^1 = e\bar{\psi}\gamma^1\psi \;\rightarrow\; -\partial_0 E = e(|\psi_R|^2 - |\psi_L|^2) \\ i(\partial_0 + \partial_1)\psi_R = e(A^0 - A^1)\psi_R \;\text{ and }\; i(\partial_0 + \partial_1)\psi_R^* = -e(A^0 - A^1)\psi_R^* \\ i(\partial_0 - \partial_1)\psi_L = e(A^0 + A^1)\psi_L \;\text{ and }\; i(\partial_0 - \partial_1)\psi_L^* = -e(A^0 + A^1)\psi_L^*\end{cases} \tag{1.40}$$

where we see that the dynamics of matter depend on the electric field and viceversa.

Concretely, considering the previous Green's functions analysis in eq.(1.19), its posterior solutions and the matter dynamics found in section 1.2.2, we can now see the behaviour of the system directly from the equations:

- From the **first equation** we see that the total quantity of matter increases the electric field in that position, so there will be an increase of electric field to the right of matter and a decrease to the left:

$$E(t,x) = E(t,x_0) + \int_{x_0}^{x} e\left(|\psi_R(x',t)|^2 + |\psi_L(x',t)|^2\right) dx' \tag{1.41}$$

  We see that all fields, right moving, left moving, or their complex conjugate contribute to this increases with the same positive charge, meaning that if we go from $-\infty$ to ∞ every matter that we cross will increase the electric field to its right, making a stair that goes up to the right. In better terms, the electric field contribution from this equation has to be a monotone function of x at any given t [9]. In the presence of matter the right infinity (∞) will always have more electric field contribution from this equation than the left infinity ($-\infty$):

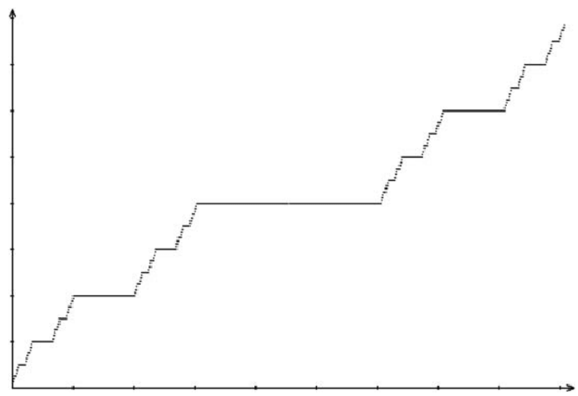


Figure 1.7: Analogy of the electric field space distribution due to matter, with an ever increasing cantor staircase.

  and because Q_V is conserved, the total quantity of matter will always be the same, meaning that the difference in the electric field between $-$ and ∞, due to the contribution of this equation, will remain the same, the shape of this "cumulative distribution function" will change with time but the total accumulation will always arrive at the same value.

- From the **second equation**, we see that a difference in right and left moving fields increases the electric field with time, an increasing background field:

$$E(t,x) = E(t_0,x) - \int_{t_0}^{t} e\left(|\psi_R(x,t')|^2 - |\psi_L(x,t')|^2\right) dt' \tag{1.42}$$

In the classical case, because the Q_A charge is conserved the difference in left and right moving will remain constant, so the total background electric field contribution from this equation will increase or decrease at a constant rate.

When the quantity of left and right moving fields in the model is equal this electric field will simply redistribute increasing to the left by the left moving infinitesimal $H(x)$ functions and decreasing to the right by the right moving infinitesimal $H(x)$ functions, but the mean will be constant.

Such case in the limit to infinite time will look something like this:

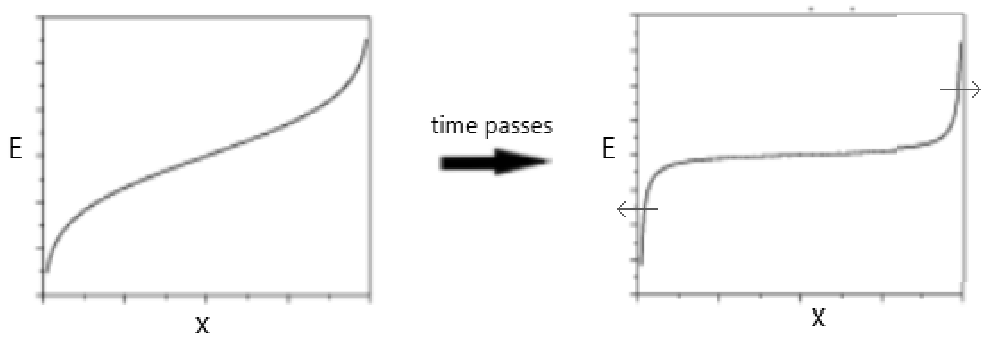


Figure 1.8: Dynamics of the electric field in the classical model as times goes to infinity.

since all the increase will have ended in the right and all the decreases in the left, being both equal in height, so that the left moving "front wave" in the bottom-left of the graph creates the same quantity of electric field than the right moving "front wave" in the top-right of the graph destroys.

In the other cases, when there is a difference in the total quantity of right and left moving fields the total electric field will increase/decrease for ever at the same rate, since what will happen is that we will have more $H(x)$ moving to one direction or to the other, so the total area under them will increase or decrease as more height is being moved to one direction or the other.

In the limit to infinite time we can picture this with one of the "front waves" creating electric field to the left, or destructing electric field to the right is higher than the other, so when they move to their respective directions they create more or less total area under them, with a constant rate for ever, being the difference of this heights!

Check this simulation where we show that in the places where there is more right moving fields than left, the electric field decreases at a constant rate, as if a $H(x)$ is disapearing to the right, as we expected!

Finally also mention that as we already said this symmetry will have an anomaly at the quantum level (we will see this in sec. 2.2), meaning that the difference of left and right moving fields will change in time, changing the dynamics of electric field increases/decreases, and preventing this "boring" infinite time ending with left matter in the left and right matter in the right with a constant electric field in between.

- And then from the **last two equations**, we again see that we have a transport equation, for right and left moving waves at the speed of light, but this time with an imaginary reaction source term, that depends on the values of A^0, A^1. We see that these imaginary reaction source terms are opposite for the normal fields ψ and for the complex conjugate ones ψ^*, telling us that they rotate in the complex plane in opposite directions, which we can understand as the normal and complex conjugated fields interacting through their phases, $\psi \rightleftharpoons \psi^*$ (their relative phases change!).

  They will propagate at the speed of light while rotating in the complex plane with angular velocity proportional to the gauge fields, check this simulation, where we can see the rotation of the normal and complex conjugated field in the left where there is gauge field, and how it stops in the right part where the gauge field is 0 [9]:

$$\begin{cases}\psi_R = e^{-ieK(A^0-A^1)}G_R(t-x) \\ \psi_L = e^{-ieL(A^0+A^1)}G_L(t+x)\end{cases} \quad \text{and} \quad \begin{cases}\psi_R^* = e^{ieK(A^0-A^1)}G_R(t-x)^* \\ \psi_L^* = e^{ieL(A^0+A^1)}G_L(t+x)^*\end{cases} \tag{1.43}$$

  where $K()$ and $L()$ are some functions we don't care about right now, and G_R and G_L are arbitrary complex functions that move to the right and to the left respectively. This results will be more clear and easy to compute, when we do the spectrum and the states of the quantized model in sec. 2.4, where we will find exactly this!

  Now we can also see the effect of each gauge component, by fixing the other component to zero. Doing so, we observe that both components rotate a field and its conjugate field in opposite ways, but while doing so the A^0 component rotates the right and left moving fields equally between them, while the A^1 component rotates the right and left moving fields in the opposite way.

So if we now put all these results together, we see that the way ρ_V and $\mathbf{j}_V$ behave is intimately related since both have the same components $|\psi_R|^2$ and $|\psi_L|^2$ moving equally, but with a relative sign in their contribution. We have right and left moving waves, that add constructively for ρ_V contribution and destructively for the $\mathbf{j}_V$ contribution, and because we only care about the total modulus squared, for all we care their movement will be equal to the free case, to the right and to the left at the speed of light (we only care about the G_R and G_L part) .

This is pretty hard to visualize and it's even harder to visualize the effect it has on the electric field. But if we take the equations of how E is modified, now that we know the behaviour of ρ_V and $\mathbf{j}_V$, we get the new equations of motion:

$$\begin{cases}\partial_1 E = e(|G_R(t-x)|^2 + |G_L(t+x)|^2) \\ \partial_0 E = -e(|G_R(t-x)|^2 - |G_L(t+x)|^2)\end{cases} \tag{1.44}$$

and the second equation integrated now gives:

$$E(t,x) = E(t_0,x) - e\int_{t_0}^{t}\left(|G_R(t'-x)|^2 - |G_L(t'+x)|^2\right)dt' \tag{1.45}$$

and integrating the first one in $E(t_0,x)$ finally gives:

$$\begin{aligned}E(t,x) = E(t_0,x_0) + e\int_{x_0}^{x}\left(|G_R(t_0-x')|^2 + |G_L(t_0+x')|^2\right)dx' \\ -e\int_{t_0}^{t}\left(|G_R(t'-x)|^2 - |G_L(t'+x)|^2\right)dt'\end{aligned}$$

which gives us the field propagator $\Delta_E(t,x,t_0,x_0) = E(t,x) - E(t_0,x_0)$ we were expecting all this time, an integration in time followed by an integration in space, and if we switch the order in which we apply the equations the result would be the other one, an integration in space follow by one in time: in function of the matter distribution:

$$\Delta^1_E(t,x,t_0,x_0) = e\left(\int_{x_0}^{x}\left(|\psi_R(t_0,x')|^2+|\psi_L(t_0,x')|^2\right)dx' - \int_{t_0}^{t}\left(|\psi_R(t',x)|^2-|\psi_L(t',x)|^2\right)dt'\right)$$

$$\Delta^2_E(t,x,t_0,x_0) = e\left(-\int_{t_0}^{t}\left(|\psi_R(t',x_0)|^2-|\psi_L(t',x_0)|^2\right)dt' + \int_{x_0}^{x}\left(|\psi_R(t,x')|^2+|\psi_L(t,x')|^2\right)dx'\right)$$

which we can schematically see represented here:

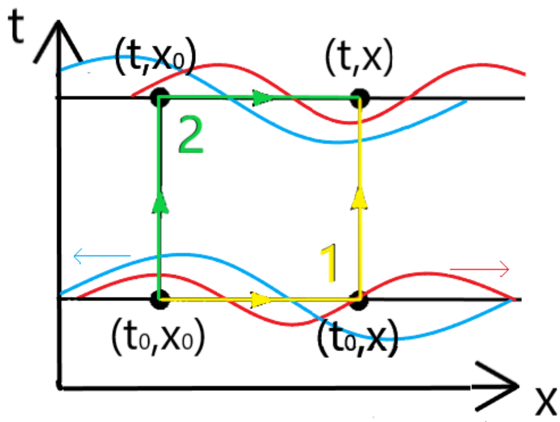
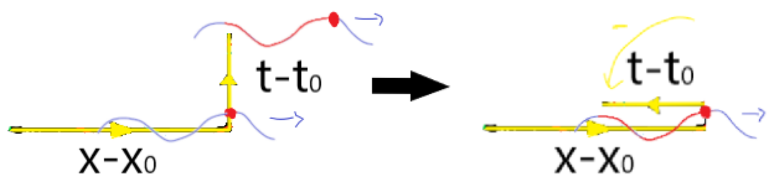


Figure 1.9: Scheme of the integration for the propagators $\Delta^1_E(t,x,t_0,x_0)$ and $\Delta^2_E(t,x,t_0,x_0)$.

where the yellow line is the path for the first propagator $\Delta^1_E(t,x,t_0,x_0)$ that goes from $(t_0,x_0) \to (t,x)$ and the green one is the path for the second propagator $\Delta^2_E(t,x,t_0,x_0)$, the red line represents the modulus of right moving fields and the blue ones, the modulus of left moving fields, as wee can see from they movements in the two time frames of the image.

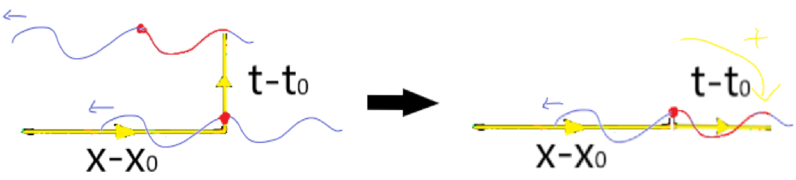
The amazing result comes when you realize that because they always go at a fixed speed c, they can really be parameterized with one variable time or position, so both the integration in time or a space integration give the same results (you can imagine a wave packet traveling to the right at constant speed to which you measure the changing height in time remaining still, or you can stop time and measure the height while you move to the left at the same speed the wave was coming, both results will give the same), taking into account that:

- integrating in time **right moving** waves is the same as integrate them to the left in space, since when you advance in time they will start coming to you from your left (the ones in your right will keeep going to the right, so they will never pass your possition):



and since right moving waves contribute with different sign to $\partial_1 E$ and $\partial_0 E$ as shown in eq.(1.44), we have to substract both integrations.

- and integrating in time **left moving** waves is the same as integrate them to the right in space, since when you advance in time, the waves at your right will travel to you (and the ones at your left will never pass you).



but this time since left moving waves contribute wiht the same sign to $\partial_1 E$ and $\partial_0 E$ as shown in eq.(1.44), we have to add both integrations.

Intuitively then the propagator can be written only integrated in space (obviously we could also have a propagator only integrated in time) like:

$$\Delta_E^x(t,x,t_0,x_0) = e\left(\int_{x_0}^{x-(t-t_0)} |\psi_R(t_0,x')|^2 dx' + \int_{x_0}^{x+(t-t_0)} |\psi_L(t_0,x')|^2 dx'\right)$$

which can be easily computed from the previous propagators $\Delta_E^{1/2}(t,x,t_0,x_0)$ doing the correct changes of variables for the right and left moving waves (different changes in each).

This tells us that the system is exactly solvable and totally deterministic! We can compute it exactly from the matter distribution $|\psi_{R/L}(x)|^2$ in a single moment of time t_0 and knowing the value of the electric field in a single point of spacetime $E(t_0',x_0')$ (as an initial/background electric field check), or alternatively having the electric field check in a single spacetime point also, but then knowing in a single space point (x_0), the value of the fields for every time instead $|\psi_{R/L}(t)|^2$, as if having a toll for cars that always go at constant speed in a infinite and straight highway, from where you can reconstruct their exact positions at anytime.

We could already have predicted this, since the evolution of E only depends on the modulus of matter, and the modulus of matter movement doesn't depend on the gauge field, so we don't really have a coupled system!

Finally we are going to give some visualization for the evolution that the electric field makes in a Δt, if we already know the configuration of the electric field in the present:

$$E(t+\Delta t,x) = E(t,x) - e\int_{x-\Delta t}^{x} |\psi_R(t,x')|^2 dx' + e\int_{x}^{x+\Delta t} |\psi_L(t,x')|^2 dx' \tag{1.46}$$

which tells us that in the electric field in a point will evolve as simply as subtracting the quantity of $|\psi_R|^2$ at a distance Δt at its left, and adding the quantity of $|\psi_L|^2$ at a distance Δt at its right.

Or in other words, the electric field in a point will evolve as simply as adding (subtracting) the left (right) moving waves that will be able pass the point in that time interval. Which makes total sense, since a left moving wave passing the point, will increase the amount of charge ρ_V in your left, moving infinitesimal step functions $H(x)$ of electric field through you to the left that will remain going to the left for ever, and a right wave passing will decrease the electric field in the point since it will bring the infinitesimal step functions $H(x)$ from the left to the right, which that point will never see again.

1.3 Final discussion of the classical model

From the complete Lagrangian we have seen that there are no such negative charges in the model so the electric fields increases monotonously as we move to the right and the discussion of the pairs of positive-negative charges being confined is either applicable then since there are not negative charges to which the positive can get linked.

But as we know, electrons have a different "charge" than positrons, meaning they increase/decrease the electric in opposite ways, so what is happening here? Also we didn't showed that earlier, but if we had computed the energy for fermions fields and antifermion fields we would have had a similar problem, so?

Well the explanation is simple, we were describing the fermion fields in the classical limit with complex numbers as if they were scalar fields, but fermions are particles with spin, and we know they obey anti-commutation rules ($\{\psi_1, \psi_2\} = 0$), instead of commutation rules ($[\phi_1, \phi_2] = 0$) as the scalar fields do. In the case of scalars the resulting charges with complex valued numbers for the fields are [14]:

$$j^{\mu}_{\phi} = e(\partial^{\mu}\phi^*)\phi - e(\partial^{\mu}\phi)\phi^* \longrightarrow \begin{cases} \rho_{\phi} = (\partial_0\phi^*)\phi - (\partial_0\phi)\phi^* \\ \mathbf{j}_{\phi} = -(\partial_1\phi^*)\phi + (\partial_1\phi)\phi^* \end{cases} \tag{1.47}$$

which work great, and give different contributions to the gauge fields coupling for the fields and anti-fields. But that simply doesn't happen naturally for spin particles in classical theories.

This problem will be fixated when we quantize, because we give the commutator/anti-commutator relations to the fields in doing so, but this arises the question: how should we have done it better, what is the actual classical limit of the theory quantum theory? The answer is, that fermion and antifermions fields should have been Grassmann variables instead of simple complex variables [15] [16] :

$$\begin{array}{ccc} \underline{\text{Quantum theory}} & \xrightarrow{\text{classical limit}} & \underline{\text{Classical theory}} \\ [\phi_1, \phi_2] = 0 & & \phi_1, \phi_2 \;\; \text{Complex variables} \\ \{\psi_1, \psi_2\} = 0 & & \psi_1, \psi_2 \;\; \text{Grassmann variables} \end{array}$$

which doing so, would gives new results for charges and currents:

$$\begin{cases} \rho_G = -e[(\psi_e^*\psi_e)_R + (\psi_e^*\psi_e)_L] + e[(\psi_p^*\psi_p)_R + (\psi_p^*\psi_p)_L] \\ \mathbf{j}_G = -e[(\psi_e^*\psi_e)_R - (\psi_e^*\psi_e)_L] + e[(\psi_p^*\psi_p)_R - (\psi_p^*\psi_p)_L] \end{cases} \tag{1.48}$$

where the used fields during the chapter were: $\psi = \psi_e + \psi_p^*$ and $\psi^* = \psi_e^* + \psi_p$. The majority of the literature doesn't do this, and they wait until quantization, because doing so makes you lose the intuition that $|\psi|^2$ represents the amplitude of the matter fields, since Grassmann numbers are more complex abstract objects [17]. Which tells us the intrinsic quantum behaviour of spin particles.

At the end of the day both models have almost the same behaviour, we only need to change some details in our results, and all the intuition we have already build will be useful as well. Concretely we will have negative and positive charges in the model, which is the reason we explained all the confining and screening sections even though, in principle there were going to be no opposite charges.

To update the last section to this facts, when we want to know how the dynamics of electric field in a point will change, we will still do the same, we will integrate to the right in space for the left moving particles, and to the left in space to to take into account the right moving particles, but this time, in each integration we will have two components:

- The electron component with negative charge, which will have infinitesimal $-H(x)$ Green's functions, and the field will decrease when passing from left to right over it.
- The positron component with positive charge, which will have the normal infinitesimal $H(x)$ Green's function, where the field increases as we pass from left to right.

so now instead of having monotone increase in the electric field with steps always increasing that move to the right or the left, we will have increasing and decreasing steps moving to both directions, making a quite more chaotic picture. Also instead of the total electric field increasing/decreasing infinitely depending on if there are more or less left/right moving fields, now both moving directions can increase or decrease the electric field with time, so it will depend in if we have more $|\psi_e|_L^2 + |\psi_p|_R^2$ or more $|\psi_p|_L^2 + |\psi_e|_R^2$.

And finally making an advance on what is to come, when we quantize we will have an anomaly for Q_A and the characteristic emissions or creation/annihilation of particles of the quantum theories. So instead of having an end with all the left moving particles in the left and all the right moving particles in the right with a constant background field in between like in the classical model, in the quantum model we will have fields changing from one direction to the other, moving their infinitesimals steps functions $\pm H(x)$ to left and then to right changing directions, or pair electron-positron creation, creating a "hole or mountain" between the two faces $(H(x))$ that separate in opposite directions and then annihilate, making "collisions" between the propagating faces of the $H(x)$ functions...

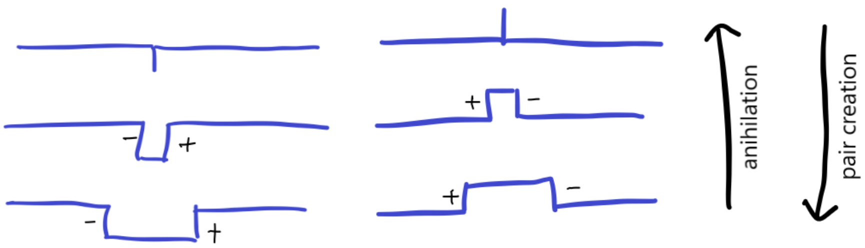


Figure 1.10: "Hole" and "mountain" creation or anhilation, depending if the time goes down or up respectively.

Also Q_V will still be conserved, so we deduce that only the above things will be able to happen, a $+H(x)$ won't change to $-H(x)$, or viceversa (electrons will remain electrons and positron, positrons, they won't mix, only annihilate), neither will two positive $H(x)$ appear from the vaccum moving in different directions (creation of pair of electrons, only particle-antiparticle creations or annihilations), since all these mentioned processes would change the total charge of the system.

We can imagine from our discussion of how charge-anticharge is energetically favorable from a certain value of the electric field, that right and left moving $\pm H(x)$, will appear from nowhere or crash into nothing, where the electric field is to high or too low (particle-antiparticle creations/anihilations to screen the electric field!).

All of this will result in quite more complex and interesting dynamics than the ones from the classical model. So now let's precisely do that, let's quantize the theory and check this thoughts.

Chapter 2

The Schwinger model, massless QED in 1+1 d

In this chapter we will quantize massless QED in 1+1 d, so we will work with the quantum Schwinger model, normally referred more simply as the Schwinger model, with:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^{\mu}(i\partial_{\mu} - eA_{\mu})\psi = \frac{1}{2}E^2 + \psi_R^{\dagger}i(D_0 + D_1)\psi_R + \psi_L^{\dagger}i(D_0 - D_1)\psi_L \quad (2.1)$$

where now the fields will gain the notion of particles and antiparticles whose quantum behaviour will create new dynamics that will make the model really interesting.

2.1 Photons develop a mass in 1+1 d

The first thing we are going to show is that even for $m = 0$, there will be a mass gap for our bound states, because the gauge field mediator will develop a mass [18] [19]:

The lowest-order vacuum polarization of QED in dimensional regularization is:

$$\begin{aligned} i\Pi^{\mu\nu}(q) =&(-ie)^2(-1)\int \frac{d^dk}{(2\pi)^d}\ tr\left[\gamma^{\mu}\frac{i}{\not{k}-m}\gamma^{\nu}\frac{i}{\not{k}+\not{q}-m}\right] = \\ =& -e^2\int \frac{d^dk}{(2\pi)^d}\ tr\left[\gamma^{\mu}\frac{\not{k}-m}{k^2-m^2}\gamma^{\nu}\frac{\not{k}+\not{q}-m}{(k+q)^2-m^2}\right] \end{aligned}$$

and for the limit of zero mass (m=0) ends up as:

$$\begin{aligned} i\Pi^{\mu\nu}(q) =& -e^2 tr\left[\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu}\gamma^{\beta}\right]\int \frac{d^dk}{(2\pi)^d}\frac{k^{\alpha}(k+q)^{\beta}}{k^2(k+q)^2} = \\ =& i\left(q^2g^{\mu\nu} - q^{\mu}q^{\nu}\right)\frac{2e^2tr[1]}{(4\pi)^{d/2}}\int_0^1 dx\ \frac{x(1-x)\ \Gamma(2-\frac{d}{2})}{(-x(1-x)q^2)^{2-d/2}} \end{aligned}$$

which for d=4 is not exactly solvable, and you need to use $\epsilon = 4 - d$ obtaining logarithmic divergences, with branch cuts at the thresholds for creation of a real electron-positron pair.

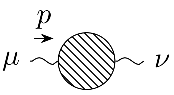
Instead, for d=2 it is exactly solvable! The 1 particle irreducible (1PI) when d=2 and m=0 is [20]:

$$i\Pi^{\mu\nu}(q) \equiv iq^2\Delta^{\mu\nu}\Pi(q^2) = i\left(q^2g^{\mu\nu} - q^{\mu}q^{\nu}\right)\frac{e^2}{\pi q^2} \quad (2.2)$$

which separating the components is:

$$\begin{cases} \Delta^{\mu\nu} = g^{\mu\nu} + \frac{q^{\mu}q^{\nu}}{q^2} \quad (\Delta^{\mu\rho}g_{\rho\sigma}\Delta^{\sigma\kappa}g_{\kappa\nu} = \Delta^{\mu}_{\rho}\Delta^{\rho}_{\nu} = \Delta^{\mu}_{\nu}) \\ \Pi(q^2) = \frac{e^2}{\pi q^2} \end{cases} \quad (2.3)$$

and we know that the infinite sum of the renormalized propagator ends up converging to:



$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2} i\Pi^{\rho\sigma}(q)\frac{-ig_{\sigma\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2} i\Pi^{\rho\kappa}(q)\frac{-ig_{\kappa\sigma}}{q^2} i\Pi^{\kappa\sigma}(q)\frac{-ig_{\sigma\nu}}{q^2} + ... =$$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-i}{q^2}\Delta_{\mu\nu}\Pi(q^2) + \frac{-i}{q^2}\Delta_{\mu\rho}\Delta^{\rho}_{\nu}\Pi^2(q^2) + ... = \frac{-ig_{\mu\nu}}{q^2} + \frac{-i\Delta_{\mu\nu}}{q^2}(\Pi(q^2) + \Pi^2(q^2) + ...) =$$

$$= \frac{-i(g_{\mu\nu} - \Delta_{\mu\nu})}{q^2} + \frac{-i\Delta_{\mu\nu}}{q^2(1-\Pi(q^2))} \xrightarrow{\text{Ward identity: } q^\mu q^\nu = 0 \text{ or } \Delta_{\mu\nu} = g_{\mu\nu}} \frac{-ig_{\mu\nu}}{q^2(1-\Pi(q^2))} = \frac{-ig_{\mu\nu}}{q^2 - \frac{e^2}{\pi}}$$

which means that there is going to be a pole at $q^2 = \frac{e^2}{\pi}$ (or $\Pi(q^2) = 1$) $\rightarrow$ $M_\gamma = \frac{e}{\sqrt{\pi}}$ from where we see that the gauge propagator in the Schwinger model behaves like that of a massive boson [1].

Also mention, that we have arrived at this result because the fermions are massless, which for the $\Pi(q^2) = \frac{Cte}{q^2}$ gives us a function with a branchcut in the imaginary Riemann surfaces, equal to when the photon splits into a pair of fermions.

2.2 ABJ, Chiral or Axial anomaly in 1+1 d

The next interesting property we are going to show is the already mentioned anomaly of the model, which shows that the axial charge won't be conserved at the quantum level.

The question is whether these operators can be defined to satisfy the quantum conservation equations [19]:

$$\begin{cases} \partial_\mu \langle j^\mu_V(x)\rangle = 0 \\ \partial_\mu \langle j^\mu_A(x)\rangle = 0 \end{cases} \tag{2.4}$$

where:

$$\langle j^\mu(x)\rangle = \frac{\int D\psi D\bar{\psi} j^\mu(x) e^{i\int d^4x\mathcal{L}}}{\int D\psi D\bar{\psi} e^{i\int d^4x\mathcal{L}}} = \frac{\int D\psi D\bar{\psi} j^\mu(x) e^{i\int d^4x(i\bar{\psi}\partial\!\!\!/\psi - eJ^\mu_V A_\mu)}}{\int D\psi D\bar{\psi} e^{i\int d^4x(i\bar{\psi}\partial\!\!\!/\psi - eJ^\mu_V A_\mu)}} \tag{2.5}$$

Also $\langle j^\mu(x)\rangle$ can be thought as the correlation function given a source, so in momentum space we can compute the expected value of the currents as:

$$\begin{cases} \langle j^\mu_V(q)\rangle = -\frac{i}{e}(i\Pi^{\mu\alpha}(q)A_\alpha(q)) = -\frac{e}{\pi}(A^\mu(q) - \frac{q^\mu q^\alpha}{q^2}A_\alpha(q)) \\ \langle j^\mu_A(q)\rangle = -\epsilon^{\mu\nu}\langle j_{V\nu}(q)\rangle = \epsilon^{\mu\nu}\frac{i}{e}(i\Pi_{\nu\alpha}(q)A^\alpha(q)) = \epsilon^{\mu\nu}\frac{e}{\pi}(A_\nu(q) - \frac{q_\nu q_\alpha}{q^2}A^\alpha(q)) \end{cases} \tag{2.6}$$

where for the axial current we have used eq.(1.34).

Now using the Ward identity, we get:

$$\begin{cases} q_\mu\langle j^\mu_V(q)\rangle = -\frac{e}{\pi}(q_\mu A^\mu(q) - \frac{q^2 q^\alpha}{q^2}A_\alpha(q)) = 0 \\ q_\mu\langle j^\mu_A(q)\rangle = \frac{e}{\pi}\epsilon^{\mu\nu}(q_\mu A_\nu(q) - \frac{q_\mu q_\nu q^\alpha}{q^2}A_\alpha(q)) = \frac{e}{\pi}\epsilon^{\mu\nu}q_\mu A_\nu(q) \end{cases} \tag{2.7}$$

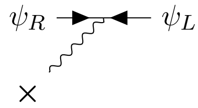
which coming back to position space tells you:

$$\begin{cases} \partial_\mu\langle j^\mu_V\rangle = 0 \\ \partial_\mu\langle j^\mu_A\rangle = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} = -\frac{eE}{\pi} \end{cases} \tag{2.8}$$

that finally shows that the axial current is not conserved, which tells us that actually we will have right moving fermions passing to be left moving fermions. Using the definition of the charge in eq.(1.37), a constant electric field would give:

$$\frac{\partial\langle Q_A\rangle}{\partial t} = -\int \frac{eE}{\pi}\, dx \quad \longrightarrow \quad \frac{\partial\langle|\psi_R|^2 - |\psi_L|^2\rangle}{\partial t} = \frac{eE}{\pi}\Delta x \tag{2.9}$$

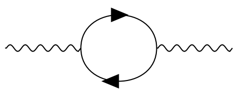
which as an operator can also be thought as:



.

where the $\times$ operator is proportional to $\epsilon^{\mu\nu}F_{\mu\nu} = 2E$, an electric field, which is the operator that mixes the right and left moving particles of the model [21] [22].

If we now make the previous process more general we will be able to see where does this anomaly actually come from. We know, from dimensional analysis that:



$$= ie^2(C_1 g^{\mu\nu} - C_2\frac{q^\mu q^\nu}{q^2})$$

for which dimensional regularization set $C_1 = C_2$ previously. If now we keep C_1 setting $C_2 = 1$ (which is unambiguously determined by the low-energy structure of the theory, since it is the residue of the pole in q^2), we will have a more general result:

$$\begin{cases} \langle j_V^\mu(q)\rangle = -\frac{i}{e}(i\Pi^{\mu\alpha}(q)A_\alpha(q)) = -\frac{e}{\pi}(C_1 A^\mu(q) - \frac{q^\mu q^\alpha}{q^2}A_\alpha(q)) \\ \langle j_A^\mu(q)\rangle = -\epsilon^{\mu\nu}\langle j_{V\nu}(q)\rangle = \epsilon^{\mu\nu}\frac{i}{e}(i\Pi_{\nu\alpha}(q)A^\alpha(q)) = \epsilon^{\mu\nu}\frac{e}{\pi}(C_1 A_\nu(q) - \frac{q_\nu q_\alpha}{q^2}A^\alpha(q)) \end{cases} \tag{2.10}$$

Which now applying again the Ward identity, gives:

$$\begin{cases} q_\mu\langle j_V^\mu(q)\rangle = -\frac{e}{\pi}(C_1 q_\mu A^\mu(q) - \frac{q^2 q^\alpha}{q^2}A_\alpha(q)) = -\frac{e}{\pi}(C_1 - 1)\; q_\mu A(q)^\mu \\ q_\mu\langle j_A^\mu(q)\rangle = \frac{e}{\pi}\epsilon^{\mu\nu}(C_1 q_\mu A_\nu(q) - \frac{q_\mu q_\nu q^\alpha}{q^2}A_\alpha(q)) = \frac{e}{\pi}C_1\epsilon^{\mu\nu}q_\mu A_\nu(q) \end{cases} \tag{2.11}$$

where we see that we can't have both Ward identities being 0 at the same time. Depending on the choice of regularization we can have the anomaly in the vector (j_V) or in the axial current (j_A).

And not having the vector current conserved would be way worse than not having the chiral current conserved. Since a not conserved vector current, would also imply the breaking of the gauge symmetry, which if our gauge field is dynamical (has kinetic terms) is fatal for consistency. So we see that the only solution is the previously found, imposing $C_1 = 1$ and getting the anomaly in the axial charge Q_A.

From the last discussion of the first chapter, we see that as we predicted, a background field appears when a left particle changes to right and viceversa, because of the $H(x)$ functions changing directions, and passing from accumulating field to evaporating it.

Finally say that because the parameter θ acts as a background field, and we just found that the operator which breaks the chiral charge is also a background field [21], this will make the θ parameter unphysical, and in next sections we will show that it can be absorbed into other parameters.

2.3 Hamiltonian formalism of the Schwinger model

Now we are going to construct the Hamiltonian formalism of the theory, to then show the form of our fermionic states in terms of creation/anihilation operators.

Once we have this, in the next section we will find the solutions and spectrum for our fermions (imposing periodic boundary conditions).

To start the Hamiltonian formalism, we first need to identify the canonical momenta of the system:

$$\pi^1 \equiv \pi_{A^1} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_1)} = -F^{01} = -E \tag{2.12}$$

$$\pi_\alpha \equiv \pi_{\psi_\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_\alpha)} = i\psi^\dagger_\alpha \tag{2.13}$$

$$\pi^\dagger_\alpha \equiv \pi_{\psi^\dagger_\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi^\dagger_\alpha)} = 0 \tag{2.14}$$

so the Hamiltonian in terms of the fields, is:

$$\begin{aligned}
\mathcal{H} =& \pi^1\partial_0 A_1 + \pi_\alpha \partial_0 \psi_\alpha - \mathcal{L} = -E\partial_0 A_1 + i\psi^\dagger_\alpha \partial_0 \psi_\alpha - \mathcal{L} = \\
=& E\left(\frac{E}{2} - \partial_1 A_0\right) - \psi^\dagger_R i(D_0 + D_1 - \partial_0)\psi_R - \psi^\dagger_L i(D_0 - D_1 - \partial_0)\psi_L \\
=& \frac{E^2}{2} - E\partial_1 A_0 - \begin{pmatrix}\psi^\dagger_R & \psi^\dagger_L\end{pmatrix} \begin{pmatrix} i\partial_1 - e(A_0 + A_1) & 0 \\ 0 & -i\partial_1 - e(A_0 - A_1)\end{pmatrix} \begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}
\end{aligned}$$

where we see that choosing the Weyl gauge ($A_0 = 0$, $A_1 \equiv A$) makes the Hamiltonian end pretty simple:

$$\mathcal{H} = \frac{E^2}{2} - \begin{pmatrix}\psi^\dagger_R & \psi^\dagger_L\end{pmatrix} \begin{pmatrix} +(i\partial_1 - eA) & 0 \\ 0 & -(i\partial_1 - eA)\end{pmatrix} \begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix} \tag{2.15}$$

and the states of our system $|\phi\rangle$ can be separated in terms of a fermionic part given by the fermionic Hamiltonian with an external source A $|\psi\rangle_A$, times the electric field part given by the pure Maxwell Hamiltonian with the same A configuration $|E\rangle_A$ [23]:

$$|\phi\rangle = \sum_A f(A)|\psi\rangle_A |E\rangle_A \tag{2.16}$$

where $|\psi\rangle_A$ and $|E\rangle_A$ are constructed via the operators $\psi(x)$, $\psi^\dagger(x)$ and $A(x)$.

We can then express these operators in terms of the eigenvectors of the corresponding Hamiltonians:

$$\begin{cases} \mathcal{H}_f \psi^k_{R/L}(x) = \pm(i\partial_x - eA(x))\psi^k_{R/L}(x) = \pm\epsilon^k_\psi \psi^k_{R/L}(x) \\ \mathcal{H}_G|n\rangle = \frac{E^2}{2}|n\rangle = \epsilon^n_E|n\rangle \end{cases} \tag{2.17}$$

and then:

$$\begin{cases} \psi(x) = \psi_R(x) + \psi_L(y) = \sum_k r(k)\psi^k_R(x) + l(k)\psi^k_L(x) \\ \psi^\dagger(x) = \psi^\dagger_R(x) + \psi^\dagger_L(y) = \sum_k r(k)^\dagger \psi^{k*}_R(x) + l(k)^\dagger \psi^{k*}_L(x) \end{cases} \tag{2.18}$$

with

$$\begin{cases} \{r_n, r^\dagger_m\} = \{l_n, l^\dagger_m\} = \delta_{nm} \\ \{r_n, r_m\} = \{r^\dagger_n, r^\dagger_m\} = \{l_n, l_m\} = \{l^\dagger_n, l^\dagger_m\} = 0 \end{cases} \tag{2.19}$$

as creation/anihilation operators.

2.4 Spectrum of the Schwinger model in a circle

Now we are going to fix our spacetime with periodic boundary conditions to find finite energy solutions of the system. So our model will live in a circle of radius r [24]:

$$\begin{cases} A(x+2\pi r) = A(x) \\ E(x+2\pi r) = E(x) \\ \psi(x+2\pi r) = e^{-i2\pi\alpha}\psi(x) \\ \psi^\dagger(x+2\pi r) = e^{+i2\pi\alpha}\psi^\dagger(x) \end{cases} \tag{2.20}$$

which since E is periodic, means that the total charge of the system has to be $Q_V = 0$ on physical states, which is exactly what we commented last chapter we would want for having finite energy states.

The gauge part of the system has one degree of freedom, since at a given time, the set of gauge inequivalent field configurations is itself a circle [22] (because of the boundary conditions we imposed). The only gauge invariant quantity then has to be a phase, which we are going to construct as the Wilson loop variable (check Appendix B):

$$e^{ie\int_0^{2\pi r} A_1(x,t)\ dx} \quad \text{with} \begin{cases} A_\mu \to\ A_\mu - \partial_\mu\Lambda \\ \psi_\alpha \to\ e^{ie\Lambda}\psi_\alpha \end{cases} \longrightarrow\ e^{ie\left(\int_0^{2\pi r} A_1(x,t)\ dx\ -\ (\Lambda(2\pi r)-\Lambda(0))\right)} \tag{2.21}$$

where from the gauge transformations with single-valued functions, we see that Λ has to be single-valued mod $2\pi r$ in space, with:

$$\frac{2\pi n}{e} = \Lambda(2\pi r) - \Lambda(0) \ \longrightarrow\ \Lambda = \frac{x}{er}n \ \longrightarrow\ A_1' = A_1 - \frac{n}{er} \tag{2.22}$$

where n is the topological winding number of the gauge manifold. This means that A_1 is restricted to the interval [0,r] since the values outside this interval are gauge equivalent to a point inside the interval (this identify what could be regarded as topologically distinct vacua), the gauge field A_1 lives on a topological circle of length $er!$.

Now let's find the spectrum of the fermions in this theory, for that we have to evaluate the eigenvectors of $\mathcal{H}_f$ from eq.(2.17), which because of the periodic conditions have a finite energy solution now:

$$\begin{cases} \mathcal{H}_f\psi_R^k(x) = +(i\partial_x - eA(x))\psi_R^k(x) = +\epsilon_\psi^k\psi_R^k(x) \\ \mathcal{H}_f\psi_L^k(x) = -(i\partial_x - eA(x))\psi_L^k(x) = -\epsilon_\psi^k\psi_L^k(x) \end{cases} \tag{2.23}$$

which has solutions:

$$\psi_{R/L}^k(x) = \frac{P_{R/L}}{\sqrt{2\pi r}}e^{-i(\,\epsilon_\psi^k x + e\int_0^x A(x')dx')} \quad \text{and} \quad \psi_{R/L}^{k\,*}(x) = \frac{P_{R/L}}{\sqrt{2\pi r}}e^{+i(\,\epsilon_\psi^k x + e\int_0^x A(x')dx')} \tag{2.24}$$

with:

$$\epsilon_\psi^k = \frac{1}{r}\left(k - \frac{e}{2\pi}\int_0^{2\pi r} A(x')dx'\right) \equiv \frac{1}{r}(k - \bar{A}) \tag{2.25}$$

from where we see that same direction particle/antiparticle share the same energy, since the eigenvalues are real $\mathcal{H}_f^*\psi^{k*} = \epsilon_\psi^{k\,*}\psi^{k*} \ \longrightarrow\ \mathcal{H}_f\psi^{k*} = \epsilon_\psi^k\psi^{k*}$.

Also doing a naive time evolution with $U(t) = \exp(-i\mathcal{H}_f t)$, we see that for $\int A(x)dx = 0$ the right and left components go in opposite directions at the speed of light as expected:

$$\begin{cases} U(t)\psi^k_{R/L}(x) = e^{-i(\pm\epsilon^k_\psi)t}\psi^k_{R/L}(x) = \frac{P_{R/L}}{\sqrt{2\pi r}}e^{-i(\,\epsilon^k_\psi(x\pm t)+e\int_0^x A(x')dx')} \\ U(t)\psi^{k\,*}_{R/L}(x) = e^{-i(\pm\epsilon^k_\psi)t}\psi^{k\,*}_{R/L}(x) = \frac{P_{R/L}}{\sqrt{2\pi r}}e^{+i(\,\epsilon^k_\psi(x\mp t)+e\int_0^x A(x')dx')} \end{cases} \tag{2.26}$$

with $\bar{A}$ rotating the particle-antiparticle wave functions in opposite ways in the complex plane, with angular velocity proportional to the value of $\bar{A}$ (check the simulation).

But then we also see that for $\int A(x)dx \neq 0$ the particles don't depend only on $t \pm x$ anymore, meaning that the particles are going slower than the speed of light, particles are becoming massive through A_1! We also see that this is affecting particles and antiparticles in opposite ways, from where we can deduce that maybe there are massive bound states of charge-anticharge through the gauge field, getting mass from the mass gap we computed at the start of this chapter. We will see that this is exactly what is happening, at the end of this chapter when talking about bosonization.

Now analyzing the energy eigenvalues, we observe the so called **spectral flow**, when we jump between gauge equivalents values of $\bar{A}$ we have opposite increases/decreases for the energy of right and left moving particles:

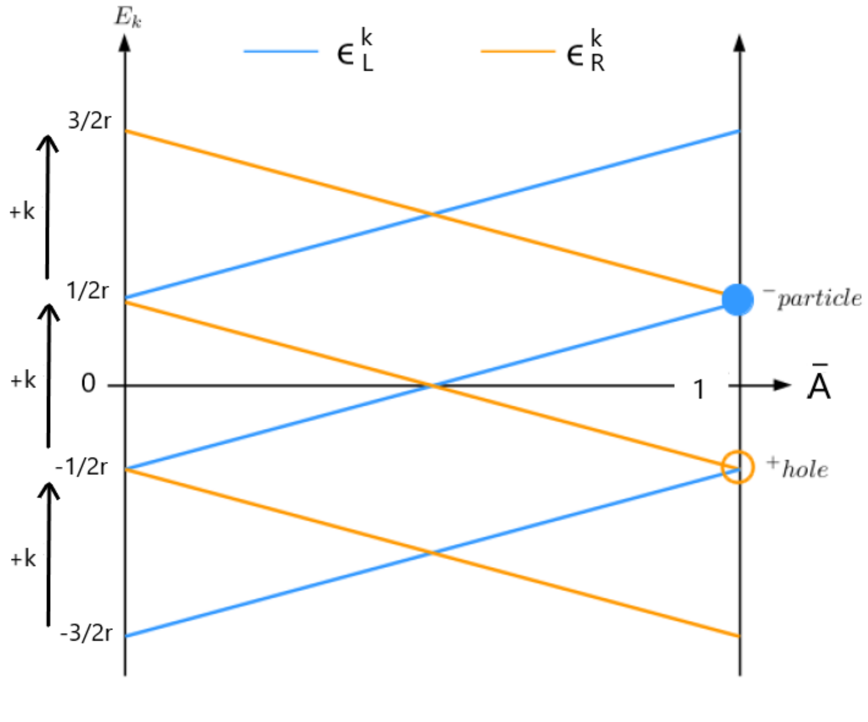


Figure 2.1: Fermion spectrum as function of $\bar{A}$ [11]

from the spectrum we see that for $\bar{A} = 0$ the energy levels of the positive and negative chirality fermions are degenerate. However as we increase $\bar{A}$ the energy levels split until we arrive to $\bar{A} = 1$ where the degeneracy occurs again. More over, because this two values of $\bar{A}$ are gauge equivalent, we should be able to go from one to the other, and when we do so, in order for the energies to remain equal (since the energy should be a gauge invariant quantity), there has to be a restructuring of the spectrum where the right and left levels shift up or down in opposte ways.

This implies that the axial charge (Q_A) changes by 2 when $\bar{A}$ jumps from one gauge equivalence to another:

$$\Delta Q_A = 2 = 2\Delta\bar{A} = \frac{eL}{\pi}\Delta A_1 \longrightarrow \dot{Q}_A = \frac{eL}{\pi}\dot{A}_1 \longrightarrow \partial_\mu j^\mu_A = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} \tag{2.27}$$

which is the same result we obtained for the anomaly from the path integral quantization. So, we now see, that this restructuring is the essence of the chiral anomaly. This is also exactly how we described the anomaly in the last discussion of first chapter with how the $H(x)$ function's would change directions.

In the next section we will try to relate this to the θ parameter we discussed in the classical chapter, which is currently missing from the quantized theory, so we will add it again and compare how the behaviour changes.

2.5 The irrelevance of the θ parameter in the massless model

Adding now the θ parameter to the theory again, we remember that it precisely contributed with a full loop of A_1, which is exactly what $\bar{A}$ does, and we have seen that the value of this is redundant, because we can always change to gauge equivalent configurations, where we reduce or increase this background field changing left moving particles for right moving particles!

Let's show this more explicitly, if we write the gauge part of the Lagrangian in terms of $\bar{A}$:

$$\mathcal{L} = \frac{1}{4\pi e^2 R}\dot{\bar{A}}^2 + \frac{\theta}{2\pi}\dot{\bar{A}} \quad\longrightarrow\quad p_{\bar{A}} = \frac{\partial\mathcal{L}}{\partial\dot{\bar{A}}} = \frac{1}{2\pi e^2 R}\dot{\bar{A}} + \frac{\theta}{2\pi}$$

and then the Hamiltonian is:

$$\mathcal{H} = p_{\bar{A}}\dot{\bar{A}} - \mathcal{L} = \frac{1}{4\pi e^2 R}\dot{\bar{A}}^2 = \pi e^2 R\left(p_{\bar{A}} - \frac{\theta}{2\pi}\right)^2$$

which is precisely a particle moving on a circle in the presence of a constant flux.

In this well known quantum system there is an important effect given by θ in the energy of the eigenstates $\psi_l = e^{il\bar{A}}$ with $l\epsilon Z$, which is:

$$\mathcal{H}\psi_l = E_l\psi_l \text{ with } E_l = \pi e^2 R\left(l - \frac{\theta}{2\pi}\right)^2$$

and we see that the spectrum is periodic in θ as expected.

- For $\theta\ \epsilon\ (-\pi,\pi)$, the ground state is $l = 0$
- For $\theta = \pm\pi$, there are two degenerate ground states, $l = 0$ and $l = \pm 1$.

If we increase $\theta \to \theta + 2\pi$, then the spectrum remains the same, but all the states shift by 1, spectral flow! It has exactly the same behaviour as the gauge redundant $\bar{A}$.

So the θ term at the end is just a background field, which can be included directly from the available operators for the Lagrangian, but we have seen that it is also closely related with the anomaly, and the redundant gauge parameter $\bar{A}$.

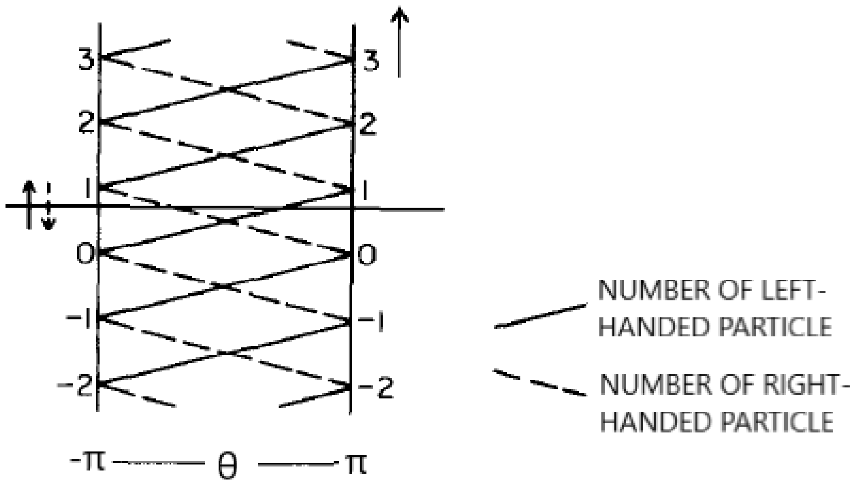


Figure 2.2: Fermion spectrum as function of θ [22]

When space is a circle, what we have discovered is an essential topological difference between pure electrodynamics and the Schwinger model. In pure electrodynamics, the configuration space is itself a circle, since the gauge field is specified, up to a gauge equivalence, by the Wilson loop phase factor in eq.(2.21), and the wavefunction must be single valued on this circle. In the Schwinger model, on the other hand, the circle of Dirac operators associated with the configuration space has a nonzero spectral flow!

The immediate consequence is that the total wavefunction of a quantum state is no longer strictly periodic, but periodic "with a twist" as one orbit the configurations space. This means that the configuration space is effectively replaced by its covering space (the line) when fermions are present. States of the Schwinger model are best described in an extended scheme, using a linear variable $\bar{A}$ rather than an angular one A_1. It is this change of topology that is responsible for the axial anomaly, and for the irrelevance of the θ parameter [22].

Actually if one had familiarity with characteristic classes and indices, he would be able see how the anomaly arises purely from the topology of the gauge manifold [5]:

$$\frac{-i}{2}\int dx\partial^{\mu}J^5_{\mu} = n_+ - n_- = \text{index}D_+ = \int_M ch(F) = -\frac{1}{4\pi}\int dx\epsilon^{\mu\nu}F_{\mu\nu} = -\frac{1}{2\pi}\int Edx$$

but this is outside the scope of this work.

And finally, what I think is the most clear and easy way to see the redundancy of the θ term, if we now do an axial rotation to our fermions proportional to θ:

$$\psi \to e^{i\theta\gamma^5/2}\psi \tag{2.28}$$

this induces a shift in the Lagrangian/Hamiltonian [25]:

$$\mathcal{H} \to \mathcal{H} + \frac{\theta}{2}\partial_{\mu}j^{\mu}_{A} = \mathcal{H} + \theta\frac{E}{2\pi} \tag{2.29}$$

which is exactly the θ term that the theory could include, coming naturally from the anomaly. A chiral rotation changes the value of θ, making the physics independent of theta!

2.6 Explicit canonical quantization of the Schwinger model

We will now quantize the theory explicitly, with the anti/commutator relations, from which we will describe all of our previous phenomena through operators and the commutations of these. For this we will maintain the same Weyl gauge and anti/periodic boundary conditions used previously

Doing this we will solve the complete model, naturally arriving to the well know behaviour of the model as free compact bosons, the bosonization.

2.6.1 Canonical anti/commutation relations and Gauge symmetry

Now, we will show the anticommutator relations for the canonical momenta and how the gauge transformations arise from them [26] [27]:

$$\begin{cases} [A_1(t,x), \pi^1(t,y)] = i\delta(x-y) \\ \{\psi_\alpha(t,x), \pi_\beta(t,y)\} = i\delta_{\beta\alpha}\delta(x-y) \end{cases} \longrightarrow \begin{cases} [A_1(t,x), E(t,y)] = -i\delta(x-y) \\ \{\psi_\alpha(t,x), \psi^\dagger_\beta(t,y)\} = \delta_{\beta\alpha}\delta(x-y) \end{cases}$$

with the rest of anti/commutators relationship being:

$$\begin{cases} [A_1(t,x), A_1(t,y)] = [E(t,x), E(t,y)] = 0 \\ \{\psi_\alpha(t,x), \psi_\beta(t,y)\} = \{\psi^\dagger_\alpha(t,x), \psi^\dagger_\beta(t,y)\} = 0 \end{cases}$$

and with the Gauss law as a constrain: $\partial_1 E(t,x) = e\psi^\dagger(t,x)\psi(t,x)$ which can be written as first class constrain operator [28]:

$$G(\alpha) = -i\int (\partial_1 E(x) - e\psi^\dagger(x)\psi(x))\alpha(x)dx \tag{2.30}$$

giving the infinitesimal gauge transformations:

$$\begin{cases} [A_1(x), dG] = [A_1(x), -i(\partial_1 E(x))\alpha(x)] = i(\partial_1\alpha(x))[A_1(x), E(x)] = -\partial^1\alpha(x) \\ [\psi(x), dG] = [\psi(x), ie\psi^\dagger(x)\psi(x)\alpha(x)] = ie\alpha(x)[\psi(x), \psi^\dagger(x)\psi(x)] = ie\alpha(x)\psi(x) \\ [\psi(x)^\dagger, dG] = [\psi(x)^\dagger, ie\psi^\dagger(x)\psi(x)\alpha(x)] = ie\alpha(x)[\psi(x)^\dagger, \psi^\dagger(x)\psi(x)] = -ie\alpha(x)\psi(x)^\dagger \end{cases}$$

that when become finite give the exact same Gauge transformations we found from the Lagrangian, in eq.(1.6):

$$\begin{cases} A_1(x) \to A_1(x) - \partial^1\alpha(x) \\ \psi(x) \to e^{ie\alpha(x)}\psi(x) \\ \psi(x)^\dagger \to e^{-ie\alpha(x)}\psi(x)^\dagger \end{cases} \tag{2.31}$$

2.6.2 Intuition and construction of the Electric field Fock space

And now we are going to introduce and construct the Fock space for the electric field, because in the process we will see lots of similarities with our classical analysis in the first chapter, and it will present the bases for solving the system.

To describe the gauge fields, we will use gauge invariant operators, which from Appendix B we know can be: the Wilson loops, $U(\gamma)$ (total electric field in the full circle, $\bar{A}$) and the Wilson line between particle-antiparticle, $L_{\alpha\beta}(x,y)$ (also proportional to the electric field between these) :

$$\begin{cases} U(x,y) \equiv e^{ie\int_x^y A_1(x')dx'} \\ U(\gamma) \equiv e^{ie\oint_\Gamma A_1(x')dx'} \\ L_{\alpha\beta}(x,y) \equiv \psi^\dagger_\alpha(x)U(x,y)\psi_\beta(y) \end{cases} \tag{2.32}$$

which represent the creation of electric field in the full circle (notice $U(\Gamma)\ \alpha\ \bar{A}$) , and the creation of separated pairs of particle-antiparticle with a field in between (the "holes and "mountains" we shown in the previous chapter, in Figure 1.10!). This can be easier seen, seeing the commutation relations they fulfill:

$$\begin{cases} [U(x,y),E(z)] = ie\int_x^y -i\delta(z-x')dx'U(x,y) = eH(x,y|z)U(x,y) \\ [U(\Gamma),E(z)] = ie\int_\Gamma -i\delta(z-x')dx'U(\Gamma) = en_\Gamma U(\Gamma) \\ [L_{\alpha\beta}(x,y),E(z)] = \psi_\alpha^\dagger(x)\psi_\beta(y)ie\int_x^y -i\delta(z-x')dx'e^{ie\int_x^y A(x')dx'} = eH(x,y|z)L_{\alpha\beta}(x,y) \end{cases}$$

where n_Γ represent the winding number of the path Γ, as "how many times does it go around the full circle", and where we recover our so beloved step functions! In this case $H(x,y|z)$ it's going to be 1 if z is in between x and y and 0 if it's outside.

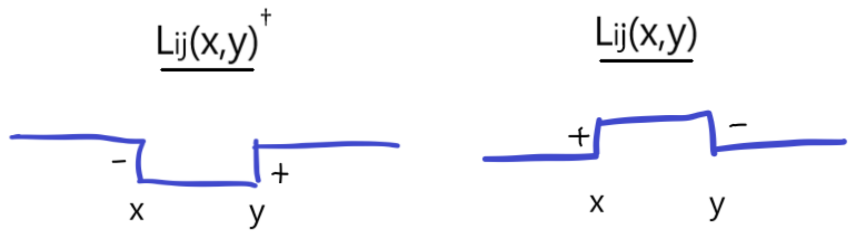


Figure 2.3: Representation of the effect of the L operators in the electric field. The +/- charges represent oposite charges with moving direction depending on the i,j components

So what we see from this equations is that the commutators of $U(\Gamma)$ and $L_{\alpha\beta}(x,y)$ are proportional to themselves, meaning they are the ladder operators, that increase the electric field! In other words, if we have an eigenstate of electric field operator $|\phi\rangle$ with eigenvalue $e(z)$, then:

$$E(z)|\phi\rangle = e(z)|\phi\rangle \quad \longrightarrow \quad \begin{cases} E(z)[U(x,y)|\phi\rangle] = (e(z) - eH(x,y|z))\,[U(x,y)|\phi\rangle] \\ E(z)[U(\Gamma)|\phi\rangle] = (e(z) - en_\Gamma)\,[U(\Gamma)|\phi\rangle] \\ E(z)[L_{\alpha\beta}(x,y)|\phi\rangle] = (e(z) - eH(x,y|z))\,[L_{\alpha\beta}(x,y)|\phi\rangle] \end{cases}$$

the integration's of A^μ around the circle are creating electric field wherever they pass over.

Other interesting commutator relationships are:

$$\begin{cases} [U(x,y),U(x',y')] = 0 \\ [U(\Gamma),L_{\alpha\beta}(x,y)] = 0 \\ [L_{\alpha\beta}(x,y),L_{\alpha'\beta'}(x',y')] = \delta_{\beta\alpha'}\delta(y-x')L_{\alpha\beta'}(x,y') - \delta_{\beta'\alpha}\delta(y'-x)L_{\alpha'\beta}(x',y) \end{cases}$$

which at the end of the day just tells us that creating fields one over the other doesn't commute, and neither does creating pairs of fermions-antifermions with field in between except when a pair of equal moving fermion-antifermion are created one over the other, as the same spacetime point anticommutation relation dictate.

We now are going to describe how will we represent the states, and how will they behave under the previous operators. If a pair (x_i, y_i) represents the start and end of a Wilson line ("mountain/hole" of electric field between x_i, y_i), and n represents the quantity of complete Wilson loops (full circle filled with field), then our states will be:

$$|n, x_1, y_1, ..., x_a, y_a\rangle = U(\Gamma_n)U(x_1,y_1)...U(x_a,y_a)|0\rangle \tag{2.33}$$

where $|0\rangle$ represents the vacuum, the state with no fermions and no electric field.

Then our operators acting on these states, have to fulfill:

- a Wilson line creates electric field between two points:

$$U(x,y)|n,x_1,y_1,...,x_a,y_a\rangle = |n,x_1,y_1,...,x_a,y_a,x,y\rangle$$

- field in all space can be transferred to the winding count of the background field $\bar{A}$:

$$U(\Gamma_m)|n,x_1,y_1,...,x_a,y_a\rangle = |n,x_1,y_1,...,x_a,y_a,0,L,..^{(m)}..,0,L\rangle = |n{+}m,x_1,y_1,...,x_a,y_a\rangle$$

- the electric field in a point will be equal to the background electric field $\pm$ in between of how many pair of charge-anticharge it is (how many "holes/mountains" is it over):

$$E(x)|n,x_1,y_1,...,x_a,y_a\rangle = -e\left(n + \sum_{i=1}^{a} H(x_i,y_i|x)\right)|n,x_1,y_1,...,x_a,y_a\rangle$$

- increase and decrease in the same point cancel each other (as we explained in the classical section with the $+H(x)$ and the $-H(x)$ overlaping):

$$|n,x_1,y_1,..,x_a,y_a,z,z\rangle = |n,x_1,y_1,..,x_a,y_a\rangle$$

- the particle-antiparticle pairing order doesn't matter, since the total number of particles/antiparticles at the right or left in a point will always be the same:

$$|n,x_i-y_i,x_j-y_j.,..,x_a,y_a\rangle = |n,x_i-y_i,x_j-y_j.,..,x_a,y_a\rangle$$

- the L's operator also add fermions to the fermion fock space, which fullfill the electric field they create in the electric field Fock space:

$$L_{\alpha\beta}(x,y)|n,x_1,y_1,..,x_a,y_a\rangle = \psi(x^{\dagger}_{\alpha},y_{\beta})|n,x_1,y_1,..,x_a,y_a,x,y\rangle$$

2.6.3 Fock space and solution of the full Hamiltonian

Now we want to find the ladder operators of the full Hamiltonian, which will be the operators from where we construct the physical states, acting on the vacuum.

For that, we first will write the Hamiltonian operator and the currents with the previous operators, so we can commute them easily. The Hamiltonian will look like:

$$\mathcal{H} = \mathcal{H}_G + \mathcal{H}_f = \frac{1}{2}E^2 - i\lim_{y\longrightarrow x}\gamma^5_{\alpha\beta}\partial_y L_{\alpha\beta}(x,y)$$

and the currents can be written as:

$$\begin{cases} j^{\mu}_V(x) = e\psi^{\dagger}(x)\gamma^{\mu}\psi(x) = (\rho_V(x),\ \mathbf{j}_V(x)) = e\gamma^{\mu}_{\alpha\beta}L_{\alpha\beta}(x,x) \\ j^{\mu}_A(x) = e\psi^{\dagger}(x)\gamma^{\mu}\gamma^5\psi(x) = (\mathbf{j}_V(x),\ \rho_V(x)) = e\gamma^{\mu}_{\alpha i}\gamma^5_{i\beta}L_{\alpha\beta}(x,x) \end{cases} \tag{2.34}$$

with:

$$\begin{cases} \rho_V(x) = e\psi^{\dagger}(x)\psi(x) = e(\psi^{\dagger}_R(x)\psi_R(x) + \psi^{\dagger}_L(x)\psi_L(x)) = eL_{\alpha\alpha}(x,x) \\ \mathbf{j}_V(x) = e\psi^{\dagger}(x)\sigma_z\psi(x) = e(\psi^{\dagger}_R(x)\psi_R(x) - \psi^{\dagger}_L(x)\psi_L(x)) = e\gamma^5_{\alpha\beta}L_{\beta\alpha}(x,x) \end{cases} \tag{2.35}$$

which now that we have quantized the fields, are well defined with the correct positive or negative charges for particles and antipartices.

The general $L_{\alpha\beta}(x,y)$ commutations relations are:

$$[L_{\alpha\beta}(x,y),\psi_i(z)] = e[\psi^\dagger_\alpha(x)U(x,y)\psi_\beta(y),\psi_i(z)] = -\delta_{\alpha i}\delta(x-z)U(x,y)\psi_\beta(y) \tag{2.36}$$
$$[L_{\alpha\beta}(x,y),\psi^\dagger_i(z)] = e[\psi^\dagger_\alpha(x)U(x,y)\psi_\beta(y),\psi^\dagger_i(z)] = +\delta_{\beta i}\delta(y-z)U(x,y)\psi^\dagger_\alpha(x) \tag{2.37}$$
$$[L_{\alpha\beta}(x,y),A^\mu(z)] = [L_{\alpha\beta}(x,y),E(z)] = 0 \tag{2.38}$$

from where we can compute now the eigenstates of the full Lagrangian!

And since the charges/currents are already ladder operators for the electric field, which mean that they will also be ladders for $\mathcal{H}_G$, lets precisely try these operators ρ^k_V and $\mathbf{j}^k_V$ (the density charge and current of the ψ^k), but with the fermionic part $\mathcal{H}_f$:

$$[\rho^k_V(z),\mathcal{H}_f(x)] = -ie\lim_{y\longrightarrow x}\partial_y\gamma^5_{\alpha\beta}[L_{\alpha'\alpha'}(z,z),L_{\alpha\beta}(y,x)] = \delta(z-x)k\mathbf{j}^k(z) \tag{2.39}$$
$$[\mathbf{j}^k_V(z),\mathcal{H}_f(x)] = -ie\lim_{y\longrightarrow x}\partial_y\gamma^5_{\alpha\beta}\gamma^5_{\alpha'\beta'}[L_{\beta'\alpha'}(z,z),L_{\alpha\beta}(y,x)] = \delta(z-x)k\rho^k(z) \tag{2.40}$$

were we see they are a kind of mixed ladder operators [29].

From where it's obvious that a sum of these will be the ladder operator:

$$[\rho^k_V+\mathbf{j}^k_V,\mathcal{H}_f] = k(\rho^k_V+\mathbf{j}^k_V) \tag{2.41}$$

meaning that the fermionic Hamiltonian can be written as:

$$\mathcal{H}_f = \frac{1}{2}\int_0^\infty\left(\rho^k_V\rho^{-k}_V+\mathbf{j}^k_V\mathbf{j}^{-k}_V\right)\,dk \tag{2.42}$$

and because the gauge part, if you replace the E field by the Coloumb interaction in terms of ρ_k's, can be written as:

$$\mathcal{H}_G = e^2\int_0^\infty\frac{\rho^k_V\rho^{-k}_V}{2\pi k^2}dk \tag{2.43}$$

the full Hamiltonian can be written as:

$$\mathcal{H} = \mathcal{H}_G+\mathcal{H}_f = \frac{1}{2}\int_0^\infty\left(\left(\frac{e^2}{\pi k^2}+1\right)\rho^k_V\rho^{-k}_V+\mathbf{j}^k_V\mathbf{j}^{-k}_V\right)\,dk \tag{2.44}$$

which because is quadratic in ρ^k_V and $\mathbf{j}^k_V$ will be solvable. What's more, going back to space base ($f(x)=\int\frac{dk}{2\pi}e^{-ikx}f^k$), we can rewrite the complete Hamiltonian as:

$$\mathcal{H} = \frac{1}{2}\int_0^\infty\left(\left(\frac{e^2}{\pi}+\partial_x^2\right)\bar{\rho}^2_V(x)+\mathbf{j}^2_V(x)\right)\,dk \tag{2.45}$$

where $\bar{\rho}=\rho/k$. This Hamiltonian reminds us a lot of that of a scalar boson, if $\mathbf{j}_V$ were the canonical momentum of $\bar{\rho_V}$.

Also the physical states will conserve the vector current ($\partial_\mu j^\mu_V=0$), which translates into, fulfilling: $\partial_0\rho_V=\partial_1\mathbf{j}_V$, which if we apply some commutators ends as:

$$\partial_0\rho = \frac{1}{i}[\rho,H] = \partial_x\mathbf{j} \;\longrightarrow\; \partial_0^2\rho = \frac{1}{i}[\partial_x\mathbf{j},H] = i\partial_x[\mathbf{j},H_E+H_L] = i\partial_x\left(i\frac{e^2x}{\pi}\rho+\frac{1}{i}\partial_x\rho\right)$$

that results in:

$$\partial_0^2\rho = \partial_x^2\rho-\frac{e^2}{\pi}\rho$$

which is the Klein-gordon equation for a free massive scalar field with mass $M_\gamma=e^2/\pi$! So, we see that our bound states are the charges moving freely without interacting which get massive from the contribution of the term $\mathcal{H}_G$ (from the electric field they produce).

2.7 Bosonization of the massless Schwinger model

Now that we have understood pretty well the dynamics of the model at the quantum level, seening that the charges behave as free compact bosons, we are now going to redefine the fields precisely to match those of a bosonic theory.

Fermions in two dimensions can be written in terms of bosons yes! After all, the difference between bosons and fermions is not so stark in d=1+1 dimensions. First, in a single spatial dimension there is no meaning to rotation, therefore no meaning to spin. Relatedly, to interchange two particles on a line, one has to pass over the other, in contrast to higher dimensions where particle position can be exchange keeping them separated.

Even after having seen the similarities in the last section, this seems like a rather remarkable claim. The Hilbert space of a single bosonic oscillator looks nothing the Hilbert space of a single fermionic oscillator, yet we claim that the theories in d = 1+1 not only have the same Hilbert space (excluding a subtle $\mathcal{Z}_2$ issue), but also the same spectrum. Furthermore, for any operator that we can construct out of fermions, there is a corresponding operator made from bosons [8].

This is actually a very well know phenomenon, in fact, in the majority of the literature, the authors directly make the field redefinitions through the bosonization dictionaries (we show some in the annex D), which takes you from our original massless [massive] lagrangian:

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\partial\!\!\!/\psi - ej^{\mu}_{V}A_{\mu}\ [-\ m\bar{\psi}\psi]$$

into the equivalent massless [massive] lagrangian in boson terms:

$$\begin{aligned}\mathcal{L}_{\text{Bosonized}} =& \frac{e^2}{2\pi}\left(\phi' + \frac{1}{2\sqrt{\pi}}\theta\right)^2 + \frac{1}{2}(\partial_\mu\phi')^2\ \ [+\ cmM_\gamma\cos(2\sqrt{\pi}\phi')] = \\ =& \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M_\gamma^2}{2}\phi^2\ \ [+\ cmM_\gamma\cos(2\sqrt{\pi}\phi - \theta)]\,.\end{aligned}$$

where $M_\gamma = e^2/\pi$ shows that the boson is massive exactly with the mass found in the mass gap and in the Klein Gordon equation for the charge density, $\phi' = \phi - \frac{1}{2\sqrt{\pi}}\theta$ give two ways of writing the Lagrangian, with the external field inside the cosine or outside and $c = e^\gamma/4\pi$ with $\gamma \approx 0.5771$.

The second form of writting the bosonic Lagrangian shows explicitly that for the massless case ($m = 0$) the system is independent of the θ parameter and can be reabsorbed, while in the massive case this won't be possible.

Figure 2.4: Bosonic field due to external test charges [30]

In our case, since we have almost arrived into this Lagrangian in terms of ρ's and **j**'s, we will continue from where we left it in the last section, and show the bosonization explicitly. After doing so, we will show how the charge gets screened in the theory [31] [32], and we will revisit some previous calculations from the bosons point of view.

2.7.1 Explicit bosonization of the Schwinger model

Starting from the originals $\rho_V(x)$ and $\mathbf{j}_V(x)$ and their momentum representations ρ_V^k and $\mathbf{j}_V^k$, we can see that they commute as:

$$[\mathbf{j}_V^k, \rho_V^q] = 2k\delta(k+q) \quad \longrightarrow \quad [\rho_V(x), \mathbf{j}_V(y)] = \frac{i}{\pi}\partial_x \delta(x-y) \tag{2.46}$$

where the + sign in the delta comes from considering how they act as creation/anihilation on the vacumm with opposite momenta. And because ρ_V^k has no k=0 term (it is not a neutral state), we can then define the new fields as:

$$\phi_k = \frac{\sqrt{\pi}}{k}\rho_V^k \qquad\qquad \Pi_k = \sqrt{\pi}\mathbf{j}_V^k \tag{2.47}$$

which because $\partial_\mu j^\mu = 0$, fulfills:

$$\Pi(x) = \partial_0 \phi(x) \qquad \text{and} \qquad [\Pi_k, \phi_q] = 2\pi\delta(k+q) \tag{2.48}$$

and the Hamiltonian takes the form:

$$\mathcal{H}_{\text{Bosonized}} = \int_0^\infty \left(\Pi_k \Pi_{-k} + \left(k^2 + \frac{e^2}{\pi}\right)\phi_k \phi_{-k}\right) \frac{dk}{2\pi} \tag{2.49}$$

from where is obvious that we are dealing with a free massive Bose field given the canonical commutation relations. If we now write this in terms of x, we get:

$$\mathcal{H}_{\text{Bosonized}} = \int_0^\infty \left(\frac{1}{2}\Pi(x)^2 - \frac{1}{2}(\partial_x\phi(x))^2 + \frac{e^2}{\pi}\phi(x)^2 \right) \, dx \tag{2.50}$$

and the Lagrangian in terms of $\phi(x)$ only, is then:

$$\mathcal{L}_{\text{Bosonized}} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M_\gamma^2}{2}\phi^2 \tag{2.51}$$

with $M_\gamma = \frac{e}{\sqrt{\pi}}$, which is exactly the Lagrangian we showed from the bosonization dictionaries for $m = 0$.

To end this subsection we want to show how does a gauge transfomation affect the boson field. It follows from the commutations relations that:

$$[\phi(x), dG] = -\partial_x[\rho, dG] = 0 \tag{2.52}$$

so since our bosons come from the gauge invariant charges, they are gauge invariant also.

So we have a real field that behave as a scalar, and which is totally gauge invariant, totally physical. This sounds pretty much as how the electric field behaves in 1+1 d! And what's more, passing the k to the other side in eq.(2.47) and going to spatial base pretty vastly, we obtain a quick/naive similarity:

$$\partial_x\phi(x) \approx i\sqrt{\pi}\rho_V(x) = i\frac{\sqrt{\pi}}{e}\partial_1 E(x) \tag{2.53}$$

where in the second equality we used a equation of motion. So we see that our boson states precisely behave as the electric field!

$$\phi(x) \approx i(2)\frac{\sqrt{\pi}}{e}E(x) + C \tag{2.54}$$

2.7.2 Screening of external charges with Bosonization

Now we are going to show how the system automatically screens charges. For that we are going to consider two external opposite charges at x_1 and x_2:

$$\rho_{\text{ext}}(x) = eQ_{\text{ext}}(\delta(x - x_1) - \delta(x - x_2)) \tag{2.55}$$

and now from the Klein Gordon equation we we discovered for the charges, we have:

$$\partial_0^2\rho(x) = \partial_x^2\rho(x) - M_\gamma^2(\rho(x) + \rho_{\text{ext}}(x)) \tag{2.56}$$

this equation implies that there is a classical, time-independent component of the charge density operator induced by these external charges:

$$\begin{aligned}\rho_{\text{ind}}(x) = & - M_\gamma^2 eQ_{\text{ext}} \int \left(\frac{cos(k|x - x_1|)}{k^2 + M_\gamma^2} + \frac{cos(k|x - x_2|)}{k^2 + M_\gamma^2} \right) \frac{dk}{2\pi} = \\ = & - \frac{eQ_{\text{ext}} M_\gamma}{2} \left(e^{-M_\gamma|x-x_1|} + e^{-M_\gamma|x-x_2|} \right)\end{aligned}$$

and this results clearly shows how the system screens external charges. We can also see, that this screening occurs mainly at distances smaller than $1/M_\gamma$, where the exponential really start growing.

And finally say, that it's easy to check that the screening appears even if we have a single charge, instead of a pair. The charges get screened individually. This is shown in [23], the value of the induced screening charges, when you precisely, place a single external test charge into the system, goes as:

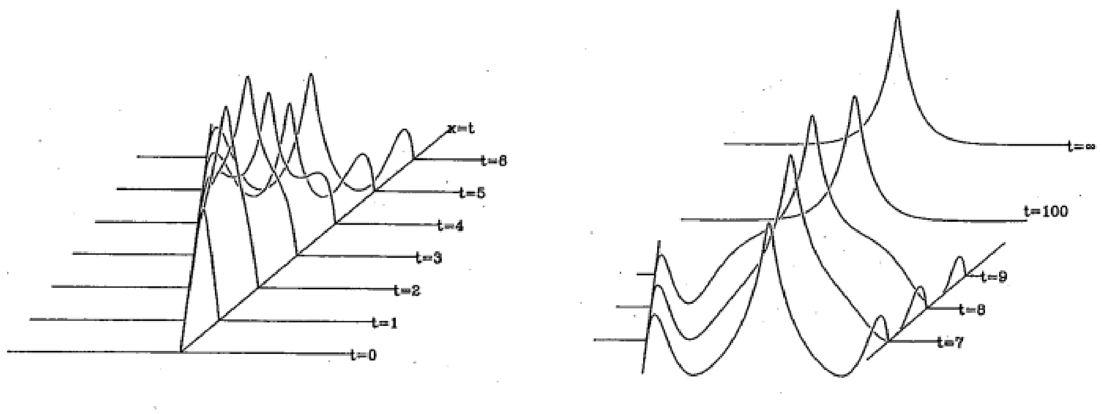


Figure 2.5: Induced current density for different times when an external charge is located at the center. [23]

from where we can see how the induced charge is zero outside the light cone, and how for $t = \infty$ the charge gets completely screened (with a exponentially small contribution left) at distances around than $1/M_\gamma$.

This can be seen explicitly, integrating the previously computed induced charge density, giving:

$$Q_{\text{ind}} = -\frac{eQ_{\text{ext}} M_\gamma}{2} \int_{-\infty}^{\infty} \left(e^{-M_\gamma|x-x_1|} + e^{-M_\gamma|x-x_2|} \right) = -\frac{eQ_{\text{ext}} M_\gamma}{2} \left[\frac{2}{M_\gamma} + \frac{2}{M_\gamma} \right] = -2eQ_{\text{ext}}$$

which is exactly the total external charge, so we have checked that the screening is total!

This also shows that an external background field, which we can consider created by two opposite charges separated to + and - infinity, don't really affect our system at all, since they are at infinite distance away, but would get screened with a finite distance, so the system would directly see no contributions from those charges, the vacuum state would totally screen it.

To finish this section, we are going to study this phenomenon from the boson point of view, integrating the total charge, because of the definition of the boson field as: $\rho = \partial_1\phi/\sqrt{\pi}$ yields to a difference in value of the boson in - and + infinity:

$$Q_{\text{ext}} = \int \rho_{\text{ind}}\ dx = \frac{1}{\sqrt{\pi}}\int \partial_1\phi\ dx = \frac{1}{\sqrt{\pi}}(\phi(\infty) - \phi(-\infty)) \tag{2.57}$$

concretely the difference in value at + and - infinite of the boson field is exactly $\sqrt{\pi}Q_{\text{ext}}$. This is the exact behaviour we explained from E(x) in the classical discussion around Fig. 1.7, when we considered that there were only positive charges in the model, which implied a difference in the electric field at $+\infty$ and $-\infty$, precisely proportional to the charge.

Finally say, that in the circle we worked before, this boson as the electric field did, would have periodic boundary conditions given that the total charge is neutral in physical states.

2.7.3 Revisit ABJ anomaly with Bosonization

To end this chapter we want to show again the axial anomaly, but from the point of view of bosonization. To do so, we first need to realize the components of this two operators:

$$\begin{cases} j_A^\mu = (\mathbf{j}, \rho) \\ \partial_\mu\phi/\sqrt{\pi} = (\mathbf{j}, \rho) \end{cases} \longrightarrow\ j_A^\mu = \partial_\mu\phi/\sqrt{\pi} \tag{2.58}$$

and now from the equations of motion for ϕ:

$$\partial^2\phi = M_\gamma^2\phi \quad\longrightarrow\quad \partial_\mu j_A^\mu = \frac{1}{\sqrt{\pi}}\partial^2\phi = \frac{e^2}{\pi^{3/2}}\phi \tag{2.59}$$

where we see that the earlier complex derivation of the anomaly in two dimensions as a subtle quantum effect, comes now easily from the equations of motions of the mesons.

More over, if we use the bosonization dictionaries, we see that $2(i)\sqrt{\pi}\phi\ \alpha\ E$, and we end with the anomaly in the exact same form we have always found:

$$\partial_\mu j_A^\mu\ \alpha\ \frac{e^2}{2\pi}E \tag{2.60}$$

but now we can understand it better from another point of view. We see that the rate of the anomaly is proportional to the value of the boson field ($\partial_\mu j_A^\mu\ \alpha\ \phi$), and we also know that the boson field is massive, which means it doesn't go at the speed of light. But our fermions are still massless, so they should be moving at the speed of light, but the charges are formed by fermions, so how can this happen?

There is a pretty simple solution for this, the bound boson state ϕ is changing the quantities of ψ_R and ψ_L through the gauge field, breaking the chiral symmetry and slowing down their effective velocities (the fermions keep going at the speed of light, but changing from moving forward and backwards), or which is the same giving them an effective mass, through a fermion condensate!
WE HAVE ELECTRIC FIELD MOVING as a massive scalar field with more or less speed depending on the strengh of the coupling /// Relate this with the propagator we found, maybe?

Chapter 3

The massive Schwinger model

3.1 Introduction to the massive Schwinger model

We will now present the massive model and some interesting properties that we can see directly from the Lagrangian:

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - ej_V^{\mu}A_{\mu}$$

which can be written in left and right components can be written as:

$$\mathcal{L}_{QED} = \frac{1}{2}F_{01}^2 + \psi_R^{\dagger}i(D_0 + D_1)\psi_R + \psi_L^{\dagger}i(D_0 - D_1)\psi_L - m(\psi_R^{\dagger}\psi_L + \psi_L^{\dagger}\psi_R)$$

where an early clear distinction appears in the fermion eq. of motion of such a system:

$$\begin{cases} \left(i\partial_0 + i\partial_1 - e(A^0 + A^1)\right)\psi_R = m\psi_L \\ \left(i\partial_0 - i\partial_1 - e(A^0 - A^1)\right)\psi_L = m\psi_R \end{cases} \quad \begin{cases} \left(i\partial_0 + i\partial_1 + e(A^0 + A^1)\right)\psi_R^{\dagger} = -m\psi_L^{\dagger} \\ \left(i\partial_0 - i\partial_1 + e(A^0 - A^1)\right)\psi_L^{\dagger} = m\psi_R^{\dagger} \end{cases}$$

each direction of movement gets an imaginary source proportional to the other one, concretely the classical solutions would look as:

$$\begin{cases} \psi(t,x)_R \ \ \alpha \ \ e^{-i(eA^0+eA^1)t} \ e^{ip_k^R(x-t)} - imt\psi_L \\ \psi(t,x)_L \ \ \alpha \ \ e^{-i(eA^0-eA^1)t} \ e^{ip_k^L(x+t)} - imt\psi_R \end{cases} \tag{3.1}$$

from where we see that they no longer depend only on $x \pm t$ meaning they won't move at the speed of light anymore, as one would expect from massive particles. We have a mixing of left and right moving waves through the mass term.

An intuitive way of thinking of this imaginary source term mixing left-right moving particles, is that now that the fields are massive, we will be able to go to one reference frame, where the left moving particles pass to be right moving and viceversa [33].

Also the fact that the fermions are now massive makes creation of particles-antiparticles harder, intuitively meaning that screening will be harder now, and flux tubes as we introduced in the section of confinement will be more resilient, without splitting into new particles.

Concretely the limit to how far flux tubes can stretch, is the limit to when the energy stored in the string is greater than the energy required to create a new particle-antiparticle pair, and this should happen for separations [8]:

$$\frac{e^2}{2}L > 2m \ \longrightarrow \ L > 4m/e^2 \tag{3.2}$$

The upshot of this argument, is that we expect the spectrum of the theory to consist of a tower of neutral meson-like states, each containing a particle and anti-particle.

Also remember from the bosonizations dictionaries, the theory is know to be equivalent to a bosonic lagrangian of the form:

$$\begin{aligned}\mathcal{L}_{\text{Bosonized}} =& \frac{e^2}{2\pi}\left(\phi' + \frac{1}{2\sqrt{\pi}}\theta\right)^2 + \frac{1}{2}(\partial_\mu \phi')^2 + cmM_\gamma \cos(2\sqrt{\pi}\phi') = \\ =& \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M_\gamma^2}{2}\phi^2 + cmM_\gamma\cos(2\sqrt{\pi}\phi - \theta).\end{aligned}$$

with $M_\gamma = e^2/\pi$ and $\phi' = \phi - \frac{1}{2\sqrt{\pi}}\theta$ Show explicitly how to get here, and check what is c exactly!.

3.2 The two regimes of the massive Schwinger model

The massive Schwinger Model is not exactly solvable, so we will need to work in two different limits regime of the dimensionless parameter e/m , where the theory behaves pretty different [25] [34] [35]:

- The weak coupling regime $\frac{e}{m} << 1$ or $m >> e$:

  In the weak coupling regime, we will have almost free massive fermions interacting, like in all the weak coupling theories with the typical perturbative expansion of the propagators in the coupling (e/m). In this case all divergences cancel each other, and we have to compute pretty few diagrams. What is more, the cancellations are so many, that they can actually be generalized for any order, and the exact form of the propagator can be found [5] [20] .

  Also because the mass is much bigger than the coupling we can work on the non-relativistic limit, for which the spectrum will be easily solvable using the Airy functions [36] [37] introduced in the Annexes E, which is what we will do in this thesis.

  In this limit, the vacuum can be taken with no fermionic exitations [34], and the vaccum energy density is given purely by the electrostatic energy term we already saw in Fig. 1.5, where there is a clear discontinuity in the slope for $\theta = \pi$, corresponding to a first-order phase transition.

- The strong coupling regime $\frac{m}{e} << 1$ or $m << e$:

  In the strong coupling regime, we can now do perturbation in the mass instead (m/e), starting from the massless Schwinger model [38]–[40].

  In this regime we will have massive highly energetic states bounded through the gauge field, which we can describe pretty well through bosonization [3] [4] [8].

  Also in this limit, as we saw for the massless case the dependence of θ can be absorbed away, so the vacuum energy density will have to remain constant as a function of θ [34].

3.3 The relevance of the θ parameter in the massive model

If we make memory, the most clear argument to show the irrelevance of the θ parameter in the massless case was that we could absorb it with a shift in the Lagrangian/Hamiltonian when we performed a chiral rotation to the fermions. So let's try that again in this case:

In the massive model, when we do a chiral rotation to the fermions, because of the mass term, the shift in the Lagrangian/Hamiltonian is now given by:

$$\mathcal{H} \longrightarrow \mathcal{H} + \frac{\theta}{2}\partial_\mu j^\mu_A + m(\psi^\dagger e^{-i\theta\gamma^5/2}\gamma^0 e^{+i\theta\gamma^5/2}\psi - \psi^\dagger\gamma^0\psi) \tag{3.3}$$

which using the Baker–Campbell–Hausdorff formula, gives:

$$\mathcal{H} \longrightarrow \mathcal{H} + \frac{\theta}{2}\partial_\mu j^\mu_A + m(\psi^\dagger \frac{i}{2}\theta[\gamma^5, \gamma^0]\psi) = \mathcal{H} + \frac{\theta}{2}\partial_\mu j^\mu_A - \theta m(\psi^*_R\psi_L - \psi^*_L\psi_R) \tag{3.4}$$

which shows, that you can't absorb the θ term anymore (absorbing it from its original term, would make it appear in the fermion mass term)!

This means that massive model then depends on the θ parameter again, and because of the discussion we have done for the vacuum energy in each regime, we know that for the weak coupling limit ($m/e >> 1$), the energy density has a phase transition for $\theta = \pi$, but for the strong coupling limit ($e/m >> 1$) as we approach the massless case this phase transition and also all the dependence on θ has to disappear.

So the Schwinger model has to develop a phase transition ($\theta = \pi$) at some critical point of the parameter $(m/e)_c$:

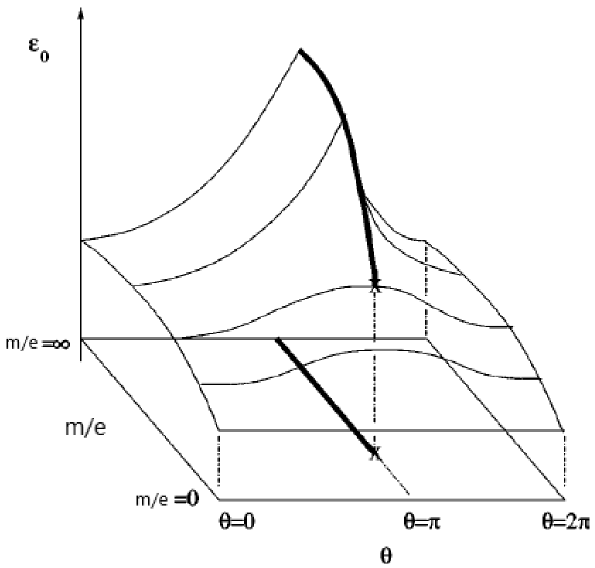


Figure 3.1: Schematic plot of the vacuum energy density as a function of m/e and θ. The heavy line marks the first-order transition line, where the energy density has a cusp, terminating at the second order critical point $(m/e)_c$, where the slope no longer has a discontinuity. [34]

where we can see the energy density of Fig. 1.5 in the deep wall with its slope discontinuity, and how it changes with the dimensionless parameter m/e until the critical point, where the discontinuity disappears, but the energy still depends on θ, to finally arrive to the massless case, where this dependence disappears completely, having a flat horizontal line as a function of θ. This behaviour has been numerically demonstrated in [30] who located the critical point at $(m/e)_c = 0.325(20)$, with an associated critical index $\nu = 0.9(1)$.

3.4 Critical point $(m/e)_c$ for the massive Schwinger model

Before doing any computations in each of the two regimes, we want to find the critical point ourselves. For that let's consider the Bosonic massive lagrangian:

$$\mathcal{L}_{\text{Bosonized}} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{M_\gamma^2}{2}\phi^2 + cmM_\gamma \cos(2\sqrt{\pi}\phi - \theta).$$

and now lets consider the case we are interested in, $\theta = \pi$, as kinetic part with an effective potential (the sign in the cosine has flipped, when setting $\theta = \pi$):

$$U(\phi) = \frac{M_\gamma^2}{2}\phi^2 + cmM_\gamma \cos(2\sqrt{\pi}\phi)$$

this potential for strong coupling $(m/e) << 1$, clearly has a unique vacuum at $\phi = 0$. But for the weak coupling $(m/e) >> 1$ however, there are two vacua, located at $\phi = \pm\frac{1}{2}\sqrt{\pi}$ and the symmetry $\phi \leftrightarrow -\phi$ suffers spontaneous breakdown.

This is a spontaneous breakdown of a Z_2 symmetry, which implies that the critical point should belong to the universality class of the (1+1)D or 2D Ising model, with critical indices $\nu = 1$ and $\beta = 1/8$, which is consistent with the value found by Hamer in [30]:

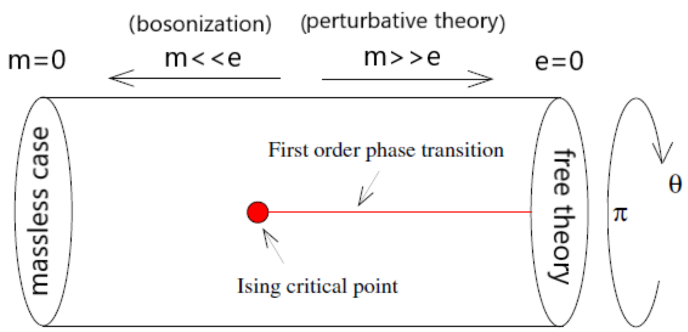


Figure 3.2: Phase diagram of the Schwinger model, based on the phase diagram of 1+1 d scalar theory of David Tong.

to find this point now that we can consider it a Ising critical point (2n order phase transition), the theory has to be a conformal theory which means that it's going to have scale invariance, telling us that the boson mass has to be zero!

So expanding our potential $U(\phi)$ around $\phi = 0$, we can see that we will have an harmonic oscillator with mass:

$$U(\phi) = \frac{M_\gamma^2}{2}\phi^2 + cmM_\gamma \left(1 - \frac{(2\sqrt{\pi}\phi)^2}{2} + ...\right) \approx cmM_\gamma + \left(\frac{M_\gamma^2}{2} - 2cmM_\gamma\pi\right)\phi^2 \quad (3.5)$$

and if the mass of the boson has to be zero then:

$$\frac{M_\gamma^2}{2} - 2cmM_\gamma\pi = 0 \longrightarrow \frac{e^2}{2\pi} = 2cme\sqrt{\pi} \longrightarrow (\frac{m}{e})_c = \frac{e^{-\gamma}}{\sqrt{\pi}} \approx 0.31679 \quad (3.6)$$

which is inside the expected value $(m/e)_c = 0.325(20)$.

3.5 The weak coupling regime (massive) (m»e)

As we said, in this regime we will solve an approximate non-relativistic Hamiltonian that has known solutions, through the Airy functions, because it is where it's easier to compute the spectrum, which is what we are interested on.

3.5.1 Non-relativistic linear potential

In this non-relativistic limit of the problem, we will have a bound state called "positronium", an electron-positron pair connected by a flux tube, which is hard to break/screen due to the high mass of the fermions in this regime.

Also because of this, the electric field will be constant between the pair, giving a potential for the electric field: $U = \frac{1}{2}\int E^2 dx = \frac{e^2}{2}L$, which gives us a Schrodinger equation for the positronium:

$$H\psi = \left(\frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{e^2}{2}|x_1 - x_2|\right)\psi = \left(\frac{p^2}{m} + \frac{e^2}{2}|x|\right)\psi = E\psi \tag{3.7}$$

where we have moved to the center of mass coordinates with $x = x_1 - x_2$. We are now in front of a linear potential Schrödinger equation, which we know can be solved using the Airy functions, as we can see in Appendix E, writing the Schrödinger equation in the following way:

$$-\frac{1}{m}\frac{\partial^2\psi}{\partial x^2} = \left(E - \frac{e^2}{2}|x|\right)\psi \;\longrightarrow\; \frac{\partial^2\psi}{\partial x^2} = m\left(\frac{e^2}{2}|x| - E\right)\psi \tag{3.8}$$

where doing the change of variable: $s = \left(\frac{4m}{e^4}\right)^{1/3}\left(\frac{e^2}{2}|x| - E\right)$, is easy to see (after some double derivative chain rules), that the equation ends up as:

$$\frac{\partial^2\psi}{\partial s^2}\left(\frac{\partial s}{\partial x}\right)^2 + \frac{\partial\psi}{\partial s}\frac{\partial^2 s}{\partial x^2} = s\psi m\left(\frac{e^4}{4m}\right)^{1/3} \longrightarrow \frac{\partial^2\psi}{\partial s^2} = s\psi \tag{3.9}$$

which is exactly the equation Airy functions solve. Concretely for our states to be normalizable we will have $\psi(x) = \sum_i c_i A_i(x)$, since $B_i(x)$ diverges in an infinite potential.

If we also assume that the all possible wave function are antisymmetric $\psi(x) = -\psi(-x)$, since we already were in the center of mass of particle-antiparticle states, then we can be sure that the i-th function we are interested in, fulfills $A_i(x = 0) = 0$. So we only need to check the zero's of the Airy function for the variable s, $A(s_i) = 0$, from where we see that:

$$s_i = \left(\frac{4m}{e^4}\right)^{1/3}\left(\frac{e^2}{2}|x = 0| - E_i\right) \;\longrightarrow\; E_i = \sqrt[3]{\frac{e^4}{4m}}(-s_i) \tag{3.10}$$

and checking the first zeros of the Airy functions, we see that the lowest energy levels have energies:

$$E_1 = 2.33810\sqrt[3]{\frac{e^4}{4m}} \;;\quad E_2 = 4.08794\sqrt[3]{\frac{e^4}{4m}} \;;\quad E_3 = 5.52055\sqrt[3]{\frac{e^4}{4m}}$$

where we see that the energy separation between levels decreases for higher levels.

Here we can see an example of the 20th antisymmetric state:

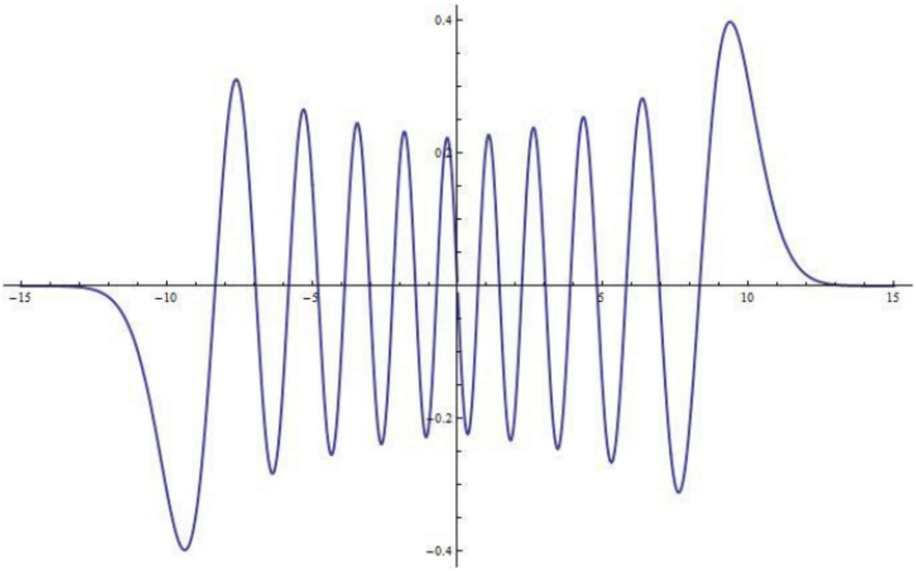


Figure 3.3: 20th antisymmetric Airy function, figure extracted from [41]

But this spectrum is not affected by θ, and we discussed that for weak coupling, the θ parameter has to affect the spectrum, and not only that, but also has to have a phase transition for $\theta = \pi$.

Well the reason for that is that we have lied, the potential of the electric field between the particle-antiparticle also has a contribution from θ term of course! And then the real Schrodinger equation is [4]:

$$H\psi = \left(\frac{p^2}{m} + \frac{e^2}{2}\left(|x| - \frac{\theta}{\pi}x\right)\right)\psi = E\psi \tag{3.11}$$

where we see that we have solved the case for $\theta = 0$.

The most interesting result, we can see at first sight, is that the potential will be linear for x for all the values of except for $\theta = \pi$! We are not going to repeat the computation, since for $\theta \neq \pi$ it will be exactly the same we did, but dragging a constant term, that will modify the final energies.

Even though we won't compute it's worth mentioning that the number of bound states with energy less than E_{max} will approximately be (it will work well for big $N(E_{\text{max}})$) [4]:

$$\begin{aligned} N(E_{\text{max}}) =& \frac{1}{2\pi}\int dpH(E_{\text{max}} - H) = \\ =& \frac{2}{e^2(1-\theta^2/\pi^2)}\int dp(E_{\text{max}} - p^2)H(E_{\text{max}} - p^2) = \\ =& \frac{8E_{\text{max}}^{3/2}}{3e^2(1-\theta^2/\pi^2)} \end{aligned}$$

where we see that when θ goes to the discontinuity $\theta = \pi$, the quantity of particles with lower energy increase.

Chapter 4

Discussion and outlook

In this thesis, we studied the Schwinger model both in its classical and quantum versions for both the massless and the massive cases. We obtained various interesting results for the classical massless model regarding its dynamics, the theta term and the confinement. Then for the quantum massless model we also obtained pretty interesting results, regarding the chiral anomaly, the mass gap, the spectrum and the dynamics of the canonical quantized model, from which naturally we then arrived at the bosonization of the theory, from where we explored the screening, the anomaly and the spectrum of the theory from another perspective.

Finally we have done an introduction to the massive Schwinger model, which is a very interesting model, that has lots of possible computations that haven't been done yet, even after so many papers published in the subject, and we have grasped the surface of these computations, computing the spectrum for the weak regime, and studying the transitions from one regime to the other through the analysis of the critical point of the model.

It is important to note that we benefited greatly from working in 1+1 dimensions where we could build very good intuitions and understandings of all the processes both at classical and at quantum level, which normally at higher dimensional theories because they carry more difficulties and subtleties, are difficult to obtain.

Perhaps the most natural continuation of this work should be towards working in the strong coupling regime and then studying the transition from one regime to the other, trying to do better approximations for each regime and taking them together to compare them, doing maybe an interpolation between both spectrum's or finding some patterns. But this sure isn't the only possibility, we have only worked with a U(1) symmetry, and working with more realistic gauge theories such as QCD or other non-abelian theories using the same methods outlined in this work could also be a really interesting option.

Finally, we could also think of continuing the venture of the simulations we did along the thesis, to work with numerical method on the lattice Schwinger model, which seems an increasing field, from where the most recent papers we read were. We even read papers about quantum simulations for computing the critical point, in which they obtained a pretty similar result to mine even.

Appendices

Appendix A

The geometry of Gauge theories [42] [43]

We stipulate that our theory should be invariant under a local phase rotation:

$$\psi(x) \to e^{i\alpha(x)}\psi(x)$$

which means gauge symmetry is not an accidental curiosity but rather the fundamental principle under which we will construct the theory.

For the mass term there is no problem, the lagrangian is invariant, but the derivatives term is not, because each term of the derivative transforms with a different $alpha(x)$:

$$n^\mu \partial_\mu \psi = \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon}[\psi(x+\epsilon n) - \psi(x)]$$

but this subtraction will depend on the gauge transformation, so this derivative doesn't have a useful geometric interpretation (we are considering $\alpha(x)$ varies like crazy even in infinitesimally separated positions).

In order to do this substraction in a meaningful way, we must introduce a factor that compensates for the difference in the phase transformation between infinitesimally separated points:

$$U(y,x) \to e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}$$

where $U(x,x) = 1$. Now the objects $\psi(y)$ and $U(y,x)\psi(x)$ transform equally and we can substract them in a meaningful way, which an invariant derivative, which is the already known covariant derivative:

$$n^\mu D_\mu \psi = \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon}[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)]$$

which to make this explicit we need and expression for the comparator $U(y,x)$. In the infinitesimal separation between two points it can be expanded as:

$$U(x+\epsilon n, x) = 1 - ie\ \epsilon n^\mu A_\mu(x) + O(\epsilon^2) \approx e^{-ie\ \epsilon n^\mu A_\mu(x) + O(\epsilon^2)} \tag{A.1}$$

where $A_\mu(x)$ are the coefficients for the phase gauge variation of the displacement in the direction. Such a field, which appears as the infinitesimal limit of a comparator of local symmetry transformations is called a **connection**. The covariant derivative then takes the form:

$$D_\mu\psi(x) = \partial_\mu\psi(x) + ieA_\mu\psi(x) \tag{A.2}$$

where A_μ transforms under this local phase rotation as:

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

We have seen that even the very existence of the vector field A_μ follows from the postulate of local phase rotation symmetry, being it the connection of that symmetry.

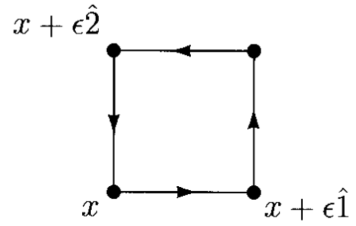


Figure A.1: Construction of the field strength by comparisons around a small square in the (1,2) plane.

Finally we are going to show that if we take the comparator U(y,x) doing closed loops in different directions, such as the ones in Figure A.1, we get values $\neq 0$, there is curvature!:

$$\begin{aligned}\mathbf{U}(x) =& U(x, x+\epsilon 2)U(x+\epsilon 2, x+\epsilon 1+\epsilon 2)U(x+\epsilon 1+\epsilon 2, x+\epsilon 1)U(x+\epsilon 1, x) = \\ =& 1 - i\epsilon^2 e[\partial_1 A_2(x) - \partial_2 A_1(x)] + O(\epsilon^3)\end{aligned}$$

Therefor the structure:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{A.3}$$

is locally invariant and represent the gauge phase gained when doing a closed loop in the directions x_μ and x_ν. This is the familiar electromagnetic field tensor, which we now see that is the first order term of the **curvature** of the Gauge symmetry manifold!!!

This can also be understood when we see that:

$$[D_\mu, D_\nu] = ieF_{\mu\nu}$$

which is nothing more than seeing that the $F_{\mu\nu}$ is the comparison of comparisors $U(y, x)$ across a square.

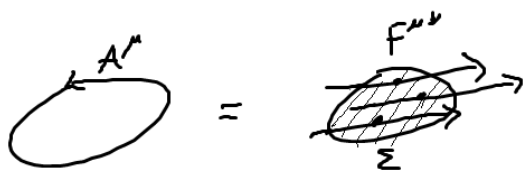
So we have seen that the comparator between two points x and y, at finite separation, depends on the path taken from x to y. If we now want to extend the comparator for any finite separation we will obtain what is called a **Wilson line**:

$$U_P(y, x) = e^{-ie\int_P dz^\mu A_\mu(x)}$$

which depends on the path. If we then close this path we obtain a **Wilson loop**:

$$U_P(y, y) = e^{-ie\oint_P dz^\mu A_\mu(x)} = e^{-i\frac{e}{2}\int_\Sigma d\sigma^{\mu\nu} F_{\mu\nu}}$$

where using the Green's theorem, we see that how the connection (A_μ) changes through a closed lines is equal to how much curved Σ is (how much $F^{\mu\nu}$ going through Σ).



Appendix B

Wilson lines and Wilson loops [44]

Definition and basic properties

QED Wilson lines (crange compensators):

$$U(\Gamma_{y,x}) = e^{ie\int_{\Gamma\ y,x} dz^\mu A_\mu(z)} \tag{B.1}$$

with a path Γ that goes from x to y.

Under the gauge transformation:

$$\begin{cases} A_\mu(z) \to \frac{1}{e}A_\mu(z) + \partial_\mu \alpha(z) \\ \psi(x) \to e^{i\alpha(x)}\psi(x) \end{cases}$$

then you can define a covariant derivative that transforms equally to the field:

$$D_\mu\psi(x) \to e^{i\alpha(x)} D_\mu\psi(x)$$

where $D_\mu = \partial_\mu + ieA\mu$ is exactly given by the connections of the gauge manifold, by the factors ieA_μ which are also the exponent of the Wilson line phase factor (that is where the gauge phase comes, after traveling through the Wilson line). This Wilson line will then transform as:

$$U_\gamma(y,x) \to e^{i\alpha(y)} U_\gamma(x,y) e^{-i\alpha(x)}$$

therefor the Wilsonian line will transform as the product:

$$U(\Gamma_{y,x}) \sim "\psi(y)\psi^\dagger(x)"$$

And a field at the point x will transform like one at the point y after a multiplication by this Wilson line phase factor:

$$U(\Gamma_{y,x})\psi(x) \sim "\psi(y)"$$

where this time we see explicitly that the Wilson line phase factor plays a role of the parallel transporter in an electromagnetic field, and to compare phases of a wave function at points x and y, we should first make a parallel transport along some contour $\Gamma_{y,x}$. This result is Γ dependent expect when $A_\mu(z)$ is a pure gauge (vanishing field strength $F_{\mu\nu}$).

We can then obviously close this $\Gamma_{y,x}$ and obtain what is called a Wilson loop:

$$U(\Gamma) = e^{ie\oint_\Gamma dz^\mu A_\mu(z)} \tag{B.2}$$

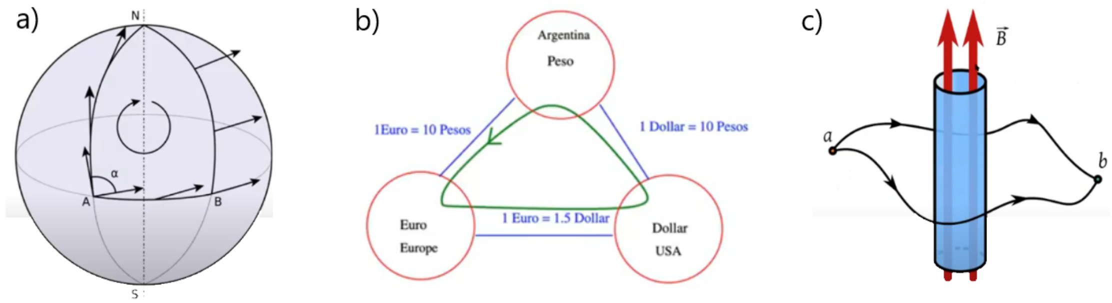


Figure B.1: a) is an analogy to the "phase" obtained to the vector with paral · lel transport in a spacetime manifold, b) is an analogy with currency where in a closed loop you obtain a gain value, c) is the actual thing where we get a phase out of a closed loop (which is constructed with the connection of the gauge manifold [the same thing with which we construct the covariant derivative ieA_μ]).

Appendix C

The Theta Angle [45]

The typical relativistic notation Maxwell Lagrangian is given by:

$$\mathcal{L}_{Maxwell} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \tag{C.1}$$

this Lagrangian is very simple, because there is very little else we can write down that is both Gauge and Lorentz invariant (There are terms or order F^4 and higher, which give rise to non-linear electrodynamics, but these will always be suppressed by some high mass scale and are unimportant and low-energies).

There is, however, one other term that we can add to the Maxwell action, **the theta term**:

$$\mathcal{L}_\theta = \frac{\theta\alpha}{\pi}\frac{1}{4}F^{*\mu\nu}F_{\mu\nu} = \frac{\theta e^2}{4\pi^2}\frac{1}{4}\left(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right)F_{\mu\nu} = -\frac{\theta e^2}{4\pi^2}\,\mathbf{E}\cdot\mathbf{B} \tag{C.2}$$

where this Lagrangian can be written down as a total derivative, so we say that the theta term is topological, it depends only on the boundary information. The theta term does not change the equations of motion, but it does lead to interesting physics involving subtle interplay of quantum mechanics and topology.

Why θ is periodic

In classical axion electrodynamics, can take any value. Indeed, as we have seen, it is only spatial and temporal variations of that play a role. However, in the quantum theory is a periodic variable:

$$\theta = [0, 2\pi) \text{ or } (-\pi, \pi]$$

this comes from the fact that **E** and **B** are gauge fields that have to fulfill that:

$$A_\mu = A_\mu + \partial_\mu\omega$$

which working in a compact Gauge manifold $\mathbf{T}^4$ (we then realize it doesn't depend on the size of the manifold, so it can be generalized to non-compact manifolds also) with some concrete conditions give for example:

$$S_\theta = \int_{\mathbf{T}^4} L_\theta\ d^4x = \frac{\alpha}{4\pi}\int_{\mathbf{T}^4}\mathbf{E}\cdot\mathbf{B}\ d^4x = N \quad\rightarrow\quad S_\theta = \theta N \text{ with N } \epsilon\mathcal{Z} \tag{C.3}$$

which means that in the partition functions the θ term contributes with $e^{iS_\theta} = e^{iN\theta}$. From where we see that the factor i persists, so that the value of θ in the partition function is only important modulo 2π.

Appendix D

Bosonization dictionaries

Here we will show two different approaches to the bosonization dictionaries, one by David Tong and another from Sidney Coleman:

D.1 First Bosonization Dictionary (David Tong) [8]

To obtain the Bosonization as David Tong does, we have to compare our theory with that of a compact boson. So we will focus on a massless, real scalar field ϕ:

$$S = \int d^2x \frac{\beta^2}{2}(\partial_\mu \phi)^2$$

then the correlation functions that are equivalent of the fermionic and bosonic theories are:

$$\langle \psi_-(x)\psi_-^\dagger(y)\rangle = \frac{i}{2\pi}\frac{1}{(x-y)+i\epsilon} \quad \text{and} \quad \langle e^{i\phi_-(x)}e^{-i\phi_-(y)}\rangle = \frac{i\epsilon}{(x-y)+i\epsilon} \tag{D.1}$$

which tells us, that we should identify:

$$\begin{cases} \psi_-(x) \longleftrightarrow \sqrt{\frac{1}{2\pi\epsilon}}e^{i\phi_-(x)} \\ \psi_+(x) \longleftrightarrow \sqrt{\frac{1}{2\pi\epsilon}}e^{-i\phi_+(x)} \end{cases} \tag{D.2}$$

so:

$$\hat{\psi}\psi \longleftrightarrow -\frac{1}{2\pi\epsilon}(e^{-i\phi(x)} + e^{i\phi(x)}) = -\frac{1}{\pi\epsilon}\cos\phi \tag{D.3}$$

and finally:

$$\begin{cases} j_V^\mu \longleftrightarrow -\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi \\ j_A^\mu \longleftrightarrow -\beta^2\partial^\mu\phi \end{cases} \tag{D.4}$$

so the Schwinger action ends up as:

$$\begin{aligned} S &= \int d^2x \ \frac{1}{2e^2}F_{01}^2 + \frac{\theta}{2\pi}F_{01} + \hat{\psi}\not{D}\psi - m\hat{\psi}\psi = \\ &= \int d^2x \ \frac{1}{2e^2}F_{01}^2 + \frac{\theta}{2\pi}F_{01} + \frac{1}{8\pi}(\partial_\mu\phi)^2 + \frac{1}{2\pi}A_\mu\epsilon^{\mu\nu}\partial_\nu\phi + \frac{m}{\pi\epsilon}\cos\phi = \\ &= \int d^2x \ \frac{1}{2e^2}F_{01}^2 + \frac{1}{2\pi}(\theta+\phi)F_{01} + \frac{1}{8\pi}(\partial_\mu\phi)^2 + \frac{m}{\pi\epsilon}\cos\phi. \end{aligned}$$

for which the equations of motion for ϕ and the gauge field are respectively:

$$\frac{1}{4\pi}\partial^2\phi = -\frac{1}{2\pi}F_{01} \quad \text{and} \quad \partial_1\left(\frac{1}{e^2}F_{01} + \frac{1}{2\pi}\phi\right) = 0 \ \xrightarrow[\text{vanishes at infinity}]{\text{if the combination}} \ F_{01} = -\frac{e^2}{2\pi}\phi$$

D.2 Second Bosonization Dictionary (S. Coleman) [3] [4]

$$: \bar{\psi}\psi : = \begin{cases} : \bar{\psi}(1+\gamma_5)\psi : \\ + \\ : \bar{\psi}(1-\gamma_5)\psi : \end{cases} \longleftrightarrow \begin{cases} ce : e^{i2\sqrt{\pi}\phi'} : U \\ + \\ ce : e^{-i2\sqrt{\pi}\phi'} : U^\dagger \end{cases} = -cm\cos(2\sqrt{\pi}\phi') \tag{D.5}$$

where U was the unitary operator that distinguished vacua as: $U|\theta\rangle = e^{i\theta}|\theta\rangle$

$$: \bar{\psi}\gamma^\mu\psi : \longleftrightarrow \frac{\epsilon^{\mu\nu}}{\sqrt{\pi}}\partial_\nu\phi' \tag{D.6}$$

so the Schwinger action ends up as:

$$\begin{aligned} S &= \int d^2x \, \frac{1}{2}F_{01}^2 + \hat{\psi}\not{D}\psi - m\hat{\psi}\psi = \\ &= \int d^2x \, \frac{1}{2}F_{01}^2 + \frac{1}{\sqrt{\pi}}\partial_\nu\epsilon^{\mu\nu}(\partial_\mu\phi') + \frac{ie}{\sqrt{\pi}}A_\mu\epsilon^{\mu\nu}\partial_\nu\phi + cm^2\cos(2\sqrt{\pi}\phi') = \\ &= \int d^2x \, \frac{e^2}{2\pi}\left(\phi' + \frac{1}{2\sqrt{\pi}}\theta\right)^2 + \frac{1}{2}(\partial_\mu\phi')^2 + cm^2\cos(2\sqrt{\pi}\phi') = \\ &= \int d^2x \, \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M^2}{2}\phi^2 + cmM\cos(2\sqrt{\pi}\phi - \theta). \end{aligned}$$

where $M = e^2/\pi$, $\phi' = \phi - \frac{1}{2\sqrt{\pi}\theta}$ and $F_{01} = \frac{e}{\sqrt{\pi}}\left(\phi' + \frac{1}{2\sqrt{\pi}\theta}\right)$, as shown in next section 2.7.2: "Revisit Theta as a Background field with Bosonization".

Comparison between both Bosonizations

Lastly say that comparing both bosonizations we see a correspondence between their parameters given by:

$$\underline{\text{David Tong}} \longleftrightarrow \underline{\text{Sidney Coleman}} \tag{D.7}$$

$$\phi = -2\sqrt{\pi}\phi' \tag{D.8}$$

$$\frac{1}{\pi\epsilon} = c\,m \tag{D.9}$$

which is useful, given that the ϵ parameter on the David Tong part we see that come from regularization, containing a divergence in the computation, and it's nice to write it in finite terms as in Coleman case, to compute things at the end.

Appendix E

Linear Potential and Airy Functions [46]–[50]

If in quantum mechanics we consider the potential as:

$$V(x) = V(x_r) + V'(x_r)(x - x_r) + ... = E + V'(x_r)(x - x_r) \tag{E.1}$$

which would mean that we aproximate the potential to first order, where the value equates that of the Energy ($V(x_r) = E$), the so called turning points (x_r).

Then the Schrodinger equation becomes:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x) \;\rightarrow\; \frac{\partial^2}{\partial x^2}\psi(x) + C\psi(x)(x - x_r) = 0 \tag{E.2}$$

where $C = -2mV'(x_r)/\hbar^2$. Now doing a change of variable $s = (x-x_r)/|C|^{1/3}$ considering the solution to be an exponential with WKB approximation, we finally get the desired differential equations:

$$\begin{cases} \psi''(s) - s\psi(s) = 0 \;\text{ if }\; V'(x_r) > 0 \\ \psi''(s) + s\psi(s) = 0 \;\text{ if }\; V'(x_r) < 0 \end{cases} \tag{E.3}$$

where we only need to sole one of this, since they are the same equation changing $s \rightarrow -s$.

Airy functions

The general solution to

$$\psi''(s) = s\psi(s)$$

is given by linear combinations of what is called the Airy functions, depending on the initial and frontiers conditions. Which take the following forms:

$$\begin{aligned} Ai(s) =& \frac{1}{\pi}\int_0^\infty \cos\left(\frac{s^3}{3} + ks\right)dk = \sqrt{\frac{s}{9}}\left[I_{-1/3}\left(\frac{2}{3}s^{3/2}\right) - I_{1/3}\left(\frac{2}{3}s^{3/2}\right)\right] \\ Bi(s) =& \frac{1}{\pi}\int_0^\infty \left[e^{-\frac{s^3}{3}+ks} + \sin\left(\frac{s^3}{3} + ks\right)\right]ds = \sqrt{\frac{s}{3}}\left[I_{-1/3}\left(\frac{2}{3}s^{3/2}\right) + I_{1/3}\left(\frac{2}{3}s^{3/2}\right)\right] \end{aligned}$$

where $I_\nu(x)$ are Bessel functions. Also we see that $Ai(s) \xrightarrow{s\rightarrow\infty} 0$ while $Bi(s) \xrightarrow{s\rightarrow\infty} \infty$:

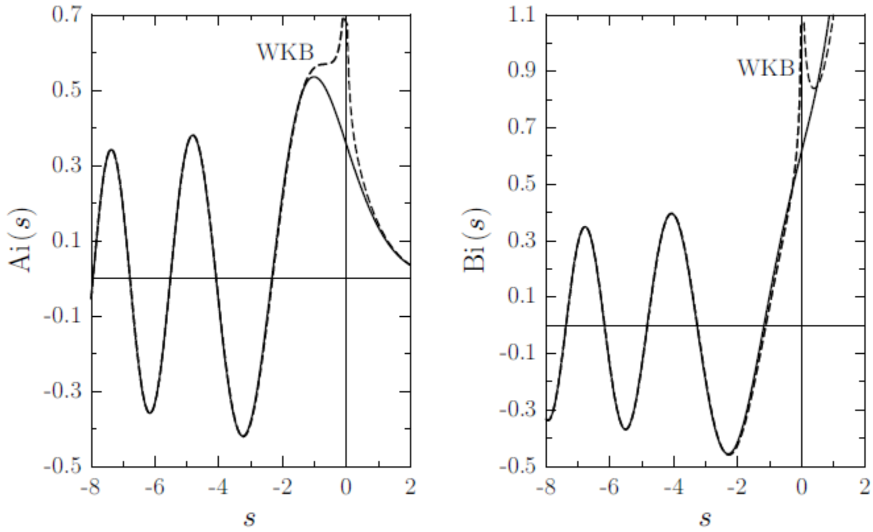


Figure E.1: Airy functions (continuous lines) and it's WKB approximations (dotted line)

The asymptotic forms of these Airy functions are:

	$s << 0$	$s >> 0$
$Ai(s)$	$\frac{1}{\sqrt{\pi}(-s)^{1/4}}\sin[\frac{2}{3}(-s)^{3/2}+\frac{\pi}{4}]$	$\frac{1}{2\sqrt{\pi}s^{1/4}}e^{-\frac{2}{3}s^{3/2}}$
$Bi(s)$	$\frac{1}{\sqrt{\pi}(-s)^{1/4}}\cos[\frac{2}{3}(-s)^{3/2}+\frac{\pi}{4}]$	$\frac{1}{\sqrt{\pi}s^{1/4}}e^{\frac{2}{3}s^{3/2}}$

Table E.1: Asymptotic freedom forms [48] [47], and there are even better approximations which we won't care to show [46]

where we see that the functions behave as oscillations or decays in each side of the turning point (x_r), as we know should happen when the potential surpasses the energy of the state, where classically should simply not trespass it.

So for having a normalizable state versus and infinite width linear potential, we can only keep $|\psi\rangle = aAi(s)$ as solution:

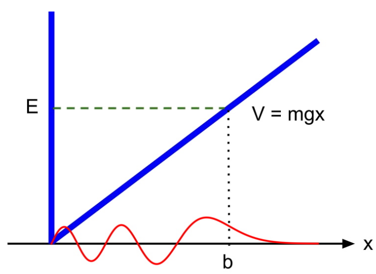


Figure E.2: Normalizable function with infinite width linear potential

The $Bi(s)$ solution will be used only when the wave function behind a finite width potential that was higher than the energy is starting to oscillate again. And what we would have to do is a matching of the amplitude coefficients comparing the free solutions given by WKB at each side with the Airy functions as Frontiers conditions.

But for our concrete case, where our potential will be pretty similar to that of and infinite width linear one, we will have the solution from Figure E.2.

Appendix F

Code used for the simulations

```
#include <string.h>
#include <stdlib.h>
#include <stdio.h>
#include <math.h>

#define Nx 4001

int main(){

  //GRID PARAMETERS
  double Ndx = 4000.0 ;
  int Time = 10000;
  double r = 0.25 ;
  double delx = 1.0/Ndx ;
  double delt = r*delx ;

  //FIELDS
  double Lold[Nx][2];
  double Lnew[Nx][2];
  double Rold[Nx][2];
  double Rnew[Nx][2];
  double Aolder[Nx];
  double Aold[Nx];
  double Anew[Nx];
  double E[Nx];

  //CONDITIONS OF THE SIMULATION
  int IICC=2;
  double e = 20.0; //COUPLING!
  double mx = 0.5 ;
  double sig = 0.03 ;

  int i,t, k=0;
  int tout=Time/100;

  //INITIAL CONDITIONS OF THE FIELDS
  for (i=0; i<Nx; i++){
    Lold[i][0]= (exp(-0.5*((i*delx-mx/2)/sig)*((i*delx-mx/2.0)/sig))+exp
   (-0.5*((i*delx-mx*3.0/2.0)/sig)*( (i*delx-mx*3.0/2.0)/sig )))/(4.0*sig/
   delx*sqrt(2*3.14159265));

    Lnew[i][0]= (exp(-0.5*((i*delx-mx/2)/sig)*( (i*delx-mx/2.0)/sig ))+exp
   (-0.5*((i*delx-mx*3.0/2.0)/sig)*( (i*delx-mx*3.0/2.0)/sig )))/(4.0*sig/
   delx*sqrt(2*3.14159265));

    Lold[i][1]= 0.0*(exp(-0.5*((i*delx-mx/2)/sig)*( (i*delx-mx/2.0)/sig ))+
   exp(-0.5*((i*delx-mx*3.0/2.0)/sig)*( (i*delx-mx*3.0/2.0)/sig )))/(4.0*
```

```
    sig/delx*sqrt(2*3.14159265));

    Lnew[i][1]= 0.0*(exp(-0.5*((i*delx-mx/2)/sig)*( (i*delx-mx/2.0)/sig ))+
    exp(-0.5*((i*delx-mx*3.0/2.0)/sig)*( (i*delx-mx*3.0/2.0)/sig )))/(4.0*
    sig/delx*sqrt(2*3.14159265));


    Rold[i][0]= (exp(-0.5*((i*delx-mx/3)/(0.5*sig))*( (i*delx-mx/3.0)/(0.5*
    sig) ))+0.0*exp(-0.5*((i*delx-mx*3.5/2.0)/(0.5*sig))*((i*delx-mx
    *3.5/2.0)/(0.5*sig) )))/(2.0*sig/delx*sqrt(2*3.14159265));

    Rnew[i][0]= (exp(-0.5*((i*delx-mx/3)/(0.5*sig))*( (i*delx-mx/3.0)/(0.5*
    sig) ))+0.0*exp(-0.5*((i*delx-mx*3.5/2.0)/(0.5*sig))*((i*delx-mx
    *3.5/2.0)/(0.5*sig) )))/(2.0*sig/delx*sqrt(2*3.14159265));

    Rold[i][1]= 0.0*(exp(-0.5*((i*delx-mx/3)/(0.5*sig))*( (i*delx-mx/3.0)
    /(0.5*sig) ))+exp(-0.5*((i*delx-mx*3.5/2.0)/(0.5*sig))*( (i*delx-mx
    *3.5/2.0)/(0.5*sig) )))/(2.0*sig/delx*sqrt(2*3.14159265));

    Rnew[i][1]= 0.0*(exp(-0.5*((i*delx-mx/3)/(0.5*sig))*( (i*delx-mx/3.0)
    /(0.5*sig) ))+exp(-0.5*((i*delx-mx*3.5/2.0)/(0.5*sig))*( (i*delx-mx
    *3.5/2.0)/(0.5*sig) )))/(2.0*sig/delx*sqrt(2*3.14159265));


    Aolder[i]=0.001;
    Aold[i]=0.001;
    Anew[i]=0.001;

  }

  //FILE PRINTING
  char buffer [100];
  FILE * sFile;
  sprintf(buffer,"mkdir T%d_N%d_sig%.3f_%d", Time, Nx, sig, IICC);
  system(buffer);

  for(t=0 ; t<Time ; t++){

    if((t%tout)==0){
      printf("%d \n",k);
      sprintf(buffer,"T%d_N%d_sig%.3f_%d/Sky%d.txt", Time, Nx, sig,IICC,k);
      k=k+1;
      sFile = fopen (buffer , "w" );

      for(i=0; i<Nx; i++){
         fprintf(sFile, "%d %.8f %.8f %.8f %.8f %.8f %.8f %.8f %.8f \n",i,
    Lnew[i][0],Lnew[i][1],sqrt(Lnew[i][0]*Lnew[i][0]+Lnew[i][1]*Lnew[i][1])
    ,Rnew[i][0],Rnew[i][1], sqrt(Rnew[i][0]*Rnew[i][0]+Rnew[i][1]*Rnew[i
    ][1]), Anew[i]*10, E[i]*100);
      }

      fclose(sFile);

    }

    //INTEGRATION OF FIELD IN ALL SPACE PARAMETERS FOR NORMALIZATION CHECKS
    double Ltotal=0.0;
    double Rtotal=0.0;
    double Atotal=0.0;

    //EQUATIONS OF MOTION
    for(i=0; i<Nx-1; i++){
      Lnew[i][0]= +r*(Lold[i+1][0]-Lold[i][0]) + Lold[i][0]-e*Aold[i]*(-
```

```
    Lold[i][1]);

      Lnew[i][1]= + r*(Lold[i+1][1] - Lold[i][1]) + Lold[i][1] - e*Aold[i]*Lold[i][0];

      Ltotal +=  sqrt(Lnew[i][0]*Lnew[i][0] + Lnew[i][1]*Lnew[i][1]);

      Rnew[i+1][0]= - r*(Rold[i+1][0] - Rold[i][0])+ Rold[i+1][0] + e*Aold[i+1]*(-Rold[i+1][1]);

      Rnew[i+1][1]= - r*(Rold[i+1][1]-Rold[i][1])+ Rold[i+1][1] + e*Aold[i+1]*Rold[i+1][0];

      Rtotal +=  sqrt(Rnew[i+1][0]*Rnew[i+1][0] + Rnew[i+1][1]*Rnew[i+1][1]);
    }


    for (i=0; i<Nx; i++){
        Anew[i]=2*Aold[i]-Aolder[i]-delt*delt*e*(Rold[i][0]*Rold[i][0] + Rold[i][1]*Rold[i][1]-Lold[i][0]*Lold[i][0]-Lold[i][1]*Lold[i][1]) ;
      Atotal += fabs(Anew[i]);
      E[i]=-(Anew[i]-Aold[i])/delt;
    }

    //PERIODIC BOUNDARY CONDITIONS
    Lnew[Nx-1][0]=+ r*(Lold[0][0]-Lold[Nx-1][0])+ Lold[Nx-1][0] - e*Aold[Nx-1]*(-Lold[Nx-1][1]);

    Lnew[Nx-1][1]=+ r*(Lold[0][1]-Lold[Nx-1][1])+ Lold[Nx-1][1] - e*Aold[Nx-1]*Lold[Nx-1][0];

    Rnew[0][0]= - r*(Rold[0][0]-Rold[Nx-1][0])+ Rold[0][0] + e*Aold[0]*(-Rold[0][1]);

    Rnew[0][1]= - r*(Rold[0][1]-Rold[Nx-1][1])+ Rold[0][1] + e*Aold[0]*Rold[0][0];

    Ltotal +=  sqrt(Lnew[Nx-1][0]*Lnew[Nx-1][0] + Lnew[Nx-1][1]*Lnew[Nx-1][1]);

    Rtotal +=  sqrt(Rnew[0][0]*Rnew[0][0] + Rnew[0][1]*Rnew[0][1]);


    //UPDATE OLD FIELDS
    for(i=0; i<Nx; i++){
      Aolder[i]=Aold[i];
    }

    for(i=0; i<Nx; i++){
      Lold[i][0]= Lnew[i][0];
      Lold[i][1]= Lnew[i][1];
      Rold[i][0]= Rnew[i][0];
      Rold[i][1]= Rnew[i][1];
      Aold[i]= Anew[i];

    }

  }

}
```

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