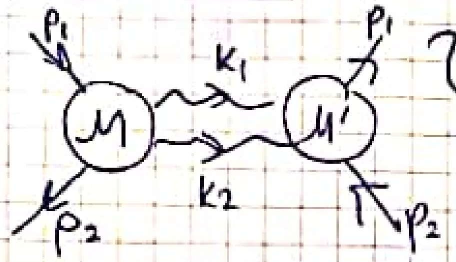
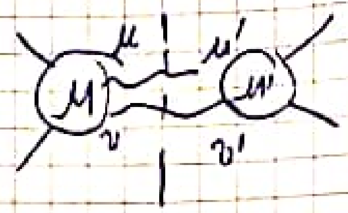
have to put :
- integral over phase space
- sum over polarizations
- momentum conserving δ fun.
- numerator of propagator

Multiparticle threshold

$2\mathrm{Im}\{-i$  $\} =$ 

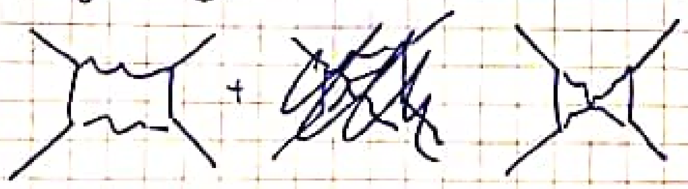
integral over k_1, k_2 + momentum conserving δ

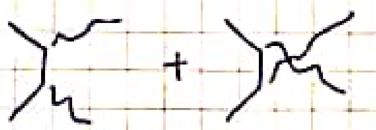
$$= \frac{1}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\mathcal{M}^{\mu\nu}\mathcal{M}'^{\mu'\nu'}\left(g_{\mu\mu'}\,2\pi\delta(k_1^2)\right)\left(g_{\nu\nu'}\,2\pi\delta(k_2^2)\right)$$

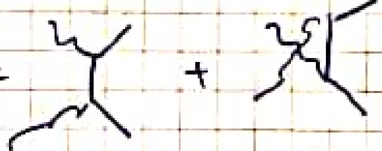
$$(2\pi)^4\delta^4(k_1+k_2-p_1-p_2)$$

momentum conservation

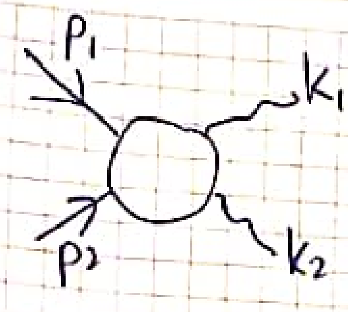
symmetry factor



$i\mathcal{M} =$  } $i\mathcal{M}\cdot i\mathcal{M}'$ double counts the two diagrams above

$i\mathcal{M}' =$ 

$$= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{2E_1 2E_2}\Big[\underbrace{\tfrac{1}{2}\mathcal{M}^{\mu\nu}\mathcal{M}'^{\mu'\nu'}g_{\mu\mu'}g_{\nu\nu'}}_{\text{has to be } \sum_{\text{polarizations}}|\mathcal{M}|^2}\Big](2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2)$$



In CM frame

$k_{1\mu} = (E,0,0,E) = E\, n_\mu$ $\qquad n^2 = \bar{n}^2 = 0$

$k_{2\mu} = (E,0,0,-E) = E\, \bar{n}_\mu$ $\qquad n\cdot\bar{n} = 2.$

We choose: $\varepsilon^+_\mu(k_1) \equiv \varepsilon^+_{1\mu} = \varepsilon_\mu(k_2) \equiv \varepsilon^-_{2\mu} = \frac{1}{\sqrt{2}} n_\mu$

$\varepsilon^-_\mu(k_1) \equiv \varepsilon^-_{1\mu} = \varepsilon^+_\mu(k_2) \equiv \varepsilon^+_{2\mu} = \frac{1}{\sqrt{2}} \bar{n}_\mu.$

$\varepsilon^T_{1\mu}(k_1) = (0,1,0,0)$

$\varepsilon^T_{2\mu}(k_1) = (0,0,1,0).$

Then $\varepsilon^+_\mu \varepsilon^-_{\mu'} + \varepsilon^-_\mu \varepsilon^+_{\mu'} = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}[1\,0\,0\,{-1}] = \begin{bmatrix}1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1\end{bmatrix}.$

$\varepsilon^T_{1\mu}\varepsilon^T_{1\mu'} + \varepsilon^T_{2\mu}\varepsilon^T_{2\mu'} = \cdots \begin{bmatrix}0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0\end{bmatrix}$

$$g_{\mu\mu'} = \varepsilon^{+*}_{1\mu}\varepsilon^-_{1\mu'} + \varepsilon^{-*}_{1\mu}\varepsilon^+_{1\mu'} - \sum_i \varepsilon^{T*}_{i\mu}\varepsilon^T_{i\mu'}$$

$$g_{\nu\nu'} = \varepsilon^{+*}_{2\nu}\varepsilon^-_{2\nu'} + \varepsilon^{-*}_{2\nu}\varepsilon^+_{2\nu'} - \sum_i \varepsilon^{+*}_{i\nu}\varepsilon^T_{i\nu'}$$

$$\mathcal{M}^{\mu\nu}\mathcal{M}^{\mu'\nu'} g_{\mu\mu'}g_{\nu\nu'} = \mathcal{M}^{++}\mathcal{M}'^{--} + \mathcal{M}^{+-}\mathcal{M}'^{-+} - \mathcal{M}^{+T}\mathcal{M}'^{-T}$$
$$+ \mathcal{M}^{-+}\mathcal{M}'^{+-} + \mathcal{M}^{--}\mathcal{M}'^{++}$$
$$- \mathcal{M}^{-T}\mathcal{M}'^{+T} + \mathcal{M}^{T+}\mathcal{M}'^{T-}$$
$$- \mathcal{M}^{T-}\mathcal{M}'^{T+} + \mathcal{M}^{TT}\mathcal{M}'^{TT}$$

$\mathcal{M}^{++} \equiv \mathcal{M}^{\mu\nu}\varepsilon_1^{+*}\varepsilon_2^+$ etc

circled are all zero.

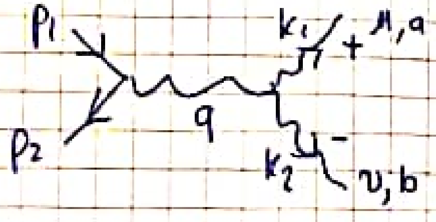
every time we have a + its the same as dotting with k. & must vanish.

but − is not transverse does not vanish.

this is the one that must survive

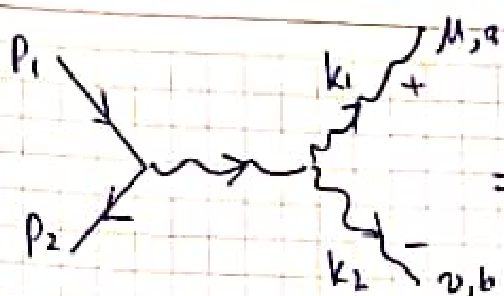
$\therefore \mathcal{M}^{+-}\mathcal{M}'^{-+} + \mathcal{M}^{-+}\mathcal{M}'^{+-}$ must cancel.

At lowest order we saw



$|\overline{\mathcal{M}}|^2 = \sum_{h=\pm} |\mathcal{M}|^2$

$= \sum (\mathcal{M}\varepsilon_1^{T*}\varepsilon_1)(\mathcal{M}^*\varepsilon^T_1 \ldots$



Feynman gauge

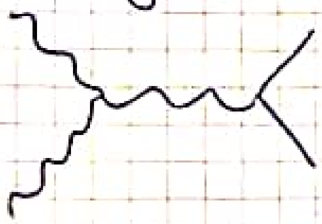
$$= (-i)\, ig\, \bar{v}(p_2)\gamma_\rho T^c u(p_1) \frac{-i}{q^2}\, g f^{abc}$$

$$\times \left[\underbrace{g^{\mu\nu}[k_2-k_1]^\rho}_{n^2=0} + \underbrace{g^{\nu\rho}[-k_1-2k_2]^\mu}_{k_1\cdot n=0} + \underbrace{g^{\rho\mu}[2k_1+k_2]^\nu}_{k \cdot n=0} \right]$$

$$\times \underbrace{\varepsilon_{1\mu}^{+\,*}\, \varepsilon_{2\nu}^{-\,*}}_{\frac{1}{2} n_\mu n_\nu}$$

$$= \boxed{ig^2 f^{abc} \frac{1}{q^2} \bar{v}(p_2) \underbrace{\cancel{k}_1}_{=q-k_2} T^c u(p_1) = \mathcal{M}^{+-}}$$

Similarly:



$$\Rightarrow \boxed{\mathcal{M}'^{-+} = -ig^2 f^{abc} \frac{1}{q^2} \bar{u}(p_1) \cancel{k}_1 T^c v(p_2)}$$

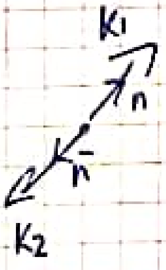
Explanation of last page:

We found in exercise $\quad k_{1\mu} \mathcal{M}^{\mu\nu} \varepsilon_2(k_2) = 0$

$\quad$ iff $k_2 \cdot \varepsilon^+(k_2) = 0$

ε_1^+ is a gauge boson polarized along its momentum.

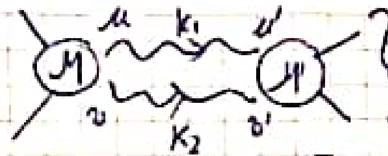
parallel to k_1 $\quad \varepsilon^+(k_1) \propto k_1$



$$\mathcal{M}^{+\square} = \mathcal{M}^{\mu\nu} \underbrace{\varepsilon^+_\mu}_{\substack{\varepsilon^+_\mu(k_1) \\ \propto k_{1\mu}}} \varepsilon^\square_\nu$$

vanishes for $\square = +, T$ but not for $\square = -$

$\varepsilon^\square_\nu(k_2) \cdot k_2 \overset{?}{=} 0$

$2\,\mathrm{Im}\{-i$  $\}$

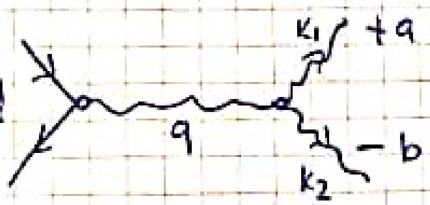
$$= \int \frac{d^3k_1\, d^3k_2}{(2\pi)^3(2\pi)^3}\frac{1}{2E_1\, 2E_2}\Big[\underbrace{\tfrac{1}{2}\mathcal{M}^{\mu\nu}\mathcal{M}'^{\mu'\nu'} g_{\mu\mu'}g_{\nu\nu'}}_{\substack{\text{Unitarity requires} \\ =\sum_{\text{polarisations}}|\mathcal{M}|^2}}\Big](2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2)$$

$$\mathcal{M}^{\mu\nu}\mathcal{M}'^{\mu'\nu'}g_{\mu\mu'}g_{\nu\nu'} = \underbrace{\mathcal{M}^{+-}\mathcal{M}'^{-+}+\mathcal{M}^{-+}\mathcal{M}'^{+-}}+\mathcal{M}^{TT}\mathcal{M}'^{TT}$$

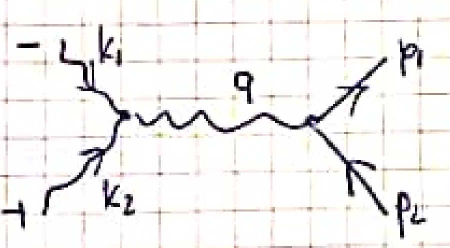
$(\mathcal{M}^{+-} = \mathcal{M}^{\mu\nu}\epsilon^{+*}_{1\mu}\epsilon^{-*}_{2\nu}$, etc$)$

Ward Identity works for + & transverse.

we calculated



$$= ig^2 f^{abc}\frac{1}{q^2}\bar{v}(p_2)\not{k}_1 T^c u(p_1) = \mathcal{M}^{+-} = \mathcal{M}^{-+}$$ ← changing k_1 for $-k_2$

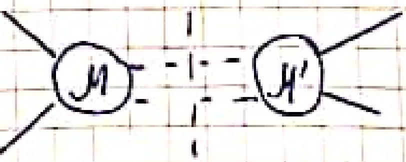


$$= ig^2 f^{abc}\frac{1}{q^2}\bar{u}(p_1)\not{k}_2 T^c v(p_2) = \mathcal{M}'^{-+} = \mathcal{M}'^{+-}$$ ← changing k_2 for $-k_1$

$$\Rightarrow \frac{1}{2}\left[\mathcal{M}^{+-}\mathcal{M}'^{-+}+\mathcal{M}^{-+}\mathcal{M}'^{+-}\right] =$$

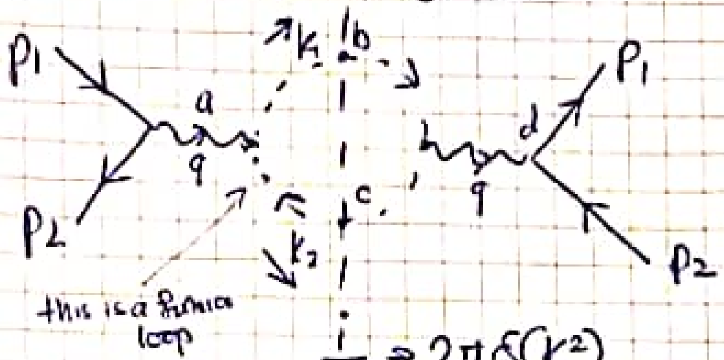
$$= -g^4 f^{abc}f^{abd}\frac{1}{q^4}\bar{u}(p_1)\not{k}_2 T^c v(p_2)\bar{v}(p_2)\not{k}_1 T^d u(p_1) \neq 0.$$

We missed one contribution: (ghosts).



we need to calculate this contribution & show it cancels.

Ghosts only couple to gauge fields. Need to calculate:



$$\frac{i}{k^2} \Rightarrow 2\pi\delta(k_1^2)$$

Check this calculation it has a wrong sign.

$$= \bar{v}(p_2)\,(ig T^a_{\gamma_\mu})\, u(p_1)\, \bar{u}(p_1)\,(ig T^d_{\gamma_\nu})\, v(p_2)$$

$$\times \frac{-i}{q^2}\cdot\frac{-i}{q^2}\,(-1)\,(-g f^{bac} k_1^\mu)(+g f^{cdb} k_2^\nu)$$

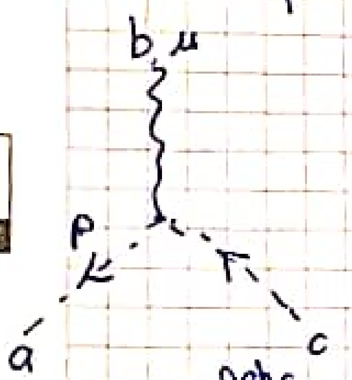
fermion loop.

important, results in the cancellation we need.

$$= g^4 f^{bac} f^{cdb} \frac{1}{q^4}$$

$$\times\ \bar{u}(p_1)\, \cancel{k}_2 T^d v(p_2)$$

$$\times\ \bar{v}(p_2)\, \cancel{k}_1 T^a u(p_1)$$



$$= -g f^{abc} p_\mu$$

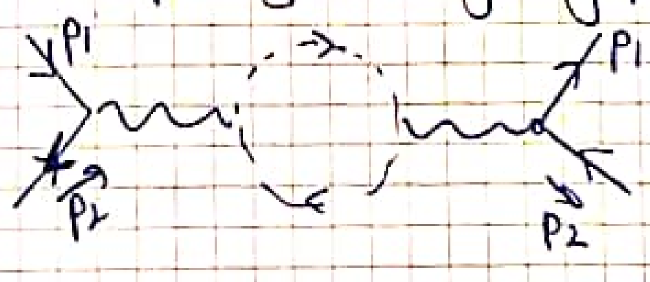
$$= g^4 f^{abc} f^{abd} \frac{1}{q^4}$$

$$\times\ \bar{u}(p_1)\, \cancel{k}_2 T^c v(p_2)$$

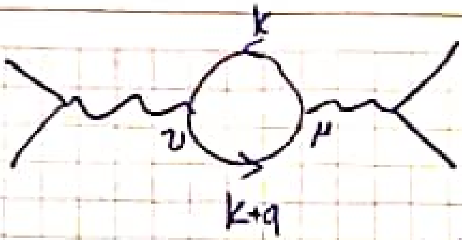
$$\times\ \bar{v}(p_2)\, \cancel{k}_1 T^d u(p_1)$$

$$= \frac{i\delta^{ab}}{p^2}$$

__Exercise__ Check explicitly the imaginary part of:



$$\#k^2+\#$$

In QED: 

$$= \int \frac{d^Dk}{(2\pi)^D} \frac{(-1)\,\mathrm{Tr}\left[(\not k+m)\gamma_\mu(\not k+\not q+m)\gamma_\nu\right]}{(k^2-m^2)((k+q)^2-m^2)}$$

$$\Rightarrow \# \int \frac{d^Dk}{(2\pi)^D} \frac{k^2}{(k^2-m^2)((k+q)^2-m^2)}$$

$$+ \# \int \frac{d^Dk}{(2\pi)^D} \frac{\quad}{(\quad)(\quad)}$$

Something like

$$\log\left[\frac{4m^2-q^2}{m^2}\right]$$

$$q^2 > (4m)^2$$

log of negative # has imaginary part. $\Rightarrow$ for small q^2, purely real, until the is enough energy to produce on shell particles.

the exercise was:

$2\mathrm{Im}\{-i$  $\} = -\frac{Ng^4 J_\mu^{a\dagger} J^{a\mu}}{96\pi q^2}$

$$J^a_\mu = \bar u(p_1)\gamma_\mu T^a v(p_2)$$

$\log(-q^2) = \pm i\pi$ (prob. choose +).

careful have to choose propagator prescription

3. GAUGE THEORIES WITH SSB.

Spontaneous symmetry breaking & the linear sigma model.

Consider a theory of N real scalar fields $\phi \equiv \phi^a \quad a=1,\dots,N$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a)(\partial^\mu\phi^a) - V(\phi^a).$$

We assume that the theory has an $O(N)$ global symmetry.

$$\phi^a \to R^{ab}\phi^b \qquad R\in O(N) \quad \text{N-dim orthogonal matrix.}$$

Then the potential is invariant:

$$V(\phi^a) = F(\phi^a\phi^a) \equiv F(\phi^2).$$

Let's expand $V(\phi^2)$ around its minimum ϕ_0^a:

$$V(\phi) = V(\phi_0) + \cancel{\left[\frac{\partial V}{\partial\phi^a}(\phi_0)\right]}^{\,0}\hat{\phi}^a + \frac{1}{2}\left[\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}(\phi_0)\right]\hat{\phi}^a\hat{\phi}^b + \dots \qquad (1).$$

with $\hat{\phi} = \phi - \phi_0$.

Since ϕ_0 is a minimum: $\left(\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}\right)(\phi_0) \equiv m^2_{ab}$ is a symmetric matrix with non-negative eigenvalues.

Note that m^2_{ab} is the mass matrix for the ϕ^a.

At the renormalizable level, V contains no derivatives and then the condition that V is minimum at every spacetime point requires ϕ_0 = constant. (P.S. if it was nonconstant would also contribute KE.)

Since $V(\phi)$ is $O(n)$ symmetric, we have that for any $R\in O(N)$

$$V(R\phi_0) = V(\phi_0) = V_{min} = V_0$$

and so for $N \geq 2$ there is a continuous set of minima

$$\phi_0^R = R\bar{\Phi}_0 \quad \text{with} \quad \bar{\Phi}_0 \equiv (0,0,\dots,v), \quad v\geq 0$$

a surface of minimum potential.

So we have a continuous family of minima.

The minimization of $V(\phi)$ only fixes

$$\phi_0^2 = v^2$$

What terms could we have in the potential? $1, \phi, \ldots$ — not Lorentz invariant etc.

In terms of $\hat{\phi}^a$:

$$V(\phi) = V_0 + \frac{1}{2} m_{ab}^2 \hat{\phi}^a \hat{\phi}^b + \ldots \quad (2)$$

the theory does not have the $O(N)$ symmetry

$$\hat{\phi} \to R\hat{\phi} \quad \text{for shifted field } \hat{\phi}$$

because this transformation is equivalent to

$$\phi \to R\phi + (R\phi_0 - \phi_0) \quad \text{with } R\phi_0 \neq \phi_0.$$

However, the theory retains the $O(N-1)$ symmetry that leaves ϕ_0 invariant:

$$R'\phi_0 = \phi_0 \Rightarrow \hat{\phi} \to R'\hat{\phi} \Leftrightarrow \phi \to R'\phi.$$

We say that the $O(N)/O(N-1)$ symmetry is HIDDEN or that the $O(N)$ symmetry has been spontaneously broken down to $O(N-1)$.

Infinitesimally, the $O(N)$ transformation is given by

$$\phi^a \to \phi^a + \alpha^A (T^A)^{ab} \phi^b$$
$$= \underbrace{(1 + \alpha^A T^A)^{ab}}_{O(N) \text{ matrix.}} \phi^b$$

with $(1+\alpha \cdot T)(1+\alpha T) = 1$

$$= 1 + \alpha(T^T + T) + O(\alpha^2) \quad \Rightarrow T \text{ is antisymmetric.}$$

The T^A are a set of $\frac{N(N-1)}{2}$ antisymmetric generators of $O(N)$.

Since $V(\phi)$ is $O(N)$ invariant we have:

$$V(\phi^a) = V(\phi^a + (\alpha\cdot T)^{ab}\phi^b)$$
$$= V(\phi^a) + \frac{\partial V}{\partial \phi^b}\alpha\cdot T^{ab}\phi^b + \dots$$

has to hold order-by-order. $\Rightarrow \frac{\partial V}{\partial \phi^a}(T^A)^{ab}\phi^b = 0$.

$$0 = \frac{\partial}{\partial \phi^c}\left[\frac{\partial V}{\partial \phi^a}(T^A)^{ab}\phi^b\right]_{\phi_0}$$

$$= \underbrace{\left(\frac{\partial^2 V}{\partial \phi^c \partial \phi^a}\right)_{\phi_0}}_{m^2_{ca}}(T^A)^{ab}\phi_0^b + \cancel{\left(\frac{\partial V}{\partial \phi^a}\right)_{\phi_0}}^{\,0}(T^A)^{ac}$$

$\Rightarrow$

$$m^2(T^A\phi_0) = 0$$

- If $T^A\phi_0 = 0$, this equation is trivially fulfilled. rotation around axis in direction of ϕ_0.
  - this is the case $\phi_0 \to \phi_0$ (N-1 rotations)
- If $T^A\phi_0 \neq 0$, $T^A\phi_0$ is an eigenvector of M^2 with eigenvalue zero.

  $\Rightarrow$ We say that T^A is a broken generator and it leads to a zero eigenvalue of m^2 which (after diagonalizing M^2) is associated with a massless field called a

  GOLDSTONE BOSON.

This is Goldstone's theorem: if a symmetry G is spontaneously broken to a subgroup H, then there are n massless particles with

$$n = \text{rank}(G/H) = \#\text{ of broken generators}.$$

In our case $G = O(N)$, $H = O(N-1)$ and so

$$n = \text{rank}\left(O(N)/O(N-1)\right) = \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N-1$$

we thus expect $N-1$ massless fields.

<u>Exercise</u> Consider $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) + \frac{1}{2}\mu^2 \phi^a \phi^a - \frac{\lambda}{4}(\phi^a\phi^a)^2$

this is called <u>linear sigma model</u>

(a) Find $v^2 = \phi_0^2$

(b) Take $\phi_0 = (0,0,\dots,0,v)$

with $\vec{\phi}(x) = (\underbrace{\phi^1(x), \phi^2(x), \dots}_{\equiv \pi^i(x)\ \text{massless}}, \underbrace{\phi^N(x) - v}_{\equiv \sigma(x)\ \text{will have mass}})$

write $\mathcal{L}$ explicitly in terms of π^i & σ and identify mass terms.

<u>Pions as Goldstone Bosons</u>

QCD with 2 quark flavors $q^i = \binom{u}{d}$ is the SU(3) gauge theory with 2 fermions in the fundamental rep.

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^2 + \bar{q}_i(i\not{D} - m_i)q_i$$

$$= -\frac{1}{4}F^2 + \bar{q}_L^i\, i\not{D}\, q_L^i + \bar{q}_R^i\, i\not{D}\, q_R^i - m_i(\bar{q}_L^i q_R^i + \bar{q}_R^i q_L^i).$$

We saw in Ch1 that when $m_i = 0$ $\mathcal{L}_{QCD}$ has a global

Same as: $G = U(1)_L \times U(1)_R \times SU(2)_L \times SU(2)_R$

$G = U(2)_L \times U(2)_R$ symmetry:

$$q^i_{L,R} \to U^{ij}_{L,R}\, q^j_{L,R} \qquad U_{L,R} \in U(2)$$

this is called (chiral) flavor symmetry.

*L, R are indistinguishable in this theory.

$q^1 = u$, $q^2 = d$

$u_L = P_L u$ also have

$u_R = P_R u$ Dirac index.

$u = u_L + u_R$

The QCD potential (very complicated due to dynamics of F^2 term, triple couplings etc) seems to have a minimum such that $\langle 0|\bar{q}^i q^i|0\rangle \neq 0$

(this is also related to confinement).

Thus if we redefine fields to express $\mathcal{L}_{QCD}$ in terms of the field

$$\phi = \bar{q}^i q^i = \bar{q}^i_L q^i_R + \bar{q}^i_R q^i_L$$

we would find (presumably) that

$$\phi_0 = v \neq 0 .$$

but under G, ϕ (and ϕ_0) transforms non-trivially:

$$\phi \Rightarrow Tr(\bar{q}_L U_L^\dagger U_R q_R) + Tr(\bar{q}_R U_R^\dagger U_L q_L)$$

this is different from ϕ if $U_L^\dagger U_R \neq 1$

if $U_L = U_R \Rightarrow \phi_0$ does not transform, and thus there is a remaining symmetry:

$$U(2)_V : q_{L,R} \Rightarrow U_V q_{L,R}$$

breaks axial part but not vector part.

A subgroup of G.

The $SU(2)_V$ part of this symmetry is called <u>ISOSPIN</u>, and the $U(1)_V$ part is called <u>baryon #</u>. $U(2)_V = U(1)_V \times SU(2$

The broken set is $G/H = U(2)_L \times U(2)_R / U(2)_V$ which has rank $2\times 4 - 4 = 4$.

The spectrum contains 4 massless Goldstones:

$$\underbrace{\pi^\pm, \pi^0}_{SU(2)_A}, \underset{U(1)_A}{\eta}$$

(anomalous)

- eigenstates of Hamiltonian with zero mass.
- So no hierarchy problem with small mass of pions.

SSB in Gauge theories: the Higgs mechanism

The $U(1)$ case:

$$\mathcal{L} = -\frac{1}{4}F^2 + (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi)$$

$U(1)$ gauge theory with a complex scalar $\phi \in \mathbb{C}$.

(Just 1 field). but two components because complex

$$iD_\mu = i\partial_\mu + gA_\mu$$

We assume that $V(\phi)$ is such that

$$V_{min} = V(\phi_0) \quad \phi_0 \neq 0.$$

As before:

$$\hat{\phi} = \phi - \phi_0$$

Under the $U(1)$ gauge symmetry:

$$\phi_0 \to e^{i\alpha}\phi_0 \neq \phi_0$$

$\Rightarrow$ We expect one massless Goldstone

This is completely analogous to the $O(2)$ linear sigma model!

Take, for example,

$$V(\phi) = -\mu^2\phi^\dagger\phi + \frac{\lambda}{2}(\phi^\dagger\phi)^2$$

nontrivial min

$$= \frac{\lambda}{2}(\phi^\dagger\phi - v^2)^2 - \frac{\lambda v^4}{2} \quad \text{complete the square.}$$

where $v^2 \equiv \frac{\mu^2}{\lambda}$

Then we write: $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$\Rightarrow \phi_0^\dagger\phi_0 = \frac{1}{2}\phi_{01}^2 + \frac{1}{2}\phi_{02}^2 = v^2$$

One possibility: $\phi_{01} = \sqrt{2}\,v$ & $\phi_{02} = 0 \Rightarrow \phi_0 = v \geq 0$

there is a continuous set of minima: $\phi_0^\alpha = e^{i\alpha}\phi_0$

In terms of:

$$\hat{\phi}(x) = \phi(x) - v = \frac{1}{\sqrt{2}}\left(\hat{\phi}_1(x) + i\hat{\phi}_2(x)\right)$$

$$\uparrow \qquad\qquad \uparrow$$

$$\phi_1(x) - \sqrt{2}v \qquad \phi_2(x)$$

then: $V(\phi) = V_0 + \lambda v^2 \hat{\phi}_1^2 + \dots$

$$(D_\mu\phi)^\dagger (D^\mu \phi) = \frac{1}{2}(\partial\hat{\phi}_1)^2 + \frac{1}{2}(\partial\hat{\phi}_2)^2 + g^2v^2A^2 - \sqrt{2}gvA^\mu(\partial_\mu\hat{\phi}_2) + \dots$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4}F^2 + \frac{1}{2}(\partial\hat{\phi}_1)^2 + \frac{1}{2}(\partial\hat{\phi}_2)^2 + \frac{1}{2}m_A^2A^2 - \underline{\sqrt{2}gvA^\mu(\partial_\mu\hat{\phi}_2)} - \frac{1}{2}(2\mu^2)\hat{\phi}_1^2 + \text{interactions}$$

vector field with spin 0.

with $\boxed{m_A^2 = 2g^2v^2}$ a gauge boson mass.

m_A term would be a problem by itself.

but we also have 

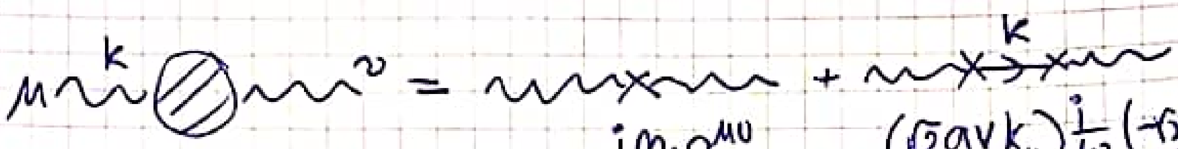
$$= -\sqrt{2}gv p_\mu$$

we have a photon mixing with Goldstone boson!

- A is massive
- $\hat{\phi}_2$ is a goldstone
- A mixes with $\hat{\phi}_2$

A is massive, but the Ward Id still holds!

Before, massive vector bosons broke Ward.
But here, for example:



$$im_A g^{\mu\nu} \qquad (\sqrt{2} g v k_\mu)\frac{i}{k^2}(-\sqrt{2} g v k_\nu)$$

$$= i m_A^2\left(g^{\mu\nu} - \frac{\cancel{2g^2v^2}^{\,1}}{\cancel{m_A^2}}\frac{k^\mu k^\nu}{k^2}\right)$$

$$= i m_A^2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \qquad \text{satisfies Ward.}$$

<u>Recap</u>

$$\mathcal{L} = -\frac{1}{4}F^2 + (D_\mu\phi)^2 - V(\phi)$$

$$D\phi = \partial\phi - igA\phi \qquad -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

$$(D\phi)^2 \sim g^2A^2\phi^2$$
$$\sim g^2 v^2 A^2$$

$V_{min} = V(\phi_0)$

$$\phi_0 \to e^{i\alpha}\phi_0 \neq \phi_0 \qquad \phi = v + \frac{1}{\sqrt{2}}(\hat{\phi}_1 + i\hat{\phi}_2)$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4}F^2 + \frac{1}{2}(\partial_\nu\hat{\phi}_1)^2 + \frac{1}{2}(\partial_\mu\hat{\phi}_2)^2$$
$$+ \frac{1}{2}\underbrace{m_A^2}_{2g^2v^2}A^2 - m_A A^\mu(\partial_\nu\hat{\phi}_2) - \frac{1}{2}(2\mu^2)\hat{\phi}_1$$

$$= i m_A^2 g_{\mu\nu} + m_A^2\frac{i k_\mu k_\nu}{k^2} = i m_A^2\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

We can choose to write: $\phi(x) = (v + \sigma(x))\, e^{i\pi(x)/v}$

Then we can do a gauge transformation:

$$\phi'(x) = e^{-i\alpha(x)}\phi(x)$$
$$A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x)$$

← factors not right. timelike. has effect of removing right amount to fix transverse polarization ↓ Ward? dont need mixing leg anymore.

and we choose $\alpha(x) = \frac{\pi(x)}{v} \Rightarrow \phi'(x) = v + \sigma(x)$.

We have removed π from ϕ and put it into A. (preserving the Lagrangian).

The goldstone boson no longer occurs as a scalar, now it is a polarization of A.

To understand this better, need to quantize.

The d.o.f has been absorbed by the gauge field. We say that the gauge boson has "eaten" the goldstone boson. This gauge is called the UNITARY GAUGE.

The relation of gauge with/without Goldstones becomes clear when we quantize the theory.

<u>General (non-abelian) case</u>

Consider a set of N scalar fields (complex.) transforming in an N-dim representation of a gauge group:

$$\phi_i \to (1 + \alpha^a T^a)_{ij}\phi_j$$

has to be invariant under Lorentz. ↓ $(\langle 0|\phi_i|0\rangle \neq 0.)$

* We use scalar fields to break symmetry because it's the only sort of VEV that has Lorentz symmetry. (But they can be composite operators.) $\boxed{\text{e.g. } \bar{q}q \text{ is a Lorentz scalar.}}$

↑ eg. technicolor

The part of the Lagrangian coupling the scalar field to the gauge field is:

$$(D_\mu\phi^\dagger)\cdot(D^\mu\phi) = (\partial_\mu\phi - igA^a_\mu T^a\phi)^\dagger \cdot (\partial^\mu\phi - igA^{b\mu}T^b\phi)$$

$$(D_\mu\phi)^\dagger(D^\mu\phi) = (\partial_\mu\phi)^2 + ig A^a_\mu\left[(T^a\phi)^\dagger(\partial^\mu\phi) - (\partial^\mu\phi)^\dagger(T^a\phi)\right]$$
$$+ g^2 (T^a\phi)^\dagger (T^b\phi) A^a_\mu A^{b,\mu}$$

Now assume the potential $V(\phi)$ is such that

$$\phi(x) = \phi_0 + \phi(x)$$

(i.e. $\phi_i(x) = \phi_{0i} + \phi_i(x)$, ϕ_{0i} nonzero for some i)

then a mass for A^a appears:

$$\underbrace{g^2 (T^a\phi_0)^\dagger (T^b\phi_0)}_{=\frac{1}{2}m^2_{ab}} A^a_\mu A^{b\mu}$$

this is a gauge boson mass matrix.

*we are not working in unitary gauge in this section.

Let's call:
(notation)
$$F^a_i = (T^a\phi_0)_i$$

The matrix m^2_{ab} is hermitian: $m_{ab}^{*} = 2g^2 F^a_i F^{b*}_i = m_{ba}$

numbers (summed over i).

And then we can take it to be symmetric, and thus real.

$$\frac{1}{2} m^2_{ab} A^a_\mu A^{b\mu} = \frac{1}{2} \underbrace{\frac{m^2_{ab} + m^2_{ba}}{2}}_{M^2_{ab}} A^a_\mu A^{b\mu}$$

real & symmetric

looks more like a mass matrix.

$$M^2_{ab} = g^2\left[F^{a*}_i F^b_i + c.c.\right].$$

Since M^2_{ab} is symmetric & real, it is diagonalizable.

$$M^2_{ab} = (O^T M^2_{Diag} O)_{ab} \quad \text{with } O \text{ orthogonal.}$$

Then we can rotate the gauge fields. (just a linear real rotation).

$$\hat{A}^a = O_{ab}A^b$$

Then: $\frac{1}{2}M^2_{ab}A_\mu A^{b\mu} = \frac{1}{2}m_a^2 \hat{A}^a_\mu \hat{A}^{a\mu}$ where $m_a^2 = \sum_{b,c} O_{ab} M^2_{bc} O_{ac}$

M^2_{bc}

finally $M_a^2 = g^2 O_{ab}[F_i^{b*} F_i^c + cc] O_{ac}$ $\boxed{\hat{F}^a = O_{ab} F^b}$

$= g^2 [\hat{F}_i^{a*} \hat{F}_i^a + cc] = 2g^2 |\hat{F}^a|^2$

and so: $M_a^2 = 2g^2 |\hat{F}^a|^2$

So: • If $\hat{F}^a = O_{ab}(T^b \phi_0) = 0 \Rightarrow \hat{A}^a$ remains massless.

• If $\hat{F}^a = O_{ab}(T^b \phi_0) \neq 0 \Rightarrow \hat{A}^a$ acquires mass $\sqrt{2} g |\hat{F}^a|$

$\therefore$ # of massive gauge bosons = # of broken generators

# of nonzero $\hat{F}^a$s.

(We still have Goldstones here.)

$M_A^2 g_{\mu\nu}$

finally $^a \sim O \sim ^b$ is transverse

$= \sim\times\sim$

$+ \sim\times\to\times\sim$

$\sim \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \delta_{ab} |\hat{F}^a|^2$

$M_A k_\mu \frac{1}{k^2} M_A k_\nu$

F_i^{a*} F_j^{b*}

<u>Exercise</u> (a) Consider the case of a scalar field in the fundamental of SU(2) (a duplet). And assume that

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

Work out the masses of the three gauge bosons & the symmetry remaining after SSB.

(b) Do the same for a field ϕ in the adjoint with VEV $\phi_0 = \frac{1}{\sqrt{2}}(0,0,v)$.

Aside: we can always treat mass terms as vertices.

$$\frac{i(\not{p} + m)}{p^2 - m^2} = \frac{i\not{p}}{p^2} + \frac{i\not{p}}{p^2}\left[(im)\frac{i\not{p}}{p^2}\right] + \dots \text{ geometric series}$$