Lecture 10: Growth of cosmological perturbations in a matter-dominated Universe.

Q: Imagine the Universe were precisely homogeneous. How would it have evolved, what would it look like today?

It would still be precisely homogeneous. After recombining at $z \simeq 1000$ when the age of the Universe was $\sim 4 \times 10^5$ years old, the medium of hydrogen and helium would still be atomic and uniform today, with nothing else happening.

L: The Universe requires rather strange initial conditions: we have evidence of nearly perfect homogeneity and isotropy (distant galaxies and radio sources, CMB), but we also know that initial fluctuations with a small amplitude were present (large-scale structure in the galaxy distribution, CMB anisotropies). These fluctuations have a homogeneous statistical distribution: the density fluctuates, but following a statistical distribution that is the same everywhere.

L: Definition: perturbation in matter density.

$$\delta(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t) - \bar{\rho}}{\bar{\rho}} \; . \tag{1}$$

Note that the minimum value of the mass density fluctuation is -1, in the most empty voids, but can be very large inside galaxies. However, we will be interested in the initial conditions, when $|\delta| \ll 1$ everywhere. In this case, a linear approximation for the evolution of δ works very well.

L: In general, we have many components in the Universe (photons, baryons, dark matter, neutrinos...), and every component can have its own perturbation δ, so we have a coupled evolution of all of them. But we will consider the most simple case: we only have a mass density perturbation, valid for all the mass.

L: We start considering a top-hat spherical perturbation in a matter-only Universe, with no pressure or any force other than gravity. Around a central point, a constant perturbation evolves inside an initial physical radius R_0: $\delta(t)$, with density $\rho(t) = \bar{\rho}(t)[1 + \delta(t)]$. The whole sphere inside R_0 will evolve as a portion of a closed Universe embedded in a flat one.

Q: What is the equation of motion for $R(t)$?

$$\ddot{R} = -\frac{GM}{R^2} = -\frac{4\pi}{3} G\bar{\rho}(1 + \delta)R \; . \tag{2}$$

$$\frac{\ddot{R}}{R} = -\frac{H^2(t)\Omega_m(t)}{2}(1 + \delta) \; . \tag{3}$$

We note that if we can write $\ddot{R}/R$ in terms of δ and its time derivatives, then we have an equation for the evolution of δ where the radius R no longer appears.

L: With this purpose, we note the condition that the mass is constant for a top-hat profile,

$$M = \frac{4\pi}{3} \bar{\rho}(t) \left[1 + \delta(t)\right] R(t)^3 = \text{constant} \; , \tag{4}$$

$$R^{-3}(t) \propto \bar{\rho}(t) \left[1 + \delta(t)\right]^{-1} \; ; \qquad R(t) \propto a(t) \left[1 + \delta(t)\right]^{-1/3} \; . \tag{5}$$

Doing the time derivatives, we find that

$$\frac{\dot{R}}{R} = \frac{\dot{a}}{a} - \frac{\dot{\delta}}{3(1 + \delta)} \; , \tag{6}$$

and

$$\frac{\ddot{R}}{R} = \frac{\dot{a}\dot{R}/R + \ddot{a} - \dot{a}^2/a}{a} - \frac{\dot{\delta}\dot{R}/R + \ddot{\delta} - \dot{\delta}^2/(1+\delta)}{3(1+\delta)} \ . \tag{7}$$

Replacing $\dot{R}/R$ in the second derivative equation,

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{2\dot{\delta}\dot{a}}{3a(1+\delta)} - \frac{\ddot{\delta}}{3(1+\delta)} + \frac{4\dot{\delta}^2}{9(1+\delta)^2} \ . \tag{8}$$

Now, we realise that the equation for the acceleration of the scale factor in the global dynamical evolution of the Universe is:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\bar{\rho}_{\rm tot}}{3} \ , \tag{9}$$

where $\bar{\rho}_{\rm tot}$ includes any uniform energy density there may be in addition to that of matter. Any difference between $\bar{\rho}_{\rm tot}$ and $\bar{\rho}$ would also be incorporated in the equation for $\ddot{R}/R$ when we generalize to a model that contains another component that remains smooth and does not follow the collapse of matter (such as radiation or dark energy). So we are left with the equation for the perturbation,

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \frac{4\dot{\delta}^2}{3(1+\delta)} = 4\pi G\bar{\rho}\delta(1+\delta) \ , \tag{10}$$

or

$$\ddot{\delta} + 2H\dot{\delta} - \frac{4\dot{\delta}^2}{3(1+\delta)} = \frac{3\Omega_m(t)H^2}{2}\delta(1+\delta) \ . \tag{11}$$

Now, we go to the linear regime, $\delta \ll 1$, where we neglect all the second-order terms, and we are left with the first-order equation

$$\ddot{\delta} + 2H\dot{\delta} = \frac{3\Omega_m(t)H^2}{2}\delta \ . \tag{12}$$

In general, this will give us the solution for a model where, in addition to the matter that is collapsing, there is also another component that remains smooth.

For the flat model with $\Omega_m = 1$, we have $H = 2/(3t)$, and

$$\ddot{\delta} + \frac{4\dot{\delta}}{3t} - \frac{4\delta}{3t^2} = 0 \ . \tag{13}$$

which has the general solution

$$\delta = D_1(t/t_0)^{2/3} + D_2(t/t_0)^{-1} = D_1 a + D_2 a^{-3/2} \ . \tag{14}$$

These are the solutions for the growing and decaying mode of cosmological perturbations, respectively, which exist in all cosmological models. These solutions are valid for the flat model with only pressureless matter.

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Now, we derive the perturbation evolution equations under more general conditions, but staying in a non-relativistic formulation, which is valid only for scales much smaller than the horizon and velocities much smaller than the speed of light. We consider a fluid with density ρ and pressure p with a general perturbation.

A fluid is a material where the mean free path between collisions among particles is much shorter than any scale of variation of the fluid properties. We can define fluid elements where fluid properties like density and pressure are approximately constant. The fluid has a bulk velocity $\vec{v}$ (or mean fluid velocity) at every position $\vec{x}$. The fluid evolution equations, in usual proper coordinates, are:

$$\text{Continuity}: \qquad \frac{\partial\rho}{\partial t} + \vec{\nabla}_x \cdot (\rho\vec{v}) = 0 \ . \tag{15}$$

$$\text{Momentum :} \qquad \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}_x)\vec{v} = -\frac{\vec{\nabla}_x p}{\rho} - \vec{\nabla}_x \Phi \ . \tag{16}$$

$$\text{Poisson :} \qquad \nabla_x^2 \Phi = 4\pi G \rho \ . \tag{17}$$

Here, p is the fluid pressure and Φ is the total gravitational potential.

We now change from proper coordinates $\vec{x}$ to comoving ones $\vec{r}$:

$$\vec{x} = a\vec{r} \ ; \qquad \vec{v} = \dot{a}\vec{r} + a\dot{\vec{r}} = \dot{a}\vec{r} + \vec{v}_p \ , \tag{18}$$

where $\vec{v_p}$ is the peculiar velocity. The time derivative is different depending on whether we do it at a fixed proper coordinate $\vec{x}$, or at a fixed comoving coordinate $\vec{r}$:

$$\left.\frac{\partial}{\partial t}\right|_{\vec{x}} = \left.\frac{\partial}{\partial t}\right|_{\vec{r}} + \left.\frac{\partial \vec{r}}{\partial t}\right|_{\vec{x}} \cdot \left.\frac{\partial}{\partial \vec{r}}\right|_{t} = \left.\frac{\partial}{\partial t}\right|_{\vec{r}} - \frac{\dot{a}}{a}\vec{r} \cdot \vec{\nabla}_r \ . \tag{19}$$

Note that the gradient with respect to the comoving coordinates is $\vec{\nabla}_r = a\vec{\nabla}_x$. From now on, we will drop the subindex r in the gradient sign, so gradients are always understood to be evaluated with respect to comoving coordinates. The continuity equation becomes:

$$\frac{\partial \rho}{\partial t} - \frac{\dot{a}}{a}\vec{r} \cdot \vec{\nabla}\rho + \frac{1}{a}\vec{\nabla} \cdot (\rho \dot{a}\vec{r} + \rho \vec{v}_p) = \frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\vec{\nabla} \cdot (\rho \vec{v}_p) = 0 \ . \tag{20}$$

The Poisson equation is modified by expressing the gravitational potential decomposed into a part due to the smooth component and a part due to the perturbation, ϕ. Note that this corresponds to a Newtonian perspective, from a certain coordinate center; in Einstein's theory, only the gravitational potential of the perturbation makes sense.

$$\Phi = \frac{2\pi}{3} G\bar{\rho}x^2 + \phi \ ; \qquad \nabla_x^2 \Phi = 4\pi G\bar{\rho} + \frac{1}{a^2}\nabla^2\phi = 4\pi G\bar{\rho}(1+\delta) \ . \tag{21}$$

So the Poisson equation for the perturbation is

$$\nabla^2 \phi = 4\pi G\bar{\rho}a^2\delta \ . \tag{22}$$

At the same time the gradient of the total potential is replaced by

$$\vec{\nabla}_x \Phi = \frac{4\pi}{3} G\bar{\rho}\vec{x} + \frac{1}{a}\vec{\nabla}\phi = \frac{H^2\Omega_m}{2}\vec{x} + \frac{1}{a}\vec{\nabla}\phi \ . \tag{23}$$

For the momentum equation in comoving coordinates, we first compute:

$$\left.\frac{\partial \vec{v}}{\partial t}\right|_{x} = \left.\frac{\partial \vec{v}}{\partial t}\right|_{r} - \frac{\dot{a}}{a}(\vec{r} \cdot \vec{\nabla})\vec{v} = \ddot{a}\vec{r} + \frac{\partial \vec{v}_p}{\partial t} - \frac{\dot{a}}{a}(\vec{r} \cdot \vec{\nabla})\dot{a}\vec{r} - \frac{\dot{a}}{a}(\vec{r} \cdot \vec{\nabla})\vec{v}_p \ . \tag{24}$$

The second term in the momentum equation is:

$$(\vec{v} \cdot \vec{\nabla}_x)\vec{v} = \frac{\dot{a}}{a}(\vec{r} \cdot \vec{\nabla})\dot{a}\vec{r} + \frac{\dot{a}}{a}(\vec{r} \cdot \vec{\nabla})\vec{v}_p + \frac{\vec{v}_p \cdot \vec{\nabla}}{a}(\dot{a}\vec{r}) + \frac{\vec{v}_p \cdot \vec{\nabla}}{a}\vec{v}_p \ . \tag{25}$$

Replacing this in the momentum equation, and cancelling four terms,

$$\ddot{a}\vec{r} + \frac{\partial \vec{v}_p}{\partial t} + \frac{\vec{v}_p \cdot \vec{\nabla}}{a}(\dot{a}\vec{r}) + \frac{\vec{v}_p \cdot \vec{\nabla}}{a}\vec{v}_p = -\frac{\vec{\nabla}p}{a\rho} - \frac{\vec{\nabla}\phi}{a} - \frac{\dot{a}^2}{a}\frac{\Omega_m}{2}\vec{r} \ . \tag{26}$$

Using also the equation for the global acceleration, $\ddot{a}/a = -(4\pi G\bar{\rho})/3 = -(\Omega_m/2)(\dot{a}/a)^2$, we obtain finally Euler's equation in comoving coordinates:

$$\frac{\partial \vec{v}_p}{\partial t} + \frac{\vec{v}_p \cdot \vec{\nabla}}{a}\vec{v}_p + \frac{\dot{a}}{a}\vec{v}_p = -\frac{\vec{\nabla}\phi}{a} - \frac{\vec{\nabla}p}{a\rho} \ . \tag{27}$$

Now we transform the continuity equation replacing $\rho = \bar{\rho}(1+\delta)$:

$$\frac{\partial \bar{\rho}}{\partial t}(1+\delta) + \bar{\rho}\frac{\partial \delta}{\partial t} + 3\frac{\dot{a}}{a}\rho + \frac{\bar{\rho}}{a}\vec{\nabla} \cdot [(1+\delta)\vec{v}_p] = 0 \ . \tag{28}$$

But $\partial\bar{\rho}/\partial t = -3(\dot{a}/a)\bar{\rho}$, so:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla} \cdot [(1+\delta)\vec{v}_p] = 0 \ . \tag{29}$$

We now consider these equations in the linear regime: we neglect second-order terms containing two perturbation variables δ or $\vec{v}_p$. The linear continuity equation, expressing the time derivative at fixed r as a top dot, is simply

$$a\dot{\delta} = -\vec{\nabla} \cdot \vec{v}_p \ . \tag{30}$$

We now take a time derivative, and replace the time derivative of $\vec{v}_p$ using the momentum equation:

$$a\ddot{\delta} + \dot{a}\dot{\delta} = -\vec{\nabla} \cdot \frac{\partial \vec{v}_p}{\partial t} = \frac{\dot{a}}{a}\vec{\nabla} \cdot \vec{v}_p + \frac{\nabla^2\phi}{a} + \frac{\nabla^2 p}{a\rho} - \frac{\vec{\nabla} p \cdot \vec{\nabla}\rho}{\rho^2 a} \ . \tag{31}$$

The last term is also second order, because it involves a product of perturbative variations of pressure and density, so we have

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = -\frac{\nabla^2\phi}{a^2} + \frac{\nabla^2 p}{a^2\rho} = 4\pi G\bar{\rho}\delta + \frac{\nabla^2 p}{a^2\rho} \ . \tag{32}$$

In the last term, we can replace to first order $\rho = \bar{\rho}$ in the denominator. Also, assuming that the fluid fluctuations behave adiabatically, the sound speed is related to any pressure fluctuations according to $c_s^2 = \partial p/\partial\rho$, at fixed entropy, so the pressure is $p = \bar{p} + \bar{\rho}c_s^2\delta$, and

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\bar{\rho}\delta + c_s^2\frac{\nabla^2\delta}{a^2} \ . \tag{33}$$

We have recovered the same equation as for the spherical collapse, adding now the effect of pressure in an adiabatic fluid. This equation is generally valid in the non-relatistic and linear regime, for a fluid. For collisionless matter things are very similar (instead of pressure we have a velocity dispersion tensor).

L: Note that in the linear regime without pressure, the perturbation stays with a constant shape, only its amplitude evolves. When we introduce pressure, we can have pressure waves, or sound.

Let us consider a particular Fourier mode with wavenumber k to see how waves evolve:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \delta\left(4\pi G\bar{\rho} - \frac{c_s^2k^2}{a^2}\right) \ . \tag{34}$$

There is a critical Jeans wavelength:

$$\frac{k_J}{a} = \frac{\sqrt{4\pi G\bar{\rho}}}{c_s} \ ; \qquad \lambda_J = a\frac{2\pi}{k_J} = c_s\sqrt{\frac{\pi}{G\bar{\rho}}} = \frac{\pi c_s}{H}\sqrt{\frac{8}{3\Omega_m}} \ . \tag{35}$$

The proper Jeans wavelength is λ_J (comoving Jeans wavelength is λ_J/a). A perturbation with $k < k_J$ will gravitationally collapse, but when $k > k_J$ it will oscillate as a wave.

L: In the linear regime without pressure, the general solution to the evolution of a perturbation is

$$\delta(\vec{r},t) = D_1(t)\delta_1(\vec{r}) + D_2(t)\delta_2(\vec{r}) \ . \tag{36}$$

At any given point $\vec{r}$, we have to find the two coefficients that match the value of δ and its time derivative to figure out which combination of growing and decaying mode we have.

Q: As an example: what growing and decaying mode components we have if, at some initial time t_i in a flat universe with matter only, we have a perturbation $\delta_1(\vec{r})$, and the time derivative of the perturbation is zero everywhere? The time derivative is zero if the velocities are zero at t_i (actually, it is enough if the velocity divergence is zero everywhere). The time derivative of δ is:

$$\dot{\delta}(\vec{r}, t_i) = D_1'(t_i)\delta_1(\vec{r}) + D_2'(t_i)\delta_2(\vec{r}) = 0 \ . \tag{37}$$

For the flat universe with only matter, $D_1'(t_i) = 2D_1/(3t_i)$, $D_2'(t_i) = -D_2/t_i$, so

$$\delta_2(\vec{r}) = \delta_1(\vec{r}) \frac{2D_1(t_i)}{3D_2(t_i)} \ , \tag{38}$$

so

$$\delta(\vec{r}, t) = D_1(t)\delta_1(\vec{r}) \cdot \left[1 + \frac{2}{3} \left(\frac{t}{t_i} \right)^{-5/3} \right] \ . \tag{39}$$

At late times, $t \gg t_i$, $\delta(\vec{r}, t) = (3/5)\, \delta(\vec{r}, t_i)(t/t_i)^{2/3}$. So the fluctuation grows according to the growing mode at late times, but initially it loses a factor 3/5 if the velocity starts out at zero.

This is a good model for what happens with a pure baryonic fluctuation at the end of recombination: if baryons are left at rest when they recombine, but with some initial fluctuation, then the fluctuation will have grown at late times by a factor $3/5(a/a_i)$

L: Finally, one can find a general solution for a Universe that contains a cosmological constant in addition to matter, and may have space curvature, for the growing and decaying modes:

$$D_1(a) \propto \frac{\dot{a}}{a} \int_0^a \frac{da'}{(\dot{a}')^3} \ ; \qquad D_2(a) \propto \frac{\dot{a}}{a} \ . \tag{40}$$

As an exercise, you can show this satisfies the equation for the evolution of δ with no pressure, when you have only mass and a cosmological constant, and space curvature.

The result for these functions D_1 and D_2 is what you expect: as curvature or the cosmological constant start dominating, the growing mode stops growing and goes asymptotically to some maximum amplitude. The decaying mode decays more slowly in the open model, and stops decaying in the cosmological constant model.