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Elementary Particles: Week 7

We have the process: $e^+ e^- \longrightarrow \pi^+ \pi^-$, which will have a differential cross section (in the center of mass frame) of:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{\alpha^2 \beta}{4 s e^4} |\mathcal{M}|^2 \quad , \text{ where } \beta = \sqrt{1 - \frac{4m_\pi^2}{s}}$$

The interaction Lagrangian is: $\mathcal{L}_I = -q_e \bar\Psi \gamma^\mu A_\mu \Psi + i q_\pi A^\mu (\partial_\mu \phi^\dagger \phi - \phi^\dagger \partial_\mu \phi) + \underbrace{+ q_\pi^2 A^\mu A_\mu \phi^\dagger \phi}_{\text{Neglect since is quadratic in } q_\pi.}$

Following the steps done for the electron-muon scattering, we obtain:

$$i\mathcal{M} = i^2 \int d^4x \, {}_\gamma\langle 0 | T\{A_\mu(0) A_\nu(x)\} | 0 \rangle_\gamma \; {}_e\langle f | j_e^\mu(0) | i \rangle_e \; {}_\pi\langle f | j^\nu_\pi(x) | i \rangle_\pi$$

where: ${}_e\langle f| = {}_e\langle 0|$; $|i\rangle_e = |e^+_{\lambda_A}(p_A)\, e^-_{\lambda_B}(p_B)\rangle_e$; ${}_\pi\langle f| = {}_\pi\langle \pi^+(p_1), \pi^-(p_2)|$; $|i\rangle_\pi = |0\rangle_\pi$

$$j_e^\mu(0) = q_e \bar\Psi_e(0) \gamma^\mu \Psi_e(0) \; ; \quad j_\pi^\nu(x) = -i q_\pi (\partial^\nu \phi^\dagger(x) \phi(x) - \phi^\dagger(x) \partial^\nu \phi(x))$$

Then:

$${}_\gamma\langle 0 | T\{A_\mu(0) A_\nu(x)\} | 0 \rangle_\gamma = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(0-x)}}{k^2 + i\eta} (-g_{\mu\nu})$$

$${}_e\langle 0 | j_e^\mu(0) | e^+_{\lambda_A}(p_A) e^-_{\lambda_B}(p_B)\rangle_e = {}_e\langle 0| q_e \underbrace{\bar\Psi_e(0)}_{\propto \bar u a^\dagger + \bar v b} \gamma^\mu \underbrace{\Psi_e(0)}_{\propto u a + v b^\dagger} | \underbrace{e^+_{\lambda_A}(p_A)}_{\propto b^\dagger} \underbrace{e^-_{\lambda_B}(p_B)}_{\propto a^\dagger} \rangle_e =$$

$$= q_e \bar v(A) \gamma^\mu u(B)$$

$${}_\pi\langle \pi^+(p_1), \pi^-(p_2) | j_\pi^\nu(x) | 0 \rangle = {}_\pi\langle \underbrace{\pi^+(p_1)}_{\propto \langle 0| c} \underbrace{\pi^-(p_2)}_{\propto \langle 0| d} | -i q_\pi [\partial^\nu \underbrace{\phi^\dagger}_{\propto c^\dagger + d} \underbrace{\phi}_{\propto c + d^\dagger} - \phi^\dagger \partial^\nu \phi] |0\rangle_\pi =$$

$$= -i q_\pi \left[(i p_1^\nu) e^{ip_1 x} e^{ip_2 x} - e^{ip_1 x} (i p_2^\nu) e^{ip_2 x} \right] = q_\pi e^{i(p_1+p_2)x} [p_1^\nu - p_2^\nu]$$

Where we have used that the contraction of the field with the states is:

$$\langle p | \phi(x) = e^{ipx} \quad ; \quad \phi(x)|p\rangle = e^{-ipx}$$

Therefore, we obtain:

$$i\mathcal{M} = \overbrace{i^2}^{-1} \int d^4x \int \frac{d^4k}{(2\pi)^4} \frac{i e^{ikx}}{k^2+i\eta} (-g_{\mu\nu})\, q_e \bar{v}(A)\gamma^\mu u(B)\, q_n e^{i(P_1+P_2)x} (P_1^\nu - P_2^\nu) =$$

$$= i q_e q_n \int d^4k \frac{(g_{\mu\nu})}{k^2+i\eta} \delta(k+P_1+P_2)\bar{v}(A)\gamma^\mu u(B)[P_1^\nu - P_2^\nu] =$$

$$= i q_e q_n \frac{g_{\mu\nu}}{(P_1+P_2)^2} \bar{v}(A)\gamma^\mu u(B)(P_1^\nu - P_2^\nu)$$

Then: $|\mathcal{M}|^2 = \frac{e^4}{s^2} |\bar{v}(A)\gamma^\mu u(B)|^2 (P_1^\nu - P_2^\nu)^2$

Where $e = q_e = q_n$, and $s = (P_1+P_2)^2$.

Since the spin states are not measured, we need to sum over all final spin states:

$$\frac{1}{4}\sum_{\lambda_A,\lambda_B=+,-} (\bar{v}(A)\gamma^\mu u(B))(\bar{v}(A)\gamma^\mu u(B))^\dagger = \frac{1}{4}\sum_{\lambda_A,\lambda_B} \bar{v}(A)\gamma^\mu u(B)\bar{u}(B)\gamma^\mu v(A) =$$

$$= \frac{1}{4} Tr(\gamma^\mu \sum_{\lambda_B} u(B)\bar{u}(B)\gamma^\nu \sum_{\lambda_A} v(A)\bar{v}(A)) = \frac{1}{4} Tr(\gamma^\mu(\not{P}_B + m_e)\gamma^\nu(\not{P}_A - m_e)) =$$

Neglect because $m_e^2 \ll 1$

$$= \frac{1}{4}[Tr(\gamma^\mu \not{P}_B \gamma^\nu \not{P}_A) - \cancel{m_e^2 Tr(\gamma^\mu\gamma^\nu)}] = \frac{4}{4}[P_B^\mu P_A^\nu - g^{\mu\nu} P_B P_A + P_B^\nu P_A^\mu] \equiv L^{\mu\nu}$$

$$|\bar{\mathcal{M}}|^2 = \frac{e^4}{s^2} L^{\mu\nu}(P_1-P_2)_\mu (P_1-P_2)_\nu = \frac{e^4}{s^2}[P_B^\mu P_A^\nu - g^{\mu\nu}P_B P_A + P_B^\nu P_A^\mu][P_{1\mu}P_{1\nu} + P_{2\mu}P_{2\nu} -$$
$$- P_{1\mu}P_{2\nu} - P_{2\mu}P_{1\nu}] =$$

$$= \frac{e^4}{s^2}[(P_B P_1)(P_A P_1) + (P_B P_2)(P_A P_2) - (P_B P_1)(P_A P_2) - (P_B P_2)(P_A P_1) -$$

$$- (P_B P_A)[(P_1P_1) + (P_2P_2) - (P_1P_2) - (P_2P_1)] + (P_AP_1)(P_BP_1) + (P_AP_2)(P_BP_2) - (P_AP_1)(P_BP_2) - (P_AP_2)(P_BP_1)] =$$

$$= \frac{e^4}{s^2}[2(P_BP_1)(P_AP_1) + 2(P_BP_2)(P_AP_2) - 2(P_BP_1)(P_AP_2) - 2(P_BP_2)(P_AP_1) +$$

$$+ (P_BP_A)[-(P_1P_1) - (P_2P_2) + (P_1P_2) + (P_2P_1)]]$$

Since we are working in the CoM frame: $p_B = \bar{p}_A$; $p_2 = \bar{p}_1$. Then:

$$|\bar{\mathcal{M}}|^2 = \frac{e^4}{s^2}\Big[2(\bar{p}_A p_1)(p_A p_1) + 2(\bar{p}_A \bar{p}_1)(p_A \bar{p}_1) - 2(\bar{p}_A p_1)(p_A \bar{p}_1) - 2(\bar{p}_A \bar{p}_1)(p_A p_1) +$$

$$+ (\bar{p}_A p_A)\big[-(p_1 p_1) - (\bar{p}_1 \bar{p}_1) + (p_1 \bar{p}_1) + (\bar{p}_1 p_1)\big]\Big] =$$

$$= \frac{e^4}{s^2}\Big[\underbrace{4(E_1 E_A + |\vec{p}_1||\vec{p}_A|)(E_1 E_A - |\vec{p}_1||\vec{p}_A|)}_{4[(E_1 E_A)^2 - (|\vec{p}_1||\vec{p}_A|)^2]} \underbrace{- 2(E_1 E_A + |\vec{p}_1||\vec{p}_A|)^2 - 2(E_1 E_A - |\vec{p}_1||\vec{p}_A|)^2}_{-2[\cancel{(E_1 E_2)^2} + (|\vec{p}_1||\vec{p}_A|)^2] - 2[\cancel{(E_1 E_A)^2} + (|\vec{p}_1||\vec{p}_A|)^2]} +$$

$$+ (E_A^2 + |\vec{p}_A|^2)\Big[\underbrace{-2(\cancel{E_1^2} - |\vec{p}_1|^2) + 2(\cancel{E_1^2} + |\vec{p}_1|^2)}_{4|\vec{p}_1|^2}\Big] =$$

$$= \frac{e^4}{s^2}\Big[-8(\vec{p}_1 \vec{p}_A)^2 + 4E_A^2|\vec{p}_1|^2 + 4|\vec{p}_A|^2|\vec{p}_1|^2\Big]$$

By using the Mandelstam variables, we should obtain that:

$$|\bar{\mathcal{M}}|^2 \simeq \frac{e^4}{s^2}\Big[\underbrace{-\frac{(u-t)^2}{2}}_{\mathrm{I}} + \underbrace{\frac{s^2}{2}}_{\mathrm{II}} - \underbrace{2m_\pi^2 s}_{\mathrm{III}}\Big]$$

Let's check that we obtain the same:

I $\;-\frac{(u-t)^2}{2} = -\frac{[(p_1-\bar{p}_A)^2 - (p_1-p_A)^2]^2}{2} = -\frac{[\cancel{p_1^2} + \cancel{p_A^2} - 2p_1\bar{p}_A - (\cancel{p_1^2} + \cancel{p_A^2} - 2p_1 p_A)]^2}{2} =$

$$= -2[p_1\bar{p}_A - p_1 p_A]^2 = -2[2(\vec{p}_1 \vec{p}_A)]^2 = \boxed{-8(\vec{p}_1 \vec{p}_A)^2}$$

II $\;\frac{s^2}{2} = \frac{(p_A + \bar{p}_A)^2}{2} = \frac{(2E_A)^4}{2} = 8E_A^4$

III $\;-2m_\pi^2 s = -2p_1^2(p_A + \bar{p}_A)^2 = -2(E_1^2 - |\vec{p}_1|^2)(2E_A)^2 = -8(E_1^2E_A^2 - |\vec{p}_1|^2E_A^2) =$

$$= 4E_A^2|\vec{p}_1|^2 - 8E_1^2E_A^2 + 4E_A^2|\vec{p}_1|^2$$

If we take into account that $E_A^2 = \cancel{m_e^2}^{\,0} + |\vec{p}_A|^2$ and sum what we obtained for II and III, then:

$$\text{II} + \text{III} = 4E_A^2|\vec{P}_1|^2 + 8E_A^2 - 8E_1^2E_A^2 + 4E_A^2|\vec{P}_1|^2 = 4E_A^2|\vec{P}_1|^2 + 8|\vec{P}_A|^2|\vec{P}_A|^2 - 8E_1^2|\vec{P}_A|^2 +$$

$$+ 4|\vec{P}_A|^2|\vec{P}_1|^2 = 4E_A^2|\vec{P}_1|^2 + 4|\vec{P}_A|^2\left[2(E_A)^2 - 2E_1^2 + |\vec{P}_1|^2\right] = \boxed{4E_A^2|\vec{P}_1|^2 + 4|\vec{P}_A|^2|\vec{P}_1|^2}$$

where we have used that:

- $E_A = E_B \rightarrow E_A^2 = \dfrac{2E_AE_B + E_A^2 + E_B^2}{3} = \dfrac{(E_A+E_B)^2}{3}$
- $E_1 = E_2 \rightarrow E_1^2 = \dfrac{2E_1E_2 + E_2^2 + E_1^2}{3} = \dfrac{(E_1+E_2)^2}{3}$

$\Big\}\ 0$

We have checked that we have obtained:

$$|\bar{\mathcal{M}}| = \frac{e^4}{s^2}\left[-\frac{(u-t)^2}{2} + \frac{s^2}{2} - 2m_\pi^2 s\right]$$

Now we express this in terms of the angle θ:

$$-\frac{(u-t)^2}{2} + \frac{s^2}{2} - 2m_\pi^2 s = -\frac{(-2E_A^2\cdot(\cancel{1}+\cos\theta) + 2E_A^2(\cancel{1}-\cos\theta))^2}{2} + \frac{(4E_A^2)^2}{2} - 2m_\pi^2(4E_A)^2 =$$

$$= -\frac{(-4E_A^2\cos\theta)^2}{2} + \frac{(4E_A^2)^2}{2} - 2m_\pi^2(4E_A^2) = 8E_A^4\underbrace{(1-\cos^2\theta)}_{\sin^2\theta} - 8m_\pi^2E_A^2$$

$$|\bar{\mathcal{M}}|_{CoM} = \frac{e^4}{16E_A^2}8E_A^2(E_A^2\sin^2\theta - m_\pi^2) = \frac{e^4}{2}\left(\sin^2\theta - \frac{m_\pi^2}{E_A^2}\right) =$$

$\dfrac{m_\pi^2}{E_A^2} = \dfrac{4m_\pi^2}{s}$; $1 = \cos^2\theta + \sin^2\theta$

$$= \frac{e^4}{2}\sin^2\theta\left[1 - \frac{4m_\pi^2}{s}\left(1 + \frac{1}{\cancel{\tan^2\theta}}\right)\right] = \frac{e^4}{2}\sin^2\theta\underbrace{\left[1 - \frac{4m_\pi^2}{s}\right]}_{\beta^2}$$

$\frac{1}{\tan^2\theta} \simeq 0$, in CoM $\to \theta \sim \frac{\pi}{2}$

Hence: $|\bar{\mathcal{M}}|_{CoM} = \dfrac{e^4\beta}{2}\sin^2\theta$

And we finally obtain:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{\alpha^2\beta}{4se^4}|\mathcal{M}|^2 = \frac{e^4}{2}\sin^2\theta\beta^2\frac{\alpha^2\beta}{4se^4} = \frac{\alpha^2\beta^3\sin^2\theta}{8s}}$$