Lecture 13: Cosmic Microwave Background fluctuations.

L: There is a relation between the primordial density fluctuations in the Universe, and the intensity fluctuations we observe in the CMB, which are often called temperature fluctuations because of the radio-astronomy vocabulary that refers to intensity as brightness temperature.

The physics governing this relation is all determined by general relativity and standard physics of thermodynamics and electromagnetism. We are therefore able to calculate how they relate, for any specific model of how the dark matter and dark energy behave.

L: There is an important scale separating two different regimes for what we observe in the CMB: the horizon at the decoupling epoch, or last scattering surface. On larger scales, fluctuations were still outside the horizon at the time of decoupling: we are observing them now when they were still not in causal contact. On smaller scales, fluctuations were already in causal contact at decoupling and so they could have started to evolve in response to their own gravity and pressure.

Q: What is the physical scale of the horizon at the epoch of decoupling, labeled by the redshift z_d?

$$r_h(z_d) = \frac{c}{1+z_d} \int_{z_d}^{\infty} \frac{dz}{H(z)} \simeq \frac{2c}{H(z_d)} \simeq \frac{2c}{H_0 \Omega_{m0}^{1/2} (1+z_d)^{3/2}} \ . \qquad (1)$$

The last expression assumes the Universe is matter dominated, and is not very accurate because the equalization epoch is at redshift $z_{\rm eq} \simeq 3500$, not much higher than the decoupling epoch. To obtain an accurate result we need to include the radiation as well. This physical scale is about 0.2 Mpc.

Q: What is the angular size of this horizon at decoupling, as we observe it on the sky?

This angular size is obtained by dividing the physical size of the horizon by the angular diameter distance at z_d. For the Benchmark model, the comoving distance to the decoupling epoch is found to be $r(z_d) = 3.18 c/H_0 \simeq 13700$ Mpc, and the angular diameter distance is $D_A(z_d) = r/(1+z_d) = 12.5$ Mpc. So the angular size is:

$$\theta_h(z_d) = \frac{r_h(z_d)}{D_A(z_d)} \simeq 1 \, \text{degree}. \qquad (2)$$

L: On scales larger than this angular size, there has not been any causal influence on the evolution of perturbations. We see the perturbations as they were created in a primordial universe, initially by some unknown process. On smaller scales, the fluctuations were already within the horizon at the epoch of decoupling and they had started evolving according to causal influences, mainly gravitational collapse and radiation pressure.

L: Sachs-Wolfe effect: this is a gravitational effect that implies we see brightness fluctuations (or temperature fluctuations) in the CMB on scales larger than θ_h, because the photons traveling to us have suffered a gravitational redshift and time-delay that fluctuates over the sky. These fluctuations could not be measured at the epoch of decoupling, because they were outside the horizon. They are produced only later as a result of the propagation of CMB photons through the gravitational fields of the cosmological linear fluctuations.

Q: What is gravitational redshift? If there is a gravitational potential on the surface of last scattering $\delta\phi$, photons will be redshifted by

$$\frac{\delta\nu}{\nu} = \frac{\delta T}{T} = \frac{\delta\phi}{c^2} \ . \qquad (3)$$

Wherever ϕ is positive, we are in an underdense region, and photons will suffer a blueshift when they get out of that region, making the CMB hotter. Photons starting from overdense regions (negative potential) will suffer a gravitational redshift and the CMB will look fainter.

L: One important thing to note here is that in the matter-dominated regime, the potential stays constant in the linear regime:

$$\frac{1}{a^2}\nabla^2\phi = 4\pi G\delta \cdot \bar{\rho} \sim \frac{1}{a^2} \ . \tag{4}$$

Because $\delta \propto a$ in the growing mode, and $\bar{\rho} \propto a^{-3}$, the potential stays constant. On the other hand, if radiation or curvature or dark energy dominate, δ grows more slowly so the gravitational potential perturbation decreases with the scale factor. In addition, the gravitational potential perturbation on the scale of the horizon, c/H, is of order δ: $\nabla^2\phi \sim H^2\phi/c^2 \sim 3H^2\delta/2$.

So this means the Sachs-Wolfe effect is always of order the amplitude of primordial perturbations, or the characteristic value of δ at horizon entry.

L: However, at the same time the evolution of the surface of last scattering will suffer a time delay: everything is seen by us to go slower when we look into a region with negative gravitational potential, whereas when we look upward to a region of positive potential everything seems to go faster, by $\delta t/t = \delta\phi/c^2$. So the last scattering surface we see actually has this perturbation in the cosmic time, which causes a fluctuation in the measured temperature $\delta T/T = -\delta a/a$, where the scale factor perturbation δa is the corresponding one to the cosmic time fluctuation δt. In the approximation of a matter-dominated universe, $\delta a/a = (2/3)\delta t/t$, so the fluctuation this gravitational time-delay causes is $\delta T/T = -(2/3)\delta\phi/c^2$.

The total Sachs-Wolfe effect due to gravitational redshift and time delay of the last-scattering surface is:

$$\frac{\delta\nu}{\nu} = \frac{\delta T}{T} = \frac{1}{3}\frac{\delta\phi}{c^2} \ . \tag{5}$$

This factor 1/3 is actually slightly larger when we properly take into account the contribution of radiation to the dynamics of expansion at the epoch of decoupling.

L: Note: the fluctuations on large scales are not caused by the different temperatures themselves on different locations at the same cosmic time! The decoupling always occurs at the same temperature, which is regulated by the physics of atomic recombination. The variation of the cosmic time at the epoch of decoupling that would be measured by a local observer is a smaller effect that the Sachs-Wolfe one.

L: Smaller scales than θ_h: as a fluctuation enters the horizon, the dark matter fluctuation grows while the baryon-photon fluid oscillates in *acoustic oscillations*. These acoustic oscillations are governed by the sound speed,

$$c_s^2 = \frac{\delta p}{\delta\rho} = \frac{\delta\rho_\gamma c^2/3}{\delta\rho_\gamma + \delta\rho_b} = \frac{c^2}{3(1+\delta\rho_b/\delta\rho_\gamma)} = \frac{c^2}{3(1+3\rho_b/4\rho_\gamma)} \ . \tag{6}$$

The physical *sound horizon* is now defined in terms of the sound speed,

$$r_s(z_d) = \frac{1}{1+z_d}\int_{z_d}^{\infty} \frac{c_s(z)dz}{H(z)} \ . \tag{7}$$

In the Benchmark model, $\theta_s \simeq 0.6$ degrees. Below this scale, the acoustic peaks are produced because of the scales at which the oscillation of these waves is caught on certain phases that make the CMB brightness fluctuations more intense, depending on both intrinsic temperature fluctuations and peculiar velocities that induce a Doppler effect.

On smaller scales, the diffusion of photons with respect to the baryons (called *Silk damping*) reduces and smooths the amplitude of the fluctuations.

Q: How do we measure the fluctuations and correlations of the CMB brightness on the sky?

After we have cleaned up all the foregrounds of synchrotron, free-free and dust emission by using the frequency dependence, we expand the CMB temperature map on the celestial sphere in harmonic oscillators:

$$T(\theta, \phi) = \sum_{lm} a_{lm} Y_{lm}(\theta, \phi) \ . \tag{8}$$

The correlation function of the CMB fluctuations is defined in analogy to the correlation function of a field in space:

$$C(\alpha) = < \frac{\delta T(\hat{\mathbf{n}})}{T} \cdot \frac{\delta T(\hat{\mathbf{n}}')}{T} >_{\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' = \cos\alpha} \ . \tag{9}$$

The correlation is computed by averaging over all the pairs of directions that are separated by an angle α. One can show that this correlation function is given in terms of the expected value of the coefficients of the expansion in spherical harmonics:

$$C(\alpha) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell + 1) C_\ell P_\ell(\cos\alpha) \ , \tag{10}$$

where $P_\ell(\cos\alpha)$ are the Legendre polynomials, and

$$C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{+\ell} |a_{\ell m}|^2 \ . \tag{11}$$

The quantities C_ℓ are equivalent to the power spectrum: they represent the power of intensity fluctuations on angular scales $\alpha \sim \pi/\ell$. As before, it is convenient to express the characteristic fluctuation amplitude contributed by modes of a typical scale corresponding to ℓ:

$$(\Delta T)_\ell = \left[\frac{\ell(\ell+1)}{2\pi} C_\ell \right]^{1/2} \ . \tag{12}$$

From the detailed measurement of the C_ℓ, we fit the best cosmological model with six parameters: τ, H_0, Ω_{b0}, Ω_{m0}, and the amplitude and slope of the original power spectrum: $P_i(k) = A(k/k_*)^{-3-\epsilon} = A(k/k_*)^{-4+n_s}$ (free parameters A, n_s). In addition, the space curvature of the universe determined by $1 - \Omega_{m0} - \Omega_{\Lambda 0} - \Omega_{r0}$ is fixed to zero, but it can be added as a 7th parameter and it is found to be consistent with zero (see the Planck collaboration latest paper in 2018).