UNIVERSITAT DE BARCELONA

Facultat de Física

SCHWINGER MODEL

Master Thesis

GUILLERMO ABAD LÓPEZ

JULY, ACADEMIC COURSE 2021/2022

Supervisor:
Bartomeu Fiol Nuñez

AUTHORSHIP STATEMENT OF THE MASTER THESIS

Me, Guillermo Abad López, with National Document of Identity (DNI) 41011884D, student of Universitat de Barcelona, in relation with the memoir of the Master Thesis presented to its defense and evaluation in July of the 2021/2022 academic course, state that:

- The presented document is original and has been realized by my person.
- The work as been done mainly with the objective of evaluating the subject of the master thesis at UB, and has not been previously presented to be evaluated in any other subject, at this one, neither at any other university.
- If any content of the thesis is from published material of third parties, the authorship is clearly given, citing the corresponding sources.
- In the cases where my thesis has been made in collaboration with other investigators and/or students, its is stated with precision, which contributions are taken from third ones and which ones are from my own contribution.
- Except for the previously mentioned points, the presented thesis is of my authorship.

Signed:

EXTENSION STATEMENT OF THE MASTER THESIS

Me, Guillermo Abad López, with National Document of Identity (DNI) 41011884D, student of Universitat de Barcelona, in relation with the memoir of the Master Thesis presented to its defense and evaluation in July of the 2020/2021 academic course, state that:

- The number of total words (according to proposed count) included in the sections from introduction to conclusions is of A words.
- The number of total figures is B.

In total the document, counts:

$$A \text{ words} + B \times 200 \text{ words/figure} = T$$

Which fulfills the normative, having less that LIMIT.

Signed:

Abstract

In physics, the Schwinger model, named after Julian Schwinger [1], is the model describing 1+1 D (1 spatial dimension + time) Lorentzian quantum electrodynamics which includes electrons, coupled to photons.

This model exhibits confinement of the fermions and as such, is a toy model for QCD. A handwaving argument why this is so is because in two dimensions, classically, the potential between two charged particles goes linearly as r, instead of 1/r in 4 dimensions, 3 spatial, 1 time.

This model also exhibits a spontaneous symmetry breaking of the U(1) symmetry due to a chiral condensate due to a pool of instantons. The photon in this model becomes a massive particle at low temperatures. This model can be solved exactly and is used as a toy model for other more complex theories.

Contents

Chapter 1

Introduction

1.1 Introduction to massless 1+1 QED

1.1.1 Equations of motion of QED [2]

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi = \frac{1}{2}A^{\mu}(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)A^{\nu} + \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi \quad (1.1)$$

where $D^\mu = \partial^\mu + ieA^\mu$ is the covariant derivative eq.(5.2), $F^{\mu\nu} = \frac{-i}{e}[D^\mu, D^\nu] = D^\mu A^\nu - D^\nu A^\mu = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength tensor eq.(5.3), $\not{X} = X^\mu \gamma_\mu$ where γ^μ is a Dirac matrix that has to satisfy the Dirac Algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\bar{\psi} = \psi^\dagger\gamma^0$.

The equations of motions from A^ν, ψ and $\bar{\psi}$ for this Lagrangian are:

$$\frac{\partial\mathcal{L}}{\partial A^\mu} - \partial^\nu \frac{\partial\mathcal{L}}{\partial(\partial^\nu A^\mu)} = 0 \quad \rightarrow \quad \partial_\nu F^{\nu\mu} = e\bar{\psi}\gamma^\mu\psi \quad (1.2)$$

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial^\mu \frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)} = 0 \quad \rightarrow \quad (i\not{D} + m)\bar{\psi} = 0 \quad (1.3)$$

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial^\mu \frac{\partial\mathcal{L}}{\partial(\partial^\mu\bar{\psi})} = 0 \quad \rightarrow \quad (i\not{D} - m)\psi = 0 \quad (1.4)$$

from where we see two conserved currents $\partial_\mu(\partial_\nu F^{\nu\mu}) = 0 = \partial_\mu(\bar{\psi}\gamma^\mu\psi)$ due to the anti-symmetry of the field strength tensor.

1.1.2 $U(1)$ Local and global gauge symmetries of QED

Directly from the QED Lagrangian (1.1), we see that this theory has a $U(1)$ local gauge symmetry:

$$\begin{cases} \psi(x) \rightarrow e^{ie\alpha(x)}\psi(x) \\ A^\mu(x) \rightarrow A^\mu(x) - \partial^\mu\alpha(x) \end{cases} \quad (1.5)$$

which, due to Noether theorem, leads to the conserved currents:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\delta A^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = F^{\mu\nu}\partial_\nu\alpha(x) - e\alpha(x)\bar{\psi}\gamma^\mu\psi \quad (1.6)$$

which if the fields fullfill the equations of motion (1.2), become:

$$j^\mu = -\partial_\nu(F^{\nu\mu}\alpha(x)) \quad \rightarrow \quad \partial_\mu j^\mu = -\partial_\mu\partial_\nu(F^{\nu\mu}\alpha(x)) = 0 \quad (1.7)$$

where it's easy to see the current is conserved due to the field strength tensor antisymmetry.

If we now restrain our selves to a $U(1)$ global gauge symmetry ($\alpha(x) \to \alpha$, so it can be factorized out of the currents), we see that current from eq. (1.7) becomes the sames we already found directly from the eq. of motion (1.2). But now we know it's origin is due to the theory being symmetric under a global gauge transformation.

So we are going to define the vector current by the conserved current of this global gauge symmetry:

$$j^\mu_V = \partial_\nu F^{\nu\mu} = \bar{\psi}\gamma^\mu\psi \tag{1.8}$$

1.1.3 Weyl spinors and chiral symmetry of QED

Now rewriting the Lagrangian (1.1) in the corresponding Weyl components:

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\not{D}\psi_R - m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)\psi \tag{1.9}$$

where $\psi_L = P_L\psi = \frac{1+\gamma^5}{2}\psi$ and $\psi_R = P_R\psi = \frac{1-\gamma^5}{2}\psi$ are projections to orthogonal subspaces of the Dirac indices ($P_LP_R = 0$), and $\gamma^5 = \gamma^0...\gamma^{D-1}$, such that we can label the components of the fermion field ψ in this spinor basis as:

$$\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \text{ and } \bar{\psi} = \begin{pmatrix}\bar{\psi}_R & \bar{\psi}_L\end{pmatrix} \tag{1.10}$$

where ψ_L and ψ_R will have, half the original dimensions.

From this new Lagrangian (1.9), we see that the mass term mixes the projections (ψ_L, ψ_R), so if we had massless QED (m=0) we would have a theory with $U(1)_L \times U(1)_R$ symmetries, one independent transformation for each projection:

$$\begin{cases}\psi_L & \to & U_L\ \psi_L \\ \psi_R & \to & U_R\ \psi_R\end{cases} \quad \text{with} \quad \begin{cases}j^\mu_L = \bar{\psi}_L\gamma^\mu\psi_L \\ j^\mu_R = \bar{\psi}_R\gamma^\mu\psi_R\end{cases} \tag{1.11}$$

but we could also express this transformations as a symmetry that transforms both components equally $j^\mu_V = j^\mu_L + j^\mu_R$, which is exactly, the one defined at equation (1.8), and then another symmetry that would transform them in the opposite way $j^\mu_A = j^\mu_L - j^\mu_R$, which is easy to see that will be given by:

$$\begin{cases}j^\mu_V = j^\mu_L + j^\mu_R = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R = \bar{\psi}\gamma^\mu\psi \\ j^\mu_A = j^\mu_L - j^\mu_R = \bar{\psi}_L\gamma^\mu\psi_L - \bar{\psi}_R\gamma^\mu\psi_R = \bar{\psi}\gamma^\mu\gamma^5\psi\end{cases} \tag{1.12}$$

which if we compute its divergence, we would get $\partial_\mu j^\mu_A = 2im\bar{\psi}\gamma^5\psi$, which confirms that such current is conserved when m=0, and that the system is also symmetric under such transformation.

So massless QED will classically have chiral symmetry, the projections of the fermion field (ψ_L, ψ_R) are totally independent, and can transform separately, conserving both currents, j^μ_V and j^μ_A.

The lagrangian of such a theory would be:

$$\mathcal{L}_{QED_{m=0}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{\partial}\psi - ej^\mu_V A_\mu \tag{1.13}$$

but as we are goFing to see in section 1.4, for even numbers of spacetime dimensions (1+1 massless QED for example), we will have a spontaneous symmetry breaking (SSB) in the conservation of the axial current at the quantum level (by radiative corrections), telling us that such symmetry is not compatible with gauge invariance.

1.1.4 Massless QED in 1+1 dimensions [3] [4] (Check $\psi_{r/l} = \psi_{+-} = \psi_0 \pm \psi_1$)

In two dimensions the gamma matrices can be chosen to be:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.14}$$

for which the Dirac fields will only have two Dirac components:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \text{ and } \bar{\psi} = \begin{pmatrix} \psi_R^* & \psi_L^* \end{pmatrix} \tag{1.15}$$

so this time ψ_L and ψ_R will be scalars eigenfunctions of γ^5 having +1 and -1 eigenvalues respectively:

$$\gamma^5\psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ -\psi_R \end{pmatrix} \to \begin{cases} \gamma^5\psi_L = \psi_L \\ \gamma^5\psi_R = -\psi_R \end{cases} \tag{1.16}$$

Using this explicit representation, we can rewrite the fermionic part of the Lagrangian (1.13), as:

$$\mathcal{L}_f = \bar{\psi} i(D_0\gamma^0 + D_1\gamma^1)\psi = \psi_L i(D_0 + D_1)\psi_L + \psi_R i(D_0 - D_1)\psi_R \tag{1.17}$$

which in the free theory would give the following eqs. of motion:

$$\begin{cases} i(\partial_0 + \partial_1)\psi_L = 0 \\ i(\partial_0 - \partial_1)\psi_R = 0 \end{cases} \tag{1.18}$$

the solution to this equations are waves that move to the right and to the left in one dimensional space, at the speed of light.

Having both currents of eq. (1.7) conserved means that the total quantity of waves is conserved ($N_V = N_L + N_R$) and that the proportion of left-right moving is also conserved ($N_A = N_L - N_R$), which is the same as having independently conservation of the quantity of left (N_L) and of the quantity of right (N_R) moving waves. As we commented at the quantum level N_A is not going to be conserved, which mean that left waves will be able to pass to right waves and vice-versa through interactions.

$$\partial_\mu j^\mu = 0 \ \to \ \partial_0 j^0 = -\partial_1 j^1 \ \to \ Q = \int dx\ j^0 \text{ with } \frac{\partial Q}{\partial t} = 0 \tag{1.19}$$

where:

$$Q_L = \int \bar{\psi_L}\gamma^0\psi_L\ dx = \int \psi_L^*\psi_L\ dx = \int |\psi_L|^2\ dx \tag{1.20}$$

$$Q_R = \int \bar{\psi_R}\gamma^0\psi_R\ dx = \int \psi_R^*\psi_R\ dx = \int |\psi_R|^2\ dx \tag{1.21}$$

$$Q_V = \int \bar{\psi}\gamma^0\psi\ dx = \int (\psi_L^*\psi_L + \psi_R^*\psi_R)\ dx = \int (|\psi_L|^2 + |\psi_R|^2)\ dx \tag{1.22}$$

$$Q_A = \int \bar{\psi}\gamma^1\psi\ dx = \int (\psi_L^*\psi_L - \psi_R^*\psi_R)\ dx = \int (|\psi_L|^2 - |\psi_R|^2)\ dx \tag{1.23}$$

It is also important to remark that 2D has the strange property that vector and axial currents are not independent of each other, but follow the relation:

$$\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu \quad \to \quad \gamma^0\gamma^5 = \gamma^1 \text{ and } \gamma^1\gamma^5 = \gamma^0 \tag{1.24}$$

which makes that the time and space components of both currents are related as:

$$j^{\mu}_{A} = -\epsilon^{\mu\nu} j_{V\,\nu} \quad \rightarrow \quad \begin{cases} j_V = \left(\bar{\psi}\gamma^0\psi \ , \ \bar{\psi}\gamma^1\psi\right) \\ j_A = \left(\bar{\psi}\gamma^1\psi \ , \ \bar{\psi}\gamma^0\psi\right) \end{cases} \tag{1.25}$$

If we now focus on the Gauge field part of the Lagrangian (1.13), which is:

$$\mathcal{L}_G = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - ej^{\mu}_{V}A_{\mu} = \frac{1}{2}F_{01} - e(j^0_V A_0 + j^1_V A_1) \tag{1.26}$$

where $F^{\mu\nu} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ and $E = -(\partial_0 A^1 + \partial_1 A^0) = -(\frac{\partial A}{\partial t} + \frac{\partial \phi}{\partial x})$. Which in the free theory would give the following eq. of motion:

$$\partial_{\nu}F^{\nu\mu} = 0 \ \rightarrow \ \ \partial_0 E = 0 \ \text{ and } \ \partial_1 E = 0. \tag{1.27}$$

which tells us that E is constant through spacetime, or what is the same, that there are no electromagnetic waves solutions in d=1+1 dimensions.

In general d dimensional spacetime, the gauge field is Aµ with the index running over $\mu = 0, 1, ..., d-1$. However, not all of these components are physical. The standard way to isolate the physical degrees of freedom is to use the gauge symmetry $A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\omega$ to set $A_0 = 0$ (only having A_1 left as a scalar field, which is the vector potential, $E = -\partial_0 A_1$). This leaves us with only the spatial gauge fields $\vec{A}$. However, we still have to impose the equation of motion for A_0 which is solved by insisting that $\vec{\nabla} \cdot A = 0$. This projects out the longitudinal fluctuations of $\vec{A}$, leaving us just with the transverse modes. The upshot is that the gauge field in d dimensions carries d − 2 physical degrees of freedom. When d = 3 + 1, these are the familiar two polarisation modes of the photon. However, in d = 1+1 dimensions, there are no transverse modes and the electromagnetic field has no propagating degrees of freedom [4].

Applying now the current of the Lagrangian (adding matter) we will have the following eq. of motion:

$$\partial_{\nu}F^{\nu\mu} = ej^{\mu}_{V} \ \rightarrow \ \begin{cases} \partial_0 E = -e\vec{j}(x) \\ \partial_1 E = e\rho(x) \end{cases} \tag{1.28}$$

which for a charge q point particle, and instant current j in all space $\left(j_V = (\rho, \vec{j})\right)$, become:

$$\begin{cases} E_j(t,x) = -e\int j\delta(t)dt = F(x) - ej\theta(t) \\ E_q(t,x) = e\int q\delta(x)dx = F(t) + eq\theta(x) \end{cases} \tag{1.29}$$

which tells us that the electric field has some dynamics given a current (given the presence of matter). Concretely:

- The first one tells us that the Electric field produced by and instant of current j in all space (movements of matter to the right for example) add a constant value for E in all space (is a time theta function that steps precisely after the currents occurs).
- The second one in the other hand tells us that the Electric field produced by a point charge q is a change of constant value over the charge (is a space theta function that precisely steps over the charge)

where F(x) and F(t) are background fields, tipically fixed by values at space and time infinities respectively.

From this discussion we see that the E field of a charge is constant in space instead of going with $1/r^2$ (such as that of an infinite uniform surface charge in $d = 3 + 1$).

If we now look at the full eq. of motion (1.2) for QED, knowing what the currents actually are, we will have:

$$\begin{cases} \partial_0 F^{01} = e\bar{\psi}\gamma^1\psi \;\rightarrow\; -\partial_0 E = e(|\psi_L|^2 - |\psi_R|^2) \\ \partial_1 F^{10} = e\bar{\psi}\gamma^0\psi \;\rightarrow\; \partial_1 E = e(|\psi_L|^2 + |\psi_R|^2) \\ i(\partial_0 + \partial_1)\psi_L = (eA^0 - eA^1)\psi_L = -e(A_0^1 - \int Edt)\psi_L \\ i(\partial_0 - \partial_1)\psi_R = (eA^0 + eA^1)\psi_R = e(A_0^1 - \int Edt)\psi_R \end{cases} \tag{1.30}$$

where wee see that the variation to matter depends on the electromagnetic field and viceversa. Concretely:

- From the first two wee see that matter makes a jump of the field in their position (between charge and anticharge a constant field appears), and that the change from ψ_L to ψ_R generates a background field (anhilation of fermion antifermion into constant electromagnetic background field).
- And then from the last two, we see that the speed at which the fermions (matter) propagates depends on the value of A^1, the value of E by the time it passes.

Finally say that the energy contained in the electric field is:

$$H = \int dx \frac{1}{2e^2} F_{01}^2 \tag{1.31}$$

This means that a classical point charge in d = 1+1 dimensions costs infinite energy. So we conclude that the finite energy states must be charge neutral.

To end this section consider a charge q at position $x = -L/2$ and a charge −q at position $x = +L/2$. We have the equations of motion:

$$\partial_1 F^{01} = qe[\delta(-L/2) - \delta(L/2)] \;\rightarrow\; F^{01} = \begin{cases} qe \text{ between the charges} \\ 0 \text{ outside} \end{cases} \tag{1.32}$$

where we have chosen the integration constant F to ensure vanishing electric field at$x = \pm\infty$. The total energy (1.31) stored in the electric field is:

$$H = \frac{q^2 e}{2} L \tag{1.33}$$

where we see that energy grows linearly with the separation. In other words, **electric charges in $d = 1 + 1$ are classically confined!** The reason is that electric field is forced to form a flux tube (we will see this more clearly after understanding the θ angle, in Figue **fig:flux**), simply because it has nowhere else to go (as happened in the infinite uniformly charged planes in d=3+1, check Figure **??**).

We will see more of this confinement in detail at section 1.8]

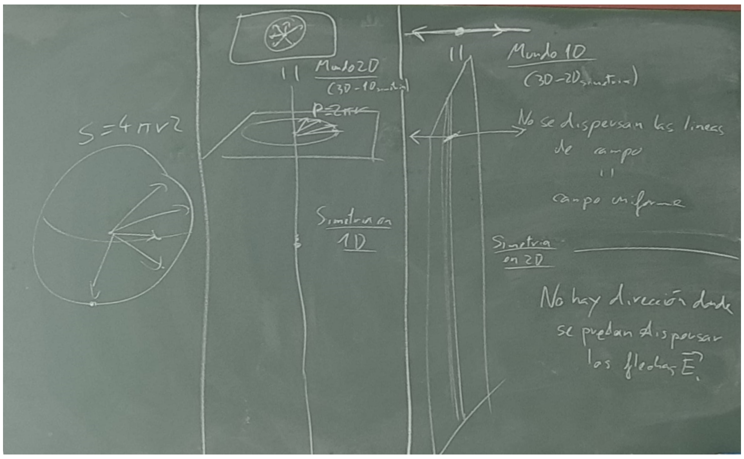


Figure 1.1: Pictures I did in my office about how electric lines can't escape in 1+1D, so the electric field has to be constant.

SAME DISCUSSION (SHORTER), but notes from a YOUTUBE VIDEO (hyperlink):

$$\psi = \begin{pmatrix} \psi_p \\ \psi_e^\dagger \end{pmatrix} \quad \bar{\psi} = \begin{pmatrix} \psi_e \\ \psi_p^\dagger \end{pmatrix} \tag{1.34}$$

where the first has the aniquilation of a positron and the creation of an electron as components, and the second one, the aniquilation of an electron and the creation o a positron.

Setting A_0 as gauge condition guives you:

$$\Pi_a = \dot{A}_1 = F_{01} = -E \tag{1.35}$$

and the equation of motion becomes:

$$\begin{aligned} 0 &= \partial_1 F^{10} - e\bar{\psi}\gamma^0\psi \\ 0 &= \nabla E - ej^0 \\ \nabla E &= e\rho \\ E(x) &= F + \int^x dx' e\rho(x') \end{aligned}$$

where F is a constant background field which is physically significant in 1 + 1 dimensions [5]. Physical states satisfying Gauss' law have zero net charge on the circle. When the space is non-compact, the constant F is naturally fixed by the electric field value at infinity. On a circle of circumference L, we pick an arbitrary point to be the origin, and take it as the lower limit of the integral in (2.3); then F = E(x = 0) = E(x = L).

The E field produced by a charge in 1D should be constant everywhere so, as we see the divergence is 0 except over the particle $\vec{\nabla}\cdot E(x) = e\delta(x) \rightarrow \begin{cases} \vec{\nabla}\cdot E(x) = e\delta(0) & \text{at the charge} \\ \vec{\nabla}\cdot E(x) = 0 & \text{else} \end{cases}$, where the field changes direction.

1.2 Theta Angle on the Line [4] [5] [6] (check factor e^2)

The pure Maxwell theory in d=1+1 has no propagating wave-like solutions, but the theory still allow for constant electric fields. This will be enough to give rise to a Hilbert space in the quantum theory.

As shown in the Annexes 5.5 for four dimensional Maxwell theory, we can add another ingredient to pure Maxwell theory, a θ term, in 1+1D the analogous term will be:

$$\mathcal{L}_\theta = \frac{\theta}{2\pi} F_{01} \quad \rightarrow \quad \mathcal{L} = \frac{1}{2e^2} F_{01}^2 + \frac{\theta}{2\pi} F_{01}$$

which is a total derivative, and does not affect the classical equations of motion, but it does affect the quantum spectrum.

Now, taking the theory to live on $R \times S^1$ we are going to introduce the "zero mode", which is:

$$U(t) = \int_0^{2\pi R} A_1(x,t)dx \tag{1.36}$$

where we see that we have a single degree of freedom rather than an infinite number (one per spatial point). Also $U(t)$ is gauge invariant, dimensionless and periodic.

The dynamics of U follows from the Lagrangian:

$$\mathcal{L} = \frac{1}{4\pi e^2 R}\dot{U}^2 + \frac{\theta}{2\pi}\dot{U} \quad \rightarrow \quad p = \frac{1}{2\pi e^2 R}\dot{U} + \frac{\theta}{2\pi} \quad \rightarrow \quad H = \frac{1}{4\pi e^2 R}\dot{U}^2 = \pi e^2 R \left(p - \frac{\theta}{2\pi}\right)^2$$

which is precisely the problem of a particle moving on a circle in the presence of flux (A flux through S^1, producing a rotating constant electric field, maybe?).

Although the classical physics remains unchanged by θ, there is an important effect in the quantum physics, the energy eigenstates $\psi_l = e^{ilU}$ with $l \epsilon Z$ is:

$$H\psi_l = E_l \psi_l \text{ with } E_l = \pi e^2 R \left(l - \frac{\theta}{2\pi}\right)^2$$

and we see that the spectrum is periodic in θ as expected.

- For $\theta \epsilon (-\pi, \pi)$, the ground state is $l = 0$

- For $\theta = \pm\pi$, there are two degenerate ground states, $l = 0$ and $l = \pm 1$.

If we increase $\theta \rightarrow \theta + 2\pi$, then the spectrum remains the same, but all the states shift by 1. (This is called spectral flow)

1.2.1 The Theta Angle is a Background Electric Field

We can interpret the θ angle in two dimensions simply as a background electric field. Concretely in $A_0 = 0$ gauge this is given by:

$$F_{01} = \frac{1}{2\pi R}\dot{U} = e^2 \left(p - \frac{\theta}{2\pi}\right) \xrightarrow{\text{evaluated on } \psi_l} F_{01} = e^2 \left(l - \frac{\theta}{2\pi}\right) \; l \epsilon \mathcal{Z} \tag{1.37}$$

where we see that this can be thought as describing integrally spaced, constant electric fields, shifted by the θ angle.

If we now expand the limit of R to infinity, where every previous result didn't depend on it, so then the action noting that the θ term is a total derivative, ends as:

$$S = -\int \frac{1}{2e^2} F_{01} F^{01} d^2x + \frac{\theta}{2\pi} \oint A_\mu dx^\mu$$

where the contour integral should be taken around the space-time boundary. Which looks as a Wilson line, with a particles:

- charge $\theta/2\pi$ at $x = -\inf$
- charge $-\theta/2\pi$ at $x = +\inf$

which results in an electric field $F_{01} = -\theta e^2/2\pi$ which agrees with the computation in 1.37

Our discussion above suggest that something happens when $\theta = \pi$: degeneration! These are the states with $l = 0$ and $l = \pm 1$ which have $F_{01} = \pm e^2/\pi$.

If we were to change θ slowly, passing through the value $\theta = \pi$, we jump discontinuously from the background field $F_{01} = -e^2/2$ to $F_{01} = +e^2/2$ which is a first order phase transition. (CHECK SPECTRUM HERE!)

Finally if we include dynamical matter, we will have the fluxes mentioned previously, which are gonna get canceled by this background field, giving totally free particles/antiparticles for some configurations, and totally bounded states for others:

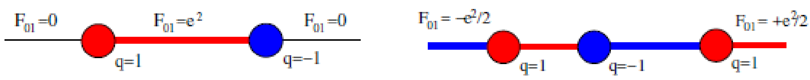


Figure 51: When $\theta = 0$, there is a confining string between particles and anti-particles

Figure 52: When $\theta = \pi$, the string tensions cancel on either side and alternating particles/anti-particles feel no long-distance force.

Figure 1.2: Flux tubes between particles antiparticles, and their effect as θ varies. (In the left the flux is confining, and in the right one no (because the θ field has equated the external electric field to the flux one.

(Read Coleman papers and start computing things!!!)

1.3 Photon develops a mass in 1+1D [7]

The lowest-order vacuum polarization of QED in dimensional regularization in the limit of zero mass is:

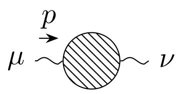
$$i\Pi^{\mu\nu}(q) = i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\frac{2e^2}{(4\pi)^{d/2}}tr[1]\int_0^1 dx\ x(1-x)\frac{\Gamma(2-\frac{d}{2})}{(-x(1-x)q^2)^{2-d/2}} \tag{1.38}$$

which for d=4 is not exactly solvable, and you need to use $\epsilon = 4 - d$ obtaining logarithmic divergences, with branch cuts at the thresholds for creation of a real electron-positron pair.

Instead, for d=2 it is exactly solvable! The 1 particle irreducible (1PI) when d=2 and m=0 is:

$$\mu \sim \text{1PI} \sim \nu \quad = i\Pi^{\mu\nu}(q) = i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\frac{e^2}{\pi q^2} \ \rightarrow \ \Pi(q^2) = \frac{e^2}{\pi q^2} \tag{1.39}$$

and we know that the renormalized propagator



$$= \mu \sim\sim \nu \ + \ \mu \sim \text{1PI} \sim \nu \ + \ \mu \sim \text{1PI}_1 \sim \text{1PI}_2 \sim \nu \ + \ \ldots =$$

$$= \frac{-ig_{\mu\nu}}{q^2(1-\Pi(q^2))} = \frac{-ig_{\mu\nu}}{q^2 - \frac{e^2}{\pi}} \ \rightarrow \ \text{pole at: } \ 1 = \Pi(q^2)\ ;\ q^2 = \frac{e^2}{\pi} \rightarrow \ m_\gamma = \frac{e}{\sqrt{\pi}}$$

from where we have seen that the photon of two dimensional QED is a free massive boson [1].

Also mention, that we have arrived at this result because the fermions are masless, which for the $\Pi(q^2) = \frac{Cte}{q^2}$ gives us a function with a branchcut in the imaginary rieman surfaces, equal to when the photon splits into a pair of fermions.

1.4 ABJ, Chiral or Axial anomaly in 1+1D [7]

In the quantum (massless) theory, both the vector and the axial-vector currents are composite operators that need to be defined.

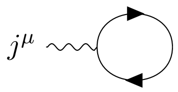
The question is whether these operators can be defined to satisfy the quantum conservation equations:

$$\begin{cases} \partial_\mu \langle j_V^\mu(x)\rangle = 0 \\ \partial_\mu \langle j_A^\mu(x)\rangle = 0 \end{cases} \tag{1.40}$$

where:

$$\langle j^\mu(x)\rangle = \frac{\int D\psi D\bar{\psi} j^\mu(x) e^{i\int d^4x \mathcal{L}}}{\int D\psi D\bar{\psi} e^{i\int d^4x\mathcal{L}}} = \frac{\int D\psi D\bar{\psi} j^\mu(x) e^{i\int d^4x (i\bar{\psi}\partial\!\!\!/\psi - eJ_V^\mu A_\mu)}}{\int D\psi D\bar{\psi} e^{i\int d^4x (i\bar{\psi}\partial\!\!\!/\psi - eJ_V^\mu A_\mu)}} \tag{1.41}$$

Also $\langle j^\mu(x)\rangle$ can be thought as the correlation function given a source:



In momentum space we can compute the excepted value of both currents as:

$$\begin{cases} \langle j_V^\mu(q)\rangle = -\frac{i}{e}(i\Pi^{\mu\alpha}(q)A_\alpha(q)) = -\frac{e}{\pi}(A^\mu(q) - \frac{q^\mu q^\alpha}{q^2}A_\alpha(q)) \\ \langle j_A^\mu(q)\rangle = -\epsilon^{\mu\nu}\langle j_{V\nu}(q)\rangle = \epsilon^{\mu\nu}\frac{i}{e}(i\Pi_{\nu\alpha}(q)A^\alpha(q)) = \epsilon^{\mu\nu}\frac{e}{\pi}(A_\nu(q) - \frac{q_\nu q_\alpha}{q^2}A^\alpha(q)) \end{cases} \tag{1.42}$$

where for the axial current we have used eq. (1.25). Now we are going to try the Ward identity, getting:

$$\begin{cases} q_\mu\langle j_V^\mu(q)\rangle = -\frac{e}{\pi}(q_\mu A^\mu(q) - \frac{q^2 q^\alpha}{q^2}A_\alpha(q)) = 0 \\ q_\mu\langle j_A^\mu(q)\rangle = \frac{e}{\pi}\epsilon^{\mu\nu}(q_\mu A_\nu(q) - \frac{q_\mu q_\nu q^\alpha}{q^2}A_\alpha(q)) = \frac{e}{\pi}\epsilon^{\mu\nu}q_\mu A_\nu(q) \end{cases} \tag{1.43}$$

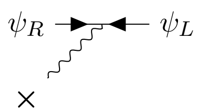
which coming back to position space is:

$$\begin{cases} \partial_\mu\langle j_V^\mu\rangle = 0 \\ \partial_\mu\langle j_A^\mu\rangle = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} = -\frac{eE}{\pi} \end{cases} \tag{1.44}$$

that finally shows us that the axial current is not conserved, which tells us that actually we will have right moving fermions passing to be left moving fermions. As we can see from eq. (1.19) and eq. (1.23):

$$\frac{\partial\langle Q_A\rangle}{\partial t} = -\int \frac{eE}{\pi}\,dx \quad\rightarrow\quad \frac{\partial\langle|\psi_R|^2 - |\psi_L|^2\rangle}{\partial t} = \frac{eE}{\pi} \tag{1.45}$$

which as an operator can also be thought as:



where the $\times$ operator is something like $\epsilon^{\mu\nu}F_{\mu\nu}$, we will see this in more detail in the next chapter [8] [9].

An Example with Fermion Number Nonconservation

To complete our discussion of the two-dimensional axial vector current, we will show that the nonconservation equation (19.18) also has a global aspect. In free fermion theory, the integral of the axial current conservation law gives

$$\int d^2x\, \partial_\mu j^{\mu 5} = N_R - N_L = 0. \tag{19.30}$$

This relation implies that the difference in the number of right-moving and left-moving fermions cannot be changed in any possible process. Combining this with the conservation law for the vector current, we conclude that the number of each type of fermion is separately conserved. From (19.8), we might conclude that these separate conservation laws hold also in two-dimensional QED. However, we have already found that we must be careful in making statements about the axial current.

In two-dimensional QED, the conservation equation for the axial current is replaced by the anomalous nonconservation equation (19.18). If the right-hand side of this equation were the total derivative of a quantity falling off sufficiently rapidly at infinity, its integral would vanish and we would still retain the global conservation law. In fact, $\epsilon^{\mu\nu}F_{\mu\nu}$ is a total derivative:

$$\epsilon^{\mu\nu}F_{\mu\nu} = 2\partial_\mu(\epsilon^{\mu\nu}A_\nu). \tag{19.31}$$

However, it is easy to imagine examples where the integral of this quantity does not vanish, for example, a world with a constant background electric field. In such a world, the conservation law (19.30) must be violated. But how can this happen?

Let us analyze this problem by thinking about fermions in one space dimension in a background A^1 field that is constant in space and has a very slow time dependence. We will assume that the system has a finite length L, with periodic boundary conditions. Notice that the constant A^1 field cannot be removed by a gauge transformation that satisfies the periodic boundary conditions. One way to see this is to note that the system gives a nonzero value to the Wilson line

$$\exp\Big[-ie\int_0^L dx\, A_1(x)\Big], \tag{19.32}$$

which forms a gauge-invariant closed loop due to the periodic boundary conditions.

Following the derivation of the three-dimensional Hamiltonian, Eq. (3.84), we find that the Hamiltonian of this one-dimensional system is

$$H = \int dx\, \psi^\dagger(-i\alpha^1 D_1)\psi, \tag{19.33}$$

where $\alpha = \gamma^0\gamma^1 = \gamma^5$. In the components (19.7),

$$H = \int dx\Big\{-i\psi_+^\dagger(\partial_1 - ieA^1)\psi_+ + i\psi_-^\dagger(\partial_1 - ieA^1)\psi_-\Big\}. \tag{19.34}$$

For a constant A^1 field, it is easy to diagonalize this Hamiltonian. The eigenstates of the covariant derivatives are wavefunctions

$$e^{ik_n x}, \quad \text{with} \quad k_n = \frac{2\pi n}{L}, \quad n = -\infty, \ldots, \infty, \tag{19.35}$$

to satisfy the periodic boundary conditions. Then the single-particle eigenstates of H have energies

$$\begin{aligned} \psi_+ :& \quad E_n = +(k_n - eA^1), \\ \psi_- :& \quad E_n = -(k_n - eA^1). \end{aligned} \tag{19.36}$$

Each type of fermion has an infinite tower of equally spaced levels. To find the ground state of H, we fill the negative energy levels and interpret holes created among these filled states as antiparticles.

Now adiabatically change the value of A^1. The fermion energy levels slowly shift in accord with the relations (19.36). If A^1 changes by the finite amount

$$\Delta A^1 = \frac{2\pi}{eL}, \tag{19.37}$$

which brings the Wilson loop (19.32) back to its original value, the spectrum of H returns to its original form. In this process, each level of ψ_+ moves down to the next position, and each level of ψ_- moves up to the next position, as shown in Fig. 19.2. The occupation numbers of levels should be maintained in this adiabatic process. Thus, remarkably, one right-moving fermion disappears from the vacuum and one extra left-moving fermion appears. At the same time,

$$\begin{aligned} \int d^2x\,\Big(\frac{e}{\pi}\epsilon^{\mu\nu}F_{\mu\nu}\Big) &= \int dt\, dx\, \frac{e}{\pi}\partial_0 A_1 \\ &= \frac{e}{\pi}L(-\Delta A^1) \\ &= -2, \end{aligned} \tag{19.38}$$

where we have inserted (19.37) in the last line. Thus the integrated form of the anomalous nonconservation equation (19.18) is indeed satisfied:

$$N_R - N_L = \int d^2x\,\Big(\frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}\Big). \tag{19.39}$$

Even in this simple example, we see that it is not possible to escape the question of ultraviolet regularization in analyzing the chiral conservation law. Right-moving fermions are lost and left-moving fermions appear from the depths of the fermionic spectrum, $E \to -\infty$. In computing the changes in the separate fermion numbers, we have assumed that the vacuum cannot change the charge it contains at large negative energies. This prescription is gauge invariant, but it leads to the nonconservation of the axial vector current.

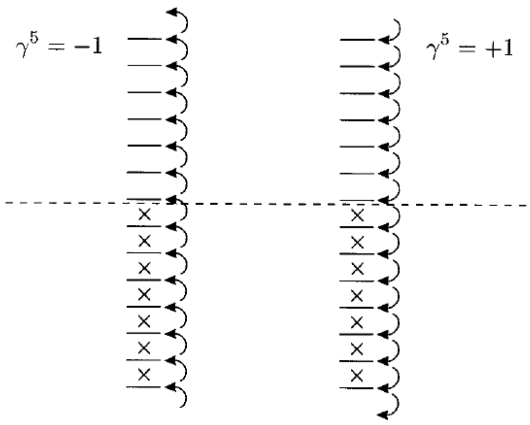
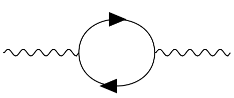


Figure 19.2. Effect on the vacuum state of the Hamiltonian H of one-dimensional QED due to an adiabatic change in the background A^1 field.

If we make this more general we know, from dimensional analisis that:



$$= ie^2(C_1 g^{\mu\nu} - C_2\frac{q^\mu q^\nu}{q^2})$$

for which dimensional regularization set $C_1 = C_2$ previously. If now we keep C_1 setting

$C_2 = 1$ (which is unambiguously determined by the low-energy structure of the theory, since it is the residue of the pole in q^2), we will have:

$$\begin{cases} \langle j_V^\mu(q)\rangle = -\frac{i}{e}(i\Pi^{\mu\alpha}(q)A_\alpha(q)) = -\frac{e}{\pi}(C_1A^\mu(q) - \frac{q^\mu q^\alpha}{q^2}A_\alpha(q)) \\ \langle j_A^\mu(q)\rangle = -\epsilon^{\mu\nu}\langle j_{V\nu}(q)\rangle = \epsilon^{\mu\nu}\frac{i}{e}(i\Pi_{\nu\alpha}(q)A^\alpha(q)) = \epsilon^{\mu\nu}\frac{e}{\pi}(C_1A_\nu(q) - \frac{q_\nu q_\alpha}{q^2}A^\alpha(q)) \end{cases} \tag{1.46}$$

Now trying the Ward identity, we get:

$$\begin{cases} q_\mu\langle j_V^\mu(q)\rangle = -\frac{e}{\pi}(C_1q_\mu A^\mu(q) - \frac{q^2q^\alpha}{q^2}A_\alpha(q)) = -\frac{e}{\pi}(C_1-1)\ q_\mu A(q)^\mu \\ q_\mu\langle j_A^\mu(q)\rangle = \frac{e}{\pi}\epsilon^{\mu\nu}(C_1q_\mu A_\nu(q) - \frac{q_\mu q_\nu q^\alpha}{q^2}A_\alpha(q)) = \frac{e}{\pi}C_1\epsilon^{\mu\nu}q_\mu A_\nu(q) \end{cases} \tag{1.47}$$

where we see that we can't have both Ward Identities being 0 at the same time. Depending on the chose of regularization we can have the anomaly in the vector (j_V) or in the axial current (j_A).

And not having the vector current conserved would be way worse, than the original problem, so after imposing the $C_1 = 1$ the only solution is the previously found.

1.5 Bosonization

1.5.1 Revisit photons develops a mass with Bosonization

1.5.2 Revisit ABJ anomaly with Bosonization

1.6 Phase transition and critical point

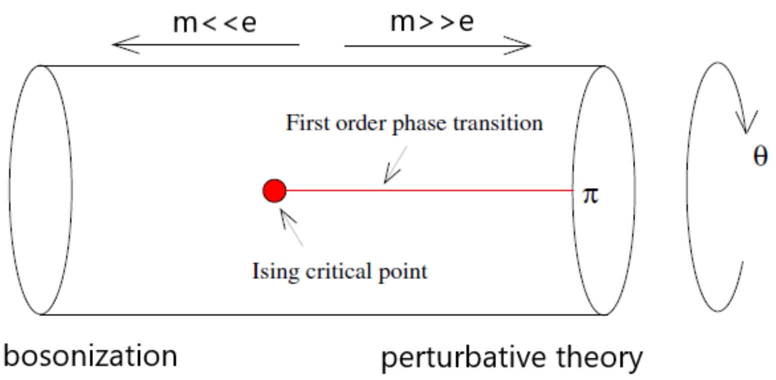


Figure 1.3: The phase diagram of the 2d Schwinger model

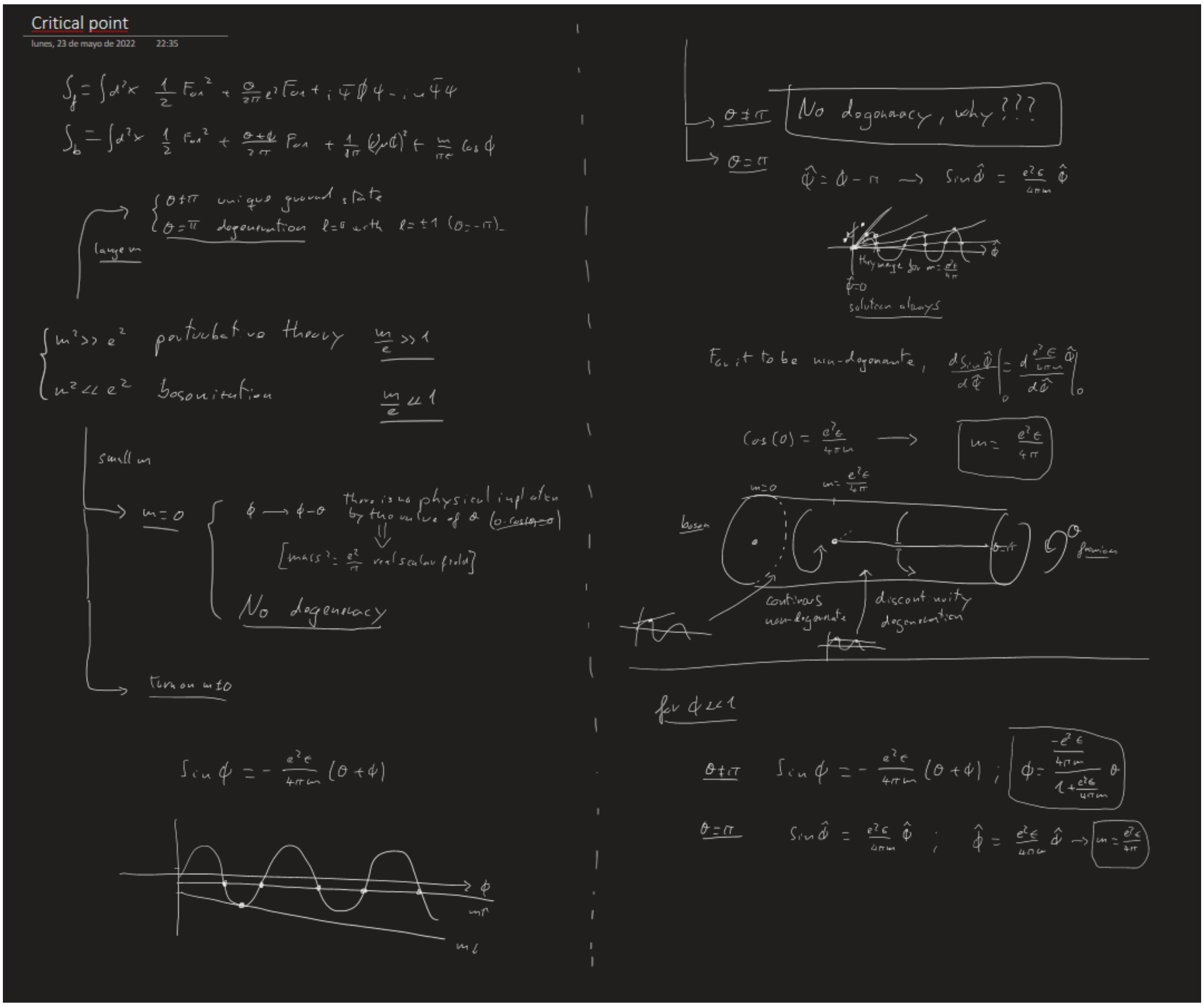


Figure 1.4: Phase transitions, OneNote

1.7 Massless QED in a circle [8] [9] (OLD SECTION (NEEDS UPDATE)!)

Now we are going to fix our spacetime with periodic boundary condition, with period 2π, our model will live in a circle of radius 1.

This system has one degree of freedom, since at a given time, the set of gauge inequivalent field configurations is itself a circle [9]. The only gauge invariant quantity then has to be a phase, which we are going to construct as the Wilson loop variable:

$$e^{i\int_0^{2\pi} A_1(x,t)\ dx} \tag{1.48}$$

and from gauge transformations, with single-valued functions, we can remove the dependence with x, so $A_1(t)$ determines the configuration space, even more we can bring A_1 to the interval [0,1] (this identify what could be regarded as topologically disctinct vacua).

Since $2\pi A_1$ is an angular variable, the wave function $\psi(A_x)$ must be periodic with period 1, a stationary state of the form:

$$\psi(A_1) = e^{2\pi i n A_1(t)} \tag{1.49}$$

.

The Lagrangian density of the model is:

$$\mathcal{L} = \frac{1}{2e^2}(\partial_0 A_1 - \partial_1 A_0)^2 + \bar{\psi} i \not{D} \psi \tag{1.50}$$

The Hamiltonian, in the momentum representation, takes the form

$$H = -\frac{e^2}{4\pi}\frac{\partial^2}{\partial A_x^2} + \sum_p a_{1,p}^\dagger a_{1,p}(p + A_x) + \sum_p a_{2,p}^\dagger a_{2,p}(-p - A_x) + \frac{e^2}{4\pi}\sum_{p\neq 0} j^0(p)\frac{1}{p^2}j^0(-p). \tag{3.14}$$

The fact that A_x is independent of x has been used here. Also, since the total electric charge is zero, the longitudinal part of the electric field $E_x^{\text{long}} = -\partial_x A_t$ has no $p = 0$

226 N. S. MANTON

component, and the potentially singular $p = 0$ component of the Coulomb energy is absent. This is important later.

For fixed A_x, $a_{1,p}^\dagger$ and $a_{1,p}$ are creation and destruction operators for a positive chirality particle of momentum p and energy $p + A_x$. Similarly $a_{2,p}^\dagger$ and $a_{2,p}$ are creation and destruction operators for a negative chirality particle of momentum p and energy $-p - A_x$. We shall refer to these types of particles as left- and right-handed, respectively.

Figure 1.5: Caption

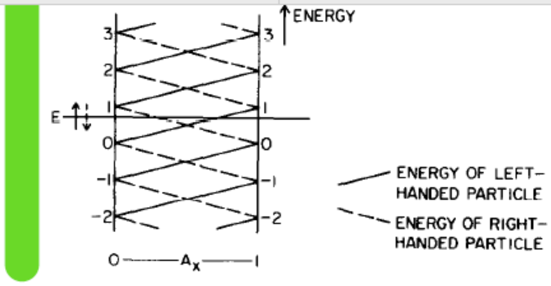


FIG. 2. The one-particle fermionic spectrum as a function of the background gauge potential A_x. The spectral flow at the arbitrarily chosen energy E is two.

However, as A_x increases from 0 to 1, the energies of the left-handed particles each increase by one while the energies of the right-handed particles decrease by one (see Fig. 2). The numbers of energy levels which pass through any given energy value, weighted by the direction of flow, are $+1$ and -1, respectively, for the left- and right-handed particles. The spectral flow [5] associated with the circle of one-dimensional Dirac operators parametrized by A_x is defined to be the difference of these numbers. It equals two.

The spectral flow induces a nontrivial periodicity on the amplitudes. To illustrate this, let us consider unexcited states. Let $\psi_{M,N}(A_x)$ denote the amplitude of the basis state in which left-handed particles of all momenta $\leqslant M$ and right-handed particles of all momenta $\geqslant N$ are present. The boundary conditions that should be imposed are

$$\psi_{M,N}(1) = \psi_{M+1,N+1}(0), \tag{3.15a}$$

$$\partial_{A_x}\psi_{M,N}(1) = \partial_{A_x}\psi_{M+1,N+1}(0). \tag{3.15b}$$

The topologically nontrivial gauge transformation $g(x) = \exp ix$ decreases A_x from 1 to 0, but at the same time it changes the eigenstate of the Dirac operator $\exp ipx$ to $\exp i(p+1)x$, thereby increasing the momenta of all particles by one. Equation (3.15a) therefore identifies the amplitudes for gauge equivalent states. More physically, the Fermi surface $E^{\mathrm{F}}_{\mathrm{L}}$ ($E^{\mathrm{F}}_{\mathrm{R}}$) of a basis state increases (decreases) by one as A_x increases continuously from 0 to 1, because of the spectral flow, and Eq. (3.15a) identifies the amplitudes of basis states with the same Fermi surfaces. $\psi_{M,N}$ near $A_x = 1$ should actually join smoothly on to $\psi_{M+1,N+1}$ near $A_x = 0$, because there is nothing special about these values of A_x if $2\pi A_x$ is regarded as an angle. Equation (3.15b) matches first derivatives, and the Schrödinger equation will ensure that all higher derivatives match too.

Figure 1.6: Caption

For unexcited states $N = M + 1$. Note that A_x cancels in these equations, so that there is a constraint on the basis states that can occur in a physical state, but no constraint on A_x. If A_x had been constrained, the theory would have been inconsistent (cf. Section 5).

The axial charge is

$$\begin{aligned} Q_5^{\mathrm{reg}} &= Q_{\mathrm{L}}^{\mathrm{reg}} - Q_{\mathrm{R}}^{\mathrm{reg}} \\ &= 2E_{\mathrm{L}}^{\mathrm{F}}. \end{aligned} \tag{3.22}$$

For unexcited states

$$Q_5^{\mathrm{reg}} = 2M + 2A_x + 1. \tag{3.23}$$

As A_x increases from 0 to 1, the axial charge changes by 2, as one left-handed particle is created and one right-handed particle is destroyed.

The $\frac{1}{2}$ that occurs in the definitions (3.20) of $Q_{\mathrm{L}}^{\mathrm{reg}}$ and $Q_{\mathrm{R}}^{\mathrm{reg}}$ may appear arbitrary, but is justified on symmetry grounds. One consequence is that an unexcited state with zero electric charge has zero momentum, because the momenta of the left- and right-handed particles together take all integer values precisely once. Furthermore, the momentum of an excited state, which is generally nonzero, is determined by the nature of the excitation, and is independent both of the value of A_x and of the location of the Fermi surfaces.

By considering its momentum, one can also see that a state with nonzero electric charge is unphysical. In an unexcited state of charge one, for example, the particle momenta take all integer values once, and one value occurs twice. The net momentum is this particular value, say, P. But it is not gauge invariant, since it can be changed to $P + 1$ by a topologically nontrivial gauge transformation. There is therefore a momentum anomaly, which is unacceptable.

Figure 1.7: Caption

1.8 Charge shielding and Confinement [6] [10] [11] (OLD SECTION (NEEDS UPDATE)!)

then, unlike free electrodynamics, there is no long-range force between widely separated external charges. This is connected with the Higgs phenomenon. If we were in four dimensions, we would say the photon acquires a mass. However, in two dimensions, there is no photon, even for free electrodynamics, because there are no transverse directions, and all we can say is that the long-range Coulomb force disappears.

(4) Quark trapping occurs. Here, "quark," with some nomenclatural presumption, refers to the fundamental fermion of the theory. If we consider the space of gauge-invariant states, those states that are obtained by applying gauge-invariant operators to the vacua, then this space contains no states that correspond to widely separated quarks. Indeed, the only particle in the theory is a free meson of mass $e/\pi^{1/2}$, which can be thought of as a quark–antiquark bound state. This is connected with charge shielding. If we attempt to separate a quark–antiquark pair, when the separation is sufficiently great, it is energetically favorable for a new pair to materialize from the vacuum. The new quark is attracted to the original antiquark and the new antiquark is attracted to the original quark. This both shields the long-range force and insures that what we are separating is not a quark and an antiquark but two quark–antiquark bound states.

In this paper, we investigate to what extent these properties persist in a generalization of the model in which the fermions are given a mass,

$$\mathscr{L} \to \mathscr{L} - m_o \bar{\psi}\psi. \tag{1.4}$$

gauge-invariant quantities, we do not see the spontaneous breakdown of gauge invariance. However, the spontaneous breakdown of chirality is announced by the appearance of the infinitely degenerate vacua. In the vacuum labeled by θ, the expectation values of $\sigma_\pm$ are $\exp(\pm i\theta)$. Global chiral transformations rotate one vacuum into another. There are no fermions in the theory, nor are there Goldstone bosons; the only particle is a free scalar meson of mass $e/\pi^{1/2}$, etc.

We now turn to the massive model. Eqs. (3.6a) and (3.6b) tell us how to write

Figure 1.8: from paper [6]

FIG. 7. Configuration-space visualization of Fig. 3.

Now consider the space-time development of the final state. Initially the original partons are receding from one another at almost the speed of light. After separating a certain finite average distance, the first generation pair is produced between them (Fig. 7). It is convenient to think of this pair as a dipole $g(r_+ - r_-)$ where $r_\pm$ refers to the position of the quark (antiquark) and g is the quark-gauge-field coupling constant. Subsequently, more pairs are produced and the region containing polarized pairs between the receding original partons spreads out along a line between them (Fig. 8). This type of process can be thought of as an "inside-outside" cascade. It will be convenient to define a dipole density on this line $\phi(z,t)$. The polarization charge density and current are then,

$$\rho = \frac{\partial\phi}{\partial z}, \quad j = -\frac{\partial\phi}{\partial t}. \tag{1}$$

The outer ends of the line of polarized pairs carry net polarization charge which eventually catches up to the original receding charges and neutralizes them (Fig. 9). The process we have described must not only be probable, but it must occur with probability one. We will argue that this is indeed the case when the Schwinger phenomenon[3] occurs.

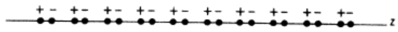


FIG. 9. After time duration $\approx Q$ the line of dipoles catches the outgoing pair. This is the configuration-space visualization of Fig. 6.

The definition of the electric current requires some care in order to satisfy gauge invariance. Define the point-separated current[3]

$$j^\mu(x) = \text{sym} \lim_{\epsilon\to 0} g\bar\psi(x+\epsilon)\gamma^\mu \times \exp\left(ig\int_x^{x+\epsilon} A^\mu dx_\mu\right)\psi(x), \tag{4}$$

where ϵ is a spacelike vector. The equation of motion for j^μ follows[3]:

$$(\Box + m^2) j^\mu(x) = 0, \tag{5}$$

where $m^2 = g^2/\pi$. Thus, the spectrum of the theory contains free massive bosons. If an external current $j^\mu_{ext}(x)$ is introduced into the theory, the equation of motion for j^μ becomes

$$(\Box + m^2) j^\mu(x) = -m^2 j^\mu_{ext}(x). \tag{6}$$

We will introduce a dipole density $\phi(x)$ in analogy with Eq. (1),

$$j^\mu = \epsilon^{\mu\nu}\partial_\nu\phi, \quad j^\mu_{ext} = \epsilon^{\mu\nu}\partial_\nu\phi_{ext},$$
$$\phi(z,t) = \int_{-\infty}^{z} j^0(z',t)dz'. \tag{7}$$

It follows that ϕ satisfies the equation

Figure 1.9: from paper [10]

In order that quarks exist as separate final-state particles it must be posible to have quarks-antiquarks loops with well-separated quarks and antiquarks lines, at least when x and 0 are far apart. This is illustrated in Fig. 6(a). If the quark-antiquark paths are unlikely to separate beyond a fixed size, say 10 cm [see Fig. 6(b)], then clearly no detector will see a quark or antiquark in isolation:

$$\exp\left[-g^2\oint ds^\mu \oint ds'^\nu D_{\mu\nu}(y-y')\right],$$

where $D_{\mu\nu}(y-y')$ is the free gauge-field propagator.

The quark binding mechanism can be seen by comparing the above expression for one space dimension and three space dimensions. In three space dimensions this calculation gives no binding (the binding occurs only with a modified gauge-field action: see Sec. IV), while there is binding in one space dimension. In three space dimensions $D_{\mu\nu}(y-y')$ behaves as $(y-y')^{-2}$. In con-

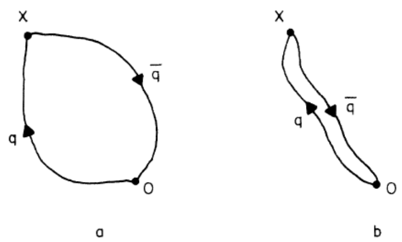


FIG. 6. (a) Loop with well-separated quark and antiquark. (b) Loop with small separation between quark and antiquark.

Figure 1.10: Wilson paper[11]

1.9 Massive model [6] [5] [12] (OLD SECTION (NEEDS UPDATE)!

The massless model which is exactly solvable, has this 4 main interesting properties:

---

(1) The Higgs phenomenon occurs. Local electric charge conservation is spontaneously broken, but no Goldstone boson appears because the Goldstone mode may be gauged away.

(2) The Nambu–Goldstone phenomenon occurs. Global chiral symmetry is spontaneously broken, and the vacuum is infinitely degenerate. Here also no Goldstone boson appears, but for a different reason: The local chiral current is afflicted with an anomaly, and a conserved local current is needed for the Goldstone theorem [3].

(3) Charge shielding occurs. If we couple an external *c*-number current, J_μ, to the theory,

$$\mathscr{L} \to \mathscr{L} - J_\mu A^\mu, \tag{1.3}$$

then, unlike free electrodynamics, there is no long-range force between widely separated external charges. This is connected with the Higgs phenomenon. If we were in four dimensions, we would say the photon acquires a mass. However, in two dimensions, there is no photon, even for free electrodynamics, because there are no transverse directions, and all we can say is that the long-range Coulomb force disappears.

(4) Quark trapping occurs. Here, "quark," with some nomenclatural presumption, refers to the fundamental fermion of the theory. If we consider the space of gauge-invariant states, those states that are obtained by applying gauge-invariant operators to the vacua, then this space contains no states that correspond to widely separated quarks. Indeed, the only particle in the theory is a free meson of mass $e/\pi^{1/2}$, which can be thought of as a quark–antiquark bound state. This is connected with charge shielding. If we attempt to separate a quark–antiquark pair, when the separation is sufficiently great, it is energetically favorable for a new pair to materialize from the vacuum. The new quark is attracted to the original antiquark and the new antiquark is attracted to the original quark. This both shields the long-range force and insures that what we are separating is not a quark and an antiquark but two quark–antiquark bound states.

Figure 1.11: From paper: [6]

Adding a mass to this makes the model unsolvable, but we can do perturbation theory on the mass parameter, and we obtain the following results:

(i) In mass perturbation theory, the long-range force between external charges of arbitrary magnitude does not disappear. From this we infer that the Higgs phenomenon does not occur.

(ii) However, if the external charges are integral multiples of the fundamental charge e then the long-range force *does* disappear. This is an exact result, independent of mass perturbation theory. (Of course, it is also true in mass perturbation theory.) From this, we infer that quark trapping *does* take place.

These conclusions, quark trapping but no Higgs phenomenon, are the same as those that have recently been found for the strong-coupling approximation to lattice gauge theories [5].

Additional insight into the contrast between the massless and the massive models can be obtained by expressing our results diagrammatically. Fig. 1 shows all the Feynman diagrams that contribute to the interaction between external charges. The direct exchange of an electromagnetic propagator, Fig. 1a, gives a long-range force in both the massless and massive models. In the massless model, this force is cancelled by the first graph in Fig. 1b; the vacuum polarization function has a pole at zero momentum that cancels the pole in Fig. 1a. The

guiablo

[5] is Wilson quark confinement paper

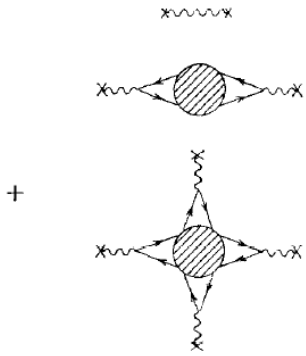


Figure 1.12: From paper: [6]

270 COLEMAN, JACKIW AND SUSSKIND

remaining diagrams in Fig. 1b make no contribution to the long-range force. (Indeed, they vanish identically.) The situation is very different for the massive model. Here, Fig. 1a is cancelled not just by the first graph in Fig. 1b, but by the whole sum, and then only for special values of the external charges. In particular, this implies that the vacuum polarization function has no pole at zero momentum, in agreement with the results of ordinary perturbation theory (in e).

Figure 1.13: From paper: [6]

Chapter 2

Hands on

2.1 Computation of critical point for $\theta = \pi$

Expand: $M = c_0(\frac{m}{e})^0 + c_1(\frac{m}{e})^1 + c_2(\frac{m}{e})^2$ and then set $M = 0$ in the critical point, due the scale invariance of the 2nd order phase transition.

Which will give $m/e \approx 0.3335$

2.2 Computation of the full spectrum for Bosonization (m«e)

2.3 Computation of the full spectrum for typical perturbation theory (m»e)

2.4 Comparison of the spectrums of the two regimes.

Chapter 3

Results and analysis

Chapter 4

Conclusions

Chapter 5

Apendices

5.1 The geometry of Gauge Invariance [13] [14]

We stipulate that our theory should be invariant under a local phase rotation:

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x)$$

which means gauge symmmetry is not an accidental curiosity but rather the fundamental principle under which we will construct the theory.

For the mass term there is no problem, the lagrangian is invariant, but the derivatives term is not, because each term of the derivative transforms with a different $alpha(x)$:

$$n^\mu \partial_\mu \psi = \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon}[\psi(x + \epsilon n) - \psi(x)]$$

but this subtraction will depend on the gauge transformation, so this derivative doesn't have a useful geometric interpretation (we are considering $\alpha(x)$ varies like crazy even in infinitesimally separated positions).

In order to do this substraction in a meaningful wy we need, we must introduce a factor that compensates for the difference in the phase transformation between infinitesimally separated points:

$$U(y, x) \rightarrow e^{i\alpha(y)}U(y, x)e^{-i\alpha(x)}$$

where $U(x, x) = 1$. Now the objects $\psi(y)$ and $U(y, x)\psi(x)$ transform equally and we can substract them in a meaningful way, which an invariant derivative, which is the already known covariant derivative:

$$n^\mu D_\mu \psi = \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon}[\psi(x + \epsilon n) - U(x + \epsilon n, x)\psi(x)]$$

which to make this explicit we need and expression for the comparator $U(y, x)$. In the infinitesimal separation between two points it can be expanded as:

$$U(x + \epsilon n, x) = 1 - ie\ \epsilon n^\mu A_\mu(x) + O(\epsilon^2) \approx e^{-ie\ \epsilon n^\mu A_\mu(x) + O(\epsilon^2)} \tag{5.1}$$

where $A_\mu(x)$ are the coeficients for the phase gauge variation of the displacement in the direction. Such a field, which appears as the infinitesimal limit of a comparator of local symmetry transformations is called a **connection**. The covaraint derivative then takes the form:

$$D_\mu \psi(x) = \partial_\mu \psi(x) + ieA_\mu \psi(x) \tag{5.2}$$

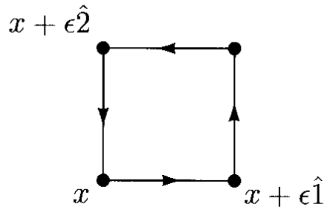


Figure 5.1: Construction fo the field strenght by comparisons around a small square in the (1,2) plane.

where A_μ transforms under this local phase rotation as:

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

We have seen that even the very existance of the vector field A_μ follows from the postulate of local phase rotation symmetry, being it the connection of that symmetry.

Finally we are going to show that if we take the comparator U(y,x) doing closed loops in different directions, such as the ones in Figure 5.1, we get values $\neq 0$, there is curvature!:

$$\begin{aligned}\mathbf{U}(x) =& U(x, x+\epsilon 2)U(x+\epsilon 2, x+\epsilon 1+\epsilon 2)U(x+\epsilon 1+\epsilon 2, x+\epsilon 1)U(x+\epsilon 1, x) = \\ =& 1 - i\epsilon^2 e[\partial_1 A_2(x) - \partial_2 A_1(x)] + O(\epsilon^3)\end{aligned}$$

Therefor the structure:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.3}$$

is locally invariant and represent the gauge phase gained when doing a closed loop in the directions x_μ and x_ν. This is the familiar electromagnetic field tensor, which we now see that is the first order term of the **curvature** of the Gauge symmetry manifold!!!

This can also be understood when we see that:

$$[D_\mu, D_\nu] = ieF_{\mu\nu}$$

which is nothing more than seeing that the $F_{\mu\nu}$ is the comparison of comparisors $U(y,x)$ across a square.

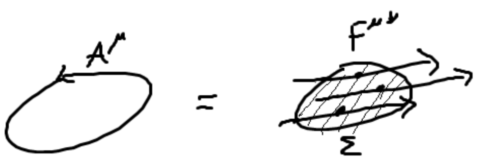
So we have seen that the comparator between two point x and y at finite separation depends on the path taken from x to y. If we now want to extend the comparator for any finite separation we will obtain what is called a **Wilson line**:

$$U_P(y,x) = e^{-ie\int_P dz^\mu A_\mu(x)}$$

which depends on the path. If we then close this path we obtain a **Wilson loop**:

$$U_P(y,y) = e^{-ie\oint_P dz^\mu A_\mu(x)} = e^{-i\frac{e}{2}\int_\Sigma d\sigma^{\mu\nu}F_{\mu\nu}}$$

Where we see that how the connection (A_μ) changes through a closed lines is equal to how much curved Σ is ($F^{\mu\nu}$ travesing Σ)!!!



5.2 Wilson lines and Wilson loops [15]

5.2.1 Definition and basic properties

QED Wilson lines (crange compensators):

$$U(\Gamma_{y,x}) = e^{ie\int_{\Gamma}{}_{y,x} dz^{\mu} A_{\mu}(z)} \tag{5.4}$$

with a path Γ that goes from x to y.

Under the gauge transformation:

$$\begin{cases} A_{\mu}(z) \to \frac{1}{e} A_{\mu}(z) + \partial_{\mu}\alpha(z) \\ \psi(x) \to e^{i\alpha(x)}\psi(x) \end{cases}$$

then you can define a covariant derivative that transforms equally to the field:

$$D_{\mu}\psi(x) \to e^{i\alpha(x)} D_{\mu}\psi(x)$$

where $D_{\mu} = \partial_{\mu} + ieA\mu$ is exactly given by the connections of the gauge manifold, by the factors ieA_{μ} which are also the exponent of the Wilson line phase factor (that is where the gauge phase comes, after traveling through the Wilson line). This Wilson line will then transform as:

$$U_{\gamma}(y,x) \to e^{i\alpha(y)} U_{\gamma}(x,y) e^{-i\alpha(x)}$$

therefor the Wilsonian line will transform as the product:

$$U(\Gamma_{y,x}) \sim \text{"}\psi(y)\psi^{\dagger}(x)\text{"}$$

And a field at the point x will transform like one at the point y after a multiplication by this Wilson line phase factor:

$$U(\Gamma_{y,x})\psi(x) \sim \text{"}\psi(y)\text{"}$$

where this time we see excplicitly that the Wilson line phase factor plays a role of the parallel transporter in an electromagnetic field, and to comparre phases of a wave function at points x and y, we should first make a parallel transport along some contour $\Gamma_{y,x}$. This result is Γ dependent expect when $A_{\mu}(z)$ is a pure gauge (vanishin field strnegh $F_{\mu\nu}$).

We can then obviously close this $\Gamma_{y,x}$ and obtain what is called a Wilson loop:

$$U(\Gamma) = e^{ie\oint_{\Gamma} dz^{\mu} A_{\mu}(z)} \tag{5.5}$$

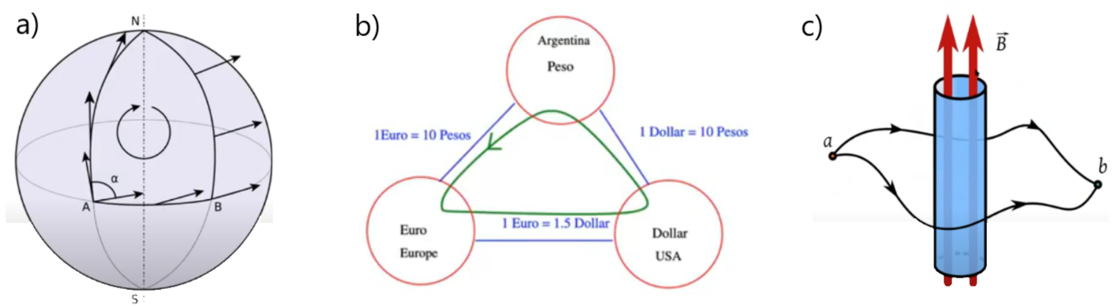


Figure 5.2: a) is an analogy to the "phase" obtained to the vector with paral · lel transport in a spacetime manifold, b) is an analogy with currency where in a closed loop you obtain a gain value, c) is the actual thing where we get a phase out of a closed loop (which is constructed with the connection of the gauge manifold [the same thing with which we construct the covariant derivative ieA_{μ}]).

5.2.2 Propagator in external field and Aharonov-Bohm effect

For the propagator we get:

$$G(x,y;A) = \sum_{\Gamma_{y,x}} e^{-S_{\text{free}}(\Gamma_{y,x})+ie\int_{\Gamma_{y,x}} dz^\mu A_\mu(z)} \tag{5.6}$$

where the exponent is just the classical (Euclidean) action of a particle in an external electromagnetic field.

If we now consider the system shown in 5.2 b), with more detail:

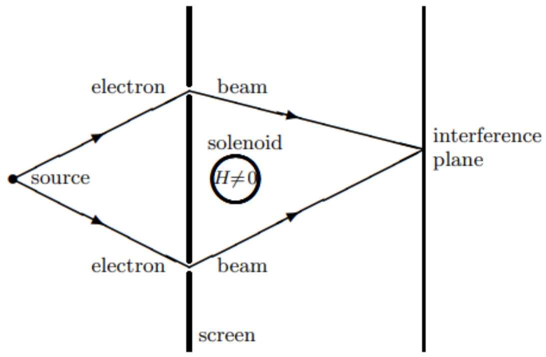


Figure 5.3

Transverse components of the electromagnetic field describe photons. Longitudinal components are related to gauging the phase of a wave function, i.e. permit one to compare its values at different space-time points when an electron is placed in an external electromagnetic field.

In quantum mechanics, the wave-function phase itself is unobservable. Only the phase differences are observable, e.g. via interference phenomena. The phase difference depends on the value of the phase factor for a given path yx along which the parallel transport is performed.

The phase factors are observable in quantum theory, in contrast to classical theory. This is seen in the Aharonov–Bohm effect [2] whose scheme is depicted in **??**.

Electrons do not pass inside the solenoid where the magnetic field is concentrated. Nevertheless, a phase difference arises between the electron beams passing through the two slits. The interference picture changes with the value of the electric current.

The phase difference (which will be the obtained phase on doing a full Wilson loop) depends on (the real part of):

$$e^{ie\int_{\Gamma^+_{y,x}} dz^\mu A_\mu(z)} e^{-ie\int_{\Gamma^-_{y,x}} dz^\mu A_\mu(z)} = e^{ie\oint_\Gamma dz^\mu A_\mu(z)} = e^{ie\int d\sigma^{\mu\nu}F_{\mu\nu}} = e^{ieHS}$$

where the closed contour Γ is the composed contour from $\Gamma^+_{y,x}$ and $\Gamma^-_{y,x}$. It does not depend on the shape of $\Gamma^+_{y,x}$ or $\Gamma^-_{y,x}$ but depends only on HS, the magnetic flux through the solenoid.

Physical meaning:

$$\langle U(\Gamma)\rangle = \frac{1}{z}\int DA\; e^{iS_{\text{Maxwell}}+ie\int_\Gamma A_\mu dz^\mu} \tag{5.7}$$

where this expectation value is the phase adquired by a particle which has charge e going along Γ interacting with an external Gauge Field.

Important, Area Law and confinement (there is a chapter in the paper, tomeu paper also, maybe?):

It's important to explain that doing Wilson loops in rectangular path one can evaluate the confinement of a theory (which is the only know way to show mathematically that QCD is confined, and it's what we are going to use to show that the Schwinger model also is confining.

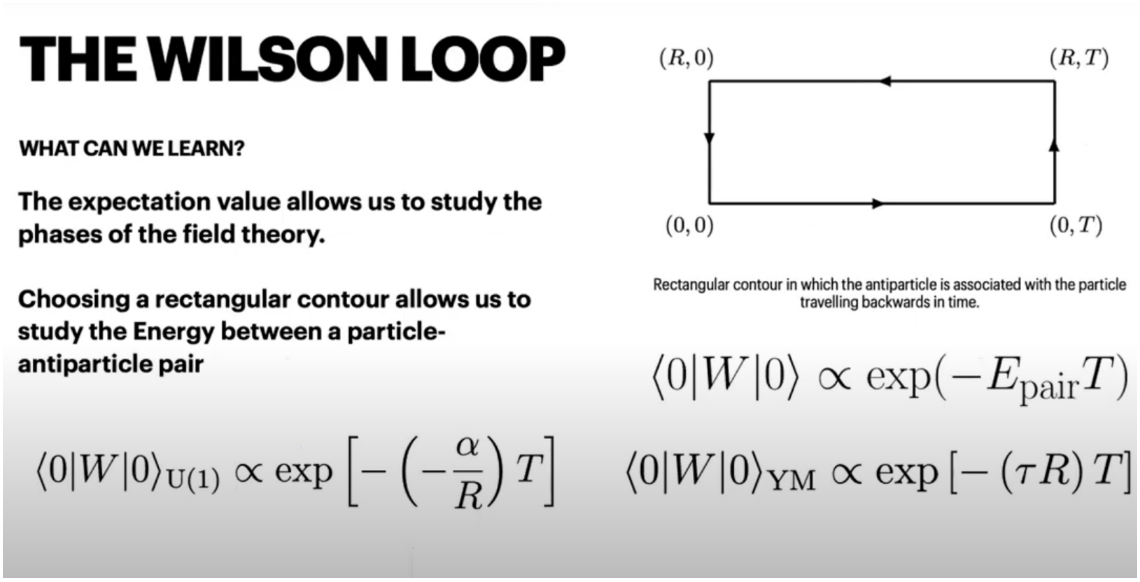


Figure 5.4

It has a lot of sense, because it would be evaluating at two different times, the increase of the gauge phase by the particle antiparticle, field. Which will increase or decrease as the particles get near or apart with time. So a Wilson loop that goes with T/r means that the "potential" is getting weaker in the R with time and they are getting apart.

5.3 Wick Rotation from Minkoswky space to Euclidean space

If we let time be imaginary, we can express it with two different real quantities:

$$T = t + i\tau = \begin{pmatrix} t \\ \tau \end{pmatrix}$$

where we see that if we rotate T in the imaginary plane we pass from a Minkowsky space to an Euclidean one, and viceversa, we can imagine this as being in a Complex manifold where the real part is a Minkowsky one and the real one is Euclidean.

$$\mathbb{C} = \begin{pmatrix} \mathbb{M} \\ \mathbb{E} \end{pmatrix}$$

where we see that if we rotate from $t \to \tau$, to have the same T we need $\tau = -it$, because then $T = i\tau = i(-it) = t$.

In our case the Wick Rotation we will use will be:

$$\begin{aligned} t &\xrightarrow{+WR} \tau = -it \\ -dt^2 + dx^2 &\longrightarrow -d\tau^2 + dx^2 = dt^2 + dx^2 \\ \mathbb{M}(t,x) &\longrightarrow \mathbb{M}(\tau,x) = \mathbb{E}(t,x) \\ \mathbb{S}(t,x) &\longrightarrow \mathbb{S}(\tau,x) = \mathbb{S}_{\mathbb{E}}(t,x) \end{aligned}$$

where it will be easier to do some computations, but after those, we need to rotate the answer back into our original problem, with:

$$\tau = -it \;\to\; t$$

which is the same as doing another Wick rotation in the other direction:

$$\begin{aligned} t' &\xrightarrow{-WR} \tau' = +it' \\ dt'^2 + dx^2 &\longrightarrow d\tau'^2 + dx^2 = -dt'^2 + dx^2 \\ \mathbb{E}(t',x) &\longrightarrow \mathbb{E}(\tau',x) = \mathbb{M}(t',x) \\ \mathbb{S}_{\mathbb{E}}(t',x) &\longrightarrow \mathbb{S}_{\mathbb{E}}(\tau',x) = \mathbb{S}(t',x) \end{aligned}$$

Sumarizing the process we will follow is:

$$\text{Problem in } \mathbb{M} \xrightarrow{+WR} \text{Solve problem, get answer in } \mathbb{E} \xrightarrow{-WR} \text{get answer in } \mathbb{M}$$

5.4 Instantons [16]

In spite of the suggestive name, instantons are not a kind of particle. In particle physics, they are part of an approach for solving quantum field theory problems using classical equations of motion in Euclidean space-time. Basically, the instanton approach first involves transforming a problem from four-dimensional Minkowski spacetime (where the space and time axes are all real) to Euclidean space (where the time axis is imaginary, but we can use QM to solve the problem).

(1,d) Minkowsky spacetime (t,x1,..) $\rightarrow$ (d+1) Euclidean space (-it,x1,...)

After this transformation, it is often possible to find one or many classical solutions to the equations of motion that can be solved in a straightforward way. These classical solutions are called instantons. Once these classical solutions have been solved, the problem is transformed back to Minkowski spacetime to see the physical results.

(For example, to find solutions to a (1,0) QFT we can Wick rotate it into a 1d QM problem where we treat our imaginary time as a space coordinate and solve QM problem at semi-classical orders to then wick rotate back to our (1,0) QFT solutions.)

Instantons: Time-independent finite energy solutions to the E-L eqns.

- d=1 $\rightarrow$ Kink soliton
- d=2 $\rightarrow$ Vortices
- d=3 $\rightarrow$ Monopoles
- d=4 $\rightarrow$ Euclidean Instantons

When we perform the Wick rotation of sec. 5.3 we obtain:

$$t \ \rightarrow \ \tau = -it$$

$$\begin{aligned} iS(t,x) = i\int dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right] \ \rightarrow \ iS(\tau,x) = i\int d\tau\left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x)\right] &= \\ = -\int dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right] &= -S_E(t,x) \\ = -\int dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \ (-V(x))\right] &= -S\left(V(x)\rightarrow -V(x)\right) \end{aligned}$$

where we have achieved now that our path integrals are convergent, because the imaginary exponential will become a exponential decay, with the potential $V(x)$ reversed $-V(x)$.

$$\int [dq]\ e^{\frac{iS[q(t)]}{\hbar}} \ \rightarrow \ \int [dq]\ e^{\frac{-S_E[q(t)]}{\hbar}}$$

$$\int [dq]\ e^{i\int dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right]/\hbar} \ \rightarrow \ \int [dq]\ e^{-\int dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \ (-V(x))\right]/\hbar}$$

This tells us that the solutions of the instantons will be an integral over all the classical solutions to the upside-down potential (then Wick rotated back to normal).

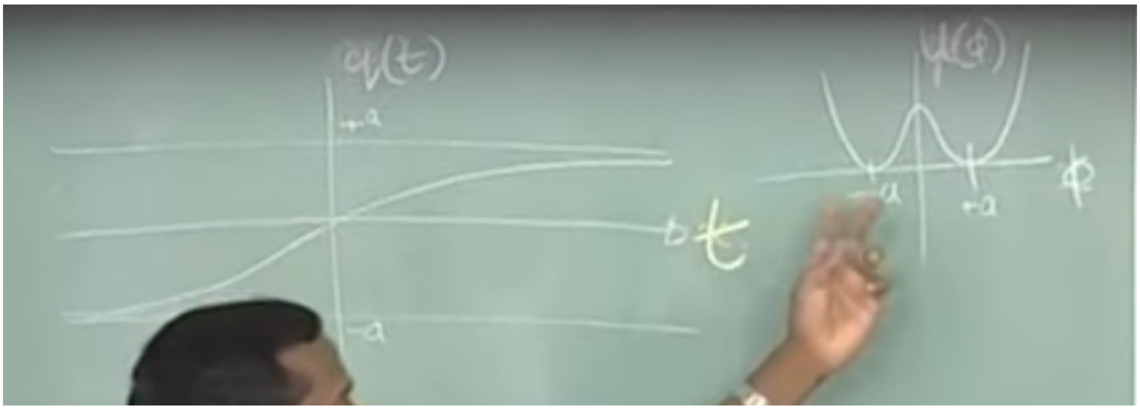
this is because if x_0 is an extreme (first order term doesn't appear in the Taylor expansion, $f'(x_0) = 0$), the negative exponential will have the property (which the imaginary exponential didn't had):

$$A = \int dx\ e^{-f(x)} = \int dx\ e^{-f(x_0) - \frac{1}{2}(x-x_0)^2 f''(x_0)} = e^{-f(x_0)} \int dx\ e^{-\frac{1}{2}(x-x_0)^2 f''(x_0)} \approx e^{-f(x_0)}$$

which is evaluating the function in the extremes. Applying this to path integrals, where the extreme of the action is nothing else than the **classical trajectories**, we have:

$$\int [dq]\ e^{\frac{-S_E[q(t)]}{\hbar}} \approx \sum_{classical(t)} e^{\frac{-S_E[q_{classical}(t)]}{\hbar}} \approx \text{classical solutions for inverted potential!}$$

In the example of the Mexican hat potential, when we invert it, we get a valley between +a and -a, whose classical solutions are particles that go from +a to -a in a way described by the left image with time (REVIEW THE T lim to INFINITY thing).



So from classical solutions to the inverted potential we have just deduced quantum tuneling! AMAZING!

Going now to Yang Mills theory, the equations of motion for the instantons are [16]:

$$*F = \pm F \tag{5.8}$$

where + is for instantons and - for anti-instantons.

Web reference: https://physics.stackexchange.com/questions/224297/what-to-a-physicist-are-instantons-and-the-donaldson-invariants

5.5 The Theta Angle [17]

The typical relativistic notation Maxwell Lagrangian is given by:

$$\mathcal{L}_{Maxwell} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \tag{5.9}$$

this Lagrangian is very simple, because there is very little else we can write down that is both Gauge and Lorentz invariant (There are terms or order F^4 and higher, which give rise to non-linear electrodynamics, but these will always be suppressed by some high mass scale and are unimportant and low-energies).

There is, however, one other term that we can add to the Maxwell action, **the theta term**:

$$\mathcal{L}_\theta = \frac{\theta\alpha}{\pi}\frac{1}{4}F^{*\mu\nu}F_{\mu\nu} = \frac{\theta e^2}{4\pi^2}\frac{1}{4}\left(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right)F_{\mu\nu} = -\frac{\theta e^2}{4\pi^2}\,\mathbf{E}\cdot\mathbf{B} \tag{5.10}$$

where this Lagrangian can be written down as a total derivative, so we say that the theta term is topological, it depends only on the boundary information. The theta term does not change the equations of motion, but it does lead to interesting physics involving subtle interplay of quantum mechanics and topology.

5.5.1 Axion Electrodynamics [17]

The θ term can affect the dynamics classically, this occurs when we have a $\theta(x,t)$ that varies in space and time:

$$\mathcal{L} = \mathcal{L}_{Maxwell} + \mathcal{L}_\theta = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\alpha}{4\pi}\theta(x,t)F^{*\mu\nu}F_{\mu\nu} \tag{5.11}$$

The deformed Maxwell equations are sometimes referred as axion electrodynamics, given by:

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= -\frac{\alpha}{\pi}(\nabla\theta\ \mathbf{B})\\
\nabla\times\mathbf{B} &= \frac{\partial\mathbf{E}}{\partial t} + \frac{\alpha}{\pi}(\dot\theta\mathbf{B} + \nabla\theta\times\mathbf{E})\\
\nabla\cdot\mathbf{B} &= 0\\
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} + 0
\end{aligned}$$

where we see that discontinuities on θ will act as sources, $\nabla\theta\ \mathbf{B}$ will act as an electric charge density, and $\dot\theta\mathbf{B} + \nabla\theta\times\mathbf{E}$ as a current density.

$$\begin{aligned}
\rho_\theta &= -\frac{\alpha}{\pi}(\nabla\theta\ \mathbf{B})\\
j_\theta &= \frac{\alpha}{\pi}(\dot\theta\mathbf{B} + \nabla\theta\times\mathbf{E})
\end{aligned}$$

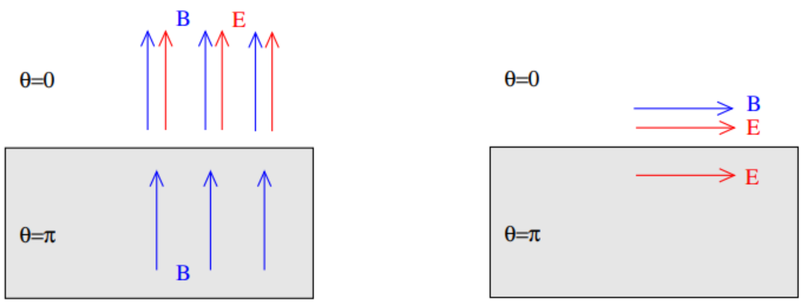


Figure 5.5: Left: applying a magnetic field, Right: applying an electric field

5.5.2 Why θ is periodic [17]

In classical axion electrodynamics, can take any value. Indeed, as we have seen, it is only spatial and temporal variations of that play a role. However, in the quantum theory is a periodic variable:

$$\theta = [0, 2\pi) \text{ or } (-\pi, \pi]$$

this comes from the fact that **E** and **B** are gauge fields that have to fulfill that:

$$A_\mu = A_\mu + \partial_\mu \omega$$

which working in a compact Gauge manifold $\mathbf{T}^4$ (we then realize it doesn't depend on the size of the manifold, so it can be generalized to non-compact manifolds also) with some concrete conditions give for example:

$$S_\theta = \int_{\mathbf{T}^4} L_\theta \; d^4x = \frac{\alpha}{4\pi} \int_{\mathbf{T}^4} \mathbf{E} \cdot \mathbf{B} \; d^4x = N \quad \rightarrow \quad S_\theta = \theta N \text{ with N } \epsilon \mathcal{Z} \tag{5.12}$$

which means that in the partition functions the θ term contributes with $e^{iS_\theta} = e^{iN\theta}$. From where we see that the factor i persists, so that the value of θ in the partition function is only important modulo 2π.

5.5.3 P, T and topological insulators [17]

We sayy that $\theta \rightarrow -\theta$ under P and T, which means that in general the theta term breaks both parity (P) and time-reversal (T) invariance. But there are two exceptions, the obvious $\theta = 0$, and $\theta = \pi$ since we know that θ is periodic with $e^{iS_{\theta=\pi}} = e^{iS_{\theta=-\pi}}$.

We define then a type of material called topological insulator, which has $\theta = \pi$, which are time-reversal invariant, given by having a band structure twisted in a particular way (dynamics seen in Fig 5.5).

References

[1] J. Schwinger, "Gauge invariance and mass. ii," *Phys. Rev.*, vol. 128, pp. 2425–2429, 5 Dec. 1962. DOI: `10.1103/PhysRev.128.2425`.

[2] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley, 1995, ISBN: 978-0-201-50397-5.

[3] ——, "The axial current in two dimensions," in *An Introduction to quantum field theory*, Reading, USA: Addison-Wesley, 1995, ch. 19.1, pp. 651–653, ISBN: 978-0-201-50397-5.

[4] D. Tong, "Quantum field theory on the line," in *Lectures on Theoretical Physics: Lectures on Gauge Theory*, Cambridge University, 2018, ch. 7, pp. 332–384. [Online]. Available: `http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html`.

[5] S. Coleman, "More about the massive schwinger model," *Annals of Physics*, vol. 101, no. 1, pp. 239–267, 1976, ISSN: 0003-4916. DOI: `10.1016/0003-4916(76)90280-3`.

[6] S. Coleman, R. Jackiw, and L. Susskind, "Charge shielding and quark confinement in the massive schwinger model," *Annals of Physics*, vol. 93, no. 1, pp. 267–275, 1975, ISSN: 0003-4916. DOI: `10.1016/0003-4916(75)90212-2`.

[7] M. E. Peskin and D. V. Schroeder, "The axial current in two dimensions, vacuum polarization diagrams," in *An Introduction to quantum field theory*, Reading, USA: Addison-Wesley, 1995, ch. 19.1, pp. 653–655, ISBN: 978-0-201-50397-5.

[8] ——, "The axial current in two dimensions, an example with fermion number non-conservation," in *An Introduction to quantum field theory*, Reading, USA: Addison-Wesley, 1995, ch. 19.1, pp. 657–659, ISBN: 978-0-201-50397-5.

[9] N. Manton, "The schwinger model and its axial anomaly," *Annals of Physics*, vol. 159, no. 1, pp. 220–251, 1985, ISSN: 0003-4916. DOI: `10.1016/0003-4916(85)90199-X`.

[10] A. Casher, J. Kogut, and L. Susskind, "Vacuum polarization and the absence of free quarks," *Phys. Rev. D*, vol. 10, pp. 732–745, 2 Jul. 1974. DOI: `10.1103/PhysRevD.10.732`.

[11] K. G. Wilson, "Confinement of quarks," *Phys. Rev. D*, vol. 10, pp. 2445–2459, 8 Oct. 1974. DOI: `10.1103/PhysRevD.10.2445`. [Online]. Available: `https://link.aps.org/doi/10.1103/PhysRevD.10.2445`.

[12] D. J. Gross, I. R. Klebanov, A. V. Matytsin, and A. V. Smilga, "Screening versus confinement in 1 + 1 dimensions," *Nuclear Physics B*, vol. 461, no. 1, pp. 109–130, 1996, ISSN: 0550-3213. DOI: `10.1016/0550-3213(95)00655-9`.

[13] M. E. Peskin and D. V. Schroeder, "The geometry of gauge invariance," in *An Introduction to quantum field theory*, Reading, USA: Addison-Wesley, 1995, ch. 15.1, pp. 483–486, ISBN: 978-0-201-50397-5.

[14] ——, "The gauge-invariant wilson loop," in *An Introduction to quantum field theory*, Reading, USA: Addison-Wesley, 1995, ch. 15.3, pp. 491–494, ISBN: 978-0-201-50397-5.

[15] Y. M. Makeenko, "Brief introduction to wilson loops and large n," *Physics of Atomic Nuclei*, vol. 73, no. 5, pp. 878–894, May 2010. DOI: 10.1134/s106377881005011x. [Online]. Available: https://doi.org/10.1134%2Fs106377881005011x.

[16] S. Coleman, "The uses of instantons," in *Aspects of Symmetry: Selected Erice Lectures*, Cambridge University Press, 1985, ch. 7, pp. 265–350. DOI: 10.1017/CBO9780511565045.

[17] D. Tong, "The theta term," in *Lectures on Theoretical Physics: Lectures on Gauge Theory*, Cambridge University, 2018, ch. 1.2, pp. 10–22. [Online]. Available: http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html.

[18] ——, *Lectures on Theoretical Physics: Lectures on Gauge Theory*. Cambridge University, 2018. [Online]. Available: http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html.