5. QED for leptons

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5.1 QED for elementary spin 1/2 particles

$$\mathcal{L}_{\rm QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\gamma^\mu D_\mu - m)\psi \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad D_\mu = \partial_\mu + iqA_\mu$$

- The Lagrangian above is for a single elementary particle of electric charge q (and its antiparticle)
- Recall that the concept of elementary depends on the energy scale we are (for instance, for $E \lesssim 1$ GeV, hadrons look like elementary particles)
- If we have n elementary particles of charges q_j then

$$\mathcal{L}_{\rm QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_j^n \bar\psi_j(i\gamma^\mu D^j_\mu - m_j)\psi_j \quad , \quad D^j_\mu = \partial_\mu + iq_jA_\mu$$

  - The Lagrangian above is invariant under $\psi_j \to e^{i\theta_j}\psi_j$, $\theta_j \neq \theta_j(x)$, $\forall j = 1, \dots, n \implies$ there are n conserved charges $\implies$ each flavor j is conserved
  - The interaction Lagrangian reads

$$\mathcal{L}_I = \sum_j^n -q_j\bar\psi_j\gamma^\mu A_\mu\psi_j \equiv -\sum_j^n j^\mu_j A_\mu \equiv -j^\mu A_\mu$$

- The simplest physical processes requires second order in $\mathcal{L}_I$
- We shall focuse on two that were relevant to the discovery of QCD
  - $e^- \mu^- \to e^- \mu^-$
  - $e^+ e^- \to \mu^+ \mu^-$
  - They are related by crossing
- The amplitude is given by

$$
\begin{aligned}
i\mathcal{M} &= \frac{i^2}{2!}\int d^4x \,\langle f|\mathrm{T}\{\mathcal{L}_I(0)\mathcal{L}_I(x)\}|i\rangle = i^2\int d^4x\,\langle f|\mathrm{T}\{j_e^{\mu}(0)A_\mu(0)j_m^{\nu}(x)A_\nu(x)\}|i\rangle \\
&= i^2\int d^4x\,\langle f|\mathrm{T}\{A_\mu(0)A_\nu(x)\}\,\mathrm{T}\{j_e^{\mu}(0)j_m^{\nu}(x)\}|i\rangle
\end{aligned}
$$

- The initial and final states may be written as

$$
|i\rangle = |0\rangle_\gamma |i\rangle_e |i\rangle_\mu \quad , \quad |f\rangle = |0\rangle_\gamma |f\rangle_e |f\rangle_\mu
$$

- Then

$$
i\mathcal{M} = i^2\int d^4x\; {}_\gamma\langle 0|\mathrm{T}\{A_\mu(0)A_\nu(x)\}|0\rangle_\gamma\; {}_e\langle f|j_e^{\mu}(0)|i\rangle_e\; {}_\mu\langle f|{j^\nu}_m(x)|i\rangle_\mu
$$

- Consider $e^- \mu^- \to e^- \mu^-$, then

$$|i\rangle_e = |\vec{p}_A \lambda_A ;\rangle_e \quad , \quad |i\rangle_\mu = |\vec{p}_B \lambda_B ;\rangle_\mu \quad , \quad |f\rangle_e = |\vec{p}_1 \lambda_1 ;\rangle_e \quad , \quad |f\rangle_\mu = |\vec{p}_2 \lambda_2 ;\rangle_\mu$$

- Hence,

$$_\gamma \langle 0| T\{A_\mu(0) A_\nu(x)\}|0\rangle_\gamma = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(0-x)}}{k^2 + i\eta}(-g_{\mu\nu}) \quad , \quad u(l) \equiv u_{\lambda_l}(\vec{p}_l)$$

$$_e \langle f| j_e^\mu(0) |i\rangle_e =_e \langle f| q_e \bar{\psi}_e(0)\gamma^\mu \psi_e(0) |i\rangle_e = q_e \, \bar{u}(1)\gamma^\mu u(A)$$

$$_\mu \langle f| j^\nu{}_m(x) |i\rangle_\mu =_\mu \langle f| q_m \bar{\psi}_\mu(x)\gamma^\nu \psi_\mu(x) |i\rangle_\mu = q_m \, \bar{u}(2)\gamma^\nu u(B) e^{-ix.(p_B - p_2)}$$

- Then $(q_e = q_m = -e)$

$$i\mathcal{M} = \frac{(-ig_{\mu\nu})}{(p_B - p_2)^2} \bar{u}(1)\left(-iq_e\gamma^\mu\right) u(A)\bar{u}(2)\left(-iq_m\gamma^\nu\right) u(B) = \frac{ie^2 \bar{u}(1)\gamma^\mu u(A)\bar{u}(2)\gamma_\mu u(B)}{(p_B - p_2)^2}$$

- In order to calculate cross-sections only $|\mathcal{M}|^2$ is relevant so global phases (and signs) can be dropped
- There is a shortcut to calculate the amplitudes: the Feynman rules, see for instance pg. 15-17 of this link or pg. 228-233 of Griffiths' e-book

- In order to calculate cross-sections only $|\mathcal{M}|^2$ is relevant so global phases (and signs) can be dropped
- $|\mathcal{M}|^2$ depends on the three momentum and third component of the spin (or helicity) of each of the particles in the initial and final state
- If the beam is not polarized $\Longrightarrow$ average over the initial spin states (polarizations)
- If spin states (polarizations) are not measured $\Longrightarrow$ sum over all final spin states (polarizations)
- Then

$$|\mathcal{M}|^2 \to |\overline{\mathcal{M}}|^2 \equiv \frac{1}{2}\sum_{\lambda_A=+,-}\frac{1}{2}\sum_{\lambda_B=+,-}\sum_{\lambda_1=+,-}\sum_{\lambda_2=+,-}|\mathcal{M}|^2$$

- Since

$$|\mathcal{M}|^2 = \frac{e^4\bar{u}(1)\gamma^\mu u(A)\bar{u}(2)\gamma_\mu u(B)\left(\bar{u}(1)\gamma^\nu u(A)\bar{u}(2)\gamma_\nu u(B)\right)^*}{(p_B - p_2)^4}$$

- Consider

$$\sum_{\lambda_A=+,-}\sum_{\lambda_1=+,-}\bar{u}(1)\gamma^\mu u(A)\left(\bar{u}(1)\gamma^\nu u(A)\right)^* = \sum_{\lambda_A=+,-}\sum_{\lambda_1=+,-}\bar{u}(1)\gamma^\mu u(A)\left(\bar{u}(1)\gamma^\nu u(A)\right)^\dagger =$$

$$\sum_{\lambda_A=+,-}\sum_{\lambda_1=+,-}\bar{u}(1)\gamma^\mu u(A)\left(\bar{u}(A)\gamma^0\gamma^{\nu\,\dagger}\gamma^0 u(1)\right) = \sum_{\lambda_A=+,-}\sum_{\lambda_1=+,-}\mathrm{tr}\left(\gamma^\mu u(A)\bar{u}(A)\gamma^\nu u(1)\bar{u}(1)\right) =$$

$$\mathrm{tr}(\gamma^\mu \sum_{\lambda_A=+,-}u(A)\bar{u}(A)\ \gamma^\nu \sum_{\lambda_1=+,-}u(1)\bar{u}(1)\ ) = \mathrm{tr}\left(\gamma^\mu\left(\not{p}_A + m_e\right)\gamma^\nu\left(\not{p}_1 + m_e\right)\right)$$

- Then

$$\mathrm{tr}\left(\gamma^{\mu}\left(\not{p}_A+m_e\right)\gamma^{\nu}\left(\not{p}_1+m_e\right)\right)=\mathrm{tr}\left(\gamma^{\mu}\not{p}_A\gamma^{\nu}\not{p}_1\right)+m_e^2\mathrm{tr}\left(\gamma^{\mu}\gamma^{\nu}\right)=$$
$$4\left(p_A^{\mu}p_1^{\nu}-g^{\mu\nu}p_A.p_1+p_A^{\nu}p_1^{\mu}\right)+4m_e^2g^{\mu\nu}\equiv 2L_e^{\mu\nu}$$

- Analogously

$$\sum_{\lambda_B=+,-}\sum_{\lambda_2=+,-}\bar{u}(2)\gamma^{\mu}u(B)\left(\bar{u}(2)\gamma^{\nu}u(B)\right)^*=4\left(p_B^{\mu}p_2^{\nu}-g^{\mu\nu}p_B.p_2+p_B^{\nu}p_2^{\mu}\right)+4m_{\mu}^2g^{\mu\nu}\equiv 2L_m^{\mu\nu}$$

- Then

$$|\overline{\mathcal{M}}|^2=\frac{e^4L_e^{\mu\nu}L_{\mu\nu\,m}}{(p_B-p_2)^4}$$

$$\begin{aligned}L_e^{\mu\nu}L_{\mu\nu\,m} &= 8\left((p_Ap_B)(p_1p_2)+(p_Ap_2)(p_1p_B)-m_e^2(p_Bp_2)-m_{\mu}^2(p_Ap_1)+2m_e^2m_{\mu}^2\right)\\ &= 4\left((s-m_e^2-m_{\mu}^2)^2+\frac{t^2}{2}+st\right)\end{aligned}$$

- We have used the Mandelstam variables:

$$\begin{aligned}s &\equiv (p_A+p_B)^2=(p_1+p_2)^2=m_e^2+m_{\mu}^2+2(p_Ap_B)=m_e^2+m_{\mu}^2+2(p_1p_2)\\ t &\equiv (p_A-p_1)^2=(p_2-p_B)^2=2m_e^2-2(p_Ap_1)=2m_{\mu}^2-2(p_Bp_2)\\ u &\equiv (p_A-p_2)^2=(p_1-p_B)^2=m_e^2+m_{\mu}^2-2(p_Ap_2)=m_e^2+m_{\mu}^2-2(p_1p_B)\\ & s+t+u=2m_e^2+2m_{\mu}^2\end{aligned}$$

- Finally (recall $m_\mu \gg m_e$)

$$|\overline{\mathcal{M}}|^2 = \frac{4e^4}{t^2}\left((s-m_e^2-m_\mu^2)^2+\frac{t^2}{2}+st\right) \simeq \frac{4e^4}{t^2}\left((s-m_\mu^2)^2+\frac{t^2}{2}+st\right)$$

- In the high energy limit $s \gg m_\mu^2$

$$|\overline{\mathcal{M}}|^2 \simeq \frac{2e^4}{t^2}\left(s^2+u^2\right)$$

- In the CoM frame, the kinematics of a $A\,B \to 1\,2$ collision is fixed except for the angle between the incoming and outgoing direction. If we keep this angle unintegrated, we have

$$\sigma = \frac{1}{4\sqrt{(p_A\cdot p_B)^2-m_A^2m_B^2}}\int\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\int\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}|\mathcal{M}|^2(2\pi)^4\delta(p_A+p_B-p_1-p_2)$$

$$\Longrightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{|\mathcal{M}|^2 p_f}{64\pi^2 s\, p_i} \equiv \text{ Differential cross section}$$

$$p_i \equiv |\vec{p}_A| = |-\vec{p}_B| = \frac{\sqrt{(s-(m_A+m_B)^2)\,(s-(m_A-m_B)^2)}}{2\sqrt{s}}$$

$$p_f \equiv |\vec{p}_1| = |-\vec{p}_2| = \frac{\sqrt{(s-(m_1+m_2)^2)\,(s-(m_1-m_2)^2)}}{2\sqrt{s}}$$

- For $e^-\mu^- \to e^-\mu^-$, $p_i = p_f$
- In the high energy limit ($p_i \gg m_\mu$)
  - $s = (p_A + p_B)^2 \simeq (|\vec{p}_A| + |\vec{p}_B|)^2 = 4p_i^2$
  - $t = (p_A - p_1)^2 \simeq -(\vec{p}_A - \vec{p}_1)^2 = -4p_i^2 \sin^2\frac{\theta}{2}$
  - $u = (p_A - p_2)^2 \simeq -(\vec{p}_A - \vec{p}_2)^2 = -4p_i^2 \cos^2\frac{\theta}{2}$
- Then

$$|\overline{\mathcal{M}}|^2 \simeq 2e^4 \frac{1 + \cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}$$

  - Independent of p_i
  - Differential cross section forward enhanced (in fact it blows up !)
- In the low energy limit ($p_i \ll m_e$)
  - $s = (p_A + p_B)^2 \simeq (m_e + m_\mu)^2 \simeq m_\mu^2 + 2m_e m_\mu$
  - $t = (p_A - p_1)^2 \simeq -(\vec{p}_A - \vec{p}_1)^2 = -4p_i^2 \sin^2\frac{\theta}{2}$
  - $u = (p_A - p_2)^2 \simeq (m_e - m_\mu)^2 \simeq m_\mu^2 - 2m_e m_\mu$
- Then

$$|\overline{\mathcal{M}}|^2 \simeq \frac{16e^4 m_e^2 m_\mu^2}{(\vec{p}_A - \vec{p}_1)^2} = \frac{e^4 m_e^2 m_\mu^2}{p_i^4 \sin^4\frac{\theta}{2}}$$

- This is nothing but Coulomb scattering with relativistic normalization
- The differential cross section (which is independent of the normalization) reads

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \simeq \frac{|\mathcal{M}|^2}{64\pi^2 (m_e + m_\mu)^2} \simeq \frac{e^4 m_e^2}{64\pi^2 p_i^4 \sin^4 \frac{\theta}{2}} = \frac{\alpha^2 m_e^2}{4 p_i^4 \sin^4 \frac{\theta}{2}}$$

which agrees with Rutherford's formula

The LAB frame

- This is the suitable frame to describe fixed target experiments

$$p_A = (E_A\,, \vec{p}_A) \quad , \quad p_B = (m_B\,, 0) \quad , \quad p_1 = (E_1\,, \vec{p}_1) \quad , \quad p_2 = (E_2\,, \vec{p}_2)$$

$$E_A + m_B = E_1 + E_2 \quad , \quad \vec{p}_A = \vec{p}_1 + \vec{p}_2$$

- The cross section may be written as

$$\begin{aligned}
\sigma &= \frac{1}{4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}} \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \int \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} |\mathcal{M}|^2 (2\pi)^4 \delta(p_A + p_B - p_1 - p_2) \\
&= \frac{1}{4 m_B |\vec{p}_A|} \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \int d^4 p_2\, \theta(p_2^0)\, \delta(p_2^2 - m_2^2)\, |\mathcal{M}|^2 (2\pi) \delta(p_A + p_B - p_1 - p_2) \\
&= \frac{1}{4 m_B |\vec{p}_A|} \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \theta(p_A^0 + p_B^0 - p_1^0) \delta\left((p_A + p_B - p_1)^2 - m_2^2\right) |\mathcal{M}|^2 (2\pi)
\end{aligned}$$

- Let us introduce

$$q \equiv p_A - p_1 \quad , \quad \nu \equiv \frac{q.p_B}{m_B} = p_A^0 - p_1^0 = E_A - E_1 = q^0$$

$$(p_A + p_B - p_1)^2 - m_2^2 = (q + p_B)^2 - m_2^2 = q^2 + 2q.p_B + m_B^2 - m_2^2$$

- Let us particularize to $e^-\mu^- \to e^-\mu^- \Longrightarrow m_A = m_1 = m_e,\ m_B = m_2 = m_\mu$, and assume that $E_A \gg m_A$

$$q^2 + 2q.p_B + m_B^2 - m_2^2 = q^2 + 2 m_B \nu \quad , \quad q^2 \simeq -4 E_A E_1 \sin^2\frac{\theta}{2}$$

$$
\begin{aligned}
\sigma &= \frac{1}{4m_B|\vec{p}_A|}\int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\theta(p_A^0+p_B^0-p_1^0)\delta\left((p_A+p_B-p_1)^2-m_2^2\right)|\mathcal{M}|^2 \\
&\simeq \frac{1}{4m_B E_A}\int d\Omega \int \frac{dE_1\, E_1}{(2\pi)^2\, 2}\theta(q^0+m_B)\frac{\delta\left(\frac{q^2}{2m_B}+\nu\right)}{2m_B}|\mathcal{M}|^2
\end{aligned}
$$

- Hence

$$
\left(\frac{d\sigma}{dE_1 d\Omega}\right)_{LAB} = \frac{|\mathcal{M}|^2 E_1}{64\pi^2 E_A m_B^2}\theta(\nu+m_B)\delta\left(\frac{q^2}{2m_B}+\nu\right)
$$

- The delta function allows to carry out the integral over E_1

$$
\delta\left(\frac{q^2}{2m_B}+\nu\right) \simeq \delta\left(-\frac{2E_A E_1}{m_B}\sin^2\frac{\theta}{2}+E_A-E_1\right) = \frac{1}{1+\frac{2E_A}{m_B}\sin^2\frac{\theta}{2}}\delta\left(E_1-\frac{E_A}{1+\frac{2E_A}{m_B}\sin^2\frac{\theta}{2}}\right)
$$

$$
\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \frac{|\mathcal{M}|^2}{64\pi^2 m_B^2}\frac{1}{\left(1+\frac{2E_A}{m_B}\sin^2\frac{\theta}{2}\right)^2} = \frac{|\mathcal{M}|^2 E_1^2}{64\pi^2 m_B^2 E_A^2} \quad , \quad E_1 = \frac{E_A}{1+\frac{2E_A}{m_B}\sin^2\frac{\theta}{2}}
$$

- Let us write in a convenient form for later use ($E_A \gg m_e$)

$$|\overline{\mathcal{M}}|^2 \simeq \frac{4e^4}{t^2}\left((s-m_\mu^2)^2+\frac{t^2}{2}+st\right)$$

$$t=q^2\simeq -4E_AE_1\sin^2\frac{\theta}{2} \quad , \quad s\simeq m_\mu^2+2E_Am_\mu \quad , \quad E_1=\frac{E_A}{1+\frac{2E_A}{m_\mu}\sin^2\frac{\theta}{2}}$$

$$\begin{aligned}
|\overline{\mathcal{M}}|^2 &= \frac{4e^4}{q^4}\left(4E_A^2m_\mu^2+\frac{q^2}{2}\left(-4E_AE_1\sin^2\frac{\theta}{2}\right)+\left(m_\mu^2+2E_Am_\mu\right)\left(-4E_AE_1\sin^2\frac{\theta}{2}\right)\right)\\
&= \frac{4e^4}{q^4}\left(4E_AE_1\left(1+\frac{2E_A}{m_\mu}\sin^2\frac{\theta}{2}\right)m_\mu^2+\frac{q^2}{2}\left(-4E_AE_1\sin^2\frac{\theta}{2}\right)\right.\\
&\quad\left.+\left(m_\mu^2+2E_Am_\mu\right)\left(-4E_AE_1\sin^2\frac{\theta}{2}\right)\right)\\
&= \frac{16e^4m_\mu^2E_AE_1}{q^4}\left(\cos^2\frac{\theta}{2}-\frac{q^2}{2m_\mu^2}\sin^2\frac{\theta}{2}\right)
\end{aligned}$$

- The differential cross sections read

$$\left(\frac{d\sigma}{dE_1 d\Omega}\right)_{LAB} = \frac{4\alpha^2 E_1^2}{q^4}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m_\mu^2}\sin^2\frac{\theta}{2}\right)\delta\left(\frac{q^2}{2m_\mu}+\nu\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \frac{\alpha^2 E_1}{4E_A^3\sin^4\frac{\theta}{2}}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m_\mu^2}\sin^2\frac{\theta}{2}\right)$$

  - This is an arbitrary (and redundant) way of writing the differential cross sections, which is convenient to experimentalist
  - Remember that the independent variables are E_A and θ
- In the limit $m_e \ll E_A \ll m_\mu \implies E_1 \simeq E_A$, Mott's formula is recovered

$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB} \simeq \frac{\alpha^2\cos^2\frac{\theta}{2}}{4E_A^2\sin^4\frac{\theta}{2}}$$

5.2 Pair creation

$e^- e^+ \to \mu^- \mu^+$

- This process is related by crossing to $e^- \mu^- \to e^- \mu^-$
- Assume the beams are unpolarized and polarizations are not measured
- Then $|\overline{\mathcal{M}}|^2(e^- e^+ \to \mu^+ \mu^-)$ is related to $|\overline{\mathcal{M}}|^2(e^- \mu^- \to e^- \mu^-)$

$$
\begin{array}{cccccc} e^- & \mu^- & \to e^- & \mu^- \\ p_A & p_B & p_1 & p_2 \end{array} \quad , \quad \begin{array}{cccc} e^- & e^+ & \to \mu^+ & \mu^- \\ p'_A & p'_B & p'_1 & p'_2 \end{array}
$$

$$
p'_A = p_A \quad , \quad p'_B = -p_1 \quad , \quad p'_1 = -p_B \quad , \quad p'_2 = p_2
$$

$$
\begin{aligned}
s' &= (p'_A + p'_B)^2 = (p_A - p_1)^2 = t \\
t' &= (p'_A - p'_1)^2 = (p_A + p_B)^2 = s \\
u' &= (p'_A - p'_2)^2 = (p_A - p_2)^2 = u
\end{aligned}
$$

- Hence $|\overline{\mathcal{M}}|^2(\underset{p_A}{e^-}\ \underset{p_B}{e^+} \to \underset{p_1}{\mu^+}\underset{p_2}{\mu^-})$ can be obtained from $|\overline{\mathcal{M}}|^2(\underset{p_A}{e^-}\ \underset{p_B}{\mu^-} \to \underset{p_1}{e^-}\underset{p_2}{\mu^-})$ by

$$
s \to t \quad , \quad t \to s \quad , \quad u \to u
$$

- Then

$$|\overline{\mathcal{M}}|^2 = \frac{4e^4}{s^2}\left((t-m_e^2-m_\mu^2)^2+\frac{s^2}{2}+st\right) \simeq \frac{4e^4}{s^2}\left((t-m_\mu^2)^2+\frac{s^2}{2}+st\right)$$

- In the high energy limit $t \gg m_\mu^2$

$$|\overline{\mathcal{M}}|^2 \simeq \frac{2e^4}{s^2}\left(t^2+u^2\right)$$

  - In the *CoM* frame

$$s \simeq 4E_A^2 \quad , \quad t \simeq -2E_A^2(1-\cos\theta) \quad , \quad u \simeq -2E_A^2(1+\cos\theta)$$

$$|\overline{\mathcal{M}}|^2_{CoM} \simeq e^4\left(1+\cos^2\theta\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} \simeq \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \quad \Longrightarrow \quad \sigma = \frac{4\pi\alpha^2}{3s}$$

  - Note that the maximum of pair production is attained at $\theta = 0, \pi$

- If we allow energies close to the muon mass ($t \gtrsim m_\mu^2$)

$$p_i = |\vec{p}_A| = |-\vec{p}_B| = \frac{\sqrt{s-4m_e^2}}{2} \simeq \frac{\sqrt{s}}{2} \simeq E_A \simeq E_1$$

$$p_f = |\vec{p}_1| = |-\vec{p}_2| = \frac{\sqrt{s-4m_\mu^2}}{2} = \frac{\sqrt{s}}{2}\sqrt{1-\frac{4m_\mu^2}{s}} \equiv \frac{\sqrt{s}}{2}\beta$$

$$t \simeq m_\mu^2 - 2(p_A p_1) = m_\mu^2 - 2(E_A E_1 - |\vec{p}_A||\vec{p}_1|\cos\theta) = m_\mu^2 - \frac{s}{2}(1-\beta\cos\theta)$$

$$|\overline{\mathcal{M}}|^2_{CoM} \simeq e^4\left(\left(1+\cos^2\theta\right) + \frac{4m_\mu^2}{s}\sin^2\theta\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} \simeq \frac{\alpha^2\beta}{4s}\left(\left(1+\cos^2\theta\right) + \frac{4m_\mu^2}{s}\sin^2\theta\right) \quad \Longrightarrow \quad \sigma = \frac{2\pi\alpha^2\beta}{3s}(1+\beta)$$

$e^- \, e^+ \to \pi^- \pi^+$

- Let us assume that $E_A \ll 1$ GeV so that the pions may be considered point-like particles
- The relevant interaction Lagrangians are the one of QED for the electron and the one of SQED for the pion

$$\mathcal{L}_I = -q_e \bar{\psi} \gamma^\mu A_\mu \psi + i q_\pi A^\mu \left(\partial_\mu \phi^* \phi - \phi^* \partial_\mu \phi \right) + q_\pi^2 A^\mu A_\mu \phi^* \phi$$

- The last term is quadratic in q_π and hence it does not contribute at leading order
- We may use the same formula as for electron-muon scattering

$$i\mathcal{M} \;=\; i^2 \int d^4x \; {}_\gamma \langle 0 | \mathrm{T}\left\{ A_\mu(0) A_\nu(x) \right\} |0\rangle_\gamma \; {}_e\langle f | j_e^\mu(0) | i \rangle_e \; {}_\pi\langle f | j^\nu{}_\pi(x) | i \rangle_\pi$$

$$j_\pi^\nu = -i q_\pi \left(\partial^\nu \phi^* \phi - \phi^* \partial^\nu \phi \right)$$

$$|i\rangle_e = |\vec{p}_A \lambda_A \, ; \vec{p}_B \lambda_b \rangle_e \quad , \quad |i\rangle_\pi = |0\rangle_\pi \quad , \quad |f\rangle_e = |0\rangle_e \quad , \quad |f\rangle_\pi = |\vec{p}_1 \, ; \vec{p}_2 \rangle_\pi$$

- For unpolarized beams and $E_A \gg m_e$ one eventually obtains (this is the exercise for this week !)

$$\overline{|\mathcal{M}|^2} \simeq \frac{e^4}{s^2} \left(-\frac{(u-t)^2}{2} + \frac{s^2}{2} - 2 m_\pi^2 s \right) \;,\; \overline{|\mathcal{M}|^2_{CoM}} \simeq \frac{e^4 \beta^2 \sin^2\theta}{2} \;,\; \beta \equiv \sqrt{1 - \frac{4 m_\pi^2}{s}}$$

- Which leads to

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} \simeq \frac{\alpha^2\beta^3}{8s}\sin^2\theta \quad \Longrightarrow \quad \sigma = \frac{\pi\alpha^2\beta^3}{3s}$$

    - The maximum occurs at $\theta = \pi/2$, versus $\theta = 0, \pi$ for muons $\Longrightarrow$ measuring the angular distribution in pair production allows to tell apart the spin of the produced particles
    - For $s \gg m_\pi^2$, $\sigma \simeq \pi\alpha^2/3s$, a fourth of the muon's one

- The following processes are related by crossing to pion pair production
    - $\pi^-\pi^+ \to e^-\, e^+$
    - $e^-\,\pi^- \to e^-\,\pi^-$
    - $e^+\,\pi^+ \to e^+\,\pi^+$

5.3 Other elementary QED processes

- Møller scattering $e^- \, e^- \to e^- \, e^-$
  - There are two contributions to the amplitude with a relative minus sign
  - The cross section must be divided by 2! because the final state has two identical particles
  - It is related by crossing to Bhabha scattering and to $e^+ \, e^+ \to e^+ \, e^+$
- Bhabha scattering $e^- \, e^+ \to e^- \, e^+$
  - There are two contributions to the amplitude with the same sign
  - It is related by crossing to Møller scattering
- Compton scattering $e^- \, \gamma \to e^- \, \gamma$
  - There are two contributions to the amplitude with the same sign
  - The photon polarizations enter in the amplitude
  - The Dirac propagators enters in the amplitude
- Electron-positron annihilation $e^- \, e^+ \to \gamma \, \gamma$
  - There are two contributions to the amplitude with the same sign
  - The photon polarizations enter in the amplitude
  - The Dirac propagators enters in the amplitude
  - It is related by crossing to Compton scattering and to $\gamma \, \gamma \to e^- \, e^+$
  - Important for positronium ($e^- \, e^+$ bound state) decay