Formal LSZ

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1 In-Out states

Assume:

- Interactions are local.
- Initial state at $t = -\infty$ consist of isolated particles: in-state $|k_1 \cdots k_r; in\rangle$, described by *free* fields ϕ_{in}, expanded as ladder operators $a^{in}_{\boldsymbol{p}}$, $a^{in\dagger}_{\boldsymbol{p}}$.
- Final state at $t = +\infty$ consist of isolated particles: out-state $|p_1 \cdots p_n; out\rangle$, described by *free* fields ϕ_{out}, expanded as ladder operators $a^{out}_{\boldsymbol{p}}$, $a^{out\dagger}_{\boldsymbol{p}}$.

The ladder operators are related to the fields as in exercise 2.1:

$$\begin{aligned}
a_{\boldsymbol{p}}^{in} &= \frac{1}{\sqrt{2E_p}}\int \mathrm{d}^3x\, e^{ipx}(i\dot{\phi}_{in}(x) + E_p\phi_{in}(x)) \\
&= \frac{1}{\sqrt{2E_p}}\int \mathrm{d}^3x\, (e^{ipx} i\partial_0(\phi_{in}(x)) - i\partial_0(e^{ipx})\phi_{in}(x)) \\
&= \frac{i}{\sqrt{2E_p}}\int \mathrm{d}^3x\, (e^{ipx} \partial_0(\phi_{in}(x)) - \partial_0(e^{ipx})\phi_{in}(x)) \\
&\equiv \frac{i}{\sqrt{2E_p}}\int \mathrm{d}^3x\, e^{ipx}\overleftrightarrow{\partial_0}\phi_{in}(x) \\
a_{\boldsymbol{p}}^{in\dagger} &= \frac{-i}{\sqrt{2E_p}}\int \mathrm{d}^3x\, e^{-ipx}\overleftrightarrow{\partial_0}\phi_{in}(x)
\end{aligned}$$

And an equivalent expression for the *out* ladder and field operators.

These ladder operators create one-particle states as usual:

$$|p; in\rangle = \sqrt{2E_p}\, a_{\boldsymbol{p}}^{in\dagger} |0\rangle$$

When $t \to -\infty$ the Heisenberg interacting field $\phi(x)$ of the full theory must have a description equivalent to $\phi_{in}(x)$, since they have the same e.o.m., however the normalization is arbitrary, so

$$\lim_{t\to-\infty} \phi(x) = \lim_{t\to-\infty} Z^{1/2}\phi_{in}(t, x)$$

and equivalent expressions for the *out* states:

$$|p; out\rangle = \sqrt{2E_p}\, a_{\boldsymbol{p}}^{out\dagger}\,|0\rangle \Rightarrow \langle p; out| = \sqrt{2E_p}\,\langle 0|\, a_{\boldsymbol{p}}^{out}$$

$$\lim_{t\to+\infty}\phi(x) = \lim_{t\to+\infty} Z^{1/2}\phi_{out}(t,x)$$

2 Transition matrix element

The transition matrix element between r particles in the *in* and n particles in the *out* state $r \to n$, can be computed as:

$$\langle p_1 \cdots p_n; out | k_1 \cdots k_r; in\rangle \equiv \langle p_1 \cdots p_n | S | k_1 \cdots k_r\rangle$$

We will assume that all *in*, *out* momenta are different:

$$p_i \neq k_j \,\forall i, j$$

since otherwise this would be an spectator particle (disconnected diagram), and the transition matrix element would correspond to $(r-1) \to (n-1)$.

Let's start with the first *in* particle k_1, and write the transition matrix element

$$\begin{aligned} \langle p_1 \cdots p_n; out | k_1 \cdots k_r; in\rangle &= \sqrt{2E_{k_1}} \langle p_1 \cdots p_n; out | a^{in\dagger}_{\boldsymbol{k_1}} | k_2 \cdots k_r; in\rangle \\ &= \frac{1}{i} \int \mathrm{d}^3 x \, \langle p_1 \cdots p_n; out | e^{-ik_1 x} \overleftrightarrow{\partial}_0 \phi_{in}(x) | k_2 \cdots k_r; in\rangle \\ &= \lim_{t \to -\infty} Z^{-1/2} \frac{1}{i} \int \mathrm{d}^3 x \, \langle p_1 \cdots p_n; out | e^{-ik_1 x} \overleftrightarrow{\partial}_0 \phi(x) | k_2 \cdots k_r; in\rangle \end{aligned} \tag{1}$$

We have an expression with an integration over the 3-space, and we would like an invariant expression with an integration over 4-space: $\int \mathrm{d}^4 x$. We can make use of the fundamental calculus theorem:

$$\lim_{t_f\to+\infty}\lim_{t_i\to-\infty}\int_{t_i}^{t_f}\mathrm{d}t\frac{\partial}{\partial t}\int \mathrm{d}^3x f(t,\boldsymbol{x}) = \lim_{t\to\infty}\int \mathrm{d}^3x f(t,\boldsymbol{x}) - \lim_{t\to-\infty}\int \mathrm{d}^3x f(t,\boldsymbol{x})$$

so we need to subtract to the above expression a term with $t\to\infty$, that is, a term related to the *out* states:

$$\sqrt{2E_{k_1}}\langle p_1\cdots p_n; out|a^{out\dagger}_{\boldsymbol{k_1}}|k_2\cdots k_r; in\rangle = 0$$

this term is 0, because the $a^{out\dagger}_{\boldsymbol{k_1}}$ annihilates a particle with k_1 from the final state, however $k_1\neq p_i\forall i$. So we can subtract this term from (1) and make the same kind of manipulations as for the *in* state, but now with $t\to\infty$:

$$\begin{aligned}
&\lim_{t\to-\infty} Z^{-1/2}\frac{1}{i}\int \mathrm{d}^3x\,\langle p_1\cdots p_n; out|e^{-ik_1x}\overleftrightarrow{\partial_0}\phi(x)|k_2\cdots k_r; in\rangle \\
&-\lim_{t\to+\infty} Z^{-1/2}\frac{1}{i}\int \mathrm{d}^3x\,\langle p_1\cdots p_n; out|e^{-ik_1x}\overleftrightarrow{\partial_0}\phi(x)|k_2\cdots k_r; in\rangle \\
=\;& Z^{-1/2}\frac{-1}{i}\int_{-\infty}^{\infty}\mathrm{d}t\int \mathrm{d}^3x\,\langle p_1\cdots p_n; out|\partial_0(e^{-ik_1x}\overleftrightarrow{\partial_0}\phi(x))|k_2\cdots k_r; in\rangle \\
=\;& iZ^{-1/2}\int \mathrm{d}^4x\,\langle p_1\cdots p_n; out|\partial_0(e^{-ik_1x}\overleftrightarrow{\partial_0}\phi(x))|k_2\cdots k_r; in\rangle
\end{aligned}\tag{2}$$

For the derivative inside we have:

$$\begin{aligned}
& \partial_0((e^{-ik_1x}\partial_0\phi(x) - \partial_0(e^{-ik_1x})\phi(x)) \\
= \; & \partial_0(e^{-ik_1x})\partial_0(\phi(x)) + e^{-ik_1x}\partial_0^2(\phi(x)) - \partial_0^2(e^{-ik_1x})\phi(x) - \partial_0(e^{-ik_1x})\partial_0(\phi(x)))
\end{aligned}$$

the crossed terms cancel, and we are left with:

$$e^{-ik_1x}\partial_0^2(\phi(x)) - \partial_0^2(e^{-ik_1x})\phi(x) \tag{3}$$

The second term in this expression:

$$-\partial_0^2(e^{-ik_1x}) = k_{1,0}^2 e^{-ik_1x} = (\boldsymbol{k}_{\mathbf{1}}^2 + k_1^2)e^{-ik_1x} = (\boldsymbol{k}_{\mathbf{1}}^2 + m^2)e^{-ik_1x} = (-\nabla^2 + m^2)e^{-ik_1x}$$

which we substitute in (3):

$$e^{-ik_1x}\partial_0^2(\phi(x)) - \nabla^2(e^{-ik_1x})\phi(x) + m^2 e^{-ik_1x}\phi(x) \tag{4}$$

we integrate twice by parts the second term:

$$\begin{aligned}
& e^{-ik_1x}\partial_0^2(\phi(x)) - e^{-ik_1x}\nabla^2\phi(x) + m^2 e^{-ik_1x}\phi(x) \\
= \; & e^{-ik_1x}(\partial_\mu\partial^\mu + m^2)\phi(x) = e^{-ik_1x}(\Box_x + m^2)\phi(x)
\end{aligned}$$

This last term contains the Klein-Gordon operator, but $\phi(x)$ is not the free field, therefore it is not zero. We substitute this expression back in our transition matrix element (2):

$$\begin{aligned}
&\langle p_1 \cdots p_n; out | k_1 \cdots k_r; in\rangle \\
= \; & iZ^{-1/2} \int \mathrm{d}^4x \, \langle p_1 \cdots p_n; out | e^{-ik_1 x} (\Box_x + m^2) \phi(x) | k_2 \cdots k_r; in\rangle \\
= \; & iZ^{-1/2} \int \mathrm{d}^4x \, e^{-ik_1 x} (\Box_x + m^2) \langle p_1 \cdots p_n; out | \phi(x) | k_2 \cdots k_r; in\rangle \qquad (5)
\end{aligned}$$

We have arrived at the LSZ reduction formula for particle k_1. Now we would have to repeat it for all initial and final states particles.

Let's do it for final state p_1:

$$\begin{aligned}
&\langle p_1 \cdots p_n; out | k_1 \cdots k_r; in\rangle \to \langle p_1 \cdots p_n; out | \phi(x) | k_2 \cdots k_r; in\rangle \\
= \; & \sqrt{2E_{p_1}} \langle p_2 \cdots p_n; out | a^{out}_{\boldsymbol{p_1}} \phi(x) | k_2 \cdots k_r; in\rangle \\
= \; & \lim_{y^0 \to \infty} iZ^{-1/2} \int \mathrm{d}^3y \, e^{ip_1 y} \overleftrightarrow{\partial}_{y^0} \langle p_2 \cdots p_n; out | \phi(y) \phi(x) | k_2 \cdots k_r; in\rangle
\end{aligned}$$

We would like to follow the same logic, and subtract $a^{in}_{\boldsymbol{p_1}}$ find the $\lim_{y^0 \to -\infty}$ and substitute the subtraction with an integration and a derivative. **However** it does not work so easy:

- To apply the subtraction $\to$ integration+derivation we need the two a operators to be to the left of $\phi(x)$:

$$(a^{out}_{\boldsymbol{p_1}} - a^{in}_{\boldsymbol{p_1}})\phi(x) \to \lim_{y^0\to\infty} \phi(y)\phi(x) - \lim_{y^0\to-\infty} \phi(y)\phi(x)$$

- to cancel the contribution, we need the $a^{in}_{\boldsymbol{p_1}}$ operator to be to the right of $\phi(x)$:

  $$0 = \phi(x)a^{in}_{\boldsymbol{p_1}}|k_2\cdots k_r; in\rangle \to \lim_{y^0\to-\infty} \phi(x)\phi(y)|k_2\cdots k_r; in\rangle \quad (\text{since } k_j \neq p_1 \forall j)$$

Since the fields do not commute, these expressions are different.

Solution: introduce the time-order operation T, and write, for finite x:

$$\begin{aligned}
&\lim_{t_f\to\infty}\lim_{t_i\to-\infty}\int_{t_i}^{t_f} \mathrm{d}y^0 \left[\frac{\partial}{\partial y^0} iZ^{-1/2}\int \mathrm{d}^3y\, e^{ip_1y}\overleftrightarrow{\partial}_{y^0}\langle p_2\cdots p_n; out|T\{\phi(y)\phi(x)\}|k_2\cdots k_r; in\rangle\right] \\
=\;& \lim_{t_f\to\infty} iZ^{-1/2}\int \mathrm{d}^3y\, e^{ip_1y}\overleftrightarrow{\partial}_{y^0}\langle p_2\cdots p_n; out|\phi(y)\phi(x)|k_2\cdots k_r; in\rangle \\
-\;& \lim_{t_i\to-\infty} iZ^{-1/2}\int \mathrm{d}^3y\, e^{ip_1y}\overleftrightarrow{\partial}_{y^0}\langle p_2\cdots p_n; out|\phi(x)\phi(y)|k_2\cdots k_r; in\rangle
\end{aligned}$$

Now we can repeat the former manipulations to the first line above to find:

$$iZ^{-1/2}\int \mathrm{d}^4y\, e^{ip_1y}(\Box_y + m^2)\langle p_2\cdots p_n; out|T\{\phi(y)\phi(x)\}|k_2\cdots k_r; in\rangle$$

We now repeat this procedure for all initial and all final states to finally find the **Lehman-Symanzik-Zimmermann (LSZ) reduction formula in position space**:

$$\boxed{\begin{aligned} & \langle p_1 \cdots p_n; out | k_1 \cdots k_r; in \rangle \equiv \langle p_1 \cdots p_n | S | k_1 \cdots k_r \rangle \\ = \; & (iZ^{-1/2})^{n+r} \int \mathrm{d}^4 y_1 \cdots \mathrm{d}^4 y_n \int \mathrm{d}^4 x_1 \cdots \mathrm{d}^4 x_r \\ \times \; & e^{ip_1 y_1} \cdots e^{ip_n y_n} e^{-ik_1 x_1} \cdots e^{-ik_r x_r} \\ \times \; & (\Box_{y_1} + m^2) \cdots (\Box_{x_r} + m^2) \langle 0 | T\{\phi(y_1) \cdots \phi(x_r\} | 0 \rangle \end{aligned}}$$

Going to momentum space:

$$(\Box_x + m^2)\phi(x) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} (-q^2 + m^2) e^{-iqx} \tilde{\phi}(q)$$

and performing the integrals on x_i, y_i as $\delta^4(p-q)$ functions:

$$\boxed{\begin{aligned} &\langle p_1 \cdots p_n; out | k_1 \cdots k_r; in\rangle \equiv \langle p_1 \cdots p_n | S | k_1 \cdots k_r\rangle \\ = \; &(iZ^{-1/2})^{n+r} \prod_{k=1}^{n} (-p_k^2 + m^2) \prod_{l=1}^{r} (-k_l^2 + m^2) \\ \times \; &\langle 0 | T\{\tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n) \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_r\} | 0\rangle \end{aligned}}$$

3 Fermions

For fermions the relations of the ladder operators with the fields is (exercise 3.2):

$$\begin{aligned} a_{\boldsymbol{k}}^{r\,in} &= \frac{1}{\sqrt{2E_k}} \int \mathrm{d}^3x\, e^{ikx} u^{r\dagger}(\boldsymbol{k}) \psi_{in}(x) = \frac{1}{\sqrt{2E_k}} \int \mathrm{d}^3x\, e^{ikx} \bar{u}^r(\boldsymbol{k}) \gamma^0 \psi_{in}(x) \\ b_{\boldsymbol{k}}^{r\,in} &= \frac{1}{\sqrt{2E_k}} \int \mathrm{d}^3x\, e^{ikx} \psi_{in}^{\dagger}(x) v^r(\boldsymbol{k}) = \frac{1}{\sqrt{2E_k}} \int \mathrm{d}^3x\, e^{ikx} \bar{\psi}_{in}(x) \gamma^0 v^r(\boldsymbol{k}) \end{aligned}$$

with the corresponding expression for $a_{\boldsymbol{k}}^{r\,in\dagger}$, $b_{\boldsymbol{k}}^{r\,in\dagger}$, and the equivalent expressions for the *out* fields. Using the same procedure as for the scalar field, and the Dirac equation for the u and v spinors:

$$(\not{p} - m) u^r(\boldsymbol{p}) = 0 \;\; ; \;\; (\not{p} + m) v^r(\boldsymbol{p}) = 0$$

with:

- r particles, r_f fermions and $r - r_f$ anti-fermions in initial state, with polarizations s_i
- n particles, n_f fermions and $n - n_f$ anti-fermions in final state, with polarization s_j

$$
\boxed{\begin{aligned}
&\langle (p_1,s_1')\cdots(p_{n_f},s_{n_f}')(p_{n_f+1},s_{n_f+1}')\cdots(p_n,s_n');out|(k_1,s_1)\cdots(k_{r_f},s_{r_f})(k_{r_f+1},s_{r_f+1})\cdots(k_r,s_r);in\rangle \\
=\;& \left(-i(Z)^{-1/2}\right)^{n_f+r_f}\left(i(Z)^{-1/2}\right)^{n-n_f+r-r_f}\int \mathrm{d}^4y_1\cdots\mathrm{d}^4y_n\int\mathrm{d}^4x_1\cdots\mathrm{d}^4x_r \\
\times\;& e^{-ik_1x_1}\cdots e^{-ik_rx_r}e^{ip_1y_1}\cdots e^{ip_nx_n} \\
\times\;& \bar{u}^{s_1'}(\boldsymbol{p_1})(i\gamma^\mu\partial_{y_1,\mu}-m)\cdots\bar{u}^{s_{n_f}'}(\boldsymbol{p_{n_f}})(i\gamma^\mu\partial_{y_{n_f},\mu}-m) \\
\times\;& \bar{v}^{s_{r_f+1}}(\boldsymbol{k_{r_f+1}})(i\gamma^\mu\partial_{x_{r_f+1},\mu}-m)\cdots\bar{v}^{s_r}(\boldsymbol{k_r})(i\gamma^\mu\partial_{x_r,\mu}-m) \\
\times\;& \langle 0|T\{\bar\psi(y_{nf+1})\cdots\bar\psi(y_n)\psi(y_1)\cdots\psi(y_n)\bar\psi(x_1)\cdots\bar\psi(x_{r_f})\psi(x_{r_f+1})\cdots\psi(x_r)\}|0\rangle \\
\times\;& (-i\gamma^\mu\overleftarrow{\partial}_{x_1,\mu}-m)u^{s_1}(\boldsymbol{k_1})\cdots(-i\gamma^\mu\overleftarrow{\partial}_{x_{r_f},\mu}-m)u^{s_{nf}}(\boldsymbol{k_{r_f}}) \\
\times\;& (-i\gamma^\mu\overleftarrow{\partial}_{y_{n_f+1},\mu}-m)v^{s_{n_f+1}'}(\boldsymbol{p_{n_f+1}})\cdots(-i\gamma^\mu\overleftarrow{\partial}_{y_n,\mu}-m)v^{s_{n_f}'}(\boldsymbol{p_n})
\end{aligned}}
$$

by taking the Fourier transform, and going to momentum space, we recover the LSZ reduction formula from the previous slides set.