Correlation Functions

Jaume Guasch, UB, November 3, 2021

1 Correlation Functions

Let us reformulate the scattering process using correlation functions. First we will take an *intuitive* view of the subject, and leave the formal proofs for the end[1]. We will start with the $\lambda\phi^4$ theory to avoid the non-fundamental troubles of dealing with spinors.

Define the n**-point correlation function** or n**-point Green's function** in position space:

$$G(x_1, x_2, \ldots, x_n) \equiv \langle 0| T\left\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\right\}|0\rangle \tag{1}$$

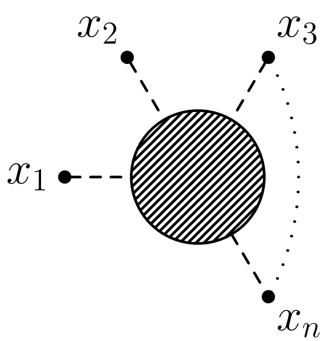
$$\phi(x) \ \text{ fields in the Heisenberg picture}$$

(**not** the fields in the interaction picture $= \phi_I(x)$).

The interpretation of the definition (1) is the following:

- The fields $\phi(x_i)$ create or absorb particles at points x_i, with interactions among them:

[1] see Peskin-Schroeder for a more formal development.



Between the x-points and the *blob* there is a propagation of the field, and the *blob* itself is like a transition matrix element

So we could write:

$$G(x_1, \ldots, x_n) \sim \Delta_F(y_1 - x_1) \cdots \Delta_F(y_n - x_n) i\mathcal{T}(y_1, \ldots, y_n)$$

Since the transition elements are usually written for initial and final states of definite momentum, it will be useful to introduce the **n-point Green's function in momentum space**:

$$\begin{aligned}
\tilde{G}(q_1, \ldots, q_n) &= \int \mathrm{d}^4 x_1 \cdots \mathrm{d}^4 x_n \, e^{iq_1 x_1} \cdots e^{iq_n x_n} G(x_1, \ldots x_n) \\
&= \int \left(\prod_{i=1}^{n} \mathrm{d}^4 x_i \, e^{iq_i x_i} \right) G(x_1, \ldots, x_n) \\
G(x_1, \ldots, x_n) &= \int \left(\prod_{i=1}^{n} \frac{\mathrm{d}^4 q_i}{(2\pi)^4} e^{-iq_i x_i} \right) \tilde{G}(q_1, \ldots, q_n) \qquad (2)
\end{aligned}$$

Imagine we want to describe the process:

$$k_1 \dots k_\ell \to p_1 \dots p_n$$

for the matrix element $\langle f|i\mathcal{T}|i\rangle$ we want to select:

- $e^{ikx} a^\dagger_{\boldsymbol{k}}|0\rangle$ for the initial state
- $\langle 0|e^{-ipx} a_{\boldsymbol{p}}$ for the final state

$$\langle f|i\mathcal{T}|i\rangle \sim \int \mathrm{d}^4 x_i \left\{ \begin{array}{l} q_i = -k_i \; (i = 1 \cdots \ell) \\ q_{i+m} = p_i \; (i = 1 \cdots n) \end{array} \right\} G(x_1, \dots, x_{n+\ell}) \sim \tilde{G}(-k_1, \dots, -k_\ell, p_1, \dots, p_n)$$

But this function contains the external propagators, so in the end:

$$\begin{aligned} \tilde{G}(-k_1, \dots, -k_\ell, p_1, \dots, p_n) \;&=\; \frac{i}{k_1^2 - m^2 + i\varepsilon} \cdots \frac{i}{k_\ell^2 - m^2 + i\varepsilon} \\ &\times\; \frac{i}{p_1^2 - m^2 + i\varepsilon} \cdots \frac{i}{p_n^2 - m^2 + i\varepsilon} \langle f|i\mathcal{T}|i\rangle \end{aligned}$$

a way to remove the external propagators is to multiply by $p^2 - m^2$ and take the limit $p^2 \to m^2$:

$$\langle f|i\mathcal{T}|i\rangle \;=\; \prod_{i=1}^{\ell} \lim_{k_i^2 \to m^2} \frac{k_i^2 - m^2}{i} \prod_{j=1}^{n} \lim_{p_j^2 \to m^2} \frac{p_j^2 - m^2}{i} \tilde{G}(-k_1, \dots, -k_\ell, p_1, \dots, p_n) \tag{3}$$

this is the *amputated* Green's function, since we have removed the external propagators.

1.1 Evolution of the correlation functions

The Green's functions are defined as time-ordered products of Heisenberg fields (1), which we don't know how to compute. We write the fields in (1) as a function of interaction fields, which we know how to treat with the relation:

$$\phi(x) = U_I^\dagger(t,t_0)\phi_I(x)U_I(t,t_0)$$

and the relation:

$$U_I(t_1,t_2)U_I^\dagger(t_3,t_2) = U_I(t_1,t_3) \tag{4}$$

Let's assume that the fields in (1) are already time-ordered: $t_1 > t_2 > \ldots t_n$, relating the fields in the H-picture to the fields in the I-picture

$$\begin{aligned}\langle 0|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|0\rangle = \langle 0|&U_I^\dagger(t_1,t_0)\phi_I(x_1)U_I(t_1,t_0)\; U_I^\dagger(t_2,t_0)\phi_I(x_2)U_I(t_2,t_0)\\ &\times U_I^\dagger(t_3,t_0)\phi_I(x_3)U_I(t_3,t_0)\cdots U_I^\dagger(t_n,t_0)\phi_I(x_n)U_I(t_n,t_0)\;|0\rangle\end{aligned}$$

We use use the property (4) to join the evolution operators between fields, and to introduce a new time t, such that[2] $t > t_1 \cdots t_n > -t$

$$\begin{aligned}&= \langle 0|U_I^\dagger(t,t_0)U_I(t,t_1)\phi_I(x_1)U_I(t_1,t_2)\phi_I(x_2)U_I(t_2,t_3)\;\phi_I(x_3)U_I(t_3,t_4)\;\cdots\\ &\cdots\; U_I(t_{n-1},t_n)\phi_I(x_n)U_I(t_n,-t)U_I(-t,t_0)\;|0\rangle\end{aligned}$$

[2]For the first term, we take eq. (4), multiplying to the left by $U_I^\dagger(t_1,t_2)$: $U_I^\dagger(t_3,t_2) = U_I^\dagger(t_1,t_2)U_I(t_1,t_3)$, for $t_3 \to t_1$, $t_1 \to t$, $t_2 \to t_0$.
For the last term, we multiply eq. (4) to the right by $U_I(t_3,t_2)$: $U_I(t_1,t_2) = U_I(t_1,t_3)U_I(t_3,t_2)$.

At this point we can re-introduce the time-order symbol:

$$\begin{aligned}
&= \langle 0|U_I^\dagger(t,t_0) T\left\{U_I(t,t_1)\phi_I(x_1)U_I(t_1,t_2)\phi_I(x_2)U_I(t_2,t_3)\ \phi_I(x_3)U_I(t_3,t_4)\ \cdots\right.\\
&\qquad \left.\cdots\ U_I(t_{n-1},t_n)\phi_I(x_n)U_I(t_n,-t)\right\} U_I(-t,t_0)\ |0\rangle\\
&= \langle 0|U_I^\dagger(t,t_0) T\left\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\times\right.\\
&\qquad \left.\times U_I(t,t_1)U_I(t_1,t_2)\cdots U_I(t_{n-1},t_n)U_I(t_n,-t)\right\} U_I(-t,t_0)|0\rangle\\
&= \langle 0|U_I^\dagger(t,t_0) T\left\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)U_I(t,-t)\right\} U_I(-t,t_0)|0\rangle\\
&= \langle 0|U_I^\dagger(t,t_0) T\left\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\exp\left[-i\int_{-t}^{t}\mathrm{d}t' H^I_{int}(t')\right]\right\} U_I(-t,t_0)|0\rangle
\end{aligned}$$

If the process starts at $t_0=-\infty$ and ends at $t=-t_0=\infty$

$$\langle 0|U_I^\dagger(\infty,-\infty) T\left\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\exp\left[-i\int_{-\infty}^{\infty}\mathrm{d}t' H^I_{int}(t')\right]\right\}|0\rangle$$

The factor inside the time-ordered product is similar to the one inside the $\mathcal{T}$ expression, but we have an extra factor: $\langle 0|U_I^\dagger(\infty,-\infty)$. This factor represents the evolution of the vacuum from $t_0=-t=-\infty$ to $t=\infty$ through the evolution of the states in the interaction image:

$$|\Psi,t\rangle_I = U_I(t,t_0)|\Psi,t_0\rangle_I \Rightarrow {}_I\langle 0,\infty| = {}_I\langle 0,-\infty|U_I^\dagger(\infty,-\infty) \equiv \langle 0|U_I^\dagger(\infty,-\infty)$$

- At $t=\infty$ the theory is non-interacting:

$$|0,\infty\rangle_I \propto |0,-\infty\rangle_I$$

- Both states are normalized to 1

$$\langle 0,\infty|0,\infty\rangle = 1 = \langle 0,-\infty|0,-\infty\rangle$$

- They are related just by a phase

$$|0,\infty\rangle = U_I(\infty,-\infty)|0,-\infty\rangle = e^{i\alpha}|0,-\infty\rangle$$

$$e^{i\alpha} = \langle 0,-\infty|0,\infty\rangle = \langle 0,-\infty|U_I(\infty,-\infty)|0,-\infty\rangle$$

$$\boxed{e^{i\alpha} = \langle 0|T\left\{\exp\left[-i\int_{-\infty}^{\infty}\mathrm{d}t' H^I_{int}(t')\right]\right\}|0\rangle}$$

this is the **vacuum to vacuum transition** or *vacuum bubbles*

In the expression appears

$$\langle 0|U_I^\dagger(\infty,-\infty) = \langle 0|e^{-i\alpha} = \left(e^{i\alpha}\right)^{-1}\langle 0|$$

so in the end we obtain:

$$\boxed{G(x_1,\ldots,x_n) = \langle 0|T\left\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\right\}|0\rangle = \frac{\langle 0|T\left\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\exp\left[-i\int_{-\infty}^{\infty}\mathrm{d}t' H^I_{int}(t')\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-\infty}^{\infty}\mathrm{d}t' H^I_{int}(t')\right]\right\}|0\rangle} \quad (5)}$$

- The technics to compute Green's functions are the same as for transition matrix elements
- The **vacuum bubbles** appear in the denominator, they will cancel any disconnected vacuum bubble in the numerator.

1.2 Two-point correlation function

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = G(x-y) \tag{6}$$

in the non-interacting theory, this object is the Feynman propagator, and is it is the Green's function of the Klein-Gordon equation:

$$\begin{aligned}(\Box_x + m^2)G(x-y) &= i\delta^4(x-y)\\(p^2-m^2)\tilde{G}(p) &= i\\\tilde{G}(p) &= \frac{i}{p^2-m^2}\end{aligned}$$

It has the following **properties**:

- Has a pole at the particle mass: $p^2 = m^2$
- The residue of the propagator is i:

$$\lim_{p^2\to m^2}(p^2-m^2)\tilde{G}(p) = i$$

- It represents **one** particle propagating from y to x

The Klein-Gordon (or any other homogeneous differential equation) does not determine the the normalization of the fields. Define:

$$\varphi(x) = A\phi(x)$$

then $\varphi(x)$ is also solution of the Klein-Gordon equation:

$$(\Box + m^2)\varphi(x) = 0$$

but the propagator is:

$$G_\varphi(x-y) = \langle 0|T\left\{\varphi(x)\varphi(y)\right\}|0\rangle = A^2\langle 0|T\left\{\phi(x)\phi(y)\right\}|0\rangle = A^2 G(x-y)$$

so that:

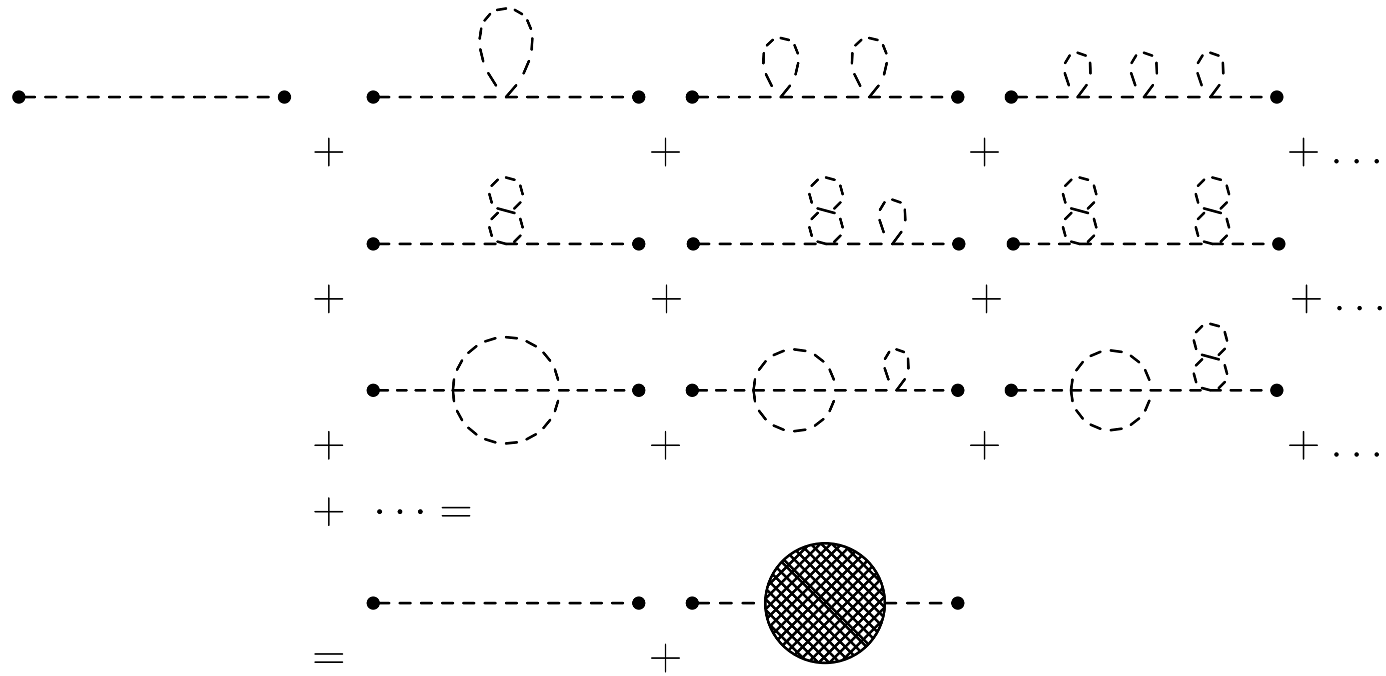
$$\begin{aligned}
(\Box_x + m^2)G_\varphi(x-y) &= iA^2\delta^4(x-y) \\
(p^2 - m^2)\tilde{G}_\varphi(p) &= iA^2 \\
\tilde{G}_\varphi(p) &= \frac{iA^2}{p^2 - m^2}
\end{aligned}$$

it has the pole in the correct place (so it also represents a particle of mass m) **but** the residue of the propagator is iA^2, so it **does not represent one particle** because the field is not correctly normalized.

Let's compute the two-point correlation function in the full theory:

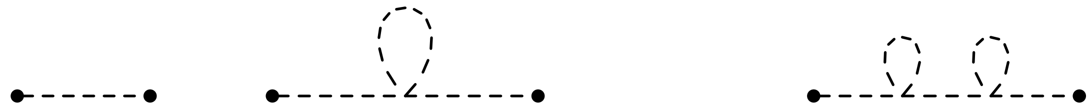
$$\langle 0|T\left\{\phi(x)\phi(y)\right\}|0\rangle = \frac{\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-\infty}^{\infty}\mathrm{d}t' H^I_{int}(t')\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-\infty}^{\infty}\mathrm{d}t' H^I_{int}(t')\right]\right\}|0\rangle}$$

which means: *draw all Feynman diagrams* $1 \to 1$, *excluding the disconnected vacuum bubbles*,

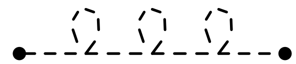


$$
\begin{aligned}
&= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left[-i\Sigma(p^2)\right]\frac{i}{p^2-m^2} \\
\Sigma(p^2) &\equiv \text{full self-energy.}
\end{aligned}
$$

$\Sigma(p^2)$ is composed of pieces with different number of $p^2 - m^2$ propagators:



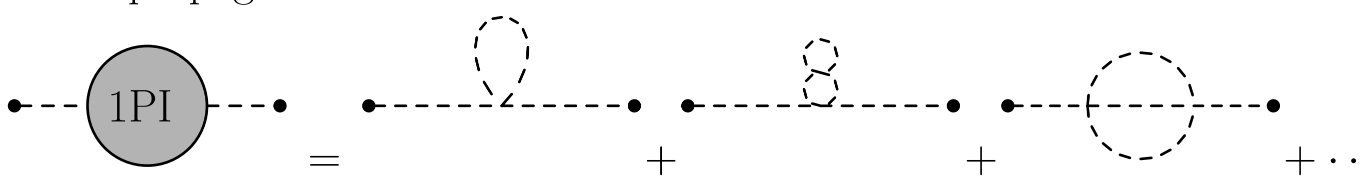
$$\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}(-i\Sigma^1)\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}(-i\Sigma^1)\frac{i}{p^2-m^2}(-i\Sigma^1)\frac{i}{p^2-m^2}$$

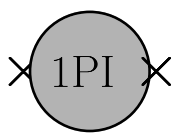


$$+ \frac{i}{p^2-m^2}(-i\Sigma^1)\frac{i}{p^2-m^2}(-i\Sigma^1)\frac{i}{p^2-m^2}(-i\Sigma^1)\frac{i}{p^2-m^2} + \cdots$$

we have an infinite series of this kind.

Define: one-particle-irreducible self-energy (1PI): has no internal $p^2 - m^2$ propagators:





$$\equiv -i\Sigma^{1PI}(p^2)$$

The full propagator is:

$$\begin{aligned}
\tilde{G}(p) &= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}(-i\Sigma^{1PI})\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}(-i\Sigma^{1PI})\frac{i}{p^2-m^2}(-i\Sigma^{1PI})\frac{i}{p^2-m^2} + \cdots \\
&= \frac{i}{p^2-m^2}\sum_{n=0}^{\infty}\left(-i\Sigma^{1PI}(p^2)\frac{i}{p^2-m^2}\right)^n
\end{aligned}$$

This series can be summed!

$$\tilde{G}(p) = \frac{i}{p^2-m^2}\frac{1}{1-\frac{\Sigma^{1PI}}{p^2-m^2}} = \frac{i}{p^2-m^2-\Sigma^{1PI}(p^2)} \tag{7}$$

- The pole of the propagator has moved: the mass of the particle is not m, but M such that:

$$p^2 - m^2 - \Sigma^{1PI}(p^2)\Big|_{p^2=M^2} = 0 \Rightarrow M^2 = m^2 + \Sigma^{1PI}(M^2)$$

- The residue of the propagator is not 1:

$$\lim_{p^2\to M^2}\frac{p^2-M^2}{i}\frac{i}{p^2-m^2-\Sigma^{1PI}(p^2)} \neq 1$$

$$\lim_{p^2\to M^2} \frac{1}{p^2-M^2}(p^2-m^2-\Sigma^{1PI}(p^2)) \equiv \frac{\mathrm{d}}{\mathrm{d}p^2}(p^2-m^2-\Sigma^{1PI}(p^2))\bigg|_{p^2=M^2}$$
$$= 1 - \frac{\mathrm{d}\Sigma^{1PI}(p^2)}{\mathrm{d}p^2}\bigg|_{p^2=M^2} \neq 1$$

Define: wave function renormalization constant of the field ϕ:

$$Z_\phi = \frac{1}{1-\Sigma^{1PI\prime}(M^2)} \tag{8}$$

This means that:

$$\lim_{p^2\to M^2} \tilde{G}(p)(p^2-M^2) = iZ_\phi$$

The correlation function:

$$\langle 0|T\left\{\phi(x)\phi(y)\right\}|0\rangle$$

does not represent one particle, the field which represents one particle states is:

$$\phi^{phys}(x) = Z_\phi^{-1/2}\phi(x)$$

so that if we want to compute the **physical correlation functions** we have to compute:

$$\left(Z_\phi^{-1/2}\right)^n \langle 0|T\left\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\right\}|0\rangle \tag{9}$$

or, in momentum space:

$$\langle f|i\mathcal{T}|i\rangle \;=\; \left(Z_\phi^{-1/2}\right)^{n+\ell}\prod_{i=1}^{\ell}\frac{k_i^2-M^2}{i}\prod_{j=1}^{n}\frac{p_j^2-M^2}{i}\tilde{G}(-k_1,\ldots,-k_\ell,p_1,\ldots,p_n)$$

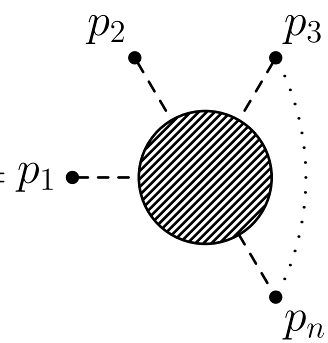
This is the **Lehman-Symanzik-Zimmermann (LSZ) reduction formula**, going to position space:

$$\begin{aligned}
&\prod_{i=1}^{\ell}\frac{i\sqrt{Z_\phi}}{k_i^2-M^2}\prod_{j=1}^{n}\frac{i\sqrt{Z_\phi}}{p_j^2-M^2}\langle p_1\ldots p_n|i\mathcal{T}|k_1\ldots k_\ell\rangle\\
&=\tilde{G}(-k_1,\ldots,-k_\ell,p_1,\ldots,p_n)\\
&=\int\prod_{i=1}^{\ell}\mathrm{d}^4x_i e^{-ik_ix_i}\prod_{j=1}^{n}\mathrm{d}^4y_j e^{ip_jy_j}\langle 0|T\left\{\phi(x_1)\cdots\phi(x_\ell)\phi(y_1)\cdots\phi(y_n)\right\}|0\rangle
\end{aligned}$$

1.3 Application of the LSZ formula

Imagine we have some n-point Green's function in momentum space

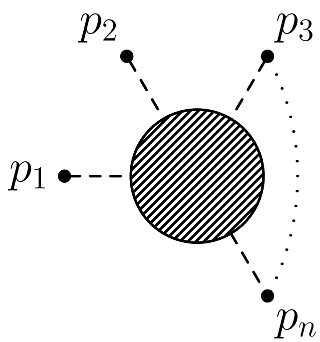
$$\tilde{G}(p_1, \ldots, p_n) = \qquad (10)$$



(At this moment we don't distinguish between initial and final states). To find the corresponding transition matrix element we must, for every external line, multiply by $Z^{-1/2}$ the inverse propagator, and take the limit:

$$\lim_{p_i^2 \to M^2} \frac{p_i^2 - M^2}{i} Z^{-1/2} \qquad (11)$$

Let's do this process for the momentum p_1: The full Green's function (10) connects the external p_1 to the interaction region through the particle propagator, and it has the following expression:



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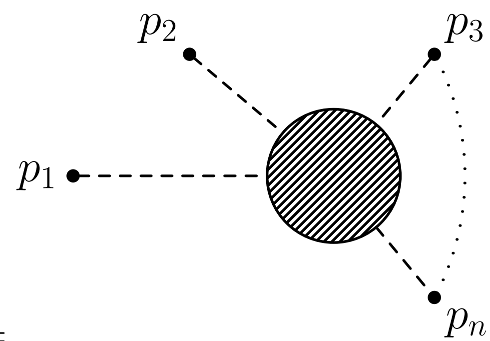

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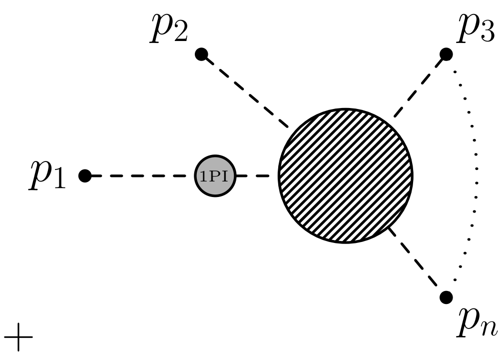

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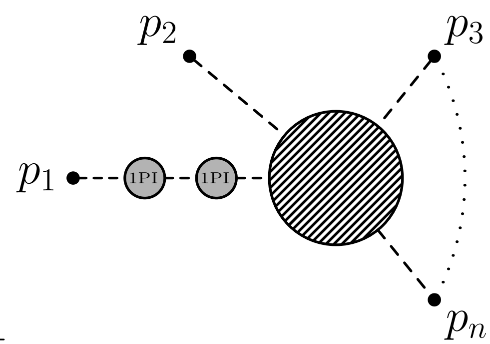

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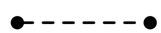
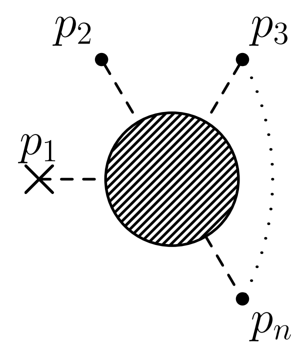
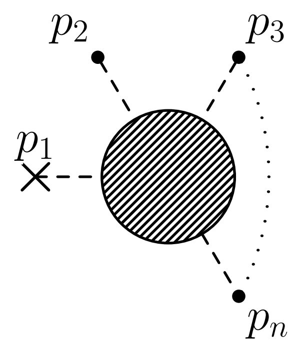
$=$  $\times \left(1 + (-i\Sigma^{full}(p_1^2))\frac{i}{p_1^2 - m^2}\right) \times$ 

$$= \frac{i}{p_1^2 - m^2 - \Sigma^{1PI}(p_1^2)} \times$$

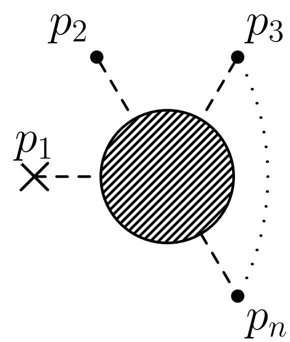


where the cross in the 1 line means that we have *amputated* its propagator, which means that between the cross and the interaction region there is no propagator.

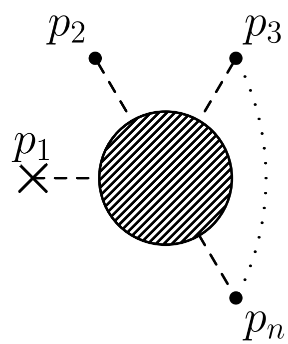
Now lets take the operation (11) on the p_1 line of the Green's function (10):

$$\lim_{p_1^2 \to M^2} \frac{p_1^2 - M^2}{i} Z^{-1/2} \tilde{G}(p_1, \dots, p_n) =$$

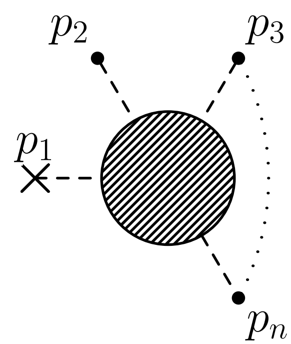
$$= \lim_{p_1^2 \to M^2} \frac{p_1^2 - M^2}{i} Z^{-1/2} \frac{i}{p_1^2 - m^2 - \Sigma^{1PI}(p_1^2)} \times$$



$$= Z^{-1/2} Z$$



$$= Z^{1/2}$$



(12)

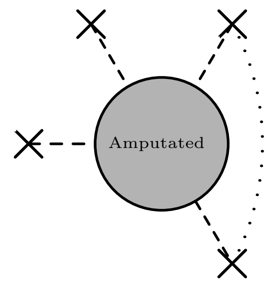
This means that to compute the transition matrix element, for each external line we have to:

- Amputate the external propagator and multiply by $Z^{1/2}$

so at the end:

$$\langle p_1 \dots p_n | i\mathcal{T} | k_1 \dots k_\ell \rangle = Z^{\frac{n+\ell}{2}} \tilde{G}_a(-k_1, \dots, -k_\ell, p_1, \dots, p_n)$$

$$= Z^{\frac{n+\ell}{2}} \tag{13}$$



where $\tilde{G}_a$ is the *amputated* n-point Green's function.

1.4 LSZ reduction formula for fermions

For fermions we have different propagators, and we have to take into account the spinors. A fermion propagator is:

$$\frac{i(\not{p}+M)}{p^2-M^2} = \frac{i\sum_r u^r(p)\bar{u}^r(p)}{p^2-M^2} = \frac{i}{\not{p}-M} \tag{14}$$

then, for an anti-fermion, we can write:

$$\frac{i\sum_r v^r(p)\bar{v}^r(p)}{p^2-M^2} = \frac{i(\not{p}-M)}{p^2-M^2} = \frac{i}{\not{p}+M} \tag{15}$$

and we have to take into account the u and v spinors of the external fields, so if one has:

- m initial states, of which m_f are fermions and $(m-m_f)$ anti-fermions, with polarizations r_i
- n final states, of with n_f are fermions and $n-n_f$ anti-fermions, with polarizations r'_i

one arrives at the **LSZ reduction formula for fermions**:

$$\prod_{i=1}^{m_f} \frac{i\sqrt{Z}}{\not{k}_i - M} \prod_{j=m_f+1}^{m} \frac{i\sqrt{Z}}{\not{k}_j + M} \prod_{l=1}^{n_f} \frac{i\sqrt{Z}}{\not{p}_l - M} \prod_{s=n_f+1}^{n} \frac{i\sqrt{Z}}{\not{p}_s + M} \langle p_1 \dots p_n | i\mathcal{T} | k_1 \dots k_m \rangle =$$
$$= \int \left(\prod_{i=1}^{m} \mathrm{d}^4 x_i e^{-ik_i x_i} \right) \left(\prod_{l=1}^{n} \mathrm{d}^4 y_l e^{+ip_l y_l} \right) \prod_{j=m_f+1}^{m} \bar{v}_{\alpha_j}^{r_j}(k_j) \prod_{l=1}^{n_f} \bar{u}_{\beta_l}^{r'_l}(p_l)$$
$$\times \langle 0 | T \left\{ \prod_{i=1}^{m_f} \bar{\psi}_{\gamma_i}(x_i) \prod_{j=m_f+1}^{m} \psi_{\alpha_j}(x_j) \prod_{l=1}^{n_f} \psi_{\beta_l}(y_l) \prod_{s=n_f+1}^{n} \bar{\psi}_{\delta_s}(y_s) \right\} |0\rangle$$
$$\times \prod_{i=1}^{m_f} u_{\gamma_i}^{r_i}(k_i) \prod_{s=n_f+1}^{n} v_{\delta_s}^{r'_s}(p_s) \qquad (16)$$

Again: to use this formula, one goes to the momentum-space, and removes the external propagator by amputating the Green's function and taking the on-shell limit: $p^2 \to M^2$, the result is the multiplication by one factor of wave-function renormalization constant $Z^{1/2}$ for each external leg.