Lecture 11: Non-linear collapse of a top-hat spherical perturbation.

L: A very simple model for the formation of a halo (for example, the halo of a galaxy or a cluster of galaxies) is the spherical top-hat collapse: a region with spherical symmetry where $\delta > 0$ is constant within an initial comoving radius r_0. At $r > r_0$, the overdensity maybe zero or some other value lower than δ, but spherical symmetry is assumed to be maintained so that there are no external tidal forces.

In this case, the top-hat region will behave like a closed universe if the internal density is above the critical density: it will slow down its Hubble expansion, turn around and collapse. If the rest of the universe has $\Omega_m = 1$, the top-hat will always collapse no matter how small δ is. All spherical shells collapse self-similarly, so the overdensity remains contant. The solution for the trajectory of the shells at $r < r_0$ is the same as that of the scale factor of a closed universe.

Q: In a flat $\Omega_m = 1$ universe, what is the overdensity of the top-hat region when it reaches zero total velocity and turns around, starting the collapse?

Let the time at which the top-hat turns around be t_0. The Hubble constant at this time is $H_0 = 2/(3t_0)$. The mass at initial radius $r_i = a_0 r_0$ has changed its position to a turnaround radius $r_t < r_i$, because of the gravitational collapse. The mass of the top-hat region is

$$M = \frac{4\pi}{3}\bar{\rho} r_i^3 = \frac{H_0^2}{2G} r_i^3 \ . \tag{1}$$

If we look at a mass element at initial comoving radius r_0, it falls by the attraction of a constant mass M, reaching turnaround at time t_0, and then it will collapse and reach back to the origin at time $2t_0$. The motion must be just like that of a Keplerian orbit around a fixed mass M in the limit of very high eccentricity, with a semimajor axis equal to $r_t/2$ and an orbital period $2t_0$. So, third Kepler's law tells us that:

$$\left(\frac{r_t}{2}\right)^3 = \frac{GM}{4\pi^2}(2t_0)^2 = \frac{H_0^2}{8\pi^2} r_i^3 \left(\frac{4}{3H_0}\right)^2 = \frac{2}{9\pi^2} r_i^3 \ . \tag{2}$$

So the turnaround radius is

$$r_t = \left(\frac{16}{9\pi^2}\right)^{1/3} r_i \ , \tag{3}$$

and the density of the top-hat at turnaround is $\rho_t = (9\pi^2/16)\rho_0 \simeq 5.55\rho_0$.

Q: What is the time t_0 at which this turnaround stage will be reached, in terms of the density fluctuation of the top-hat, δ_1, at an early time $t_1 \ll t_0$, when the fluctuation was still linear?

We can figure this out requiring conservation of energy of a particle that ends up at r_t at t_0, which has an energy per unit mass that has to be conserved of:

$$E = -\frac{GM}{r_t} = -\left(\frac{9\pi^2}{16}\right)^{1/3} \frac{H_0^2 r_i^2}{2} \ . \tag{4}$$

At the early time t_1, in the absence of the top-hat perturbation (in the uniform universe), a particle ending at r_i is at $r_1 = r_0(t_1/t_0)^{2/3}$, with Hubble velocity $v_1 = 2r_1/(3t_1) = (2GM/r_1)^{1/2}$. In the presence of the perturbation, the same particle will end up at r_t at t_0, and at time t_1 it will be at a radius $r_1 - \Delta r_1$, with

$$\frac{\Delta r_1}{r_1} = \frac{\delta_1}{3} \ , \tag{5}$$

by requiring mass conservation. We can also derive the peculiar velocity this particle will have from $\dot{\delta}_1 = -\overrightarrow{\nabla} \cdot v_{p1}$. Using Gauss theorem, we know that for a spherical top-hat perturbation, $\overrightarrow{\nabla} \cdot v_{p1} = 3v_{p1}/r_1$. So we find,

$$v_{p1} = -\frac{2}{9}\frac{\delta_1 r_1}{t_1} \ . \tag{6}$$

The total velocity change of the particle due to the presence of the top-hat perturbation is the sum of the differential Hubble velocity due to the displacement Δr_1 and the peculiar velocity:

$$\Delta v_1 = \frac{2\Delta r_1}{3t_1} + v_{p1} = -\frac{4}{9}\frac{\delta_1 r_1}{t_1} \ . \tag{7}$$

While without the top-hat perturbation the particle always has a total energy per unit mass equal to zero, with the perturbation the conserved energy is:

$$E = v_1 \Delta v_1 - \frac{GM}{r_1^2}\Delta r_1 = -\frac{2r_1}{3t_1}\frac{4\delta_1 r_1}{9t_1} - \frac{2r_1^2}{9t_1^2}\frac{\delta_1}{3} = -\frac{10\delta_1}{27}\frac{r_1^2}{t_1^2} \ . \tag{8}$$

Equating this to the final energy per unit mass of the particle at the surface of the top-hat when it turns around at t_0 at r_t, we find

$$\frac{H_0^2 r_0^2}{2}\left(\frac{9\pi^2}{16}\right)^{1/3} = \frac{10\delta_1}{27}\frac{r_1^2}{t_1^2} = \frac{10\delta_1}{27}\frac{r_0^2}{t_0^2}\left(\frac{t_0}{t_1}\right)^{2/3} , \tag{9}$$

which leads to the final result for the linearly extrapolated density perturbation at the time of turnaround of a top-hat,

$$\delta_1 \left(\frac{t_0}{t_1}\right)^{2/3} = \frac{3}{5}\left(\frac{9\pi^2}{16}\right)^{1/3} \simeq 1.0624 \ . \tag{10}$$

If the final gravitational collapse to a halo occurs at $t = 2t_0$, the extrapolated linear overdensity when halo formation occurs is

$$\delta_1 \left(\frac{2t_0}{t_1}\right)^{2/3} = \frac{3}{5}\left(\frac{9\pi^2}{4}\right)^{1/3} \simeq 1.6865 \ . \tag{11}$$

If the collapse were perfectly spherical and there were absolutely no pressure or other perturbing force, and the top-hat region had perfectly constant density, then the perturbation would collapse to a black hole. However, if r_0 is very small compared to the horizon size, any small perturbation away from spherical symmetry will cause particles to deviate from perfectly radial trajectories. In a realistic collapse, this will lead to "virialization", or the formation of a collapsed halo which is approximately in dynamical equilibrium.

L: At the turnaround radius, the top-hat region has a negative gravitational potential energy:

$$W_t = -\frac{GM^2}{r_t}\int_0^1 dx\, 3x^2\, x^2 = -\frac{3GM^2}{5r_t} \ . \tag{12}$$

Q: If this same mass M collapses to an object in virial equilibrium, what is the relation between its final potential and kinetic energies, W_f and K_f?

The virial theorem tells us that $W_f = -2K_f$.

Q: If the total energy has been conserved, what is the relation of the final virialized radius to the turnaround radius?

The total initial energy is W_t, so $W_t = W_f + K_f = W_f/2$. So, $W_f = 2W$, implying that the typical radius of the object has been reduced by a factor 2. We could think of the virialized object as a uniform sphere of radius $r_v = r_t/2$, even though the density profile is something closer to an isothermal sphere.

Q: What is the typical final overdensity of the collapsed object?

If we take a characteristic radius $r_v = r_t/2$ when the age of the universe is already $2t_0$, and the mean density has therefore decreased by a factor 4 compared to the time t_0, while the density of the virialized object has increased by a factor 8 compared to the turnaround time, we have a mean density of the virialized halo $\bar{\rho}_{\rm vir} = 18\pi^2\bar{\rho}(2t_0)$.

Q: What is the velocity dispersion of the halo that is formed at $t \simeq 2t_0$, if it is close to virial equilibrium?

If the one-dimensional halo velocity dispersion is σ_h, the kinetic energy is

$$K_f = \frac{3M\sigma_h^2}{2} = -\frac{W_f}{2} = -W_t = \frac{3GM^2}{5r_i}\left(\frac{3\pi}{4}\right)^{2/3} . \tag{13}$$

From this we find,

$$\sigma_h = \left(\frac{3\pi}{4}\right)^{1/3} \frac{H_0 r_i}{\sqrt{5}} \simeq 0.5951 H_0 r_i . \tag{14}$$

We can also express this in terms of the age of the universe at the time of collapse $2t_0$, and the radius of the original comoving region that contained the mass M that formed the halo at $2t_0$, $2^{2/3} r_i$:

$$\sigma_h = 5^{-1/2}\left(\frac{4\pi}{9}\right)^{1/3} \frac{2^{2/3} r_i}{2t_0} \simeq 0.5\, \frac{2^{2/3} r_i}{2t_0} . \tag{15}$$

L: This what is often used in cosmology for a model of halo formation: when the linearly extrapolated overdensity reaches $\delta = (3/5)(3\pi/2)^{2/3}$, a halo forms and reaches virial equilibrium with an overdensity relative to the background universe $\rho_{\rm vir} = 18\pi^2\bar{\rho}$. For the open and cosmological model universe, generalized versions of this can be derived.

These numbers can also be fitted to the halos that are formed in numerical simulations from random initial conditions, and the results are in fairly good agreement even though this was all derived for a spherical model.

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The Zeldovich approximation and the formation of the cosmic web

The Zeldovich approximation is a useful way to intuitively understand the most basic aspects of the formation of structure in the Universe beyond linear theory.

In general, matter at an initial comoving position $\mathbf{q}$ will move at cosmic time t at a new comoving position $\mathbf{x}$, with a displacement that grows according to the growth function. Assuming we are in the growing mode (and the decaying mode has long decayed), with pure gravity and no pressure,

$$\mathbf{x} = \mathbf{q} + D_1(t)\mathbf{s}(\mathbf{q}) . \tag{16}$$

The Lagrangian coordinate $\mathbf{q}$ of any matter particle does not depend on time, and labels the initial position, whereas the Eulerian comoving coordinate $\mathbf{x}$ does. The Zeldovich approximations consists of assuming linear theory for the displacement: the vector $\mathbf{s}(\mathbf{q})$ does not depend on time, and the time dependence of the comoving displacement is absorbed in the growth factor $D_1(t)$. The peculiar velocity is

$$\mathbf{v}_p(\mathbf{q}) = a\dot{\mathbf{x}} = a\dot{D}_1\mathbf{s}(\mathbf{q}) . \tag{17}$$

The gravitational acceleration is

$$\dot{\mathbf{v}}_p + \frac{\dot{a}}{a}\mathbf{v}_p = 2\frac{\dot{a}}{a}\mathbf{v}_p + a\ddot{D}_1\mathbf{s} = -\frac{1}{a}\nabla\phi . \tag{18}$$

$$(2\dot{a}\dot{D}_1 + a\ddot{D}_1)\mathbf{s} = 4\pi G\bar{\rho} a D_1 \mathbf{s} = -\frac{1}{a}\nabla\phi . \tag{19}$$

So in the end,

$$\mathbf{x} = \mathbf{q} + \frac{1}{4\pi G\bar{\rho}a^2}\nabla\phi . \tag{20}$$

Note that in general, the fluctuation potential ϕ evolves proportionally to D_1/a. In the Zeldovich approximation, we use this displacement from linear theory but then derive a new density from the Jacobian of the transformation from the Lagrangian to Eulerian coordinates. The deformation tensor is defined as

$$\frac{\partial x_i}{\partial q_j} = \delta_{ij} + \frac{1}{4\pi G\bar{\rho}a^2}\phi_{,ij} \ . \tag{21}$$

If we diagonalize this deformation tensor at any point, we generally find a matrix with three eigenvalues. Eigenvalues greater than 1 lead to expansion and those smaller than one lead to contraction. As the growing mode D_1, increases, the eigenvalues less than one will become negative, indicating shell crossing and collapse.

In general, the eigenvalues can be: all contracting (locations of halos with full collapse), two contracting and one expanding (filaments), one contracting and two expanding (pancakes), and all three expanding (voids). This essentially leads to the formation of the cosmic web.

In reality, this deformation tensor can be computed only when small scales are smoothed, allowing for smooth second derivatives of the linear potential. In reality halos start forming at small scales, and then at larger scales the structure of pancakes, filaments and halos is formed by a combination of mergers of previously formed halos and accretion of material in smaller voids, pancakes and filaments. Filaments contain small, previously collapsed small halos, which expand along the filament, gradually falling into the halos at the filament edges. Small halos are also present in pancakes, flowing toward the filaments that connect them.