HW Sheet Javier, GUILLERMO ABAD

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1. Generators of $SU(5)$ and Weinberg's angle in $SU(5)$ GUT's

The simplest model for a unified theory containing the standard model is based on $SU(5)$.

(a) How many generators does $SU(5)$ have?

(b) An easy choice is to start writing the following 11 independent generators:

$$T^a = \begin{pmatrix} T^a_{SU(3)} & 0 \\ 0 & 0 \end{pmatrix}, \qquad T^{b+8} = \begin{pmatrix} 0 & 0 \\ 0 & T^b_{SU(2)} \end{pmatrix},$$

with $a = 1,..,8$, $b = 1,..,3$. Convince yourself that this is correct, and a good idea. Build 12 of the remaining generators of $SU(N)$

$$T^{c+11} = \begin{pmatrix} 0_{3\times3} & \begin{matrix} * & * \\ * & * \\ * & * \end{matrix} \\ \begin{matrix} * & * & * \\ * & * & * \end{matrix} & 0_{2\times2} \end{pmatrix} \qquad c = 1, \ldots, 12,$$

explicitly out of Pauli matrices, normalized such that $\mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}$. Do they commute with the other generators T^a and T^{b+8}? Should they commute?

(c) The last generator of $SU(5)$ is diagonal. Construct a diagonal generator T^{24} which commutes with T^a and T^{b+8} (but not with T^{c+11}), is traceless, hermitian and is normalized according to $\mathrm{Tr}[T^iT^{24}] = \frac{1}{2}\delta^{i\,24}$, for $i = 1,..,24$.

(d) The $SU(5)$ gauge field is given by the 5×5 matrix $A_\mu = A^a_\mu T^a$. Write down explicitly A_μ in matrix form using the following very very very convenient notation: $A^a_\mu \equiv G^a_\mu$ for $a = 1,..,8$, $A^{b+8}_\mu \equiv W^b_\mu$ for $b = 1,..,3$ and $A^{24}_\mu \equiv B_\mu$. (Why is this convenient?)

(e) Couple the gauge field A_μ calculated in part (d) to a fermion Ψ in the fundamental representation of $SU(5)$:

$$\mathcal{L}_\Psi = g_5 \bar{\Psi} \not{A} \Psi$$

where

$$\Psi_k = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \end{pmatrix}.$$

Find the couplings of Ψ_4 and Ψ_5 to B_μ and W^3_μ in terms of the $SU(5)$ coupling g_5.

(f) The couplings of B_μ and W^3_μ to Ψ should be identified with g' and g of the SM. Calculate the Weinberg angle

$$\sin^2\theta_\mathrm{w} = \frac{g'^2}{g'^2 + g^2},$$

and compare the value with the experimental result $\sin^2\theta_\mathrm{w} \simeq 0.23 \pm 0.01$. What could be the reason for the discrepancy?

a)

$SU(N)$ has N^2-1 dof $\longrightarrow$ $SU(5)$ has $5^2-1 = \boxed{24 \text{ generators}}$

b)

It's easy to see that if we need a base of complex hermitian matrices, we just need to fill the right-top triangle with 1's and i's, and then put the complex conjugate in the left-bottom triangle, such as:

$$\boxed{\frac{1}{\sqrt{2}}\left\{
\begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 1/i & \\ & \\ & \end{pmatrix} \\ \begin{pmatrix} 1/-i & & \\ & & \end{pmatrix} & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{pmatrix},
\begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} & 1/i \\ & \\ & \end{pmatrix} \\ \begin{pmatrix} & & \\ 1/-i & & \end{pmatrix} & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{pmatrix},
\begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} & \\ 1/i & \\ & \end{pmatrix} \\ \begin{pmatrix} & 1/-i & \\ & & \end{pmatrix} & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{pmatrix},
\begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} & \\ & 1/i \\ & \end{pmatrix} \\ \begin{pmatrix} & & \\ & 1/-i & \end{pmatrix} & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{pmatrix},
\begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} & \\ & \\ 1/i & \end{pmatrix} \\ \begin{pmatrix} & & 1/-i \\ & & \end{pmatrix} & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{pmatrix},
\begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} & \\ & \\ & 1/i \end{pmatrix} \\ \begin{pmatrix} & & \\ & & 1/-i \end{pmatrix} & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{pmatrix}
\right\}}$$

[Actually this method is how you construct the non-diagonal generators for any $SU(N)$!]

- The 1's give 6 elements, and the $i/-i$ gives the other 6 elements. $\longrightarrow$ 12 generators ✓✓
- They are clearly orthogonal with the other non-diagonal generators, and when multiplied with the diagonals generators the result won't be in the diagonal and $tr(\;)=0$, so we only need to check the normalization:

$$tr(t^a t^a) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2} \quad \text{and} \quad tr(t^a, t^{b\neq a}) = 0 \quad ✓✓$$

- They don't need to commute, because they are a mixing of the direct sums:

$$\left[\begin{pmatrix} \bigcirc & \\ & \end{pmatrix}, \begin{pmatrix} & \\ & \bigcirc \end{pmatrix}\right] = 0 \iff \left.\begin{matrix} (V_3 \oplus 1\!\!1_2)(1\!\!1_3 \oplus V_2') = V_3 \oplus V_2' \\ (1\!\!1_3 \oplus V_2')(V_3 \oplus 1\!\!1_2) = V_3 \oplus V_2' \end{matrix}\right\} \text{same!}$$

$$\left[\begin{pmatrix} & \bigcirc \\ & \end{pmatrix}, \begin{pmatrix} \bigcirc & \\ & \end{pmatrix} \text{or} \begin{pmatrix} & \\ & \bigcirc \end{pmatrix}\right] \neq 0 \implies \left.\begin{matrix} V_S(1\!\!1_3 \oplus V_2') = V_{S_{(3)}} \oplus (V_S V_2')_{(2)} \\ (1\!\!1_3 \oplus V_2') V_S = V_{S_{(3)}} \oplus (V_2' V_S)_{(2)} \end{matrix}\right\} \begin{matrix}\text{the projections} \\ \text{don't need} \\ \text{to be the same!}\end{matrix}$$

- We can think of this as a generalization of Pauli matrices (or Gelmann), to bigger N, at least for the non-diagonal. And non-diagonal λ_i do not commute, so it makes sense that these neither.

$$\left[\begin{matrix} \begin{pmatrix} 1&0&0 \\ 0&-1&0 \\ 0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}\begin{pmatrix} 1&0&0 \\ 0&-1&0 \\ 0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix} \end{matrix} \;\; \begin{matrix} \text{different } PA \neq AP \\ \text{where } P \text{ is a projector} \end{matrix}\right]$$

c)

For T^{24} to commute with T^a and T^{b+8}, we just need it to be proportional to the $1\!\!1$ in both subspaces, which also solves $Tr(T^a, T^{24}) = Tr(T^{b+8}, T^{24}) = \underbrace{Tr(T^{c+11}, T^{24})}$ $= 0 \quad \forall a,b,c$:

$$T^{24} = \begin{pmatrix} a & & & & \\ & a & & 0 & \\ & & a & & \\ & 0 & & b & \\ & & & & b \end{pmatrix}$$

(this last is fullfilled because T^{c+11} have no elements in the diagonal, and so: Diagonal · Nodiagonal = Nodiagonal')

To be hermitian we need $a,b \in \mathbb{R}$, and traceless and normilized are fullfilled if:

$$\left\{\begin{matrix} tr(T^{24}) = 0 \\ Tr(T^{24}T^{24}) = \frac{1}{2} \end{matrix}\right. \implies \left\{\begin{matrix} 3a + 2b = 0 \\ 3a^2 + 2b^2 = \frac{1}{2} \end{matrix}\right. \implies a = \frac{1}{\sqrt{15}}, \quad b = \frac{-3}{2\sqrt{15}}$$

so finally, we get:

$$\boxed{T^{24} = \frac{1}{2\sqrt{15}}\begin{pmatrix} 2 & & & & \\ & 2 & & 0 & \\ & & 2 & & \\ & 0 & & -3 & \\ & & & & -3 \end{pmatrix}}$$

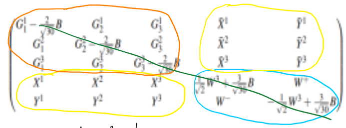
d)

If $\quad A_\mu = A_\mu^a T^a \quad \left\{\begin{matrix} A_\mu^a = G_\mu^a & \text{for } a = 1,\dots,8 \\ A_\mu^{b+8} = W_\mu^b & \text{for } b = 1,\dots,3 \\ A_\mu^{24} = B_\mu & \end{matrix}\right.$, $\vec{T}$ is going to be the T^{c+11} base found in b)

then $\quad \boxed{A_\mu = \left(\vec{G}_\mu \cdot \frac{\vec{\lambda}_{3\times3}}{2} \oplus 1\!\!1_{2\times2}\right) + \left(1\!\!1_{3\times3} \oplus \vec{W}_\mu \frac{\vec{\sigma}_{2\times2}}{2}\right) + B_\mu T^{24} + \vec{X}_\mu \vec{T}} =$

$$= \left(\vec{G}_\mu \vec{T}_{SU(3)} \oplus \mathbb{1}_{2\times 2}\right) + \left(\mathbb{1}_{3\times 3} \oplus \vec{W}_\mu \vec{T}_{SU(2)}\right) + B_\mu T^{24} + \vec{X}_\mu \vec{T} =$$

$$= \left(\vec{G}_\mu \vec{T}_{SU(3)} \oplus \vec{W}_\mu \vec{T}_{SU(2)}\right) + B_\mu T^{24} + \vec{X}_\mu \vec{T} =$$



(some cte factors are wrong, but the basic structure is this one.)

Where it is convinient because $\vec{G}$ are going to be our gluons and $\vec{W}, B_\mu$ our $W^\pm, Z^0, \gamma$ bosons, and then we also have the mixing fields $\vec{X}_\mu$. So we have seen a pretty similar theory to the typical SM $SU(3)\otimes SU(2)\otimes U(1)$ with $SU(5)$, making it seem as a posible candidate for unification! (But sadly no proton decay have been observed :( )

e)

$$\mathcal{L} = g_s \bar{\Psi} \not{A} \Psi \qquad \text{where} \qquad \Psi_\alpha = (\Psi_1 \Psi_2 \Psi_3 \Psi_4 \Psi_5)^T$$

- $\mathcal{L}_4 = g_s \bar{\Psi}_4 \Psi_4 A_{44} = g_s \bar{\Psi}_4 \Psi_4 \left(\frac{(\sigma_3)_{11}}{2} W_\mu^3 + (T^{24})_{44} B_\mu\right) = g_s \bar{\Psi}_4 \Psi_4 \left(\frac{1}{2} W_\mu^3 - \frac{3}{2\sqrt{15}} B_\mu\right)$
- $\mathcal{L}_5 = g_s \bar{\Psi}_5 \Psi_5 A_{55} = g_s \bar{\Psi}_4 \Psi_4 \left(\frac{(\sigma_3)_{22}}{2} W_\mu^3 + (T^{24})_{55} B_\mu\right) = g_s \bar{\Psi}_5 \Psi_5 \left(-\frac{1}{2} W_\mu^3 - \frac{3}{2\sqrt{15}} B_\mu\right)$

The couplings are then:

	W_μ^3	B_μ
Ψ_4	$\frac{1}{2}$	$-\frac{3}{2\sqrt{15}}$
Ψ_5	$-\frac{1}{2}$	$-\frac{3}{2\sqrt{15}}$

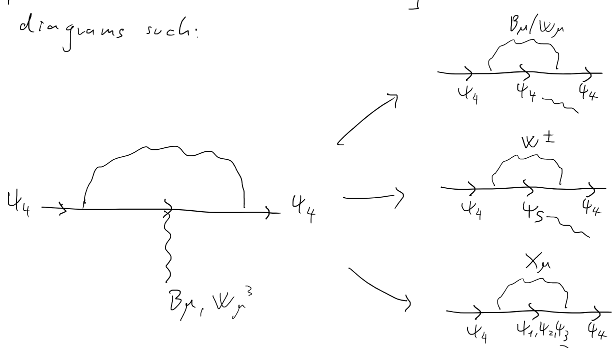
in units of g_s

f)

If g is the W_μ^3 coupling and g' the B_μ coupling, the Weinberg angle is:

$$\boxed{\sin^2\theta_W} = \frac{g'^2}{g^2+g'^2} = \frac{\left(\frac{3}{2\sqrt{15}}\right)^2}{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2\sqrt{15}}\right)^2}\left(\frac{\cancel{g_s}}{\cancel{g_s}}\right)^{\cancel{2}\,1} = \frac{\frac{9}{60}}{\frac{1}{4}+\frac{9}{60}} = \frac{9}{15+9} = \frac{9}{24} = \boxed{\frac{3}{8} = 0{,}375}$$

Which compared to the experimental $\sin^2\theta_W \simeq 0.23 \pm 0.01$, give a discrepancy. The reason for this discrepancy can be that we are not taking into account the mixing's, which would give loop diagrams such:



(Couples to B_μ but not to W_μ^3 for example)

Another cause for this discrepancy would be that this theory is not the reality, where we live. Because even when we implement the mixing X's fields, and use the full SU(5) theory, even considering the coupling scale-dependence, the theory gives: $\sin^2\theta_W = 0{,}21$ which is outside $0{,}23 \pm 0{,}01$.

2. Spontaneous breaking of $SU(5)$ by fields in the Adjoint

Consider a gauge theory with the gauge group $SU(5)$, coupled to a scalar field Φ in the adjoint representation.

(a) The adjoint representation of $SU(N)$ is the real representation of dimension N^2-1. The generators are given by the structure constants of the group. A field in the adjoint representation is a (N^2-1)-vector Φ^a. However it is very convenient to arrange the N^2-1 components of this vector into a $N \times N$ matrix Φ defined as:

$$\Phi_{\alpha\beta} \equiv \Phi^a T^a_{\alpha\beta} \, ,$$

where T^a are the (N^2-1) generators in the **fundamental** representation. We know that under a gauge transformation $\Phi^a \to (U_{\text{adj.}})^{ab}\Phi^b$, and that the covariant derivative is $D_\mu \Phi^a = [\delta^{ab}\partial_\mu - ig(A^{\text{adj.}}_\mu)^{ab}]\Phi^b$, where $(A^{\text{adj.}}_\mu)^{ab} = A^c_\mu (t^c_{\text{adj.}})^{ab} = -if^{abc}A^c_\mu$. Show that:

- (i) The matrix Φ transforms as $\Phi \to U\Phi U^\dagger$, with U in the fundamental.
- (ii) The covariant derivative of Φ is given by $D_\mu\Phi = \partial_\mu\Phi - ig[A_\mu, \Phi]$.
- (iii) The covariant derivative is covariant, that is, transforms exactly like Φ.
- (iv) The only allowed kinetic term for the adjoint scalar Φ^a is $\mathcal{L}^{\text{kin}}_\Phi = \frac{1}{2}\text{Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)]$.

(b) Assume that the potential for this scalar field forces it to acquire a nonzero vacuum expectation value. Two possible choices for this expectation values are

$$\langle\Phi\rangle = A\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix} \qquad \langle\Phi\rangle = B\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

where A and B are arbitrary constants. For each case, work out the spectrum of gauge bosons and the unbroken symmetry group. For this you should identify the relevant terms in $\mathcal{L}^{\text{kin}}_\Phi$, and notice that in the matrix notation an unbroken generator is defined by $[T^a, \langle\Phi\rangle] = 0$. Start by proving this statement.

a)

i)

$$\boxed{\Phi_{\alpha\beta}} \longrightarrow \Phi'_{\alpha\beta} = \Phi'^a T^a_{\alpha\beta} = (U^{ab}_{adj}\Phi^b)T^a_{\alpha\beta} \overset{(*)}{=} (\Phi^a + \theta^c f^{cab}\Phi^b)T^a_{\alpha\beta} =$$

$$\overset{(**)}{=} \Phi_{\alpha\beta} - i\theta^c\Phi^b[T^b, T^c]_{\alpha\beta} = \Phi^a(T^a_{\alpha\beta} + i\theta^c T^c_{\alpha\kappa}T^a_{\kappa\beta} - iT^a_{\alpha\gamma}\theta^c T^c_{\gamma\beta}) \overset{O(\theta^2)\simeq 0}{=}$$

$$= \Phi^a(1_{\alpha\kappa} + i\theta^c T^c_{\alpha\kappa})T^a_{\kappa\gamma}(1_{\gamma\beta} - i\theta^c T^c_{\gamma\beta}) = U_{f\,\alpha\kappa}\Phi^a T^a_{\kappa\gamma}U^\dagger_{f\,\gamma\beta} = \boxed{U_f\Phi U^\dagger_f{}_{\alpha\beta}}$$

$$\left((*)\ \Phi^a \to U^{ab}_{adj}\Phi^b = (1^{ab} + i\theta^c T^{cab}_{adj})\Phi^b = \Phi^a + \theta^c f^{cab}\Phi^b \right)$$

$$\left((**)\ [T^a, T^b]_{\alpha\beta} = i f^{abc} T^c_{\alpha\beta} \right)$$

ii)

$$\left(D_\mu\Phi^a = \partial_\mu\Phi^a - igA^{adj\,ab}_\mu\Phi^b = \partial_\mu\Phi^a - gf^{abc}A^c_\mu\Phi^b \right)$$

$$\boxed{D_\mu\Phi_{\alpha\beta}} = D_\mu(\Phi^a T^a_{\alpha\beta}) = (\partial_\mu\Phi^a - gf^{abc}A^c_\mu\Phi^b)T^a_{\alpha\beta} = \partial_\mu\Phi_{\alpha\beta} + ig[T^b, T^c]_{\alpha\beta}A^c_\mu\Phi^b =$$

$$= \partial_\mu\Phi_{\alpha\beta} + ig[\Phi, A_\mu]_{\alpha\beta} = \boxed{\partial_\mu\Phi_{\alpha\beta} - ig[A_\mu, \Phi]_{\alpha\beta}}$$

iii)

$$D_\mu \Phi = \partial_\mu \Phi - ig[A_\mu, \Phi] =$$

$$= \partial_\mu \Phi - ig(A_\mu \Phi - \Phi A_\mu)$$

$$D_\mu \Phi \longrightarrow \partial_\mu (U\Phi U^\dagger) - ig\left[(U A_\mu \cancel{U^\dagger})(\cancel{U}\Phi U^\dagger) - (U\Phi \cancel{U^\dagger})(\cancel{U} A_\mu U^\dagger)\right] =$$

$$= U \partial_\mu \Phi U^\dagger - ig\, U[A_\mu \Phi - \Phi A_\mu] U^\dagger = U(D_\mu \Phi)U^\dagger \quad ✓✓$$

iv)

$$K = \frac{1}{2}(D_\mu \phi)(D^\mu \phi)^\dagger \longrightarrow \frac{1}{2} U (D_\mu \phi) \cancel{U^\dagger} \cancel{U} (D^\mu \phi)^\dagger U^\dagger = \frac{1}{2} U (D_\mu \phi)(D^\mu \phi)^\dagger U^\dagger = UKU^\dagger$$

Not invariant!

We need to multiply the end with the start, so that $UKU^\dagger \to K\cancel{U^\dagger U} = K$

$$\boxed{\frac{1}{2} tr\left((D_\mu \phi)(D^\mu \phi)^\dagger\right) \text{ will be invariant!}}$$

b) i) First let's proof that if $[T_a, \langle \phi \rangle] = 0$ the generator is unbroken:

$(\hat{\phi} = \phi - \langle \phi \rangle)$

$$S(\hat{\phi}) = \frac{1}{2} tr\left[(D_\mu \hat{\phi})(D^\mu \hat{\phi})^\dagger\right] = \frac{1}{2} tr\left[(D_\mu \phi - D_\mu \langle \phi \rangle)(D^\mu \phi - D^\mu \langle \phi \rangle)^\dagger\right] =$$

$$= \frac{1}{2}\Big\{ \underbrace{tr\left[(D_\mu \phi)(D^\mu \phi)^\dagger\right]}_{1} + \underbrace{tr\left[(D_\mu \langle \phi \rangle)(D^\mu \langle \phi \rangle)^\dagger\right]}_{2} \underbrace{- tr\left[(D_\mu \phi)(D^\mu \langle \phi \rangle)^\dagger\right] - tr\left[(D_\mu \langle \phi \rangle)(D^\mu \phi)^\dagger\right]}_{3} \Big\}$$

For U to be a symmetry, the new action has to remain invariant under it:

$$S(U\phi U^\dagger, U\langle \phi \rangle U^\dagger) = S(\phi, \langle \phi \rangle) \qquad (1 \text{ and } 2 \text{ already are invariant})$$

So we will need that 3 is invariant aswell under U (given by some T^a):

$$\boxed{[\langle \phi \rangle, U] = 0}$$

$$tr\left[(D_\mu \langle \phi \rangle)(D^\mu \phi)^\dagger\right] \xrightarrow{U} tr\left[(D_\mu \langle \phi \rangle)(D^\mu (U\phi U^\dagger))^\dagger\right] = tr\left[(D_\mu \langle \phi \rangle) U (D^\mu \phi)^\dagger U^\dagger\right] =$$

$$= tr\left[\cancel{U}(D_\mu \langle \phi \rangle)(D^\mu \phi)^\dagger \cancel{U^\dagger}\right] = tr\left[(D_\mu \langle \phi \rangle)(D^\mu \phi)^\dagger\right] \quad ✓$$

ii) We have seen that for a T^a to be unbroken, we need that $[\langle \phi \rangle, T^a] = 0$, with this now let's proced to find the unbroken generators of the two $\langle \phi \rangle$ examples provided:

SU(4) generators invariant (*), and also $\langle \phi \rangle \in U(1)$ (**)

- First case: $\langle \phi \rangle = A \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix}$ $\qquad \boxed{SU(5) \longrightarrow SU(4) \times U(1)}$

• Second case: $\langle\phi\rangle = B\begin{pmatrix} \boxed{\begin{matrix}2&&\\&2&\\&&2\end{matrix}} & \\ & \boxed{\begin{matrix}-3&\\&-3\end{matrix}} \end{pmatrix}$ $\qquad \boxed{SU(5) \longrightarrow SU(3)\times SU(2)\times U(1)}$

$\to$ SU(2) generators invariant ⊛, and also $\langle\phi\rangle \in U(1)$ ⊛

$\to$ SU(3) generators invariant ⊛

iii)

And finally let's find the spectrum of gauge bosons using:

$$\frac{1}{2}m_{ab}^2 = -g^2\,\mathrm{Tr}\left([T^a,\langle\phi\rangle][T^b,\langle\phi\rangle]\right)$$

which for each case gives:

• First case:

- a or b $\in SU(4)$ or $U(1)$: 0 because $[T^a,\langle\phi\rangle]=0$ ⊛
- a,b $\notin SU(4)$ or $U(1)$: (Only 8 generators remaining $\begin{pmatrix} \boxed{0} & \begin{matrix}x\\y\\z\\k\end{matrix} \\ x^*\,y^*\,z^*\,k^* & \boxed{0}\end{pmatrix}$) $\left(\begin{matrix}x,y,z,k \text{ are}\\ \tfrac{1}{2} \text{ or } \tfrac{i}{2} \text{ for}\\ \text{each generator}\end{matrix}\right)$

$$\left.\begin{matrix} T^a\langle\phi\rangle = A\begin{pmatrix} \boxed{0} & \begin{matrix}-4x\\-4y\\-4z\\-4k\end{matrix} \\ x^*\,y^*\,z^*\,k^* & \boxed{0}\end{pmatrix} \\ \langle\phi\rangle T^a = A\begin{pmatrix} \boxed{0} & \begin{matrix}x\\y\\z\\k\end{matrix} \\ -4x^*\,-4y^*\,-4z^*\,-4k^* & \boxed{0}\end{pmatrix}\end{matrix}\right\} [T^a,\langle\phi\rangle] = A\begin{pmatrix} \boxed{0} & \begin{matrix}-5x\\-5y\\-5z\\-5k\end{matrix} \\ 5x^*\,5y^*\,5z^*\,5k^* & \boxed{0}\end{pmatrix}$$

$$\Downarrow$$

$$\mathrm{Tr}\left([T^a,\langle\phi\rangle][T^b,\langle\phi\rangle]\right) = A^2\,\mathrm{Tr}\begin{pmatrix} -25x^2 & -25xy & -25xz & -25xk & 0 \\ -25yx & -25y^2 & -25yz & -25yk & 0 \\ -25zx & -25zy & -25z^2 & -25zk & 0 \\ -25kx & -25ky & -25kz & -25k^2 & 0 \\ 0 & 0 & 0 & 0 & M \end{pmatrix} = \frac{-25-25}{4}A^2\delta^{ab}$$

$(M = -25x^2-25y^2-25z^2-25k^2)$

so finally $\frac{1}{\cancel{2}}m_{ab}^2 = -g^2\,\frac{-25}{\cancel{2}}A^2\delta^{ab} \longrightarrow$ $\boxed{\begin{cases} m_a = 5Ag & \text{for the 8 broken generators.} \\ m_a = 0 & \text{for } SU(4) \text{ and } \langle\phi\rangle \text{ generators.}\end{cases}}$

• Second case:

- a or b $\in SU(3)$, $SU(2)$ or $U(1)$: 0 because $[T^a,\langle\phi\rangle]=0$ ⊛
- a,b $\notin SU(3), SU(2)$ or $U(1)$: (Only 12 generators remaining $\begin{pmatrix} \boxed{0} & \begin{matrix}k&x\\l&y\\m&z\end{matrix} \\ \begin{matrix}k^*&l^*&m^*\\x^*&y^*&z^*\end{matrix} & \boxed{0}\end{pmatrix}$) $\left(\begin{matrix}x,y,z,k,l,m\\ \text{are } \tfrac{1}{2} \text{ or } \tfrac{i}{2} \text{ for}\\ \text{each generator}\end{matrix}\right)$

$$\left.\begin{matrix} T^a\langle\phi\rangle = B\begin{pmatrix} \boxed{0} & \begin{matrix}-3k&-3x\\-3l&-3y\\-3m&-3z\end{matrix} \\ \begin{matrix}2k^*&2l^*&2m^*\\2x^*&2y^*&2z^*\end{matrix} & \boxed{0}\end{pmatrix} \\ \langle\phi\rangle T^a = B\begin{pmatrix} \boxed{0} & \begin{matrix}2k&2x\\2l&2y\\2m&2z\end{matrix} \\ \begin{matrix}-3k^*&-3l^*&-3m^*\\-3x^*&-3y^*&-3z^*\end{matrix} & \boxed{0}\end{pmatrix}\end{matrix}\right\} [T^a,\langle\phi\rangle] = B\begin{pmatrix} \boxed{0} & \begin{matrix}-5k&-5x\\-5l&-5y\\-5m&-5z\end{matrix} \\ \begin{matrix}5k^*&5l^*&5m^*\\5x^*&5y^*&5z^*\end{matrix} & \boxed{0}\end{pmatrix}$$

$$\Downarrow$$

$$\mathrm{Tr}\left([T^a,\langle\phi\rangle][T^b,\langle\phi\rangle]\right) = B^2\,\mathrm{Tr}\begin{pmatrix} -25(k^2+x^2) &&&& \\ & -25(l^2+y^2) &&& \\ && -25(m^2+z^2) && \\ &&& -25(k^2+l^2+m^2) & \\ &&&& -25(x^2+y^2+z^2) \end{pmatrix} =$$

$$= B^2\,\frac{-25-25}{4}\,\delta^{ab}$$

so finally $\frac{1}{\cancel{2}}m_{ab}^2 = -g^2\,\frac{-25}{\cancel{2}}B^2\delta^{ab} \longrightarrow$ $\boxed{\begin{cases} m_a = 5Bg & \text{for the 12 broken generators.} \\ m_a = 0 & \text{for } SU(3), SU(2) \text{ and } \langle\phi\rangle \text{ generators.}\end{cases}}$

(⊛ It is clear that because $\langle\phi\rangle$ is proportional to the identity $\mathbb{1}$ in that subspace, all the generators within it will commute with $\langle\phi\rangle$. $[T_{subspace}, \langle\phi\rangle]=0$)

(⊛⊛ We always are gonna have an unbroken generator that will give $U(1)$ symmetry, given by $\langle\phi\rangle$ itself, $[\langle\phi\rangle, \langle\phi\rangle]=0$ (Always that the VEV ($\langle\phi\rangle$) is traceless))

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3. Baryon-Number violating operators in the SM and in $SU(5)$ GUT's

(a) Write a couple (or more) Lorentz-invariant dimension 6 local operators built out of SM fields, invariant under the SM gauge group and which break Baryon Number.

(b) Now consider an $SU(5)$ gauge theory coupled to a fermion Ψ in the $\bar{\mathbf{5}}$ representation of $SU(5)$ and a fermion Φ in the $\mathbf{10}$. The $\bar{\mathbf{5}}$-field Ψ can be represented by a column 5-vector Ψ_i and the $\mathbf{10}$-field can be represented by an antisymmetric 10×10 matrix Φ_{ij}, transforming as: $\Psi_i \to U^\dagger_{ij}\Psi_j$ and $\Phi_{ij}\to U_{ik}\Phi_{kj}$, where U is the gauge transformation matrix in the fundamental. $\psi^i \to U_{adj}^{ij}\psi^j$, $\phi_{ij} \to U_{ik}\,\phi_{kl}\,U_{lj}$

Write all possible 4-fermion $SU(5)$- (and Lorentz-) invariant dimension-6 operators.

(c) We arrange all known fermions in $SU(5)$ representations in the following way:

$$\Psi = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ -\nu \end{pmatrix}_L , \quad \Phi = \begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{pmatrix}_L .$$

Expand the operators in part (b) in terms of u, d, e, ν fields. Do you recover the SM operators of part (a)?

(d) Draw a tree-level Feynman diagram for a Baryon-number-violating process mediated by an $SU(5)$ gauge boson. Compute the corresponding amplitude in the limit where the CM energy is much smaller than the mass of the gauge boson M_X. In this limit the propagator can be written as $\mathcal{P} = -i/M_X^2$. This amplitude is equal to the matrix element of a dimension-six operator times some coefficient. Find the operator and the coefficient. What is needed to suppress the rate of such Baryon-number-violating processes?

a)

And the dimension-six operators are

$$-\mathcal{L} = \frac{1}{M_*^2}\left(y^u_{i3}U_i^c F_3 \tilde{H} U_T^\dagger U_D + y^d_{i3} D_i^c F_3 H U_T^\dagger U_D^\dagger + y^u_{3i} F_3 Q_i \Phi^\dagger U_T U_D \right.$$
$$\left. + y^{\prime d}_{3i}\overline{f}_3 Q_i \Phi U_T U_D\right) + \text{H.C.} .$$

explain our convention. We denote the first two family quark doublets, right-handed up-type quarks, right-handed down-type quarks, lepton doublets, right-handed neutrinos, right-handed charged leptons, and the corresponding Higgs field respectively as Q_i, U_i^c, D_i^c, L_i, N_i^c, E_i^c, and H, as in the supersymmetric SM convention. We denote the third family SM fermions as F_3, $\overline{f}_3$, and N_3^c. To give the masses to the third family of the SM fermions, we introduce a $SU(5)$ anti-fundamental Higgs field $\Phi \equiv (H_T', H')$. We also need to introduce

b)

Transformations:

$\psi_i \to U_{ij}^\dagger \psi_j = U_{ji}^* \psi_j$ and $\phi_{ij} \to U_{ik} U_{jl} \Phi_{kl} = U_{ik}\Phi_{kl} U_{lj}^T$

$\psi^T \to \psi^T U^* / \psi \to U^\dagger \psi$ $\phi \to U\phi U^T$

Invariant terms:

$\psi^\dagger \phi\phi\psi \to (\psi^\dagger U)(U\phi U^\dagger)(U\phi U^\dagger)U^\dagger\psi = \psi^\dagger U^2 \phi\phi U^{\dagger 2}\psi$ ✗

$\psi^\dagger\psi\psi^\dagger\psi \to (\psi^\dagger U)(U^\dagger\psi)(\psi^\dagger U)(U^\dagger\psi) = \psi^\dagger\psi\psi^\dagger\psi$ ✓

⋮

c)

$$\bar{\psi}\psi\bar{\psi}\psi = \left(|\psi|^2\right)^2 = \left(\bar{d}_1^{\,2} + \bar{d}_2^{\,2} + \bar{d}^{\,2}\right.$$

$$\bar{\psi}\phi\phi\psi = \dots$$