Lecture 9:
The Big Bang, the horizon and space-time diagrams of the expanding Universe.

L: The Friedmann-Robertson-Walker metric is:

$$-ds^2 = c^2\,dt^2 - a^2(t)\left[dr^2 + S_k^2(r)(d\theta^2 + \sin^2\theta d\phi^2)\right] \ . \tag{1}$$

Q: What is the history of the expansion $a(t)$? Draw the function $a(t)$ versus $H_0 t$, in a diagram where we fix at the present time, $a(t_0) = 1$ and $\dot{a}(t_0) = H_0$.

L: Simple examples: $\Omega_m = 1$, $a(t) = (3H_0 t/2)^{2/3}$. Empty Universe, $\Omega = 0$: $a(t) = H_0 t$. Radiation dominated Universe: $a(t) = (2H_0 t)^{1/2}$. Draw also the cases for the open and closed Universe with no radiation or dark energy.

Q: For the same value of H_0, which Universe is older and younger, the open or the closed one? The matter or the radiation dominated one?

Q: What happens for the Benchmark model with $\Omega_{m0} = 0.3$, $\Omega_{\Lambda 0} = 0.7$? Is this Universe older or younger than without the cosmological constant?

L: Now, we want to draw a space-time diagram of the Universe, which can be conveniently reduced to two dimensions if we eliminate the θ, ϕ coordinates, so we only look at radial trajectories. Every point in this $r - t$ diagram represents a whole spherical surface. We would like light rays to be straight lines moving at an angle of 45 degrees, and for that we use units like second and light-second.

L: Using comoving coordinates in this diagram is very useful, then all galaxies following the Hubble flow are represented as world lines moving vertically up in cosmic time. But we have a problem: the time-varying scale factor $a(t)$ means that light-rays no longer move at 45 degrees because, in a light second, they move over 1 comoving light year divided by $a(t)$.

L: To solve this, we define conformal time, a kind of "comoving time": $d\eta = dt/a(t)$. So now, let us draw the whole history of the Universe in a $c\eta - r$ diagram. Light rays always move at 45 degrees in this diagram.

Note: these are not exactly like Penrose diagrams, for that you need to define new coordinates to bring infinity into a finite space-time map.

Q: What is $\eta(t)$ in the matter-only Universe?

$$d\eta = (t/t_0)^{-2/3}\,dt; \qquad \eta = 3t_0\,(t/t_0)^{1/3} = \frac{2}{H_0} a^{1/2} \ . \tag{2}$$

L: In general, $d\eta = da/(Ha^2)$.

Q: What is η for the radiation-dominated epoch of the Universe? We use $H \simeq H_0 \Omega_{r0}^{1/2}/a^2$ (assuming as usual $a_0 = 1$ at present, and so $a = 1/(1+z)$), and $t = 1/(2H)$, to find

$$\eta = \frac{a}{H_0 \Omega_{r0}^{1/2}} = \left(\frac{2t}{H_0}\right)^{1/2} \Omega_{r0}^{1/4} \ . \tag{3}$$

Q: What happens in an open model with only matter and curvature?

$$d\eta = \frac{1}{H_0} \frac{da}{a(\Omega_0/a + 1 - \Omega_0)^{1/2}} \ . \tag{4}$$

For very large a (in the long-term future, when matter becomes highly diluted and negligible compared to curvature), we can approximate the integral to:

$$\eta \simeq \frac{1}{H_0\sqrt{\Omega_0}} \log a + constant \ . \tag{5}$$

Note that for large times, $a \sim t$.

In all these cases, as t goes to zero (Big Bang), η goes to zero. As t goes to infinity, η goes to infinity. When looking to the past, we can only see out to the horizon; to see the Universe further, we have to wait a long time, although we'll be able to see as far as we want if we just wait long enough.

Q: The Big Bang is not an event. It is a beginning that took place simultaneously in every point of space. This brings up the horizon problem: why was the Universe so close to homogeneous, without causal communication? And yet, not exactly homogeneous, in such a way that present galaxies formed? In the Big Bang model without anything else, the only answer is that God simply made the Universe like that.

L: Now we look at the Benchmark model, which contains dark energy.

$$d\eta = \frac{1}{H_0} \frac{da}{a^2(\Omega_0/a^3 + 1 - \Omega_0)^{1/2}} \ . \tag{6}$$

For large a, the result of integrating is:

$$\eta \simeq \eta_{\max} - \frac{1}{H_0\sqrt{1-\Omega_0}a} \ . \tag{7}$$

As a function of time, we can reexpress this as:

$$dt \simeq \frac{da}{H_0\sqrt{1-\Omega_0}a} \ ; \qquad t \simeq \frac{1}{H_0\sqrt{1-\Omega_0}}(\log a + \text{constant}) \ ; \tag{8}$$

$$\eta \simeq \eta_{\max} - \frac{\exp\left[-H_0\sqrt{1-\Omega_0}(t-t_1)\right]}{H_0\sqrt{1-\Omega_0}} \ . \tag{9}$$

This diagram looks quite different. The infinitely far future corresponds to a finite conformal time $\eta_{\max}$. Without any matter (only cosmological constant), as a becomes small η goes to negative infinity (there is no Big Bang). With finite matter, there is a Big Bang but then also a maximum $\eta_{\max}$ which is never exceeded.