Propagators

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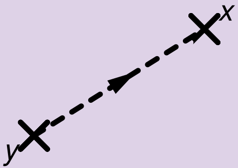
Propagators

- functions that represent the propagation of particles from point y to point x.
- Closely related to commutators.

A particle that is created at point y, propagates to point x, and is annihilated



$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle &= \langle 0|\phi^+(x)\phi^+(y)+\phi^-(x)\phi^-(y)+\phi^+(x)\phi^-(y)+\phi^-(x)\phi^+(y)|0\rangle \\ &= \langle 0|\phi^+(x)\phi^-(y)|0\rangle \\ &= \langle 0|\phi^-(y)\phi^+(x)+[\phi^+(x),\phi^-(y)]|0\rangle \\ &= \boxed{\langle 0|0\rangle[\phi^+(x),\phi^-(y)]\equiv D(x-y)\equiv \Delta^+(x-y)} \end{aligned} \tag{1}$$

- Definition:

$$\Delta^-(x-y)\equiv[\phi^-(x),\phi^+(y)] = -[\phi^+(y),\phi^-(x)] = -\Delta^+(y-x) = -D(y-x) \tag{2}$$

$$
\begin{aligned}
D(x-y) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3 \sqrt{2E_q}} e^{-i(px-qy)} [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}^\dagger] \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3 \sqrt{2E_q}} e^{-i(px-qy)} (2\pi)^3 \delta^3(\boldsymbol{p}-\boldsymbol{q}) \\
&\quad \text{since } \boldsymbol{p} = \boldsymbol{q} \Rightarrow E_p = E_q \\
&= \boxed{\int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} e^{-ip(x-y)} = D(x-y) = \Delta^+(x-y)} \qquad (3)
\end{aligned}
$$

Expression (3) is Lorentz-invariant.

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} e^{-ip(x-y)} = D(x-y) = \Delta^+(x-y)$$

- If $x-y$ is time-like, $(x-y)^2 > 0$:
  - choose a reference frame in which $\boldsymbol{x}-\boldsymbol{y}=\mathbf{0}$,
  - define $t \equiv x^0 - y^0$

$$\begin{aligned}
D(x-y) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} e^{-iE_p t} \\
\int \mathrm{d}^3 p &= 4\pi \int_0^\infty |\boldsymbol{p}|^2 \mathrm{d}|\boldsymbol{p}| = 4\pi \int_m^\infty |\boldsymbol{p}| E_p \mathrm{d}E_p \\
D(x-y) &= \frac{4\pi}{2(2\pi)^3} \int_m^\infty |\boldsymbol{p}| e^{-iE_p t} \mathrm{d}E_p = \frac{1}{4\pi^2} \int_m^\infty \sqrt{E^2 - m^2} e^{-iEt} \mathrm{d}E
\end{aligned}$$

for $t \to \infty$, e^{-iEt} is largely oscillating, and only the smallest values of E survive the integration:

$$\sim e^{-imt} \Rightarrow \text{time evolution of a wave function of E=m}$$

$$\int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)} = D(x-y) = \Delta^+(x-y)$$

- if $x - y$ is space-like $(x-y)^2 < 0$:
  - choose a reference frame: $x^0 = y^0$
  - define $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$:

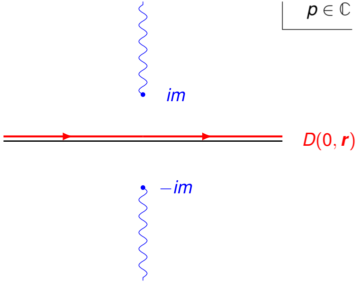
$$\begin{aligned}
D(x-y) &= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{i\boldsymbol{p}\cdot\boldsymbol{r}} \\
&= \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \int_0^\infty \frac{|\boldsymbol{p}|^2 d|\boldsymbol{p}|}{(2\pi)^3 2E} e^{i|\boldsymbol{p}||\boldsymbol{r}|\cos\theta} \\
&= \frac{-i}{2(2\pi)^2|\boldsymbol{r}|} \int_0^\infty \frac{|\boldsymbol{p}| d|\boldsymbol{p}|}{E} (e^{i|\boldsymbol{p}||\boldsymbol{r}|} - e^{-i|\boldsymbol{p}||\boldsymbol{r}|})
\end{aligned}$$

to easy the notation we define $p \equiv |\boldsymbol{p}|$; $r \equiv |\boldsymbol{r}|$ then

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{p^2+m^2}} p\, e^{ipr}$$

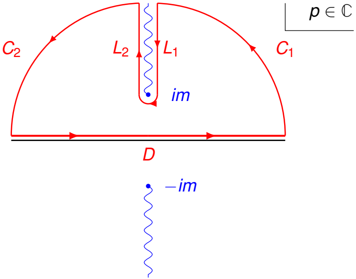
$\Rightarrow$ move to the complex plane

$\Rightarrow$ two branch cuts: $\sqrt{p^2+m^2} = 0 \Rightarrow p = \pm im$

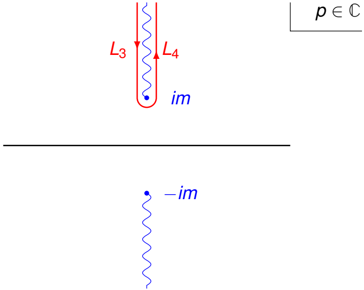


$$D(0,r) = \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\sqrt{p^2+m^2}}\, p\, e^{ipr}$$

$r > 0 \Rightarrow$ close the contour in the upper-side:



$$\begin{aligned}
0 &= \oint = \int_D + \int_{C_1} + \int_{L_1} + \int_{L_2} + \int_{C_2} \\
\int_D &= D(x-y) \\
\int_{C_1} &= \int_{C_2} = 0 \ \text{ since } p = i\rho \to e^{ipr} = e^{-\rho r} \to 0 \\
D(x-y) &= -\int_{L_1} - \int_{L_2} = +\int_{L_3} + \int_{L_4}
\end{aligned}$$



$$
\begin{aligned}
D(x-y) &= -\int_{L_1} - \int_{L_2} = +\int_{L_3} + \int_{L_4} \\
\int_{L_4} &= \frac{-i}{2(2\pi)^2 r} \int_{im}^{i\infty} \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} p e^{ipr} \quad , \quad p = i\rho \\
&= \frac{-i}{2(2\pi)^2 r} \int_{m}^{\infty} \frac{-\mathrm{d}\rho}{\sqrt{-\rho^2+m^2}} \rho e^{-\rho r} \\
&= \frac{1}{2(2\pi)^2 r} \int_{m}^{\infty} \frac{\mathrm{d}\rho}{\sqrt{\rho^2-m^2}} \rho e^{-\rho r}
\end{aligned}
$$

The branch jump picks up a factor 2:

$$D(x-y) = \int_{L_4} + \int_{L_3} = 2 \int_{L_4}$$

in the end

$$D(x-y) = \frac{1}{4\pi^2 r} \int_{m}^{\infty} \frac{\rho e^{-\rho r} \mathrm{d}\rho}{\sqrt{\rho^2 - m^2}} \rightsquigarrow e^{-mr} \neq 0$$

$\Rightarrow$ The propagation of a field is $\neq 0$ over space-like regions!

$\Rightarrow$ Causality????

Quantum Mechanics

Two operators can influence each other if:

$$[A, B] \neq 0$$

if $[A, B] = 0 \Rightarrow$ the results of measurements of B do not influence the measurements of A.

$\Rightarrow$ compute **NOT** the field propagation

$$\langle 0|\phi(x)\phi(y)|0\rangle$$

but the fields commutators at two different points:

$$\begin{aligned} [\phi(x), \phi(y)] &= 0 \Rightarrow \text{ can not influence each other} \\ [\phi(x), \phi(y)] &\neq 0 \Rightarrow \text{ influence each other} \end{aligned}$$

up to now we only know the e.t.c. commutators:

$$[\phi(x), \phi(y)] \quad ; \quad x^0 = y^0$$

For $x^0 \neq y^0$ we can obtain the propagator

$$
\begin{aligned}
[\phi(x), \phi(y)] &= [\phi^+(x) + \phi^-(x), \phi^+(y) + \phi^-(y)] \\
&= \underbrace{\cancel{[\phi^+(x), \phi^+(y)]}}_{[a,a]=0} + \underbrace{\cancel{[\phi^-(x), \phi^-(y)]}}_{[a^\dagger, a^\dagger]=0} + [\phi^+(x), \phi^-(y)] + [\phi^-(x), \phi^+(y)] \\
&= D(x-y) - D(y-x) \equiv \Delta^+(x-y) + \Delta^-(x-y) \\
&\equiv \Delta(x-y) \qquad (4)
\end{aligned}
$$

- $(x-y)$ space-like: $(x-y)^2 < 0$

  $\Rightarrow$ go to a ref. frame where $x^{0\prime} = y^{0\prime}$:

$$
\begin{aligned}
D(x'-y') &= D(x^{0\prime} - y^{0\prime}, \boldsymbol{x}' - \boldsymbol{y}') = D(0, \boldsymbol{x}' - \boldsymbol{y}') \\
D(y'-x') &= D(y^{0\prime} - x^{0\prime}, \boldsymbol{y}' - \boldsymbol{x}') = D(0, \boldsymbol{y}' - \boldsymbol{x}')
\end{aligned}
$$

$D(0, \boldsymbol{y}' - \boldsymbol{x}')$: Rotation of angle π: $\boldsymbol{y}'' - \boldsymbol{x}'' = \boldsymbol{x}' - \boldsymbol{y}'$

$\Rightarrow$ this is a Lorentz transformation

$$
D(0, \boldsymbol{y}' - \boldsymbol{x}') = D(0, \boldsymbol{y}'' - \boldsymbol{x}'') = D(0, \boldsymbol{x}' - \boldsymbol{y}')
$$

$$
[\phi(x), \phi(y)] = D(0, \boldsymbol{x}' - \boldsymbol{y}') - D(0, \boldsymbol{x}' - \boldsymbol{y}') = 0 \quad \text{for} \quad (x-y)^2 < 0
$$

$\Rightarrow$ micro-causality.

- $(x-y)$ time-like $(x-y)^2 > 0$: $\Rightarrow$ not the same computation
  - no (proper) Lorentz transformation changes: $x-y \to y-x$.
  - In a ref. frame $\boldsymbol{x}-\boldsymbol{y}=\mathbf{0}$: $x-y=(t,\mathbf{0})$: $y-x=(-t,\mathbf{0})$,
  - and Lorentz transformations do not change the time sign!

Propagator properties

Green's function

Define: $\Box_x = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}$:

$$(\Box_x + m^2)\Delta(x-y) = [(\Box_x + m^2)\phi(x), \phi(y)] = [0, \phi(y)] = 0$$

Green's function differential equation for the Klein-Gordon differential operator.

Expression

$$
\begin{aligned}
\Delta(x-y) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right) \\
&= \frac{-2i}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2E_p}\sin(p(x-y)) \\
&= \frac{-2i}{(2\pi)^4}\int \mathrm{d}^4p\ (2\pi)\delta(p^2-m^2)\,\Theta(p^0)\ \sin(p(x-y)) \qquad (5)
\end{aligned}
$$

- Transform 3-D Lorentz invariant momentum integration $\rightarrow$ 4-D
- Use Heaviside Θ to select $p^0 > 0$

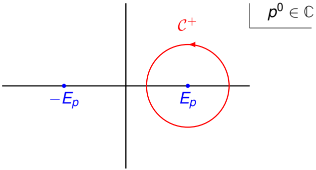
Complex-integral representation

- Job of the δ-function in the 4-D integration (5): is to pick up the point $p^0 = \pm\sqrt{\boldsymbol{p}^2 + m^2}$
- the $\Theta(p^0)$ chooses $p^0 > 0$

  $\Rightarrow$ Same effect by using the
  residue theorem of complex integrals,
  by choosing and appropriate function with a
  pole at $p^0 = +\sqrt{\boldsymbol{p}^2 + m^2}$

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \frac{\mathrm{d}p^0}{(2\pi)} \frac{e^{-ipx}}{(p^0)^2 - \boldsymbol{p}^2 - m^2}$$

$p^0 \in \mathbb{C}$: two single poles at: $p^0 = \pm\sqrt{\boldsymbol{p}^2 + m^2} \equiv \pm E_p$

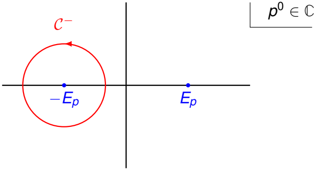


$$f(p^0) = \frac{e^{-ipx}}{(p^0)^2 - \boldsymbol{p}^2 - m^2} = \frac{e^{-ipx}}{(p^0 - E_p)(p^0 + E_p)}$$

Residue at $p^0 = E_p$: $\lim\limits_{p^0 \to E_p} (p^0 - E_p) f(p^0) = \left.\dfrac{e^{-ipx}}{(p^0 + E_p)}\right|_{p^0 = E_p} = \dfrac{e^{-ipx}}{2E_p}$

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int_{\mathcal{C}^+} \frac{\mathrm{d}p^0}{2\pi} \frac{e^{-ipx}}{p^2 - m^2} = 2\pi i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\pi} \frac{e^{-ipx}}{2E_p} = i\Delta^+(x)$$

$$\Delta^+(x) = -i \int_{\mathcal{C}^+} \mathrm{d}p^0 \int \frac{\mathrm{d}^3 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2} \tag{6}$$



Residue at $p^0 = -E_p$:

$$\lim_{p^0 \to -E_p} (p^0 + E_p) f(p^0) = \left. \frac{e^{-ipx}}{(p^0 - E_p)} \right|_{p^0=-E_p} = \left. \frac{e^{-ip^0 t} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{(p^0 - E_p)} \right|_{p^0=-E_p} = \frac{e^{iE_p t} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{-2E_p}$$

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int_{\mathcal{C}^-} \frac{\mathrm{d}p^0}{2\pi} \frac{e^{-ipx}}{p^2 - m^2} = 2\pi i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\pi} \frac{e^{iE_p t} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{-2E_p}$$

convert the argument $\to$ scalar product $qy = q^0 y_0 - \boldsymbol{q} \cdot \boldsymbol{y}$
(a $-$ sign between the time and space part)‘

$\Rightarrow$ integral over the full $\boldsymbol{p}$-space 3-volume

$\Rightarrow$ variable change: $\boldsymbol{q} = -\boldsymbol{p}$, $E_q = E_p$, jacobian=1

$$-i \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{e^{i(E_q(t) - \boldsymbol{q}\cdot\boldsymbol{x})}}{2E_q} = -i \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{e^{iqx}}{2E_q} = -i \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{e^{-i(q(-x))}}{2E_q}$$

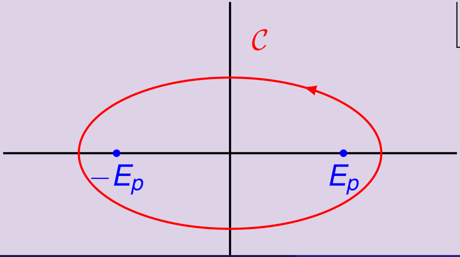
$$= -i\Delta^+(-x) = i\Delta^-(x)$$

$$\Delta^-(x) = -\Delta^+(-x) = -i \int_{\mathcal{C}^-} \frac{\mathrm{d}p^0}{2\pi} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{-ipx}}{p^2 - m^2}$$

$$\Delta^+(x) = -i \int_{\mathcal{C}^+} \mathrm{d}p^0 \int \frac{\mathrm{d}^3 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2}$$

$\Delta(x) = \Delta^+(x) + \Delta^-(x) \Rightarrow$ Take a circuit $\mathcal{C}$ including both poles

$$\boxed{\Delta(x) = \Delta^+(x) + \Delta^-(x) = -i \int_{\mathcal{C}} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2}} \tag{7}$$



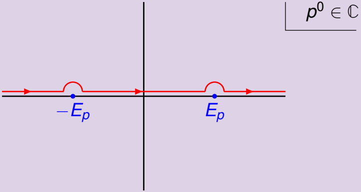
$p^0 \in \mathbb{C}$

Other kind of propagators

Different circuit integrations $\Rightarrow$ different propagators.

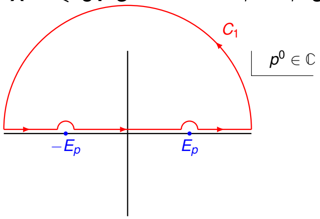
Example 1:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{-ipx}}{p^2 - m^2}$$



p^0 integration circuit *slightly above* the real axis

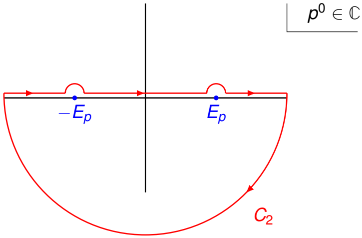
- if $x^0 < 0$: $e^{-ip^0x^0} \xrightarrow{p^0 \to iR} e^{Rx^0} \xrightarrow{R\to\infty} 0$



$$0 = \oint = \int_{-\infty}^{+\infty} + \underbrace{\int_{C_1}}_{=0}$$

$$\int_{-\infty}^{+\infty} = 0 \text{ for } x^0 < 0$$

- if $x^0 > 0$: $e^{-ip^0x^0} \xrightarrow{p^0 \to -iR} e^{-Rx^0} \xrightarrow{R\to\infty} 0$



$$\oint = -\int_{\mathcal{C}}$$

$$\oint = \int_{-\infty}^{+\infty} + \underbrace{\int_{C_2}}_{=0}$$

$$\int_{-\infty}^{+\infty} = -\int_{\mathcal{C}} \text{ for } x^0 > 0$$

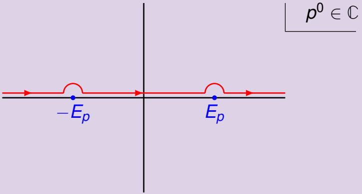
using the propagator definition (7):

$$-i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2} = \begin{cases} 0 & x^0 < y^0 \\ -\Delta(x-y) & x^0 > y^0 \end{cases}$$

$\Rightarrow$ this propagates **ONLY** when x^0 is in the future of y^0

Retarded propagator

$$\boxed{\begin{aligned} D_R(x-y) &= i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2} = \left\{ \begin{array}{ll} 0 & x^0 < y^0 \\ \Delta(x-y) & x^0 > y^0 \end{array} \right\} \\ &= \Theta(x^0-y^0)\Delta(x-y) = \Theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle \\ & p^0 \text{ integration circuit slightly above the real axis} \end{aligned}} \tag{8}$$

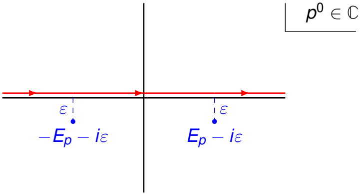


Retarded propagator (8): non-homogenous Green's function of the Klein-Gordon operator (See exercise sheet.)

$$(\Box_x + m^2) D_R(x-y) = -i\delta^4(x-y)$$

Alternative representation: move the poles *slightly below* the real axis:

$$\lim_{\varepsilon\to 0^+} : -E_p - i\varepsilon \quad ; \quad +E_p - i\varepsilon$$



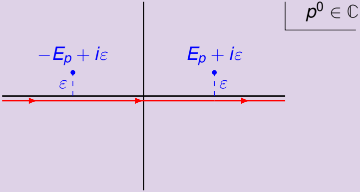
The retarded propagator in momentum-space representation

$$D_R(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \tilde{D}_R(p)\, e^{-ipx} \Longrightarrow \boxed{\tilde{D}_R(p) = \frac{i}{p^2 - m^2}}$$

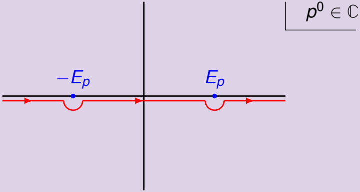
by taking the appropriate p^0-circuit or p^0 pole position.

Example 2:

Poles slightly above $\mathbb{R}$

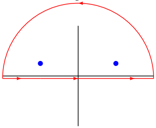


Circuit slightly below $\mathbb{R}$



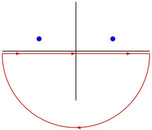
$$\lim_{\varepsilon \to 0^+} : -E_p + i\varepsilon \quad ; \quad +E_p + i\varepsilon$$

- if $x^0 - y^0 < 0$: $e^{-ip^0(x^0-y^0)} \xrightarrow{p^0 \to iR} e^{R(x^0-y^0)} \xrightarrow{R \to \infty} 0$



$$\int_{-\infty}^{+\infty} = \Delta(x-y) \text{ for } x^0 < y^0$$

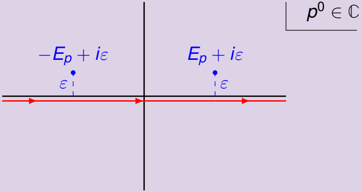
- if $x^0 - y^0 > 0$: $e^{-ip^0(x^0-y^0)} \xrightarrow{p^0 \to -iR} e^{-R(x^0-y^0)} \xrightarrow{R \to \infty} 0$

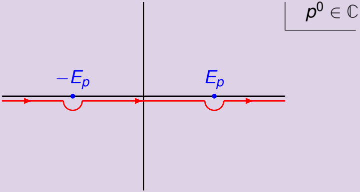


$$\int_{-\infty}^{+\infty} = 0 \text{ for } x^0 > y^0$$

Advanced propagator

$$\boxed{\begin{aligned} D_A(x-y) &= -i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2} \\ &= \Theta(y^0-x^0)\Delta(x-y) \\ &\quad p^0 \text{ poles slightly above the real axis:} \\ &\quad -E_p+i\varepsilon \quad ; \quad +E_p+i\varepsilon \quad ; \quad \varepsilon\to 0^+ \end{aligned}} \tag{9}$$





The retarded and Feynman propagators

Retarded propagator

- Propagation of a particle from point y to point x,
- when $y^0 < x^0 \Rightarrow x$ is in the future of y

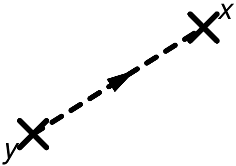
$$
\begin{aligned}
D_R(x-y) &= \Theta(x^0-y^0)[\phi(x),\phi(y)] = \Theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle = \\
&= \Theta(x^0-y^0)\langle 0|\cancelto{0=[a,a]}{[\phi^+(x),\phi^+(y)]} + \cancelto{0=[a^\dagger,a^\dagger]}{[\phi^-(x),\phi^-(y)]} \\
&\quad +[\phi^+(x),\phi^-(y)] + [\phi^-(x),\phi^+(y)]|0\rangle
\end{aligned}
$$

apply $\phi^+|0\rangle = 0$

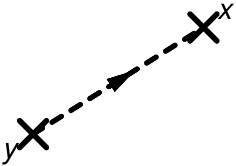
$$
\begin{aligned}
D_R(x-y) &= \Theta(x^0-y^0)\langle 0|\phi^+(x)\phi^-(y) - \phi^+(y)\phi^-(x)|0\rangle \\
&= \Theta(x^0-y^0)\left(\Delta^+(x-y) - \Delta^+(y-x)\right) \\
&= \Theta(x^0-y^0)\left(\Delta^+(x-y) + \Delta^-(x-y)\right) \\
&= \Theta(x^0-y^0)\Delta(x-y)
\end{aligned}
$$

$$D_R(x-y) = \Theta(x^0 - y^0)\left(\Delta^+(x-y) + \Delta^-(x-y)\right)$$

$\langle 0|\phi^+(x)\phi^-(y)|0\rangle \equiv \Delta^+(x-y)$



$-\langle 0|\phi^+(y)\phi^-(x)|0\rangle \equiv \Delta^-(x-y)$



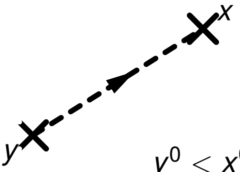
$D_R(x-y)$

represents

- a particle which moves from $y \to x$ or $x \to y$,
- but with x in the future of y.

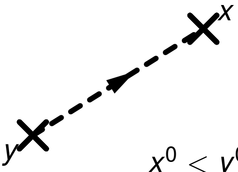
QFT: interactions are described by particle exchange

- e.g. if ϕ is a π^0 meson, and *n*, *n′* are nucleons



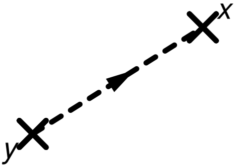
$y^0 < x^0$

+

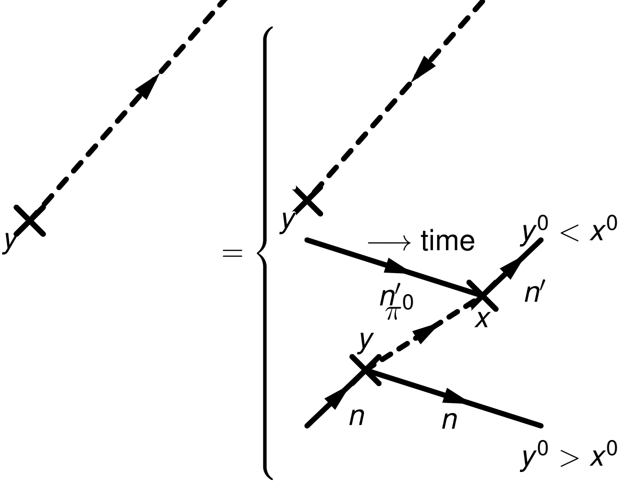


$x^0 < y^0$

Can we represent?



(10)



(11)

$$
\begin{aligned}
\Delta_F(x-y) &= \Theta(x^0-y^0)\langle 0|\phi^+(x)\phi^-(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\phi^+(y)\phi^-(x)|0\rangle \\
&= \Theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle \\
&= \Theta(x^0-y^0)\Delta^+(x-y) + \Theta(y^0-x^0)\Delta^+(y-x) \\
&= \Theta(x^0-y^0)\Delta^+(x-y) - \Theta(y^0-x^0)\Delta^-(x-y) \qquad (12)
\end{aligned}
$$

Define: the time-ordered product T:

$$T\{\phi(x),\phi(y)\} = \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & x^0 < y^0 \end{cases}$$

put the *earliest* field to the right.

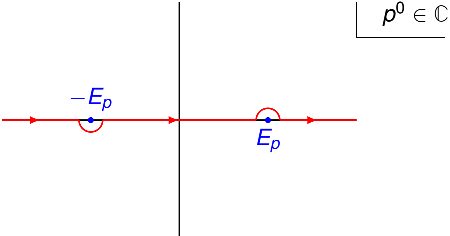
Definition: **Feynman propagator**

$$\Delta_F(x-y) = \langle 0| T\{\phi(x),\phi(y)\}|0\rangle \tag{13}$$

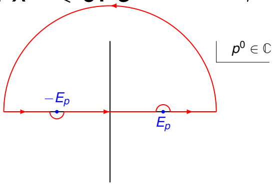
an can be computed from expression (12).

complex-integral representation:

$$\Delta_F(x) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ipx}$$

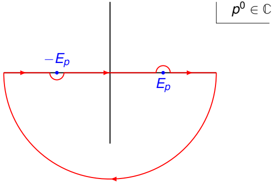


- If $x^0 < 0$: $e^{-ip^0 x^0} \xrightarrow{p^0 \to iR} e^{Rx^0} \xrightarrow{R \to \infty} 0$



$$\begin{aligned} \Delta_F(x) &= i \int_{\mathcal{C}^-} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2} = i(i\Delta^-(x)) \\ &= -\Delta^-(x) \ , \ \ (x^0 < 0) \end{aligned}$$

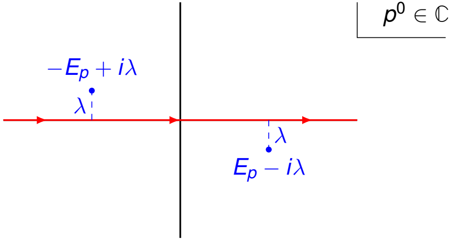
- if $x^0 > 0$: $e^{-ip^0 x^0} \xrightarrow{p^0 \to -iR} e^{-Rx^0} \xrightarrow{R \to \infty} 0$



$$\begin{aligned} \Delta_F(x) &= i \int_{-\mathcal{C}^+} \int \frac{\mathrm{d}^4 p}{(2pi)^4} \frac{e^{-ipx}}{p^2 - m^2} = i(-i\Delta^+(x)) \\ &= \Delta^+(x) \ , \ \ (x^0 > 0) \end{aligned}$$

Instead of chosing a circuit, we can *move* the poles out of the real axis:

$$\Delta_F = \lim_{\lambda \to 0^+}$$



Poles are at: $(+E_p - i\lambda) \;\; ; \;\; (-E_p + i\lambda)$

$$
\begin{aligned}
\text{Denom.} &= (p^0 - (E_p - i\lambda))(p^0 - (-E_p + i\lambda)) \\
&= (p^0 - (E_p - i\lambda))(p^0 + (E_p - i\lambda)) \\
&= (p^0)^2 - (E_p - i\lambda)^2 \\
&= (p^0)^2 - E_p^2 + 2iE_p\lambda + \lambda^2 \;\;, \;\; (\lambda \to 0^+) \\
&= (p^0)^2 - E_p^2 + 2iE_p\lambda \;\;, \;\; (\varepsilon = 2\lambda E_p) \\
&= (p^0)^2 - E_p^2 + i\varepsilon = (p^0)^2 - (\boldsymbol{p}^2 + m^2) + i\varepsilon \\
&= \boxed{p^2 - m^2 + i\varepsilon \;\;, \;\; (\varepsilon \to 0^+)}
\end{aligned}
$$

$+i\varepsilon$ prescription (Feynman prescription) for the pole placement:

$$\Delta_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ipx} \quad \text{with} \quad \varepsilon \to 0^+ \tag{14}$$