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More about the Massive Schwinger Model*

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The massive Schwinger model is quantum electrodynamics of a Dirac particle of mass m and charge e in $1 + 1$ dimensions. It is known that the physics of the model depends on an arbitrary parameter independent of e and m, an angle, θ, $|\theta| \leqslant \pi$. I give a physical explanation of this angle, and explain why a corresponding parameter does not appear in $(3 + 1)$-dimensional electrodynamics. I also compute some quantitative properties of the theory for both weak coupling, $e \ll m$, and strong coupling, $e \gg m$, and conjecture a qualitative description of the model that interpolates smoothly between weak and strong coupling. A typical quantitative result is that for weak coupling and $|\theta| \neq \pi$, the number of stable particles in the theory is

$$\frac{4m^2}{\pi e^2}\frac{1}{(1-\theta^2/\pi^2)}[2\sqrt{3} - \ln(2+\sqrt{3})] + O(1).$$

I do similar computations for a generalization of the model with "flavor $SU(2)$," i.e., with two fermions of equal charge and mass. For weak coupling the results are very much like those for the massive Schwinger model, but for strong coupling there are some surprising differences.

1. INTRODUCTION AND CONCLUSIONS

The massive Schwinger model is quantum electrodynamics in two-dimensional space-time. The model is defined by the Lagrangian density[1]

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi, \tag{1.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{1.2}$$

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[1] Notation: $g_{00} = -g_{11} = 1$. $\epsilon^{01} = -\epsilon^{10} = 1$. $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. x is always the space coordinate, x^1, and p is always the spatial momentum, p^1. $\Theta(x)$ is the unit step function; θ is an angle. Π is a canonical momentum density; π is a numerical constant.

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and e and m are positive numbers. The model requires no infinite renormalizations, other than a trivial renormalization of the zero-point energy; in particular, e and m are finite (though bare) parameters.

In a recent paper [1], Jackiw, Susskind, and I Investigated this model and argued that it displayed quark trapping. (Here, "quark" is just a pretentious name for the fundamental charged fermion.) This paper continues the investigation. Reference [1] was devoted to establishing some general qualitative properties of the model; this paper is mainly devoted to some approximate (but, I hope, reliable) quantitative computations, although some qualitative insights emerge along the way.

Perhaps the most surprising of the results of [1] is that the solution to the model involves an arbitrary parameter, an angle, θ, totally independent of e and m. The physics of the model is a periodic function of θ with period 2π. In Section 2, I give a new explanation of the origin of this angle, and explain why a similar angle does not appear in four-dimensional quantum electrodynamics.

Equation (1.1) involves a dimensionless parameter, e/m. When e/m goes to infinity, the model becomes the exactly soluble Schwinger model [2, 3]; when e/m goes to zero, the model becomes the exactly soluble free Dirac theory. Since the model is exactly soluble in both limits, it should be possible to do approximate calculations for both strong coupling, $e \gg m$, and weak coupling, $m \gg e$.

Section 3 is such an approximate computation of the particle spectrum of the model in the strong-coupling limit. By convention, I choose θ to lie in the interval $[-\pi, \pi]$. I find, that for $|\theta| > \pi/2$, there is exactly one stable particle in the theory; for $\pi/2 \geqslant |\theta| > 0$, there are two stable particles; for $\theta = 0$, there are three stable particles. In addition, for $|\theta| \leqslant \pi/2$, there is an infinite family of unstable particles, linearly spaced in mass.

Section 4 is an approximate computation of the particle spectrum in the weak-coupling limit. I find that N, the number of stable particles in the theory, is given by

$$N = \frac{4m^2}{\pi e^2} \frac{1}{1 - \theta^2/\pi^2} [2\sqrt{3} - \ln(2 + \sqrt{3})] + O(1), \tag{1.3}$$

for $|\theta| \neq \pi$. In addition, there are an infinite number of unstable particles in the theory, linearly spaced in mass squared for large mass.[2] Consistent with the results of [1], in all cases all particles are neutral bosons.

In Section 5, I consider the spacial case $|\theta| = \pi$, in the weak-coupling limit. I find that here there are two particles. They are almost (but not quite) free quarks. $|\theta| = \pi$ is also special in that it is the only value of θ for which the theory undergoes a phase transition as it passes from strong to weak coupling. (These orphic remarks will be explained in Section 5.)

[2] This is similar to the structure found in the 't Hooft model [4], and, indeed, the underlying physics is also similar.

In Section 6, I attempt to extend these approximations to a model with two fermions, of equal charge and mass,

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{2} \bar{\psi}_i(i\partial\!\!\!/ - eA\!\!\!/ - m)\,\psi_i\,. \tag{1.4}$$

This model is interesting because it has an internal SU(2) invariance (isospin). For weak coupling, the results are almost the same as for the massive Schwinger model. There are four times as many stable particles, arranged in approximately degenerate isosinglets and isotriplets. The approximate degeneracy is better for more massive particles than for less massive ones. For strong coupling, I am unable to do a complete computation. However, I am able to show that the lightest particle in the theory is an isotriplet, and the next lightest is an isosinglet. The isosinglet/isotriplet mass ratio is $\sqrt{3}$, whatever the value of θ. If there are other stable particles in the model, they must be $O([e/m]^{2/3})$ times heavier than these.

As a byproduct of this investigation, I show that the WKB formula for the mass spectrum of the quantum sine-Gordon equation, derived by Dashen, Hasslacher, and Neveu [5], is exact for a value of the coupling constant ($\beta^2 = 2\pi$) for which it has not previously been checked.

The strong-coupling methods described here are based on special tricks, and I have not been able to extend them beyond these two theories. On the other hand, the weak-coupling methods are applicable to any theory of quarks and gauge fields (colored quarks). I plan to report on the results of such an extension in a subsequent communication.

2. The Origin of θ

Let us attempt to write the massive Schwinger model in Hamiltonian form. Of course, before we can do this, we must impose a gauge condition. I will choose

$$A_1 = 0. \tag{2.1}$$

This is both the two-dimensional version of axial gauge and the two-dimensional version of radiation (Coulomb) gauge. The equation of motion for A_0,

$$\partial_1{}^2 A_0 = -e{:}\psi^+\psi{:} \equiv -ej_0\,, \tag{2.2}$$

then becomes an equation of constraint. (Note that there are no true dynamical degrees of freedom associated with the electromagnetic field. In one spatial dimension, there are no photons, even for free electromagnetism, because there are no transverse directions.) Just as in four dimensions, Eq. (3.2) is solved by defining the Coulomb Green's function,

$$\langle x' \mid \partial_1^{-2} \mid x \rangle = \tfrac{1}{2} \mid x - x' \mid. \tag{2.3}$$

The general solution to Eq. (3.2) is then

$$A_0 = -e\, \partial_1^{-2} j_0 - Fx - G. \tag{2.4}$$

where F and G are arbitrary c-number constants. G is an irrelevant parameter, because it does not enter into the expression for the electric field,

$$F_{01} = e\, \partial_1^{-1} j_0 + F, \tag{2.5}$$

and always can be gauged away. On the other hand, F represents a constant c-number background electric field and is a physically significant parameter.

At this point you should be feeling suspicious; nothing that I have said could not also have been said in four dimensions, and certainly there is no talk about a constant background electric field in standard treatments of four-dimensional quantum electrodynamics. There is good reason for this: If such a field existed at the beginning of the universe, or, if we attempted to create such a field now in a large region of space, say by erecting giant condenser plates at opposite ends of the galaxy, the vacuum would suffer dielectric breakdown. To create an electron–positron pair costs energy 2m, but when the components of the pair travel to the condensor plates, energy FD is gained, where F is the background field and D is the distance between the plates. Thus, for sufficiently widely separated plates, and, *a fortiori*, for infinitely separated plates (a constant field), it is always energetically favorable for the vacuum to emit pairs until F is brought down to zero. Thus, in four dimensions, if one wanted to, one could introduce a constant background electric field, but it wouldn't matter; the field would be cancelled by pair production from the vacuum.

However, the energetics of pair production is a bit different in one spatial dimension. Figure 1 shows a materialized quark–antiquark pair, in each of the two possible orientations. Displayed underneath the lines are the values of F_{01} in different regions of space. The difference between the electrostatic energy of these configurations and that of the vacuum without a pair is

$$\begin{aligned} \Delta E &= \tfrac{1}{2} \int dx\, [(F_{01})^2 - F^2] \\ &= \tfrac{1}{2} L[(F \pm e)^2 - F^2]. \end{aligned} \tag{2.6}$$

Thus, it is not energetically favorable for the vacuum to produce a pair if $\mid F \mid \leqslant \tfrac{1}{2}e$.

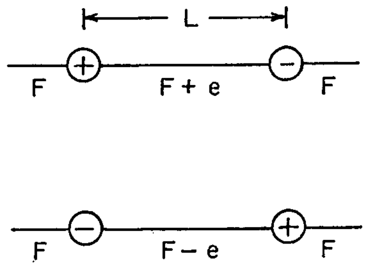

FIG. 1. The electrostatic field for two unit charges of opposite sign in a background electric field F.

If $|F| > \frac{1}{2}e$, pairs will be produced until $|F| \leqslant \frac{1}{2}e$. Physics is a periodic function of F with period e. Thus it is convenient to define an angle, θ, by

$$\theta = 2\pi F/e. \tag{2.7}$$

physics is a periodic function of θ with period 2π. By convention, I will always choose θ to lie in the interval $[-\pi, \pi]$.

We can now find the Hamiltonian density,

$$\mathscr{H} = \bar{\psi}(i\gamma_1\partial_1 + m)\psi + \tfrac{1}{2}(F_{01})^2. \tag{2.10}$$

This can be cast into another form if we restrict ourselves to states of charge zero, that is to say, states for which

$$\int j_0(x)\, dx = 0. \tag{2.9}$$

On the charge-zero subspace, trivial integration by parts yields

$$\begin{aligned} H = &\int dx\, \bar{\psi}(i\gamma_1\partial_1 + m)\, \psi \\ &- (e^2/4) \int dx\, dy\, j_0(x)\, j_0(y)\, |x - y| - eF \int dx\, x j_0(x), \end{aligned} \tag{2.10}$$

plus an irrelevant constant. Note that Eq. (2.9) is necessary for the translational invariance of this expression.

I emphasize that we lose no information by restriction ourselves to the charge-zero subspace. In particular, if the theory contains charged particles, we can discover them by searching the charge-zero subspace for states that correspond to widely-separated particle–antiparticle pairs. In fact, we shall shortly discover that the charge-zero subspace contains no such states, and therefore the theory contains no charged particles. (This statement requires some slight modification for the special case $|\theta| = \pi$.)

We are now in a position to rederive one of the main results of [1], the Bose form of the massive Schwinger model. It is known [1, 6] that the theory of a free massive Dirac field in two dimensions,

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{2.11}$$

is equivalent to the theory of a Bose field defined by

$$\mathscr{H} = N_m[\tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\partial_1\phi)^2 - cm^2 \cos 2\pi^{1/2}\phi] \tag{2.12}$$

Here, N_m denotes normal-ordering[3] with respect to the mass m, and c is a numerical constant, related to Euler's constant. (The precise value of c will be irrelevant to our investigation.) Every local charge-zero operator in the Dirac theory corresponds to some local function of θ in the Bose theory. In particular,

$$:\bar{\psi}\psi: = -cmN_m \cos 2\pi^{1/2}\phi \tag{2.13}$$

and

$$j^u \equiv :\bar{\psi}\gamma^u\psi: = \pi^{-1/2}\epsilon^{\mu\nu}\partial\nu\phi. \tag{2.14}$$

We can use Eq. (2.14) to write the interaction term in Eq. (2.8) in Bose form. Again, I will restrict myself to states of charge zero,

$$\int dx\, j^0 = \pi^{-1/2}[\phi(\infty) - \phi(-\infty)] = 0. \tag{2.15}$$

In this case,

$$F_{01} = e\pi^{-1/2}(\phi + \tfrac{1}{2}\pi^{-1/2}\theta). \tag{2.16}$$

Hence,

$$\mathscr{H} = N_m \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_1\phi)^2 - cm^2 \cos 2\pi^{1/2}\phi + \frac{e^2}{2\pi}\left(\phi + \frac{1}{2}\pi^{-1/2}\theta\right)^2\right]. \tag{2.17}$$

It will be convenient for later work to make the shift,

$$\phi \to \phi - \tfrac{1}{2}\pi^{-1/2}\theta, \tag{2.18}$$

[3] Normal-ordering with respect to a mass m is defined by Wick's theorem, with the contraction function that of a free field of mass m. Any theory of a scalar field with nonderivative interactions in two dimensions is free of divergences in every order of perturbation theory if the Hamiltonian is normal-ordered with respect to any mass m, although graphs involving contractions of two fields at the same vertex are cancelled completely only if m is the mass in the free Hamiltonian. Two useful identities: $N_m[\Pi^2 + (\partial_1\phi)^2] = N_\mu[\Pi^2 + (\partial_1\phi)^2]$, plus an irrelevant constant. $N_m \cos \beta\phi = (\mu/m)^{\beta^2/4\pi} N_\mu \cos \beta\phi$. For proofs of these, and a more detailed discussion, see [6].

and to change the normal-ordering mass to

$$\mu^2 = e^2/\pi. \tag{2.19}$$

The Hamiltonian of the theory then becomes

$$\mathscr{H} = N_\mu[\tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\partial_1 \phi)^2 + \tfrac{1}{2}\mu^2\phi^2 - cm\,\mu \cos(2\pi^{1/2}\phi - \theta)]. \tag{2.20}$$

This is the Bose form of the theory, as derived in [1].[4]

3. Particle Spectrum for Strong Coupling

The Bose form of the theory, Eq. (2.20), is most suitable for discussing strong coupling, $e \gg m$, for this displays the theory as that of a scalar meson with weak self-interactions. Thus the theory always contains at least one particle, the original meson, of mass μ (plus small corrections). If it contains other particles, they will be weakly bound n-meson states, of mass $n\mu$ (plus small corrections). Such weakly bound states always can be found by nonrelativistic reasoning.

For notational simplicity, I will choose the scale of mass such that $\mu = 1$. In the nonrelativistic limit, meson number is a good quantum number, and only the even terms in the expansion of the cosine are relevant; terms odd in ϕ have no meson-number-conserving matrix elements. Of course, the odd terms acting more than once can produce a meson-number-conserving force, as shown in Fig. 2, but this is an effect of higher order in m, and negligible for strong coupling.

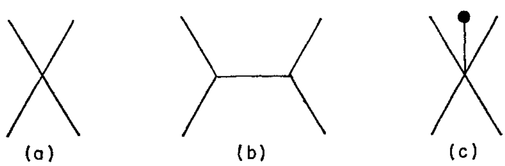

FIG. 2. Some graphs that contribute to the two-body force in a theory of massive scalar mesons with weak nonlinear self-interactions. The heavy dot in Fig. 3c is a vertex linear in the scalar field.

In the nonrelativistic limit, the even terms produce a string of many-body delta-function potentials. For example, in the two-meson subspace, only the ϕ^4 term is effective, and

$$H^{(2)} = \tfrac{1}{2}(p_1^2 + p_2^2) - mc_2 \cos\theta\,\delta(x_1 - x_2), \tag{3.1}$$

[4] This is a different definition of θ than that given in [1]. If we denote the latter by θ_1, then $\theta = \pi - \theta_1$.

where c_2 is a positive numerical constant. Likewise, in the three-meson subspace, only the ϕ^4 and ϕ^6 terms are effective, and

$$H^{(3)} = \tfrac{1}{2}(p_1{}^2 + p_2{}^2 + p_3{}^2) - mc_2 \cos\theta \sum_{i>j} \delta(x_i - x_j) + mc_3 \cos\theta\, \delta(x_1 - x_2)\, \delta(x_2 - x_3), \tag{3.2}$$

where c_3 is a positive constant. The three-body potential is negligible compared to the two-body potential. This is easily seen by introducing rescaled canonical variables,

$$\begin{aligned} p_i &= mp_i{}', \\ x_i &= x_i{}'/m. \end{aligned} \tag{3.3}$$

Equation (4.2) then becomes

$$H^{(3)} = m^2 \Big[\tfrac{1}{2}(p_1'^2 + p_2'^2 + p_3'^2) - c_2 \cos\theta \sum_{i>j} \delta(x_i{}' - x_j{}') + mc_3 \cos\theta\, \delta(x_1{}' - x_2{}')\, \delta(x_2{}' - x_3{}')\Big]. \tag{3.4}$$

This argument extends trivially to the n-meson subspace; the only important force is the two-body delta-function potential.

As we see from Eq. (3.1), when $\cos\theta$ is negative, that is to say when $|\theta|$ is greater than $\pi/2$, this potential is repulsive. Thus there are no bound states and there is only one stable particle, the meson.

On the other hand, when $\cos\theta$ is positive, that is to say when $|\theta|$ is less than $\pi/2$, the potential is attractive. It is known that in one spatial dimension an attractive delta-function potential produces one and only one n-body bound state [7]. Thus we have a sequence of particles of mass n (less small corrections).

The case $|\theta| = \pi/2$ requires special treatment. All the even terms in the expansion of the cosine vanish, and the leading two-body potential is given by Figs. 2b and 2c. Both these graphs give attractive potentials, so this case is qualitatively the same as $|\theta| < \pi/2$.

The terms we have neglected have a negligible effect on bound-state energies, but they are critical in determining bound-state stabilities. For example, the ϕ^3 term in the expansion of the cosine has a nonvanishing matrix element between an n-meson bound state and a state of $n - 1$ free mesons. Thus only the meson and the two-meson bound state are stable.

An exception occurs when θ vanishes. In this case, the cosine is an even function of ϕ, and the three-meson bound state is also stable. All higher bound states are still unstable; for example, the ϕ^4 term enables an n-meson bound state to decay into n-2 free mesons.

This concludes the analysis, and establishes all the results announced for strong coupling in Section 1.

4. Particle Spectrum for Weak Coupling

The Fermi form of the theory, Eq. (2.10), is most suitable for discussing weak coupling, $m \gg e$, for this displays the theory as that of quarks with a weak Coulomb interaction. For notational simplicity, I will choose the scale of mass such that $m = 1$.

To orient ourselves, let us begin by studying a simpler system, a nonrelativistic quark–antiquark pair with the Coulomb interaction of Eq. (2.6). In the center-of-mass frame,

$$H = 2 + p^2 + (e^2/2)(| x | - (\theta/\pi)x). \tag{4.1}$$

(I have retained the mc^2 energy to facilitate subsequent comparison with a fully relativistic computation.) If $| \theta | \neq \pi$, this potential increases linearly with x, and thus there are an infinite number of bound states.

For high E, we can compute $N(E)$, the number of bound states with energy less than E, by the semiclassical formula,

$$N(E) = (2\pi)^{-1} \int dp\, dx\, \Theta(E - H). \tag{4.2}$$

The integral is trivial:

$$\begin{aligned} N(E) &= \frac{2}{e^2\pi(1 - \theta^2/\pi^2)} \int dp\, (E - 2 - p^2)\, \Theta(E - 2 - p^2) \\ &= \frac{8(E - 2)^{3/2}}{3e^2\pi(1 - \theta^2/\pi^2)} . \end{aligned} \tag{4.3}$$

For a general potential, the error in Eq. (4.2) is $O(1)$; the semiclassical formula is accurate if $N(E)$ is large. In the case at hand, this means that for any fixed E, we have a good approximation for sufficiently-small e^2.

This simple nonrelativistic calculation is a prototype of the relativistic calculation. The relativistic calculation has several steps, and I will outline the argument before doing the detailed computations:

(1) We shall begin by dividing the Hilbert space of the theory into the vacuum, a two-particle (quark–antiquark pair) subspace, a four-particle subspace, etc. We

shall compute the matrix elements of H between states in the two-particle subspace. Of course, there are also particle-number-changing terms in H, but we shall ignore these in our lowest approximation, and take account of their effects later, using perturbation theory.

(2) We shall analyze H (restricted to the two-particle subspace) semiclassically, just as above. Just as in Eq. (4.3), $N(E)$ will turn out to be proportional to e^{-2}, and the semiclassical approximation will be justified for small e.

(3) The semiclassical formula will predict that the spacing between successive energy levels is proportional to e^2. Since the particle-number-changing terms in H are all proportional to e^2, and since they affect two-particle energy levels only in second order of pertubation theory, they shift the energies by amounts proportional to e^4. This is neglibible on the scale of the two-particle energy spectrum, and thus we shall not need to do any explicit perturbative computations.

(4) Although these terms have a negligible effect on bound-state energies, they are critical in determining bound-state stabilities. In particular, H contains terms that change the particle number by two; these enable a sufficiently excited two-particle bound state to decay into two two-particle ground states. This is energetically possible if E is greater than 4 (plus negligible corrections). All two-particle bound states of lower energy are absolutely stable; they cannot decay into free quarks, because there are no free quarks, and they cannot decay into lower bound states by the emission of photons, because there are no photons. Thus, the total number of stable two-particle bound states is $N(4)$, with an $O(1)$ error.

(5) We shall not need to search for four-particle bound states; if these exist, they can all decay into two two-particle ground states. (This may be energetically forbidden for the lowest four-particle bound states, but ignoring these states just makes another $O(1)$ error.). Likewise, bound states made of six or more particles are irrelevant.

The upshot of all this is that we need explicit computations only for the two-particle bound states. We shall now do these computations.

The first step is to compute H restricted to the two-particle subspace. A general two-particle state, $|\, p, q\rangle$ consists of a quark with momentum p and an antiquark with momentum q. (There is no need to specify the spins because there are no spins in two dimensions.) I normalize these states such that

$$\langle p', q' \,|\, p, q\rangle = \delta(p' - p)\, \delta(q' - q), \tag{4.4}$$

and define the reduced (center-of-mass) Hamiltonian by

$$\langle p', q' \,|\, H \,|\, p, -p\rangle = \delta(p' + q')\langle p' \,|\, H^R \,|\, p\rangle. \tag{4.5}$$

The easiest matrix element to compute is that of the free Hamiltonian

$$\langle p' \mid H_0^R \mid p \rangle = \delta(p' - p)\, 2(p^2 + 1)^{1/2}. \tag{4.6}$$

In operator form,

$$H_0^R = 2(p^2 + 1)^{1/2} \tag{4.7}$$

The interaction term in Eq. (3.10) is best handled by reducing it to a sum of normal-ordered operators, using Wick's theorem. Such a reduction is shown in Fig. 3. It is then straightforward to compute the contributions of the individual graphs.[5]

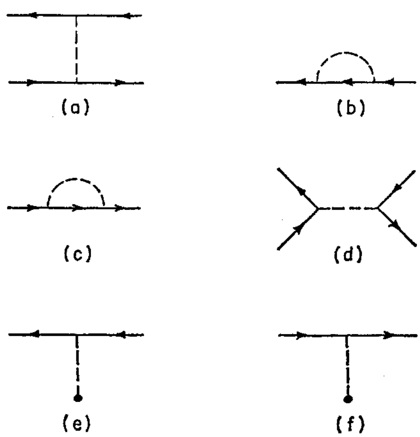

FIG. 3. The expansion of the interaction Hamiltonian in terms of annihilation and creation operators. Annihilation operators are on the right, creation operators on the left.

Figure 3a contributes

$$\begin{aligned}\langle p' \mid H_1^R \mid p \rangle = &-\frac{e^2}{8\pi} (p^2 + 1)^{-1/2} (p'^2 + 1)^{-1/2} \\ &\times [(p^2 + 1)^{1/2} (p'^2 + 1)^{1/2} + pp' + 1] \\ &\times \left| \frac{1}{(p - p' + i\epsilon)^2} + \frac{1}{(p - p' - i\epsilon)^2} \right| \end{aligned} \tag{4.8}$$

If we define an x operator in the usual way,

$$\langle p' \mid x \mid p \rangle = -i\delta'(p - p'), \tag{4.9}$$

[5] For a sketch of the computations, see the Appendix.

We can write Eq. (4.8) in operator form,

$$H_1{}^R = \frac{e^2}{4}\,[\,|\,x\,| + p(p^2+1)^{-1/2}\,|\,x\,|\,(p^2+1)^{-1/2}\,p + (p^2+1)^{-1/2}\,|\,x\,|\,(p^2+1)^{-1/2}]. \tag{4.10}$$

Figures 3b and 3c contribute identical amounts; their sum is

$$\langle p' \mid H_2{}^R \mid p\rangle = -\frac{e^2}{\pi}\,(p^2+1)^{-1/2}\,\delta(p-p'). \tag{4.11}$$

In operator form,

$$H_2{}^R = -\frac{e^2}{\pi}\,(p^2+1)^{-1/2}. \tag{4.12}$$

As one would expect from the form of the graphs, this can be interpreted as a mass renormalization,

$$2\left(p^2+1-\frac{e^2}{\pi}\right)^{1/2} = 2(p^2+1)^{1/2} - \frac{e^2}{\pi}\,(p^2+1)^{-1/2} + O(e^4). \tag{4.13}$$

The annihilation graph, Fig. 3d, contributes

$$\langle p' \mid H_3{}^R \mid p\rangle = \frac{e^2}{8\pi}\,(p^2+1)^{-1}\,(p'^2+1)^{-1}. \tag{4.14}$$

In operator form,

$$H_3{}^R = \frac{e^2}{4}\,(p^2+1)^{-1}\,\delta(x)(p^2+1)^{-1}. \tag{4.15}$$

Figures 3e and 3f, the interaction with the background field, contribute

$$\langle p' \mid H_4{}^R \mid p\rangle = ieF\delta'(p-p'). \tag{4.16}$$

In operator form,

$$H_4{}^R = -\frac{e^2\theta}{2\pi}\,x. \tag{4.17}$$

Thus,

$$H^R = H_0{}^R + H_1{}^R + H_2{}^R + H_3{}^R + H_4{}^R. \tag{4.18}$$

This is an ugly mess. Fortunately, we merely wish to do a semiclassical computation, that is to say, a computation to lowest order in $\hbar$, and most of the terms in H^R are of order $\hbar$. This is obscured by the fact that we are working in units in which

$\hbar$ is one; however, it is easy enough to restore the missing $\hbar$'s by dimensional analysis (still keeping $m = c = 1$) Thus,

$$[H^R] = [p] = 1, \tag{4.19a}$$

and

$$[x] = [e^{-2}] = [\hbar]. \tag{4.19b}$$

Hence, $H_3{}^R$ is proportional to $\hbar^2$ and $H_2{}^R$ to $\hbar$, and we can discard both of them. Likewise, in $H_1{}^R$, we can blithely commute x's and p's and ignore thecommutators. Thus,

$$H^R = H_c{}^R + O(\hbar), \tag{4.20}$$

where the classical (zeroth order in $\hbar$) Hamiltonian is given by

$$\begin{aligned} H_c{}^R &= 2(p^2 + 1)^{1/2} + (e^2/2)(|\,x\,| - (\theta/\pi)\,x) \\ &= T + V(x, \theta). \end{aligned} \tag{4.21}$$

This is simply the nonrelativistic Hamiltonian, Eq. (4.1), corrected to take account of relativistic kinematics. What we have learned by long argument and tedious computation is that all the other effects of relativistic field theory (pair production, velocity-dependent forces, etc.) are negligible in the approximation in which we are working.

Now for the semiclassical computation:

$$N(E) = (2\pi)^{-1} \int dp\, dx\, \Theta(E - H_c{}^R). \tag{4.22}$$

The x-integral is the same as before,

$$N(E) = \frac{2}{e^2\pi(1 - \theta^2/\pi^2)} \int dp(E - T)\, \Theta(E - T). \tag{4.23}$$

To do the p integral, we change variables,

$$E = 2\cosh Y, \qquad p = \sinh y. \tag{4.24}$$

Whence,

$$\begin{aligned} N(E) &= \frac{4}{e^2\pi(1 - \theta^2/\pi^2)} \int_{-Y}^{Y} dy \cosh y(\cosh Y - \cosh y) \\ &= \frac{4}{e^2\pi(1 - \theta^2/\pi^2)} [\cosh Y \sinh Y - Y]. \end{aligned} \tag{4.25}$$

For high Y, E is approximately e^Y, and

$$N(E) = \frac{E^2}{e^2\pi(1 - \theta^2/\pi^2)}\,. \tag{4.26}$$

Thus, at high mass, the bound states are linearly spaced in mass squared, as announced in Section 1.

Of course, the high-mass bound states are not stable. The number of stable particles is given by Eq. (4.25) with $E = 4$, that is to say, with $\cosh Y = 2$, $\sinh Y = \sqrt{3}$,

$$N(4) = \frac{4}{e^2\pi(1 - \theta^2/\pi^2)} [2\sqrt{3} - \ln(2 + \sqrt{3})]. \tag{4.27}$$

This is Eq. (1.3).

It is worth remarking that the simple nonrelativistic Eq. (4.3) is not at all bad for computing the number of stable particles,

$$\frac{N(4) \text{ from Eq. (4.3)}}{N(4) \text{ from Eq. (4.27)}} = 0.88. \tag{4.28}$$

Of course, Eq. (4.3) is terrible for predicting the spectrum of high-mass unstable bound states.

This establishes all the results announced for weak coupling in Section 1. However, I would like to spend a few paragraphs on the connection between the strong-coupling and weak-coupling limits. For the moment, I will restrict myself to the case $\theta = 0$. In this case, the background field, which normally breaks both C and P invariance, is absent, and the stable particles must be eigenstates of C and P.

For weak coupling, the lowest bound states are well described by the nonrelativistic Hamiltonian, Eq. (4.1). As for any even one-dimensional potential, the ground states wavefunction is even, that of the first excited state is odd, that of the second excited state is even, etc. Since the particles being bound are a fermion and an antifermion, this means the ground state is parity odd, the first excited state is parity even, etc. All the bound states are CP even.

We can transport the definitions of P and C from the Fermi form of the theory to the Bose form, using Eqs. (2.13) and (2.14). We see that the meson is pseudoscalar and CP even. Thus, in the strong-coupling description of the theory, the meson is parity odd, the two-meson bound state is parity even, etc. All these states are CP even. This is the same patterns as in weak coupling.

This is consistent with a very simple picture of the transition from weak coupling to strong coupling. For small e, the quark and antiquark sit in a shallow linear potential, and thus have very many closely spaced bound states. As we increase e, the potential becomes steeper and the bound state energies are pushed up. Thus the high-lying bound states become unstable, one after the other, as e is increased. Finally, for very large e, only the three lowest bound states remain as stable particles.

Of course, a similar picture is consistent with everything we have found for general θ (excluding $|\theta| = \pi$). The only special feature of $\theta = 0$ is that parity invariance gives us an extra consistency check.

5. Half-Asymptotic Particles

In the weak-coupling limit, $|\theta| = \pi$ is very special. For example,

$$\begin{aligned} V(x, \pi) &= 0, \qquad & x > 0, \\ &= -e^2 x, \qquad & x < 0. \end{aligned} \tag{5.1}$$

Thus if a quark is to the left of an antiquark, there is an attractive linear potential between them, but if the quark is to the right, there is no force at all. This statement is a slight exaggeration, because V is only an approximation to the interaction. Nevertheless, it does contain the electrostatic potential energy, which is the only thing that contributes to the long-range force between a quark and an antiquark. Thus, it is accurate to say that there is no long-range force between a quark and an antiquark when the quark is on the right.

Indeed, it is easy to see that if we have a string of alternating quarks and antiquarks, with the last particle on the right a quark, then there are no long-range forces between the particles. This situation is shown in Fig. 4a. As we traverse the line, whenever we pass a particle, F_{01} changes sign, but the electrostatic energy $(F_{01})^2$, is unchanged. This is not the case for other configurations of the same particles. For example, in Fig. 4b, there is a long-range attractive force between the central quark and antiquark.

In a theory with quark trapping, like the massive Schwinger model for $|\theta| \neq \pi$, widely separated quarks do not appear at all in asymptotic states. In a theory with only short-range interactions between fermions, like Yukawa theory, the fermions are genuine asymptotic particles. There are asymptotic states which can be described in the far future (or far past) as states consisting of widely-separated fermions, in arbitrary order. We have here an intermediate situation; widely separated quarks and antiquarks can appear in asymptotic states, but only in a certain order. For this reason I call the quarks and antiquarks in this case half-asymptotic particles.

There is certainly no trace of such half-asymptotic particles in the strong-coupling limit. Nevertheless, I can give hand-waving arguments that suggest the nature of the transition from weak to strong coupling.

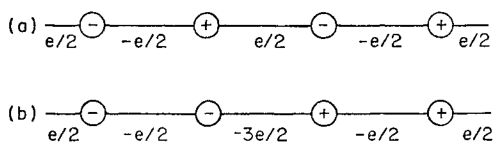

Fig. 4. The electrostatic field for some configurations of quarks and antiquarks, with F equal to $e/2$.

Let us consider the Bose form of the theory for $|\theta| = \pi$,

$$\mathscr{H} = N_\mu[\tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\partial_1\phi)^2 + \tfrac{1}{2}\mu^2\phi^2 + cm\mu\cos 2\pi^{1/2}\phi], \tag{5.2}$$

and let us treat this by semiclassical methods.[6] (I stress that I have no reason to believe that semiclassical methods will yield a good approximation for this system; that is why this is just a hand-waving argument.)

As in all semiclassical computations, we begin by finding the minima of

$$U(\phi) = \tfrac{1}{2}\mu^2\phi^2 + cm\mu\cos 2\pi^{1/2}\phi. \tag{5.3}$$

These minima are identified with the vacuum states of the quantum theory. For m much less than e, there is a unique vacuum, $\phi = 0$. However, for m much larger than e, there are two vacua, approximately located at $\phi = \pm\frac{1}{2}\pi^{1/2}$. (Note that this is special to $|\theta| = \pi$; for all other θ's, U has a unique minimum for any positive e and m.) Thus for sufficiently large e, the symmetry $\phi \to -\phi$ suffers spontaneous breakdown.

Whenever a theory of a single scalar field in two-dimensional space–time has two ground states, there exist time-independent finite-energy solutions of the classical field equations that pass monotonically from one ground state to the other as x traverses the real axis. I will call the increasing solution "a lump" and the decreasing one "an antilump". Lumps and antilumps, as classical extended systems, are half-asymptotic particles; to continuously match the field when constructing a state of several widely separated lumps and antilumps, the lumps and antilumps must alternate. In the semiclassical approximation, the lumps and antilumps are identified with quantum half-asymptotic particles.

As I have said, there is no reason to trust the quantitative details of the semiclassical approximation for this system. Nevertheless, its qualitative features offer a plausible picture of the transition from strong to weak coupling: For strong coupling, the symmetry $\phi \to -\phi$ is unbroken; there is only a single vacuum, and there are no half-asymptotic particles. Somewhere on the way to weak coupling, a phase transition occurs; the symmetry is broken, and there are two vacua and half-asymptotic particles. This is nothing but a translation into Bose language of the situation we found earlier using the Fermi form of the theory. As we can see from Eq. (2.14), the symmetry $\phi \to -\phi$ is charge conjugation in Fermi language. Thus, the two vacua are the two values of the background field, $e/2$ and $-e/2$, and the lump and antilump are the quark and antiquark.

[6] "Semiclassical" here means methods that use classical field theory as the initial approximation. In Section 4, "semiclassical" meant methods that used classical particle theory as the initial approximation. These are very different approximations, and should not be confused.

6. A MODEL WITH FLAVOR

General Remarks

I now turn to electrodynamics with two Dirac fields:

$$\mathscr{L} = \sum_{i=1,2} \bar{\psi}_i(i\partial\!\!\!/ - m - eA\!\!\!/)\,\psi_i - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{6.1}$$

This theory has an internal SU(2) symmetry, which I will call isospin. The Dirac fields form an isodoublet; the electromagnetic field is an isosinglet. It is known [8] that in two dimensions there is no spontaneous breakdown of continuous internal symmetries, unless the current-conservation equations are afflicted with anomalies, or the Higgs mechanism occurs. Neither is possible here in electromagnetism: vector currents do not develop anomalies, and there are no gauge fields coupled to the isospin currents. Thus the particles of the theory must belong to isospin multiplets.

Weak Coupling and Fine Structure

The weak-coupling analysis is a trivial extension of that of Sections 4 and 5. For $\theta = \pi$, the particles of the theory are quark–antiquark bound states, held together by Coulomb interactions. In the leading approximation of Section 4,

$$H^R = H_e{}^R = 2(p^2 + 1)^{1/2} + V(x, \theta), \tag{6.2}$$

in units where $m = 1$, independent of the isospin channel. Thus the bound states form degenerate isotriplets and isosinglets, and there are four times as many stable particles as before. The only term in H^R that removes this degeneracy is the annihilation force, Fig. 3d, which acts only in the isosinglet channel. From Eq. (4.15), this is

$$H_3{}^R = \frac{e^2}{4}(p^2 + 1)^{-1}\,\delta(x)(p^2 + 1)^{-1}. \tag{6.3}$$

This is a positive operator; thus the isosinglet states are shifted above the isotriplet states.

It is easy to estimate this energy shift.

For the low-lying states, $(E - 2) \ll 1$, we may safely make the nonrelativistic approximation,

$$H = 2 + p^2 + \frac{e^2}{2}\left(|\,x\,| - \frac{\theta}{\pi}\,x\right) + \frac{e^2}{4}\,\delta(x). \tag{6.4}$$

If we introduce the rescaled variables, $x' = e^{2/3}x$ and $p' = e^{-2/3}p$, then

$$H = 2 + e^{4/3}\left[p'^2 + \frac{1}{2}\left(|\,x'\,| - \frac{\theta}{\pi}\,x'\right)\right] + \frac{e^{8/3}}{4}\,\delta(x'). \tag{6.5}$$

Thus, the separation between the ground state and the first excited state is $O(e^{4/3})$, while the isosinglet–isotriplet splitting is $O(e^{8/3})$. The annihilation force contributes only to the fine structure of the energy spectrum.

For the higher excited states, we cannot use nonrelativistic reasoning, but we can obtain an estimate of the energy shift from the correspondence principle. For any one-dimensional dynamical system. let $F(p, x)$ be any function of the dynamical variables, without explicit dependence on $\hbar$. Let E be any energy for which classical bound motion occurs. Within any fixed small energy interval containing E, there are $O(\hbar^{-1})$ quantum bound states for small $\hbar$. The correspondence principle states that it is possible to form wave packets from these states that simulate the classical motion of energy E. Thus, if $\langle F(p, x)\rangle$ denotes the expectation value of F, averaged over all the bound states in the fixed interval, then

$$\langle F(p, x)\rangle = (1/T) \int_0^T dt\, F(p, x) + O(\hbar), \tag{6.6}$$

where the integral on the right is over one period, T, of the classical motion. In the case at hand, this implies that, to leading order,

$$\langle \delta E\rangle = \langle H_3{}^R\rangle = \frac{2}{T}\frac{e^2}{4}\,|(p^2+1)^{-2}\,(dx/dt)^{-1}\,|_{x=0}\,. \tag{6.7}$$

(The factor of 2 arises because the particle passes through the origin twice in each period.) It is trivial to compute the terms on the right-hand side of this equation from Eq. (6.2):

$$(p^2+1)\,|_{x=0} = E^2/4, \tag{6.8a}$$

$$dx/dt = 2p(p^2+1)^{-1/2}, \tag{6.8b}$$

and,

$$T = \frac{8}{e^2(1-\theta^2/\pi^2)}\,|\,p\,|_{x=0}\,. \tag{6.8c}$$

Hence,

$$\langle \delta E\rangle = \frac{e^4(1-\theta^2/\pi^2)}{E^3(E^2-4)}\,. \tag{6.9}$$

The spacing between successive isosinglet states is

$$\frac{dE}{dN} = \frac{e^2\pi(1-\theta^2/\pi^2)}{2(E^2-4)^{1/2}}\,. \tag{6.10}$$

Hence,

$$\frac{\langle \delta E\rangle}{(dE/dN)} = \frac{2e^2}{\pi E^3(E^2-4)^{1/2}}\,. \tag{6.11}$$

Once again, this is fine structure; indeed, the heavier the state, the finer the iso-singlet–isotriplet splitting. Note that Eq. (6.11) is perfectly consistent with our earlier estimate for low-lying states, because the low-lying states have energies equal to $2 + O(e^{4/3})$.

Bose Form and Strong Coupling

To study the strong-coupling limit, we need to write the theory in Bose form. Since we have two Fermi fields, we introduce two Bose fields, as in Section 2:

$$:\bar{\psi}_i \gamma^\mu \psi_i: = \pi^{-1/2} \epsilon^{\mu\nu} \partial_\nu \phi_i , \tag{6.12}$$

$$:\bar{\psi}_i \psi_i: = cmN_m \cos(2\pi^{1/2}\phi_i), \tag{6.13}$$

where $i = 1, 2$, and there is no sum on i. The electric charge density is

$$j_0 = :\psi_1{}^+\psi_1 + \psi_2{}^+\psi_2: = \pi^{-1/2}\partial_1(\phi_1 + \phi_2). \tag{6.14}$$

Thus, by the same arguments as in Section 2, the Bose form of the theory is

$$\begin{aligned}\mathscr{H} = N_m \Big[&\frac{1}{2}\Pi_1{}^2 + \frac{1}{2}\Pi_2{}^2 + \frac{1}{2}(\partial_1\phi_1)^2 + \frac{1}{2}(\partial_1\phi_2)^2 + cm^2\cos(2\pi^{1/2}\phi_1) \\ &+ cm^2\cos(2\pi^{1/2}\phi_2) + \frac{e^2}{2\pi}\left(\phi_1 + \phi_2 + \frac{1}{2}\pi^{-1/2}\theta\right)^2\Big]\end{aligned} \tag{6.15}$$

If we define

$$\phi_+ = 2^{-1/2}(\phi_1 + \phi_2 + \tfrac{1}{2}\pi^{-1/2}\theta), \tag{6.16a}$$

$$\phi_- = 2^{-1/2}(\phi_1 - \phi_2), \tag{6.16b}$$

and

$$\mu^2 = 2e^2/\pi, \tag{6.16c}$$

this becomes

$$\begin{aligned}\mathscr{H} = N_m \Big\{&\frac{1}{2}\Pi_+{}^2 + \frac{1}{2}(\partial_1\phi_+)^2 + \frac{\mu^2}{2}\phi_+{}^2 + \frac{1}{2}\Pi_-{}^2 + \frac{1}{2}(\partial_1\phi_-)^2 \\ &- 2cm^2\cos\left[(2\pi)^{1/2}\phi_+ - \frac{1}{2}\theta\right]\cos[(2\pi)^{1/2}\phi_-]\Big\}.\end{aligned} \tag{6.17}$$

This expression will be the starting point of our computations for $m \ll \mu$; it displays the theory as that of two scalar fields, one heavy and one light, with weak self-interactions.

The isospin–invariance of the theory has been obscured by these transformations, but it is still present. ϕ_+ is an isosinglet, as is clear from Eq. (6.14); ϕ_-, on the other hand, has complicated nonlinear transformation properties under a general isospin transformation. All three isospin currents can be written as functions of ϕ_-, but only the third is represented simply,

$$j_3^\mu = \tfrac{1}{2}{:}\bar{\psi}_1\gamma^\mu\psi_1 - \bar{\psi}_2\gamma^\mu\psi_2{:} = (2\pi)^{1/2}\epsilon^{\mu\nu}\partial_\nu\phi_- . \tag{6.18}$$

The others are complicated nonlinear nonlocal functions[7] of ϕ_-.

Likewise, the periodicity in θ (with period 2π) has been obscured, but it is still present. A shift in θ by 2π can be compensated for by a shift in ϕ_- by $(\pi/2)^{1/2}$.

In contrast to the one-quark theory of Section 3, we can not analyze this theory by straightforward perturbative methods. The scale of the ϕ_- mass and the magnitude of the ϕ_- couplings are given by the same small parameter, m; thus, for example, even a quite complicated radiative correction to the ϕ_+ self energy, like that shown in Fig. 5a, is proportional to m^2, because m is the only mass in the graph. For the same reason, the correction to ϕ_+ scattering of Fig. 5c is proportional to m^2, that is to say, of the same order of magnitude as the tree approximation for the same process, Fig. 5b. Thus, although we can be fairly sure there is a ϕ_+ meson (possibly unstable) with mass close to μ, it is no easy matter to compute the corrections to its mass, or to search for possible multimeson bound states. For this reason I will not attempt here to investigate the particle spectrum of the theory for masses of the order of μ or greater.

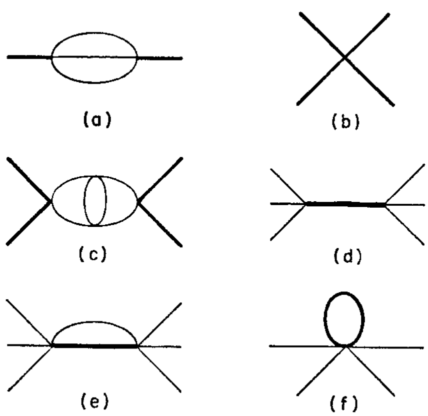

FIG. 5. Some graphs that arise in the perturbation expansion of a theory of heavy and light mesons. The heavy mesons are represented by heavy lines; the light mesons by light lines.

However, the situation is much simpler if we look for particles of mass much less than μ, and restrict ourselves to ϕ_- Green's functions. In this case, graphs with internal ϕ_+ lines, like those shown in Figs. 5d and 5e, are down by powers of

[7] These functions can easily be constructed by the methods of Mandelstam [9] and Halpern [10].

m/μ, and can reasonably be neglected. The one exception to this statement are graphs involving the emission and absorption of a virtual ϕ_+ meson from the same vertex, as in Fig. 5f. These graphs involve factors of $\ln(m/\mu)$, and are possible sources of trouble. Fortunately, they can be removed by renormal-ordering:

$$N_m \cos[(2\pi)^{1/2} \phi_+ - \tfrac{1}{2}\theta] = (\mu/m)^{1/2} N_\mu \cos[(2\pi)^{1/2} \phi_+ - \tfrac{1}{2}\theta]. \tag{6.19}$$

We can now ignore ϕ_+ altogether, and obtain the reduced form of our theory,

$$\mathscr{H} = N_m\{\tfrac{1}{2}\Pi_-^2 + \tfrac{1}{2}(\partial_1\phi_-)^2 + 2cm^{3/2}\mu^{1/2} \cos \tfrac{1}{2}\theta \cos[(2\pi)^{1/2}\phi_-]\}. \tag{6.20}$$

(As usual, $|\theta| = \pi$ is an exceptional case, and we will have to treat it separately later.) Equation (6.20) apparently involves two mass parameters, m and the coefficient of the interaction. This is an illusion which can be dispelled by renormal-ordering,

$$\mathscr{H} = N_{m'}\{\tfrac{1}{2}\Pi_-^2 + \tfrac{1}{2}(\partial_1\phi_-)^2 + m'^2 \cos[(2\pi)^{1/2} \phi_-]\} \tag{6.21}$$

where

$$m' = (2cm\mu^{1/2} \cos \tfrac{1}{2}\theta)^{2/3} \tag{6.22}$$

Some remarks are in order:

(1) The only mass parameter in Eq. (6.21) is m'. Thus, the only effect of θ on the spectrum and interactions of the lightest particles in the theory is through an overall mass scale. This is in striking contrast to the one-quark theory of Section 3. By the same token, the particles we have thrown away by our approximation are $O(\mu/m') = O(e/m)^{2/3}$ times more massive than the particles we have retained.

(2) Because ϕ_+ is an isosinglet, our approximation does *not* destroy isospin invariance. Equation (6.21) defines a (very nonobviously) isospin–invariant theory. As we shall see shortly, this put an important consistency check on our further analysis.

(3) It is easy to check that if we had given the two quarks different masses, we would still have obtained Eq. (6.21), except, of course, for a change in m'. Thus, in the strong-coupling limit, the lightest particles in the theory fall into degenerate isomultiplets and have isospin–invariant interactions, even if there is a very large isospin-breaking quark mass ratio. This is very different from the weak-coupling limit, and also from the naive quark model.

Equation (6.21) is a special case of the sine-Gordon equation, the theory of a scalar field in two space–time dimension with interaction proportional to $\cos(\beta\phi)$, where β is a positive number. This equation was the object of extensive analysis by Dashen, Hasslacher, and Neveu [5] who computed the particle spectrum of the theory for arbitrary β using a method based on the WKB approximation. Their results are:

(1) The theory contains two particles of equal mass, M, called the solition and antisolition. If one defines the conserved current,

$$j^\mu = (\beta/2\pi)\,\epsilon^{\mu\nu}\partial_\nu\phi, \tag{6.23}$$

then the soliton and antisoliton have "charge" plus and minus one. Dashen *et al.* do not claim to be able to compute M except approximately.

(2) In addition, for $\beta^2 < 4\pi$, the theory possesses soliton–antisoliton bound states, all with charge zero and with masses given by

$$M_n = 2M\sin(\beta'^2 n/16), \tag{6.24}$$

where

$$\beta'^2 = \frac{\beta^2}{1 - \beta^2/8\pi}, \tag{6.25}$$

and $n = 1, 2 \cdots < 8\pi/\beta'^2$. Dashen *et al.* have checked Eq. (6.24) against several nontrivial orders of perturbation theory and conjecture that it is exact.

Let me provisionally assume Eq. (6.24) is indeed exact, and use it to compute the mass spectrum in the case at hand, $\beta^2 = 2\pi$. The current of Eq. (6.23) is the I_3 current of Eq. (6.18); thus the solition and antisoliton carry I_3 plus and minus one. This is a reassuringly reasonable assignment for bound states of isospinor quarks and antiquarks; of course, since our theory is isospin invariant, an $I_3 = 0$ state must be around someplace to complete the isotriplet. The only possibility is that it is somewhere among the soliton-antisoliton bound states. This is easy enough to investigate: For $\beta^2 = 2\pi$,

$$\beta'^2/16 = \pi/6, \tag{6.26}$$

whence the mass of the lightest bound state is

$$M_1 = 2M\sin(\pi/6) = M. \tag{6.27}$$

We have completed the isotriplet. At the same time, we have checked Eq. (6.24). If we had not been able to complete the isotriplet, we would have known that Eq. (6.24) was wrong, for, as we have seen, at $\beta^2 = 2\pi$ the sine-Gordon equation is isospin invariant.

Equation (6.24) predicts one more bound state,

$$M_2 = 2M\sin(\pi/3) = \sqrt{3}M. \tag{6.28}$$

Since this has no partners with nonzero I_3, it must be an isosinglet.

I have now derived all the results for strong coupling stated in Section 1.

The Passage from Weak to Strong Coupling

Just as for the one-quark theory, the passage from weak to strong coupling is most easily discussed for the case $\theta = 0$, for here we can use P and C conservation (or, equivalently, P and G conservation) to keep track of energy levels.

In the weak-coupling regime, the lightest bound state has $I^{PG} = 1^{-+}$. Displaced very slightly above it (by the annihilation force) is a 0^{--} state. Above this set of levels there are a 1^{+-} state and a 0^{++} state. These states have Schrödinger wave-functions that vanish at the origin; thus the annihilation force does not shift their energies.

From Eqs. (6.12) and (6.16), it follows that ϕ_- is pseudoscalar and G even, while the ϕ_+ is pseudoscalar and G odd. The analysis of Dashen *et al.* implies that P and C transformation properties of the lightest soliton–antisoliton bound state are those of the fundamental meson field of the sine-Gordon equation; those of the next lightest bound state are those of a two-meson bound state, etc. Thus, for strong coupling, the lightest state is 1^{-+}, just as in the weak-coupling case. However, the next state up is 0^{++}, *not* 0^{--}. There is a 0^{--} state in the theory, but it is way up in mass, near mass μ. It is the fundamental ϕ_+ meson, whose stability is guaranteed by G conservation (or, equivalently, by invariance under $\phi_+ \to -\phi_+$).

I am tempted to explain this by saying that the annihilation force, though negligible for weak coupling, becomes dominant for strong coupling, and pushes the originally lightest isosinglet state far above the originally next-lightest. However, I am reluctant to yield to this temptation: if the annihilation force is so important in the two-quark theory, it should be just as important in the one-quark theory. But, if it were, it would push all the parity-odd states above the parity-even ones in the strong-coupling limit, and, as we have seen, this is not what happens.

There is something going on here that I do not understand.

The Case $|\theta| = \pi$

When $|\theta| = \pi$ our analysis collapses; the terms we have identified as dominant in the ϕ_- dynamics dissappear, and the dominant graphs are those with one internal ϕ_+ line, like those of Figs. 5d and 5e. (Of course, if I were doing a really thorough analysis, I would also consider these graphs for $|\theta|$ very close to π, where they become competitive with the terms we have already considered. I won't bother to do this here.)

These graphs produce an effective interaction of the form

$$iS_{\text{eff}} \propto -m^3\mu \int d^2x\, d^2y\, [N_m \cos(2\pi)^{1/2}\, \phi_-(x)] \times \Delta_F(x-y;\, \mu)[N_m \cos(2\pi)^{1/2}\, \phi_-(y)], \tag{6.29}$$

where Δ_F is the Feynman propagator for a free field of mass μ, and the proportionality sign indicates a positive constant. (It will turn out that there is nothing to be gained in this computation by keeping track of all the c's and π's. I will still keep track of the m's and μ's, though, because they tell us the power of e/m.) It is convenient to rotate this expression into Euclidean space ($x^2 = -ix^0$):

$$iS_{\text{eff}} \propto m^3\mu \int d^2x\, d^2y\, [N_m \cos(2\pi)^{1/2}\, \phi_-(x)] \times \Delta_E(x-y;\mu)[N_m \cos(2\pi)^{1/2}\, \phi_-(y)], \tag{6.30}$$

where

$$\Delta_E(x-y;\mu) = \int \frac{d^2k\, e^{ik\cdot x}}{k^2+\mu^2}, \tag{6.31}$$

and I have switched to the Euclidean (positive signature) metric.

For our purposes, the significant property of Δ_E is

$$\int d^2x\, \Delta_E(x-y;\mu)\, |\, x\,|^n \propto \mu^{-(n+2)}. \tag{6.32}$$

Thus we may expand the product of the two cosines in a power series in $(x-y)$, and discard all but the first nontrivial terms.

By Wick's theorem,

$$\begin{aligned} &N_m \exp[i\beta_1\phi(x)]\, N_m \exp[i\beta_2\phi(0)] \\ &\quad = N_m \exp[i\beta_1\phi(x) + i\beta_2\phi(0) - \beta_1\beta_2\Delta_E(x;m)] \\ &\quad \propto |\, mx\,|^{\beta_1\beta_2/2\pi}\, [1 + O(m^2x^2)]\, N_m \exp[i\beta_1\phi(x) + i\beta_2\phi(0)]. \end{aligned} \tag{6.33}$$

Whence

$$\begin{aligned} &N_m \cos[(2\pi)^{1/2}\phi(x)] N_m \cos[(2\pi)^{1/2}\phi(0)] \\ &\quad \propto |\, mx\,|^{-1}[1 + O(m^2x^2)]\, N_m[1 - \pi(x_\mu\partial_\mu\phi)^2 + m^2x^2\cos[(8\pi)^{1/2}\phi] + O(x^4)]. \end{aligned} \tag{6.34}$$

The first term contributes only an irrelevant constant to S_{eff}, and thus may be dropped. Furthermore, upon integration over angle,

$$(x_\mu\partial_\mu\phi)^2 \to \tfrac{1}{2}x^2(\partial_\mu\phi)^2. \tag{6.35}$$

Thus, to leading order for strong coupling,

$$iS_{\text{eff}} \propto m^2\mu^{-2}N_m \int d^2x\, [-\pi(\partial_\mu\phi_-)^2 + m^2\cos(8\pi)^{1/2}\, \phi_-]. \tag{6.36}$$

Of course, this must be added to the free action for the ϕ_- field. In Euclidean space, this is

$$iS_0 = -\int d^2x\, N_m(\partial_\mu\phi_-)^2. \tag{6.37}$$

The derivative term in the sum can be brought into standard form by a wave-function renormalization; thus we obtain the sine-Gordon equation again, this time with

$$\beta^2 = 8\pi - O(m/e)^2. \tag{6.38}$$

It is important that the small correction to β^2 is negative, for it is known [6] that the sine-Gordon equation is sick if β^2 exceeds 8π. Once this has been established, though, there is no point in retaining the correction to β^2, because it is of the same magnitude as other terms which I have already thrown away, and I will neglect it from now on.

The analysis of the particle spectrum of the theory (for masses much less than μ), goes much the same as before. According to Dashen *et al.*, for β^2 greater than 4π, the only particles in the theory are the soliton and the antisoliton. These carry "charge" plus and minus one, where charge is defined by Eq. (6.23). Since β is now twice as large as it was in our previous analysis, the charge is now twice I_3. Thus the two particles have I_3 plus and minus $\frac{1}{2}$, that is to say, they are an isodoublet. Since these particles are pure ϕ_- excitations, they have no electric charge.

There is a strong qualitative similarity between the situation we have found here and the one we found earlier for the one-quark theory with $|\theta| = \pi$. There, for weak coupling, there were two half-asymptotic particles, quark and antiquark, carrying electric charge plus and minus one; for strong coupling, there was only one asymptotic particle, and it was electrically neutral. Here, for weak coupling, there are two half-asymptotic isodoublets, quark and antiquark, carrying electric charges plus and minus one; for strong coupling, there is only one asymptotic isodoublet, and it is electrically neutral. This leads me to believe that there is a phase transition here much like that in the one-quark theory, but I lack the analytic tools needed to form a more detailed picture of the nature of the transition.

Three Things I Don't Understand

In the one-quark theory, everything that happened, even for strong coupling, was qualitatively understandable in terms of the basic ideas of the naive quark model, the picture of quarks confined in a linear potential. For the two-quark theory, there are three strong-coupling phenomena that I cannot understand in these terms:

(1) Why are the lightest particles in the theory a degenerate isotriplet, even if one quark is 10 times heavier than the other?

(2) Why does the next-lightest particle have $I^{PG} = 0^{++}$, rather than 0^{--}?

(3) For $|\theta| = \pi$, how can an isodoublet quark and an isodoublet antiquark, carrying opposite electric charges, make an isodoublet bound state with electric charge zero?

I think it is important to answer these questions. It is possible that the answers will turn out to involve uninteresting peculiarities of two-dimensional physics, but it is also possible that the answers will lead to new insights into the behavior of strongly interacting quarks in four dimensions.

APPENDIX: Dull Calculations

This Appendix is a brief description of the calculations that lead to the reduced Hamiltonian of Section 4.

Fundamental Formulas

The expansion of a canonical Dirac field of unit mass at time zero in terms of annihilation and creation operators is

$$\psi(x) = (2\pi)^{-1/2} \int dp\, [a(p)\, u(p)\, e^{ipx} + b^\dagger(p)\, v(p)\, e^{-ipx}]. \tag{A1}$$

Here the a's and b's are conventionally normalized annihilation operators,

$$\{a(p), a(p')^\dagger\} = \{b(p), b(p')^\dagger\} = \delta(p - p'), \tag{A2}$$

and the u's and v's are the solutions to the free Dirac equation, normalized such that $u(p)^\dagger\, u(p) = v(p)^\dagger\, v(p) = 1$. In a basis in which $\beta = \sigma_z$ and $\alpha = \sigma_x$,

$$\begin{aligned} u(p) &= (2E)^{-1/2}\, (E + 1)^{-1/2} \begin{pmatrix} E + 1 \\ p \end{pmatrix}, \\ v(p) &= (2E)^{-1/2}\, (E + 1)^{-1/2} \begin{pmatrix} p \\ E + 1 \end{pmatrix}. \end{aligned} \tag{A3}$$

where $E = (p^2 + 1)^{1/2}$.

We will also need the Fourier transform of the Coulomb potential,

$$\begin{aligned} \int dx\, e^{ipx}\, | x | &= -\frac{1}{(p + i\epsilon)^2} - \frac{1}{(p - i\epsilon)^2} \\ &\equiv -2\, \frac{P}{p^2}. \end{aligned} \tag{A4}$$

For the self-energy computation, we will need the equal-time quark propagator,

$$\overline{\psi(x)\psi^\dagger}(y) = \int \frac{dp}{4\pi}\, e^{ip(x-y)}\, \frac{E + p\alpha + m\beta}{E}. \tag{A5}$$

Finally, for convenience, I repeat here the interaction Hamiltonian,

$$H' = -\frac{e^2}{4}\int dx\, dy\, j_0(x)\, j_0(y) \mid x - y \mid - eF \int dx\, x j_0(x). \tag{A6}$$

Coulomb Force

The Coulomb force, Fig. 3a, leads by standard manipulations to

$$\langle p' \mid H_1^R \mid p\rangle = -\frac{e^2}{2\pi}\frac{P}{(p'-p)^2}\, u^\dagger(p')\, u(p)\, v^\dagger(-p)\, v(-p'). \tag{A7}$$

Substitution of the explicit expressions for the u's and v's, Eq. (A3), leads immediately to Eq. (4.8).

Self-Mass

Figures 3b and 3c make identical contributions; their sum yields

$$\langle p' \mid H_2^R \mid p\rangle = 2A\delta(p - p'), \tag{A8}$$

where

$$\begin{aligned} A &= e^2 \int \frac{dq}{4\pi}\, u^\dagger(p)\, \frac{E(q) + \alpha q + \beta}{E(q)}\, u(p)\, \frac{P}{(p-q)^2} \\ &= \frac{e^2}{E(p)} \int \frac{dq}{4\pi}\, \frac{E(p)\, E(q) + pq + 1}{E(q)}\, \frac{P}{(p-q)^2}. \end{aligned} \tag{A9}$$

The first term in the numerator trivially integrates to zero; thus we may change its sign without affecting the integral. The numerator now vanishes quadratically when $p = q$, and we can drop the principal-value symbol. Thus,

$$A = \frac{e^2}{E(p)} \int \frac{dq}{4\pi}\, \frac{-E(p)\, E(q) + pq + 1}{E(q)}\, \frac{1}{(p-q)^2}. \tag{A10}$$

Introducing the new variables,

$$p = \sinh Y, \qquad q = \sinh y, \tag{A11}$$

we find

$$\begin{aligned} A &= \frac{e^2}{4\pi E(p)} \int dy\, \frac{\cosh(y - Y) - 1}{[\sinh y - \sinh Y]^2} \\ &= -\frac{e^2}{4\pi E(p)} \int \frac{dy}{2\cosh^2 \frac{1}{2}(y + Y)} \\ &= -\frac{e^2}{2\pi E(p)}. \end{aligned} \tag{A12}$$

This yields Eq. (4.11).

Annihilation Force

At first glance, one might think that the annihilation force, Fig. 3d, vanishes, since $u(p)^\dagger v(-p) = 0$. However, this zero is cancelled by a simultaneous zero in the denominator of the Coulomb propagator, Eq. (A4). To obtain an unambiguous answer we must evaluate the matrix element when the quark-antiquark pair has nonzero total momentum, Δ, and then let Δ go to zero. Thus,

$$\langle p' \mid H_3{}^R \mid p\rangle = \lim_{\Delta\to 0} (e^2/2\pi\Delta^2)\, u^\dagger(p' + \Delta)\, v(-p')\, v^\dagger(-p)\, u(p + \Delta). \tag{A13}$$

From Eq. (A3),

$$v^\dagger(-p)\, u(p + \Delta) = (\Delta/2E^2) + O(\Delta^2), \tag{A14}$$

from which Eq. (4.14) follows directly.

Background Field

Figures 3e and 3f are not independently invariant under translations; therefore it is wisest to compute their matrix elements between general quark–antiquark states, and specialize to states of total momentum zero afterwards. Thus we compute

$$\begin{aligned}\langle p', q' \mid H_4 \mid p, q\rangle &\\ &= ieF[\delta'(p - p')\, \delta(q - q')\, u^\dagger(p')\, u(p) - \delta'(q - q')\, \delta(p - p')\, v^\dagger(q)\, v(q')]\\ &= ieF[\delta'(p - p')\, \delta(q - q') - \delta'(q - q')\, \delta(p - p')].\end{aligned} \tag{A15}$$

If we use the identity

$$\delta'(a)\, \delta(a + b) = \delta'(a)\, \delta(b) - \delta(a)\, \delta'(b), \tag{A16}$$

this becomes

$$\langle p', q' \mid H_4 \mid p, q\rangle = ieF\delta'(p - p')\, \delta(p - p' + q - q'), \tag{A17}$$

from which Eq. (4.16) follows directly.

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