1 The Concept of the Renormalization Group

The key idea of the renormalization group results from comparing phenomena at different length/energy scales. A QFT is defined by an action functional of the fields $S[\phi; g_i]$ which depends *a priori* on an infinite number of parameters: the coupling constants in a general sense so including mass parameters, *etc.*. The set of couplings $\{g_i\}$ can be thought of as a set of coordinates on *theory space*. The functional integral takes the form

$$\mathscr{Z} = \int [d\phi]\, e^{iS[\phi;g_i]} \tag{1.1}$$

and so, as well as the action, we have to define the measure $\int[d\phi]$. This is a very tricky issue since a classical field has an infinite number of degrees-of-freedom and it is by no means a trivial matter to integrate over such an infinite set of variables. In perturbation theory a symptom of the difficulties in defining the measure shows up as the divergences that occur in loop integrals. These UV divergences occur when the momenta on internal lines become large so they are intimately bound up with the fact that the field has an infinite number of degrees-of-freedom and can fluctuate on all energy scales. Of course there may also be IR divergences, however these are not as conceptually serious as the UV ones: in reality in a real experiment one is working in some finite region of spacetime and this provides a natural IR cut-off.

At least initially, in order to make sense of $\int[d\phi]$, we have to implement some UV cut-off procedure in order to properly define the measure, or equivalently, in perturbation theory regulate the infinities that occur in loop diagrams. As we have said above, these UV, high energy divergences occur because the fields can fluctuate at arbitrarily small distances and in order to regulate the theory we have to somehow suppress these high energy modes. Whatever way this is done inevitably introduces a new energy scale μ, the *cut-off*, into the theory.

Cut-offs or Regulators

There are many ways of introducing a cut-off, or regulator, into a QFT. For example, one can define the theory on a spatial lattice (after Wick rotation to Euclidean space). In this case μ^{-1} is the physical lattice spacing. Or one can suppress the high momentum modes by modifying the action or the measure. Or in perturbation theory one can analytically continue the spacetime dimension, a procedure known as dimensional regularization.

Suppose we have some physical quantity $\mathscr{F}(g_i; \ell)_\mu$ which can depend in general on a (or possibly several) length scale ℓ (or equivalently an energy scale $1/\ell$). The theory of RG postulates that one can change the cut-off of the theory in such a way that the physics on energy scales $< \mu$ remains constant. In order that this is possible the couplings must change with μ. This idea can be summed up in the

RG equation:[1]

$$\mathscr{F}(g_i(\mu);\ell)_\mu = \mathscr{F}(g_i(\mu');\ell)_{\mu'} \ . \tag{1.2}$$

The functions $g_i(\mu)$ with defines the RG flow of the theory in the space of couplings. The RG flow is conventionally thought of as being towards the IR, *i.e.* decreasing μ, but we shall often think about it in the other direction as well, towards the UV with μ increasing. In order that the RG equation (1.2) can hold it is necessary that the space of couplings includes *all* possible couplings (necessarily an infinite number). The RG is non-trivial because in order to lower the cut-off we somehow have to "integrate out" the degrees-of-freedom of the theory that lie between energy scales μ and μ'. In general this is a difficult step, however, as we shall see, in QFT we are in a very lucky situation due to the remarkable focusing properties of RG flows.

The RG energy scale μ plays a central rôle in the theory and it is important to understand what exactly it is. To start with we have identifed μ with the physical cut-off; however, there is another way to interpret μ. The point is that if we wish to describe physical process at the energy scales E_{phys} or below, or distance scales greater than $\ell = E_{\text{phys}}^{-1}$, then there is no reason why we cannot take the cut-off to be at the scale $\mu = E_{\text{phys}}$. In fact this would be the optimal choice since the efffective description would then only involve modes with energies $\leq E_{\text{phys}}$, *i.e.* the ones directly involved in the physical process. So another to think of RG flow is that the couplings run with the typical energy scale of the process being investigated, $g_i(\mu)$, where μ is that energy scale.

Since the physical observables of the theory can be determined once the action is known, the RG transformation itself follows from following how the action changes as the cut-off changes. The action at a particular cut-off is known as the Wilsonian Effective Action $S[\phi;\mu,g_i]$ and since it depends on the fields, as well as the couplings, the RG transformation must be generalized to

The Key RG Equation

$$S\big[Z(\mu)^{1/2}\phi;\mu,g_i(\mu)\big] = S\big[Z(\mu')^{1/2}\phi;\mu',g_i(\mu')\big] \ , \tag{1.3}$$

where $Z(\mu)$ is known as "wavefunction renormalization" of the field. In the general case with many fields, $Z(\mu)$ is a matrix quantity that can mix all the fields. The action of a QFT can be written as the sum of a kinetic term and linear combination of "operators" $\mathcal{O}_i(x)$ which are powers of the fields and their derivatives, *e.g.* ϕ^n, $\phi^n\partial_\mu\phi\partial^\mu\phi$, *etc*:[2]

$$S[\phi;\mu,g_i] = \int d^dx\,\Big[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_i \mu^{d-d_i}g_i\mathcal{O}_i(x)\Big] \tag{1.4}$$

where d_i is the classical dimension of $\mathcal{O}_i(x)$. Notice that we chosen the couplings to be dimensionless by inserting the appropriate power of the cut-off to soak up the dimensions. This is because it is really the value of a coupling relative to the cut-off that is physically relevant. The wavefunction renormalization factor $Z(\mu)$ can be thought of as the coupling to the kinetic term since[3]

$$S[Z^{1/2}\phi;\mu,g_i]=\int d^dx\left[\frac{Z}{2}\partial_\mu\phi\partial^\mu\phi+\cdots\right]. \tag{1.5}$$

It is often useful to think about infinitesimal RG transformations, in which case we define the *beta function* of a theory

$$\mu\frac{dg_i(\mu)}{d\mu}. \tag{1.6}$$

The running couplings then follow by integration of the beta-function equations above. Notice that due to the fact that couplings always appear as combinations $\mu^{d-d_i}g_i$ means that the beta functions always have the form

$$\mu\frac{dg_i}{d\mu}=(d_i-d)g_i+\beta_{g_i}^{\text{quant.}}. \tag{1.7}$$

One can think of the first term as arising from classical scaling and the second piece as arising from the non-trivial integrating-out part of the RG transformation. If $\hbar$ were re-introduced $\exp[iS]\to\exp[iS/\hbar]$, then the quantum piece would indeed vanish in the limit $\hbar\to 0$. We can also define the *anomalous dimension* of a field ϕ as

$$\gamma_\phi=-\frac{\mu}{2}\frac{d\log Z(\mu)}{d\mu}. \tag{1.8}$$

In particle physics the ultimate physical observables are the probabilities for particular things to happen. However, it is often useful, especially in massless theories where the S-matrix is problematic to formulate because of long-range interactions, to consider the Green functions of fields. Schematically,

$$\big\langle\phi(x_1)\cdots\phi_n(x_n)\big\rangle_{g_i(\mu),\mu}=\frac{\int_\mu[d\phi]\,e^{iS[\phi;\mu,g_i(\mu)]}\phi(x_1)\cdots\phi(x_n)}{\int_\mu[d\phi]\,e^{iS[\phi;\mu,g_i(\mu)]}}. \tag{1.9}$$

It follows from (1.3) that for these quantities that depend on the field we must generalize (1.2) to take account of wavefunction renormalization, giving

$$Z(\mu)^{-n/2}\big\langle\phi(x_1)\cdots\phi_n(x_n)\big\rangle_{g_i(\mu),\mu}=Z(\mu')^{-n/2}\big\langle\phi(x_1)\cdots\phi_n(x_n)\big\rangle_{g_i(\mu'),\mu'}, \tag{1.10}$$

What is particularly important about RG flows are their IR and UV limits; namely $\mu\to 0$ and $\mu\to\infty$, respectively. As we flow towards the IR, all masses relative to the cut-off, that is m/μ, increase. If a theory has a mass-gap (no massless particles) then as $\mu\to 0$ all physical masses are become infinitely heavy relative to the cut-off and there is nothing left to propagate in the IR. Hence, in the IR limit we have an empty, trivial or null theory. The other possibility is when the RG flow starts on the:

Critical Surface

The infinite dimensional subspace in the space-of-theories for which the mass gap vanishes. These theories consequently have a non-trivial IR limit in which only the massless degrees-of-freedom remain.

In this case, as $\mu \to 0$ the massless particles will remain and in all known cases the couplings flow to a fixed point of the RG $g_i(\mu) \to g_i^*$ as $\mu \to 0$ where the beta functions vanish:[4]

Equation for a fixed point or conformal field theory

$$\mu \frac{dg_i}{d\mu}\Big|_{g_j^*} = 0 \ . \tag{1.11}$$

The theories at the fixed points are very special because as well only having massless states particles they have no dimension-full parameters at all. This means that they are scale invariant. However, this scale invariance is naturally promoted to the group of conformal transformations and so the fixed point theories are also "conformal field theories" (CFTs).[5]

In the neighbourhood of a fixed point, or CFT, $g_i = g_i^* + \delta g_i$, we can alway linearize the RG flows:

$$\mu \frac{dg_i}{d\mu}\Big|_{g_j^*+\delta g_j} = A_{ij}\delta g_j + \mathcal{O}(\delta g_j^2) \ . \tag{1.12}$$

In a suitable diagonal basis for $\{\delta g_i\}$ which we denote $\{\sigma_i\}$,

$$\mu \frac{d\sigma_i}{d\mu} = (\Delta_i - d)\sigma_i + \mathcal{O}(\sigma^2) \tag{1.13}$$

and so to linear order the RG flow is simply

$$\sigma_i(\mu) = \left(\frac{\mu}{\mu'}\right)^{\Delta_i - d} \sigma_i(\mu') \ . \tag{1.14}$$

The quantity Δ_i is called the scaling (or conformal) dimension of the operator associated to σ_i. In general in an interacting QFT it will not be the classical scaling dimension and the difference

$$\gamma_i = \Delta_i - d_i \tag{1.15}$$

is known as the *anomalous dimension* of the operator.

In a CFT the Green functions are covariant under scale transformations and this provides non-trivial constraints. As an example, consider the 2-point Green function $\langle\phi(x)\phi(0)\rangle$. This satisfies the more general RG equation (1.10)

$$Z(\mu)^{-1}\big\langle\phi(x)\phi(0)\big\rangle_{g_i(\mu),\mu} = Z(\mu')^{-1}\big\langle\phi(x)\phi(0)\big\rangle_{g_i(\mu'),\mu'} \ . \tag{1.16}$$

At a fixed point $g_i(\mu) = g_i(\mu') = g_i^*$ and we have $Z(\mu) = (\mu'/\mu)^{2\gamma_\phi^*} Z(\mu')$, where $\gamma_\phi^* = \gamma_\phi(g_i^*)$. Using dimensional analysis we must have

$$\big\langle \phi(x)\phi(0) \big\rangle_{g_i^*,\mu} = \mu^{2d_\phi} \mathcal{G}(x\mu) \ , \tag{1.17}$$

where d_ϕ is the classical dimension of the field ϕ. Substituting into the RG equation allows us to solve for the unknown function $\mathcal{G}$, up to an overall multiplicative constant, yielding

$$\big\langle \phi(x)\phi(0) \big\rangle_{g_i^*,\mu} = \frac{c}{\mu^{2\gamma_\phi^*} x^{2d_\phi + 2\gamma_\phi^*}} \propto \frac{1}{x^{2\Delta_\phi^*}} \tag{1.18}$$

where c is a constant. This is the typical power-law behaviour characteristic of correlation functions in a CFT. In the problem for this lecture you will see that using the whole of the conformal group provides even more information.

Relevant, Irrelevant and Marginal

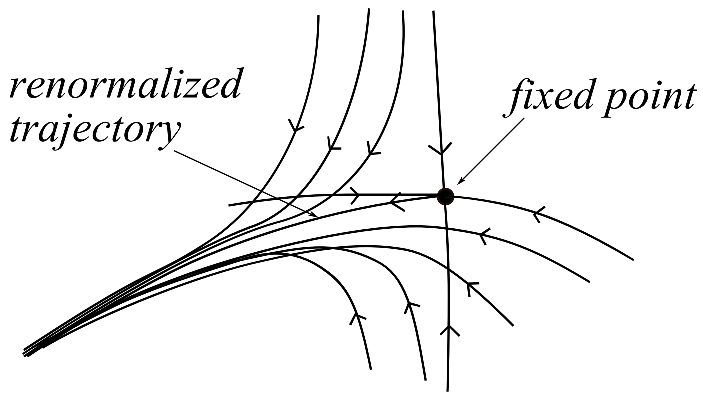
Couplings in the neighbourhood of a fixed point flow as (1.14) and are classified in the following way:

(i) If a coupling has $\Delta_i < d$ the flow diverges away from the fixed point into the IR as μ decreases and is therefore known as a *relevant* coupling.

(ii) If $\Delta_i > d$ the coupling flows into the fixed point and is known as *irrelevant*.

(iii) The case $\Delta_i = d$ is a marginal coupling for which one has to go to higher order to find out the behaviour. If, due to the higher order terms, a coupling diverges away/converges towards from the fixed point it is *marginally relevant/irrelevant*. The final possibility is that the coupling does not run to all orders. In this case it is *truly marginal* coupling and implies that the original fixed point is actually part of a whole line of fixed points.

When we follow an RG flow backwards towards the UV all particle masses decrease relative to cut-off and the so trajectory of a theory with a mass gap must approach the critical surface. In typical cases, with or without a mass-gap, the trajectory either diverges off to infinity for finite μ or approaches a fixed point lying on the critical surface in the limit $\mu \to \infty$. Below we show the RG flows around a fixed point with 2 irrelevant directions and 1 relevant direction.



Notice that the flows lying off the critical surface naturally focus onto the "renormalized trajectory", which is defined as the flow that comes out of the fixed point. The focusing effect, along with the fact that there are only a finite number of relevant directions, leads to the property of *Universality* which is the most important feature of RG flows. Universality also arises for flows starting on the critical surface since in this case they all flow into the fixed point.

Universality

CFTs only have a finite (and usually small) number of relevant couplings. This means that the domain-of-attraction of a fixed point (the set of all points in theory-space) that flow into a fixed point is infinite dimensional (this dimension is the number of irrelevant couplings). This also means that RG flows of a theory lying off the critical surface strongly focus onto finite dimensional subspaces parameterized by the relevant couplings of a fixed point as in the figure above. The implication of this is that the behaviour of theories in the IR is determined by only a small number of relevant couplings and not by the infinite set of couplings g_i. This means that IR behaviour of a given theory with couplings g_i can lie in a small set of "universality classes" which are determined by the domain of attraction of the set of fixed points.

Notice that the RG is directly relevant to the problem of taking a continuum limit of a QFT:

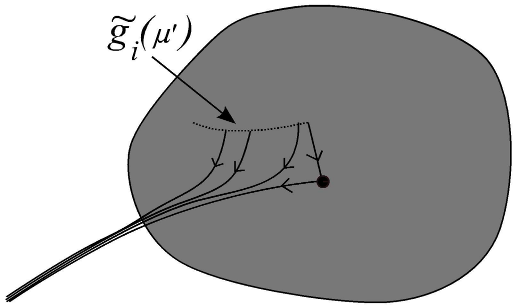
The Continuum Limit

This is the process of taking the cut-off from its original value μ to ∞ whilst keeping the keeping the physics at any energy less than the original μ fixed. Whether such a limit exists is a highly non-trivial issue and central to our story. Notice that taking a continuum limit involves the inverse RG flow, that is $g_i(\mu)$ with μ increasing.

The RG Equation (1.2) shows how this can be achieved. We can send $\mu \to \infty$, as long as the UV limit of $g_i(\mu)$ is suitably well-defined which in practice means that $g_i(\infty)$ is a fixed-point of RG (this is what Weinberg calls "asymptotic safety). The resulting $g_i(\mu)$ is known as a "renormalized trajectory" since it defines a theory on all length scales. Clearly a renormalized trajectory has to have the infinite set of irrelevant couplings at the UV fixed-point vanishing. Searching for a renormalized trajectory would seem to involve searching for a needle in an infinite haystack. Fortunately, however, universality comes to our rescue:

Taking a Continuum Limit

We do not need to actually sit precisely on the renormalized trajectory in order to define a continuum theory. All that is required is a one-parameter set of theories defined with cut-off μ' and with couplings $g_i = \tilde{g}_i(\mu')$ (which need *not* necessarily be an RG flow, since this would mean sitting on the renormalized trajectory) for which $\tilde{g}_i(\infty)$ lies in the domain of attraction of the UV CFT, as illustrated below, where the domain of attraction is the shaded area



The limit $\mu' \to \infty$ is defined in such a way that the IR physics at the original cut-off scale μ is fixed. In particular, the number of parameters that must be specified in order to take a continuum limit, *i.e.* which fix the IR physics, equals the number of relevant couplings of the CFT. However, both the way that relevant couplings are fixed at the scale μ and the values of the irrelevant couplings as $\tilde{g}_i(\mu')$ as $\mu' \to \infty$ can be defined in many different ways. So there are many ways to take a continuum limit, or many "schemes", which all lead to the same continuum theory. In particular, in particle physics this always allows us to take very simple forms for the action with a small number of operators (equal to the number of relevant coupling of the UV CFT). However, at the same time it allows us to describe the same QFT by using, say, a lattice cut-off.

Notes

1 In the literature, RG equations like (1.2) are also often written in infinitesimal form as

$$\Big[\mu\frac{\partial}{\partial\mu} + \beta_{g_i}\frac{\partial}{\partial g_i}\Big]\mathscr{F}(g_i;\ell)_\mu = 0\ . \tag{1.19}$$

where $\beta_{g_i} = \mu dg_i/d\mu$.

2 The use of the term "operator" or "composite operator" derives from a canonical quantization approach to QFT in which one builds a Hilbert space and on which the fields becomes operators. It is conventional to use this language when using the functional integral approach when strictly-speaking the quantities are not operators.

3 You may have noticed that wavefunction renormalization of the fields in QFT plays a special role relative to the other coupling in the action. In a sense the definition of RG is ambiguous since the beta-functions of the couplings depend implicitly on $Z(\mu)$? Another more concrete way to phrase this question is why did we *choose* the wavefunction renormalization in order to keep the kinetic term intact after RG flow? The intuitive answer is that we want our theories to describe particles which, at least in the case of a theory with a mass gap, when well separated are approximately free and so have a free field propagator $\sim i/(p^2 - m_{\text{ph}}^2)$. This is why we re-scaled the field in the effective action to keep the kinetic term like that for a free particle. The physical mass of the particle m_{ph} (which is not the parameter m in the original action) is then identified by the position of the pole in the propagator.

In the statistic physics interpretation of the functional integral, other choices of wavefunction renormalization are sometimes appropriate. For instance, in $d \geq 4$ one can impose a choice where the kinetic term is relevant leaving a theory with no propagating field. In this case, non-zero momentum modes of the field are suppressed in the IR and only the zero mode, which is constant in space, survives. This limit is known as *mean field theory* in statistical physics.

4 Other more exotic possibilities, like limit cycles, have been found in some very bizarre theories in two spacetime dimensions.

5 One potential point of confusion regarding fixed points of the RG flow is that the fields can still have non-trivial wavefunction renormalization factors $Z(\phi)$. However, remember that the fields are in some sense just dummy variables. So a fixed point theory, or CFT, is really scale *covariant* rather than scale *invariant*.

The (connected part of the) conformal group consists of Poincaré transformations along with scale transformations, or "dilatations" $x \to sx$, and special conformal transformations

$$x \longrightarrow \frac{x + x^2 b}{1 + 2b \cdot x + x^2 b^2} \tag{1.20}$$

The infinitesimal transformations for Lorentz, dilatations and special conformal transformations are

$$\delta x^\mu = \epsilon^\mu{}_\nu x^\nu \ , \qquad \delta x^\mu = s x^\mu \ , \qquad \delta x^\mu = x^2 b^\mu - 2x^\mu (x \cdot b) \ , \tag{1.21}$$

where $\epsilon^{\mu\nu} = -\epsilon^{\mu\nu}$. In any local QFT there exists an energy-momentum tensor $T_{\mu\nu}$ and correlation functions satisfy the Ward identity

$$\sum_{p=1}^{n} \big\langle \phi_1(x_1) \cdots \delta\phi_p(x_p) \cdots \phi_n(x_n) \big\rangle = -\int d^d x \, \big\langle \phi_1(x_1) \cdots \phi_n(x_n) T^\mu{}_\nu(x) \big\rangle \, \partial_\mu \big(\delta x^\nu\big) \ . \tag{1.22}$$

Invariance of the QFT under Lorentz transformations requires $T_{\mu\nu} = T_{\nu\mu}$ while invariance under dilatations $T^\mu{}_\mu = 0$. From this it follows that these two conditions are sufficient to imply invariance under infinitesimal special conformal transformations.

2 Scalar Field Theories

We now illustrate RG theory in the context of the QFT of a single scalar field. Usually we write down simple actions like

$$S[\phi] = \int d^d x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 \right) , \tag{2.1}$$

however, in the spirit of RG we should allow all possible operators consistent with spacetime symmetries. In the case of a scalar field, all powers of the field and its derivatives, where the latter are contracted in a Lorentz invariant way. For simplicity we shall restrict to operators even in $\phi \to -\phi$.[1]. Simple scaling analysis shows that a "composite operator" $\mathcal{O}$ containing p derivatives and $2n$ powers of the field, schematically $\partial^p \phi^{2n}$, has classical dimension

$$d_{\mathcal{O}} = n(d-2) + p \ . \tag{2.2}$$

Even at the classical level we see that the number of relevant/marginal couplings, those with $d_{\mathcal{O}} \leq d$ is small. The classical dimensions of various operators are given in the table below.

$\mathcal{O}$	$d > 4$	$d = 4$	$d = 3$	$d = 2$
ϕ^2	rel	rel	rel	rel
ϕ^4	irrel	marg	rel	rel
ϕ^6	irrel	irrel	marg	rel
ϕ^{2n}	irrel	irrel	irrel	rel
$\partial \phi_\mu \partial^\mu \phi$	marg	marg	marg	marg

The classical scaling suggests that we only need keep track of the kinetic term and potential,[2]

$$\mathscr{L} = \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) \ , \tag{2.3}$$

where we take

$$V(\phi) = \sum_n \mu^{d-n(d-2)} \frac{g_{2n}}{(2n)!} \phi^{2n} \ , \tag{2.4}$$

and where we have used the cut-off to define dimensionless couplings g_{2n}.

Now we come to crux of the problem, that of finding the RG flow, or concretely the beta functions, of the couplings. In order to do this we must apply the RG equation (1.3) to the *Wilsonian Effective Action* $S[Z(\mu)^{1/2}\varphi; \mu, g_i(\mu)]$ defined for the theory with cut-off μ in such a way that the phenomena on energy scales below the cut-off is fixed as μ is varied.

Before we can describe how to relate the theories with cut-off μ and μ' let us first choose a cut-off procedure. The most basic and conceptually simple way is to introduce a sharp momentum cut-off on the Fourier modes after Wick rotation to Euclidean space. In Euclidean space the Lagrangian (2.3) has the form

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 + V(\phi) \tag{2.5}$$

and the functional integral becomes $\int[d\phi]e^{-S}$.[3] The momentum cut-off involves Fourier transforming the field

$$\phi(x) = \int \frac{d^dp}{(2\pi)^d}\tilde{\phi}(p)e^{ip\cdot x} \tag{2.6}$$

and then limiting the momentum vector by a sharp cut-off $|p| \leq \mu$. The resulting theory is manifestly UV finite since loop integrals can never diverge. In addition, we have a very concrete way of performing the RG transformation. Namely, we split the field ϕ defined with cut-off μ' into

$$\phi = \varphi + \hat{\phi} \ , \tag{2.7}$$

where φ has modes with $|p| \leq \mu$ while $\hat{\phi}$ are modes with $\mu \leq |p| \leq \mu'$. In order to extract the beta-function it is sufficient to consider the infinitesimal transformation with $\mu' = \mu + \delta\mu$. We can then obtain the RG flow by considering how the action changes when we integrating out $\hat{\phi}$, so concretely

$$\exp\big\{ - S_{\text{eff}}[\varphi]\big\} = \int [d\hat{\phi}]\ \exp\big\{ - S[\varphi + \hat{\phi}; \mu', g_{2n}(\mu')]\big\} \ . \tag{2.8}$$

On the left-hand side we have the *Wilsonian Effective Action* which is to be identified with

$$S_{\text{eff}}[\varphi] = S[Z(\mu)^{1/2}\varphi; \mu, g_{2n}(\mu)] \ . \tag{2.9}$$

Notice that we have taken $Z(\mu') = 1$ since we only need the variation as μ changes in order to extract the anomalous dimension γ_ϕ.

Expanding the action on the right-hand side in powers of $\hat{\phi}$:[4]

$$S[\varphi + \hat{\phi}] = S[\varphi] + \int d^dx\, \Big(\frac{1}{2}(\partial_\mu\hat{\phi})^2 + \frac{1}{2}V''(\varphi)\hat{\phi}^2 + \frac{1}{6}V'''(\varphi)\hat{\phi}^3 + \cdots\Big) \ . \tag{2.10}$$

Feynman Diagram Interpretation

Contributions to the effective action can be interpreted in term of Feynman diagrams with only $\hat{\phi}$ on internal lines with a propagator $1/(p^2 + g_2\mu^2)$, with p integrated over the shell $\mu \leq |p| \leq \mu'$, and with only φ on external lines (but amputated meaning no propagators on the external lines). The vertices are provided by the interaction terms in $V(\varphi + \hat{\phi})$. Each loop involves an integral over the momentum of $\hat{\phi}$ which lies in a shell between radii μ and μ' in momentum space:

$$\int_{\mu\leq|p|\leq\mu'} \frac{d^dp}{(2\pi)^d}\, f(p) \ . \tag{2.11}$$

However, if we are only interested in an infinitesimal RG transformation $\mu' = \mu + \delta\mu$ then the integrals over internal momenta (2.11) become much simpler:

$$\int_{\mu\leq|p|\leq\mu+\delta\mu} \frac{d^dp}{(2\pi)^d} f(p) = \frac{\mu^{d-1}}{(2\pi)^d} \int d^{d-1}\hat{\Omega}\, f(\mu\hat{\Omega})\, \delta\mu \ , \tag{2.12}$$

where $\hat{\Omega}$ is a unit d vector integrated over a unit S^{d-1}. In addition, since each loop integral brings a factor of $\delta\mu$, so to linear order in $\delta\mu$ only one loop diagrams are needed. This is equivalent to saying that we only need the term quadratic in $\hat{\phi}$ in (2.10). The resulting integral over $\hat{\phi}$ is Gaussian and yields[5]

$$S_{\text{eff}}[\varphi] = S[\varphi] + \frac{1}{2} \log\det\big(-\Box + V''(\varphi)\big) \ . \tag{2.13}$$

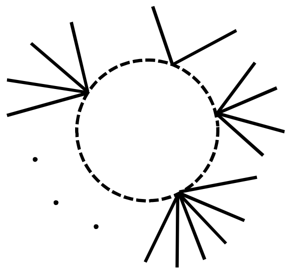
In order to extract the RG transformation we identify the left-hand side with the Wilsonian effective action (2.9). In this case, one finds that no wavefunction renormalization is required.[6] In order to extract the effective potential (that is the part of the Lagrangian not involving derivatives of the field) we can temporarily assume that φ is constant, in which case

$$\frac{1}{2} \log\det\big(-\Box + V''(\varphi)\big) = a\mu^{d-1} \int d^dx\, \log(\mu^2 + V''(\varphi))\, \delta\mu \ , \tag{2.14}$$

where $a = \mathrm{Vol}(S^{d-1})/(2(2\pi)^d) = 2^{-d}\pi^{-d/2}/\Gamma(d/2)$. From this expanding in powers of φ it follows that

$$\boxed{\mu\frac{dg_{2n}}{d\mu} = (n(d-2)-d)g_{2n} - a\mu^{2n}\frac{d^{2n}}{d\varphi^{2n}} \log(\mu^2 + V''(\varphi))\Big|_{\varphi=0} \ . \qquad (2.15)}$$

The contributions on the right-hand side are identified with one-loop diagrams of the form



with $2n$ external legs.

From (2.15), it follows, for example, that

$$\begin{aligned}
\mu\frac{dg_2}{d\mu} &= -2g_2 - \frac{ag_4}{1+g_2} \ , \\
\mu\frac{dg_4}{d\mu} &= (d-4)g_4 + \frac{3ag_4^2}{(1+g_2)^2} - \frac{ag_6}{1+g_2} \ , \\
\mu\frac{dg_6}{d\mu} &= (2d-6)g_6 - \frac{30ag_4^3}{(1+g_2)^3} + \frac{15ag_4g_6}{(1+g_2)^2} - \frac{ag_8}{1+g_2} \ .
\end{aligned} \tag{2.16}$$

Notice that the quantum contributions involve inverse powers of the factor $1 + g_2$ which physically is $m^2/\mu^2 + 1$. So when $m \gg \mu$, the quantum terms are suppressed as one would expect on the basis of decoupling.

Decoupling

Decoupling expresses the intuition that a particle of mass m cannot directly affect the physics on energy scales $\ll m$. For instance, the potential due to the exchange of massive particle in 4 dimensions is $\sim e^{-mr}/r$. This is exponentially suppressed on distances scales $\gg m^{-1}$.

The beta functions allow us to map-out RG flow on theory space. The first thing to do is to find the RG fixed points corresponding to the CFTs. The "Gaussian" fixed point is the trivial fixed point where all the couplings vanish. Linearizing around this point, the beta-functions are

$$\mu\frac{dg_{2n}}{d\mu} = \big(n(d-2)-d\big)g_{2n} - ag_{2n+2} \ . \tag{2.17}$$

So the scaling dimensions are the classical dimensions $\Delta_{2n} = d_{2n} = n(d-2)$, *i.e.* the anomalous dimensions vanish, although the couplings that diagonalize the matrix of scaling dimensions σ_{2n} are not precisely equal to g_{2n} due to the second term in (2.17). In particular, $\sigma_2 = g_2$ is always relevant, $\sigma_4 = g_4 + ag_2/(2-d)$ is relevant for $d < 4$, irrelevant for $d > 4$ and marginally irrelevant for $d = 4$. In this latter case we need to go beyond the linear approximation. Since g_6 is irrelevant in $d = 4$, we shall ignore it, in which case since $a = 1/(16\pi^2)$ we have

$$\mu\frac{dg_4}{d\mu} = \mu\frac{dg_4}{d\mu} = \frac{3}{16\pi^2}g_4^2 \ , \tag{2.18}$$

whose solution is

$$\frac{1}{g_4(\mu)} = C - \frac{3}{16\pi^2}\log\mu \ . \tag{2.19}$$

This shows that g_4 is actually *marginally irrelevant* at the Gaussian fixed point because it gets smaller as μ decreases. We usually write the integration constant in terms of a dimensionful parameter Λ as

$$g_4(\mu) = \frac{16\pi^2}{3\log(\Lambda/\mu)} \ , \tag{2.20}$$

with $\mu < \Lambda$. This is our first example of *dimensional transmutation* where the degree-of-freedom of a dimensionless coupling g_4 has changed into a dimensionful quantity, namely Λ.

To find other non-trivial fixed points is difficult and the only way we can make progress is to work perturbatively in the couplings. This turns out to be consistent only if we accept the perversion of working in arbitrary non-integer dimension and regard $\epsilon = 4 - d$ as a small parameter. In that case, we find a new non-trivial fixed point known as the Wilson-Fischer fixed point at

$$g_2^* = -\frac{1}{6}\epsilon + \cdots \ , \qquad g_4^* = \frac{1}{3a}\epsilon + \cdots \ , \qquad g_{2n>4}^* \sim \epsilon^n + \cdots \tag{2.21}$$

In particular, the Wilson-Fischer fixed point is only physically acceptable if $\epsilon > 0$, or $d < 4$, since otherwise the couplings g_{2n}^* are all negative and the potential of the theory would not be bounded from below. In the neighbourhood of the fixed point in the g_2, g_4 subspace we have to linear order in ϵ

$$\mu\frac{d}{d\mu}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix} = \begin{pmatrix}\epsilon/3-2 & -a(1+\epsilon/6)\\ 0 & \epsilon\end{pmatrix}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix}\ , \tag{2.22}$$

with

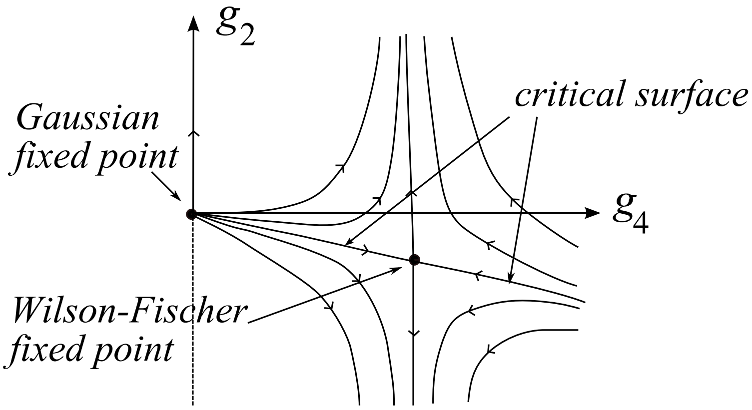
$$a = 1/(16\pi^2) + \epsilon(1-\gamma_E+\log 4\pi)/32\pi^2 + \mathcal{O}(\epsilon^2)\ . \tag{2.23}$$

So the scaling dimensions of the associated operators and the associated couplings are

$$\begin{aligned}&\Delta_2 = 2-2\epsilon/3\ , \qquad \sigma_2 = \delta g_2\ ,\\ &\Delta_4 = 4\ , \qquad \sigma_4 = \delta g_4 - \frac{a}{2+\epsilon/3}\delta g_2\ .\end{aligned} \tag{2.24}$$

So at this fixed point only the mass coupling is relevant.

The flows in the (g_2, g_4) subspace of scalar QFT for small $\epsilon > 0$ are shown below

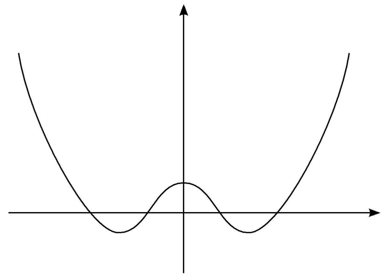


The Gaussian and Wilson-Fischer fixed points are shown and all the other couplings are irrelevant and so flow to the (g_2, g_4) subspace. Notice that the critical surface intersects this subspace in the line that joins the two fixed points as shown

Although we have only proved the existence of the Wilson-Fischer fixed point for small ϵ, it is thought to exist in both $d = 3$ and $d = 2$. In the language of statistical physics it lies in the universality class of the Ising Model.[7] What our simple analysis fails to show is that in $d = 2$ there are actually an infinite sequence of additional fixed points.[8]

Vacuum Expectation Values

In a scalar QFT, the field can develop a non-trivial Vacuum Expectation Value (VEV) $\langle\phi\rangle \neq 0$. This possibility is determined by finding the minima of the *effective potential*. This is the potential on the constant (or zero) mode of the field after all the non-zero modes have been integrated out. In other words, this is the potential in the Wilsonian effective action in the limit $\mu \to 0$.[9] A VEV develops when the effective potential develops minima away from the origin as in

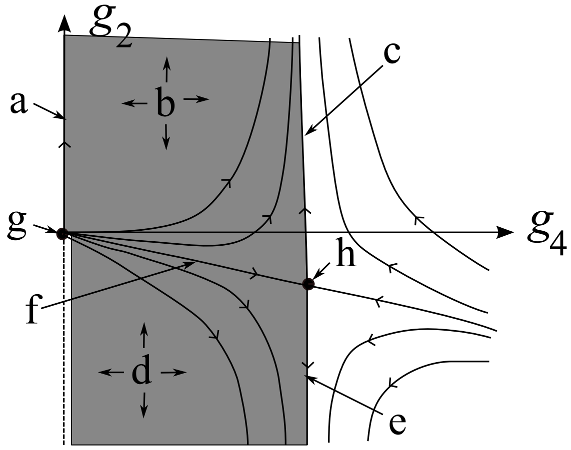


Notice that since we started with a theory symmetric under $\phi \to -\phi$, there will necessarily two possible vacuum states with opposite values of $\langle\phi\rangle$. A QFT must choose one or the other and so we say that the symmetry $\phi \to -\phi$ is *spontaneously broken*.[10]

Now that we have a qualitative picture of the RG flows, it is possible to describe the possible continuum limits of scalar field theories:

$d \geq 4$**:** In this case, only the Gaussian fixed point exists and this fixed point only has one relevant direction, namely the mass coupling g_2. Hence there is a single renormalized trajectory on which $g_2(\mu) = (\mu'/\mu)^2 g_2(\mu')$ while all the other couplings vanish. This renormalized trajectory describes the free massive scalar field. If we sit precisely at the fixed point we have a free massless scalar field. In particular, according to this analysis there is no interacting continuum theory in $d = 4$.

$d < 4$**:** At least for small enough ϵ (whatever that means) there are two fixed points and a two-dimensional space of renormalized trajectories parameterized by the couplings g_2 and g_4 on which g_{2n}, $n > 2$ have some values fixed by g_2 and g_4. In particular, if we parameterize our continuum theories by the values of g_2 and g_4 then they are limited to the region shown below



In particular, theory (a) is free and massive (and must have $g_2 > 0$); (b,d) is interacting and massive and in the UV becomes free since the trajectories originate from the Gaussian fixed point. In case (d) $g_2 < 0$ and the field has a VEV; (c,e) describe a massive interacting theory that becomes the Ising model universality class in the UV, case (e) has $g_2 < 0$ and a VEV; (f) describes a massless interacting theory that interpolates between a free theory in the UV and the Ising Model in the IR; (g) is a free massless theory; and (h) is the Ising model CFT.

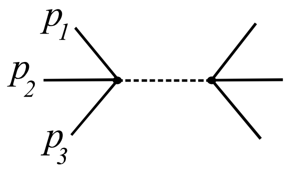
Notes

1 This is an example of using a symmetry to restrict theory space. The important point is that the symmetry is respected by RG flow and so is self-consistent.

2 Note, however, that, in principle, there is no difficulty in keeping track of all the couplings including the higher derivative terms.

3 We take it as established fact that one can move between the Minkowski and Euclidean versions of the theory without difficulty. In our conventions, when we Wick rotate $g_{\mu\nu}a^\mu b^\nu = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 \to -a_\mu b_\mu = -a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3$. In Euclidean space the functional integral $\int [d\phi] e^{-S[\phi]}$ can be interpreted as a probably measure (when properly normalized) on the the field configuration space. This is why Euclidean QFT is intimately related to systems in statistical physics.

4 In the following we have ignored the term which is linear in $\hat{\phi}$. The reason is that we are after an effective action for φ which is local (that is a spacetime integral over terms which are powers of the field and its derivatives). The linear coupling in $\hat{\phi}$ gives rise to graphs which are "1-particle reducible", the simplest of which is shown below



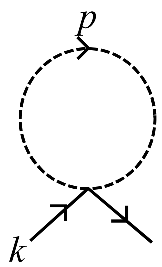
The contribution from this graph is only non-vanishing if $\mu \leq |p_1 + p_2 + p_3| \leq \mu + \delta\mu$. In particular as long as we work with local actions which are expansions in derivatives, *i.e.* momenta, we can ignore such contributions. These non-analytic contributions are the price we pay for working with a sharp momentum cut-off. It is important to understand that although the effective action would have these non-analytic contributions the observables of the theory, namely the Green functions or the S-matrix, would be perfectly well behaved. For a discussion of this in much greater detail, see the excellent lecture notes of Weinberg.

5 Here we used the identity

$$\int d^n x \, e^{-x\cdot A\cdot x} = \frac{\pi^{n/2}}{\det\, A} = \pi^{n/2} e^{-\frac{1}{2}\log\det\, A} \tag{2.25}$$

for a finite matrix A and extended it to the functional case.

6 The terms quadratic in ϕ come from the diagram below



which gives a contribution

$$\sim \int \frac{d^d k}{(2\pi)^d} \tilde{\phi}(k)\tilde{\phi}(-k) \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + g_2\mu^2} = \text{const.} \times \int d^d x \, \phi(x)^2 \ . \tag{2.26}$$

Notice that the resulting expression does not involve derivatives of the field because, due to momentum conservation, no external momentum can flow in the loop. Hence, there is no wavefunction renormalization.

7 The Ising Model is a statistical model defined on a square lattice with spins $\sigma_i \in \{+1, -1\}$ at each site and with an energy (which we identify with the Euclidean action)

$$\mathscr{E} = -\frac{1}{T} \sum_{(i,j)} \sigma_i \sigma_j \ . \tag{2.27}$$

The sum is over all nearest-neighbour pairs (i, j) and T is the temperature. Notice that at low temperatures, the action/energy favours alignment of all the spins, while at high temperatures thermal fluctuations are large and the long-range order is destroyed. This can be viewed as a competition between energy and entropy. There is a 2nd order phase transition at a critical temperature $T = T_c$ at which there are long-distance power-law correlations. This critical point is in the same universality class as the Wilson-Fischer fixed point and the water-steam critical point.

8 In $d = 2$ there are powerful methods for analyzing CFTs because in $d = 2$ the conformal group is infinite dimensional since it consists of any holomorphic transformation $t \pm x \to f(t \pm x)$.

9 It can be shown that the effective potential defined in terms of the Wilsonian effective action is equal to the effective potential extracted from the more familiar 1-Particle Irreducible (1-PI) effective action defined in perturbation theory—at least for a QFT with a mass gap. In the massless case the latter quantity is ill-defined due to IR divergences.

10 The reason why spontaneous symmetry breaking occurs is that the zero mode of a scalar field is not part of the variables that are integrated over in the measure $\int [d\phi]$, rather it acts as a boundary condition on the scalar field at spatial infinity. However, this is only true in spacetime dimensions $d > 2$: in $d = 2$ one must integrate over the zero mode and so spontaneous symmetry breaking of *continuous* symmetries cannot occur. Of course $\phi \to -\phi$ is a discrete symmetry that can still be spontaneously broken even in $d = 2$.