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Solution of a Field Theoretical Model in One Space-One Time Dimension*†

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A model of massless fermions interacting with massive vector particles in two dimensions is solved. The solutions are first given in terms of Green's functions and then in terms of operators which reproduce the Green's functions. The operators span a larger Hilbert space and it is shown that the field equations hold only in a subspace. The solution is also discussed in terms of Feynman diagrams.

I. INTRODUCTION

In the following we shall investigate a theory of massless fermions interacting with a massive vector particle in two dimensions. The latter being coupled to a conserved current, it is not necessary to require the Lorentz condition. It is well known[1] that in this case the quanta of negative energy are effectively uncoupled and could be transformed away. This theory has the advantage that

a. All components of the vector field are canonical quantities and hence the equations for the generator of the Green's functions can be directly derived.

b. All Green's functions are finite functions of space and time and no problems of how to handle infinite quantities arise. This is at the expense of the redundance of the Hilbert space which also contains unphysical quanta of negative energy. We shall discuss the solution of this theory from three points of view. First we solve the equations for the generator of the Green's functions; secondly, we discuss what this means in terms of Feynman diagrams; and thirdly, we give an operator solution which reproduces the Green's functions.

The solution shows some interesting features. First of all the results of the theory depend on the limiting process by which the current is defined. We shall adopt the one which was advocated in particular by Schwinger (*2*), namely,

$$j_n(x) = e \lim_{\epsilon \to \pm 0} \psi^\dagger(x + \epsilon/2)\alpha_n\psi(x - \epsilon/2) \exp\,(ieA^\rho(x)\epsilon_\rho) \tag{1}$$

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[1] See, e.g., ref. *1*.

ϵ is a vector which approaches zero from one direction and the opposite one to ensure the hermiticity of the current. In our case the result is independent of the direction of ϵ. It turns out that with this definition of j all gauge and Lorentz variant terms disappear in the vacuum polarization and do not have to be subtracted by regularization. In two dimensions (1) leads to a mass renormalization of the vector boson. This is to be contrasted with the treatment of Glaser and Jaksic (3) who do not use the limiting procedure and do not obtain a mass change of the boson. In our solution the consequences of current conservation and other consistency checks will be satisfied, which is not the case for other definitions of j.

II. THE GENERATING FUNCTIONAL

For the convenience of the reader we shall first recapitulate the formal solution for the generating functional in an arbitrary number of dimensions. Our theory is derived from the Lagrangian:

$$L = -\tfrac{1}{2}(A^{i}_{,k}A_{i}^{,k} - \mu_0^2 A_i A^i) + \psi^\dagger \alpha_k \partial^k \psi - A_i j^i \tag{2}$$

with the ensuing field equations

$$(\Box + \mu_0^2)A_i = j_i \qquad \alpha^k(i\partial_k - eA_k)\psi = 0 \tag{3}$$

and commutation relations:

$$\begin{gathered} [A_i(x), \dot{A}_k(x')]_{t=t'} = -ig_{ik}\delta(\mathbf{x} - \mathbf{x}') \qquad \{\psi^\dagger(x), \psi(x')\}_{t=t'} = \delta(\mathbf{x} - \mathbf{x}') \\ [A_i A_k]_{t=t'} = [\dot{A}_i \dot{A}_k]_{t=t'} = [A_i \psi]_{t=t'} = 0 \end{gathered} \tag{4}$$

The content of the theory is contained in the functional:

$$\begin{aligned} &G(J, \eta, \eta^\dagger) \\ &\quad = T\langle 0 | \exp\left[i\int dy(A^k(y)J_k(y) + \eta^\dagger(y)\psi(y) + \psi^\dagger(y)\eta(y))\right] | 0\rangle \end{aligned} \tag{5}$$

from which all Green's functions can be derived by functional differentiation.

From (3) and (4) it follows[2]:

$$\begin{gathered} \alpha^k\left(i\partial_k + ie\frac{\delta}{\delta J^k(x)}\right)\frac{\delta G}{\delta\eta^\dagger(x)} = -i\eta(x)G \\ (\Box + \mu_0^2)\frac{\delta G}{\delta J^j(x)} = \left(-iJ_j(x) + ie\,\mathrm{tr}\,\frac{\delta}{\delta\eta(x)}\,\alpha_j\frac{\delta}{\delta\eta^\dagger(x)}\right)G \end{gathered} \tag{6}$$

[2] In the following we shall always understand that the limiting procedure (1) for the current is observed.

These equations for G can be solved if the Green's function S for the Dirac field in the presence of an arbitrary external field is known.[3]

$$\alpha^k(i\partial_k - eA_k(x))S(x,x',A) = \delta(x - x') \tag{7}$$

With the help of $S(xx'A)$ the first of Eqs. (6) can be integrated immediately:

$$G(J, \eta, \eta^\dagger) = \exp\left[-i\int dx''\, dx'\eta^\dagger(x'')S\left(x''x'\frac{\delta}{i\delta J}\right)\eta(x')\right]G_J \tag{8}$$

where G_J is a functional of J only.

In order to solve the second of Eqs. (6) we have to introduce the functional $F(A)$ corresponding to all closed loop diagrams. This functional can be constructed in terms of $S(xx'A)$ and satisfies (4):

$$\frac{\delta F(A)}{\delta A^j(x)} = e\lim_{\epsilon\to\pm 0}\operatorname{tr}\alpha_j\, S\left(x+\frac{\epsilon}{2}, x-\frac{\epsilon}{2} A\right)e^{ie\epsilon^\rho A_\rho} \tag{9}$$

Using the Ward identity

$$\frac{\delta}{\delta A^j(x)}\, S(x''x'A) = e\int S(x''xA)\alpha_j\, S(xx'A) \tag{10}$$

one verifies that G is finally given by

$$\begin{aligned} G = {} & \exp\left[-i\int dx''\, dx'\eta(x'')S\left(x''x'\frac{1}{i}\frac{\delta}{\delta J}\right)\eta(x')\right]\exp\left[-F\left(\frac{\delta}{i\delta J}\right)\right] \\ & \cdot\exp\left[-\frac{1}{2}\int dz''\, dz' J^i(z'')\Delta_0(z''-z')J_i(z')\right][e^{-F(\delta/i\delta J')}e^{-\frac{1}{2}J'\Delta_0 J'/2}]_{J'=0}^{-1} \end{aligned} \tag{11}$$

Here Δ_0 is a free boson propagator with mass μ_0.

In general, S is given by an expansion in A. It is only in the two-dimensional model that we have a closed solution of (7) and the simple structure of S and F allows us to evaluate (11) further. One readily sees that in two dimensions

$$S(xx'A) = S_0(x - x')\exp\left[e\int dy(S_0(x - y) - S_0(x' - y))\alpha^k A_k(y)\right] \tag{12}$$

where S_0 is the free fermion propagator. That in the exponent the Feynman propagator has to be taken follows, e.g., from perturbation theory. The simple form of (11) is due to the fact that in two dimensions all α commute and can be treated as ordinary numbers. For this S the limit of the right hand side of (9) can be taken and using

$$\frac{\epsilon^\mu\epsilon^\nu}{\epsilon^2} + \frac{\bar\epsilon^\mu\bar\epsilon^\nu}{\bar\epsilon^2} = g^{\mu\nu}, \qquad \epsilon\bar\epsilon = 0$$

[3] This equation is to be solved using Feynman boundary conditions.

we find that due to the exponential factor the limit is independent of ϵ

$$\frac{\delta F(A)}{\delta A^j(x)} = -\frac{ie^2}{\pi} \int dx' P_{jk}(x - x') A^k(x') \tag{9b}$$

where P is the transversal projection operator

$$P_{jk}(x - x') = g_{jk}\delta(x - x') - \partial_j\partial_k D(x - x') \tag{13}$$

The functional $F(A)$ is therefore given by:

$$F(A) = -\frac{ie^2}{2\pi} \int dx\, dx' A^j(x) P_{jk}(x - x') A^k(x') \tag{14}$$

Inserting (12) and (14) into (11) we can carry out the functional differentiations involved in F and S by elementary methods.[4] We find:

$$\begin{aligned} G(\eta, \eta^\dagger, J) = {} & \exp\left\{-i \int dx\, dx' \eta^\dagger(x) S_0(x - x') \right. \\ & \cdot \exp\left[-ie \int dy (S_0(x - y) - S_0(x' - y))\alpha^k \frac{\delta}{\delta J_k}\right] \left. \eta(x') \right\} \\ & \cdot \exp\left[-i/2 \int dz\, dz' J^j(z) \mathfrak{h}_{jm}(z - z') J^m(z')\right] \end{aligned} \tag{15}$$

$$\begin{aligned} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \eta^\dagger(x_1) S_0(x_1 - x_1') \eta(x_1') \cdots \eta^\dagger(x_n) S_0(x_n - x_n') \eta(x_n') \\ \cdot \exp\left[-\tfrac{1}{2} \int \mathcal{J}^j_{(n)}(z) \mathfrak{h}_{jm}(z - z') \mathcal{J}^m_{(n)}(z')\right] \end{aligned}$$

where

$$\mathcal{J}^j_{(n)}(z) = J^j(z) + ie \sum_{m=1}^{n} (S(z - x_m) - S(z - x_m'))\alpha^j \tag{16}$$

and

$$\mathfrak{h} = \left[\Box + \mu_0^2 + \frac{e^2}{\pi} P\right]^{-1} = [(\Box + \mu^2)P + (\Box + \mu_0^2)(1 - P)]^{-1} \tag{17}$$

In momentum space where P is diagonal we have

$$\mathfrak{h}_{jm}(k) = \frac{P_{jm}}{\mu^2 - k^2} + \frac{(1 - P)_{jm}}{\mu_0^2 - k^2} = \left(g_{jm} - \frac{k_j\, k_m}{k^2}\right) \frac{1}{\mu^2 - k^2} + \frac{k_j\, k_m}{k^2(\mu_0^2 - k^2)} \tag{18}$$

$$\mu^2 = \mu_0^2 + (e^2/\pi)$$

[4] $e^{ic\partial^2/\partial x^2} e^{iBx^2} = \frac{1}{(1 + 4Bc)^{\frac{1}{2}}} e^{i[B/(1+4BC)]x^2}$

III. DISCUSSION

Having the explicit form of G given by (14) we shall now discuss some features of the Green's functions. The boson propagator is

$$T\langle 0 \mid A_i(x) A_k(x') \mid 0\rangle = \frac{-\delta^2 G}{\delta J_i(x) \delta J_k(x')} = +i\mathfrak{h}_{ik}(x - x') \tag{18}$$

It is consistent with the canonical commutation rules (4) which imply in a spectral representation (see ref. *1*).

$$\mathfrak{h}_{ik}(x - x') = \int_0^\infty da^2 \left[\pi(a)(a^2 g_{ik} - \partial_i \partial_k') + \frac{\partial_i \partial_k'}{\mu_0^2} \delta(\mu_0^2 - a^2)\right] \Delta(x - x', a) \tag{19}$$

the sum rules

$$\int da^2 \pi = 1/\mu_0^2, \qquad \int da^2 a^2 \pi = 1 \tag{20}$$

In our case we have

$$\pi(a) = \frac{\delta(a^2 - \mu^2)}{\mu^2} + \delta(a^2) \frac{\mu^2 - \mu_0^2}{\mu^2 \mu_0^2} \tag{21}$$

which satisfies the sum rules and is positive definite. The $\delta(a^2)$-term yields the possibility that the physical mass μ^2 is $> \mu_0^2$. Diagrammatically the simplicity of G is due to the fact that all closed loop diagrams vanish except the lower order one. This gives a contribution $(e^2/\pi)P$ to the proper vacuum polarization part as exhibited by (16).

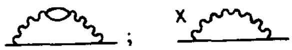
The fermion propagator is found to be

$$\begin{aligned} T\langle 0 \mid \psi(x)\psi^\dagger(x') \mid 0\rangle &= \frac{\delta^2 G}{\delta\eta^\dagger(x)\delta\eta(x')} = iS_0(x - x') \\ &\exp\left[\frac{ie^2}{\pi}(S(z - x) - S(z - x'))\alpha_\mu \mathfrak{h}^{\mu\nu}(z - z')\alpha_\nu(S(z' - x) - S(z' - x'))\right] \\ &= iS_0(x - x')e^{ie^2(D(x-x') - D(0))} \end{aligned} \tag{22}$$

where

$$D(x) = \frac{1}{\mu_0^2}(\Delta(x, \mu_0) - \Delta(x, 0)) - \frac{1}{\mu^2}(\Delta(x, \mu) - \Delta(x, 0)) \tag{23}$$

It should be noted that D together with its first three derivatives is finite and continuous for all values of x. Hence there is no difficulty in having D in the exponent, and the canonical commutation rules for ψ are satisfied. In a spectral representation the zero mass part in D removes the pole of the spectral function for mass zero in the same way as in the model studied by Schroer (*5*). It appears from (23) that the next correction to S_0 is $\sim e^4$ corresponding to the diagram



The lowest order diagram vanishes because $\alpha^k\beta\alpha_k = 0$. Hence only the part of the boson propagator $k_x k_y$ contributes and the form of the propagator (22) equals the one of a fermion coupled to the gradient of a scalar. From the above it appears that from the present model no conclusions should be drawn regarding the convergence of perturbation theory or the existence of a more realistic field theory. Only a negligibly small member of the totality of diagrams contribute in the two-dimensional case.

IV. OPERATOR SOLUTION

In this section we shall construct a solution in terms of free fields. This is most conveniently done in a larger Hilbert space where the two components of A are expressed in terms of 4 canonical independent fields: a_σ, b, B, C:

$$\begin{aligned} A_\sigma &= a_\sigma + \frac{1}{\mu}\epsilon_{\sigma\tau} C^{,\tau} + \frac{1}{\mu_0}(B_{,\sigma} + b_{,\sigma}) \\ (\Box + \mu^2)a_\sigma &= 0, \qquad a^{\sigma}_{,\sigma} = 0 \\ (\Box + \mu_0^2)B &= \Box C = \Box b = 0 \end{aligned} \tag{24}$$

The fields a and b have the usual commutation rules whereas the ones B and C have a sign corresponding to negative energies. ψ is expressed in terms of these and of a free canonical spinor χ:

$$\begin{aligned} \psi(x) &= e^{i\phi(x)}\chi(x) \qquad i\alpha_\nu\,\partial^\nu\chi = 0 \\ \phi(x) &= -\frac{\gamma_5}{\mu}\left(\frac{1}{\mu}\epsilon^{\sigma\tau}\alpha_{\sigma,\tau} + C\right) - \frac{1}{\mu_0}(B + b) \end{aligned} \tag{25}$$

Substituting in (5) and working out the expectation value one arrives at the expression (15). This suggests that (24) and (25) furnish an operator solution of our problem. One verifies easily that (4) and the Dirac equation are satisfied. To evaluate Maxwell's equation we remove the ambiguity of $P_{\mu\nu}\partial^\nu f$ in the case $\Box f = 0$, putting $P_{\mu\nu}\partial^\nu f = 0$. Maxwell's equation then reduces to:

$$\mu_0 b_{,j} + \mu\epsilon_{ji}c^{,i} - e\chi^\dagger\alpha_j\chi = \mathcal{R}_j = 0 \tag{26}$$

Since the free fields are supposed to be independent, (26) cannot hold in operator form. The answer to this puzzle lies in the observation that (26) holds for all matrix elements in the physical Hilbert space which is generated by applying powers of A, ψ, $\psi^\dagger$ onto the vacuum. To show this we realize that $\mathcal{R}_j$ can be split into positive and negative frequences $\mathcal{R}_j^+$ and $\mathcal{R}_j^-$ and that according to the definition

$$\mathcal{R}_j^- \mid 0 \rangle = \langle 0 \mid \mathcal{R}_j^+ = 0$$

A direct calculation shows that

$$[\mathcal{R}_j^{(x)}, A_j(y)] = [\mathcal{R}_j(x)\psi(y)] = [\mathcal{R}_j(x)\psi^\dagger(y)] = 0 \qquad (27)$$

Therefore there holds a super selection rule in the Hilbert space of A, ψ, $\psi^\dagger$, and Eq. (26) is valid in the subspace mentioned above. The statement is also demonstrated by the fact that:

$$T\left\langle 0 \left| \exp\left[i \int dy \{ A^j(y) J_j(y) + \eta^\dagger(y)\psi(y) + \psi^\dagger(y)\eta(y) \} \right] \mathcal{R}_j(x) \right| 0 \right\rangle = 0 \quad (28)$$

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