2. Fields for Free Particles. Discrete Symmetries.

Joan Soto

Universitat de Barcelona)
Departament de Física Quàntica i Astrofísica
Institut de Ciències del Cosmos

2.0 Overview

We need a formalism that:

- Combines Quantum Mechanics (QM) and Special Relativity (SR)
- Allows to describe processes with an arbitrary number of particles

The outcome is called Quantum Field Theory (QFT)

How to build a QFT for free particles?

- Find a suitable relativistic QM wave equation
- Find a Lagrangian the equation of motion of which (Lagrange equations) leads to it
- Apply canonical quantization rules to this Lagrangian

As a consequence wave functions in QM become operators in QFT

- The arbitrary functions in the general solutions of the wave equations become creation and annihilation operators

We shall display the outcome of this procedure for:

- Schrödinger field (non-relativistic)
- Scalar field (spin 0)
- Dirac field (spin 1/2)
- Vector field (spin 1)

We shall discuss in each case the implementation of Parity (P), Charge Conjugation (C) and Time Reversal (T)

How to include interactions?

- Add to the Lagrangian local terms that respect the symmetries one observes in nature
- We shall do it in the cases above for the interaction with an electromagnetic field

2.1 The Schrödinger field

The Schrödinger equation:

$$\left(i\frac{\partial}{\partial t} + \frac{\vec{\nabla}^2}{2m} \right) \psi(t,\vec{x}) = 0$$

- If the particle has spin s, each of the $2s+1$ components fulfils the equation above
- It corresponds to the equations of motion (Euler-Lagrange equations) of

$$S = \int dtL \quad , \quad L = \int d^3\vec{x}\mathcal{L} \quad , \quad \mathcal{L}(t,\vec{x}) = \psi^\dagger(t,\vec{x}) \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m} \right) \psi(t,\vec{x})$$

- Canonical quantization
  - Canonical momentum $(a, b = -s, \ldots, s)$

$$\Pi_a(x) \equiv \frac{\partial L}{\partial(\partial_0\psi_a(x))} = i\psi_a^*(x) \qquad \Pi_a^*(x) \equiv \frac{\partial L}{\partial(\partial_0\psi_a^*(x))} = 0$$

  - Quantization rules $(t = 0)$

$$\psi_a(\vec{x}) \to \hat{\psi}_a(\vec{x}) \quad , \quad \Pi_a(\vec{x}) \to \hat{\Pi}_a(\vec{x})$$

$$[\hat{\psi}_a(\vec{x}), \hat{\Pi}_b(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\delta_{ab} \quad , [\hat{\psi}_a(\vec{x}), \hat{\psi}_b(\vec{y})] = [\hat{\Pi}_a(\vec{x}), \hat{\Pi}_b(\vec{y})] = 0$$

  - For fermions, commutators must be replaced by anticommutators

- Hats are usually not displayed
- The general solution to the Schrödinger equation becomes

$$\hat{\psi}_a(\vec{x},t)=\int \frac{d^3\vec{p}}{(2\pi)^3}\hat{a}_a(\vec{p})e^{-iEt+i\vec{p}\cdot\vec{x}}$$

$$[\hat{a}_a(\vec{p}),\hat{a}_b^\dagger(\vec{p}\,')]=(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}\,')\delta_{ab} \qquad [\hat{a}_a(\vec{p}),\hat{a}_b(\vec{p}\,')]=[\hat{a}_a^\dagger(\vec{p}),\hat{a}_b^\dagger(\vec{p}\,')]=0$$

  - For fermions, commutators must be replaced by anticommutators
  - $\hat{a}_a(\vec{p})$ and $\hat{a}_a^\dagger(\vec{p})$ are called annihilation and creation operators respectively
- One assumes that the ground state $|0\rangle$ exists, and fulfils $\langle 0|0\rangle=1$ and $\hat{a}_a(\vec{p})\,|0\rangle=0$
- A one-particle state is defined as

$$\hat{a}_a^\dagger(\vec{p})\,|0\rangle\equiv|\vec{p}a\rangle \quad , \quad \langle 0|\,a_{a'}(\vec{p}\,')\equiv\langle\vec{p}\,'\,a'| \quad \Rightarrow \quad \langle\vec{p}\,'a'|\vec{p}a\rangle=(2\pi)^3\delta(\vec{p}-\vec{p}\,')\delta_{aa'}$$

- A two-particle state is defined $|\vec{p}_1a_1,\vec{p}_2a_2\rangle\equiv\hat{a}_{a_1}^\dagger(\vec{p}_1)\hat{a}_{a_2}^\dagger(\vec{p}_2)\,|0\rangle$, and so on
- The space on which the fields act is called the Fock space, and a basis of it is,

$$\{|0\rangle\,,|\vec{p}_1a_1\rangle\,,|\vec{p}_1a_1,\vec{p}_2a_2\rangle\,,\dots,|\vec{p}_1a_1,\vec{p}_2a_2,\dots,\vec{p}_na_n\rangle\,,\dots\}$$

Discrete Symmetries

- Parity

$$\vec{x} \to -\vec{x} \quad , \quad \vec{p} \to -\vec{p} \quad , \quad \vec{L} \to \vec{L} \quad , \quad \vec{S} \to \vec{S}$$

  - In QM, and hence in QFT, it is implemented by a unitary operator P,

$$\langle B|A\rangle = \langle P\,B|P\,A\rangle$$

  - Furthermore, P can be chosen such that $P^2 = 1 \implies P^{-1} = P^\dagger = P$
  - $P\psi(t,\vec{x})P^{-1} = \pm\psi(t,-\vec{x}) \implies PLP^{-1} = L$
  - $P\psi(t,\vec{x})P^{-1} = \pm\psi(t,-\vec{x}) \implies Pa(\vec{p})P^{-1} = \pm a(-\vec{p})$
  - $Pa(\vec{p})P^{-1} = \pm a(-\vec{p}) \implies P\,|\vec{p}\rangle = \pm\,|-\vec{p}\rangle$, if $P\,|0\rangle = |0\rangle$ is assumed
  - Spin indices do not transform under parity and are not displayed

- Charge Conjugation is not a symmetry in non-relativistic systems

- Time reversal

$$\vec{x} \to \vec{x} \quad , \quad \vec{p} \to -\vec{p} \quad , \quad \vec{L} \to -\vec{L} \quad , \quad \vec{S} \to -\vec{S}$$

  - In QM, and hence in QFT, it is implemented by an antiunitary operator T,

    $$\langle A|B\rangle = \langle T\,B|T\,A\rangle = \langle T^\dagger T\,A|B\rangle = \langle A|T^\dagger T\,B\rangle \quad \Longrightarrow \quad T(c\,|A\rangle) = c^*\,T\,|A\rangle$$

  - $T^\dagger = T^{-1}$
  - $T^2 \neq 1$ in general
  - For $s = 0$, $T\psi(t,\vec{x})T^{-1} = \eta_T\psi(-t,\vec{x})$, $|\eta_T| = 1 \Longrightarrow TST^{-1} = S$
  - For $s \neq 0$, s_3 labels must be mapped into $-s_3$. If the generators of the rotations in spin space S_1 and S_3 are taken real and S_2 purely imaginary then

    $$T\psi(t,\vec{x})T^{-1} = \eta_T e^{-i\pi S_2}\psi(-t,\vec{x})$$

  - Note that $T^2\psi(t,\vec{x})T^{-2} = e^{-i2\pi S_2}\psi(t,\vec{x}) = (-1)^{2s}\psi(t,\vec{x})$
  - Note that the transformation of $\psi(t,\vec{x})$ under time reversal are different in QFT than in QM (for the Schrödinger equation to be invariant under time reversal in QM, one needs $\psi(t,\vec{x}) \to \eta_T e^{-i\pi S_2}\psi^*(-t,\vec{x})$). This is because $\psi(t,\vec{x})$ is an operator in QFT rather than a state as in QM.

Coupling to electromagnetism

The Maxwell equations in the vacuum $\partial_\mu F^{\mu\nu} = 0$, can be obtained from the following Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- $F_{\mu\nu}$ is invariant under $A_\mu \to A_\mu - \partial_\mu \theta$, $\theta = \theta(x)$
- The coupling to matter fields must respect this symmetry
- On the Schrödinger field, it is implemented as $\psi(x) \to e^{iq\theta(x)}\psi(x)$
- $\mathcal{L}$ becomes invariant if we replace $\partial_\mu \to D_\mu \equiv \partial_\mu + iqA_\mu$. This is called minimal coupling
- Non-minimal couplings to $F_{\mu\nu}$ may also exist. For instance the magnetic moment, $\vec{\mu}\vec{B}$ if $s \neq 0$, $B^k = -\frac{1}{2}\varepsilon^{klm} F_{lm}$, $\vec{\mu}$ is a $(2s+1) \times (2s+1)$ matrix.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \psi^\dagger \left(iD_0 + \frac{\vec{D}^2}{2m} + \vec{\mu}\vec{B} + \cdots \right) \psi$$

2.2 Klein-Gordon field ($s = 0$)

The simplest relativistic wave equation is the Klein-Gordon (KG) equation,

$$\left(\partial_\mu \partial^\mu + m^2\right) \phi(x) = 0$$

- For ϕ complex, it corresponds to the equations of motion of

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

- The general solution of the KG equation reads upon quantization

$$\hat{\phi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E}} (e^{-ipx} \hat{a}(\vec{p}) + e^{ipx} \hat{b}^\dagger(\vec{p})) \quad , \quad E = p^0 = \sqrt{\vec{p}^2 + m^2}$$

  - $\hat{a}(\vec{p})$ and $\hat{b}(\vec{p})$ are the annihilation operators of a spin zero particle and its antiparticle
  - These particles are necessarily bosons as the quantization using anticommutators is inconsistent
  - The commutation relations are the same as in the Schrödinger case ($\hat{a}(\vec{p})$ and $\hat{a}^\dagger(\vec{p})$ commute with $\hat{b}(\vec{p})$ and $\hat{b}^\dagger(\vec{p})$)

- The ground state $|0\rangle$ is called vaccum, $\langle 0|0\rangle = 1$, $\hat{a}(\vec{p})\,|0\rangle = 0$ and $\hat{b}(\vec{p})\,|0\rangle = 0$
- The n-particle m-antiparticle state is defined

$$|\vec{p}_1 \dots \vec{p}_n; \vec{p}_1', \dots \vec{p}_m'\rangle = \sqrt{2E_1} \dots \sqrt{2E_n}\sqrt{2E_1'} \dots \sqrt{2E_m'}\hat{a}^\dagger(\vec{p}_1) \dots \hat{a}^\dagger(\vec{p}_n)\hat{b}^\dagger(\vec{p}_1') \dots \hat{b}^\dagger(\vec{p}_m')\,|0\rangle$$

  - The funny factors $\sqrt{2E}$ above are to ensure standard relativistic normalization $\langle \vec{p}|\vec{p}'\rangle = 2E(2\pi)^3\delta(\vec{p} - \vec{p}')$
- For ϕ real, it corresponds to the equations of motion of

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2$$

  - In this case $\hat{a}(\vec{p}) = \hat{b}(\vec{p})$, the antiparticle coincides with the particle.

Discrete Symmetries

- Parity
  - $P\phi(t,\vec{x})P^{-1} = \pm\phi(t,-\vec{x}) \implies PLP^{-1} = L$
  - $P\phi(t,\vec{x})P^{-1} = \pm\phi(t,-\vec{x}) \implies Pa(\vec{p})P^{-1} = \pm a(-\vec{p}),$
    $Pb(\vec{p})P^{-1} = \pm b(-\vec{p})$
    - ★ Note that particle and antiparticle have the same parity
  - $P\,|\vec{p};\rangle = \pm\,|-\vec{p};\rangle$, $P\,|;\vec{p}\rangle = \pm\,|;-\vec{p}\rangle$, if $P\,|0\rangle = |0\rangle$ is assumed
- Charge conjugation (C-parity)

$$Ca(\vec{p})C^{-1} = b(\vec{p}) \quad , \quad Cb(\vec{p})C^{-1} = a(\vec{p}) \quad ,$$

  - $C^2 = 1$, $C = C^{-1} = C^{\dagger}$ (unitary implementation)
  - Then $C\phi(x)C^{-1} = \phi^*(x) \implies C\mathcal{L}C^{-1} = \mathcal{L}$
  - If $\phi(x)$ is real, $Ca(\vec{p})C^{-1} = \pm a(\vec{p}) \implies C\phi(x)C^{-1} = \pm\phi(x)$
    - ★ $C\,|\vec{p}\rangle = \pm\,|\vec{p}\rangle$, if $C\,|0\rangle = |0\rangle$ is assumed (e. g. $C\,|\pi^0\rangle = +\,|\pi^0\rangle$)
- Time reversal
  - $T\phi(t,\vec{x})T^{-1} = \eta_T\phi(-t,\vec{x})$, $|\eta_T| = 1 \implies TST^{-1} = S$
  - $T\phi(t,\vec{x})T^{-1} = \eta_T\phi(-t,\vec{x}), \implies Ta(\vec{p})T^{-1} = \eta_T a(-\vec{p}),$
    $Tb(\vec{p})T^{-1} = \eta_T^* b(-\vec{p})$

Coupling to electromagnetism

- Analogously to the Schrödinger case, we assign the following gauge transformation $\phi(x) \to e^{iq\theta(x)}\phi(x)$
- Minimal coupling ($\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi$$

- Note that minimal coupling implies that particle (p) and antiparticle (a) have opposite electric charge

$$\phi(x) = \psi_p(x) + \psi_a^*(x) \quad , \quad \psi_p(x) \to e^{iq\theta(x)}\psi_p(x) \quad , \quad \psi_a(x) \to e^{-iq\theta(x)}\psi_a(x)$$

$\psi_p(x)$ and $\psi_a^*(x)$ contain the particle annihilation and the antiparticle creation operators respectively

- If $\phi(x)$ is real, only non-minimal couplings are allowed (e.g. $F_{\mu\nu}F^{\mu\nu}\phi$)

2.3 Dirac field ($s = 1/2$)

The suitable relativistic wave equation is the Dirac equation,

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

- $\psi(x)$ is a complex 4-vector, and γ^μ, $\mu = 0, \dots, 3$, 4×4 complex matrices that fulfil
$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$
- It corresponds to the equations of motion of
$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$
$\bar{\psi} \equiv \psi^\dagger \gamma^0$, $\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$
- The general solution of the Dirac equation reads upon quantization
$$\hat{\psi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E}} \sum_{\lambda=+,-} \left[e^{-ipx} u_\lambda(\vec{p}) \hat{a}_\lambda(\vec{p}) + e^{ipx} v_\lambda(\vec{p}) \hat{b}^\dagger_\lambda(\vec{p}) \right]$$
  - $u_\lambda(\vec{p})$ and $v_\lambda(\vec{p})$ are 4-vectors called Dirac spinors that fulfil
$$(p\!\!\!/ - m)u_\lambda(\vec{p}) = 0 \quad , \quad (p\!\!\!/ + m)v_\lambda(\vec{p}) = 0 \quad , \quad p\!\!\!/ = \gamma^\mu p_\mu$$

- The form of $u_\lambda(\vec{p})$ and $v_\lambda(\vec{p})$ depends on the representation of γ^μ
- There are several equivalent representations of the γ^μ matrices $(\gamma^{\mu\prime} = S\gamma^\mu S^{-1})$
  - Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}$$

  - Chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}$$

- γ^5 will be important later on, it is defined as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta \quad , \quad \epsilon_{0123} = 1$$

and fulfils in any representation

$$(\gamma^5)^2 = 1 \qquad \{\gamma^5, \gamma^\mu\} = 0$$

- In the Dirac representation

$$u_\lambda(\vec{p}) = \sqrt{E+m}\begin{pmatrix} \chi_\lambda \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_\lambda \end{pmatrix} \quad , \quad v_\lambda(\vec{p}) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\tilde{\chi}_\lambda \\ \tilde{\chi}_\lambda \end{pmatrix}$$

$$\tilde{\chi}_\lambda^\dagger \tilde{\chi}_{\lambda'} = \chi_\lambda^\dagger \chi_{\lambda'} = \delta_{\lambda\lambda'} \qquad \sum_\lambda \tilde{\chi}_\lambda \tilde{\chi}_\lambda^\dagger = \sum_\lambda \chi_\lambda \chi_\lambda^\dagger = \mathbb{I}_2$$

  - The choice $\tilde{\chi}_\lambda = -i\sigma^2\chi_\lambda^*$ ensures that λ corresponds to the same spin for the particle and its antiparticle
  - The choice

$$\chi_+ = \frac{1}{\sqrt{2(1+n^3)}}\begin{pmatrix} 1+n^3 \\ n^+ \end{pmatrix} , \ \chi_- = \frac{1}{\sqrt{2(1+n^3)}}\begin{pmatrix} -n^- \\ 1+n^3 \end{pmatrix} , \ n^\pm = n^1 \pm in^2$$

$\hat{n} = (n^1, n^2, n^3)$, $\hat{n}^2 = 1$ ensures that $\lambda = +(-)$ corresponds to the spin in the direction $\hat{n}$ $(-\hat{n})$

$$\chi_\pm \chi_\pm^\dagger = \frac{1}{2} \pm \frac{\hat{n}\vec{\sigma}}{2} \quad , \quad \frac{\hat{n}\vec{\sigma}}{2}\chi_\pm = \pm\frac{1}{2}\chi_\pm$$

If $\hat{n} = \hat{p} = \vec{p}/|\vec{p}|$, then $\lambda = +(-)$ corresponds to positive (negative) helicity

$$\hat{\psi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E}} \sum_{\lambda=+,-} \left[e^{-ipx} u_\lambda(\vec{p}) \hat{a}_\lambda(\vec{p}) + e^{ipx} v_\lambda(\vec{p}) \hat{b}^\dagger_\lambda(\vec{p}) \right]$$

- $\hat{a}_\lambda(\vec{p})$ and $\hat{b}_\lambda(\vec{p})$ are the annihilation operators of a particle of spin/helicity λ and its antiparticle
- These particles are necessarily fermions as the quantization using commutators is inconsistent
- The anticommutation relations are the same as in the Schrödinger case ($\hat{a}_\lambda(\vec{p})$ and $\hat{a}^\dagger_\lambda(\vec{p})$ anticommute with $\hat{b}_\lambda(\vec{p})$ and $\hat{b}^\dagger_\lambda(\vec{p})$)
- The ground state $|0\rangle$ is called vaccum, $\langle 0|0\rangle = 1$, $\hat{a}_\lambda(\vec{p})\,|0\rangle = 0$ and $\hat{b}_\lambda(\vec{p})\,|0\rangle = 0$
- The *n*-particle *m*-antiparticle state is defined

$$|\vec{p}_1\,\lambda_1 \dots \vec{p}_n\,\lambda_n; \vec{p}\,'_1\,\lambda'_1, \dots \vec{p}\,'_m\,\lambda_m{}'\rangle =$$
$$\sqrt{2E_1}\dots\sqrt{2E_n}\sqrt{2E'_1}\dots\sqrt{2E'_m}\hat{a}^\dagger_{\lambda_1}(\vec{p}_1)\dots\hat{a}^\dagger_{\lambda_n}(\vec{p}_n)\hat{b}^\dagger_{\lambda'_1}(\vec{p}\,'_1)\dots\hat{b}^\dagger_{\lambda'_m}(\vec{p}\,'_m)\,|0\rangle$$

- Complete basis of 4×4 matrices

$$\mathbb{I}_4 \quad , \quad \gamma^5 \quad , \quad \gamma^\mu \quad , \quad \gamma^5\gamma^\mu \quad , \quad \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu , \gamma^\nu]$$

- Trace properties:
  - The trace of the product of an odd number of Dirac matrices vanishes.
  - $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$
  - $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$
  - $\mathrm{tr}(\gamma^5) = 0$
  - The trace of the product of γ^5 with an odd number of Dirac matrices vanishes.
  - $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$
  - $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4i\epsilon^{\mu\nu\rho\sigma}$
- Dirac spinor properties:
  - $u_\lambda^\dagger(\mathbf{p})u_{\lambda'}(\mathbf{p}) = v_\lambda^\dagger(\mathbf{p})v_{\lambda'}(\mathbf{p}) = 2E\delta_{\lambda\lambda'}$
  - $\bar{u}_\lambda(\mathbf{p})u_{\lambda'}(\mathbf{p}) = -\bar{v}_\lambda(\mathbf{p})v_{\lambda'}(\mathbf{p}) = 2m\delta_{\lambda\lambda'}$
  - $\bar{u}_\lambda(\mathbf{p})v_{\lambda'}(\mathbf{p}) = \bar{v}_\lambda(\mathbf{p})u_{\lambda'}(\mathbf{p}) = 0$
  - $u^\dagger{}_\lambda(-\mathbf{p})v_{\lambda'}(\mathbf{p}) = v^\dagger{}_\lambda(\mathbf{p})u_{\lambda'}(-\mathbf{p}) = 0$
  - $\sum_\lambda u_\lambda(\mathbf{p})\bar{u}_\lambda(\mathbf{p}) = \gamma^\mu p_\mu + m,\ \sum_\lambda v_\lambda(\mathbf{p})\bar{v}_\lambda(\mathbf{p}) = \gamma^\mu p_\mu - m$

Discrete Symmetries

- Parity
  - $P\psi(t,\vec{x})P^{-1} = \gamma^0\psi(t,-\vec{x}) \implies PLP^{-1} = L$
  - $P\psi(t,\vec{x})P^{-1} = \gamma^0\psi(t,-\vec{x}) \implies Pa_\lambda(\vec{p})P^{-1} = a_\lambda(-\vec{p}),$
    $Pb_\lambda(\vec{p})P^{-1} = -b_\lambda(-\vec{p})$
    - ★ Note that particle and antiparticle have opposite parity
  - $P\,|\vec{p};\rangle = |-\vec{p};\rangle$, $P\,|;\vec{p}\rangle = -\,|;-\vec{p}\rangle$, if $P\,|0\rangle = |0\rangle$ is assumed
- Charge conjugation (C-parity)

$$Ca_\lambda(\vec{p})C^{-1} = b_\lambda(\vec{p}) \quad , \quad Cb_\lambda(\vec{p})C^{-1} = a_\lambda(\vec{p}) \quad ,$$

  - $C^2 = 1$, $C = C^{-1} = C^\dagger$ (unitary implementation)
  - Then $C\psi(x)C^{-1} = i\gamma^2\psi^*(x) \equiv \psi^c(x) \implies C\,S\,C^{-1} = S$
- Time reversal ($\eta_T = 1$ for simplicity)
  - $T\psi(t,\vec{x})T^{-1} = \gamma^1\gamma^3\psi(-t,\vec{x}) \implies TST^{-1} = S$
  - $T\psi(t,\vec{x})T^{-1} = \gamma^1\gamma^3\psi(-t,\vec{x}), \implies Ta_\lambda(\vec{p})T^{-1} = \lambda a_{-\lambda}(-\vec{p}),$
    $Tb_\lambda(\vec{p})T^{-1} = \lambda b_{-\lambda}(-\vec{p})$
  - $\gamma^1\gamma^3 = -\begin{pmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} = -e^{-i\pi\begin{pmatrix} \frac{\sigma^2}{2} & 0 \\ 0 & \frac{\sigma^2}{2} \end{pmatrix}}$

Coupling to electromagnetism

- Analogously to the Schrödinger case, we assign the following gauge transformation $\psi(x) \to e^{iq\theta(x)}\psi(x)$
- Minimal coupling ($\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

- Note that minimal coupling implies that particle (p) and antiparticle (a) have opposite electric charge

Chirality

$$P_R \equiv \frac{1+\gamma^5}{2} = \frac{1}{2}\begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & \mathbb{I} \end{pmatrix} \qquad P_L \equiv \frac{1-\gamma^5}{2} = \frac{1}{2}\begin{pmatrix} \mathbb{I} & -\mathbb{I} \\ -\mathbb{I} & \mathbb{I} \end{pmatrix}$$

- $P_{R,L}$ are projectors

$$P_R + P_L = 1 \qquad P_R^2 = P_R \qquad P_L^2 = P_L \qquad P_R P_L = P_L P_R = 0$$

- In the massless limit ($\Leftrightarrow$ high energy limit), right (R) and left (L) components decouple, $\psi_L \equiv P_L\psi$, $\psi_R \equiv P_R\psi$,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \bar{\psi}_R i\partial\!\!\!/ \psi_R + \bar{\psi}_L i\partial\!\!\!/ \psi_L - m\bar{\psi}_R\psi_L - m\bar{\psi}_L\psi_R \simeq \bar{\psi}_R i\partial\!\!\!/ \psi_R + \bar{\psi}_L i\partial\!\!\!/ \psi_L$$

- Right and left $u_\lambda(\vec{p})$ spinors read for $E \gg m$

$$u_\lambda^R(\vec{p}) = P_R u_\lambda(\vec{p}) = \sqrt{E+m}\begin{pmatrix} \frac{1}{2}\left(1 + \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)\chi_\lambda \\ \frac{1}{2}\left(1 + \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)\chi_\lambda \end{pmatrix} \simeq \sqrt{E}\begin{pmatrix} \frac{1}{2}\left(1 + \hat{p}\cdot\vec{\sigma}\right)\chi_\lambda \\ \frac{1}{2}\left(1 + \hat{p}\cdot\vec{\sigma}\right)\chi_\lambda \end{pmatrix}$$

$$u_\lambda^L(\vec{p}) = P_L u_\lambda(\vec{p}) = \sqrt{E+m}\begin{pmatrix} \frac{1}{2}\left(1 - \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)\chi_\lambda \\ -\frac{1}{2}\left(1 - \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)\chi_\lambda \end{pmatrix} \simeq \sqrt{E}\begin{pmatrix} \frac{1}{2}\left(1 - \hat{p}\cdot\vec{\sigma}\right)\chi_\lambda \\ -\frac{1}{2}\left(1 - \hat{p}\cdot\vec{\sigma}\right)\chi_\lambda \end{pmatrix}$$

- In the helicity basis ($\hat{n} = \hat{p}$)

$$u^R_+(\vec{p}) = \sqrt{E}\begin{pmatrix}\chi_+\\ \chi_+\end{pmatrix} \quad u^R_-(\vec{p}) = 0 \qquad u^L_+(\vec{p}) = 0 \quad u^L_-(\vec{p}) = \sqrt{E}\begin{pmatrix}\chi_-\\ -\chi_-\end{pmatrix}$$

- Analogously for the $v_\lambda(\vec{p})$ spinor

$$v^L_+(\vec{p}) = \sqrt{E}\begin{pmatrix}-\chi_+\\ \chi_+\end{pmatrix} \quad v^L_-(\vec{p}) = 0 \qquad v^R_+(\vec{p}) = 0 \quad v^R_-(\vec{p}) = \sqrt{E}\begin{pmatrix}\chi_-\\ \chi_-\end{pmatrix}$$

- Hence, when $E \gg m$ right handed fields describe particles (antiparticles) with positive (negative) helicity whereas left handed fields describe particles (antiparticles) with negative (positive) helicity
- Note that in this limit the upper and lower components of the Dirac spinors are linearly dependent
- The upper (or lower) components are solutions of the Weyl equation

$$i\sigma^\mu \partial_\mu \psi_R = 0 \quad , \quad i\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \quad , \quad \sigma^\mu = (\mathbb{I}, \vec{\sigma}) \quad , \quad \bar{\sigma}^\mu = (-\mathbb{I}, \vec{\sigma})$$

- Weyl (or chiral) fermions are the building blocks of the electroweak theory

Discrete symmetries of chiral fermions

$x=(t\,,\vec{x})\ ,\ \tilde{x}=(t\,,-\vec{x})$

- Parity

$$P\psi_L(x)P^{-1}=PP_L\psi(x)P^{-1}=P_LP\psi(x)P^{-1}=P_L\gamma^0\psi(\tilde{x})=\gamma^0P_R\psi(\tilde{x})=\gamma^0\psi_R(\tilde{x})$$

$$P\psi_R(x)P^{-1}=PP_R\psi(x)P^{-1}=P_RP\psi(x)P^{-1}=P_R\gamma^0\psi(\tilde{x})=\gamma^0P_L\psi(\tilde{x})=\gamma^0\psi_L(\tilde{x})$$

- Charge conjugation

$$C\psi_L(x)C^{-1}=CP_L\psi(x)C^{-1}=P_LC\psi(x)C^{-1}=P_Li\gamma^2\psi^*(x)=i\gamma^2P_R\psi^*(x)=i\gamma^2\psi_R^*(x)$$

$$C\psi_R(x)C^{-1}=CP_R\psi(x)C^{-1}=P_RC\psi(x)C^{-1}=P_Ri\gamma^2\psi^*(x)=i\gamma^2P_L\psi^*(x)=i\gamma^2\psi_L^*(x)$$

- CP

$$CP\psi_L(x)(CP)^{-1}=C(P\psi_L(x)P^{-1})C^{-1}=C\gamma^0\psi_R(\tilde{x})C^{-1}=\gamma^0C\psi_R(\tilde{x})C^{-1}=i\gamma^0\gamma^2\psi_L^*(\tilde{x})$$

$$CP\psi_R(x)(CP)^{-1}=C(P\psi_R(x)P^{-1})C^{-1}=C\gamma^0\psi_L(\tilde{x})C^{-1}=\gamma^0C\psi_L(\tilde{x})C^{-1}=i\gamma^0\gamma^2\psi_R^*(\tilde{x})$$

- Time reversal

$$T\psi_L(x)T^{-1}=TP_L\psi(x)T^{-1}=P_LT\psi(x)T^{-1}=P_L\gamma^1\gamma^3\psi(-\tilde{x})=\gamma^1\gamma^3P_L\psi(-\tilde{x})=\gamma^1\gamma^3\psi_L(-\tilde{x})$$

$$T\psi_R(x)T^{-1}=TP_R\psi(x)T^{-1}=P_RT\psi(x)T^{-1}=P_R\gamma^1\gamma^3\psi(-\tilde{x})=\gamma^1\gamma^3P_R\psi(-\tilde{x})=\gamma^1\gamma^3\psi_R(-\tilde{x})$$

Majorana Masses

A different kind of mass term (Majorana mass) can be added to the Dirac Lagrangian ,

$$\delta\mathcal{L} = -\delta m\,(\bar{\psi}\psi^c + \bar{\psi}^c\psi)$$

- It violates the $U(1)$ global symmetry $\psi \to e^{i\theta}\psi$, $\theta \in \mathbb{R}$ $\Longrightarrow$ the number of particles minus the number of antiparticles will not be conserved
- It allows to provide masses to chiral fermions, for instance to left-handed fields

$$\delta\mathcal{L} = -\delta m\,(\bar{\psi}_L\psi_L^c + \bar{\psi}_L^c\psi_L)$$

Majorana fermions

Majorana fermion: $a_\lambda(\vec{p}) = b_\lambda(\vec{p})$

- $\Longrightarrow$ $\psi = \psi^c = i\gamma^2\psi^*$ $\Longrightarrow$ $\psi_R = i\gamma^2\psi_L^*$, $\psi_L = i\gamma^2\psi_R^*$
- $\Longrightarrow$ ψ and ψ^* are not independent

2.3 Vector fields ($s = 1$)

The relativistic wave equation reads

$$\partial_\mu B^{\mu\nu} + m^2 B^\nu = 0 \quad , \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

It is called Proca equation for $m \neq 0$ and Maxwell equation for $m = 0$

- For B^μ real, it corresponds to the equations of motion of

$$\mathcal{L} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} m^2 B_\mu B^\mu$$

- For $m \neq 0$, the general solution of the real Proca equation reads upon quantization (hats have been dropped)

$$B^\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E}} \sum_{h=0,+,-} \left[e^{-ipx} \varepsilon_h^\mu(\vec{p}) a_h(\vec{p}) + e^{ipx} \varepsilon_h^{\mu *}(\vec{p}) a_h^\dagger(\vec{p}) \right]$$

$p_\mu \varepsilon_h^\mu(\vec{p}) = 0$, h stands for the 3rd component of the spin/helicity

- Let us take the spin quantization axis $\hat{n}$, and choose the basis $\vec{\varepsilon}_l$, $l = 1, 2, 3$

$$\vec{\varepsilon}_3 = \hat{n} \quad , \quad \vec{n} = \vec{\varepsilon}_1 \times \vec{\varepsilon}_2 \quad , \quad \vec{\varepsilon}_l \vec{\varepsilon}_r = \delta_{lr} \quad , \quad \sum_{l=1,2,3} \varepsilon_l^i \varepsilon_l^j = \delta^{ij}$$

$$\vec{\varepsilon}_0 \equiv \vec{\varepsilon}_3 \quad , \quad \vec{\varepsilon}_\pm \equiv \mp \frac{1}{\sqrt{2}} \left(\vec{\varepsilon}_1 \pm i \vec{\varepsilon}_2 \right) \quad , \quad \vec{\varepsilon}_h^* \vec{\varepsilon}_{h'} = \delta_{hh'} \quad , \quad \sum_{h=0,+,-} \varepsilon_h^i \varepsilon_h^{j*} = \delta^{ij}$$

$$\varepsilon_h^\mu(\vec{p}) = \left(\frac{\vec{\varepsilon}_h \vec{p}}{m}, \vec{\varepsilon}_h + \frac{(\vec{\varepsilon}_h \vec{p}) \vec{p}}{m(E+m)} \right) \quad , \quad \varepsilon_h^\mu(-\vec{p}) = -\varepsilon_{h\,\mu}(\vec{p})$$

$$\varepsilon_h^{\mu *}(\vec{p}) \varepsilon_{\mu\, h'}(\vec{p}) = -\delta_{hh'} \qquad \sum_{h=0,+,-} \varepsilon_h^\mu(\vec{p}) \varepsilon_h^{\nu *}(\vec{p}) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}$$

- Helicity basis: take $\hat{n} = \hat{p}$

$$\varepsilon_\pm^\mu(\vec{p}) = (0, \vec{\varepsilon}_\pm(\vec{p})) \quad , \quad \varepsilon_0^\mu(\vec{p}) = \left(\frac{|\vec{p}|}{m}, E \frac{\hat{p}}{m} \right)$$

- For B^μ complex, it corresponds to the equations of motion of

$$\mathcal{L} = -\frac{1}{2}B^*_{\mu\nu}B^{\mu\nu} + m^2 B^*_\mu B^\mu$$

- For $m \neq 0$, the general solution of the complex Proca equation reads upon quantization (hats have been dropped)

$$B^\mu(x) \quad = \quad \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_{h=0,+,-}\left[e^{-ipx}\varepsilon^\mu_h(\vec{p})a_h(\vec{p}) + e^{ipx}\varepsilon^{\mu *}_h(\vec{p})b^\dagger_h(\vec{p})\right]$$

$p_\mu\varepsilon^\mu_h(\vec{p}) = 0$, h stands for the 3rd commponent of the spin/helicity

  - $\hat{a}_h(\vec{p})$ and $\hat{b}_h(\vec{p})$ are the annihilation operators of a particle of 3rd component of the spin/helicity h and its antiparticle
  - These particles are necessarily bosons as the quantization using anticommutators is inconsistent
  - The commutation relations are the same as in the Schrödinger case ($\hat{a}_h(\vec{p})$ and $\hat{a}^\dagger_h(\vec{p})$ commute with $\hat{b}_h(\vec{p})$ and $\hat{b}^\dagger_h(\vec{p})$)

Discrete Symmetries

- Parity
  - $PB^\mu(t,\vec{x})P^{-1} = \pm B_\mu(t,-\vec{x}) \implies PLP^{-1} = L$
  - $PB^\mu(t,\vec{x})P^{-1} = \pm B_\mu(t,-\vec{x}) \implies Pa_h(\vec{p})P^{-1} = \mp a_h(-\vec{p}),$
    $Pb_h(\vec{p})P^{-1} = \mp b_h(-\vec{p})$
    - ★ Note that particle and antiparticle have the same parity
  - $P\,|\vec{p}\,h;\rangle = \mp\,|-\vec{p}\,h;\rangle$, $P\,|;\vec{p}\,h\rangle = \mp\,|;-\vec{p}\,h\rangle$, if $P\,|0\rangle = |0\rangle$ is assumed
- Charge conjugation (C-parity)

$$Ca_h(\vec{p})C^{-1} = b_h(\vec{p}) \quad , \quad Cb_h(\vec{p})C^{-1} = a_h(\vec{p}) \quad ,$$

  - $C^2 = 1$, $C = C^{-1} = C^\dagger$ (unitary implementation)
  - Then $CB^\mu(x)C^{-1} = B^{\mu\,*}(x) \implies C\mathcal{L}C^{-1} = \mathcal{L}$
  - If $B^\mu(x)$ is real, $Ca_h(\vec{p})C^{-1} = \pm a_h(\vec{p}) \implies CB^\mu(x)C^{-1} = \pm B^\mu(x)$
    - ★ $C\,|\vec{p}\,h\rangle = \pm\,|\vec{p}\,h\rangle$, if $C\,|0\rangle = |0\rangle$ is assumed (e. g. $C\,|\rho^0\rangle = -\,|\rho^0\rangle$)
- Time reversal
  - $TB^\mu(t,\vec{x})T^{-1} = \eta_T B_\mu(-t,\vec{x})$, $|\eta_T| = 1 \implies TST^{-1} = S$
  - $TB^\mu(t,\vec{x})T^{-1} = \eta_T B_\mu(-t,\vec{x}), \implies Ta_h(\vec{p})T^{-1} = \eta_T(-1)(-1)^h\, a_{-h}(-\vec{p}),$
    $Tb_h(\vec{p})T^{-1} = \eta_T^*(-1)(-1)^h b_{-h}(-\vec{p})$
  - If $B^\mu(x)$ is real $\implies \eta_T = \pm 1$

Coupling to electromagnetism

- For B^μ complex, analogously to the Schrödinger case, we assign the following gauge transformation $B^\mu(x) \to e^{iq\theta(x)} B^\mu(x)$
- Minimal coupling ($\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \tilde{B}^*_{\mu\nu} \tilde{B}^{\mu\nu} + m^2 B^*_\mu B^\mu$$

$$\tilde{B}^{\mu\nu} \equiv D_\mu B_\nu - D_\nu B_\mu$$

- Note that minimal coupling implies that particle (p) and antiparticle (a) have opposite electric charge
- For B^μ real, only non-minimal couplings are allowed. If we restrict ourselves to dimensionless couplings, we have

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} m^2 B_\mu B^\mu + c\, F_{\mu\nu} B^{\mu\nu}$$

- For the last term to be allowed $B^\mu(x)$ must transform as the photon field $A^\mu(x)$ under parity, charge conjugation and time reversal

Massless vector field (the photon field)

- In the massless case, the Proca Lagrangian reduces to Maxwell one

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

  - $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under $A_\mu(x) \to A_\mu(x) - \partial_\mu \theta(x)$ gauge transformation
  - In the $m \neq 0$ case, the mass term was not invariant under this transformation
  - The quantization of the photon field is not the $m \to 0$ of the quantization of the Proca field

- The quantization of theories with local gauge invariance is complicated
  - The Maxwell equations only determine the evolution of the gauge invariant part of $A^\mu(x)$, the transverse part $A^j_T(x)$

$$A^j_L = \frac{\partial_j \partial_i}{\vec{\nabla}^2} A^i \qquad A^j_T = A^j - \frac{\partial_j \partial_i}{\vec{\nabla}^2} A^i \qquad A^j = A^j_T + A^j_L$$

$$
\begin{aligned}
A^j_T(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_{\lambda=1,2}\left(\varepsilon^j_\lambda(\vec{p})a_\lambda(\vec{p})e^{-ipx}+\varepsilon^j_\lambda(\vec{p})a^\dagger_\lambda(\vec{p})e^{ipx}\right)\\
&= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_{h=+,-}\left(\varepsilon^j_h(\vec{p})a_h(\vec{p})e^{-ipx}+\varepsilon^{j\,*}_h(\vec{p})a^\dagger_h(\vec{p})e^{ipx}\right)
\end{aligned}
$$

- Polarization basis ($\varepsilon^\mu_\lambda(\vec{p})\in\mathbb{R}$)

$$
\vec{p}\cdot\vec{\varepsilon}_\lambda(\vec{p})=0 \quad , \quad \hat{p}=\vec{\varepsilon}_1(\vec{p})\times\vec{\varepsilon}_2(\vec{p})
$$

$$
\vec{\varepsilon}_\lambda(\vec{p})\cdot\vec{\varepsilon}_{\lambda'}(\vec{p})=\delta_{\lambda\lambda'} \qquad \sum_\lambda \varepsilon^i_\lambda(\vec{p})\varepsilon^j_\lambda(\vec{p})=\delta^{ij}-\frac{p^ip^j}{\vec{p}^2}
$$

$$
\vec{\varepsilon}_1(-\vec{p})=-\vec{\varepsilon}_1(\vec{p}) \qquad \vec{\varepsilon}_2(-\vec{p})=\vec{\varepsilon}_2(\vec{p})
$$

- Helicity basis ($\varepsilon^\mu_h(\vec{p})\in\mathbb{C}$)

$$
\vec{\varepsilon}_\pm(\vec{p})=\mp\frac{1}{\sqrt{2}}(\vec{\varepsilon}_1(\vec{p})\pm i\vec{\varepsilon}_2(\vec{p})) \quad , \quad \vec{\varepsilon}_\pm(-\vec{p})=\vec{\varepsilon}_\mp(\vec{p})
$$

$$
\vec{\varepsilon}^{\,*}_h(\vec{p})\cdot\vec{\varepsilon}_{h'}(\vec{p})=\delta_{hh'} \qquad \sum_h \varepsilon^i_h(\vec{p})\varepsilon^{j\,*}_h(\vec{p})=\delta^{ij}-\frac{p^ip^j}{\vec{p}^2}
$$

$$A^j_T(x) \;=\; \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_{h=+,-}\left(\varepsilon^j_h(\vec{p})a_h(\vec{p})e^{-ipx} + \varepsilon^{j\,*}_h(\vec{p})a^\dagger_h(\vec{p})e^{ipx}\right)$$

- $\hat{a}_h(\vec{p})$ is the annihilation operator of a photon of helicity h
- The photons are necessarily bosons as the quantization using anticommutators is inconsistent
- The commutation relations are the same as in the Schrödinger case ($[\hat{a}_h(\vec{p})\,,\hat{a}^\dagger_{h'}(\vec{p}\,')] = (2\pi)^3\delta(\vec{p}-\vec{p}\,')\delta_{hh'}$)
- The ground state $|0\rangle$ is called vaccum, $\langle 0|0\rangle = 1$, and $\hat{a}_h(\vec{p})\,|0\rangle = 0$
- The n-photon state is built with $h_i = \pm$, $i = 1\,,\ldots\,,n$

$$|\vec{p}_1\,h_1\ldots\vec{p}_n\,h_n\rangle = \sqrt{2E_1}\ldots\sqrt{2E_n}\hat{a}^\dagger_{h_1}(\vec{p}_1)\ldots\hat{a}^\dagger_{h_n}(\vec{p}_n)\,|0\rangle$$

Covariant quantization

- In order to get Lorentz invariant expressions in the intermediate steps of the calculations it is convenient to choose the Lorentz-Feynman gauge

$$\mathcal{L} \quad \to \quad \mathcal{L}_{gf} = \mathcal{L} - \frac{1}{2}(\partial_\mu A^\mu)^2$$

  - This modification does not change the dynamics of the gauge invariant part of the e.m. field $A^j_T(x)$
  - It introduces a dynamics for the unphysical fields $A^0(x)$ and $A^j_L(x)$
  - Upon quantization one gets

$$A^\mu(x) \quad = \quad \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_{h=0,+,-,3}\left(\varepsilon^\mu_h(\vec{p})a_h(\vec{p})e^{-ipx} + \varepsilon^{\mu\,*}_h(\vec{p})a^\dagger_h(\vec{p})e^{ipx}\right)$$

$$\varepsilon^\mu_\pm(\vec{p}) = (0\,,\vec{\varepsilon}_\pm(\vec{p})) \quad \varepsilon^\mu_0(\vec{p}) = (1\,,\vec{0}) \qquad \varepsilon^\mu_3(\vec{p}) = (0\,,\hat{p})$$

$$\varepsilon^{\mu\,*}_h(\vec{p})\varepsilon_{\mu\,h'}(\vec{p}) = g_{hh'} \qquad \sum_{h\,,h'=0,+,-,3}\varepsilon^\mu_h(\vec{p})\varepsilon^{\nu\,*}_{h'}(\vec{p})g^{hh'} = g^{\mu\nu}$$

$$-g_{00} = g_{++} = g_{--} = g_{33} = -1,\; g_{hh'} = 0 \text{ if } h \neq h'$$

$$[\hat{a}_h(\vec{p}), \hat{a}^\dagger_{h'}(\vec{p}\,')] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}\,')(-g_{hh'}) \qquad [\hat{a}_h(\vec{p}), \hat{a}_{h'}(\vec{p}\,')] = [\hat{a}^\dagger_h(\vec{p}), \hat{a}^\dagger_{h'}(\vec{p}\,')] = 0$$

- The ground state $|0\rangle$ is called vaccum, $\langle 0|0\rangle = 1$, $\hat{a}_h(\vec{p})\,|0\rangle = 0$ and $\hat{b}_h(\vec{p})\,|0\rangle = 0$, $h = 0, +, -, 3$
- Note that the commutation relation for $h = 0$ has the sign reversed $\Longrightarrow$ negative norm states !(e.g. $\hat{a}^\dagger_0(\vec{p})\,|0\rangle$)
- The n-photon state must be built with $h_i = \pm$, $i = 1, \ldots, n$ only

Discrete Symmetries

If we wish that the interaction with the e.m. field does not spoil the symmetries of the free Klein-Gordon and Dirac Lagrangians, the covariant derivative $D_\mu = \partial_\mu + iqA_\mu(x)$ must transform as the partial derivative ∂_μ under those symmetries. Let $x = (t, \vec{x})$, $\tilde{x} = (t, -\vec{x})$

- Parity

$$P\partial_\mu P^{-1} = \partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \tilde{x}_\mu}$$

$$PD_\mu(x)P^{-1} = D^\mu(\tilde{x}) \Longrightarrow PA_\mu(x)P^{-1} = A^\mu(\tilde{x})$$

- Charge conjugation

$$CD_\mu(x)C^{-1} = D_\mu(x)^* \Longrightarrow CA_\mu(x)C^{-1} = -A_\mu(x)$$

- Time reversal

$$T\partial_\mu T^{-1} = \partial_\mu = \frac{\partial}{\partial x^\mu} = -\frac{\partial}{\partial(-\tilde{x})_\mu}$$

$$TD_\mu(x)T^{-1} = -D^\mu(-\tilde{x}) \Longrightarrow TA_\mu(x)T^{-1} = A^\mu(-\tilde{x})$$

For the physical photons we have:

- Parity

$$PA_\mu(x)P^{-1} = A^\mu(\tilde{x}) \implies PA^i_T(x)P^{-1} = -A^i_T(\tilde{x}) \implies Pa_h(\vec{p})P^{-1} = -a_{-h}(-\vec{p})$$

- Charge conjugation

$$CA_\mu(x)C^{-1} = -A_\mu(x) \implies CA^i_T(x)C^{-1} = -A^i_T(x) \implies Ca_h(\vec{p})C^{-1} = -a_h(\vec{p})$$

- Time reversal

$$TA_\mu(x)T^{-1} = A^\mu(-\tilde{x}) \implies TA^i_T(x)T^{-1} = -A^i_T(-\tilde{x}) \implies Ta_h(\vec{p})T^{-1} = a_h(-\vec{p})$$

  - Note that there is no arbitrary phase η_T anymore

- h stands for helicity here, namely $\hat{p}\vec{S}$, hence it changes sign under parity but it does not under time reversal

2.4 Scattering and decay

Scattering and decay processes are characterized by:

- At $t \to -\infty$ we have an initial state $|i\rangle$ made out of free particles
- At $t \to \infty$ we have a final state $|f\rangle$ made out of free particles
- At finite times interactions occur that may turn $|i\rangle$ into $|f\rangle$ ($|i\rangle \neq |f\rangle$ is always assumed)
- The probability amplitude that $|i\rangle$ turns into $|f\rangle$ is given by $\langle f| \, S \, |i\rangle$, where S is an operator called S-matrix
- In the QFT course you will see that

$$S = \mathrm{T}\left\{ e^{i\int_{-\infty}^{\infty} d^4x \mathcal{L}_I} \right\}$$

  - T means time-ordering, namely operators on the left must always be at a later time than operators on the right
  - $\mathcal{L}_I$ is the interaction Lagrangian density, namely the full Lagrangian density minus the free part

- Examples of interaction Lagrangians are:

$$\mathcal{L}_{\rm SQED} = D_\mu\phi^* D^\mu\phi - m^2\phi^*\phi \quad \to \quad \mathcal{L}_I = iqA^\mu\left(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi\right) + q^2 A^\mu A_\mu \phi^*\phi$$

$$\mathcal{L}_{\rm QED} = \bar\psi(i\gamma^\mu D_\mu - m)\psi \quad \to \quad \mathcal{L}_I = -q\bar\psi\gamma^\mu A_\mu\psi$$

$$\mathcal{L}_{NRQED} = \psi^\dagger\left(iD_0 + \frac{\vec{D}^2}{2m} + \vec\mu\vec{B} + \cdots\right)\psi \to \mathcal{L}_I = -qA_0 - \frac{iq}{2m}\{\vec\nabla,\vec{A}\} - \frac{q^2}{2m}\vec{A}^2 + \vec\mu\vec{B}\ ,\ \vec\mu \sim \frac{q}{m}\vec{S}$$

- If $|i\rangle$ ($|f\rangle$) has total momentum p_i (p_f), space-time translation invariance implies

$$\langle f|S|i\rangle = i\mathcal{M}(i\to f)(2\pi)^4\delta^{(4)}(p_i - p_f)$$

- Decay width ($|i\rangle = |\vec{p}_A\rangle$, $|f\rangle = |\vec{p}_1\dots\vec{p}_n\rangle$, for spinless particles)

$$\Gamma_{A\to 1\dots n} = \frac{1}{2E_A}\int\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\cdots\int\frac{d^3\vec{p}_n}{(2\pi)^3 2E_n}|\mathcal{M}(\vec{p}_A\to\vec{p}_1\dots\vec{p}_n)|^2(2\pi)^4\delta(p_A - p_1 - \dots - p_n)$$

- Cross section ($|i\rangle = |\vec{p}_A\,\vec{p}_B\rangle$, $|f\rangle = |\vec{p}_1\dots\vec{p}_n\rangle$, for spinless particles)

$$\sigma = \frac{1}{4\sqrt{(p_A\cdot p_B)^2 - m_A^2 m_B^2}}\int\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\cdots\int\frac{d^3\vec{p}_n}{(2\pi)^3 2E_n}|\mathcal{M}(\vec{p}_A\vec{p}_B\to\vec{p}_1\dots\vec{p}_n)|^2$$
$$\times(2\pi)^4\delta(p_A + p_B - \sum_{i=1}^n p_i)$$

The remarks below hold both for the decay width and for the cross section:

- For a given process to be possible, no only the dynamics must allow it ($\mathcal{M} \neq 0$) but also the kinematics ($p_i = p_f$)
- If some particles in the initial or final states have nonzero spin, spin/helicity labels must be included
- If the initial state is unpolarized (unknown spin state), one must **average** over all possible spin/helicity states in the initial state
- If the spin/helicity of the final state is not measured, one must **sum** over all possible spin/helicity states in the final state
- If there are n identical particles in the final state, one must divide by $n!$ (this is due to way in which n-particle states are defined)
- The formulas as displayed correspond to the **total** decay width and to the **total** cross section to a given collection of particles in the final state
- When total decay width or total cross section are used with no reference to the final state, they mean the sum of them over any final state
- **Partial** decay widths and **partial** cross sections may be obtained by leaving some of (or combinations of) the momenta unintegrated

Amplitude calculations

- When the interaction Lagrangian is small, one can expand the exponential,

$$S = \mathrm{T}\left\{ e^{i\int_{-\infty}^{\infty} d^4x \mathcal{L}_I(x)} \right\} = 1 + i\int_{-\infty}^{\infty} d^4x \mathcal{L}_I(x) + \frac{i^2}{2!}\int_{-\infty}^{\infty} d^4x_1 d^4x_2 \mathrm{T}\left\{\mathcal{L}_I(x_1)\mathcal{L}_I(x_2)\right\} + \cdots$$

$$\langle f|S|i\rangle = i\mathcal{M}(i\to f)(2\pi)^4\delta^{(4)}(p_i - p_f) \Longrightarrow \mathcal{M}(p_i - p_f) = \mathcal{M}^{(1)}(p_i - p_f) + \mathcal{M}^{(2)}(p_i - p_f) + \cdots$$

- At firts order one obtains

$$i\mathcal{M}^{(1)}(\vec{p}_A\vec{p}_B \to \vec{p}_1 \dots \vec{p}_n) = \langle \vec{p}_1 \dots \vec{p}_n | i\mathcal{L}_I(0) | \vec{p}_A\vec{p}_B\rangle$$

- And at second order

$$i\mathcal{M}^{(2)}(\vec{p}_A\vec{p}_B \to \vec{p}_1 \dots \vec{p}_n) = \frac{i^2}{2!}\int d^4x \langle \vec{p}_1 \dots \vec{p}_n | \mathrm{T}\left\{\mathcal{L}_I(0)\mathcal{L}_I(x)\right\} | \vec{p}_A\vec{p}_B\rangle$$

- For amplitudes related to the decay width the same formula holds dropping $\vec{p}_B$ from the initial state

QED (e. g. for electrons)

$$\mathcal{L}_I = -q\bar{\psi}\gamma^\mu A_\mu \psi$$

- There is only a small parameter in $\mathcal{L}_I$, q
- At first order in $\mathcal{L}_I$ the dynamics allows
  - Create a photon, an electron and a positron $(|0\rangle \to |\gamma\, e^-\, e^+\rangle$
  - Create a photon and an electron and annihilate an electron $(|e^-\rangle \to |\gamma\, e^-\rangle$
  - Create a photon and a positron and annihilate a positron $(|e^+\rangle \to |\gamma\, e^+\rangle$
  - Create an electron and a positron and annihilate a photon $(|\gamma\rangle \to |e^-\, e^+\rangle$
  - Create a photon and annihilate an electron and a positron $(|e^-\, e^+\rangle \to |\gamma\rangle$
  - Create an electron and annihilate a photon and an electron $(|\gamma\, e^-\rangle \to |e^-\rangle$
  - Create a positron and annihilate a photon and a positron $(|\gamma\, e^+\rangle \to |e^+\rangle$
  - Annihilate a photon, an electron and a positron $(|\gamma\, e^-\, e^+\rangle \to |0\rangle$
- When we deal with particles described by different fields, the vacuum is the tensor product of the vacua corresponding to each field, in this case $|0\rangle = |0\rangle_e \otimes |0\rangle_\gamma$
- None of of the processes above are kinematically allowed (they do not fulfil energy momentum conservation)
- The simplest physical processes in QED require second order in $\mathcal{L}_I$

SQED (e. g. for charged pions)

$$\mathcal{L}_I = iqA^\mu \left(\partial_\mu \phi^* \phi - \phi^* \partial_\mu \phi\right) + q^2 A^\mu A_\mu \phi^* \phi$$

- There is only a small parameter in $\mathcal{L}_I$, q, but now there are two terms $\mathcal{O}(q)$ and one term $\mathcal{O}(q^2)$
- The terms $\mathcal{O}(q)$ allow the same processes as the terms in QED, exchanging $e^\pm \leftrightarrow \pi^\pm$
- All these processes are kinematically forbidden
- The term $\mathcal{O}(q^2)$ allows for 12 new processes
  - $|0\rangle \to |\gamma\,\gamma\,\pi^-\,\pi^+\rangle \quad , \quad |\gamma\,\gamma\,\pi^-\,\pi^+\rangle \to |0\rangle$
  - $|\gamma\rangle \to |\gamma\,\pi^-\,\pi^+\rangle \quad , \quad |\gamma\,\pi^-\,\pi^+\rangle \to |\gamma\rangle$
  - $|\pi^+\rangle \to |\gamma\,\gamma\,\pi^+\rangle \quad , \quad |\gamma\,\gamma\,\pi^+\rangle \to |\pi^+\rangle$
  - $|\pi^-\rangle \to |\gamma\,\gamma\,\pi^-\rangle \quad , \quad |\gamma\,\gamma\,\pi^-\rangle \to |\pi^-\rangle$
  - $|\gamma\,\gamma\rangle \to |\pi^-\,\pi^+\rangle \quad , \quad |\pi^-\,\pi^+\rangle \to |\gamma\,\gamma\rangle$
  - $|\gamma\,\pi^-\rangle \to |\gamma\,\pi^-\rangle \quad , \quad |\gamma\,\pi^+\rangle \to |\gamma\,\pi^+\rangle$
- Only the processes in the two last rows are allowed by the kinematics
- These processes get contributions at the same order in q from the second order term in $\mathcal{L}_I$

NRQED (e. g. for electrons)

$$\mathcal{L}_I = \psi^\dagger \left(-qA_0 - \frac{iq}{2m}\{\vec{\nabla}, \vec{A}\} - \frac{q^2}{2m}\vec{A}^2 + \vec{\mu}\vec{B} \right) \psi \quad , \quad \vec{\mu} \sim \frac{q}{m}\vec{S}$$

- For spinless particles the last term does not exist
- Here we have a small parameter q and small ratios of scales $\vec{A}/m \sim -i\vec{\nabla}/m \sim \vec{p}/m$
- The most important term is qA_0, but it does not describe physical photons. It is relevant beyond first order.
- The second most important terms are the remaining ones proportional to a single q, they allow
  - $|\gamma\, e^-\rangle \rightarrow |e^-\rangle \quad , \quad |e^-\rangle \rightarrow |\gamma\, e^-\rangle$
- None of this two processes is allowed by the kinematics
- The term proportional to q^2 allows
  - $|e^-\rangle \rightarrow |\gamma\,\gamma\, e^-\rangle \quad , \quad |\gamma\,\gamma\, e^-\rangle \rightarrow |e^-\rangle$
  - $|\gamma\, e^-\rangle \rightarrow |\gamma\, e^-\rangle$
- Only the last process is allowed by the kinematics
- It provides the dominant contribution to Compton scattering at low energy

Crossing

- We have seen in the case of QED and SQED that a given term in $\mathcal{L}_I$ gives rise to several dynamically allowed processes
- This is a generic feature of relativistic QFT called **crossing**
- **Crossing**: if a given dynamically allowed process has a particle A in the final state, then the same process with the antiparticle $\bar{A}$ in the initial state and the particle A removed from the final state is also dynamically allowed.
- If p is the four-momentum of the particle A, the crossed amplitude is related by the analytic continuation $(p \to -p)$ to the original one
- If $A \to B\,C\,D$ is dynamically allowed, then the following processes also are

$$A\,\bar{B} \to C\,D \quad , \quad \bar{B} \to \bar{A}\,C\,D \quad , \quad \bar{B}\,\bar{C} \to \bar{A}\,D \quad , \quad \cdots$$

- Some of the processes related by crossing may not be kinematically allowed
- For instance, if we know the amplitude for neutron β-decay $n \to p\,e^-\,\bar{\nu}_e$, we can get the amplitude for:

$$n\,\bar{p} \to e^-\,\bar{\nu}_e \quad , \quad n\,e^+ \to \bar{p}\,\bar{\nu}_e \quad , \quad n\,\nu_e \to e^-\,p$$

$$e^+\,\bar{p} \to \bar{n}\,\bar{\nu}_e \quad , \quad \nu_e\,\bar{p} \to e^-\,\bar{n} \quad , \quad e^+\,\nu_e \to p\,\bar{n}$$

Warm up calculations

- The simplest calculation is the total decay width of a two-body decay at first order in $\mathcal{L}_I$
- The kinematics in a two body decay is fixed $\Longrightarrow$ the phase space integrals in the formula of the decay width can be done for any $\mathcal{M}$

$$\Gamma(A \to 1\,2) = \frac{|\mathcal{M}|^2 |\vec{p}|}{8\pi m_A^2} \quad , \quad \vec{p} = \text{particle 1 (or 2) momentum in the rest frame of } A$$

- If $m_1 = m_2 \equiv m$, then reduces to

$$\Gamma(A \to 1\,2) = \frac{|\mathcal{M}|^2}{16\pi m_A} \sqrt{1 - \frac{4m^2}{m_A^2}}$$

- There are no two body decays at first order in $\mathcal{L}_I$ in QED or SQED, but there are several interesting ones in the electroweak theory that you may use as warm up calculations
  - $h \to l^+ \, l^- \quad , \quad \mathcal{L}_I = \lambda_l \phi_h \bar{\psi}_l \psi_l$
  - $Z^0 \to l^+ \, l^- \quad , \quad \mathcal{L}_I = -\frac{g_Z}{2} B_Z^\mu \bar{\psi}_l \gamma_\mu \left(c_V^l - c_A^l \gamma^5 \right) \psi_l$
  - $W^- \to l^- \, \bar{\nu}_l \quad , \quad \mathcal{L}_I = -\frac{g}{\sqrt{2}} B_{W^-}^\mu \bar{\psi}_l \gamma_\mu P_L \psi_{\nu_l} + \text{H.c.}$
  - $\lambda_l \, , g_Z \, , c_V^l \, , c_A^l \, , g$ are real constants