Quantum Field Theory, 2021/2022
Exercise sheet 2: Real Klein-Gordon Field
Hand-in: October 6, 2021

2.1. Consider the real, quantum, Klein-Gordon field. From the equal-time-commutation relations:

$$[\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \;\; ; \;\; [\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})] = 0 \;\; ; \;\; [\pi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})] = 0 \;\; ;$$

find the commutation relations for the creation-annihilation operators:

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] \;\; ; \;\; [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] \;\; ; \;\; [a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] \;\; ;$$

Hint: prove first that:

$$a_{\boldsymbol{p}} = \frac{1}{\sqrt{2E_p}} \int \mathrm{d}^3x \, e^{ipx}(i\dot{\phi}(x) + E_p\phi(x))$$

2.2. The parity transformation for the real Klein-Gordon field $\phi(x)$ is defined by:

$$\begin{aligned} (t,\boldsymbol{x}) &\;\to\; (t,-\boldsymbol{x}) \\ \phi(t,\boldsymbol{x}) &\;\to\; \mathcal{P}\phi(t,\boldsymbol{x})\mathcal{P}^{-1} = \eta_p\phi(t,-\boldsymbol{x}) \end{aligned}$$

where the parity operator $\mathcal{P}$ is a unitary operator which leaves the vacuum invariant $\mathcal{P}|0\rangle = |0\rangle$, and $\eta_p = \pm 1$ is called the intrinsic parity of the field.

(a) show that $\mathcal{P}$ leaves the Lagrangian density invariant

(b) show that

$$\mathcal{P}|\boldsymbol{k}_1,\boldsymbol{k}_2,\ldots\boldsymbol{k}_n\rangle = (\eta_p)^n|-\boldsymbol{k}_1,-\boldsymbol{k}_2,\ldots-\boldsymbol{k}_n\rangle$$

2.3. At class, it was sketched the computation of the Hamiltonian H for the real Klein-Gordon field. Following the same procedure, compute the linear-momentum operator:

$$: P_k : = \int \mathrm{d}^3x \; : \pi(x)\partial_k\phi(x) :$$

as a function of $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$.
NOTE: Note that it is normal-ordered.

2.4. Define the following function:

$$D_R(x-y) = \Theta(x^0-y^0)[\phi(x),\phi(y)]$$

Where Θ is the Heaviside theta function. Prove that it is the non-homogeneous Green's function of the Klein-Gordon equation, that is:

$$\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + m^2\right) D_R(x-y) = -i\delta^4(x-y)$$

NOTE: We will see that this function is the *retarded propagator*, but to do this exercise you just need to use the expression above and the properties of derivatives and commutators.