# Real Klein-Gordon field

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# Classical solution

Simplest representation: **scalar real** field: $\phi = \phi^*$

$\Rightarrow$ Real Klein-Gordon field

Lagrangian density: $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right)$

e.o.m.: $\partial_\mu \partial^\mu \phi + m^2 \phi = 0$

Solution:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} \left(a_{\boldsymbol{p}} e^{-ipx} + a^*_{\boldsymbol{p}} e^{ipx}\right) \quad ; \quad p^0 = E_p = \sqrt{\boldsymbol{p}^2 + m^2} \tag{1}$$

Normalization factor: arbitrary (but convenient).

The canonical conjugate momentum:

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} = \dot{\phi}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} (-iE_p) \left(a_{\boldsymbol{p}} e^{-ipx} - a^*_{\boldsymbol{p}} e^{ipx}\right) \tag{2}$$

Quantization: $\phi(x), \Pi(x) \rightarrow$ operators on some Hilbert space

$\phi(x) = \phi(t, x)$ usual quantization rules operators do not depend on time!

$\Rightarrow$ Small detour

# Schrödinger & Heisenberg images

## Schrödinger image of Quantum Mechanics

- Operators do not carry time-dependence
- all time-dependence is in the states

$$
\begin{aligned}
&\mathcal{O}_S \quad ; \quad |A, t\rangle_S \\
&i \frac{\mathrm{d}}{\mathrm{d}t} |A, t\rangle_S = H |A, t\rangle_S \\
&|A, t\rangle_S = U |A, 0\rangle_S \quad ; \quad U(t, t_0) = e^{-iH(t-t_0)}
\end{aligned}
$$

- quantization:

  $\Rightarrow$ impose canonical commutation relation among pairs of conjugate coordinates and momenta

$$[q_i, p_j] = i\delta_{ij} \quad ; \quad [q_i, q_j] = 0 \quad ; \quad [p_i, p_j] = 0 \tag{3}$$

Field theory: $q_i \to \phi(x)$, but:

- $q_i$ time independent
- $\phi(x) = \phi(t, \boldsymbol{x})$

## Heisenberg image of Quantum Mechanics

- time evolution is transported to the operators

$$|A,t\rangle_H \equiv |A,0\rangle_S = U^\dagger |A,t\rangle_S$$

- (time-independent) operator in the Schrödinger image $\rightarrow$ operator in the Heisenberg image:

$$\mathcal{O}_H(t) \equiv U^\dagger \mathcal{O}_S U$$

- Probability amplitudes stay invariant:

$$_S\langle B,t|\mathcal{O}_S|A,t\rangle_S = {}_S\langle B,0|U^\dagger \mathcal{O}_S U|A,0\rangle_S = {}_H\langle B|\mathcal{O}_H(t)|A\rangle_H$$

- time-evolution of the operator is:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}_H(t) = [\mathcal{O}_H, H] \qquad (\text{if } \partial_t \mathcal{O}_S = 0)$$

- Quantization rules (3) are true if:
  operators $q_i^H(t)$ and $p_j^H(t)$ are evaluated at the same time

## equal-time-commutation relations

$$[q_i^H(t), p_j^H(t)] = i\delta_{ij} \;\; ; \;\; [q_i^H(t), q_j^H(t)] = 0 \;\; ; \;\; [p_i^H(t), p_j^H(t)] = 0 \qquad (4)$$

So the fields and momenta in eqs. (1), (2) are coordinates and momenta in the Heisenberg image, and we will need to impose equal-time-commutation relations for their quantization.

# Quantum Mechanics Harmonic Oscillator

Hamiltonian of the one-dimensional harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \quad ; \quad p = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x}$$

introduce operators:

$$\left.\begin{array}{rcl} a & = & \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right) \\ a^\dagger & = & \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} x & = & \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \\ p & = & -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger) \end{array}\right.$$

with the canonical quantization rules:

$$[x, p] = i\hbar \;\; ; \;\; [x, x] = [p, p] = 0 \quad \Rightarrow [a, a^\dagger] = 1 \;\; ; \;\; [a, a] = [a^\dagger, a^\dagger] = 0$$

we can write the Hamiltonian:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

and find the commutation relations:

$$[H, a^\dagger] = \hbar\omega a^\dagger \quad ; \quad [H, a] = -\hbar\omega a$$

$$H|\psi\rangle = E|\psi\rangle$$

Define new states: $a|\psi\rangle$ and $a^\dagger|\psi\rangle$: $\begin{cases} Ha|\psi\rangle & = (E - \hbar\omega)a|\psi\rangle \\ Ha^\dagger|\psi\rangle & = (E + \hbar\omega)a^\dagger|\psi\rangle \end{cases}$

$a/a^\dagger|\psi\rangle$ are Hamiltonian eigenstates with decreased/increased energy.

**Define: vacuum $|0\rangle$**

State of minimal energy, normalized $\langle 0|0\rangle = 1$

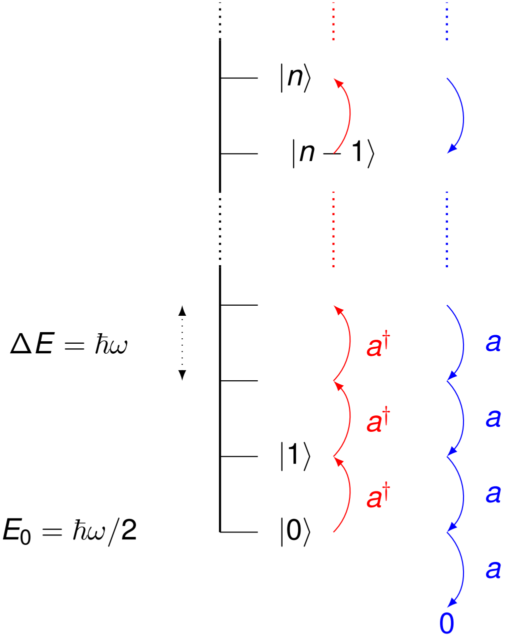
$$a|0\rangle = 0 \quad ; \quad a^\dagger|0\rangle \propto |1\rangle \ \text{(one excited state)}$$

Normalized states:

$$a|n\rangle = \sqrt{n}|n-1\rangle \ ; \ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ ; \ a^\dagger a|n\rangle = n|n\rangle \ ; \ |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

- $a^\dagger a = n$: number operator
- $|n\rangle$: eigenstates of the Hamiltonian with energy $E_n = \hbar\omega(n + 1/2)$
- $|0\rangle$ has non-zero energy $E_0 = \hbar\omega/2$

Hilbert space: $\{|n\rangle\}$ Fock space of the harmonic oscillator.




ladder operators

$a \equiv$ lowering, anihilation

$a^\dagger \equiv$ rising, creation

# Quantum Hermitic Klein-Gordon field

- Quantization rules: $\phi(x) \to$ Hermitic Heisenberg Operator with

**canonical equal-time-commutation (e.t.c.) rules**

$$
\begin{aligned}
[\phi(t,\boldsymbol{x}),\Pi(t,\boldsymbol{y})] &= [\phi(t,\boldsymbol{x}),\dot{\phi}(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\
[\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})] &= 0 \\
[\Pi(t,\boldsymbol{x}),\Pi(t,\boldsymbol{y})] &= [\dot{\phi}(t,\boldsymbol{x}),\dot{\phi}(t,\boldsymbol{y})] = 0
\end{aligned}
$$

- $a_{\boldsymbol{p}}$, $a^*_{\boldsymbol{p}}$ in eq. (1) $\Rightarrow$ operators $a_{\boldsymbol{p}}$, $a^\dagger_{\boldsymbol{p}}$,
- Commutations rules: (See exercise sheet!)

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q}) \quad ; \quad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = [a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = 0$$

$\Rightarrow$ $a_{\boldsymbol{p}}$ follow same commutation rules as harmonic oscillator!

$\Rightarrow$ $a^\dagger_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}$: rising and lowering operators for an harmonic oscillator labeled by $\boldsymbol{p}$

$\Rightarrow$ $\phi(x)$ is a combination of **infinite** harmonic oscillators, for each possible value of $\boldsymbol{p}$.

define a vacuum state $|0\rangle$:

$$a_{\boldsymbol{p}}|0\rangle = 0 \quad \forall \boldsymbol{p}$$

physical states are constructed by successive application of rising operators:

$$a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} a^\dagger_{\boldsymbol{p}_3} \cdots a^\dagger_{\boldsymbol{p}_n}|0\rangle$$

compute the **Hamiltonian** and the momentum:

$$
\begin{aligned}
H &= \int \mathrm{d}^3x\,\mathcal{H} = \int \mathrm{d}^3x \frac{1}{2}\left(\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) \\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_q}} \frac{1}{2} \times \Bigg\{ \\
&\quad -E_pE_q\left(a_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{-i(p+q)x} + a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{i(p+q)x} - a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{-i(p-q)x} - a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{i(p-q)x}\right) \\
&\quad -\boldsymbol{p}\cdot\boldsymbol{q}\left(a_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{-i(p+q)x} + a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{i(p+q)x} - a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{-i(p-q)x} - a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{i(p-q)x}\right) \\
&\quad +m^2\left(a_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{-i(p+q)x} + a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{i(p+q)x} + a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{-i(p-q)x} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{i(p-q)x}\right)\Bigg\}
\end{aligned}
$$

first integrate the $x$: $\int \mathrm{d}^3x\, e^{i\boldsymbol{k}\boldsymbol{x}} = (2\pi)^3\delta^3(\boldsymbol{k})$

$$
\begin{aligned}
&\int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3\sqrt{2E_q}} \frac{1}{2} \times \Bigg\{ \\
&-E_p E_q \Big( \{a_{\boldsymbol{p}} a_{\boldsymbol{q}} e^{-i(E_p+E_q)t} + a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} e^{i(E_p+E_q)t}\} (2\pi)^3 \delta^3(\boldsymbol{p}+\boldsymbol{q}) \\
&\qquad - \{a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} e^{-i(E_p-E_q)t} + a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{q}} e^{i(E_p-E_q)t}\} (2\pi)^3 \delta^3(\boldsymbol{p}-\boldsymbol{q}) \Big) \\
&-\boldsymbol{p}\cdot\boldsymbol{q} \Big( \{a_{\boldsymbol{p}} a_{\boldsymbol{q}} e^{-i(E_p+E_q)t} + a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} e^{i(E_p+E_q)t}\} (2\pi)^3 \delta^3(\boldsymbol{p}+\boldsymbol{q}) \\
&\qquad - \{a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} e^{-i(E_p-E_q)t} + a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{q}} e^{i(E_p-E_q)t}\} (2\pi)^3 \delta^3(\boldsymbol{p}-\boldsymbol{q}) \Big) \\
&+m^2 \Big( \{a_{\boldsymbol{p}} a_{\boldsymbol{q}} e^{-i(E_p+E_q)t} + a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} e^{i(E_p+E_q)t}\} (2\pi)^3 \delta^3(\boldsymbol{p}+\boldsymbol{q}) \\
&\qquad + \{a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} e^{-i(E_p-E_q)t} + a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{q}} e^{i(E_p-E_q)t}\} (2\pi)^3 \delta^3(\boldsymbol{p}-\boldsymbol{q}) \Big) \Bigg\}
\end{aligned}
$$

now integrate over $\boldsymbol{q} \Rightarrow \boldsymbol{p} = \pm\boldsymbol{q} \Rightarrow E_q = E_p$

$$
\begin{aligned}
H &= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\frac{1}{2} \times \Bigg\{ \\
&-E_p^2\left(a_{\boldsymbol{p}}a_{-\boldsymbol{p}}e^{-i(2E_p)t} + a^\dagger_{\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}e^{i(2E_p)t} - a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} - a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}\right) \\
&-\boldsymbol{p}^2\left(-a_{\boldsymbol{p}}a_{-\boldsymbol{p}}e^{-i(2E_p)t} - a^\dagger_{\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}e^{i(2E_p)t} - a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} - a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}\right) \\
&+m^2\left(a_{\boldsymbol{p}}a_{-\boldsymbol{p}}e^{-i(2E_p)t} + a^\dagger_{\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}e^{i(2E_p)t} + a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}\right)\Bigg\} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\frac{1}{2} \times \Bigg\{ \\
&\cancelto{0}{(m^2 + \boldsymbol{p}^2 - E_p^2)}(a_{\boldsymbol{p}}a_{-\boldsymbol{p}}e^{-i(2E_p)t} + a^\dagger_{\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}e^{i(2E_p)t}) \\
&+\cancelto{2E_p^2}{(m^2 + \boldsymbol{p}^2 + E_p^2)}(a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}})\Bigg\} \quad ; \quad [E_p^2 = m^2 + \boldsymbol{p}^2] \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\frac{1}{2} \times 2E_p^2(a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}})
\end{aligned}
$$

## Hamiltonian

$$H = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p (a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}})$$

## Linear momentum

$$P_k = \int \mathrm{d}^3 x\, \Pi(x) \partial_k \phi(x) = \cdots = \boxed{\frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} p_k (a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}) = P_k}$$

Commutators of $a_{\boldsymbol{q}}$, $a_{\boldsymbol{q}}^\dagger$ with $H$ and $P_k$:

$$
\begin{aligned}
[a_{\boldsymbol{q}}, H] &= \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p([a_{\boldsymbol{q}}, a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}] + [a_{\boldsymbol{q}}, a_{\boldsymbol{p}} a_{\boldsymbol{p}}^\dagger]) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p([a_{\boldsymbol{q}}, a_{\boldsymbol{p}}^\dagger] a_{\boldsymbol{p}} + a_{\boldsymbol{p}}[a_{\boldsymbol{q}}, a_{\boldsymbol{p}}^\dagger]) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p((2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q}) a_{\boldsymbol{p}} + a_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})) \\
&= \frac{1}{2} E_q(a_{\boldsymbol{q}} + a_{\boldsymbol{q}}) = E_q a_{\boldsymbol{q}}
\end{aligned}
$$

$$
[a_{\boldsymbol{q}}^\dagger, H] = [H, a_{\boldsymbol{q}}]^\dagger = -[a_{\boldsymbol{q}}, H]^\dagger = -E_q a_{\boldsymbol{q}}^\dagger
$$

by a similar procedure:

$$
\begin{aligned}
[a_{\boldsymbol{q}}, P_k] &= \cdots = q_k a_{\boldsymbol{q}} \\
[a_{\boldsymbol{q}}^\dagger, P_k] &= \cdots = -q_k a_{\boldsymbol{q}}^\dagger
\end{aligned}
$$

- Assume $|\psi\rangle$:

$$H|\psi\rangle = E|\psi\rangle \quad ; \quad \boldsymbol{P}|\psi\rangle = \boldsymbol{k}|\psi\rangle$$

- Build: $a^\dagger_{\boldsymbol{q}}|\psi\rangle$:

$$Ha^\dagger_{\boldsymbol{q}}|\psi\rangle = a^\dagger_{\boldsymbol{q}}H|\psi\rangle + [H, a^\dagger_{\boldsymbol{q}}]|\psi\rangle = a^\dagger_{\boldsymbol{q}}E|\psi\rangle + a^\dagger_{\boldsymbol{q}}E_q|\psi\rangle = (E + E_q)a^\dagger_{\boldsymbol{q}}|\psi\rangle$$

The energy rises by $E_q$!

$$P_ia^\dagger_{\boldsymbol{q}}|\psi\rangle = a^\dagger_{\boldsymbol{q}}P_i|\psi\rangle + [P_i, a^\dagger_{\boldsymbol{q}}]|\psi\rangle = a^\dagger_{\boldsymbol{q}}k_i|\psi\rangle + a^\dagger_{\boldsymbol{q}}q_i|\psi\rangle = (k_i + q_i)a^\dagger_{\boldsymbol{q}}|\psi\rangle$$

The linear momentum rises by $q_i$!

- $|\psi_1\rangle \equiv a^\dagger_{\boldsymbol{q}}|\psi\rangle$ is an eigenstate of $H$ and $\boldsymbol{P}$,
- it has an energy and momentum equal to that of $|\psi\rangle$ plus the energy and momentum of a particle with momentum $\boldsymbol{q}$ and energy $E_q = \sqrt{m^2 + \boldsymbol{q}^2}$:
  - $\Rightarrow$ $a^\dagger_{\boldsymbol{q}}$ creates a particle with mass $m$ and momentum $\boldsymbol{q}$

- Build: $a_{\boldsymbol{q}}|\psi\rangle$

$$\begin{aligned} H a_{\boldsymbol{q}}|\psi\rangle &= (E - E_q) a_{\boldsymbol{q}}|\psi\rangle \\ P_i a_{\boldsymbol{q}}|\psi\rangle &= (k_i - q_i) a_{\boldsymbol{q}}|\psi\rangle \end{aligned}$$

$\Rightarrow$ $a_{\boldsymbol{q}}$ removes from $|\psi\rangle$ a particle with mass $m$, momentum $\boldsymbol{q}$, and energy $E_q = \sqrt{m^2 + \boldsymbol{q}^2}$.

# Fock space

Fock space: Hilbert space of the quantum system.
Define the vacuum:

$$|0\rangle \text{ such that } a_{\boldsymbol{p}}|0\rangle = 0 \ \ \forall \boldsymbol{p} \text{ and } \langle 0|0\rangle = 1$$

- $a^\dagger_{\boldsymbol{k}}|0\rangle$ represents a state with 1 particle of momentum $\boldsymbol{k}$
- $a^\dagger_{\boldsymbol{k}}a^\dagger_{\boldsymbol{q}}|0\rangle$ represents a state with 1 particle of momentum $\boldsymbol{k}$ and 1 particle of momentum $\boldsymbol{q}$
- $\frac{1}{\sqrt{2!}}a^\dagger_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}|0\rangle \Rightarrow$ 2 particles of momentum $\boldsymbol{k}$
- $\frac{1}{\sqrt{n!}}(a^\dagger_{\boldsymbol{k}})^n|0\rangle \Rightarrow n$ particles of momentum $\boldsymbol{k}$

since $[a^\dagger_{\boldsymbol{q}}, a^\dagger_{\boldsymbol{k}}] = 0$:

$$a^\dagger_{\boldsymbol{k}}a^\dagger_{\boldsymbol{q}}|0\rangle = a^\dagger_{\boldsymbol{q}}a^\dagger_{\boldsymbol{k}}|0\rangle \Rightarrow \text{symmetric under particle exchange} \Rightarrow \text{bosons}$$

commutation relations $\Longleftrightarrow$ bosons

- these states are not Lorentz-invariant. Normalization:

$$\langle 0|a_{\boldsymbol{q}} a_{\boldsymbol{k}}^\dagger|0\rangle = \langle 0|a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{q}}|0\rangle + \langle 0|[a_{\boldsymbol{q}}, a_{\boldsymbol{k}}^\dagger]|0\rangle = 0 + \langle 0|0\rangle (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{q})$$

- Take a boost in the $x^3$ direction

$$\begin{aligned}
x^{1\prime} &= x^1 \quad ; \quad p^{1\prime} = p^1 \\
x^{2\prime} &= x^2 \quad ; \quad p^{2\prime} = p^2 \\
x^{3\prime} &= \gamma(x^3 + \beta x^0) \quad ; \quad p^{3\prime} = \gamma(p^3 + \beta E) \\
x^{0\prime} &= \gamma(x^0 + \beta x^3) \quad ; \quad E' = \gamma(E + \beta p^3)
\end{aligned}$$

together with: $\delta(f(x) - f(x_0)) = \dfrac{\delta(x - x_0)}{|\mathrm{d}f/\mathrm{d}x|_{x=x_0}}$

$$\begin{aligned}
\delta^3(\boldsymbol{k}' - \boldsymbol{q}') &= \frac{\delta^3(\boldsymbol{k}-\boldsymbol{q})}{|\mathrm{d}p^{3\prime}/\mathrm{d}p^3|} = \frac{\delta^3(\boldsymbol{k}-\boldsymbol{q})}{\gamma(1 + \beta(\partial E/\partial p^3))} \\
\frac{\partial E}{\partial p^3} &= \frac{p^3}{\sqrt{\boldsymbol{p}^2 + m^2}} = \frac{p^3}{E} \\
\delta^3(\boldsymbol{k}' - \boldsymbol{q}') &= \frac{\delta^3(\boldsymbol{k}-\boldsymbol{q})}{\gamma(1 + \beta(p^3/E))} = E\frac{\delta^3(\boldsymbol{k}-\boldsymbol{q})}{\gamma(E + \beta p^3)} = \frac{E}{E'}\delta^3(\boldsymbol{k}-\boldsymbol{q})
\end{aligned}$$

$\Rightarrow E\delta^3(\boldsymbol{k}-\boldsymbol{q})$ is a Lorentz-invariant quantity.

We choose to normalize:

$$|\boldsymbol{p_1}, \boldsymbol{p_2} \cdots \boldsymbol{p_n}\rangle = \sqrt{2E_1}\sqrt{2E_2}\cdots\sqrt{2E_n}\, a^\dagger_{\boldsymbol{p_1}} a^\dagger_{\boldsymbol{p_2}} \cdots a^\dagger_{\boldsymbol{p_n}} |0\rangle$$

such that

$$\langle \boldsymbol{p}|\boldsymbol{q}\rangle = (2\pi)^3 \sqrt{2E_p}\sqrt{2E_q}\,\delta^3(\boldsymbol{p}-\boldsymbol{q})$$

$\Rightarrow$ We must divide by $2E_p$ in other places

Projector operator over 1-particle states:

$$\mathbb{1} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} |\boldsymbol{p}\rangle \frac{1}{2E_p} \langle \boldsymbol{p}|$$

- This factor is quite common, and it is Lorentz-invariant.
- Take the Lorentz-invariant 4-D integral

$$I_1 = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,(2\pi)\delta(p^2 - m^2)\Big|_{p^0>0}$$

where we are choosing the positive energy:

$$f(E) = E^2 - \boldsymbol{p}^2 - m^2 \;;\; \frac{\mathrm{d}f}{\mathrm{d}E} = 2E \quad \text{at } p^2 = m^2 \;\rightarrow\; E = E_p = \sqrt{\boldsymbol{p}^2 + m^2}$$

$$I_1 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_p} \Leftarrow \text{Lorentz-invariant 3-momentum integration}$$

# Energy and linear momentum

$$H|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{E_p}{2}(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + a_{\boldsymbol{p}} a_{\boldsymbol{p}}^\dagger)|0\rangle$$

$a_{\boldsymbol{p}}|0\rangle = 0$, but not the second. Apply commutation rules:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + \frac{1}{2}[a_{\boldsymbol{p}}, a_{\boldsymbol{p}}^\dagger])$$

- $n_{\boldsymbol{p}} = a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}$ counts the number of particles with momentum $\boldsymbol{p}$
- $[a_{\boldsymbol{p}}, a_{\boldsymbol{p}}^\dagger] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p})$ (!!!!):

$$\lim_{\boldsymbol{p}\to\boldsymbol{q}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q}) = \lim_{\boldsymbol{p}\to\boldsymbol{q}}\int \mathrm{d}^3x e^{-i(\boldsymbol{p}-\boldsymbol{q})\boldsymbol{x}} = \int \mathrm{d}^3x \cdot 1$$

$$\equiv \text{Volume of space} \equiv V \quad (\to \infty)$$

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p\left(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + \frac{1}{2}V\right)$$

Vacuum energy:

$$\langle 0|H|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p V \frac{1}{2} \equiv E_{vac}$$

- $E = \infty$ energy, because $V = \infty$
  $\Rightarrow$ not a big problem
- but the energy density:

$$\rho_{vac} = \frac{E_{vac}}{V} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{E_p}{2} = \infty(!!!)$$

- each harmonic oscillator has a vacuum energy $\frac{1}{2}\omega_p = \frac{1}{2}E_p$, and there are $\infty$ oscillators!!
- Since zero-point energy is not physical, only energy differences matter:
  - We can subtract the zero-point energy[1] and define a new Hamiltonian:

$$H' = H - E_{vac}$$

[1] unless we deal with gravity!!

The problem is

- in classical field theory, we had products of two fields, which are commuting:
$$A \cdot B = B \cdot A$$
so we **did not** care on the order when we write $\phi^2$, $\Pi^2$, $\phi\Pi$, etc.
- in quantum theory we have operators $\hat{A}$, $\hat{B}$ and:
$$\hat{A} \cdot \hat{B} \neq \hat{B} \cdot \hat{A}$$
which is the *correct* order????
$\Rightarrow$ it is not given!!!
- Classically:
$\phi^2 = a^* a^* + aa + aa^* + a^* a = a^* a^* + aa + 2aa^* = a^* a^* + aa + 2a^* a$
- But quantum: $a^\dagger a + aa^\dagger \neq 2aa^\dagger \neq 2a^\dagger a$
$\Rightarrow$ we need some rules

## Definition: Wick ordering or Normal ordering

In a product $A = \prod a^\dagger \cdots a a^\dagger \cdots a^\dagger$ the normal-ordered or Wick-ordered product:

$$N(A) \equiv : A :$$

consists on putting all annihilation operators to the right-hand side:

$$: a a^\dagger a^\dagger a \cdots a a^\dagger \cdots a a a^\dagger a^\dagger := a^\dagger a^\dagger a^\dagger \cdots a^\dagger a a a \cdots a$$

Example:

$$: H :=: \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p (a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}}) := \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p \, a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}$$

now:

$$\langle 0 | : H : | 0 \rangle = 0$$

and

$$: H : | \boldsymbol{p_1} \boldsymbol{p_2} \cdots \boldsymbol{p_n} \rangle = (E_1 + E_2 + \cdots + E_n) | \boldsymbol{p_1} \boldsymbol{p_2} \cdots \boldsymbol{p_n} \rangle$$

We define all physics-related observables as normal-ordered operators:

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2} : \partial^{\mu}\phi\partial_{\mu}\phi - m^2\phi^2 : \\
H &= \int \mathrm{d}^3x : \frac{1}{2}(\Pi^2 + (\nabla\phi)^2 + m^2\phi^2 := \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p\, a^{\dagger}_{\boldsymbol{p}} a_{\boldsymbol{p}} \\
P_i &= \int \mathrm{d}^3x : \Pi\partial_i\phi := \int \frac{\mathrm{d}^3p}{(2\pi)^3} p_i\, a^{\dagger}_{\boldsymbol{p}} a_{\boldsymbol{p}}
\end{aligned}
$$

$$
\begin{aligned}
\langle 0|H|0\rangle &= 0 \\
H|\boldsymbol{p_1}\boldsymbol{p_2}\cdots\boldsymbol{p_n}\rangle &= (E_1 + E_2 + \cdots + E_n)|\boldsymbol{p_1}\boldsymbol{p_2}\cdots\boldsymbol{p_n}\rangle \\
P_i|\boldsymbol{p_1}\boldsymbol{p_2}\cdots\boldsymbol{p_n}\rangle &= (p_{i1} + p_{i2} + \cdots + p_{in})|\boldsymbol{p_1}\boldsymbol{p_2}\cdots\boldsymbol{p_n}\rangle
\end{aligned}
$$

## Define: *positive* & *negative* energy part of the fields

$$\phi = \phi^+ + \phi^-$$

$$\phi^+ = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}} a_{\boldsymbol{p}} e^{-ipx} \quad ; \quad \phi^- = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}} a^\dagger_{\boldsymbol{p}} e^{ipx} \tag{5}$$

$$\begin{aligned} :\phi(x)\phi(y): &= :(\phi^+(x)+\phi^-(x))(\phi^+(y)+\phi^-(y)): \\ &= \phi^+(x)\phi^+(y)+\phi^-(x)\phi^-(y)+\underbrace{\phi^-(x)\phi^+(y)+\phi^-(y)\phi^+(x)}_{\substack{\text{note the order!!} \\ \phi^+ \equiv a_{\boldsymbol{p}} \text{ to the right!}}} \end{aligned}$$

# Interpretation of the quantum field $\phi(x)$

$$
\begin{aligned}
\phi(x)|0\rangle &= \phi^-(x)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} e^{ipx} a^\dagger_{\boldsymbol{p}}|0\rangle \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} e^{ipx} \frac{1}{\sqrt{2E_p}} |\boldsymbol{p}\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} e^{ipx} |\boldsymbol{p}\rangle \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} e^{iE_p t} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} |\boldsymbol{p}\rangle
\end{aligned}
$$

$\Rightarrow$ creation of a particle at (point $\boldsymbol{x}$, time $t$) $\equiv x \Rightarrow |x\rangle$

Note that:

$$\begin{aligned}
\langle x|p\rangle &= \langle 0|\phi(x)|p\rangle = \langle 0|\int \frac{\mathrm{d}^3 q}{(2\pi)^3\sqrt{2E_q}}(a_{\boldsymbol{q}} e^{-iqx} + a^\dagger_{\boldsymbol{q}} e^{iqx})\sqrt{2E_p}a^\dagger_{\boldsymbol{p}}|0\rangle \\
&= \langle 0|\int \frac{\mathrm{d}^3 q}{(2\pi)^3\sqrt{2E_q}}\, a_{\boldsymbol{q}} a^\dagger_{\boldsymbol{p}} e^{-iqx}\,\sqrt{2E_p}|0\rangle \\
&= \langle 0|\int \frac{\mathrm{d}^3 q}{(2\pi)^3\sqrt{2E_q}} e^{-iqx}\,\sqrt{2E_p}\,(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{q}} + [a_{\boldsymbol{q}}, a^\dagger_{\boldsymbol{p}}])|0\rangle \\
&= \langle 0|\int \frac{\mathrm{d}^3 q}{(2\pi)^3\sqrt{2E_q}} e^{-iqx}\,\sqrt{2E_p}\,(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})|0\rangle \\
&= e^{-ipx}\langle 0|0\rangle = e^{-iEt}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}
\end{aligned}$$

$\equiv$ position-space representation of a single-particle wave function for definite momentum $\boldsymbol{p}$

On the other hand

$$\phi^+(x) = \text{annihilates a particle at point } x = (t, \boldsymbol{x})$$