Complex Klein-Gordon Field

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Quantization

- Same concepts as Real Klein-Gordon Field but: $\phi \neq \phi^\dagger$
  $\Rightarrow$ ϕ and $\phi^\dagger$ count as different degrees of freedom:
  $$\mathcal{L} =: \partial_\mu \phi(x) \partial^\mu \phi^\dagger(x) - m^2 \phi(x) \phi^\dagger(x) : \tag{1}$$
- NOTE normal ordering: $: \phi\phi^\dagger :=: \phi^\dagger \phi :$
- The Euler-Lagrange e.o.m.:
  $$\begin{aligned} \phi^\dagger &: \quad \partial^\mu \partial_\mu \phi + m^2 \phi = 0 \\ \phi &: \quad \partial^\mu \partial_\mu \phi^\dagger + m^2 \phi^\dagger = 0 \end{aligned}$$
- solution
  $$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} (a_{\boldsymbol{p}} e^{-ipx} + b^\dagger_{\boldsymbol{p}} e^{ipx}) \quad ; \quad p^0 = E_p = \sqrt{\boldsymbol{p}^2 + m^2}$$
  $a \neq b$ since the field is not hermitic:
  $$\phi^\dagger(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} (b_{\boldsymbol{p}} e^{-ipx} + a^\dagger_{\boldsymbol{p}} e^{ipx}) \quad ; \quad p^0 = E_p = \sqrt{\boldsymbol{p}^2 + m^2}$$

Conjugate momenta

$$\begin{aligned}\Pi_\phi(x) &= \frac{\partial\mathcal{L}}{\partial\dot\phi} = \frac{\partial\phi^\dagger}{\partial x^0} = \dot\phi^\dagger \equiv \Pi(x)\\ \Pi_{\phi^\dagger}(x) &= \frac{\partial\mathcal{L}}{\partial\dot\phi^\dagger} = \frac{\partial\phi}{\partial x^0} = \dot\phi \equiv \Pi^\dagger(x)\end{aligned}$$

Equal-time-commutation relations

$$\begin{aligned}[\phi(t,\boldsymbol{x}),\Pi(t,\boldsymbol{y})] &= [\phi(t,\boldsymbol{x}),\dot\phi^\dagger(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\\ [\phi^\dagger(t,\boldsymbol{x}),\Pi^\dagger(t,\boldsymbol{y})] &= [\phi^\dagger(t,\boldsymbol{x}),\dot\phi(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\end{aligned}$$

2nd = hermitic-conjugate of 1st.

$$\begin{aligned}[\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})] &= [\phi^\dagger(t,\boldsymbol{x}),\phi^\dagger(t,\boldsymbol{y})] = [\phi(t,\boldsymbol{x}),\phi^\dagger(t,\boldsymbol{y})] = 0\\ [\dot\phi(t,\boldsymbol{x}),\dot\phi(t,\boldsymbol{y})] &= [\dot\phi^\dagger(t,\boldsymbol{x}),\dot\phi^\dagger(t,\boldsymbol{y})] = [\dot\phi(t,\boldsymbol{x}),\dot\phi^\dagger(t,\boldsymbol{y})] = 0\\ [\phi(t,\boldsymbol{x}),\dot\phi(t,\boldsymbol{y})] &= [\phi^\dagger(t,\boldsymbol{x}),\dot\phi^\dagger(t,\boldsymbol{y})] = 0\end{aligned}$$

We can compute:

$$\begin{aligned}[a_{\boldsymbol{p}},a^\dagger_{\boldsymbol{q}}] &= (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\\ [b_{\boldsymbol{p}},b^\dagger_{\boldsymbol{q}}] &= (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\\ [a_{\boldsymbol{p}},a_{\boldsymbol{q}}] &= [a^\dagger_{\boldsymbol{p}},a^\dagger_{\boldsymbol{q}}] = [b_{\boldsymbol{p}},b_{\boldsymbol{q}}] = [b^\dagger_{\boldsymbol{p}},b^\dagger_{\boldsymbol{q}}] = [a_{\boldsymbol{p}},b_{\boldsymbol{q}}] = [a_{\boldsymbol{p}},b^\dagger_{\boldsymbol{q}}] = 0\end{aligned}$$

$\Rightarrow$ **two independent harmonic oscillators** one with the *a*-operators, and the other with the *b*-operators.

Define the vacuum

$$\left.\begin{aligned}a_{\boldsymbol{p}}|0\rangle = 0\\ b_{\boldsymbol{p}}|0\rangle = 0\end{aligned}\right\}\forall\boldsymbol{p}$$

Fock space

Vectors created by application of the $a^\dagger_{\boldsymbol{p}}$, $b^\dagger_{\boldsymbol{p}}$ operators:

$$\left.\begin{array}{l} a^\dagger_{\boldsymbol{p}}|0\rangle = |1_{\boldsymbol{p}};0\rangle \\ b^\dagger_{\boldsymbol{p}}|0\rangle = |0;1_{\boldsymbol{p}}\rangle \end{array}\right\} \text{create different kind of particles}$$

normalising as the real Klein-Gordon field:

$$\begin{aligned} |\boldsymbol{p}_1,\boldsymbol{p}_2,\dots\boldsymbol{p}_n;\boldsymbol{k}_1,\boldsymbol{k}_2,\dots\boldsymbol{k}_l\rangle &= \sqrt{2E_1}\sqrt{2E_2}\dots\sqrt{2E_n}\sqrt{2\omega_1}\sqrt{2\omega_2}\dots\sqrt{2\omega_l}\times \\ &\quad \times a^\dagger_{\boldsymbol{p}_1}a^\dagger_{\boldsymbol{p}_2}\dots a^\dagger_{\boldsymbol{p}_n}b^\dagger_{\boldsymbol{k}_1}b^\dagger_{\boldsymbol{k}_2}\dots b^\dagger_{\boldsymbol{k}_l}|0\rangle \end{aligned}$$

$E_i = \sqrt{\boldsymbol{p}_i^2 + m^2}$, $\omega_i = \sqrt{\boldsymbol{k}_i^2 + m^2}$.
r particles in a given state:

$$|r_{\boldsymbol{p}};0\rangle = (\sqrt{2E_p})^r\frac{(a^\dagger_{\boldsymbol{p}})^r}{\sqrt{r!}}|0\rangle \quad ; \quad |0;r_{\boldsymbol{k}}\rangle = (\sqrt{2E_k})^r\frac{(b^\dagger_{\boldsymbol{k}})^r}{\sqrt{r!}}|0\rangle$$

The number operators: $n^a_{\boldsymbol{p}} = a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} \quad ; \quad n^b_{\boldsymbol{p}} = b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}$
count the number of a- and b-particles for each momentum $\boldsymbol{p}$.

$$\begin{aligned} H &= \int \mathrm{d}^3x \; :\Pi(x)\Pi^\dagger(x) + \partial_i\phi(x)\partial_i\phi^\dagger(x) + m^2\phi(x)\phi^\dagger(x): \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_p(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}) \\ P_k &= \int \mathrm{d}^3x :\Pi(x)\partial_k\phi(x) + \Pi^\dagger(x)\partial_k\phi^\dagger(x): \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}p_k(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}) \end{aligned}$$

$\Rightarrow$ energy and the momentum is the same for both kind of particles.
$\Rightarrow$ What is different?

$U(1)$ symmetry

Complex Klein-Gordon lagrangian (1) has a $U(1)$ symmetry

$$\phi(x) \to e^{-i\alpha}\phi(x)$$

- conserved current

$$J^\mu = i : (\phi^\dagger(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^\dagger(x)) :$$

- conserved charge:

$$\begin{aligned} Q &= \int \mathrm{d}^3x J^0 = \int \mathrm{d}^3x : i(\phi^\dagger(x)\partial^0\phi(x) - \phi(x)\partial^0\phi^\dagger(x)) : \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b^\dagger_{\mathbf{p}}b_{\mathbf{p}}) = N_a - N_b \end{aligned}$$

$\Rightarrow$ a- and b-particles have **opposite** charge under the $U(1)$ transformation!

b-particles: **anti-particles** of the a-particles.
a-particles: **anti-particles** of the b-particles!

Commutators & propagators of the complex Klein-Gordon field

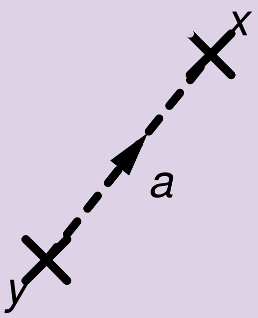
$\phi \neq \phi^\dagger \Rightarrow [\phi(x), \phi(y)] = [\phi^\dagger(x), \phi^\dagger(y)] = 0$
only contain $a + b^\dagger$ or $a^\dagger + b$, $\Rightarrow$ these combinations commute

Define:

positive and *negative* energy fields:

$$\begin{aligned} \phi^+(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} a_{\mathbf{p}} e^{-ipx} \\ \phi^-(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} b^\dagger_{\mathbf{p}} e^{ipx} \\ \phi^{\dagger+}(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} b_{\mathbf{p}} e^{-ipx} = (\phi^-(x))^\dagger \\ \phi^{\dagger-}(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} a^\dagger_{\mathbf{p}} e^{ipx} = (\phi^+(x))^\dagger \end{aligned}$$

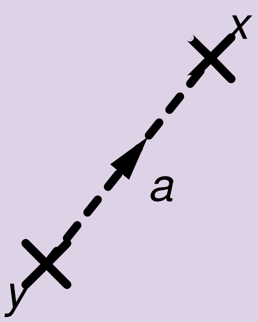
Particle *a* created at *y* and absorbed at *x*



$$
\begin{aligned}
& \langle 0|\phi(x)\phi^\dagger(y)|0\rangle \\
= \; & \langle 0|(\phi^+(x) + \cancel{\phi^-(x)})(\cancel{\phi^{\dagger+}(y)} + \phi^{\dagger-}(y))|0\rangle \\
= \; & \langle 0|\phi^+(x)\phi^{\dagger-}(y)|0\rangle \\
= \; & \langle 0|\phi^{\dagger-}(y)\phi^+(x) + [\phi^+(x), \phi^{\dagger-}(y)]|0\rangle \\
= \; & \langle 0|0\rangle \underbrace{[\phi^+(x), \phi^{\dagger-}(y)]} = \langle 0|0\rangle \Delta^+(x-y)
\end{aligned}
$$

Same expression than the corresponding real Klein-Gordon commutator in terms of *a*-operators

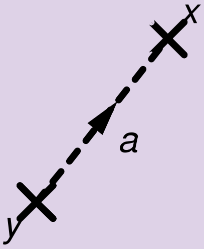
Anti-particle *b* created at *y* an absorbed at *x*

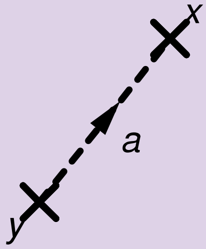


$$
\begin{aligned}
& \langle 0|\phi^\dagger(x)\phi(y)|0\rangle \\
= \; & \langle 0|(\phi^{\dagger+}(x) + \cancel{\phi^{\dagger-}(x)})(\cancel{\phi^+(y)} + \phi^-(y))|0\rangle \\
= \; & \langle 0|\phi^{\dagger+}(x)\phi^-(y)|0\rangle \\
= \; & \langle 0|\phi^-(y)\phi^{\dagger+}(x) + [\phi^{\dagger+}(x), \phi^-(y)]|0\rangle \\
= \; & \langle 0|0\rangle \underbrace{[\phi^{\dagger+}(x), \phi^-(y)]}_{\text{Same as real K-G in } b} = \langle 0|0\rangle \Delta^+(x-y)
\end{aligned}
$$

Commutator

$$
\begin{aligned}
[\phi(x), \phi^\dagger(y)] &= [\phi^+(x) + \phi^-(x), \phi^{\dagger+}(y) + \phi^{\dagger-}(y)] \\
&= \cancel{[\phi^+(x), \phi^{\dagger+}(y)]} + [\phi^+(x), \phi^{\dagger-}(y)] + [\phi^-(x), \phi^{\dagger+}(y)] + \cancel{[\phi^-(x), \phi^{\dagger-}(y)]} \\
&= [\phi^+(x), \phi^{\dagger-}(y)] + [\phi^-(x), \phi^{\dagger+}(y)] \\
&= [\phi^+(x), \phi^{\dagger-}(y)] - [\phi^{\dagger+}(y), \phi^-(x)] \\
&= \Delta^+(x-y) - \Delta^+(y-x) \\
&= \text{Propagation } a\ y \to x - \text{Propagation } b\ x \to y
\end{aligned}
$$

$=$ 

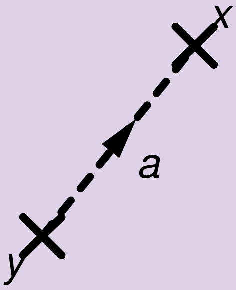
$-$ 

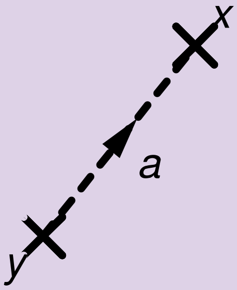
Micro-causality restored because of the cancellation of the first and second term in space-like intervals ($(x-y)^2 < 0$)

micro-causality exists thanks to anti-particles

The Feynman propagator

$$
\begin{aligned}
\Delta_F(x-y) &= \langle 0|T\{\phi(x)\phi^\dagger(y)\}|0\rangle \\
&= \Theta(x^0 - y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle + \Theta(y^0 - x^0)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle
\end{aligned}
$$

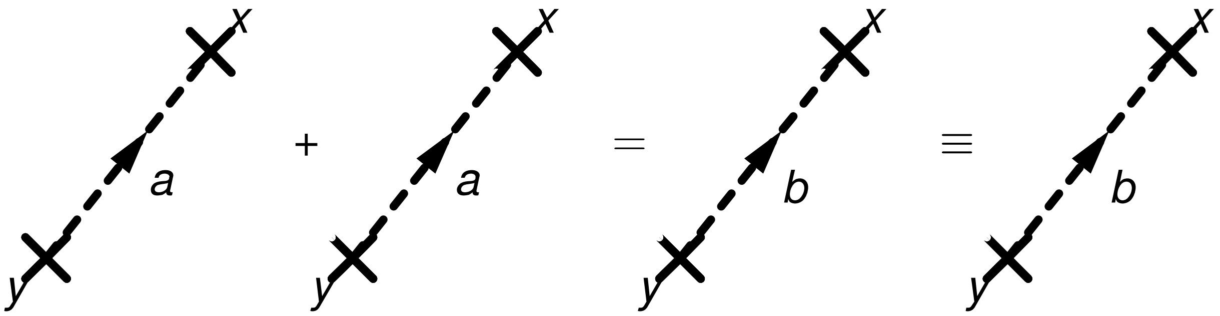
$=$ 

$+$ 

$$
\begin{aligned}
&= \Theta(x^0 - y^0)\langle 0|\phi^+(x)\phi^{\dagger-}(y)|0\rangle + \Theta(y^0 - x^0)\langle 0|\phi^{\dagger+}(y)\phi(x)^-|0\rangle \\
&= \Theta(x^0 - y^0)\Delta^+(x-y) + \Theta(y^0 - x^0)\Delta^+(y-x)
\end{aligned}
$$

$\Rightarrow$ Same expression as for the real Klein-Gordon field

$$
\Delta_F(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ipx}
$$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