Noether's theorem and space-time transformations

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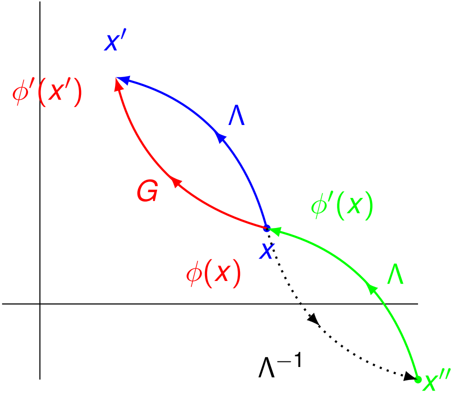
Noether's theorem

$$
\begin{aligned}
x^\mu \quad &\to \quad x'^\mu = \Lambda(x) \simeq x^\mu + \delta x^\mu \equiv x^\mu + \varepsilon^a A_a^\mu(x) \\
\phi_i(x) \quad &\to \quad \phi_i'(x') = G(\phi(x)) \simeq \phi_i(x) + \varepsilon^a F_{ia}(\phi, \partial_\mu \phi)
\end{aligned} \tag{1}
$$



$$
\begin{aligned}
\delta\phi_i &= \phi_i'(x) - \phi_i(x) \\
\phi_i'(x) &= G(\phi(x'')) = G(\phi(\Lambda^{-1}(x))
\end{aligned}
$$

Expand the last equation[1]:

$$\begin{aligned}
\phi'_i(x) &= G(\phi(\Lambda^{-1}(x))) \simeq \phi_i(\Lambda^{-1}(x)) + \varepsilon^a F_{ia}(\phi(\Lambda^{-1}(x)), \partial_\mu\phi(\Lambda^{-1}(x))) \\
&\simeq \phi_i(x^\mu - \delta x^\mu) + \varepsilon^a F_{ia}(\phi(x), \partial_\mu\phi(x)) + \mathcal{O}(\varepsilon^2) \\
&\simeq \phi_i(x) - \delta x^\mu \partial_\mu\phi_i(x) + \varepsilon^a F_{ia}(\phi(x), \partial_\mu\phi(x)) \\
&= \phi_i(x) - \varepsilon^a A^\mu_a(x)\partial_\mu\phi_i(x) + \varepsilon^a F_{ia}(\phi(x), \partial_\mu\phi(x)) \\
&\equiv \phi_i(x) + \delta\phi_i(x) \\
\delta\phi_i &\equiv \phi'_i(x) - \phi_i(x) = -\varepsilon^a A^\mu_a(x)\partial_\mu\phi_i(x) + \varepsilon^a F_{ia}(\phi(x), \partial_\mu\phi(x)) \qquad (2)
\end{aligned}$$

$$\delta S = S' - S = \int \mathrm{d}^4x'\,\mathcal{L}'(x') - \int \mathrm{d}^4x\,\mathcal{L}(x)$$

where $\mathcal{L}'(x') = \mathcal{L}(\phi'(x'), \partial_{\mu'}\phi'(x'))$.

[1] Same formalism as P. Ramond. Notation change: our eqs. (1), (2) with P. Ramond's (1.4.1), (1.4.2): $\delta^{our}\phi \equiv \delta_0^{Ramond}\phi$; $\phi'(x') - \phi(x) = \varepsilon^a F_{ia}(\phi, \partial_\mu\phi) \equiv \delta^{Ramond}\phi$.

Side note:

Two ways of analyzing this expression:

$$\delta S = S' - S = \int \mathrm{d}^4 x' \, \mathcal{L}'(x') - \int \mathrm{d}^4 x \, \mathcal{L}(x)$$

1. Taking $x' = \Lambda(x)$ as an integral variable change, and using the transformation Jacobian (as e.g. in P. Ramond)
   - Longer deduction
   - Easier application
2. Taking x' as just a silent variable and change its name $x' \to x$, then the Jacobian does not appear:
   - Easier deduction
   - Longer & more complicated application

$\Rightarrow$ In these notes we use the first formalism (using the Jacobian).
$\Rightarrow$ Be careful when reading the literature.

$$\delta S = S' - S = \int \mathrm{d}^4x'\, \mathcal{L}'(x') - \int \mathrm{d}^4x\, \mathcal{L}(x)$$

where $\mathcal{L}'(x') = \mathcal{L}(\phi'(x'), \partial_{\mu'}\phi'(x'))$.

Variable change in the first integral $x' = \Lambda(x)$. Jacobian:

$$\mathrm{d}^4x' = \left|\frac{\partial\, x'^\mu}{\partial\, x^\nu}\right| \mathrm{d}^4x \;\; ; \;\; \left|\frac{\partial\, x'^\mu}{\partial\, x^\nu}\right| = \begin{vmatrix} 1+\frac{\partial\delta x^0}{\partial x^0} & \frac{\partial\delta x^0}{\partial x^1} & \cdots \\ \frac{\partial\delta x^1}{\partial x^0} & 1+\frac{\partial\delta x^1}{\partial x^1} & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix} = 1 + \partial_\mu \delta x^\mu + \mathcal{O}(\delta x)^2$$

$$\delta S = S' - S = \int \mathrm{d}^4x \left\{\mathcal{L}'(x') + \mathcal{L}'(x')\partial_\mu\delta x^\mu - \mathcal{L}(x)\right\}$$

$$\mathcal{L}'(x') \simeq \mathcal{L}'(x) + \delta x^\mu \partial_\mu \mathcal{L}'(x) + \mathcal{O}(\delta x)^2 \simeq \mathcal{L}'(x) + \delta x^\mu \partial_\mu \mathcal{L}(x) + \mathcal{O}(\delta x)^2 \quad (3$$

Keep terms only up to order δx

$$\begin{aligned} \mathcal{L}'(x') + \mathcal{L}'(x')\partial_\mu\delta x^\mu &\simeq \mathcal{L}'(x') + \mathcal{L}(x)\partial_\mu\delta x^\mu \simeq \mathcal{L}'(x) + \underbrace{\delta x^\mu\partial_\mu\mathcal{L}(x) + \mathcal{L}(x)\partial_\mu\delta x^\mu}_{\text{product derivative}} \\ &= \mathcal{L}'(x) + \partial_\mu[\delta x^\mu \mathcal{L}(x)] \end{aligned}$$

$$
\begin{aligned}
\delta S &= \int \mathrm{d}^4x \left\{\mathcal{L}'(x) - \mathcal{L}(x) + \partial_\mu\left[\delta x^\mu \mathcal{L}(x)\right]\right\} \\
&= \int \mathrm{d}^4x \left\{\delta\mathcal{L}(x) + \partial_\mu\left[\delta x^\mu \mathcal{L}(x)\right]\right\}
\end{aligned}
$$

Definition: $\delta\mathcal{L}(x) = \mathcal{L}'(x) - \mathcal{L}(x) = \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\partial_\mu\phi_i$

$\delta\partial_\mu\phi = \partial_\mu\delta\phi$: integration by parts of the second term

$$
\begin{aligned}
\delta\mathcal{L}(x) &= \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\partial_\mu\delta\phi_i \\
&= \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right] - \delta\phi_i\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} \\
&= \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\right]}_{\text{e.o.m.}}\delta\phi_i + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right]
\end{aligned}
$$

If $\phi_i(x)$ is a solution of the equations of motion: e.o.m.$= 0$

$$\delta\mathcal{L}(x) = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right] \quad , \quad \text{for } \phi_i(x) \text{ a solution of the e.o.m.}$$

$$\delta S = \int \mathrm{d}^4x\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i + \delta x^\mu\mathcal{L}(x)\right]$$

If the transformation is a symmetry: $\delta S = 0$,

Conserved current for the solutions of the e.o.m.

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i + \delta x^\mu\mathcal{L}(x) \quad ; \quad \partial_\mu j^\mu = 0$$

As a function of transformation parameters in eq. (1):

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\varepsilon^a\left[-A^\nu_a(x)\partial_\nu\phi_i + F_{ia}(\phi,\partial_\nu\phi)\right] + \varepsilon^a A^\mu_a(x)\mathcal{L}(x)$$

Proportional to $\varepsilon^a \Rightarrow$ drop it from definition:

$$j^\mu_a = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\left[-A^\nu_a(x)\partial_\nu\phi_i + F_{ia}(\phi,\partial_\nu\phi)\right] + A^\mu_a(x)\mathcal{L}(x) \quad ; \quad j^\mu = \varepsilon^a j^\mu_a$$

Space-time translations

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu} \equiv x^{\mu} + \delta^{\mu}_{\nu}\varepsilon^{\nu} \Rightarrow A^{\mu}_{\nu}(x) = \delta^{\mu}_{\nu}$$

ε parameters: 4 indices $\equiv$ space-time ($a = \nu$).

- Space is homogeneous:

$$\phi'(x') = \phi(x) \Rightarrow F_{i\nu} = 0$$

- Change in the fields (2)

$$\phi'(x) = \phi(x^{\mu} - \varepsilon^{\mu}) = \phi(x) - \varepsilon^{\mu}\partial_{\mu}\phi(x)$$

- Conserved current:

$$\begin{aligned} j^{\mu} &= \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi_i}\delta\phi_i + \delta x^{\mu}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi_i}(-\varepsilon^{\nu}\partial_{\nu}\phi_i) + \varepsilon^{\nu}\delta^{\mu}_{\nu}\mathcal{L} \\ &= -\varepsilon^{\nu}\left(\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi_i}\partial_{\nu}\phi_i - \delta^{\mu}_{\nu}\mathcal{L}\right) \end{aligned}$$

  - ε^{ν} common factor
    $\Rightarrow$ 4 conserved currents

Energy-momentum tensor

$$T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \partial_\nu \phi_i - \delta^{\mu}_{\nu} \mathcal{L} \;\; ; \;\; \partial_\mu T^{\mu}{}_{\nu} = 0 \qquad (4)$$

$\Rightarrow$ 4 conserved currents
$\Rightarrow$ 4 conserved charges

$$Q_\nu = \int \mathrm{d}^3 x \, T^{0}{}_{\nu}$$

Time component

$$T^{0}{}_{0} = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \partial_0 \phi_i - \mathcal{L} = \mathcal{H} \text{ Hamiltonian density}$$

$$Q_0 = H = \int \mathrm{d}^3 x \, T^{0}{}_{0} = \int \mathrm{d}^3 x \, \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \partial_0 \phi_i - \mathcal{L} \text{ Energy}$$

Space components

$$T^0{}_k = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \partial_k \phi_i$$

$$Q_k = P_k = \int \mathrm{d}^3 x \, \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_i} \partial_k \phi_i = \int \mathrm{d}^3 x \, \Pi_i \partial_k \phi_i$$ linear 3-momentum

Conservation of linear 4-momentum

$$Q_\mu = P_\mu$$

Lorentz transformations

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{5}$$

$$\text{Infinitesimal: } x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = x^{\mu} + \frac{1}{2}(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}) x_{\nu} \omega^{\rho\sigma} \tag{6}$$

- $\omega^{\rho\sigma} = -\omega^{\sigma\rho}$ infinitesimal parameters.
- 6 independent parameters: $\begin{cases} 3 & \text{rotations} \\ 3 & \text{boosts} \end{cases}$

Field transformation under Lorentz transformations $G(\phi) =$???

$$F_{ia}(\phi, \partial_{\mu}\phi) = ???$$

Scalar fields are invariant

$$
\begin{aligned}
\phi'(x') &= \phi(x) \\
\phi'(x) &= \phi(\Lambda^{-1}(x)) = \phi(x) - \frac{1}{2}(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) x_\nu \omega^{\rho\sigma} \partial_\mu \phi
\end{aligned}
$$

Conserved current:

$$
\begin{aligned}
j^\mu &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi + \delta x^\mu \mathcal{L} \\
&= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \left(-\frac{1}{2}\right) \omega^{\rho\sigma} (\delta^\lambda_\rho \delta^\nu_\sigma - \delta^\lambda_\sigma \delta^\nu_\rho) x_\nu \partial_\lambda \phi + \frac{1}{2} \omega^{\rho\sigma} (\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho) \mathcal{L} \\
&= -\frac{1}{2} \omega^{\rho\sigma} \left\{ \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} (x_\sigma \partial_\rho \phi - x_\rho \partial_\sigma \phi) - (\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho) \mathcal{L} \right\}
\end{aligned}
$$

Combine to energy-momentum tensor:

$$
\begin{aligned}
j^\mu &= -\frac{1}{2} \omega^{\rho\sigma} \left\{ x_\sigma \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\rho \phi - \delta^\mu_\rho \mathcal{L} \right) - x_\rho \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\sigma \phi - \delta^\mu_\sigma \mathcal{L} \right) \right\} \\
&= -\frac{1}{2} \omega^{\rho\sigma} \left(T^\mu{}_\rho x_\sigma - T^\mu{}_\sigma x_\rho \right)
\end{aligned}
$$

Take out $\omega^{\rho\sigma}$

$$J^{\mu}{}_{\rho\sigma} = T^{\mu}{}_{\rho} x_{\sigma} - T^{\mu}{}_{\sigma} x_{\rho} \tag{7}$$

- Anti-symmetric in the 2-3 components
- 6 conserved currents:
$$\partial_{\mu} J^{\mu}{}_{\rho\sigma} = 0$$
- 6 conserved charges:
$$M^{\rho\sigma} = \int \mathrm{d}^3 x \, J^{0\rho\sigma}$$

Spacial (ij) conserved charges

$$M^{ij} = \int \mathrm{d}^3 x \, J^{0ij} = \int \mathrm{d}^3 x \, (T^{0i} x^j - T^{0j} x^i) = \text{“} P^i x^j - P^j x^i \text{”}$$

Orbital angular momentum.

Boosts conserved charges

$$\mathcal{K}^k = M^{0k} = \int \mathrm{d}^3x\, J^{00k} = \int \mathrm{d}^3x \left(T^{00} x^k - T^{0k} x^0 \right) = \text{“}Hx^k - P^k x^0\text{”}$$

Meaning? Particle mechanics:

$$E x^k - P^k x^0 = \text{constant} = a_0^k \Rightarrow x^k = \frac{a_0^k}{E} + \frac{P^k}{E} t$$

- equation of motion of a free particle
- $P^k / E = v^k$ velocity
- a_0^k / E: initial particle position

Conservation of $\begin{cases} \text{energy-momentum (4)} \\ \text{rotation-boost (7)} \end{cases}$

$$0 = \partial_\mu J^{\mu\rho\sigma} = x^\sigma \cancel{\partial_\mu T^{\mu\rho}} + T^{\mu\rho}\delta^\sigma_\mu - x^\rho \cancel{\partial_\mu T^{\mu\sigma}} - T^{\mu\sigma}\delta^\rho_\mu = T^{\sigma\rho} - T^{\rho\sigma} = 0$$

$T^{\rho\sigma}$ is symmetric: $T^{\rho\sigma} = T^{\sigma\rho}$

But canonical definition of the tensor (4) is not necessarily symmetric!!.

$\Rightarrow$ Define a different energy-momentum tensor.

Field function 3-tensor anti-symmetric in the first two components

$$f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$$

Define a new tensor: $\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}$

$$\partial_\mu \tilde{T}^{\mu\nu} = \cancel{\partial_\mu T^{\mu\nu}} + \cancel{\partial_\mu \partial_\lambda f^{\lambda\mu\nu}} = 0$$

conserved charges of new term

$$\int \mathrm{d}^3x\, \partial_\lambda f^{\lambda 0\nu} = \int \mathrm{d}^3x\, \partial_i f^{i0\nu} = \int_\infty \mathrm{d}^2x\, n^i f^{i0\nu} = 0$$

($\phi = 0$ at space ∞)

charges of $\tilde{T}^{\mu\nu}$ = charges of $T^{\mu\nu}$

$$\tilde{P}^{\nu} = \int \mathrm{d}^3x\, \tilde{T}^{0\nu} = \int \mathrm{d}^3x\, T^{0\nu} = P^{\nu}$$

$\Rightarrow$ Same values for the energy and the linear momentum.

Coleman-Mandula theorem

S. Coleman, J. Mandula, Phys. Rev. 159 (1967) 1251, DOI: 10.1103/PhysRev.159.1251:

In a relativistic theory of interacting particles, the only possible Lie group symmetries are direct products of the Poincaré group and an internal symmetry group.

Neither the statement, nor the proof, use the full apparatus of QFT

See also article by J. Mandula: `http://www.scholarpedia.org/article/Coleman-Mandula_theorem`

- Lie groups: Continous groups with commutation relations:

$$[T^a, T^b] = if^{abc}T^c$$

  $\Rightarrow$ Generators of internal symmetries & space-time do **not** mix

  $\Rightarrow$ It is not possible $[T^a, T^b] \sim P^\mu$

- way out????:

  $\Rightarrow$ If the symmetry is not a Lie group

  $\Rightarrow$ Supersymmetry: anti-commuting operators instead of commuting:

$$\{Q_{a_r}, Q^\dagger_{b_s}\} = 2\delta_{rs}\sigma^\mu_{ab}P_\mu$$

Outlook

For the analysis of this sub-chapter we have used only scalar fields,

$\Rightarrow$ Do not transform under space-time transformations:

$$\phi'(x') = \phi(x)$$

but there are more types of fields

- vectors
- tensors
- spinors

$\Rightarrow$ How do they transform under Lorentz transformations?

$\Rightarrow$ We need to study the symmetries, and the irreducible representations of the Lorentz Group!