3. Continuum Symmetries in Particle Physics

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3.1 Symmetry groups and conservation laws

Noether's theorem

Suppose that we have:

- A local Lagrangian $\mathcal{L} = \mathcal{L}(\phi_r(x)\,, \partial_\mu \phi_r(x))$
- A continuum set of transformations $\phi_r(x) \to \phi'_r(x) = \phi_r(x) + \delta\phi_r(x)$
- $S[\phi_r] = S[\phi'_r] \quad , \quad S[\phi_r] = \int d^4x\, \mathcal{L}(\phi_r(x)\,, \partial_\mu \phi_r(x))$

Then, for $\phi_r(x)$ that fulfil the equation of motion:

- There exist a conserved current j^μ that leads to a conserved charge Q

$$\partial_\mu j^\mu = 0 \quad , \quad Q = \int d^3\vec{x}\, j^0(\vec{x})$$

- The expression for j^μ is obtained by taking $\delta\phi_r(x)$ infinitessimal

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \delta\phi_r - J^\mu \quad , \quad \delta\mathcal{L} = \partial_\mu J^\mu$$

Continuum symmetries are usually divided into space-time symmetries and internal symmetries

- Space-time symmetries (related to the equivalence of inertial frames, the space-time coordinates x^μ transform):
  - Galilean group (non-relativistic systems)
    - Space-time translations $\Longrightarrow$ conservation of energy and three-momentum
    - Rotations $\Longrightarrow$ conservation of angular momentum
    - Boosts
  - Poincaré group (relativistic systems)
    - Space-time translations $\Longrightarrow$ conservation of energy and three-momentum
    - Lorentz transformations (rotations+boosts) $\Longrightarrow$ conservation of angular momentum
- Internal symmetries (the space-time coordinates x^μ do not transform)
  - $\delta\mathcal{L} = 0 \quad \Longrightarrow \quad J^\mu = 0 \quad \Longrightarrow \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\delta\phi_r$
  - Whether we have them or not depends on the particular $\mathcal{L}$ we have

Schrödinger field coupled to e.m.

$$\mathcal{L}_{NRQED} = \psi^\dagger \left(iD_0 + \frac{\vec{D}^2}{2m} + \vec{\mu}\vec{B} + \cdots \right) \psi \ , \ \vec{\mu} \sim \frac{q}{m}\vec{S}$$

- $\psi \to e^{i\theta}\psi$, $\theta \in \mathbb{R}$, $\theta \neq \theta(x)$ is an exact continuum internal symmetry
- The associated Noether charge N reads

$$N = \int d^3\vec{x}\, \psi^\dagger \psi = \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{m=-s,\dots,s} a^\dagger_m(\vec{p}) a_m(\vec{p})$$

$$N \, |\vec{p}_1\, m_1 \dots \vec{p}_n\, m_n\rangle = n \, |\vec{p}_1\, m_1 \dots \vec{p}_n\, m_n\rangle$$

- $\Longrightarrow$ The number of particles is conserved
- $\psi \to e^{i\vec{\theta}\vec{S}}\psi$, $\vec{\theta} \in \mathbb{R}^3$, $\vec{\theta} \neq \vec{\theta}(x)$, $\vec{S}^\dagger = \vec{S}$, $[S^i, S^j] = i\varepsilon^{ijk}S^k$, is an approximate continuum internal symmetry

- Indeed, at leading order in q and $\vec{p}/m$

$$\mathcal{L}_{NRQED} \simeq \psi^\dagger \left(iD_0 + \frac{\vec{\nabla}^2}{2m} \right) \psi$$

- The associated Noether charge $\hat{S}^i$ reads

$$\hat{S}^i = \int d^3\vec{x}\, \psi^\dagger S^i \psi \quad , \quad [\hat{S}^i, \hat{S}^j] = i\varepsilon^{ijk}\hat{S}^k$$

- Spin is approximately conserved in non-relativistic e.m. interactions

Complex Klein-Gordon field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

- $\phi \to e^{i\theta}\phi$, $\theta \in \mathbb{R}$, $\theta \neq \theta(x)$ is an exact continuum internal symmetry
- The associated Noether current reads

$$j^\mu = -i \left(\partial^\mu \phi^* \phi - \partial^\mu \phi \phi^* \right)$$

- And the associated Noether charge N reads

$$N = \int d^3\vec{x}\, j^0 = \int \frac{d^3\vec{p}}{(2\pi)^3} \left(a^\dagger(\vec{p})a(\vec{p}) - b^\dagger(\vec{p})b(\vec{p}) \right) + N_0$$

  N_0 is an ill-defined constant. $:N: \equiv N - N_0$ is also conserved,

$$:N:\, |\vec{p}_1 \dots \vec{p}_n\,;\vec{p}_1^{\,\prime} \dots \vec{p}_m^{\,\prime}\rangle = (n-m)\, |\vec{p}_1 \dots \vec{p}_n\,;\vec{p}_1^{\,\prime} \dots \vec{p}_m^{\,\prime}\rangle$$

- $\Longrightarrow$ The number of particles minus the number of antiparticles is conserved
- The $U(1)$ symmetry is mantained when minimal coupling to the e.m. field is introduced

Dirac field

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

- $\psi \to e^{i\theta}\psi$, $\theta \in \mathbb{R}$, $\theta \neq \theta(x)$ is an exact continuum internal symmetry
- The associated Noether current reads

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

- And the associated Noether charge N reads

$$N = \int d^3\vec{x}\, j^0 = \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{\lambda=+\,-} \Big(a^\dagger_\lambda(\vec{p})a_\lambda(\vec{p}) - b^\dagger_\lambda(\vec{p})b_\lambda(\vec{p}) \Big) + N_0$$

  N_0 is an ill-defined constant. $:N: \equiv N - N_0$ is also conserved,

  $$:N:\, |\vec{p}_1\,\lambda_1 \dots \vec{p}_n\,\lambda_n; \vec{p}'_1\,\lambda'_1 \dots \vec{p}'_m\,\lambda'_m\rangle = (n-m)\,|\vec{p}_1\,\lambda_1 \dots \vec{p}_n\,\lambda_n\,; \vec{p}'_1\,\lambda'_1 \dots \vec{p}'_m\,\lambda'_m\rangle$$

- $\Longrightarrow$ The number of particles minus the number of antiparticlesis conserved
- The $U(1)$ symmetry is mantained when minimal coupling to the e.m. field is introduced

Chiral fermions

When $m \simeq 0$ (high energy limit)

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \simeq \bar{\psi}i\gamma^\mu D_\mu\psi = \bar{\psi}_R i\gamma^\mu D_\mu\psi_R + \bar{\psi}_L i\gamma^\mu D_\mu\psi_L$$

- $\psi_R \to e^{i\theta_R}\psi_R$, $\psi_L \to e^{i\theta_L}\psi_L$, $\theta_{R\,,L} \in \mathbb{R}$, $\theta_{R\,,L} \neq \theta_{R\,,L}(x)$ are approximate continuum internal symmetries (exact when $m = 0$)
- The associated Noether currents read

$$j^\mu_R = \bar{\psi}_R\gamma^\mu\psi_R \quad , \quad j^\mu_L = \bar{\psi}_L\gamma^\mu\psi_L$$

- And the associated Noether charge $N_{R\,,L}$ read

$$N_R = \int d^3\vec{x}\, j^0_R = \int \frac{d^3\vec{p}}{(2\pi)^3} \Big(a^\dagger_+(\vec{p})a_+(\vec{p}) - b^\dagger_-(\vec{p})b_-(\vec{p}) \Big) + N_{0\,R}$$

$$N_L = \int d^3\vec{x}\, j^0_L = \int \frac{d^3\vec{p}}{(2\pi)^3} \Big(a^\dagger_-(\vec{p})a_-(\vec{p}) - b^\dagger_+(\vec{p})b_+(\vec{p}) \Big) + N_{0\,L}$$

  $N_{0\,R\,,L}$ are ill-defined constants. $:N_{R\,,L}: \equiv N_{R\,,L} - N_{0\,R\,,L}$ are also conserved,

Chiral fermions

- $\Longrightarrow$ The number of particles with $+$ helicity minus the number of antiparticles with $-$ helicity is conserved
- $\Longrightarrow$ The number of particles with $-$ helicity minus the number of antiparticles with $+$ helicity is conserved
- The $U_L(1) \times U_R(1)$ symmetry is mantained when minimal coupling to the e.m. field is introduced

Isospin

Consider a free proton ($m_p \simeq 938.3$ MeV) and a free neutron ($m_n \simeq 939.6$ MeV)

$$\mathcal{L} = \bar{\psi}_p(i\gamma^\mu\partial_\mu - m_p)\psi_p + \bar{\psi}_n(i\gamma^\mu\partial_\mu - m_n)\psi_n$$

Let us introduce

$$N = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \quad , \quad m = \frac{m_p + m_n}{2} \quad , \quad \Delta m = \frac{m_n - m_p}{2}$$

$$\mathcal{L} = \bar{N}\left((i\gamma^\mu\partial_\mu - m)\,\mathbb{I}_2 - \Delta m\,\tau^3\right)N \quad , \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Since $m \gg \Delta m$

$$\mathcal{L} \simeq \bar{N}\,(i\gamma^\mu\partial_\mu - m)\,N$$

- This Lagrangian is invariant under $N \to g\,N$, $g \neq g(x)$, $g \in U(2)$

$$U(N) = \left\{N \times N \text{ complex matrices such that } g^\dagger g = \mathbb{I}_N\right\}$$

$$\Longrightarrow \det g = e^{i\theta}\ ,\ \theta \in \mathbb{R} \Longrightarrow U(N) = U(1) \otimes SU(N)$$

$$SU(N) = \{g \in U(N) \text{ such that } \det g = 1\}$$

- In our case the $U(1)$ piece leads to baryon number conservation, and the $SU(2)$ piece to isospin conservation
- Since isospin multiplets are observed in nuclei $\Longrightarrow$ isospin $SU(2)$ must be (approximately) respected by nuclear interactions
- In the quark model $p = (uud)$ and $n = (udd)$, hence the origin of isospin might be due to $m_u \simeq m_d$

$$\mathcal{L} = \bar{\psi}_u(i\gamma^\mu\partial_\mu - m_u)\psi_u + \bar{\psi}_d(i\gamma^\mu\partial_\mu - m_d)\psi_d \simeq \bar{q}\,(i\gamma^\mu\partial_\mu - m)\,q$$

$$q = \begin{pmatrix}\psi_u \\ \psi_d\end{pmatrix} \quad , \quad m = \frac{m_u + m_d}{2} \quad , \quad \Delta m = \frac{m_d - m_u}{2}$$

- Since p and n form an isospin multiplet $\Longrightarrow$ the interactions between u and d quarks must (approximately) respect isospin $SU(2)$
- Since strange baryons, and the remaining hadrons also form isospin multiplets $\Longrightarrow$ the interactions between u, d, s and the remaining quarks must (approximately) respect isospin $SU(2)$

Flavor $SU(3)$

Since the mass of the lightest strange baryon $m_\Lambda \simeq 1116$ MeV is not much larger than the nucleon mass $m_N \simeq 940$ MeV

$$m = \frac{m_\Lambda + m_N}{2} \simeq 1028\,\text{MeV} \quad , \quad \Delta m = \frac{m_\Lambda - m_N}{2} \simeq 88\,\text{MeV}$$

$m \gg \Delta m$ is still a reasonable assumption, one can generalize the idea of isospin $SU(2)$ in the quark model to $SU(3)$

$$\mathcal{L} = \bar{q}\left((i\gamma^\mu\partial_\mu)\,\mathbb{I}_3 - \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}\right) q \quad , \quad q = \begin{pmatrix}\psi_u \\ \psi_d \\ \psi_s\end{pmatrix}$$

$$\mathcal{L} = \bar{q}\left(\left(i\gamma^\mu\partial_\mu - \frac{m_u + m_d + m_s}{3}\right)\mathbb{I}_3 - \frac{m_u + m_d - 2m_s}{6}\sqrt{3}\lambda_8 - \frac{m_u - m_d}{2}\lambda_3\right) q$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- If $m_u \simeq m_d \simeq m_s$ then

$$\mathcal{L} \simeq \bar{q}\left(i\gamma^\mu\partial_\mu - \frac{m_u + m_d + m_s}{3}\right) q$$

- If the interactions between quarks respect $SU(3)$ one should observed (approximate) $SU(3)$ multiplets

Is $m_u \simeq m_d \simeq m_s$ consistent with our current knowledge of quarks masses?

$$m_u \simeq 2.2\,\text{MeV} \quad , \quad m_d \simeq 4.7\,\text{MeV} \quad , \quad m_s \simeq 93\,\text{MeV}$$

$$\frac{m_u + m_d + m_s}{3} \simeq 33\,\text{MeV} \quad , \quad \frac{m_u + m_d - 2m_s}{6}\,\text{MeV} \simeq -30\,\text{MeV}$$
$$\frac{m_u - m_d}{2} \simeq -1.2\,\text{MeV} \quad , \quad \frac{m_u + m_d}{2} \simeq 3.4\,\text{MeV}$$

- The hypothesis that originally motivated the introduction of isospin $SU(2)$, and later on flavor $SU(3)$, in the quark model are not actually fulfilled
- But they are reasonably good approximate symmetries, we will see later on in the course what is the actual reason for it

3.2 Lie groups and Lie algebras

A Lie group or continuum grup G is:

- A group
- A differential (or smooth) manifold

As a group, $\forall g\,, g'\, g'' \in G$

- $g.g' \equiv gg' \in G$
- $(gg')g'' = g(g'g'') = gg'g''$
- $\exists e \in G\ e \equiv 1$, the neutral element, $e.g = g.e = g$
- $\exists g^{-1} \in G$, the inverse element, $gg^{-1} = g^{-1}g = e = 1$

As a differential manifold $\forall g \in G$

- $g = g(\theta)\ ,\ \theta = (\theta_1\,, \dots\,, \theta_n)\ ,\ \theta_i \in \mathbb{R}\ ,\ i = 1\,, \dots\,, n$, local coordinates
- g is a smooth function of θ_i $i = 1\,, \dots\,, n$, namely all partial derivatives at any order exist
- The local coordinates are choosen such that $g(0) = e = 1$

- We shall restrict ourselves to matrix groups, so you can think of $g(\theta)$ as a matrix the matrix elements of which depend on n real parameters $\theta_i\,,\ i=1\,,\ldots\,,n$ in a smooth way
- Then for θ close to $\theta=0$

$$g(\theta)=g(0)+\left.\frac{\partial g(\theta)}{\partial\theta_i}\right|_{\theta=0}\theta_i+\cdots\quad,\quad\left.\frac{\partial g(\theta)}{\partial\theta_i}\right|_{\theta=0}\equiv iT^i$$

- $g(\theta)=1+iT^i\theta_i+\cdots$, $<iT^i>$ spans a vector space called the Lie algebra L associated to G
- The imaginary unit in front of T^i is coventional in the physics literature but absent in the mathematical one
- Consider

$$g(\theta)g(\theta')-g(\theta')g(\theta)=g(\theta''(\theta,\theta'))-g(\theta''(\theta',\theta))$$

By expanding up to 2nd order in θ and θ' around 0 one gets

$$\left[T^i\,,T^j\right]=if^{kij}T^k\quad,\quad f^{kij}=\left.\left(\frac{\partial^2\theta''^k}{\partial\theta_i\partial\theta'_j}-\frac{\partial^2\theta''^k}{\partial\theta_j\partial\theta'_i}\right)\right|_{\theta=\theta'=0}$$

f^{kij} are called structure functions:

- $f^{kij}=-f^{kji}$ by construction
- For compact groups, f^{kij} can be choosen totally antisymmetric
- The Jacobi identity $[T^i\,,[T^j\,,T^k]]+[T^j\,,[T^k\,,T^i]]+[T^k\,,[T^i\,,T^j]]=0$

$$\Longrightarrow\qquad f^{imn}f^{jkm}+f^{jmn}f^{kim}+f^{kmn}f^{ijm}=0$$

- For $g(\theta)$ near the identity, clearly $g(\theta)\simeq e^{i\theta_iT^i}$
- For compact groups $g(\theta)=e^{i\theta_iT^i}$ always holds
- Recall that $i\,,j\,,k=1\,,\ldots\,,n$ above. n is the dimension of G and the dimension of L
- In the physics literature $\{T^i\}$ are called the generators of the G , in the mathematical one $\{iT^i\}$ are just a basis of the L
- For $SU(2)$, $T^i=\sigma^i/2$, $f^{ijk}=\epsilon^{ijk}$, $n=3$
- If G is abelian $(gg'=g'g\ \forall g\,,g'\in G)$ $\Longrightarrow$ $f^{ijk}=0$

Supose we have $\mathcal{L} = \mathcal{L}(\phi_r(x)\,,\partial_\mu\phi_r(x))$, that is invariant under the transformation $\phi_r(x) \to \phi'_r(x) = g(\theta)_r^{\ s}\phi_s(x)$, where $g(\theta) \in G$, a compact Lie group

- $g(\theta) = e^{i\theta_a T^a} \simeq 1 + i\theta_a T^a + \dots \quad \Longrightarrow \delta\phi_r(x) = i\theta_a\,(T^a)_r^{\ s}\,\phi_s(x)$
- Noether's theorem implies that the following currents are conserved

$$j^{\mu\, a} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)} i\,(T^a)_r^{\ s}\,\phi_s(x)$$

- And the following charges are conserved

$$Q^a = \int d^3\vec{x}\Pi^r(x) i\,(T^a)_r^{\ s}\,\phi_s(x) \quad , \quad \Pi^r(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_r(x))}$$

- Using canonical commutation (or anticommutation) relations

$$[\phi_s(x)\,,\Pi^r(y)]|_{x^0=y^0} = i\delta_s^r\delta(\vec{x}-\vec{y}) \quad \text{or} \quad \{\phi_s(x)\,,\Pi^r(y)\}|_{x^0=y^0} = i\delta_s^r\delta(\vec{x}-\vec{y})$$

$$\Longrightarrow \quad \left[Q^a\,,Q^b\right] = if^{abc}Q^c$$

- $\hat{g}(\theta) = e^{i\theta_a Q^a}$ is the representation of the group in terms of operators that act on the Fock space

Suppose we have a set of states $\{|\alpha\rangle\}$ that form a finite dimensional representation of G, that is $\hat{g}(\theta)\,|\alpha\rangle = M(\theta)_\alpha^{\ \beta}\,|\beta\rangle$, where $M(\theta)$ is a matrix

- If H is the Hamiltonian, since $\{Q^a\}$ are conserved $\Longrightarrow$ $[H\,,Q^a] = 0$ $\Longrightarrow$ $\hat{g}(\theta)H\hat{g}^{-1}(\theta) = H$
- Then, if $H\,|\alpha\rangle = E\,|\alpha\rangle$, $E \in \mathbb{R}$

$$\hat{g}(\theta)H\,|\alpha\rangle = E\hat{g}(\theta)\,|\alpha\rangle = E\,M(\theta)_\alpha^{\ \beta}\,|\beta\rangle$$

$$\hat{g}(\theta)H\,|\alpha\rangle = \hat{g}(\theta)H\hat{g}^{-1}(\theta)\hat{g}(\theta)\,|\alpha\rangle = H\,M(\theta)_\alpha^{\ \beta}\,|\beta\rangle = M(\theta)_\alpha^{\ \beta}H\,|\beta\rangle$$

$$\Longrightarrow \quad H\,|\beta\rangle = E\,|\beta\rangle$$

- All the states that transform into each other under the action of the group have the same energy $\Longrightarrow$ degeneracies in the spectrum
- Finding the states that transform into each other means finding the irreducible representations of the groups, a well posed mathematical problem

Usually $\{|\alpha\rangle\}$ are obtained by applying a set of operators $\hat{O}_\alpha$ that transform according to some representation of the group on the vaccum:

$$\hat{g}(\theta)\hat{O}_\alpha\hat{g}^{-1}(\theta) = M(\theta)_\alpha^\beta\hat{O}_\beta \quad , \quad |\alpha\rangle = \hat{O}_\alpha|0\rangle$$

For instance, the up and down quark fields in the case of isospin $SU(2)$

- $\hat{g}(\theta)|\alpha\rangle = \hat{g}(\theta)\hat{O}_\alpha|0\rangle = \hat{g}(\theta)\hat{O}_\alpha\hat{g}^{-1}(\theta)\hat{g}(\theta)|0\rangle = M(\theta)_\alpha^\beta\hat{O}_\beta\hat{g}(\theta)|0\rangle$
  - If $\hat{g}(\theta)|0\rangle = |0\rangle \Leftrightarrow Q^a|0\rangle = 0\ \forall Q^a \implies \hat{g}(\theta)|\alpha\rangle = M(\theta)_\alpha^\beta|\beta\rangle$
  - If $\hat{g}(\theta)|0\rangle \neq |0\rangle \implies \exists Q^a\,,\ Q^a|0\rangle \neq 0$, one says that the symmetry is **spontaneously broken** and the states do not form multiplets of G
- Both possibilities are realized in nature

Finite dimensional representations

A finite dimensional representation of G is a mapping from $G \to GL(m,\mathbb{C})$, the group of the $m\times m$ invertible complex matrices, such that if $g \to M(g)$, $g\in G$, $M(g)\in GL(m,\mathbb{C})$ preserves the properties of group and of differenciable manifold, namely:

- $M(gg') = M(g)M(g')$
- $M(e) = \mathbb{I}_m$
- $M(g^{-1}) = M(g)^{-1}$
- $M(g(\theta))$ is a smooth function of θ_i $i = 1, \ldots, n$, namely all partial derivatives at any order exist
- m is called the dimension of the representation

A finite dimensional representation of L is a mapping from $L \to gl(m,\mathbb{C})$, the vector space of the $m\times m$ complex matrices, such that if $T \to M(T)$, $T\in L$, $M(T)\in gl(m,\mathbb{C})$ preserves the properties of Lie algebra, namely:

- $M(T)$ is linear
- $M([T,T']) = [M(T),M(T')]$
- m is called the dimension of the representation

The following statements are easy to proof:

- If $\{T^a\}$ is a basis of L, $\left[T^a\,,\,T^b\right] = if^{abc}T^c$ and we find a set of matrices $\{M(T^a)\}$ such that $\left[M(T^a)\,,M(T^b)\right] = if^{abc}M(T^c)$, then the vector space generated by $< M(T^a) >$ is a representation of L
- If we have a representation of G, then the vector space generated by $< M(T^a) >$,

$$\left.\frac{\partial M(g(\theta))}{\partial \theta_a}\right|_{\theta=0} \equiv iM(T^a)\,,$$

  is a representation of L
- For compact G, if we have a representation of L, then

$$M(g(\theta)) \equiv e^{i\theta_a M(T^a)}$$

  is a representation of G

Usually, in order to make the notation lighter, one does not write $M(T^a)$ but just T^a and one must keep in mind that T^a is the generator in an arbitrary representation

- If $\left[T^a\,,\,T^b\right] = if^{abc}T^c \implies \left[-T^{a\,*}\,,-T^{b\,*}\right] = if^{abc}\,(-T^{c\,*})$, hence $\{-T^{a\,*}\}$ is also a representation called the **complex conjugate** of $\{T^a\}$
- A representation is called **real** if an S exists such that $-T^{a\,*} = ST^aS^{-1}\ \ \forall a$, otherwise it is called **complex**
- If $\{g(\theta)\}$ is a representation of $G \implies \{g(\theta)^*\}$ is also a representation, called the **complex conjugate**
- If an S exists such that $g(\theta)^* = Sg(\theta)S^{-1}\ \ \forall\theta$, the representation is called **real**, otherwise it is called **complex**.
- A representation of G is real $\iff$ if the corresponding representation of L is real
- As a consequence of the Jacobi identity, $(T^a)^m_n \equiv -if^{amn}$ is always a representation of L, called the **adjoint** representation
- The **adjoint** representation is real
- The dimension of the adjoint representation is the dimension of G

$SU(2)$

$$G = SU(2) = \left\{2 \times 2 \text{ complex matrices such that } g^\dagger g = \mathbb{I}_2 \text{ and } \det g = 1\right\}$$

Near the identity $g \simeq 1 + iT + \mathcal{O}(T^2)$

- $g^\dagger g = 1 \implies T^\dagger = T$
- $\det g = 1 \implies 1 = \det g = e^{\mathrm{tr}\log g} \simeq e^{\mathrm{tr} iT} \simeq 1 + \mathrm{tr} iT \implies \mathrm{tr}\, T = 0$

Hence,

$$L = su(2) = \left\{2 \times 2 \text{ complex matrices such that } T^\dagger = T \text{ and } \mathrm{tr}\, T = 0\right\}$$

- Standard basis: $T^a = \sigma^a/2$, $a = 1, 2, 3$, $\sigma^a =$ Pauli matrices
- $-T^{a*} = ST^aS^{-1}\ \forall a$, with $S = S^{-1} = \sigma^2 \implies$ the defining representation is **real**
- The **adjoint** representation is $(T^a)^m_n \equiv -i\epsilon^{amn}$

$$T^1 = -i\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix} \quad , \quad T^2 = -i\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix} \quad , \quad T^3 = -i\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Irreducible representations

How many *different* reprentations are there?

- The tensor product of two representations is also a representation:
  - G: $(M \otimes M')(g) \equiv M(g) \otimes M'(g)$
  - L: $(M \otimes M')(T) \equiv M(T) \otimes 1 + 1 \otimes M'(T)$
- *Different* must be made precise:
  - M and M' are said to be **equivalent** representations if S exists such that $M(g) = SM'(g)S^{-1}\ \forall g \in G \Leftrightarrow M(T) = SM'(T)S^{-1}\ \forall T \in L$
  - V is called an invariant subspace if $M(g)V \subset V\ \forall g \in G \Leftrightarrow M(T)V \subset V\ \forall T \in L$
  - M is called **irreducible** if it has no invariant subspaces (beyond 0 and the full space, which are always invariant), otherwise it is called **reducible**
  - **Theorem:** For compact G, reducible representations are equivalent to the tensor product of irreducible ones
  - *Different* $\equiv$ inequivalent irreducible

Finite dimensional irreducible representations of $SU(2)$

$$[T^1, T^2] = iT^3 \quad , \quad [T^2, T^3] = iT^1 \quad , \quad [T^3, T^1] = iT^2$$

- They cannot be diagonalized simultaneously, lets take T^3 diagonal in the basis $|j\, m\rangle$, j just labels the representation

$$T^3 |j\, m\rangle = m |j\, m\rangle \ , \ \langle j\, m | j'\, m'\rangle = \delta_{jj'}\delta_{mm'} \ , \ T^{\pm} \equiv \frac{1}{\sqrt{2}}\left(T^1 \pm iT^2\right) \ , \ (T^{\pm})^{\dagger} = T^{\mp}$$

- In the new basis,

$$[T^3, T^{\pm}] = \pm T^{\pm} \quad , \quad [T^+, T^-] = T^3$$

$$\Rightarrow \quad T^3 T^{\pm} |j\, m\rangle = \left(T^{\pm} T^3 \pm T^{\pm}\right) |j\, m\rangle = (m \pm 1) T^{\pm} |j\, m\rangle$$

$$\Rightarrow \quad T^{\pm} |j\, m\rangle = N_m^{\pm} |j\, m \pm 1\rangle \quad , \quad N_m^{\pm} \in \mathbb{C}$$

- Then

$$\begin{array}{ll} \langle j\, m | T^+ T^- | j\, m\rangle & = N_m^- \langle j\, m | T^+ | j\, m-1\rangle = N_m^- N_{m-1}^+ \\ \quad \| & \\ \langle T^-\, j\, m | T^- | j\, m\rangle & = |N_m^-|^2 \quad \Rightarrow \quad N_{m-1}^+ = N_m^{-\,*} \end{array}$$

- Consider

$$\begin{array}{ll} \langle j\, m | [T^+, T^-] | j\, m\rangle & = \langle j\, m | T^3 | j\, m\rangle = m \\ \quad \| & \\ \langle j\, m | T^+ T^- - T^- T^+ | j\, m\rangle & = |N_m^-|^2 - |N_m^+|^2 = |N_{m-1}^+|^2 - |N_m^+|^2 \end{array}$$

- Finite dimensional $\Rightarrow$ j, q $\exists$ such that $N_j^+ = 0\, , N_q^- = N_{q-1}^+ = 0$, $j \geq q$
- Then

$$\begin{array}{lll} \displaystyle\sum_{m=q}^{j} m & = & \underbrace{|N_{q-1}^+|^2}_{|N_q^-|^2=0} - |N_q^+|^2 + |N_q^+|^2 - |N_{q+1}^+|^2 + \cdots + |N_{j-1}^+|^2 - \underbrace{|N_j^+|^2}_{0} = 0 \\ \quad \| & & \\ \left(\dfrac{q+j}{2}\right)(j - q + 1) & & \Rightarrow \quad q = -j\, , \quad q = j - k\, , k \in \mathbb{N} \quad \Rightarrow \quad k = 2j \quad \Rightarrow \quad j \in \mathbb{N}/2 \end{array}$$

$SU(3)$

$$G = SU(3) = \left\{ 3\times 3 \text{ complex matrices such that } g^\dagger g = \mathbb{I}_3 \text{ and } \det g = 1 \right\}$$
$$L = su(3) = \left\{ 3\times 3 \text{ complex matrices such that } T^\dagger = T \text{ and } \mathrm{tr}\, T = 0 \right\}$$

- Standard basis: $T^a = \lambda^a/2$, $a = 1, \dots, 8$, $\lambda^a =$ Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad , \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad , \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \mathrm{tr}(T^a T^b) = \frac{\delta^{ab}}{2}$$

- **Cartan subalgebra** $\equiv \{H_i\}$ $\equiv$ maximal abelian subalgebra ($[H_i\, , H_j] = 0$), for $su(3)$: $H_1 = T^3$, $H_2 = T^8$
- **Rank** of L=dim$\{H_i\}$ $\Rightarrow$ rank of $su(2) = 1$, rank of $su(3) = 2$

- The eigenvalues of $\{H_i\}$, $H_i\,|\mu\rangle = \mu_i\,|\mu\rangle$, are called the weights of the representation. For the defining representation:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to |\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle \quad , \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to |-\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle \quad , \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to |0, -\frac{1}{\sqrt{3}}\rangle$$

- In analogy to $su(2)$ we introduce rising and lowering generators,

$$E_{(\pm 1, 0)} \equiv \tfrac{1}{\sqrt{2}}\left(T^1 \pm i T^2\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$E_{(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})} \equiv \tfrac{1}{\sqrt{2}}\left(T^4 \pm i T^5\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$E_{(\mp\frac{1}{2}, \pm\frac{\sqrt{3}}{2})} \equiv \tfrac{1}{\sqrt{2}}\left(T^6 \pm i T^7\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- $\alpha = (\pm 1, 0)\,, (\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})\,, (\mp\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ are called **roots**
- Note that $E_\alpha^\dagger = E_{-\alpha}$

- Let us call E_α the rising/lowering generators. Then,

$$[H_i\,,E_\alpha]=\alpha_i E_\alpha \quad , \quad [E_\alpha\,,E_{-\alpha}]=\alpha_i H_i$$

- For a given α, we have a set of equations similar to those for $su(2)$

$$H_i E_\alpha\,|\mu\rangle=([H_i\,,E_\alpha]+E_\alpha H_i)\,|\mu\rangle=(\alpha_i+\mu_i)\,E_\alpha\,|\mu\rangle$$

$$\Rightarrow \quad E_\alpha\,|\mu\rangle=N_{\alpha\,,\mu}\,|\mu+\alpha\rangle \quad , \quad N_{\alpha\,,\mu}\in\mathbb{C}$$

- Note that $N_{-\alpha\,,\mu}=\langle\mu-\alpha|E_{-\alpha}|\mu\rangle=\langle E_\alpha\,,\mu-\alpha|\mu\rangle=N^*_{\alpha\,,\mu-\alpha}$. Then,

$$\begin{array}{ccc}\langle\mu|[E_\alpha\,,E_{-\alpha}]|\mu\rangle & =\langle\mu|\alpha_i H_i|\mu\rangle= & \alpha_i\mu_i\equiv\alpha.\mu \\ \| & & \\ \langle\mu|E_\alpha E_{-\alpha}-E_{-\alpha}E_\alpha|\mu\rangle & = & |N_{-\alpha\,,\mu}|^2-|N_{\alpha\,,\mu}|^2=|N_{\alpha\,,\mu-\alpha}|^2-|N_{\alpha\,,\mu}|^2\end{array}$$

- Finite dimensional $\Rightarrow$ $\exists p,q\in\mathbb{N}$ such that $N_{\alpha\,,\mu+p\alpha}=N_{-\alpha\,,\mu-q\alpha}=0$
- Then $\alpha.\mu=|N_{\alpha\,,\mu-\alpha}|^2-|N_{\alpha\,,\mu}|^2$ with $\mu\to\mu+r\alpha$, $r\in\mathbb{Z}$ leads to

$$\sum_{r=-q}^{p}\alpha.(\mu+r\alpha)=\underbrace{|N_{\alpha\,,\mu-(q+1)\alpha}|^2}_{|N_{-\alpha\,,\mu-q\alpha}|^2=0}-|N_{\alpha\,,\mu-q\alpha}|^2+\cdots+|N_{\alpha\,,\mu+(p-1)\alpha}|^2-\underbrace{|N_{\alpha\,,\mu+p\alpha}|^2}_{0}=0$$

$$\|$$

$$(p+q+1)\alpha.\mu+\frac{(p+q+1)(p-q)}{2}\alpha.\alpha \quad \Rightarrow \quad \frac{\alpha.\mu}{\alpha.\alpha}=-\frac{p-q}{2}$$

Weights of the complex conjugate representation

- The complex conjugate representation is obtained $T^a\to-T^{a\,*}$
- In the Cartan basis $H_i\to-H_i^*=-H_i$ $\Rightarrow$ $\mu\to-\mu$
- For a real representation, if μ is a weight $\Rightarrow$ $-\mu$ is also a weight
- The weights of the complex conjugate of the $su(3)$ defining representation are:

$$|-\frac{1}{2},-\frac{1}{2\sqrt{3}}\rangle \quad , \quad |\frac{1}{2},-\frac{1}{2\sqrt{3}}\rangle \quad , \quad |0,\frac{1}{\sqrt{3}}\rangle$$

Weights of the adjoint representation

- The adjoint representation may be defined as

$$T^a\,|T^b\rangle\equiv|[T^a\,,T^b]\rangle=if^{abc}\,|T^c\rangle$$

- In the Cartan basis

$$H_i\,|H_j\rangle=|[H_i,H_j]\rangle=0 \quad , \quad H_i\,|E_\alpha\rangle=|[H_i,E_\alpha]\rangle=\alpha_i\,|E_\alpha\rangle$$

- There are l zero weights, l = rank of L
- The roots are weights of the adjoint representation
- The adjoint representation is always real

- **Positive root** $\equiv$ a root with the first non-zero component positive
- **Simple root** $\equiv$ a positive root that cannot be written as a sum of positive roots
- Theorems:
  - $\alpha\,,\beta \in \{$ Simple roots$\}$ $\Rightarrow$ $\alpha-\beta$ is not a root

$$\Rightarrow \quad \frac{\alpha.\beta}{\alpha.\alpha} = -\frac{p}{2} \quad , \quad \frac{\alpha.\beta}{\beta.\beta} = -\frac{p'}{2} \quad , \quad p\,,p' \in \mathbb{N}$$

  - Simple roots are lineraly independent
  - A positive root is a linear combination of simple roots with non-negative integers
  - The number of simple roots coincides with the rank of L
- For $su(3)$:
  - Roots: $(\pm 1, 0)\,,(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})\,,(\mp\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$
  - Positive roots: $(1,0)\,,(\frac{1}{2}, \frac{\sqrt{3}}{2})\,,(\frac{1}{2}, -\frac{\sqrt{3}}{2})$
  - Simple roots: $\alpha^1 \equiv (\frac{1}{2}, \frac{\sqrt{3}}{2})\,,\alpha^2 \equiv (\frac{1}{2}, -\frac{\sqrt{3}}{2})$
  - $\alpha^1.\alpha^2/\alpha^1.\alpha^1 = \alpha^1.\alpha^2/\alpha^2.\alpha^2 = -1/2 \quad \Rightarrow \quad p\,,p' = 1$
  - Note that $(1,0) = (\frac{1}{2}, \frac{\sqrt{3}}{2}) + (\frac{1}{2}, -\frac{\sqrt{3}}{2})$

- The same concept of positivity for roots can be extended to weights
- It stablishes an order in the root and weight spaces: $\mu > \mu' \Leftrightarrow \mu - \mu' > 0$
- **Highest weight** $\equiv$ the largest of the weights in a given representation
- The highest weight μ characterizes the representation:
  - $\{\alpha^i\}$ simple roots $\Rightarrow$ $E_{\alpha^i}\,|\mu\rangle = 0$

$$\Rightarrow \quad \frac{\alpha^i.\mu}{\alpha^i.\alpha^i} = \frac{q^i}{2} \quad , \quad q^i \in \mathbb{N} \quad \{q^i\} \text{ characterize the representation}$$

  - A weight μ^j such that

$$\frac{\alpha^i.\mu^j}{\alpha^i.\alpha^i} = \frac{\delta^{ij}}{2} \quad \text{is called a \textbf{fundamental} weight}$$

  - A representation with $q^j = 1\,,\ q^i = 0\,,\ i \neq j$, is called a **fundamental** representation
  - Then, any highest weight μ can be written as $\mu = \sum_{i=1}^{k} q^i \mu^i$
  - The representations are labeled as $(q^1, q^2 \ldots, q^k)$, $k =$ rank of L

For $su(3)$ we have,

$$\alpha^1 = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \quad , \quad \alpha^2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$$
$$\mu^1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}) \quad , \quad \mu^2 = (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$$

- Since the weights of the defining representation are

$$|_1\rangle \equiv \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \quad , \quad |_2\rangle \equiv \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \quad , \quad |_3\rangle \equiv \left(0, -\frac{1}{\sqrt{3}}\right) ,$$

  the highest weight is $\mu = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$ and hence it corresponds to the fundamental representation $(1,0)$ $(\mu = \mu^1)$. Physicist call it the 3 representation

- Since the weights of the complex conjugate representation of the defining one are

$$|^1\rangle \equiv \left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right) \quad , \quad |^2\rangle \equiv \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right) \quad , \quad |^3\rangle \equiv \left(0, \frac{1}{\sqrt{3}}\right) ,$$

  the highest weight is $\mu = \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$ and hence it corresponds to the fundamental representation $(0,1)$ $(\mu = \mu^2)$. Physicist call it the 3^* representation

- Since the weights of the adjoint representation are

$$2\times (0,0) \quad , \quad (\pm 1, 0) \quad , \quad \left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) \quad , \quad \left(\mp\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$$

  The highest weight is $\mu = (1,0)$ and hence it corresponds to the representation $(1,1)$ $(\mu = \mu^1 + \mu^2)$. Physicist call it the 8 representation

- The $(0,0)$ representation is the trivial representation (for L all elements are 0, for G all elements are 1)

- We are going to build the remaining representations through tensor products of the two fundamental representations

Tensor methods

- Consider the states $|_i\rangle$ the eigenstates of the Cartan subalgebra in the 3 representation, $T^a\,|_i\rangle = (T^a)^j{}_i\,|_j\rangle$
- Consider the states $|^i\rangle$ the eigenstates of the Cartan subalgebra in the 3^* representation, $T^a\,|^i\rangle = -(T^{a\,*})^i_j\,|^j\rangle = -(T^a)^i_j\,|^j\rangle$
- Let us denote the tensor product,

$$|^{i_1\dots i_m}_{j_1\dots j_n}\rangle \equiv |^{i_1}\rangle \dots |^{i_m}\rangle\,|_{j_1}\rangle \dots |_{j_n}\rangle$$

- T^a on the tensor product reads,

$$T^a\,|^{i_1\dots i_m}_{j_1\dots j_n}\rangle = \sum_{l=1}^{n} |^{i_1\dots i_m}_{j_1\dots j_{l-1}kj_{l+1}\dots j_n}\rangle\,(T^a)^k_{j_l} - \sum_{l=1}^{m} |^{i_1\dots i_{l-1}ki_{l+1}\dots i_m}_{j_1\dots j_n}\rangle\,(T^a)^{i_l}_k$$

- T^a on the tensor product coordinates reads,

$$|v\rangle = |^{i_1\dots i_m}_{j_1\dots j_n}\rangle\,v^{j_1\dots j_n}_{i_1\dots i_m}$$

$$(T^a v)^{j_1\dots j_n}_{i_1\dots i_m} = \sum_{l=1}^{n}(T^a)^{j_l}_k\,v^{j_1\dots j_{l-1}kj_{l+1}\dots j_n}_{i_1\dots i_m} - \sum_{l=1}^{m}(T^a)^k_{i_l}\,v^{j_1\dots j_n}_{i_1\dots i_{l-1}ki_{l+1}\dots i_m}$$

- The weight of a tensor product state is the sum of the weights of each of the fundamental states that form it

Consider the tensor product of m states in the 3 representation and n states in the 3^* representation

- The highest weight in the tensor product is $m\mu^1 + n\mu^2$, and hence it contains the irreducible representation (m, n)
- The state with the highest weight is $|v_H\rangle = |^{2\dots 2}_{1\dots 1}\rangle$, which has components, $v^{j_1\dots j_n}_{H\,i_1\dots i_m} = \delta^{j_1 1}\dots\delta^{j_n 1}\delta_{i_1 2}\dots\delta_{i_m 2}$
  - It is totally symmetric under the exchange of upper indices
  - It is totally symmetric under the exchange of lower indices
  - It vanishes upon the contraction with $\delta^{i_1}_{j_1}$
- This properties are mantained upon the application of $T^a \implies$ the (n, m) representation leads to traceless tensors which are totally symmetric upon the exchange of upper indices and lower indices
- Finding the dimension of (m, n) is now a combinatorial problem:

$$d(m, n) = \frac{(m+1)(n+1)(m+n+2)}{2}$$

Basic tensor product decomposition

- $3 \otimes 3^* = (1,0) \otimes (0,1) \supset (1,1)$. Since $d(1,1) = 8$ and the dimension of the tensor product is 9, there is only room for the trivial representation $(0,0)$. Hence,

$$(1,0) \otimes (0,1) = (1,1) \oplus (0,0) \quad \Leftrightarrow \quad 3 \otimes 3^* = 8 \oplus 1$$

  - Color $SU(3)$ $\Longrightarrow$ mesons are physical states
  - Flavor $SU(3)$ $\Longrightarrow$ mesons with light quarks only form octuplets and singlets
- $3 \otimes 3 = (1,0) \otimes (1,0) \supset (2,0)$. Since $d(2,0) = 6$ and the dimension of the tensor product is 9, there is room for either the $(1,0)$ representation, or the $(0,1)$ representation, or three times the $(0,0)$ representation. We need more tools to tell apart these three cases.

Invariant tensors

- Consider $\delta^n_{\ m}$: $(T^a\delta)^n_{\ m} = (T^a)^n_{\ l}\,\delta^l_{\ m} - (T^a)^l_{\ m}\,\delta^n_{\ l} = 0$
  - This is a consequence of $g^n_{\ l}\, g^{*\ s}_{\ m}\delta^l_{\ s} = g^n_{\ l}\, g*^{\ l}_{m} = g^n_{\ l}\, g^{\dagger\ l}_{\ \ m} = \delta^n_{\ m}$
- Consider ϵ^{ijk}: $(T^a\epsilon)^{ijk} = (T^a)^i_{\ l}\,\epsilon^{ljk} + (T^a)^j_{\ l}\,\epsilon^{ilk} + (T^a)^k_{\ l}\,\epsilon^{ijl} = 0$
  - This is a consequence of $\det g = 1$ $\Longrightarrow$ $g^i_{\ i'}\, g^j_{\ j'}\, g^k_{\ k'}\,\epsilon^{i'j'k'} = \epsilon^{ijk}$
- The $SU(3)$ invariant tensors are then:

$$\delta^i_{\ j} \quad , \quad \epsilon^{ijk} \quad , \quad \epsilon_{ijk}$$

  - Color $SU(3)$ $\Longrightarrow$ baryons are physical states (color singlet), and the color wave function is antisymmetric.
- The $SU(2)$ invariant tensors are then:

$$\delta^i_{\ j} \quad , \quad \epsilon^{ij} \quad , \quad \epsilon_{ij}$$

Tensor product decomposition

- $3 \otimes 3 = (1,0) \otimes (1,0) \supset (2,0)$. We know now that $(2,0)$ is totally symmetric:

$$v^i w^j = \underbrace{\frac{1}{2}\left(v^i w^j + v^j w^i\right)}_{(2,0)} + \frac{1}{2}\left(v^i w^j - v^j w^i\right)$$

$$\left(v^i w^j - v^j w^i\right) = \epsilon^{ijk} \underbrace{\epsilon_{klm} v^l w^m}_{(0,1)}$$

$$\Longrightarrow \quad 3 \otimes 3 = (1,0) \otimes (1,0) = (2,0) \oplus (0,1) = 6 \oplus 3^*$$

  - Color $SU(3)$: quark-quark states do not exist (no singlet $(0,0)$ in the tensor decomposition)
  - Flavor $SU(3)$: baryons with a single heavy quark (charm or bottom) form tripplets and sextets

- $3 \otimes 3 \otimes 3 = (6 \oplus 3^*) \otimes 3 = (6 \otimes 3) \oplus (3^* \otimes 3) = (6 \otimes 3) \oplus 8 \oplus 1$, $6 \otimes 3 = (2,0) \otimes (1,0) \supset (3,0)$, totally symmetric. Consider $v^{ij} \in (2,0)$, totally symmetric:

$$v^{ij} w^k = \underbrace{\frac{1}{3}\left(v^{ij} w^k + v^{ik} w^j + v^{kj} w^i\right)}_{(3,0)=10} + \frac{2}{3} v^{ij} w^k - \frac{1}{3}\left(v^{ik} w^j + v^{kj} w^i\right)$$

$$\frac{2}{3} v^{ij} w^k - \frac{1}{3}\left(v^{ik} w^j + v^{kj} w^i\right) = \frac{1}{3}\left(\epsilon^{ikl} \underbrace{\epsilon_{lmn} v^{jm} w^n}_{(1,1)=8} + (i \leftrightarrow j)\right)$$

$$\Longrightarrow \quad 6 \otimes 3 = 10 \oplus 8$$

$$\Longrightarrow \quad 3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

  - Color $SU(3)$: baryons are physical states (there is a singlet $(0,0)$ in the tensor product)
  - Flavor $SU(3)$: baryons made out of three light quarks may form singlets, octets $(1,1)$ and decuplets $(3,0)$
    - The Pauli principle restricts these possibilities to an octet and a decuplet for the ground states
    - It also forces the spin of the octet (decuplet) to be 1/2 (3/2)

Exotic hadrons

Can there be hadrons beyond mesons and baryons?

- Glueballs (gg)
  - In QCD the gluons belong to the 8 representation. Can there be physical states made out of gluons only?$\Leftrightarrow$ Is there a singlet in $8 \otimes 8$?

$$v^i_j \, , w^k_l \in (1,1) = 8 \quad , \quad v^i_j w^k_l \delta^l_i \delta^j_k = v^i_j w^j_i \quad \Longrightarrow \quad \text{Yes!}$$

- Tetraquarks ($qq\bar{q}\bar{q}$)
  - $3 \otimes 3^* \otimes 3 \otimes 3^* = (1 \oplus 8) \otimes (1 \oplus 8) = 1 \oplus 8 \oplus 8 \oplus (8 \otimes 8)$
  - The 1 is interpreted as two mesons put together
  - Since there is a singlet in $(8 \otimes 8)$ $\Longrightarrow$ non-trivial tetraquark states may exist
- Pentaquarks ($qqqq\bar{q}$)
  - $3 \otimes 3 \otimes 3 \otimes 3 \otimes 3^* = (10 \oplus 8 \oplus 8 \oplus 1) \otimes (1 \oplus 8) =$
    $10 \oplus 8 \oplus 8 \oplus 1 \oplus (10 \otimes 8) \oplus (8 \otimes 8) \oplus (8 \otimes 8) \oplus 8$
  - The 1 is interpreted as a baryon and a meson put together
  - Since there is a singlet in $(8 \otimes 8)$ $\Longrightarrow$ non-trivial pentaquark states may exist

$8 \otimes 8 =?$

- Use symmetrization and the invariant tensors to work out the representations in $8 \otimes 8$
- $v^i_j \, , w^l_m \in 8$ ($\Longrightarrow$ $v^i_i = w^l_l = 0$), $v^i_j w^l_m$:

$$\begin{aligned}
v^{\{i}_{\{j} w^{l\}}_{m\}} - \text{traces} \quad &\rightarrow (2,2) \quad , \quad d(2,2) = 27 \\
v^{\{i}_j w^l_m \epsilon^{n\}jm} \quad &\rightarrow (3,0) \quad , \quad d(3,0) = 10 \\
v^i_{\{j} w^l_m \epsilon_{n\}il} \quad &\rightarrow (0,3) \quad , \quad d(0,3) = 10 \\
v^i_j w^l_i - \frac{1}{3}\delta^l_j v^i_k w^k_i \quad &\rightarrow (1,1) \quad , \quad d(1,1) = 8 \\
v^i_j w^j_m - \frac{1}{3}\delta^i_m v^k_j w^j_k \quad &\rightarrow (1,1) \quad , \quad d(1,1) = 8 \\
v^i_j w^j_i \quad &\rightarrow (0,0) \quad , \quad d(0,0) = 1
\end{aligned}$$

Flavor $SU(3)$

Let us try to assign hadrons to each weight in the multiplets

- The 1st component of the weight vector corresponds to I_3 of isospin
- The second component of the weight vectors is related to strangeness
- The octuplet $(1,1) = 8$

$$
\begin{array}{rcl}
\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right) & = |_1\rangle\, |^3\rangle \rightarrow & p\,,K^+ \\
\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right) & = |_2\rangle\, |^3\rangle \rightarrow & n\,,K^0 \\
(1,0) & = |_1\rangle\, |^2\rangle \rightarrow & \Sigma^+\,,\pi^+ \\
(0,0) & = |_1\rangle\, |^1\rangle\,,|_2\rangle\, |^2\rangle\,,|_3\rangle\, |^3\rangle \rightarrow & \Sigma^0\,,\pi^0 \\
(0,0) & = |_1\rangle\, |^1\rangle\,,|_2\rangle\, |^2\rangle\,,|_3\rangle\, |^3\rangle \rightarrow & \Lambda\,,\eta \\
(-1,0) & = |_2\rangle\, |^1\rangle \rightarrow & \Sigma^-\,,\pi^- \\
\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right) & = |_3\rangle\, |^2\rangle \rightarrow & \Xi^0\,,\bar{K}^0 \\
\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right) & = |_3\rangle\, |^1\rangle \rightarrow & \Xi^-\,,K^-
\end{array}
$$

- The baryon octet $(1/2^+)$

$$
B = \begin{pmatrix}
\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\
\Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\
\Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}}
\end{pmatrix}
$$

  - The objects in the matrix are Dirac fields
  - The normalization is choosen such that $\bar{B} = B^\dagger \gamma^0$

$$
\mathrm{tr}(\bar{B}B) = \bar{\Sigma}^+\Sigma^+ + \bar{\Sigma}^0\Sigma^0 + \bar{\Sigma}^-\Sigma^- + \bar{\Xi}^-\Xi^- + \bar{\Xi}^0\Xi^0 + \bar{\Lambda}\Lambda + \bar{p}p + \bar{n}n
$$

  - Under $SU(3)$: $B \rightarrow gBg^\dagger$, $\bar{B} \rightarrow g\bar{B}g^\dagger$, the following Lagrangian is invariant

$$
\mathcal{L} = \mathrm{tr}\left(\bar{B}\left(i\partial\!\!\!/ - m\right)B\right)
$$

and leads to the Dirac Lagrangian for each of the fields. The trace above is over $SU(3)$ matrices only.

- The pseudoscalar meson octet (0^-)

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} = M^\dagger$$

  - The objects in the matrix are Klein-Gordon fields.
  - The normalization is choosen such that

$$\mathrm{tr}(MM) = 2\pi^-\pi^+ + \pi^0\pi^0 + 2K^-K^+ + 2\bar{K}^0K^0 + \eta\eta \quad , \quad \pi^- = \pi^{+\dagger} \; , \; \bar{K}^0 = K^{0\dagger}$$

  - Under $SU(3)$: $M \to gMg^\dagger$, the following Lagrangian is invariant

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}\left(\partial_\mu M \partial^\mu M - m^2 M^2\right)$$

    and leads to the Klein-Gordon Lagrangian for each of the fields

- The baryon decuplet $(3/2^+)$
  - It should be described by a field Δ^{ijk}, $i, j, k = 1, 2, 3$ totally symmetric under the exchange of the flavor indices
  - Since we have not studied relativistic equations for spin $3/2$ particles, we shall ignore the Lorentz structure
  - We expect the mass term in the Lagrangian to be of the form

$$\mathcal{L}_m = -m\Delta^\dagger_{ijk}\Delta^{ijk} = -m\left(\Delta^\dagger_{111}\Delta^{111} + 3\Delta^\dagger_{112}\Delta^{112} + 3\Delta^\dagger_{113}\Delta^{113} + 3\Delta^\dagger_{122}\Delta^{122}\right.$$
$$\left. + 6\Delta^\dagger_{123}\Delta^{123} + 3\Delta^\dagger_{133}\Delta^{133} + \Delta^\dagger_{222}\Delta^{222} + 3\Delta^\dagger_{223}\Delta^{223} + 3\Delta^\dagger_{233}\Delta^{233} + \Delta^\dagger_{333}\Delta^{333}\right)$$

  - The simplest way to identify the physical baryons is by recalling the indices $1, 2, 3$ correspond to the quarks u, d, s, then

$$\Delta^{111} = \Delta^{++} \quad , \quad \Delta^{112} = \frac{1}{\sqrt{3}}\Delta^+ \quad , \quad \Delta^{122} = \frac{1}{\sqrt{3}}\Delta^0 \quad , \quad \Delta^{222} = \Delta^-$$

$$\Delta^{113} = \frac{1}{\sqrt{3}}\Sigma^{*+} \quad , \quad \Delta^{123} = \frac{1}{\sqrt{6}}\Sigma^{*0} \quad , \quad \Delta^{223} = \frac{1}{\sqrt{3}}\Sigma^{*-}$$

$$\Delta^{133} = \frac{1}{\sqrt{3}}\Xi^{*0} \quad , \quad \Delta^{233} = \frac{1}{\sqrt{3}}\Xi^{*-}$$

$$\Delta^{333} = \Omega^-$$

Flavor $SU(3)$ breaking

- Flavor $SU(3)$ is broken because the up, down and strange quarks do not have exactly the same masses
- Since the strange hadrons are heavier than the non-strange ones, the breaking is mainly due to the strange quark mass
- If we keep isospin as an exact symmetry, the breaking must be proportional to T^8
- We may then add to the corresponding Lagrangians all possible terms that are linear in T^8
- Baryon octet:

$$\delta\mathcal{L} = -a\,\mathrm{tr}\left(\bar{B}BT^8\right) - b\,\mathrm{tr}\left(\bar{B}T^8B\right) \quad ,,\ a\,, b \in \mathbb{R}$$

  - Upon calculating the traces this leads to the following formulas

    $m_\Sigma = m+\tilde{a}+\tilde{b} \quad , \quad m_\Lambda = m-\tilde{a}-\tilde{b} \quad , \quad m_\Xi = m+\tilde{a}-2\tilde{b} \quad , \quad m_N = m-2\tilde{a}+\tilde{b}$

    $\tilde{a} = \frac{a}{2\sqrt{3}}$, $\tilde{b} = \frac{a}{2\sqrt{3}}$, which leads to the Gell-Mann-Okubo formula,

$$2(m_\Xi + m_N) = m_\Sigma + 3m_\Lambda$$

  - This formula is fulfilled within a 0.7% error:

$$4508.8\,\mathrm{MeV} \simeq 2(m_\Xi + m_N) = m_\Sigma + 3m_\Lambda \simeq 4539.7\,\mathrm{MeV}$$

- Baryon decuplet

$$\delta\mathcal{L} = -c\,\Delta^\dagger_{mjk}\left(T^8\right)^m_n \Delta^{njk} \quad ,,\ c \in \mathbb{R}$$

  - A similar procedure leads to

$$m_{\Sigma^*} - m_\Delta = m_{\Xi^*} - m_{\Sigma^*} = m_\Omega - m_{\Xi^*}$$

  - This formula is fulfilled within a 4% error

$$153\,\mathrm{MeV} \simeq 145\,\mathrm{MeV} \simeq 142\,\mathrm{MeV}$$

  - Historically, it allowed to predict the mass of the Ω^-

- Pseudoscalar meson octet

$$\delta\mathcal{L} = -a\,\mathrm{tr}\left(MMT^8\right) \quad ,\ a \in \mathbb{R}$$

  - A similar procedure leads to

$$4m_K^2 = m_\pi^2 + 3m_\eta^2$$

  - This formula is fulfilled within an 8% error

$$990423\,\mathrm{MeV}^2 \simeq 4m_K^2 = m_\pi^2 + 3m_\eta^2 \simeq 927100\,\mathrm{MeV}^2$$

- The vector meson octet (1^-)

$$V_\mu = \begin{pmatrix} \frac{\rho^0_\mu}{\sqrt{2}} + \frac{\omega_\mu}{\sqrt{6}} & \rho^+_\mu & K^{*+}_\mu \\ \rho^-_\mu & -\frac{\rho^0_\mu}{\sqrt{2}} + \frac{\omega_\mu}{\sqrt{6}} & K^{*0}_\mu \\ K^{*-}_\mu & \bar{K}^{*0}_\mu & -\frac{2\omega_\mu}{\sqrt{6}} \end{pmatrix} = V^\dagger_\mu$$

  - Under $SU(3)$, $V_\mu \to gV_\mu g^\dagger$
  - The mass term in the Lagrangian, including $SU(3)$ breaking, reads

$$\mathcal{L}_m = \frac{m^2}{2}\mathrm{tr}\left(V_\mu V^\mu\right) - c\,\mathrm{tr}\left(V_\mu V^\mu T^8\right) \quad , c \in \mathbb{R}$$

  - In analogy to the pseudoscalar meson case, we obtain

$$4m^2_{K^*} = m^2_\rho + 3m^2_\omega$$

  - This formula is badly fulfilled:

$$3182656\,\mathrm{MeV}^2 \simeq 2427472\,\mathrm{MeV}^2$$

- Why Gell-Mann-Okubo formulas are well fulfilled for baryons but not so well for mesons?

Mixing

- Since $3 \otimes 3^* = 8 \oplus 1$, in addition to the octuplet there should also be a singlet $SU(3)$ meson, both for pseudoscalars and for vectors:
  - 0^{-+}: η', $m_{\eta'} = 958$ MeV
  - 1^{--}: ϕ, $m_\phi = 1020$ MeV
- In the exact $SU(3)$ limit, we should just add to the Lagrangians we had, the corresponding term for an extra pseudoscalar or vector particle with an arbitrary mass.
- However, when we consider $SU(3)$ breaking by linear terms in T^8, a new quadratic term can be written down. In the vector case it reads,

$$\delta\mathcal{L}_m = -d\,\mathrm{tr}\left(V_\mu T^8\right) S^\mu = -\frac{d}{\sqrt{2}}\omega_\mu S^\mu \quad , d \in \mathbb{R}$$

  where S^μ is the field of the $SU(3)$ singlet
- We see that the singlet field mixes with the isospin zero field of the octet
- Hence neither the isospin zero field in the octuplet nor the singlet field correspond to the physical particles (the ω and the ϕ in the vector case).
- ω^μ in the octuplet is renamed as ω^μ_8

- The physical fields and the physical masses for ω and ϕ are obtained by diagonalizing the quadratic terms

$$\mathcal{L}_m = \frac{1}{2} m_8^2 \omega_{8\,\mu}\omega_8^\mu + \frac{1}{2} m_1^2 S_\mu S^\mu - \frac{d}{\sqrt{2}}\omega_{8\,\mu} S^\mu = \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu + \frac{1}{2} m_\phi^2 \phi_\mu \phi^\mu$$

$$\begin{aligned} \omega_8^\mu &= \phi^\mu \cos\theta_V + \omega^\mu \sin\theta_V \\ S^\mu &= -\phi^\mu \sin\theta_V + \omega^\mu \cos\theta_V \end{aligned}$$

- $m_8 \simeq 929$ MeV is obtained from the Gell-Mann-Okubo formula, d and m_1 are unknown
- Since the outcome of the diagonalization are the experimentally known masses of the ω and ϕ ($m_\omega \simeq 782$ MeV, $m_\phi \simeq 1020$ MeV), one can get the (uninteresting) values of d and m_1, and the value mixing angle θ_V

$$\cos 2\theta_V = \frac{2m_8^2 - m_\omega^2 - m_\phi^2}{m_\omega^2 - m_\phi^2} \quad \Longrightarrow \quad \theta_V \simeq 36.4^{\mathrm{o}}$$

- An analogous exercise for the pseudoscalar mesons leads to

$$\left.\begin{aligned} S &= \eta' \cos\theta_P + \eta \sin\theta_P \\ \eta_8 &= -\eta' \sin\theta_P + \eta \cos\theta_P \end{aligned}\right\} \quad \Longrightarrow \quad \theta_P \simeq 10.7^{\mathrm{o}}$$

- The mixing is then small in the pseudoscalar case but large in the vector case.
- It turns out that in the vector meson case we have

$$\sin\theta_V \simeq 0.64 \simeq 0.57 \simeq \frac{1}{\sqrt{3}} \quad , \quad \cos\theta_V \simeq 0.77 \simeq 0.81 \simeq \sqrt{\frac{2}{3}}$$

$$\begin{aligned} \omega_8^\mu &= \phi^\mu \sqrt{\frac{2}{3}} + \omega^\mu \frac{1}{\sqrt{3}} \\ S^\mu &= -\phi^\mu \frac{1}{\sqrt{3}} + \omega^\mu \sqrt{\frac{2}{3}} \end{aligned}$$

- If we put together the octet and the singlet in a nonet,

$$V_\mu = V_{8\,\mu} + \frac{S_\mu}{\sqrt{3}}\mathbb{I}_3 = \begin{pmatrix} \frac{\rho_\mu^0}{\sqrt{2}} + \frac{\omega_{8\,\mu}}{\sqrt{6}} + \frac{S_\mu}{\sqrt{3}} & \rho_\mu^+ & K^{*\,+}_{\ \mu} \\ \rho_\mu^- & -\frac{\rho_\mu^0}{\sqrt{2}} + \frac{\omega_{8\,\mu}}{\sqrt{6}} + \frac{S_\mu}{\sqrt{3}} & K^{*\,0}_{\ \mu} \\ K^{*\,-}_{\ \mu} & \bar{K}^{*\,0}_{\ \mu} & -\frac{2\omega_{8\,\mu}}{\sqrt{6}} + \frac{S_\mu}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\rho_\mu^0 + \omega_\mu}{\sqrt{2}} & \rho_\mu^+ & K^{*\,+}_{\ \mu} \\ \rho_\mu^- & \frac{-\rho_\mu^0 + \omega_\mu}{\sqrt{2}} & K^{*\,0}_{\ \mu} \\ K^{*\,-}_{\ \mu} & \bar{K}^{*\,0}_{\ \mu} & -\phi_\mu \end{pmatrix}$$

Leptonic decays of neutral vector mesons

- This should have consequences in electromagnetic decays:
  - $\Longrightarrow \quad \phi \sim |_3\rangle\, |^3\rangle \sim |s\bar{s}\rangle$
  - $\Longrightarrow \quad \omega \sim \frac{1}{\sqrt{2}}\left(|_1\rangle\, |^1\rangle + |_2\rangle\, |^2\rangle\right) \sim \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle\right)$
- Note that with no mixing
  - $\phi \sim S \sim \frac{1}{\sqrt{3}}\left(|_1\rangle\, |^1\rangle + |_2\rangle\, |^2\rangle + |_3\rangle\, |^3\rangle\right) \sim \frac{1}{\sqrt{3}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle\right)$
  - $\omega \sim \omega_8 \sim \frac{1}{\sqrt{6}}\left(|_1\rangle\, |^1\rangle + |_2\rangle\, |^2\rangle - 2\, |_3\rangle\, |^3\rangle\right) \sim \frac{1}{\sqrt{6}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle - 2\, |s\bar{s}\rangle\right)$
- Recall that the coupling of neutral vector mesons to the e. m. field was $\sim qV_{\mu\nu}F^{\mu\nu}$, $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$
- In $SU(3)$ language, V_μ becomes a matrix

$$\sim \mathrm{tr}\left(QV^{\mu\nu}\right) F_{\mu\nu} \quad , \quad Q = e\begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

- We have

$$\Gamma\left(\rho^0 \to \gamma^* \to e^+e^-\right) \sim \left(\frac{1}{\sqrt{2}}\left(1.\frac{2}{3} + (-1).\left(-\frac{1}{3}\right)\right)\right)^2 \sim \frac{1}{2}$$

- With no mixing:

$$\Gamma\left(\omega \sim \omega_8 \to \gamma^* \to e^+e^-\right) \sim \left(\frac{1}{\sqrt{6}}\left(1.\frac{2}{3} + 1.\left(-\frac{1}{3}\right) + (-2)\left(-\frac{1}{3}\right)\right)\right)^2 \sim \frac{1}{6}$$

$$\Gamma\left(\phi \sim S \to \gamma^* \to e^+e^-\right) \sim \left(\frac{1}{\sqrt{3}}\left(1.\frac{2}{3} + 1.\left(-\frac{1}{3}\right) + 1.\left(-\frac{1}{3}\right)\right)\right)^2 \sim 0$$

$$\frac{\Gamma\left(\omega \to e^+e^-\right)}{\Gamma\left(\rho^0 \to e^+e^-\right)} \sim \frac{1}{3} \sim 0.33 \quad , \quad \frac{\Gamma\left(\phi \to e^+e^-\right)}{\Gamma\left(\rho^0 \to e^+e^-\right)} \sim 0$$

- With mixing:

$$\Gamma\left(\omega \to \gamma^* \to e^+e^-\right) \sim \left(\frac{1}{\sqrt{2}}\left(1.\frac{2}{3} + 1.\left(-\frac{1}{3}\right)\right)\right)^2 \sim \frac{1}{18}$$

$$\Gamma\left(\phi \to \gamma^* \to e^+e^-\right) \sim \left(1.\left(-\frac{1}{3}\right)\right)^2 \sim \frac{1}{9}$$

$$\frac{\Gamma\left(\omega \to e^+e^-\right)}{\Gamma\left(\rho^0 \to e^+e^-\right)} \sim \frac{1}{9} \sim 0.11 \quad , \quad \frac{\Gamma\left(\phi \to e^+e^-\right)}{\Gamma\left(\rho^0 \to e^+e^-\right)} \sim \frac{2}{9} \sim 0.22$$

- The experimental results are:

$$\Gamma\left(\omega \to e^+e^-\right)/\Gamma\left(\rho^0 \to e^+e^-\right) \sim 0.086 \quad , \quad \Gamma\left(\phi \to e^+e^-\right)/\Gamma\left(\rho^0 \to e^+e^-\right) \sim 0.18$$