Advanced General Relativity

Killing trajectories, hypersurfaces and extrinsic curvature

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Acceleration of Killing orbits

In a spacetime with a timelike Killing vector k, a particle moves on an trajectory—not necessarily a geodesic—with four-velocity $u = k/|k|$ where $|k| = \sqrt{-k^2}$.

We begin by noting that, since $\nabla_\nu k_\mu = -\nabla_\mu k_\nu$, we have:

$$\nabla_k k^2 = k^\mu \nabla_\mu (k^\nu k_\nu) = 2k^\mu k^\nu \nabla_\mu k_\nu = -2k^\mu k^\nu \nabla_\nu k_\mu = -2k^\nu k^\mu \nabla_\mu k_\nu = -\nabla_k k^2$$

and so $\nabla_k k^2 = -\nabla_k |k|^2 = 0$. But then $\nabla_k |k| = 0$ and furthermore:

$$\nabla_u |k| = \frac{k^\mu}{|k|} \nabla_\mu |k| = \frac{1}{|k|} \nabla_k |k| = 0$$

That is, the scalar $|k|$ is constant of motion.

But the acceleration a of the particle is:

$$a_\mu = \nabla_u u_\mu = \nabla_u \frac{k_\mu}{|k|} = \frac{\nabla_u k_\mu}{|k|} = \frac{k^\nu \nabla_\nu k_\mu}{|k|^2} = -\frac{k^\nu \nabla_\mu k_\nu}{|k|^2} = -\frac{\nabla_\mu k^2}{2|k|^2} = \frac{\nabla_\mu |k|^2}{2|k|^2} = \nabla_\mu \log |k|$$

Interpreted in terms of Newtonian gravity, $\Phi = \log |k|$ may be viewed as a gravitational potential.

The Schwarzschild metric has no explicit dependence on the time coordinate, and so ∂_0 is a Killing vector. But:

$$|\partial_0| = \sqrt{-g_{00}} = \sqrt{1 - \frac{2M}{r}}$$

And so the corresponding potential is:

$$\Phi = \log |\partial_0| = \frac{1}{2} \log \left(1 - \frac{2M}{r}\right) \simeq -\frac{M}{r}$$

where the approximation becomes exact for $r \to \infty$, and agrees with the Newtonian potential for a point mass.

The acceleration of a particle moving on orbits with $u = \partial_0$ is:

$$a_\mu = \nabla_\mu \Phi = \nabla_\mu \frac{1}{2} \log \left(1 - \frac{2M}{r}\right) = \frac{1}{2} \partial_\mu \log \left(1 - \frac{2M}{r}\right)$$

Then the only nonzero component of a_μ is:

$$a_r = \frac{M}{r(r - 2M)}$$

Raising the index by multiplying by $g^{rr} = 1 - 2M/r$, we finally arrive at:

$$a = \frac{M}{r^2} \partial_r$$

This is the radial acceleration required to maintain a particle at a constant distance from the black hole, assuming that there is no angular motion. Notice that it's the opposite of the acceleration of a free-falling body in Newtonian gravity, which is exactly what we expect.

Gaussian normal coordinates

In Gaussian normal coordinates, the metric takes the form:

$$ds^2 = dr^2 + g_{ij}(r,x)\,dx^i dx^j$$

In this coordinate system, the components of the unit normal one-form $n = dr$ are constant everywhere since $g^{rr} = 1$ and $g^{ri} = 0$, and so $n_\mu = \delta^r_\mu$. We may also read off the elements of the induced metric $h_{ij} = g_{ij}$.

The extrinsic curvature K is given by:

$$K_{ij} = h_i{}^k h_j{}^\ell \nabla_k n_\ell = \delta_i^k \delta_j^\ell \nabla_k n_\ell = \nabla_i n_j = \partial_i n_j - \Gamma^\rho_{ij} n_\rho = -\Gamma^r_{ij}$$

and $K_{rr} = K_{ri} = 0$ since K is orthogonal to n.

But, considering that the metric has $g_{ri} = 0$ and $g^{rr} = 1$, and using a well-known formula, we see that:

$$K_{ij} = -\Gamma^r_{ij} = -\frac{1}{2} g^{r\rho}(\partial_i g_{\rho j} + \partial_j g_{i\rho} - \partial_\rho g_{ij}) = \frac{1}{2}\partial_r g_{ij}$$

Intrinsic curvature of hypersurfaces

In what follows, h is the induced metric on a hypersurface Σ with unit normal n, and D is the induced covariant derivative, defined by projecting the covariant derivative of the containing manifold onto the hypersurface.

First, note that:

$$\begin{aligned}
D_a h_{bc} &\equiv h_a{}^{a'} h_b{}^{b'} h_c{}^{c'} \nabla_{a'} h_{b'c'} \\
&= h_a{}^{a'} h_b{}^{b'} h_c{}^{c'} [\nabla_{a'} g_{b'c'} - \sigma \nabla_{a'}(n_{b'} n_{c'})] \\
&= -\sigma h_a{}^{a'} h_b{}^{b'} h_c{}^{c'} \nabla_{a'}(n_{b'} n_{c'}) \\
&= -\sigma h_a{}^{a'} \left(h_c{}^{c'} h_b{}^{b'} n_{b'} \nabla_{a'} n_{c'} + h_b{}^{b'} h_c{}^{c'} n_{c'} \nabla_{a'} n_{b'} \right) \\
&= 0
\end{aligned}$$

since $\nabla g = 0$ for the metric-compatible connection of the containing manifold, and $h_a{}^b n_b = 0$ because the normal vector has no projection onto the hypersurface. Thus, the derivative D is compatible with the metric h of Σ.

Next, given u tangent to Σ, we evaluate:

$$\begin{aligned}
D_a D_b u_c &= h_a^{a'} h_b^{b'} h_c^{c'} \nabla_{a'} \left(h_{b'}^{b''} h_{c'}^{c''} \nabla_{b''} u_{c''} \right) \\
&= h_a^{a'} h_b^{b'} h_c^{c'} h_{b'}^{b''} h_{c'}^{c''} \nabla_{a'} \nabla_{b''} u_{c''} + h_a^{a'} h_b^{b'} h_c^{c'} \left(\nabla_{a'} h_{b'}^{b''} \right) h_{c'}^{c''} \nabla_{b''} u_{c''} + h_a^{a'} h_b^{b'} h_c^{c'} \left(\nabla_{a'} h_{c'}^{c''} \right) h_{b'}^{b''} \nabla_{b''} u_{c''} \\
&= h_a^{a'} h_b^{b''} h_c^{c''} \nabla_{a'} \nabla_{b''} u_{c''} + h_a^{a'} h_b^{b'} h_c^{c''} \left(\nabla_{a'} h_{b'}^{b''} \right) \nabla_{b''} u_{c''} + h_a^{a'} h_b^{b''} h_c^{c'} \left(\nabla_{a'} h_{c'}^{c''} \right) \nabla_{b''} u_{c''} \\
&= h_a^{a'} h_b^{b''} h_c^{c''} \nabla_{a'} \nabla_{b''} u_{c''} - \sigma h_a^{a'} h_b^{b'} h_c^{c''} \left(\nabla_{a'} n_{b'} n^{b''} \right) \nabla_{b''} u_{c''} - \sigma h_a^{a'} h_b^{b''} h_c^{c'} \left(\nabla_{a'} n_{c'} n^{c''} \right) \nabla_{b''} u_{c''} \\
&= h_a^{a'} h_b^{b''} h_c^{c''} \nabla_{a'} \nabla_{b''} u_{c''} \\
&\qquad - \sigma h_a^{a'} h_b^{b'} h_c^{c''} n_{b'} \left(\nabla_{a'} n^{b''} \right) \nabla_{b''} u_{c''} - \sigma h_a^{a'} h_b^{b'} h_c^{c''} n^{b''} (\nabla_{a'} n_{b'}) \nabla_{b''} u_{c''} \\
&\qquad - \sigma h_a^{a'} h_b^{b''} h_c^{c'} n_{c'} \left(\nabla_{a'} n^{c''} \right) \nabla_{b''} u_{c''} - \sigma h_a^{a'} h_b^{b''} h_c^{c'} n^{c''} (\nabla_{a'} n_{c'}) \nabla_{b''} u_{c''} \\
&= h_a^{a'} h_b^{b''} h_c^{c''} \nabla_{a'} \nabla_{b''} u_{c''} - \sigma h_a^{a'} h_b^{b'} h_c^{c''} n^{b''} (\nabla_{a'} n_{b'}) \nabla_{b''} u_{c''} - \sigma h_a^{a'} h_b^{b''} h_c^{c'} n^{c''} (\nabla_{a'} n_{c'}) \nabla_{b''} u_{c''} \\
&= h_a^{a'} h_b^{b''} h_c^{c''} \nabla_{a'} \nabla_{b''} u_{c''} - \sigma K_{ab'} h_b^{b'} h_c^{c''} n^{b''} \nabla_{b''} u_{c''} - \sigma K_{ac'} h_b^{b''} h_c^{c'} n^{c''} \nabla_{b''} u_{c''} \\
&= h_a^{a'} h_b^{b''} h_c^{c''} \nabla_{a'} \nabla_{b''} u_{c''} - \sigma K_{ab'} h_b^{b'} h_c^{c''} n^{b''} \nabla_{b''} u_{c''} + \sigma K_{ac'} h_b^{b''} h_c^{c'} u^{c''} \nabla_{b''} n_{c''} \\
&= h_a^{a'} h_b^{b''} h_c^{c''} \nabla_{a'} \nabla_{b''} u_{c''} - \sigma K_{ab'} h_b^{b'} h_c^{c''} n^{b''} \nabla_{b''} u_{c''} + \sigma K_{ac'} K_{bc''} h_c^{c'} u^{c''} \\
&= h_a^{a'} h_b^{b'} h_{cd} \nabla_{a'} \nabla_{b'} u^d - \sigma h_{cd} K_{ab} n^e \nabla_e u^d + \sigma K_{ac} K_{bd} u^d
\end{aligned}$$

where we used $h_a{}^b n_b = 0$ and $h_a{}^b u_b = u_a$, and that $n^b \nabla_a u_b = -u^b \nabla_a n_b$ since n and u are orthogonal, along with the definition of the extrinsic curvature $K_{ab} = h_a{}^c \nabla_c n_b$, and the defining property of a projector that $h_a{}^{a'} h_{a'}{}^{a''} = h_a{}^{a''}$.

But then, since the second term is symmetric in a, b:

$$\begin{aligned}
\bar{R}_{abcd}u^d &\equiv [D_a, D_b]u_c \\
&= D_a D_b u_c - D_b D_a u_c \\
&= h_a^{a'} h_b^{b'} h_{cd} [\nabla_{a'}, \nabla_{b'}] u^d + \sigma(K_{ac}K_{bd} - K_{bc}K_{ad})u^d \\
&= h_a^{a'} h_b^{b'} h_c^{c'} R_{a'b'c'd} u^d + \sigma(K_{ac}K_{bd} - K_{bc}K_{ad})u^d \\
&= h_a^{a'} h_b^{b'} h_c^{c'} R_{a'b'c'd'} h_d^{d'} u^d + \sigma(K_{ac}K_{bd} - K_{bc}K_{ad})u^d
\end{aligned}$$

and so, since u is an arbitrary vector in the tangent space of Σ, it must be that:

$$\bar{R}_{abcd} = h_a^{a'} h_b^{b'} h_c^{c'} h_d^{d'} R_{a'b'c'd'} + \sigma(K_{ac}K_{bd} - K_{bc}K_{ad})$$

In particular, if the containing manifold is flat, so that R vanishes, then:

$$\bar{R}_{abcd} = \sigma(K_{ac}K_{bd} - K_{bc}K_{ad})$$

and the Ricci scalar is:

$$\bar{R} = \bar{R}^{ab}{}_{ab} = \sigma\left(K_a^a K_b^b - K_a^b K_b^a\right) = \sigma\left(K^2 - K_{ab}K^{ab}\right)$$

where it simply doesn't matter if we use h or g to raise indices, since K is tangential to Σ and so therefore so is $\bar{R}$.

Gauss' theorema egregium

Above, we defined $\bar{R}$ as an *intrinsic* property of the hypersurface Σ, that is, we defined it only in terms of the connection D of the submanifold, and vectors belonging to the tangent space of Σ.

In case this is unclear, we may consider a coordinate system with one coordinate running parallel to the normal one-form n of Σ, and the other coordinates agreeing with some coordinate system defined on the submanifold Σ where they intersect with the hypersurface. (Such coordinate systems exist: Gaussian normal coordinates are one such coordinate system.) We will write $|_\Sigma$ to mean an equality between tensors that ignores the coordinate running perpendicular to Σ.

In any such coordinate system it is manifest that the induced connection D is just the usual Levi-Civita connection Δ_Σ of the submanifold Σ with metric g_Σ induced from the containing manifold. That is, if we specify:

$$g_\Sigma \equiv h|_\Sigma$$

Then, by a fundamental theorem, there is exactly one torsion-free connection compatible with g_Σ. But we already showed that D is compatible with h, and it's easy to show that D is also torsion-free. For u, v tangent to Σ we have:

$$\begin{aligned}
(D_u v - D_v u - [u, v])^b &= u^a D_a v^b - v^a D_a u^b - [u, v]^b \\
&= u^a h_{b'}^b h_a^{a'} \nabla_{a'} v^{b'} - v^a h_{b'}^b h_a^{a'} \nabla_{a'} u^{b'} - [u, v]^b \\
&= h_{b'}^b \left(u^a \nabla_a v^{b'} - v^a \nabla_a u^{b'}\right) - \left(u^a \partial_a v^b - v^a \partial_a u^b\right) \\
&= h_{b'}^b \left(u^a \nabla_a v^{b'} - v^a \nabla_a u^{b'} - \left(u^a \partial_a v^{b'} - v^a \partial_a u^{b'}\right)\right) \\
&= h_{b'}^b (\nabla_u v - \nabla_v u - [u, v])^{b'} \\
&= 0
\end{aligned}$$

Therefore:

$$\Delta_\Sigma = D|_\Sigma$$

and then, since Δ_Σ is an object intrinsic to the submanifold, so is the projected curvature tensor $\bar{R}$, or, more precisely:

$$R_\Sigma = \bar{R}|_\Sigma$$

Now, we also showed above that for a hypersurface embedded in a flat manifold, $\bar{R}$ is related to the extrinsic curvature K by the formula:

$$\bar{R}_{abcd} = \sigma(K_{ac}K_{bd} - K_{bc}K_{ad})$$

We now consider a surface in three-dimensional Euclidean space. We may choose coordinates x, y tangent to the surface, and then the quantity:

$$(R_\Sigma)_{xyxy} = \bar{R}_{xyxy} = K_{xx}K_{yy} - K_{xy}K_{xy} = \det K$$

is still an intrinsic quantity of the two-dimensional surface, though certainly not manifestly so!

But the determinant of a matrix is the product of its eigenvalues. And the eigenvalues of K are the principal curvatures of the surface. Therefore, the product of principal curvatures depends only on the intrinsic quantity $\bar{R}_{xyxy}$ and not on the embedding of the surface in the containing three-dimensional space, except via the induced metric.