Real Klein-Gordon field

Jaume Guasch

Departament de Física Quàntica i Astrofísica
Universitat de Barcelona

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Classical solution

Simplest representation: **scalar real** field: $\phi = \phi^*$

$\Rightarrow$ Real Klein-Gordon field

Lagrangian density: $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right)$

e.o.m.: $\partial_\mu \partial^\mu \phi + m^2 \phi = 0$

Solution:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} \left(a_{\boldsymbol{p}} e^{-ipx} + a^*_{\boldsymbol{p}} e^{ipx}\right) \quad ; \quad p^0 = E_p = \sqrt{\boldsymbol{p}^2 + m^2} \tag{1}$$

Normalization factor: arbitrary (but convenient).

The canonical conjugate momentum:

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} = \dot{\phi}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} (-iE_p) \left(a_{\boldsymbol{p}} e^{-ipx} - a^*_{\boldsymbol{p}} e^{ipx}\right) \tag{2}$$

Quantization: $\phi(x), \Pi(x) \to$ operators on some Hilbert space

$\phi(x) = \phi(t, x)$ usual quantization rules operators do not depend on time!

$\Rightarrow$ Small detour

Schrödinger & Heisenberg images

Schrödinger image of Quantum Mechanics

- Operators do not carry time-dependence
- all time-dependence is in the states

$$\mathcal{O}_S \quad ; \quad |A,t\rangle_S$$
$$i\frac{\mathrm{d}}{\mathrm{d}t}|A,t\rangle_S = H|A,t\rangle_S$$
$$|A,t\rangle_S = U|A,0\rangle_S \quad ; \quad U(t,t_0) = e^{-iH(t-t_0)}$$

- quantization:
  - ⇒ impose canonical commutation relation among pairs of conjugate coordinates and momenta

$$[q_i, p_j] = i\delta_{ij} \quad ; \quad [q_i, q_j] = 0 \quad ; \quad [p_i, p_j] = 0 \tag{3}$$

Field theory: $q_i \to \phi(x)$, but:

- q_i time independent
- $\phi(x) = \phi(t, \boldsymbol{x})$

Heisenberg image of Quantum Mechanics

- time evolution is transported to the operators

$$|A,t\rangle_H \equiv |A,0\rangle_S = U^\dagger|A,t\rangle_S$$

- (time-independent) operator in the Schrödinger image → operator in the Heisenberg image:

$$\mathcal{O}_H(t) \equiv U^\dagger \mathcal{O}_S U$$

- Probability amplitudes stay invariant:

$${}_S\langle B,t|\mathcal{O}_S|A,t\rangle_S = {}_S\langle B,0|U^\dagger\mathcal{O}_S U|A,0\rangle_S = {}_H\langle B|\mathcal{O}_H(t)|A\rangle_H$$

- time-evolution of the operator is:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}_H(t) = [\mathcal{O}_H, H] \qquad (\text{if } \partial_t\mathcal{O}_S = 0)$$

- Quantization rules (3) are true if:
  operators $q_i^H(t)$ and $p_j^H(t)$ are evaluated at the same time

equal-time-commutation relations

$$[q_i^H(t), p_j^H(t)] = i\delta_{ij} \quad ; \quad [q_i^H(t), q_j^H(t)] = 0 \quad ; \quad [p_i^H(t), p_j^H(t)] = 0 \qquad (4)$$

So the fields and momenta in eqs. (1), (2) are coordinates and momenta in the Heisenberg image, and we will need to impose equal-time-commutation relations for their quantization.

Quantum Mechanics Harmonic Oscillator

Hamiltonian of the one-dimensional harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \quad ; \quad p = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}$$

introduce operators:

$$\left.\begin{array}{rcl} a &=& \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right) \\ a^\dagger &=& \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} x &=& \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \\ p &=& -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger) \end{array}\right.$$

with the canonical quantization rules:

$$[x, p] = i\hbar \quad ; \quad [x, x] = [p, p] = 0 \quad \Rightarrow [a, a^\dagger] = 1 \quad ; \quad [a, a] = [a^\dagger, a^\dagger] = 0$$

we can write the Hamiltonian:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

and find the commutation relations:

$$[H, a^\dagger] = \hbar\omega a^\dagger \quad ; \quad [H, a] = -\hbar\omega a$$

$$H|\psi\rangle = E|\psi\rangle$$

Define new states: $a|\psi\rangle$ and $a^\dagger|\psi\rangle$: $\begin{cases} Ha|\psi\rangle & = (E - \hbar\omega)a|\psi\rangle \\ Ha^\dagger|\psi\rangle & = (E + \hbar\omega)a^\dagger|\psi\rangle \end{cases}$

$a/a^\dagger|\psi\rangle$ are Hamiltonian eigenstates with decreased/increased energy.

Define: vacuum $|0\rangle$

State of minimal energy, normalized $\langle 0|0\rangle = 1$

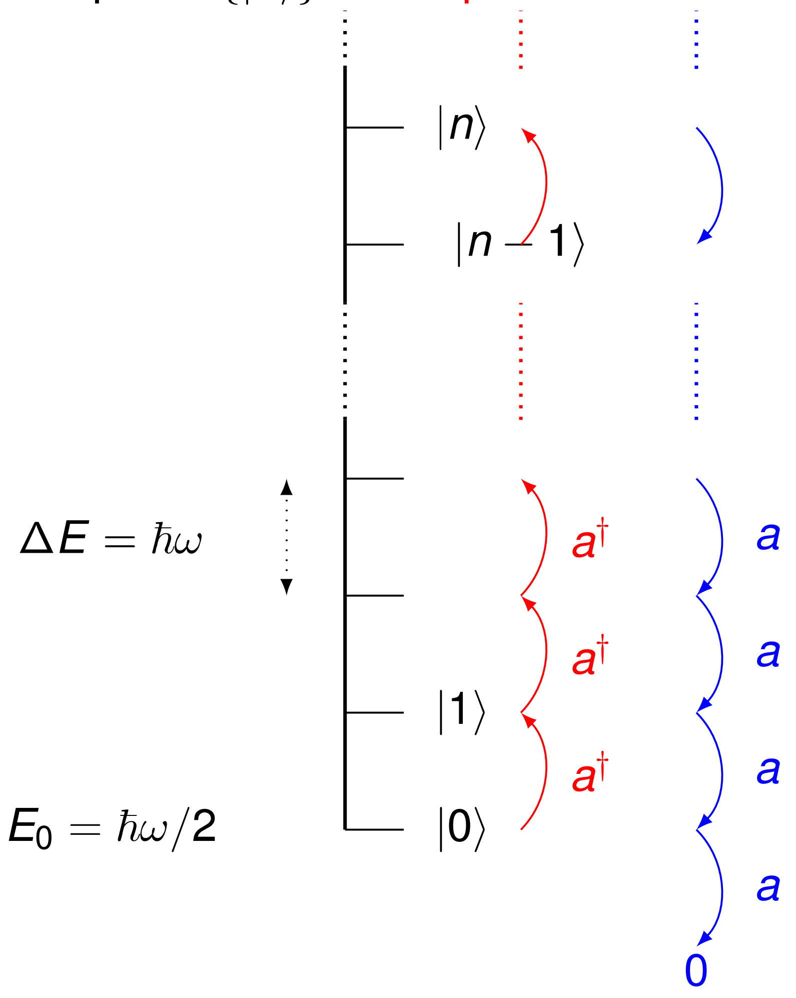
$$a|0\rangle = 0 \quad ; \quad a^\dagger|0\rangle \propto |1\rangle \text{ (one excited state)}$$

Normalized states:

$a|n\rangle = \sqrt{n}|n-1\rangle$; $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$; $a^\dagger a|n\rangle = n|n\rangle$; $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$

- $a^\dagger a = n$: number operator
- $|n\rangle$: eigenstates of the Hamiltonian with energy $E_n = \hbar\omega(n + 1/2)$
- $|0\rangle$ has non-zero energy $E_0 = \hbar\omega/2$

Hilbert space: $\{|n\rangle\}$ Fock space of the harmonic oscillator.



ladder operators

$a \equiv$ lowering, anihilation

$a^\dagger \equiv$ rising, creation

Quantum Hermitic Klein-Gordon field

- Quantization rules: $\phi(x) \to$ Hermitic Heisenberg Operator with

canonical equal-time-commutation (e.t.c.) rules

$$
\begin{aligned}
[\phi(t,\boldsymbol{x}), \Pi(t,\boldsymbol{y})] &= [\phi(t,\boldsymbol{x}), \dot{\phi}(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\
[\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})] &= 0 \\
[\Pi(t,\boldsymbol{x}), \Pi(t,\boldsymbol{y})] &= [\dot{\phi}(t,\boldsymbol{x}), \dot{\phi}(t,\boldsymbol{y})] = 0
\end{aligned}
$$

- $a_{\boldsymbol{p}}$, $a^*_{\boldsymbol{p}}$ in eq. (1) $\Rightarrow$ operators $a_{\boldsymbol{p}}$, $a^\dagger_{\boldsymbol{p}}$,
- Commutations rules: (See exercise sheet!)

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q}) \quad ; \quad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = [a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = 0$$

$\Rightarrow$ $a_{\boldsymbol{p}}$ follow same commutation rules as harmonic oscillator!

$\Rightarrow$ $a^\dagger_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}$: rising and lowering operators for an harmonic oscillator labeled by $\boldsymbol{p}$

$\Rightarrow$ $\phi(x)$ is a combination of **infinite** harmonic oscillators, for each possible value of $\boldsymbol{p}$.

define a vacuum state $|0\rangle$:

$$a_{\boldsymbol{p}}|0\rangle = 0 \quad \forall \boldsymbol{p}$$

physical states are constructed by successive application of rising operators:

$$a^\dagger_{\boldsymbol{p_1}} a^\dagger_{\boldsymbol{p_2}} a^\dagger_{\boldsymbol{p_3}} \cdots a^\dagger_{\boldsymbol{p_n}}|0\rangle$$

compute the **Hamiltonian** and the momentum:

$$
\begin{aligned}
H &= \int \mathrm{d}^3x\, \mathcal{H} = \int \mathrm{d}^3x \frac{1}{2}\left(\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) \\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_q}} \frac{1}{2} \times \Big\{ \\
&\quad -E_pE_q\left(a_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{-i(p+q)x} + a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{i(p+q)x} - a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{-i(p-q)x} - a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{i(p-q)x}\right) \\
&\quad -\boldsymbol{p}\cdot\boldsymbol{q}\left(a_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{-i(p+q)x} + a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{i(p+q)x} - a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{-i(p-q)x} - a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{i(p-q)x}\right) \\
&\quad +m^2\left(a_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{-i(p+q)x} + a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{i(p+q)x} + a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{-i(p-q)x} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{i(p-q)x}\right)\Big\}
\end{aligned}
$$

first integrate the x: $\int \mathrm{d}^3x\, e^{i\boldsymbol{kx}} = (2\pi)^3\delta^3(\boldsymbol{k})$

$$
\begin{aligned}
&\int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_q}} \frac{1}{2} \times \Big\{ \\
&-E_pE_q\Big(\{a_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{-i(E_p+E_q)t} + a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{i(E_p+E_q)t}\}(2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q}) \\
&\qquad -\{a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{-i(E_p-E_q)t} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{i(E_p-E_q)t}\}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\Big) \\
&-\boldsymbol{p}\cdot\boldsymbol{q}\Big(\{a_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{-i(E_p+E_q)t} + a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{i(E_p+E_q)t}\}(2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q}) \\
&\qquad -\{a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{-i(E_p-E_q)t} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{i(E_p-E_q)t}\}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\Big) \\
&+m^2\Big(\{a_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{-i(E_p+E_q)t} + a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{i(E_p+E_q)t}\}(2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q}) \\
&\qquad +\{a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}e^{-i(E_p-E_q)t} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{q}}e^{i(E_p-E_q)t}\}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\Big)\Big\}
\end{aligned}
$$

now integrate over $\boldsymbol{q} \Rightarrow \boldsymbol{p} = \pm\boldsymbol{q} \Rightarrow E_q = E_p$

$$
\begin{aligned}
H &= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\frac{1}{2}\times\Bigg\{ \\
&-E_p^2\left(a_{\mathbf{p}}a_{-\mathbf{p}}e^{-i(2E_p)t}+a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{i(2E_p)t}-a_{\mathbf{p}}a_{\mathbf{p}}^\dagger-a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \\
&-\mathbf{p}^2\left(-a_{\mathbf{p}}a_{-\mathbf{p}}e^{-i(2E_p)t}-a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{i(2E_p)t}-a_{\mathbf{p}}a_{\mathbf{p}}^\dagger-a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \\
&+m^2\left(a_{\mathbf{p}}a_{-\mathbf{p}}e^{-i(2E_p)t}+a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{i(2E_p)t}+a_{\mathbf{p}}a_{\mathbf{p}}^\dagger+a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right)\Bigg\} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\frac{1}{2}\times\Bigg\{ \\
&\cancelto{0}{(m^2+\mathbf{p}^2-E_p^2)}(a_{\mathbf{p}}a_{-\mathbf{p}}e^{-i(2E_p)t}+a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{i(2E_p)t}) \\
&+\cancelto{2E_p^2}{(m^2+\mathbf{p}^2+E_p^2)}(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger+a_{\mathbf{p}}^\dagger a_{\mathbf{p}})\Bigg\} \quad ; \quad [E_p^2=m^2+\mathbf{p}^2] \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\frac{1}{2}\times 2E_p^2(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger+a_{\mathbf{p}}^\dagger a_{\mathbf{p}})
\end{aligned}
$$

Hamiltonian

$$
H = \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_p(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger+a_{\mathbf{p}}^\dagger a_{\mathbf{p}})
$$

Linear momentum

$$
P_k=\int \mathrm{d}^3x\,\Pi(x)\partial_k\phi(x)=\cdots=\boxed{\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}p_k(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger+a_{\mathbf{p}}^\dagger a_{\mathbf{p}})=P_k}
$$

Commutators of $a_{\boldsymbol{q}}$, $a^\dagger_{\boldsymbol{q}}$ with H and P_k:

$$\begin{aligned}[a_{\boldsymbol{q}}, H] &= \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_p([a_{\boldsymbol{q}}, a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}] + [a_{\boldsymbol{q}}, a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}]) \\ &= \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_p([a_{\boldsymbol{q}}, a^\dagger_{\boldsymbol{p}}]a_{\boldsymbol{p}} + a_{\boldsymbol{p}}[a_{\boldsymbol{q}}, a^\dagger_{\boldsymbol{p}}]) \\ &= \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_p((2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})a_{\boldsymbol{p}} + a_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})) \\ &= \frac{1}{2}E_q(a_{\boldsymbol{q}} + a_{\boldsymbol{q}}) = E_q a_{\boldsymbol{q}}\end{aligned}$$

$$[a^\dagger_{\boldsymbol{q}}, H] = [H, a_{\boldsymbol{q}}]^\dagger = -[a_{\boldsymbol{q}}, H]^\dagger = -E_q a^\dagger_{\boldsymbol{q}}$$

by a similar procedure:

$$\begin{aligned}[a_{\boldsymbol{q}}, P_k] &= \cdots = q_k a_{\boldsymbol{q}} \\ [a^\dagger_{\boldsymbol{q}}, P_k] &= \cdots = -q_k a^\dagger_{\boldsymbol{q}}\end{aligned}$$

- Assume $|\psi\rangle$:

$$H|\psi\rangle = E|\psi\rangle \quad ; \quad \boldsymbol{P}|\psi\rangle = \boldsymbol{k}|\psi\rangle$$

- Build: $a^\dagger_{\boldsymbol{q}}|\psi\rangle$:

$$Ha^\dagger_{\boldsymbol{q}}|\psi\rangle = a^\dagger_{\boldsymbol{q}}H|\psi\rangle + [H, a^\dagger_{\boldsymbol{q}}]|\psi\rangle = a^\dagger_{\boldsymbol{q}}E|\psi\rangle + a^\dagger_{\boldsymbol{q}}E_q|\psi\rangle = (E + E_q)a^\dagger_{\boldsymbol{q}}|\psi\rangle$$

The energy rises by E_q!

$$P_i a^\dagger_{\boldsymbol{q}}|\psi\rangle = a^\dagger_{\boldsymbol{q}}P_i|\psi\rangle + [P_i, a^\dagger_{\boldsymbol{q}}]|\psi\rangle = a^\dagger_{\boldsymbol{q}}k_i|\psi\rangle + a^\dagger_{\boldsymbol{q}}q_i|\psi\rangle = (k_i + q_i)a^\dagger_{\boldsymbol{q}}|\psi\rangle$$

The linear momentum rises by q_i!

- $|\psi_1\rangle \equiv a^\dagger_{\boldsymbol{q}}|\psi\rangle$ is an eigenstate of H and $\boldsymbol{P}$,
- it has an energy and momentum equal to that of $|\psi\rangle$ plus the energy and momentum of a particle with momentum $\boldsymbol{q}$ and energy $E_q = \sqrt{m^2 + \boldsymbol{q}^2}$:

  $\Rightarrow$ $a^\dagger_{\boldsymbol{q}}$ creates a particle with mass m and momentum $\boldsymbol{q}$

- Build: $a_{\boldsymbol{q}}|\psi\rangle$

$$\begin{aligned} H a_{\boldsymbol{q}}|\psi\rangle &= (E - E_q) a_{\boldsymbol{q}}|\psi\rangle \\ P_i a_{\boldsymbol{q}}|\psi\rangle &= (k_i - q_i) a_{\boldsymbol{q}}|\psi\rangle \end{aligned}$$

$\Rightarrow$ $a_{\boldsymbol{q}}$ removes from $|\psi\rangle$ a particle with mass m, momentum $\boldsymbol{q}$, and energy $E_q = \sqrt{m^2 + \boldsymbol{q}^2}$.

Fock space

Fock space: Hilbert space of the quantum system.
Define the vacuum:

$$|0\rangle \text{ such that } a_{\boldsymbol{p}}|0\rangle = 0 \;\; \forall \boldsymbol{p} \text{ and } \langle 0|0\rangle = 1$$

- $a^\dagger_{\boldsymbol{k}}|0\rangle$ represents a state with 1 particle of momentum $\boldsymbol{k}$
- $a^\dagger_{\boldsymbol{k}} a^\dagger_{\boldsymbol{q}}|0\rangle$ represents a state with 1 particle of momentum $\boldsymbol{k}$ and 1 particle of momentum $\boldsymbol{q}$
- $\frac{1}{\sqrt{2!}} a^\dagger_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}}|0\rangle \Rightarrow$ 2 particles of momentum $\boldsymbol{k}$
- $\frac{1}{\sqrt{n!}} (a^\dagger_{\boldsymbol{k}})^n|0\rangle \Rightarrow$ n particles of momentum $\boldsymbol{k}$

since $[a^\dagger_{\boldsymbol{q}}, a^\dagger_{\boldsymbol{k}}] = 0$:

$$a^\dagger_{\boldsymbol{k}} a^\dagger_{\boldsymbol{q}}|0\rangle = a^\dagger_{\boldsymbol{q}} a^\dagger_{\boldsymbol{k}}|0\rangle \Rightarrow \text{symmetric under particle exchange} \Rightarrow \text{bosons}$$

commutation relations $\Longleftrightarrow$ bosons

- these states are not Lorentz-invariant. Normalization:

$$\langle 0|a_{\boldsymbol{q}} a^\dagger_{\boldsymbol{k}}|0\rangle = \langle 0|a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{q}}|0\rangle + \langle 0|[a_{\boldsymbol{q}}, a^\dagger_{\boldsymbol{k}}]|0\rangle = 0 + \langle 0|0\rangle (2\pi)^3 \delta^3(\boldsymbol{k}-\boldsymbol{q})$$

- Take a boost in the x^3 direction

$$\begin{aligned} x^{1\prime} &= x^1 &;\quad p^{1\prime} &= p^1 \\ x^{2\prime} &= x^2 &;\quad p^{2\prime} &= p^2 \\ x^{3\prime} &= \gamma(x^3 + \beta x^0) &;\quad p^{3\prime} &= \gamma(p^3 + \beta E) \\ x^{0\prime} &= \gamma(x^0 + \beta x^3) &;\quad E' &= \gamma(E + \beta p^3) \end{aligned}$$

together with: $\delta(f(x) - f(x_0)) = \dfrac{\delta(x - x_0)}{|\mathrm{d}f/\mathrm{d}x|_{x=x_0}}$

$$\begin{aligned} \delta^3(\boldsymbol{k}' - \boldsymbol{q}') &= \frac{\delta^3(\boldsymbol{k}-\boldsymbol{q})}{|\mathrm{d}p^{3\prime}/\mathrm{d}p^3|} = \frac{\delta^3(\boldsymbol{k}-\boldsymbol{q})}{\gamma(1 + \beta(\partial E/\partial p^3))} \\ \frac{\partial E}{\partial p^3} &= \frac{p^3}{\sqrt{\boldsymbol{p}^2 + m^2}} = \frac{p^3}{E} \\ \delta^3(\boldsymbol{k}' - \boldsymbol{q}') &= \frac{\delta^3(\boldsymbol{k}-\boldsymbol{q})}{\gamma(1 + \beta(p^3/E))} = E\frac{\delta^3(\boldsymbol{k}-\boldsymbol{q})}{\gamma(E + \beta p^3)} = \frac{E}{E'}\delta^3(\boldsymbol{k}-\boldsymbol{q}) \end{aligned}$$

$\Rightarrow E\delta^3(\boldsymbol{k}-\boldsymbol{q})$ is a Lorentz-invariant quantity.

We choose to normalize:

$$|\boldsymbol{p_1}, \boldsymbol{p_2} \cdots \boldsymbol{p_n}\rangle = \sqrt{2E_1}\sqrt{2E_2}\cdots\sqrt{2E_n}\, a^\dagger_{\boldsymbol{p_1}} a^\dagger_{\boldsymbol{p_2}} \cdots a^\dagger_{\boldsymbol{p_n}}|0\rangle$$

such that

$$\langle \boldsymbol{p}|\boldsymbol{q}\rangle = (2\pi)^3 \sqrt{2E_p}\sqrt{2E_q}\,\delta^3(\boldsymbol{p}-\boldsymbol{q})$$

$\Rightarrow$ We must divide by $2E_p$ in other places

Projector operator over 1-particle states:

$$\mathbb{1} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} |\boldsymbol{p}\rangle \frac{1}{2E_p} \langle \boldsymbol{p}|$$

- This factor is quite common, and it is Lorentz-invariant.
- Take the Lorentz-invariant 4-D integral

$$I_1 = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} (2\pi)\delta(p^2 - m^2)\Big|_{p^0>0}$$

where we are choosing the positive energy:

$$f(E) = E^2 - \boldsymbol{p}^2 - m^2 \ ; \ \frac{\mathrm{d}f}{\mathrm{d}E} = 2E \quad \text{at } p^2 = m^2 \ \rightarrow \ E = E_p = \sqrt{\boldsymbol{p}^2 + m^2}$$

$$I_1 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} \Leftarrow \text{Lorentz-invariant 3-momentum integration}$$

Energy and linear momentum

$$H|0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{E_p}{2} (a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}})|0\rangle$$

$a_{\boldsymbol{p}}|0\rangle = 0$, but not the second. Apply commutation rules:

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p (a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + \frac{1}{2}[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}])$$

- $n_{\boldsymbol{p}} = a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}$ counts the number of particles with momentum $\boldsymbol{p}$
- $[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p})$ (!!!!):

$$\lim_{\boldsymbol{p}\to\boldsymbol{q}} (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) = \lim_{\boldsymbol{p}\to\boldsymbol{q}} \int \mathrm{d}^3 x e^{-i(\boldsymbol{p}-\boldsymbol{q})\boldsymbol{x}} = \int \mathrm{d}^3 x \cdot 1$$

$$\equiv \text{Volume of space} \equiv V \ \ (\rightarrow \infty)$$

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p \left(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + \frac{1}{2} V \right)$$

Vacuum energy:

$$\langle 0|H|0\rangle = \int \frac{d^3p}{(2\pi)^3} E_p V \frac{1}{2} \equiv E_{vac}$$

- $E = \infty$ energy, because $V = \infty$
  $\Rightarrow$ not a big problem
- but the energy density:

$$\rho_{vac} = \frac{E_{vac}}{V} = \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2} = \infty(!!!)$$

- each harmonic oscillator has a vacuum energy $\frac{1}{2}\omega_p = \frac{1}{2}E_p$, and there are ∞ oscillators!!
- Since zero-point energy is not physical, only energy differences matter:
  - We can subtract the zero-point energy[1] and define a new Hamiltonian:

$$H' = H - E_{vac}$$

[1] unless we deal with gravity!!

The problem is

- in classical field theory, we had products of two fields, which are commuting:

$$A \cdot B = B \cdot A$$

  so we **did not** care on the order when we write ϕ^2, Π^2, $\phi\Pi$, etc.
- in quantum theory we have operators $\hat{A}$, $\hat{B}$ and:

$$\hat{A} \cdot \hat{B} \neq \hat{B} \cdot \hat{A}$$

  which is the *correct* order????
  $\Rightarrow$ it is not given!!!
- Classically:
  $\phi^2 = a^* a^* + aa + aa^* + a^* a = a^* a^* + aa + 2aa^* = a^* a^* + aa + 2a^* a$
- But quantum: $a^\dagger a + aa^\dagger \neq 2aa^\dagger \neq 2a^\dagger a$
  $\Rightarrow$ we need some rules

Definition: Wick ordering or Normal ordering

In a product $A = \prod a^\dagger \cdots a a^\dagger \cdots a^\dagger$ the normal-ordered or Wick-ordered product:

$$N(A) \equiv : A :$$

consists on putting all annihilation operators to the right-hand side:

$$: a a^\dagger a^\dagger a \cdots a a^\dagger \cdots a a a^\dagger a^\dagger := a^\dagger a^\dagger a^\dagger \cdots a^\dagger a a a \cdots a$$

Example:

$$: H :=: \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p (a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}}) := \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p \, a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}$$

now:

$$\langle 0 | : H : | 0 \rangle = 0$$

and

$$: H : | \boldsymbol{p_1 p_2} \cdots \boldsymbol{p_n} \rangle = (E_1 + E_2 + \cdots + E_n) | \boldsymbol{p_1 p_2} \cdots \boldsymbol{p_n} \rangle$$

We define all physics-related observables as normal-ordered operators:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} : \partial^\mu \phi \partial_\mu \phi - m^2 \phi^2 : \\
H &= \int \mathrm{d}^3 x : \frac{1}{2} (\Pi^2 + (\nabla \phi)^2 + m^2 \phi^2 := \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p \, a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \\
P_i &= \int \mathrm{d}^3 x : \Pi \partial_i \phi := \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p_i \, a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}
\end{aligned}$$

$$\begin{aligned}
\langle 0 | H | 0 \rangle &= 0 \\
H | \boldsymbol{p_1 p_2} \cdots \boldsymbol{p_n} \rangle &= (E_1 + E_2 + \cdots + E_n) | \boldsymbol{p_1 p_2} \cdots \boldsymbol{p_n} \rangle \\
P_i | \boldsymbol{p_1 p_2} \cdots \boldsymbol{p_n} \rangle &= (p_{i1} + p_{i2} + \cdots + p_{in}) | \boldsymbol{p_1 p_2} \cdots \boldsymbol{p_n} \rangle
\end{aligned}$$

Define: *positive* & *negative* energy part of the fields

$$\phi = \phi^+ + \phi^-$$

$$\phi^+ = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} a_{\boldsymbol{p}} e^{-ipx} \quad ; \quad \phi^- = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} a^\dagger_{\boldsymbol{p}} e^{ipx} \tag{5}$$

$$\begin{aligned} :\phi(x)\phi(y): &= :(\phi^+(x)+\phi^-(x))(\phi^+(y)+\phi^-(y)): \\ &= \phi^+(x)\phi^+(y)+\phi^-(x)\phi^-(y)+\underbrace{\phi^-(x)\phi^+(y)+\phi^-(y)\phi^+(x)} \end{aligned}$$

note the order!!

$\phi^+ \equiv a_{\boldsymbol{p}}$ to the right!

Interpretation of the quantum field $\phi(x)$

$$\begin{aligned} \phi(x)|0\rangle &= \phi^-(x)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} e^{ipx} a^\dagger_{\boldsymbol{p}}|0\rangle \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} e^{ipx} \frac{1}{\sqrt{2E_p}}|\boldsymbol{p}\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} e^{ipx}|\boldsymbol{p}\rangle \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} e^{iE_pt} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}|\boldsymbol{p}\rangle \end{aligned}$$

$\Rightarrow$ creation of a particle at (point $\boldsymbol{x}$, time t) $\equiv x \Rightarrow |x\rangle$

Note that:

$$
\begin{aligned}
\langle x|p\rangle &= \langle 0|\phi(x)|p\rangle = \langle 0|\int \frac{\mathrm{d}^3 q}{(2\pi)^3\sqrt{2E_q}}\left(a_{\boldsymbol{q}}e^{-iqx} + a_{\boldsymbol{q}}^\dagger e^{iqx}\right)\sqrt{2E_p}a_{\boldsymbol{p}}^\dagger|0\rangle \\
&= \langle 0|\int \frac{\mathrm{d}^3 q}{(2\pi)^3\sqrt{2E_q}}\, a_{\boldsymbol{q}}a_{\boldsymbol{p}}^\dagger e^{-iqx}\,\sqrt{2E_p}|0\rangle \\
&= \langle 0|\int \frac{\mathrm{d}^3 q}{(2\pi)^3\sqrt{2E_q}} e^{-iqx}\,\sqrt{2E_p}\,(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{q}} + [a_{\boldsymbol{q}}, a_{\boldsymbol{p}}^\dagger])|0\rangle \\
&= \langle 0|\int \frac{\mathrm{d}^3 q}{(2\pi)^3\sqrt{2E_q}} e^{-iqx}\,\sqrt{2E_p}\,(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})|0\rangle \\
&= e^{-ipx}\langle 0|0\rangle = e^{-iEt}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}
\end{aligned}
$$

$\equiv$ position-space representation of a single-particle wave function for definite momentum $\boldsymbol{p}$

On the other hand

$\phi^+(x)$ = annihilates a particle at point $x = (t, \boldsymbol{x})$