QUANTUM FIELD THEORY

Fall 2021

Superficial degree of divergence

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1) Let's think about Lagrangian Quantum Field Theory in d spacetime dimensions. Consider an interaction term with a coupling, involving b bosonic fields, f fermionic fields and k derivatives of any of these fields (b, f, k are either zero or positive integers).

- Write the mass dimension of the coupling in the interaction above, in terms of d, b, f, k (in units where $\hbar = c = 1$). **(1 point)**
- There is an integer dimension of spacetime d_*, such that for $d \geq d_*$, any of the couplings above are irrelevant. Find d_*. Your answer should be an integer (e.g. 10), not a formula. **(1 point)**

2) *Effective Lagrangian* Consider a 0+0 theory with two scalar fields, ϕ and χ, with action

$$S = \frac{1}{2}m^2\phi^2 + \frac{1}{2}M^2\chi^2 + \frac{\lambda}{2!2!}\phi^2\chi^2$$

- Write the corresponding Feynman rules and compute $\langle\phi^2\rangle_c$ up to order λ^2. **(1 point)**
- Now imagine that $M \gg m$. In this limit, experiments at energy smaller than M will not detect the heavy particles of χ. It makes sense to construct an *effective Lagrangian* in terms of ϕ only, but not χ. To do so in 0+0 dimensions, write the Euclidean path integral for this theory, and integrate out the χ field. After discarding a field-independent constant, you should arrive at an effective action that depends only on ϕ. Expanding it up to ϕ^4, it is of the form,

$$S_{eff} = \frac{1}{2}m_{eff}^2\phi^2 + \frac{\lambda_{eff}}{4!}\phi^4 + \ldots$$

  Write m_{eff} and λ_{eff} in terms of m, M, λ. **(1 point)**
- Compute again $\langle\phi^2\rangle_c$, now with the effective action S_{eff}. Expanding up to order λ^2, you should recover the result obtained above. **(1 point)**

This simple exercise contains many important lessons: experimentally, we only have access to S_{eff}, which has an infinite number of terms, increasingly suppressed by higher powers of M; in 4d, these would be irrelevant operators. If we measure only a finite number of terms of S_{eff}, we cannot uniquely determine S. On the other hand, S_{eff} is less aesthetically pleasant than S, but it is more efficient in computing $\langle\phi^2\rangle_c$: effective Lagrangians are useful!

3) *One-loop amplitudes in QED*. We have argued in class that some superficially divergent n-photon amplitudes in QED must actually be finite or even zero due to various symmetries. The goal of this assignment is to explicitly verify these claims at one-loop order for the four-photon amplitude. Note that you are not being asked to compute any integral.

- There are six one-loop diagrams contributing to the four-photon amplitude. Draw them. **(1 point)**
- Consider one of these diagrams. Write the corresponding integral in momentum space, and isolate the logarithmically divergent part of the integral $\mathcal{I}_{div}$. You can forget about the finite part of the integral for the rest of the assignment. **(1 point)**
- The divergent integral you isolated, $\mathcal{I}_{div}$, should have a product of components of the internal momentum k^μ in the numerator. The result of the integral over momentum (that you are not being asked to compute) has to preserve the tensor structure of this product of components of k^μ. The only tensor available after the integral over momentum is the Minkowski metric; argue that the symmetries of the integrand fix completely the particular combination of elements of the Minkowski metric that must appear in the answer. Finally, compute the trace over gamma matrices and find the tensor structure of $\mathcal{I}_{div}$.**(2 points)**
- Consider now the remaining five diagrams, and argue that the tensor structures of their divergent parts are such that when you add the six contributions, the divergent part of the full amplitude vanishes. **(1 point)**