Guillermo Abad

QFT: HW3

1)

a) first let's start with:

$$[S]=1 \longrightarrow \left[\int d^dx\, \mathcal{L}\right]=1 \longrightarrow [\mathcal{L}]=\left[\frac{1}{d^dx}\right]=M^d$$

so, it means:

$$[\lambda\, \phi^b \psi^f \partial^k]=M^d$$

to continue we need to know the units of our fields:

- fermions: $\mathcal{L}\propto \bar{\psi} m \psi$, so $M^d=[\bar{\psi} m \psi] \longrightarrow \boxed{[\psi]=M^{\frac{d-1}{2}}}$

- bosons: $\mathcal{L}\propto \begin{matrix}\bar{\phi} m^2 \phi \\ \bar{\phi}\partial^2\phi\end{matrix}$, so $M^d=[\bar{\phi} m^2 \phi] \longrightarrow \boxed{[\phi]=M^{\frac{d-2}{2}}}$

with this, we can continue, to obtain:

$$[\lambda][\phi^b][\psi^f][\partial^k]=M^d \quad ; \quad [\lambda]\left(M^{\frac{d-2}{2}}\right)^b\left(M^{\frac{d-1}{2}}\right)^f M^k=M^d$$

so:

$$\boxed{[\lambda]=M^{d-\left[\frac{d-2}{2}b+\frac{d-1}{2}f+k\right]}}$$

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b)

for $[\lambda]<1$ we need $d-\frac{d-2}{2}b-\frac{d-1}{2}f-k<0$

which can be written as:

$$d\left(1-\frac{b}{2}-\frac{f}{2}\right)+b+\frac{f}{2}-k<0$$

or

$$\left(\frac{b}{2}+\frac{f}{2}-1\right)d > b+\frac{f}{2}-k \longrightarrow \begin{cases} \frac{b}{2}+\frac{f}{2}-1>0 & (1) \\ \frac{b}{2}+\frac{f}{2}-1=0 & (2) \\ \frac{b}{2}+\frac{f}{2}-1<0 & (3) \end{cases}$$

• ① will happen when $b+f \geq 3$ which would be where all the couplings are, and where we are going to focus our atention first, later i'll also comment on ② and ③.

But for now, if $b+f \geq 3$ (couplings), then:

$$\left(\frac{b}{2}+\frac{f}{2}-1\right)d > b+\frac{f}{2}-K \longrightarrow d > \frac{b+\frac{f}{2}-K}{\frac{b}{2}+\frac{f}{2}-1}$$

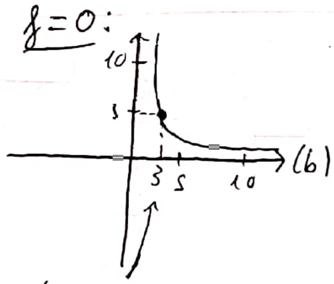
So now we only need to find the right hand side maximum:

$$d > \frac{b+\frac{f}{2}-K}{\frac{b}{2}+\frac{f}{2}-1} > \frac{b+f/2}{\frac{b}{2}+\frac{f}{2}-1} = 1 + \frac{\frac{b}{2}+1}{\frac{b}{2}+\frac{f}{2}-1}$$

Maximizing the function $\dfrac{\frac{b}{2}+1}{\frac{b}{2}+\frac{f}{2}-1}$ for b, being $b,f \in \mathbb{Z}$ with $b+f \geq 3$

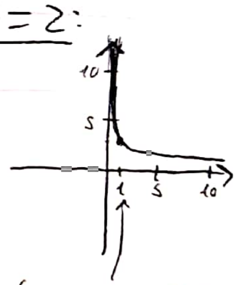
gives a maximum at $f=0$ and $b=3$

$f=0$:



(minimum possible $b=3$)
$[0+3 \geq 3]$

$f=2$:



(minimum posible $b=1$
$[1+2 \geq 3]$

The value of this maximum gives then, that:

$$\boxed{d >} 1 + \frac{\frac{3}{2}+1}{\frac{3}{2}-1} = 1 + \frac{\frac{5}{2}}{\frac{1}{2}} = 1+5 \boxed{= 6}$$

so $\boxed{d_* = 7}$, for which all coupling are irrelevant for $d \geq d_*$, $(d \geq 7)$.

Conditions for irrelevancy:

• Case ③ contains no couplings, only (4 —• or ϕ —•) $\begin{cases} b=0,\ f=1 \longrightarrow d < 2K-1 \longrightarrow \boxed{K=1 \text{ and } d=0} \\ b=1,\ f=0 \longrightarrow d < 2K-2 \longrightarrow \boxed{\text{can't be irrelevant, only can be marginal at most}} \end{cases}$

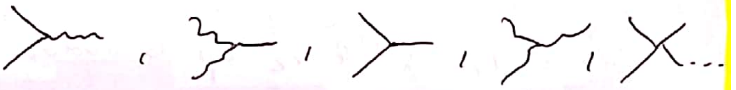
• Case ② contains the propagators term (4 —•— 4 or ϕ —•— ϕ) $\begin{cases} b=0,\ f=2 \\ b=2,\ f=0 \end{cases} \longrightarrow b+\frac{f}{2}-K < 0 \begin{array}{l} \longrightarrow 1 < K \leq 2\ \boxed{K=2} \\ \longrightarrow 2 < K \leq 2\ \boxed{\text{Only marginal}} \end{array}$

- Case ② also contains another term, the only possible exception, the $b=1, f=1$ term  , which would be irrelevant for $b + \frac{f}{2} - K \leq 0$, which means:

$$K > \frac{3}{2} \quad \text{and because } K \text{ is derivative from either}$$

$$\psi \text{ or } \phi \qquad K \leq b+f, \text{ so} \qquad \frac{3}{2} < K \leq 2 \longrightarrow \boxed{K=2}$$

so this "coupling", only would be irrelevant if we had both terms ψ and ϕ derived, independently of the dimension d. $\left(\begin{array}{l} \lambda (\partial\psi)(\partial\phi) \longrightarrow \underline{\text{irrelevant}} \\ \lambda \psi\phi \longrightarrow \underline{\text{relevant}} \end{array}, \text{independently of } d \right)$

If we don't consider this term a "coupling", and we only consider the normal couplings of $b+f \geq 3$ 

we find that $d_* = 7$, for which all couplings are irrelevant if $d \geq d_*$.

($d_* = 6$ if we include marginal relevance.)

Guillermo Abad López

2)

$$S = \frac{1}{2}m^2\phi^2 + \frac{1}{2}M^2\chi^2 + \frac{\lambda}{4}\phi^2\chi^2$$

a)

1. For each propagator: $\frac{1}{m^2}$ or $\frac{1}{M^2}$

2. For each vertex: $-\lambda$

3. For each source: j or l

4. Divide by symmetry factor

Computing $\langle\phi^2\rangle_c$ up to order λ^2:

χ field $\left(l \longrightarrow \frac{1}{M^2}\right)$

ϕ field $\left(j \longrightarrow \frac{1}{m^2}\right)$

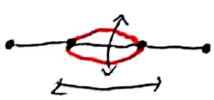
- Order λ^0:

$S=2 \longrightarrow \frac{j^2}{2m^2} \xrightarrow{\left(\frac{d^2}{dj^2}\right)} \boxed{\frac{1}{m^2}}$

- Order λ^1:

$S=4 \longrightarrow \frac{j^2(-\lambda)}{4m^4M^2} \longrightarrow \boxed{\frac{-\lambda}{2m^4M^2}}$

- Order λ^2:



$S=4 \longrightarrow \frac{j^2(-\lambda)^2}{4m^6M^4} \longrightarrow \frac{\lambda^2}{2m^6M^4}$

$S=2^3 \longrightarrow \frac{j^2(-\lambda)^2}{8m^6M^4} \longrightarrow \frac{\lambda^2}{4m^6M^4}$

$S=2^3 \longrightarrow \frac{j^2(-\lambda)^2}{8m^6M^4} \longrightarrow \frac{\lambda^2}{4m^6M^4}$

$$\left.\begin{matrix}\frac{\lambda^2}{2m^6M^4}\\ \frac{\lambda^2}{4m^6M^4}\\ \frac{\lambda^2}{4m^6M^4}\end{matrix}\right\} \boxed{\frac{\lambda^2}{m^6M^4}}$$

So finally $\boxed{\langle\phi^2\rangle_c = \frac{1}{m^2} - \frac{\lambda}{2m^4M^2} + \frac{\lambda^2}{m^6M^4}}$

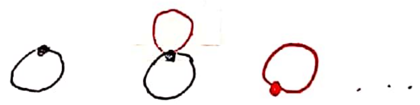
També podem calcular $\langle\phi^2\rangle_c$ amb el logaritme del quocient de les funcions de partició:

$$\boxed{\log\left(\frac{Z(\lambda,j,0)}{Z(\lambda,0,0)}\right) \overset{1}{=} \log\left(\frac{Z(\lambda,j,l)}{Z(\lambda,0,l)}\right)\Bigg|_{l=0} \overset{2}{=} \log\left(\frac{Z(\lambda,j,l)}{Z(\lambda,0,0)}\right)\Bigg|_{l=0} = \langle\phi^2\rangle_c}$$

where $Z(\lambda,j,l) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}m^2\phi - \frac{1}{2}M^2\chi^2 - \frac{\lambda}{4}\phi^2\chi^2 + j\phi + l\chi}\,\frac{d\phi\, d\chi}{2\pi}$

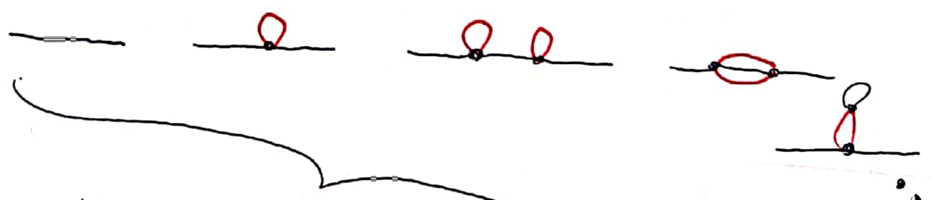
Explanation:

because we take the quocient between sets of diagrams to be left with the ones that interest us, and then the logarithm takes the connected ones, giving us only the j^2 terms:

$\log(Z(\lambda,0,0))$: 

$\log(Z(\lambda,j,0))$:  and 

$\log(Z(\lambda,j,l))$:  and 

so $\log\left(\frac{Z(\lambda,j,0)}{Z(\lambda,0,0)}\right)$: 

the ones we want!

and $\log\left(\frac{Z(\lambda,j,l)}{Z(\lambda,0,l)}\right) =$ 

(not the ones that only start l^2)

if we set $l=0 \longrightarrow$ the ones we want

③

So doing these computations, we get:

(M'havia deixat aquests factorials de l'exponencial Tomeu, però això no hem sortia!)

1.

$$\log\left(\frac{\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(-\frac{\lambda\phi^2x^2}{4}\right)^n e^{-\frac{m^2\phi^2}{2}-\frac{M^2x^2}{2}+j\phi}\,d\phi\,dx\,/n!}{\sum_{K=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(-\frac{\lambda\phi^2x^2}{4}\right)^K e^{-\frac{m^2\phi^2}{2}-\frac{M^2x^2}{2}}\,d\phi\,dx\,/K!}\right)=$$

$$=\frac{j^2}{2m^2}-\frac{j^2}{4(m^4M^2)}\lambda+\frac{j^4+8j^2m^2}{16m^8M^4}\lambda^2+O(\lambda^3)\ldots$$

which for j^2 terms give: $\left(\frac{d^2}{dj^2}(\,)\Big|_{j=0}\right)$

$$\boxed{\frac{1}{m^2}-\frac{\lambda}{2m^4M^2}+\frac{\lambda^2}{m^6M^4}+O(\lambda^3)\ldots}$$

✓✓ (The same!)

2.

$$\log\left(\frac{\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(-\frac{\lambda\phi^2x^2}{4}\right)^n e^{-\frac{m^2\phi^2}{2}-\frac{M^2x^2}{2}+j\phi+lx}\,d\phi\,dx\,/n!}{\sum_{K=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(-\frac{\lambda\phi^2x^2}{4}\right)^K e^{-\frac{m^2\phi^2}{2}-\frac{M^2x^2}{2}+lx}\,d\phi\,dx\,/K!}\right)\Bigg|_{l=0}=$$

$$=\frac{j^2}{2m^2}-\frac{j^2(l^2+M^2)}{4m^4M^4}\lambda+\frac{j^2(2l^4m^2+2l^2j^2M^2+16l^2m^2M^2+\ldots}{16m^8M^8}\ldots$$

$$\ldots\frac{j^2M^4+8m^2M^4}{}\lambda^2+O(\lambda^3)\Big|_{l=0}$$

which substituting $l=0$ and taking the j^2 terms we get:

$$\boxed{\frac{1}{m^2}-\frac{\lambda}{2m^4M^2}+\frac{\lambda^2}{m^6M^4}+O(\lambda^3)\ldots}$$

✓✓ (The same!)

Last comment, we see from the expressions how the terms always have the same units, and how if we lose a j^2 it's in favor of a m^2 or if we lose a l^2 it's in favor of a M^2 and viceversa. Giving all the diagrams we previously mentioned.

b)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(\frac{m^2}{2}\phi^2+\frac{M^2}{2}x^2+\frac{\lambda}{4}\phi^2x^2\right)}\frac{d\phi\,dx}{2\pi} = \int_{-\infty}^{\infty} e^{-\frac{m^2}{2}\phi^2}\int_{-\infty}^{\infty} e^{-\left(\frac{M^2}{2}+\frac{\lambda\phi^2}{4}\right)x^2}\frac{dx}{\sqrt{2\pi}}\frac{d\phi}{\sqrt{2\pi}} =$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{m^2}{2}\phi^2}}{\sqrt{M^2+\frac{\lambda\phi^2}{2}}}\frac{d\phi}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} e^{-L_{eff}(\phi)}\frac{d\phi}{\sqrt{2\pi}}$$

so our effective L_{eff} must be:

$$\boxed{L_{eff} = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\log\left(M^2+\frac{\lambda\phi^2}{2}\right) =}$$

$$\boxed{= \cancel{\log(M)} + \frac{1}{2}\underbrace{\left(m^2+\frac{\lambda}{2M^2}\right)}_{m_{eff}^2}\phi^2 - \underbrace{\left(\frac{\frac{3}{2}\frac{\lambda^2}{M^4}}{4!}\right)}_{\lambda_{eff}}\phi^4}$$

so our terms are:

- $\boxed{m_{eff} = \sqrt{m^2+\frac{\lambda}{2M^2}}}$
- $\boxed{\lambda_{eff} = -\frac{3}{2}\frac{\lambda^2}{M^4}}$

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c)

Let's compute again $\langle\phi^2\rangle_c$ with the effective L_{eff}:

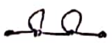
- **Order λ^0:** $\longrightarrow \frac{j^2}{2m_{eff}^2} \xRightarrow{\left(\frac{d^2}{dj^2}\right)} \boxed{\frac{1}{m_{eff}^2} = \frac{1}{m^2+\frac{\lambda}{2M^2}}}$

- **Order λ^1:**  $\longrightarrow \frac{j^2(-\lambda_{eff})}{4m_{eff}^6} \Longrightarrow \boxed{\frac{-\lambda_{eff}}{2m_{eff}^6} = \frac{\frac{3}{2}\frac{\lambda^2}{M^4}}{2\left(m^2+\frac{\lambda}{2M^2}\right)^3}}$

- **Order λ^2:**

 $\longrightarrow \frac{j^2\lambda_{eff}^2}{2\cdot 3!\, m_{eff}^{10}}$

 $\longrightarrow \frac{j^2\lambda_{eff}^2}{2^3 m_{eff}^{10}}$

 $\longrightarrow \frac{j^2\lambda_{eff}^2}{2^3 m_{eff}^{10}}$

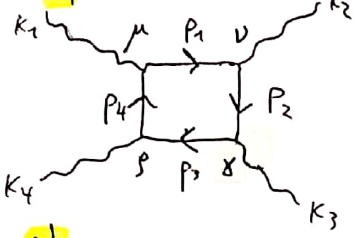
$\Longrightarrow \boxed{\frac{2}{3}\frac{\lambda_{eff}^2}{m_{eff}^{10}} = \frac{2\,\frac{9}{4}\frac{\lambda^4}{M^8}}{3\left(m^2+\frac{\lambda}{2M^2}\right)^5}}$

So finally: $\boxed{\langle\phi^2\rangle_c =} \frac{1}{m_{eff}^2} - \frac{\lambda}{2m_{eff}^6} + \frac{2}{3}\frac{\lambda_{eff}^2}{m_{eff}^{10}} + \ldots = \boxed{\frac{1}{m^2} - \frac{\lambda}{2M^2m^4} + \frac{\lambda^2}{M^4m^6} + \ldots}$

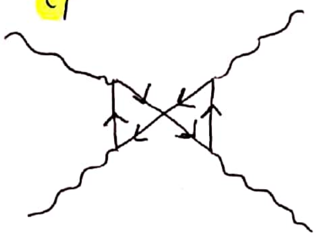
3)

a)

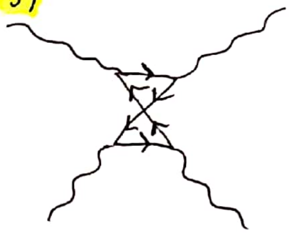
1)



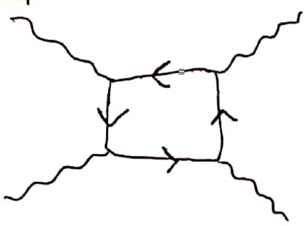
2)



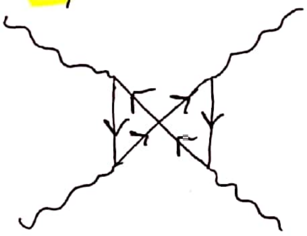
3)



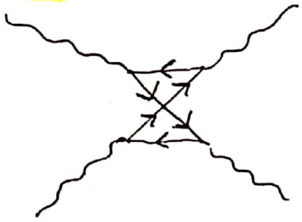
4)



5)



6)



b)

We will take the 1) diagram, for which the integral is:

$$\iiiint (-ie\gamma^\mu)\frac{i(\not p_1+m)}{p_1^2-m^2}(-ie\gamma^\nu)\frac{i(\not p_2+m)}{p_2^2-m^2}(-ie\gamma^\gamma)\frac{i(\not p_3+m)}{p_3^2-m^2}(-ie\gamma^\rho)\frac{i(\not p_4+m)}{p_4^2-m^2}\cdot$$

$$\cdot\frac{d^4p_1\, d^4p_2\, d^4p_3\, d^4p_4}{\cancel{(2\pi)^4(2\pi)^4(2\pi)^4(2\pi)^4}}\cdot\cancel{(2\pi)^4}^4\,\delta^4(p_1-k_1-p_4)\,\delta^4(p_2+k_2-p_1)\,\delta^4(p_3+k_3-p_2)\,\delta^4(p_4-k_4-p_3)=$$

$$=\delta(k_1+k_4-k_2-k_3)\,e^4\int\gamma^\mu\frac{\not p_1+m}{p_1^2-m^2}\gamma^\nu\frac{\not p_1-\not k_2+m}{(p_1-k_2)^2-m^2}\gamma^\gamma\frac{\not p_1-\not k_2-\not k_3+m}{(p_1-k_2-k_3)^2-m^2}\gamma^\rho\frac{\not p_1+\not k_1+m}{(p_1+k_1)^2-m^2}\,d^4p_1=$$

$$=\text{Finite part}+I_{div}$$

$$\left(\delta\left(\sum_f p_f-\sum_i p_i\right)=\delta(\,)=\delta(k_1+k_4-k_2-k_3)\right)$$

Where the $\boxed{I_{div}}=\delta(\,)e^4\int\frac{\gamma^\mu\not p_1\gamma^\nu\not p_1\gamma^\gamma\not p_1\gamma^\rho\not p_1}{(p_1^2)^4}d^4p_1=\boxed{\delta(\,)e^4\int\frac{Tr(\gamma^\mu\not p_1\gamma^\nu\not p_1\gamma^\gamma\not p_1\gamma^\rho\not p_1)}{(p_1^2)^4}d^4p_1}$

Which is $I_{div}\propto\int\frac{p_1^4}{p_1^8}d^4p_1\sim\int\frac{p_1^7}{p_1^8}dp_1\sim\int\frac{dp_1}{p_1}\rightarrow$ logarithmic divergence !!

Computing the expansions of $\frac{\frac{1}{x}+m}{(\frac{1}{x}+a)^2-m^2}$ when $x\to0$ $(p\to\infty)$, gives the same divergence on the integral as $\frac{\frac{1}{x}+0}{(\frac{1}{x}+0)^2-0^2}=x$, so we only need to keep the p_1 summands in the numerator and denominator $\not p_1-\not k_2+m\to\not p_1$, $(p_1-k_2)^2-m^2\to p_1^2$

c)

$$I_{div} = \delta() e^4 \int \frac{tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_1 \gamma^\gamma \not{p}_1 \gamma^3 \not{p}_1)}{(p_1^2)^4} d^4 p_1 =$$

$$= \delta() e^4 \, tr(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\gamma \gamma^\delta \gamma^3 \gamma^\tau) \int \frac{p_{1\alpha} p_{1\beta} p_{1\delta} p_{1\tau}}{(p_1^2)^4} d^4 p_1$$

And these $p_{1\alpha} p_{1\beta} p_{1\delta} p_{1\tau}$ are symmetric respect all the possible interchanges of indices, so the combination of g's they give must follow these symmetries too.

We have a "base" of the $\{g \cdot g\}_{\alpha\beta\delta\tau}$ given by 3 elements, the 3 diferent possible pairings between the indices. (We have 4 indices to make groups of 2, so that is pair α with either β, δ or τ and the other pairing is given $\rightarrow$ 3 possibilities)

The elements of this base are, then: $g_{\alpha\beta} g_{\delta\tau}$, $g_{\alpha\delta} g_{\beta\tau}$, $g_{\alpha\tau} g_{\beta\delta}$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}_{\alpha\ldots} \quad \begin{pmatrix}0\\1\\0\end{pmatrix}_{\alpha\ldots} \quad \begin{pmatrix}0\\0\\1\end{pmatrix}_{\alpha\ldots}$$

And if we change two indices an element can remain invariant $g_{\alpha\beta} g_{\delta\tau}$ with $\alpha \leftrightarrow \beta$, or can change with another element: $g_{\alpha\beta} g_{\delta\tau}$ with $\beta \leftrightarrow \delta$ $g_{\alpha\delta} g_{\beta\tau}$

$$g_{\alpha\delta} g_{\beta\tau} \updownarrow g_{\alpha\beta} g_{\delta\tau}.$$

So the only possible combination which is going to be symmetric under any index interchang will be a sum of all the elements of this base with the same factor in front, so when we change two elements, the sum stays the same:

$$\boxed{p_{1\alpha} p_{1\beta} p_{1\delta} p_{1\tau} \propto g_{\alpha\beta} g_{\delta\tau} + g_{\alpha\delta} g_{\beta\tau} + g_{\alpha\tau} g_{\beta\delta}} = \begin{pmatrix}1\\1\\1\end{pmatrix}_{\alpha,\beta,\delta,\tau \text{ base}}$$

Actually, we know from Peskin and Schrod. that in 4 dimensions we actually have:

$$\boxed{p_{1\alpha} p_{1\beta} p_{1\delta} p_{1\tau} = \frac{1}{4(4+2)} (p_1^2)^2 \left[g_{\alpha\beta} g_{\delta\tau} + g_{\alpha\delta} g_{\beta\tau} + g_{\alpha\tau} g_{\beta\delta} \right]}$$

Now we only need to compute the trace, which we actually are going to do with each element of the base in front to use contractions:

$(\gamma^\alpha \gamma^a \gamma_\alpha = -2\gamma^a)$

- **In front $\begin{pmatrix}1\\0\\0\end{pmatrix}_\alpha$:**

$$g_{\alpha\beta}\, g_{\delta\tau}\, tr(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\gamma \gamma^\delta \gamma^\rho \gamma^\tau) = Tr(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma_\alpha \gamma^\gamma \gamma^\delta \gamma^\rho \gamma_\delta) =$$

$$= 4\, Tr(\gamma^\mu \gamma^\nu \gamma^\gamma \gamma^\rho) = 16\, [\, g^{\mu\nu} g^{\gamma\rho} - g^{\mu\gamma} g^{\nu\rho} + g^{\mu\rho} g^{\nu\gamma}]$$

Which in a new base of the indices $\mu\nu\gamma\rho$, just as before can be expressed as $\boxed{16\begin{pmatrix}1\\-1\\1\end{pmatrix}_\mu}$

- **In front $\begin{pmatrix}0\\1\\0\end{pmatrix}_{\alpha..}$:**

$$g_{\alpha\delta}\, g_{\beta\tau}\, tr(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\gamma \gamma^\delta \gamma^\rho \gamma^\tau) = Tr(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\gamma \gamma_\alpha \gamma^\rho \gamma_\beta) =$$

$$= -2\, tr(\gamma^\mu \gamma^\gamma \gamma^\beta \gamma^\nu \gamma^\rho \gamma_\beta) = -8\, Tr(\gamma^\mu \gamma^\gamma g^{\nu\rho}) = -32\, g^{\mu\gamma} g^{\nu\rho}$$

$(\gamma^\alpha \gamma^\beta \gamma^\delta \gamma^\rho \gamma_\alpha = -2\gamma^\rho \gamma^\delta \gamma^\beta)$ $\quad$ $(\gamma^\alpha \gamma^\beta \gamma^\delta \gamma_\alpha = 4 g^{\beta\delta})$ $\quad$ $(Tr(\gamma^\mu \gamma^\gamma) = 4 g^{\mu\gamma})$

Which can be expressed as $\boxed{16 \cdot \begin{pmatrix}0\\-2\\0\end{pmatrix}_\mu}$

- **In front $\begin{pmatrix}0\\0\\1\end{pmatrix}_{\alpha..}$:**

is exactly as case 1)

$$g_{\alpha\tau}\, g_{\beta\delta}\, tr(\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\gamma \gamma^\delta \gamma^\rho \gamma^\tau) = \boxed{16\begin{pmatrix}1\\-1\\1\end{pmatrix}_{\mu..}}$$

So finally the total summ of $\begin{pmatrix}1\\0\\0\end{pmatrix}_\alpha + \begin{pmatrix}0\\1\\0\end{pmatrix}_\alpha + \begin{pmatrix}0\\0\\1\end{pmatrix}_\alpha$ which is $p_{1\alpha} p_{1\beta} p_{1\delta} p_{1\tau}$, gives:

$$\boxed{16\left[\begin{pmatrix}1\\-1\\1\end{pmatrix}_{\mu..} + \begin{pmatrix}0\\-2\\0\end{pmatrix}_{\mu..} + \begin{pmatrix}1\\-1\\1\end{pmatrix}_\mu\right] = 32\begin{pmatrix}1\\-2\\1\end{pmatrix}_{\mu..} = 32\left[g^{\mu\nu} g^{\gamma\rho} - 2 g^{\mu\gamma} g^{\nu\rho} + g^{\mu\rho} g^{\nu\gamma}\right]}$$

so:

$$I_{div} = \delta()\, e^4 \cdot \cancel{32}\left[g^{\mu\nu} g^{\gamma\rho} - 2 g^{\mu\gamma} g^{\nu\rho} + g^{\mu\rho} g^{\nu\gamma}\right] \frac{1}{\cancel{32}} \int \frac{\cancel{(p_1^2)^2}}{(p_1^2)^{4-2}}\, d^4 p_1$$

giving a final result for I_{div}:

$$\boxed{I_{div} = \delta(1)\, e^4 \int \frac{d^4p_1}{(p_1^2)^2}\left[g^{\mu\nu}g^{\gamma\rho} - 2g^{\mu\gamma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\gamma}\right] = \delta(1)\, e^4 \int \frac{d^4p_1}{p_1^4}\begin{pmatrix}1\\-2\\1\end{pmatrix}_{\mu\ldots}}$$

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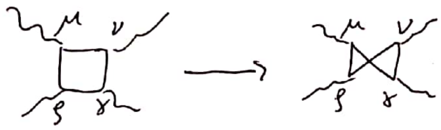
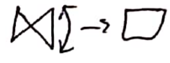
d)

Let's now consider diagrams 2) and 3), we can easily see that to go from:

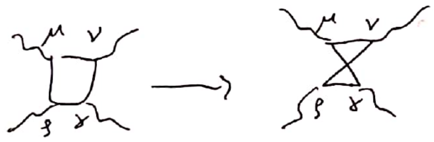


we only need to change $\nu \leftrightarrow \gamma$

(unwrapp it)

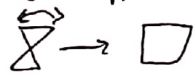


and to go from:



we only need to change $\mu \leftrightarrow \nu$

(unwrapp it)



This is the equivalent, to if we had written explicitly in the integral, the contributions from the external legs: $\varepsilon^{\mu}, \varepsilon^{*\nu} \ldots$ etc

And now we want to switch between them.

Well, if we do this, the integral goes to:

- $\nu \leftrightarrow \gamma$: $I_{div} = \delta(1)\, e^4 \int \frac{d^4p_1}{p_1^4}\left[g^{\mu\gamma}g^{\nu\rho} - 2g^{\mu\nu}g^{\gamma\rho} + g^{\mu\rho}g^{\gamma\nu}\right] = \delta(1)\, e^4 \int \frac{d^4p_1}{p_1^4}\begin{pmatrix}-2\\1\\1\end{pmatrix}_{\mu\ldots}$
- $\mu \leftrightarrow \nu$: $I_{div} = \delta(1)\, e^4 \int \frac{d^4p_1}{p_1^4}\left[g^{\nu\mu}g^{\gamma\rho} - 2g^{\nu\gamma}g^{\mu\rho} + g^{\nu\rho}g^{\mu\gamma}\right] = \delta(1)\, e^4 \int \frac{d^4p_1}{p_1^4}\begin{pmatrix}1\\1\\-2\end{pmatrix}_{\mu\ldots}$

So we easily see that adding diagrams 1) + 2) + 3) gives:

$$\delta(1)\, e^4 \int \frac{d^4p_1}{p_1^4}\cdot\left[\begin{pmatrix}1\\-2\\1\end{pmatrix}_{\mu\ldots} + \begin{pmatrix}-2\\1\\1\end{pmatrix}_{\mu\ldots} + \begin{pmatrix}1\\1\\-2\end{pmatrix}_{\mu\ldots}\right] = \begin{pmatrix}0\\0\\0\end{pmatrix}_{\mu\ldots} = \boxed{0}$$

same would happen for the sum of 4)+5)+6), they would cancel each other, giving 0 in total