Complex Klein-Gordon Field

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Quantization

- Same concepts as Real Klein-Gordon Field but: $\phi \neq \phi^\dagger$
  $\Rightarrow$ ϕ and $\phi^\dagger$ count as different degrees of freedom:
  $$\mathcal{L} =: \partial_\mu \phi(x) \partial^\mu \phi^\dagger(x) - m^2 \phi(x) \phi^\dagger(x) : \tag{1}$$
- NOTE normal ordering: $:\phi\phi^\dagger: = :\phi^\dagger\phi:$
- The Euler-Lagrange e.o.m.:
  $$\begin{aligned} \phi^\dagger &: \quad \partial^\mu \partial_\mu \phi + m^2 \phi = 0 \\ \phi &: \quad \partial^\mu \partial_\mu \phi^\dagger + m^2 \phi^\dagger = 0 \end{aligned}$$
- solution
  $$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} (a_{\boldsymbol{p}} e^{-ipx} + b^\dagger_{\boldsymbol{p}} e^{ipx}) \quad ; \quad p^0 = E_p = \sqrt{\boldsymbol{p}^2 + m^2}$$
  $a \neq b$ since the field is not hermitic:
  $$\phi^\dagger(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} (b_{\boldsymbol{p}} e^{-ipx} + a^\dagger_{\boldsymbol{p}} e^{ipx}) \quad ; \quad p^0 = E_p = \sqrt{\boldsymbol{p}^2 + m^2}$$

Conjugate momenta

$$\begin{aligned}\Pi_{\phi}(x) &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\phi^\dagger}{\partial x^0} = \dot{\phi}^\dagger \equiv \Pi(x) \\ \Pi_{\phi^\dagger}(x) &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}^\dagger} = \frac{\partial\phi}{\partial x^0} = \dot{\phi} \equiv \Pi^\dagger(x)\end{aligned}$$

Equal-time-commutation relations

$$\begin{aligned}[\phi(t,\boldsymbol{x}),\Pi(t,\boldsymbol{y})] &= [\phi(t,\boldsymbol{x}),\dot{\phi}^\dagger(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\ [\phi^\dagger(t,\boldsymbol{x}),\Pi^\dagger(t,\boldsymbol{y})] &= [\phi^\dagger(t,\boldsymbol{x}),\dot{\phi}(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\end{aligned}$$

2nd = hermitic-conjugate of 1st.

$$\begin{aligned}[\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})] &= [\phi^\dagger(t,\boldsymbol{x}),\phi^\dagger(t,\boldsymbol{y})] = [\phi(t,\boldsymbol{x}),\phi^\dagger(t,\boldsymbol{y})] = 0 \\ [\dot{\phi}(t,\boldsymbol{x}),\dot{\phi}(t,\boldsymbol{y})] &= [\dot{\phi}^\dagger(t,\boldsymbol{x}),\dot{\phi}^\dagger(t,\boldsymbol{y})] = [\dot{\phi}(t,\boldsymbol{x}),\dot{\phi}^\dagger(t,\boldsymbol{y})] = 0 \\ [\phi(t,\boldsymbol{x}),\dot{\phi}(t,\boldsymbol{y})] &= [\phi^\dagger(t,\boldsymbol{x}),\dot{\phi}^\dagger(t,\boldsymbol{y})] = 0\end{aligned}$$

We can compute:

$$
\begin{aligned}
[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] &= (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\\
[b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}] &= (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\\
[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] &= [a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = [b_{\boldsymbol{p}}, b_{\boldsymbol{q}}] = [b^\dagger_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}] = [a_{\boldsymbol{p}}, b_{\boldsymbol{q}}] = [a_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}] = 0
\end{aligned}
$$

$\Rightarrow$ **two independent harmonic oscillators** one with the *a*-operators, and the other with the *b*-operators.

Define the vacuum

$$
\left.\begin{aligned} a_{\boldsymbol{p}}|0\rangle = 0 \\ b_{\boldsymbol{p}}|0\rangle = 0 \end{aligned}\right\} \forall \boldsymbol{p}
$$

Fock space

Vectors created by application of the $a^\dagger_{\boldsymbol{p}}$, $b^\dagger_{\boldsymbol{p}}$ operators:

$$\left.\begin{array}{l} a^\dagger_{\boldsymbol{p}}|0\rangle = |1_{\boldsymbol{p}};0\rangle \\ b^\dagger_{\boldsymbol{p}}|0\rangle = |0;1_{\boldsymbol{p}}\rangle \end{array}\right\} \text{ create different kind of particles}$$

normalising as the real Klein-Gordon field:

$$\begin{aligned} |\boldsymbol{p}_1,\boldsymbol{p}_2,\dots\boldsymbol{p}_n;\boldsymbol{k}_1,\boldsymbol{k}_2,\dots\boldsymbol{k}_l\rangle &= \sqrt{2E_1}\sqrt{2E_2}\dots\sqrt{2E_n}\sqrt{2\omega_1}\sqrt{2\omega_2}\dots\sqrt{2\omega_l}\times \\ &\times a^\dagger_{\boldsymbol{p}_1}a^\dagger_{\boldsymbol{p}_2}\dots a^\dagger_{\boldsymbol{p}_n}b^\dagger_{\boldsymbol{k}_1}b^\dagger_{\boldsymbol{k}_2}\dots b^\dagger_{\boldsymbol{k}_l}|0\rangle \end{aligned}$$

$E_i = \sqrt{\boldsymbol{p}_i^2 + m^2}$, $\omega_i = \sqrt{\boldsymbol{k}_i^2 + m^2}$.
r particles in a given state:

$$|r_{\boldsymbol{p}};0\rangle = (\sqrt{2E_p})^r\frac{(a^\dagger_{\boldsymbol{p}})^r}{\sqrt{r!}}|0\rangle \quad ; \quad |0;r_{\boldsymbol{k}}\rangle = (\sqrt{2E_k})^r\frac{(b^\dagger_{\boldsymbol{k}})^r}{\sqrt{r!}}|0\rangle$$

The number operators: $n^a_{\boldsymbol{p}} = a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} \quad ; \quad n^b_{\boldsymbol{p}} = b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}$
count the number of a- and b-particles for each momentum $\boldsymbol{p}$.

$$
\begin{aligned}
H &= \int \mathrm{d}^3x \; :\Pi(x)\Pi^\dagger(x) + \partial_i\phi(x)\partial_i\phi^\dagger(x) + m^2\phi(x)\phi^\dagger(x): \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + b_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}}) \\
P_k &= \int \mathrm{d}^3x : \Pi(x)\partial_k\phi(x) + \Pi^\dagger(x)\partial_k\phi^\dagger(x) : \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3} p_k(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + b_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}})
\end{aligned}
$$

$\Rightarrow$ energy and the momentum is the same for both kind of particles.

$\Rightarrow$ What is different?

$U(1)$ symmetry

Complex Klein-Gordon lagrangian (1) has a $U(1)$ symmetry

$$\phi(x) \to e^{-i\alpha}\phi(x)$$

- conserved current

$$J^\mu = i : (\phi^\dagger(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^\dagger(x)) :$$

- conserved charge:

$$\begin{aligned} Q &= \int \mathrm{d}^3x J^0 = \int \mathrm{d}^3x : i(\phi^\dagger(x)\partial^0\phi(x) - \phi(x)\partial^0\phi^\dagger(x)) : \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} - b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}) = N_a - N_b \end{aligned}$$

$\Rightarrow$ *a*- and *b*-particles have **opposite** charge under the $U(1)$ transformation!

b-particles: **anti-particles** of the *a*-particles.
a-particles: **anti-particles** of the *b*-particles!

Commutators & propagators of the complex Klein-Gordon field

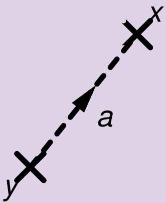
$\phi \neq \phi^\dagger \Rightarrow [\phi(x), \phi(y)] = [\phi^\dagger(x), \phi^\dagger(y)] = 0$
only contain $a + b^\dagger$ or $a^\dagger + b$, $\Rightarrow$ these combinations commute

Define:

positive and *negative* energy fields:

$$
\begin{aligned}
\phi^{+}(x) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} a_{\boldsymbol{p}} e^{-ipx} \\
\phi^{-}(x) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} b_{\boldsymbol{p}}^\dagger e^{ipx} \\
\phi^{\dagger +}(x) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} b_{\boldsymbol{p}} e^{-ipx} = (\phi^{-}(x))^\dagger \\
\phi^{\dagger -}(x) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} a_{\boldsymbol{p}}^\dagger e^{ipx} = (\phi^{+}(x))^\dagger
\end{aligned}
$$

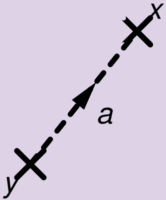
Particle *a* created at *y* and absorbed at *x*



$$
\begin{aligned}
& \langle 0|\phi(x)\phi^{\dagger}(y)|0\rangle \\
= \; & \langle 0|(\phi^{+}(x) + \cancel{\phi^{-}(x)})(\cancel{\phi^{\dagger +}(y)} + \phi^{\dagger -}(y))|0\rangle \\
= \; & \langle 0|\phi^{+}(x)\phi^{\dagger -}(y)|0\rangle \\
= \; & \langle 0|\phi^{\dagger -}(y)\phi^{+}(x) + [\phi^{+}(x), \phi^{\dagger -}(y)]|0\rangle \\
= \; & \langle 0|0\rangle \underbrace{[\phi^{+}(x), \phi^{\dagger -}(y)]} = \langle 0|0\rangle \Delta^{+}(x-y)
\end{aligned}
$$

Same expression than the corresponding real Klein-Gordon commutator in terms of *a*-operators

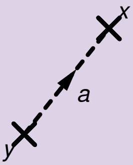
Anti-particle b created at y an absorbed at x

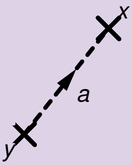


$$
\begin{aligned}
& \langle 0|\phi^\dagger(x)\phi(y)|0\rangle \\
= \; & \langle 0|(\phi^{\dagger +}(x) + \cancel{\phi^{\dagger -}(x)})(\cancel{\phi^{+}(y)} + \phi^{-}(y))|0\rangle \\
= \; & \langle 0|\phi^{\dagger +}(x)\phi^{-}(y)|0\rangle \\
= \; & \langle 0|\phi^{-}(y)\phi^{\dagger +}(x) + [\phi^{\dagger +}(x), \phi^{-}(y)]|0\rangle \\
= \; & \langle 0|0\rangle \quad \underbrace{[\phi^{\dagger +}(x), \phi^{-}(y)]}_{\text{Same as real K-G in } b} \quad = \langle 0|0\rangle \Delta^{+}(x-y)
\end{aligned}
$$

Commutator

$$
\begin{aligned}
[\phi(x),\phi^\dagger(y)] &= [\phi^+(x)+\phi^-(x),\phi^{\dagger+}(y)+\phi^{\dagger-}(y)] \\
&= \cancel{[\phi^+(x),\phi^{\dagger+}(y)]} + [\phi^+(x),\phi^{\dagger-}(y)] + [\phi^-(x),\phi^{\dagger+}(y)] + \cancel{[\phi^-(x),\phi^{\dagger-}(y)]} \\
&= [\phi^+(x),\phi^{\dagger-}(y)] + [\phi^-(x),\phi^{\dagger+}(y)] \\
&= [\phi^+(x),\phi^{\dagger-}(y)] - [\phi^{\dagger+}(y),\phi^-(x)] \\
&= \Delta^+(x-y) - \Delta^+(y-x) \\
&= \text{Propagation } a\ y \to x - \text{Propagation } b\ x \to y
\end{aligned}
$$

$=$ 

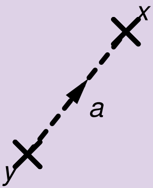
$-$ 

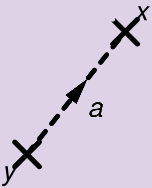
Micro-causality restored because of the cancellation of the first and second term in space-like intervals ($(x-y)^2 < 0$)

micro-causality exists thanks to anti-particles

The Feynman propagator

$$\begin{aligned}
\Delta_F(x-y) &= \langle 0|T\{\phi(x)\phi^\dagger(y)\}|0\rangle \\
&= \Theta(x^0-y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle
\end{aligned}$$

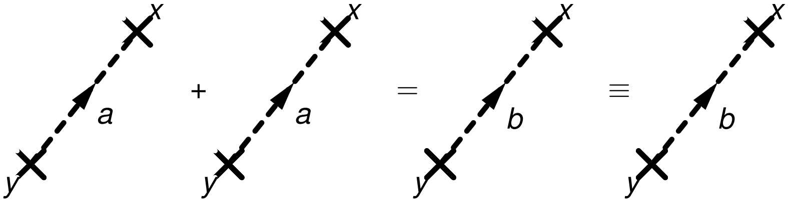
$=$ 

$+$ 

$$\begin{aligned}
&= \Theta(x^0-y^0)\langle 0|\phi^+(x)\phi^{\dagger -}(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\phi^{\dagger +}(y)\phi(x)^-|0\rangle \\
&= \Theta(x^0-y^0)\Delta^+(x-y) + \Theta(y^0-x^0)\Delta^+(y-x)
\end{aligned}$$

$\Rightarrow$ Same expression as for the real Klein-Gordon field

$$\Delta_F(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}e^{-ipx}$$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