6. QED for hadrons

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6.1 QED for composite particles. Form factors

- Hadrons are made out quarks, and quarks are spin 1/2 elementary particles
- If we have n elementary particles of charges q_j then

$$\mathcal{L}_{\rm QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_j^n \bar{\psi}_j (i\gamma^\mu D_\mu^j - m_j)\psi_j \quad , \quad D_\mu^j = \partial_\mu + i q_j A_\mu$$

  - The Lagrangian above is invariant under $\psi_j \to e^{i\theta_j}\psi_j$, $\theta_j \neq \theta_j(x)$, $\forall j = 1, \ldots, n \implies \partial_\mu j_j^\mu = 0 \implies$ there are n conserved charges $\implies$ each flavor j is conserved
  - The interaction Lagrangian reads

$$\mathcal{L}_I = \sum_j^n -q_j \bar{\psi}_j \gamma^\mu A_\mu \psi_j \equiv -\sum_j^n j_j^\mu A_\mu \equiv -j^\mu A_\mu \quad , \quad \partial_\mu j^\mu = 0$$

- How can we take into account that hadrons are made out of quarks in e.m. processes?
  - Cooking up some model (which one? How reliable is it going to be?)
  - Calculating in QCD (not known to us yet, difficult)
  - Parameterizing our ignorance in the most general way $\implies$ Form factors

Pion form factor

$e^- \, e^+ \to \pi^+ \, \pi^-$

- Suppose that in the *CoM* frame the electron energy $E_A \gtrsim 1$ GeV and we start seeing the pion structure
- Then we should substitute the e.m. current of elementary pions by the e.m. current of the quarks the pion is composed of

$$i\mathcal{M} \;=\; i^2 \int d^4x \;{}_\gamma\langle 0|\,\mathrm{T}\left\{A_\mu(0)A_\nu(x)\right\}|0\rangle_\gamma \;{}_e\langle f|\,j_e^\mu(0)\,|i\rangle_e \;{}_\pi\langle f|\,{j^\nu}_\pi(x)\,|i\rangle_\pi$$

$$j_\pi^\nu = -iq_\pi\left(\partial^\nu\phi^*\phi - \phi^*\partial^\nu\phi\right) \quad \longrightarrow \quad j_{QCD}^\nu = \sum_{j=u,d,\dots} j_j^\mu$$

$$|i\rangle_e = |\vec{p}_A\lambda_A\,;\vec{p}_B\lambda_b\rangle_e \quad , \quad |i\rangle_\pi = |0\rangle_\pi \quad , \quad |f\rangle_e = |0\rangle_e \quad , \quad |f\rangle_\pi = |\vec{p}_1\,;\vec{p}_2\rangle_\pi$$

- The problem now is that we do not know how to calculate

$${}_\pi\langle 0|\,{j^\nu}_{QCD}(x)\,|\vec{p}_1\,;\vec{p}_2\rangle_\pi$$

- Let us parameterize it using space-time translation invariance, Lorentz symmetry and current conservation

- Space-time translation invariance implies (P^μ = momentum operator)

$$ {}_\pi\langle\vec{p}_1\,;\vec{p}_2|\, j^\nu_{QCD}(x)\,|0\rangle_\pi = {}_\pi\langle\vec{p}_1\,;\vec{p}_2|\, e^{-iP.x} j^\nu_{QCD}(0)\, e^{iP.x}\,|0\rangle_\pi = e^{-i(p_1+p_2).x}\; {}_\pi\langle\vec{p}_1\,;\vec{p}_2|\, j^\nu_{QCD}(0)\,|0\rangle_\pi $$

- Lorentz invariance implies

$$ {}_\pi\langle\vec{p}_1\,;\vec{p}_2|\, j^\nu_{QCD}(0)\,|0\rangle_\pi = e\,(p_2+p_1)^\nu \tilde{F}_\pi(s) + e\,(p_2-p_1)^\nu F_\pi(s) $$

  Recall that the only scalars I can make are: $p_1^2 = m_\pi^2$, $p_2^2 = m_\pi^2$ and $p_1 p_2 = (s-2m_\pi^2)/2$

- Current conservation implies

$$ 0 = (p_1+p_2)_\nu\; {}_\pi\langle\vec{p}_1\,;\vec{p}_2|\, j^\nu_{QCD}(x)\,|0\rangle_\pi = 2(p_1+p_2)^2 \tilde{F}_\pi(s) + 0 \quad \Longrightarrow \quad \tilde{F}_\pi(s) = 0 $$

- $F_\pi(s) \equiv$ pion e.m. form factor
  - At low energy the pion looks like elementary $\Longrightarrow$ $F_\pi(s) \simeq 1$ if $s \gtrsim 4m_\pi^2$
- For this process, our ignorance on how pions are made out of quarks is encoded in a single function $F_\pi(s)$
  - It can be measured experimentally
  - It may eventually be calculated in QCD

Proton form factors

$e^- \, p \to e^- \, p$

- For elementary protons, we can just take the results from $e^- \, \mu^- \to e^- \, \mu^-$ and substitute $\mu \to p$

$$i\mathcal{M} \quad = \quad i^2 \int d^4x \; {}_\gamma \langle 0 | \mathrm{T}\{A_\mu(0) A_\nu(x)\} |0\rangle_\gamma \; {}_e\langle f | j_e^\mu(0) | i\rangle_e \; {}_p\langle f | j_p^\nu(x) | i\rangle_p$$

$$|i\rangle_e = |\vec{p}_A \lambda_A ;\rangle_e \quad , \quad |i\rangle_p = |\vec{p}_B \lambda_B ;\rangle_p \quad , \quad |f\rangle_e = |\vec{p}_1 \lambda_1 ;\rangle_e \quad , \quad |f\rangle_p = |\vec{p}_2 \lambda_2 ;\rangle_p$$

- For composite proton we have

$$j_p^\nu(x) = q_p \bar{\psi}_p(x) \gamma^\nu \psi_p(x) \quad \longrightarrow \quad j_{QCD}^\nu(x) = \sum_{j=u,d,\dots} j_j^\mu = \sum_{j=u,d,\dots} q_j \bar{\psi}_j(x) \gamma^\nu \psi_j(x)$$

- We have to parameterize

$${}_p\langle \vec{p}_2 \lambda_2 ;| \, j_{QCD}^\nu(x) \, | \vec{p}_B \lambda_B ;\rangle_p$$

  - Space-time translation invariance implies

$$\begin{aligned} {}_p\langle \vec{p}_2 \lambda_2 ;| \, j_{QCD}^\nu(x) \, | \vec{p}_B \lambda_B ;\rangle_p &= {}_p\langle \vec{p}_2 \lambda_2 ;| \, e^{-iP.x} j_{QCD}^\nu(0) e^{iP.x} \, | \vec{p}_B \lambda_B ;\rangle_p \\ &= e^{-i(p_2 - p_B).x} \, {}_p\langle \vec{p}_2 \lambda_2 ;| \, j_{QCD}^\nu(0) \, | \vec{p}_B \lambda_B ;\rangle_p \end{aligned}$$

- Lorentz invariance implies

$$_p\langle \vec{p}_2\lambda_2\,;|\, j^{\nu}_{QCD}(0)\,|\vec{p}_B\lambda_B\,;\rangle_p = \bar{u}_{\lambda_2}(\vec{p}_2)D^{\nu}(p_2\,,p_B)u_{\lambda_B}(\vec{p}_B)$$

$D^{\nu}(p_2\,,p_B)$ is the most general combination of Dirac matrices compatible with the discrete symmetries

$$\begin{aligned} D^{\nu}(p_2\,,p_B) \quad = \quad & (p_2+p_B)^{\nu}D_+ + (p_2-p_B)^{\nu}D_- + \gamma^5(p_2+p_B)^{\nu}D_{5+} + \\ & \gamma^5(p_2-p_B)^{\nu}D_{5-} + \gamma^{\nu}D_g + \gamma^5\gamma^{\nu}D_{5g} + \\ & \sigma^{\nu\rho}(p_2+p_B)_{\rho}D_{s+} + \sigma^{\nu\rho}(p_2-p_B)_{\rho}D_{s-} \end{aligned}$$

All the Ds on the r.h.s. are scalar functions of $t=(p_2-p_B)^2=q^2$

- Parity invariance implies

$$_p\langle \vec{p}_2\lambda_2\,;|\, j^{\nu}_{QCD}(0)\,|\vec{p}_B\lambda_B\,;\rangle_p =_p \langle -\vec{p}_2\lambda_2\,;|\, j_{QCD\,\nu}(0)\,|-\vec{p}_B\lambda_B\,;\rangle_p$$

$$\begin{aligned} \Longrightarrow \quad & \bar{u}_{\lambda_2}(\vec{p}_2)D^{\nu}(p_2\,,p_B)u_{\lambda_B}(\vec{p}_B) = \bar{u}_{\lambda_2}(-\vec{p}_2)D_{\nu}(\tilde{p}_2\,,\tilde{p}_B)u_{\lambda_B}(-\vec{p}_B) = \\ & = \bar{u}_{\lambda_2}(\vec{p}_2)\gamma^0 D_{\nu}(\tilde{p}_2\,,\tilde{p}_B)\gamma^0 u_{\lambda_B}(\vec{p}_B) \\ & \Longrightarrow \quad D^{\nu}(p_2\,,p_B) = \gamma^0 D_{\nu}(\tilde{p}_2\,,\tilde{p}_B)\gamma^0 \\ & \Longrightarrow \quad D_{5+} = D_{5-} = D_{5g} = 0 \end{aligned}$$

- Current conservation implies

$$(p_2 - p_B)_{\nu\, p} \langle \vec{p}_2 \lambda_2 \,;| \, j^{\nu}_{QCD}(0) \, | \vec{p}_B \lambda_B \,; \rangle_p = 0$$

$$\Longrightarrow \quad \bar{u}_{\lambda_2}(\vec{p}_2)(p_2 - p_B)_\nu D^\nu(p_2\,, p_B) u_{\lambda_B}(\vec{p}_B) = 0$$

$$\Longrightarrow \quad D_- = D_{s+} = 0$$

Note that

$$\bar{u}_{\lambda_2}(\vec{p}_2)(p_2 - p_B)_\nu \gamma^\nu u_{\lambda_B}(\vec{p}_B) = \bar{u}_{\lambda_2}(\vec{p}_2)(\not{p}_2 - \not{p}_B) u_{\lambda_B}(\vec{p}_B) = \bar{u}_{\lambda_2}(\vec{p}_2)(m_p - m_p) u_{\lambda_B}(\vec{p}_B) = 0$$

- The three terms that remain are not independent because the following equality holds

$$\bar{u}_{\lambda_2}(\vec{p}_2)\gamma^\nu u_{\lambda_B}(\vec{p}_B) = \bar{u}_{\lambda_2}(\vec{p}_2) \left(\frac{(p_2 + p_B)^\nu}{2m_p} + \frac{i\sigma^{\nu\rho}(p_2 - p_B)_\rho}{2m_p} \right) u_{\lambda_B}(\vec{p}_B)$$

- Hence only two independent terms remain, which are conventionally choosen as follows,

$$D^\nu(p_2\,, p_B) = e\, F_1(q^2)\gamma^\nu + \frac{e}{2m_p} F_2(q^2) i\sigma^{\nu\rho} q_\rho$$

- At low energy E_A, $E_B \lesssim 1$ GeV (in the *CoM* frame) we may consider the proton as elementary
    - ★ If we assumed that the proton is then described by QED, then for $q^2 \simeq 0 \implies F_1(q^2) \simeq 1,\ F_2(q^2) \simeq 0$
    - ★ However, the magnetic moment of the proton differs from the one of the Dirac equation minimally coupled to the e.m. field, like in QED
    - ★ This can be fixed by adding to the QED Lagrangian a non-minimal term

$$\delta\mathcal{L} = \frac{e\,\kappa_p}{2m_p}\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi \quad , \quad \kappa_p = \frac{g_p}{2} - 1 \simeq 1.79 \, ,$$

where $g_p \simeq 5.58$ is the proton gyromagnetic factor.

    - ★ Then for $q^2 \simeq 0 \implies F_1(q^2) \simeq 1,\ F_2(q^2) \simeq \kappa_p$

- Note that the same discussion holds for any spin 1/2 hadron
    - For instance, for the neutron, we have
        - ★ For $q^2 \simeq 0 \implies F_1(q^2) \simeq 0,\ F_2(q^2) \simeq \kappa_n = g_n/2 \simeq -1.91$
- Note that the $\delta\mathcal{L}$ above is the only dimension 5 operator that can be added to the QED Lagrangian that respects the gauge, Lorentz and discrete symmetries

$e^- \, p \to e^- \, p$ (unpolarized)

- For unpolarized beams and measurements, we need not parameterize ${}_p\langle \vec{p}_2\lambda_2\,;| \, j^{\nu}_{QCD}(x)\, |\vec{p}_B\lambda_B\,;\rangle_p$ but only

$$e^2\, L_p^{\mu\nu}(p_B\,,p_2) = \frac{1}{2}\sum_{\lambda_B=+,-}\sum_{\lambda_2=+,-} {}_p\langle \vec{p}_2\lambda_2\,;| \, j^{\mu}_{QCD}(0)\, |\vec{p}_B\lambda_B\,;\rangle_p \; {}_p\langle \vec{p}_2\lambda_2\,;| \, j^{\nu}_{QCD}(0)\, |\vec{p}_B\lambda_B\,;\rangle_p^*$$

- This is a much simpler task:
  - We have already used space-time translations to remove the x-dependence of the currents
  - Lorentz and parity invariance imply

$$L_p^{\mu\nu}(p_B\,,p_2) = -2m_p^2 G_1 g^{\mu\nu} + G_2 p^{\mu}p^{\nu} + G_3 q^{\mu}q^{\nu} + G_4(p^{\mu}q^{\nu} + q^{\mu}p^{\nu})$$

$p \equiv p_B + p_2$, $q = p_2 - p_B$, $G_i = G_i(q^2)$, $i = 1, \dots, 4$

  - Current conservation implies

$$0 = q_{\mu}L_p^{\mu\nu}(p_B\,,p_2) = -2m_p^2 G_1 q^{\nu} + 0 + G_3 q^2 q^{\nu} + G_4(0 + q^2 p^{\nu})$$

$$\Longrightarrow \quad G_4 = 0 \quad , \quad G_3 = \frac{2m_p^2}{q^2}G_1$$

  - Hence

$$L_p^{\mu\nu}(p_B\,,p_2) = 2m_p^2 G_1\left(-g^{\mu\nu} + \frac{q^{\mu}q^{\nu}}{q^2}\right) + G_2 p^{\mu}p^{\nu}$$

- As expected, it depends on two arbitrary functions G_1 and G_2
- For an elementary proton minimally coupled to the e.m. field

$$G_1 = -\frac{q^2}{2m_p^2} \quad , \quad G_2 = 1$$

- G_1 and G_2 can be calculated in terms of F_1 and F_2:

$$G_1 = -\frac{q^2}{2m_p^2}(F_1 + F_2)^2 \quad , \quad G_2 = F_1^2 - \frac{q^2}{4m_p^2}F_2^2$$

but we do not need it for unpolarized experiments

- We have

$$|\overline{\mathcal{M}}|^2 = \frac{e^4 L_e^{\mu\nu} L_{\mu\nu\, p}}{q^4}$$

$$L_e^{\mu\nu}(p_A\, , p_1) = -q^2\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + (p_A + p_1)^\mu (p_A + p_1)^\nu \quad , \quad q = p_A - p_1 = p_2 - p_B$$

$$\begin{aligned} L_e^{\mu\nu} L_{\mu\nu\, p} \;=\; & G_1\left(-6m_p^2 q^2 - 2m_p^2 (p_1 + p_A)^2\right) + G_2\left(((p_1 + p_A).(p_2 + p_B))^2 + \right. \\ & \left. + q^2 (p_2 + p_B)^2\right) \end{aligned}$$

- Early electron proton scattering experiments were carried out at fixed target (LAB frame)

$$L_e^{\mu\nu} L_{\mu\nu\, p}\big|_{\rm LAB} = 16 m_p^2 E_A E_1 \left(G_1 \sin^2\frac{\theta}{2} + G_2 \cos^2\frac{\theta}{2} \right)$$

- Lets us first analyse electron muon scattering in the LAB frame

- For the differential cross section, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB}=\frac{\alpha^2 E_1}{4E_A^3\sin^4\frac{\theta}{2}}\left(G_2\cos^2\frac{\theta}{2}+G_1\sin^2\frac{\theta}{2}\right)$$

  - In principle $G_1=G_1(q^2)$ and $G_2=G_2(q^2)$ may be obtained from the experiment
  - In practise, they are often writen as

$$G_2=\frac{G_E^2-\frac{q^2}{4m_p^2}G_M^2}{1-\frac{q^2}{4m_p^2}}\quad ,\quad G_1=-\frac{q^2}{2m_p^2}G_M^2$$

    G_E and G_M are called the electric and magnetic form factors respectively
  - For an elementary proton minimally coupled to the e.m. field $G_E=G_M=1$
  - The experimental results lead to

$$G_E\simeq\frac{1}{1+\frac{Q^2}{Q_0^2}}\quad ,\quad G_M\simeq\frac{g_p}{2}G_E\quad ,\quad Q_0\simeq 0.71\,\mathrm{GeV}\quad ,\quad Q^2\equiv -q^2$$

6.2 Deep inelastic scattering (DIS)

$e^- \, p \to e^- + \text{hadrons}$

- We can still use the formula

$$i\mathcal{M} = i^2 \int d^4x \; {}_\gamma\langle 0|\, \mathrm{T}\left\{A_\mu(0)A_\nu(x)\right\} |0\rangle_\gamma \; {}_e\langle f|\, j_e^\mu(0)\, |i\rangle_e \; {}_p\langle f|\, j_{QCD}^\nu(x)\, |i\rangle_p$$

$$|i\rangle_e = |\vec{p}_A\lambda_A\,;\rangle_e \quad , \quad |i\rangle_p = |\vec{p}_B\lambda_B\,;\rangle_p \quad , \quad |f\rangle_e = |\vec{p}_1\lambda_1\,;\rangle_e \quad , \quad |f\rangle_p$$

- The final states is left unspecified $(|f\rangle_p \sim |p\rangle \, , |p\,\pi^0\rangle \, , |p\,\pi^+\,\pi^-\rangle \, , |p\,\bar{p}\,p\rangle \, , \ldots$
- The subscript $_p$ stands now not only for the proton Fock subspace but for the whole hadronic Fock subspace
- Space-time translation invariance implies

$$\begin{aligned} {}_p\langle f|\, j_{QCD}^\nu(x)\, |\vec{p}_B\lambda_B\,;\rangle_p &= {}_p\langle f|\, e^{-iP.x} j_{QCD}^\nu(0) e^{iP.x}\, |\vec{p}_B\lambda_B\,;\rangle_p \\ &= e^{-i(p_f - p_B).x}\, {}_p\langle f|\, j_{QCD}^\nu(0)\, |\vec{p}_B\lambda_B\,;\rangle_p \end{aligned}$$

- Current conservation implies

$$(p_f - p_B)_\nu \, {}_p\langle f|\, j_{QCD}^\nu(0)\, |\vec{p}_B\lambda_B\,;\rangle_p = 0$$

$p_f - p_B = p_A - p_1 = q$

- Let us assume that we only measure the momenum of the outgoing electron and none of the properties of the outgoing hadrons
- Then, in comparison with the case that the only outgoing hadron is a single proton with momentum p_2 , we have to replace in the cross-section

$$\int \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \frac{1}{2} \sum_{\lambda_B=+,-} \sum_{\lambda_2=+,-} {}_p\langle \vec{p}_2\lambda_2 ;| j^\mu_{QCD}(0) |\vec{p}_B\lambda_B ;\rangle_p$$

$$\times \; {}_p\langle \vec{p}_2\lambda_2 ;| j^\nu_{QCD}(0) |\vec{p}_B\lambda_B ;\rangle_p^* (2\pi)^4 \delta(p_A + p_B - p_1 - p_2)$$

$$= \int \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} L_p^{\mu\nu}(p_B, p_2)(2\pi)^4 \delta(p_A + p_B - p_1 - p_2)$$

$$= L_p^{\mu\nu}(p_B, p_2)(2\pi)\theta(q^0 + m_B)\frac{\delta\left(\frac{q^2}{2m_B} + \nu\right)}{2m_B} \quad \longrightarrow$$

$$\int \left(\prod_{\{f\}} \frac{d^3\vec{p}_f}{(2\pi)^3 2E_f}\right) \frac{1}{2} \sum_{\lambda_B=+,-} \sum_{\lambda_f} {}_p\langle \{\vec{p}_f\}\{\lambda_f\}| j^\mu_{QCD}(0) |\vec{p}_B\lambda_B ;\rangle_p$$

$$\times \; {}_p\langle \{\vec{p}_f\}\{\lambda_f\}| j^\nu_{QCD}(0) |\vec{p}_B\lambda_B ;\rangle_p^* (2\pi)^4 \delta(p_A + p_B - p_1 - \sum_{\{f\}} p_f)$$

$$\equiv 4\pi m_p W^{\mu\nu}(p_B, q) \quad , \quad {}_p\langle \{\vec{p}_f\}\{\lambda_f\}| \equiv {}_p\langle f|$$

- Lorentz symmetry, parity and current conservation imply

$$W^{\mu\nu}(p_B,q)=W_1(q^2,\nu)\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)+W_2(q^2,\nu)\left(p_B^\mu-\frac{p_Bq}{q^2}q^\mu\right)\left(p_B^\nu-\frac{p_Bq}{q^2}q^\nu\right)$$

$q^2=(p_A-p_1)^2\simeq -4E_AE_1\sin^2\frac{\theta}{2}$, $\nu=\frac{qp_B}{m_B}=E_A-E_1=q^0$
$W_1(q^2,\nu)$ and $W_2(q^2,\nu)$ are called **structure functions**

- Then

$$L_e^{\mu\nu}W_{\mu\nu}=4E_AE_1\left(W_2\cos^2\frac{\theta}{2}+2W_1\sin^2\frac{\theta}{2}\right)$$

$$L_e^{\mu\nu}(p_A\,,p_1)=-q^2\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)+(p_A+p_1)^\mu(p_A+p_1)^\nu$$

- Which leads to

$$\left(\frac{d\sigma}{dE_1d\Omega}\right)_{LAB}=\frac{4\alpha^2E_1^2}{q^4}\left(W_2(q^2,\nu)\cos^2\frac{\theta}{2}+2W_1(q^2,\nu)\sin^2\frac{\theta}{2}\right)$$

- Recall that for $e^-\,\mu^-\to e^-\,\mu^-$ we had

$$\left(\frac{d\sigma}{dE_1d\Omega}\right)_{LAB}=\frac{4\alpha^2E_1^2}{q^4}\left(\cos^2\frac{\theta}{2}-\frac{q^2}{2m_\mu^2}\sin^2\frac{\theta}{2}\right)\delta\left(\frac{q^2}{2m_\mu}+\nu\right)$$

- Then, for the muon

$$W_2(q^2,\nu) = \delta\left(\frac{q^2}{2m_\mu} + \nu\right) \quad , \quad W_1(q^2,\nu) = -\frac{q^2}{4m_\mu^2}\delta\left(\frac{q^2}{2m_\mu} + \nu\right) \quad \Longrightarrow$$

$$\nu\, W_2(q^2,\nu) = \delta\left(\frac{q^2}{2m_\mu\nu} + 1\right) \quad , \quad 2m_\mu W_1(q^2,\nu) = -\frac{q^2}{2m_\mu\nu}\delta\left(\frac{q^2}{2m_\mu\nu} + 1\right)$$

- Then, for elementary fermions $\nu W_2(q^2,\nu)$ and $2m_\mu W_1(q^2,\nu)$ depend on a single variable $\omega \equiv -\frac{2m_\mu\nu}{q^2}$ rather than on two independent ones q^2 and ν. This is called **Bjorken scaling**
- The surprising thing was that experiments at SLAC for $e^- p \to e^- +$ hadrons showed Bjorken scaling for large $Q^2 \equiv -q^2 > 0$

$$Q^2 = -q^2 \simeq 4E_A E_1 \sin^2\frac{\theta}{2}$$

  - Q^2 large $\Longrightarrow$ $\theta \simeq \pi$ $\Longrightarrow$ backward outgoing electrons
  - These events are rare as the cross section goes like $\sim 1/\sin^4\frac{\theta}{2}$
- When Q^2 is large enough, photons see the proton as if it was made of free point-like constituents $\equiv$ **partons**

The parton model

$$\text{Proton} = \{\text{Free partons}\}$$

$f_i(x) \equiv$ probability that a parton of type i carries a fraction x, $x \in [0,1]$, of the proton momentum $P = p_B$, $p_i = x\,P$

	Proton	Parton i
Energy	E	xE
Momentum	$\vec{P}$	$x\vec{P}$
Mass	m_p	$m = \sqrt{(xE)^2 - (x\vec{P})^2} = xm_p$

- For a spin 1/2 parton i carrying a proton momentum fraction x, we have

$$\begin{aligned}
F_2^i(\omega) &\equiv \nu W_2(q^2,\nu) = \delta\left(\frac{q^2}{2m\nu} + 1\right) = \delta\left(-\frac{Q^2}{2xm_p\nu} + 1\right) = \delta\left(-\frac{1}{x\omega} + 1\right) \\
F_1^i(\omega) &\equiv m_p W_1(q^2,\nu) = \frac{m}{x} W_1(q^2,\nu) = \frac{1}{x}\frac{Q^2}{4m\nu}\delta\left(-\frac{Q^2}{2m\nu} + 1\right) \\
&= \frac{1}{2x^2\omega}\delta\left(1 - \frac{1}{x\omega}\right) = \frac{1}{2x}\delta\left(1 - \frac{1}{x\omega}\right)
\end{aligned}$$

- For the proton, we just sum over partons ($Q_i \equiv$ parton electric charge in units of e)

$$
\begin{aligned}
F_2(\omega) &\equiv \sum_i f_i(x) Q_i^2 F_2^i(\omega) = \sum_i f_i(x) Q_i^2 \delta\left(1 - \frac{1}{x\omega}\right) \\
F_1(\omega) &\equiv \sum_i f_i(x) Q_i^2 F_1^i(\omega) = \sum_i f_i(x) Q_i^2 \frac{1}{2x} \delta\left(1 - \frac{1}{x\omega}\right) \\
&= \sum_i f_i(x) Q_i^2 \frac{\omega}{2} \delta\left(1 - \frac{1}{x\omega}\right) = \frac{\omega}{2} F_2(\omega)
\end{aligned}
$$

$f_i(x) \equiv$ probability that the parton i carries a momentum fraction x of the proton

- One also defines

$$
\begin{aligned}
F_2(\omega) &= \sum_i f_i(x) Q_i^2 x \delta\left(x - \frac{1}{\omega}\right) \equiv F_2(x) \delta\left(x - \frac{1}{\omega}\right) \\
F_1(\omega) &= \sum_i f_i(x) Q_i^2 \frac{1}{2} \delta\left(x - \frac{1}{\omega}\right) \equiv F_1(x) \delta\left(x - \frac{1}{\omega}\right)
\end{aligned}
$$

$$
F_1(x) = \frac{F_2(x)}{2x}
$$

- Note that x is the momentum fraction that a parton carries and ω is given by the kinematics and hence measurable $\Longrightarrow$ x becomes measurable

- The differential cross section reads

$$\left(\frac{d\sigma}{dE_1 d\Omega}\right)_{LAB} = \frac{4\alpha^2 E_1^2}{q^4}\int_0^1 dx \left(\frac{F_2(\omega)}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1(\omega)}{m_p}\sin^2\frac{\theta}{2}\right)$$

$$= \frac{4\alpha^2 E_1^2}{q^4}\left(\frac{F_2(x)}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1(x)}{m_p}\sin^2\frac{\theta}{2}\right)\Bigg|_{x=\frac{1}{\omega}=\frac{Q^2}{2m_p\nu}}$$

$$F_2(x) = \sum_i f_i(x) Q_i^2 x \quad , \quad F_1(x) = \sum_i f_i(x) Q_i^2 \frac{1}{2}$$

$f_i(x)$, $i = u, \bar{u}, d, \bar{d}, \ldots$ are called **parton distribution functions** (PDFs)

- The relation $2xF_1(x) = F_2(x)$ is called Callan-Gross relation and it is particular to spin $1/2$ partons
  - If partons had spin 0 $\Longrightarrow$ $F_1(x) = 0$ (why?)
- A more elegant way to present the differential cross-section is

$$\frac{d\sigma}{dxdy} \equiv \int d\Omega \int dE_1 \left(\frac{d\sigma}{dE_1 d\Omega}\right)_{LAB} \delta\left(x + \frac{q^2}{2m_p\nu}\right)\delta\left(y - \frac{p_B q}{p_B p_A}\right)$$

- For $s \gg m_p$ (this is the exercise for this week !)

$$\frac{d\sigma}{dxdy} \simeq F_2(x)\frac{2\pi\alpha^2 s}{q^4}\left(1 + (1-y)^2\right)$$

Parton Distribution Functions (PDFs)

- Knowing the energy of the incoming electronsns E_A and measuring the energy of outgoing electrons E_1 and their angular distribution θ, $F_2(x)$ can be obtained from the DIS experiments
- Let us assume that partons are quarks, and that only u, d and s are relevant

$$\frac{F_2(x)}{x} = \left(\frac{2}{3}\right)^2 (u(x) + \bar{u}(x)) + \left(\frac{1}{3}\right)^2 \left(d(x) + \bar{d}(x)\right) + \left(\frac{1}{3}\right)^2 (s(x) + \bar{s}(x))$$

$i(x) \equiv f_i(x)$, $i = u, \bar{u}, d, \bar{d}, s, \bar{s}, \ldots$ are called **parton distribution functions**

- Let us divide the quarks into those that contribute to the hadron quantum numbers, called **valence quarks** and those that are produced in pairs, called **sea quarks** and denote by $i_v(x)$ and $i_s(x)$ their respective distribution functions
- For the proton and neutron $s(x) = s_s(x) = \bar{s}_s(x) = \bar{s}(x)$
- If we assume flavor $SU(3)$ $\Longrightarrow$ $u_s(x) = d_s(x) = s_s(x) = \bar{s}_s(x) = \bar{u}_s(x) = \bar{d}_s(x)$
- Hence, for the proton (uud) and the neutron (udd), we have

$$u_v^n(x) = d_v^p(x) \equiv d_v(x) \quad , \quad d_v^n(x) = u_v^p(x) \equiv u_v(x) \quad , \quad s_v^n(x) = s_v^p(x) = 0$$

- Using $u(x) = u_v(x) + s(x)$ and $d(x) = d_v(x) + s(x)$, we have

$$\begin{aligned}
\frac{F_2^p(x)}{x} &= \left(\frac{2}{3}\right)^2 u_v(x) + \left(\frac{1}{3}\right)^2 d_v(x) + \frac{4}{3}s(x) \\
\frac{F_2^n(x)}{x} &= \left(\frac{2}{3}\right)^2 d_v(x) + \left(\frac{1}{3}\right)^2 u_v(x) + \frac{4}{3}s(x)
\end{aligned}$$

- Let us find out the expected asymptotic behavior in $x = Q^2/2m_p\nu$, $\nu = E_A - E_1$ in the LAB frame, at fixed Q^2 (which must be large)
  - For $x \to 0$, namely large energy transfer ν, we expect lots of $q\bar{q}$ pairs to be created, and hence that sea quarks dominate $F_2(x)$,

$$\frac{F_2^p(x)}{F_2^n(x)} \underset{x\to 0}{\longrightarrow} 1$$

  - For $x \to 1$, namely small energy transfer ν, we expect few $q\bar{q}$ pairs to be created, and hence that valence quarks dominate $F_2(x)$,

$$\frac{F_2^p(x)}{F_2^n(x)} \underset{x\to 1}{\longrightarrow} \frac{4u_v(x) + d_v(x)}{4d_v(x) + u_v(x)} \longrightarrow \begin{cases} 4 \text{ if } u_v(x) \gg d_v(x) \\ \frac{1}{4} \text{ if } d_v(x) \gg u_v(x) \end{cases}$$

- Then, for $x \to 1$, we have
$$\frac{1}{4} \leq \frac{F_2^p(x)}{F_2^n(x)} \leq 4$$
- Experimental data favors the $u_v(x) \gg d_v(x)$ case

- We also have
$$\frac{F_2^p(x)}{x} - \frac{F_2^n(x)}{x} = \frac{1}{3}\left(u_v(x) - d_v(x)\right)$$

- Normalization constraints
  - The proton has u flavor number 2 (it has two u quarks):
$$\Longrightarrow \quad \int_0^1 dx\,(u(x) - \bar{u}(x)) = \int_0^1 dx u_v(x) = 2$$
  - The proton has d flavor number 1 (it has one d quark):
$$\Longrightarrow \quad \int_0^1 dx\left(d(x) - \bar{d}(x)\right) = \int_0^1 dx d_v(x) = 1$$
  - The hadron momentum must be the sum of the parton momenta
$$1 = \int_0^1 dx\, x\left(u(x) + \bar{u}(x) + d(x) + \bar{d}(x) + s(x) + \bar{s}(x) + g(x)\right) \equiv I_{u_v} + I_{d_v} + 6I_s + I_g$$
    $g(x)$ stands for the PDF of possible electrically neutral partons

- Experiment tells us

$$I_p \equiv \int_0^1 dx F_2^p(x) \simeq 0.18 \quad , \quad I_n \equiv \int_0^1 dx F_2^p(x) \simeq 0.12$$

- Then

$$\begin{aligned} I_p &= \left(\frac{2}{3}\right)^2 I_{u_v} + \left(\frac{1}{3}\right)^2 I_{d_v} + \frac{4}{3} I_s \\ I_n &= \left(\frac{2}{3}\right)^2 I_{d_v} + \left(\frac{1}{3}\right)^2 I_{u_v} + \frac{4}{3} I_s \end{aligned}$$

$$\begin{aligned} \Longrightarrow \qquad I_p + I_n &= \frac{5}{9}\left(I_{u_v} + I_{d_v}\right) + \frac{8}{3} I_s \\ &= \frac{5}{9}\left(I_{u_v} + I_{d_v}\right) + \frac{4}{9}\left(1 - I_{u_v} - I_{d_v} - I_g\right) \\ \Longrightarrow \qquad -0.14 \simeq I_p + I_n - \frac{4}{9} &= \frac{1}{9}\left(I_{u_v} + I_{d_v}\right) - \frac{4}{9} I_g \\ \Longrightarrow \qquad & 4 I_g > I_{u_v} + I_{d_v} \ ! \end{aligned}$$

- There must be electrically neutral partons that carry momentum and do not contribute to $F_2(x)$. These are the gluons in QCD

- With $e^- p$ and $e^- n$ DIS there is no enough information to disentangle $u_v(x)$, $d_v(x)$ and $s(x)$
- The same PDFs appear when one studies $\nu\, p$ DIS mediated by the Weak interaction
- The PDFs suffer from the so called **scaling violations**, namely they develop a mild dependence on Q^2, which may be understood in terms of QCD
- The dependence on Q^2 becomes strong at very low x

6.3 QCD

According to what we have learned so far, the underlying theory of the strong must have:

1. Quarks with color ($SU(3)$) and flavor quantum numbers (spin 1/2, electrically charged)
2. Only color singlet states must be allowed
3. Quark masses must be small so that we have approximate chiral symmetry $SU_L(N_f) \otimes SU_R(N_f)$, $N_f = 2, 3$
4. The chiral symmetry must be spontaneously broken to diagonal $SU(N_f)$, $N_f = 2, 3$
5. When probed at high momentum transfer the quarks in the hadrons behave as free particles $\equiv$ **Asymptotic freedom** $\alpha_s(\mu) \to 0$ when $\mu \to \infty$
6. There must be additional electrically neutral constituents in the hadrons

- Let us focus on point 2 for a single flavor
  - We have seen in QED that only the transverse field is gauge invariant. It contains the creation and annihilation operators of the photons, which build the physical states
  - We may build an analogous theory with color $SU(3)$, namely that enjoys an $SU(3)$ **local** gauge invariance rather than a $U(1)$ one

- Consider $q(x)$ in the 3 representation of $SU(3)$,

$$q(x)=\begin{pmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{pmatrix} \quad , \quad \begin{array}{c} q(x)\to g(x)q(x) \\ \bar{q}(x)\to \bar{q}(x)g^{\dagger}(x) \end{array} \quad , \quad g(x)\in SU(3)$$

- The free Dirac Lagrangian is not invariant under this transformation because the derivatives act on $g(x)$

$$\mathcal{L}=\bar{q}(x)\left(i\partial\!\!\!/ - m_q\right)q(x)$$

- In QED, in order to get it invariant we need $\partial_\mu \to D_\mu$, and the e.m. field is in $D_\mu=\partial_\mu+iq\,A_\mu(x)$
- Let us try $\partial_\mu \to D_\mu=\partial_\mu+ig_s\,G_\mu(x)$, $G_\mu(x)$ is a 3x3 matrix in color space which transforms as

$$G_\mu(x)\to g(x)G_\mu(x)g^{\dagger}(x)-\frac{i}{g_s}g(x)\partial_\mu g^{\dagger}(x) \quad , \quad G_\mu=T^aG_\mu^a$$

- G_μ^a are the **gluon** fields and live in the 8 (adjoint) representation of $SU(3)$
- Then, the following Lagrangian is invariant under local $SU(3)$ transformations

$$\mathcal{L}=\bar{q}(x)\left(iD\!\!\!\!/ - m_q\right)q(x)=\bar{q}(x)\left(i\partial\!\!\!/ - g_sG\!\!\!\!/ - m_q\right)q(x)$$

- Note that $D_\mu \to g(x)D_\mu g^{\dagger}(x)$ as an operator, namely the derivative acts not only on $g^{\dagger}(x)$ but on anything on the right of it

- Then we can construct the analogous of the Maxwell term, the Yang-Mills term,

$$ig_s G_{\mu\nu} \equiv [D_\mu,\, D_\nu] \quad , \quad G_{\mu\nu}(x) \to g(x) G_{\mu\nu} g^\dagger(x)$$

$$G_{\mu\nu} = T^a G^a_{\mu\nu} \quad , \quad G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_s f^{abc} G^b_\mu G^c_\nu \quad , \quad [T^a,\, T^b] = if^{abc} T^c$$

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{tr}\left(G_{\mu\nu} G^{\mu\nu}\right) = -\frac{1}{4} G^a_{\mu\nu} G^{a\ \mu\nu} \quad , \quad \mathrm{tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$$

- Hence, for one flavor

$$\mathcal{L}_{QCD} = -\frac{1}{2}\mathrm{tr}\left(G_{\mu\nu} G^{\mu\nu}\right) + \bar{q}(x)\left(i\not{D} - m_q\right) q(x)$$

- One flavor QCD implements point 2 and point 6 of the list
  - The quark-gluon interaction is similar to QED: $-ig_s\gamma^\mu T^a$
  - Unlike QED, in which there are no photon-photon interactions, in QCD there are gluon-gluon interactions
  - The coupling constant g_s is the same for quark-gluon and gluon-gluon interactions, unlike the electric charge, which depends on the flavor
- The generalization to an arbitrary number of quark flavors, which implements point 1, is trivial since only the quark mass depends on the flavor

$$\bar{q}(x)\left(i\not{D} - m_q\right) q(x) \quad \longrightarrow \quad \sum_{j=u,d,s,\ldots} \bar{q}_j(x)\left(i\not{D} - m_j\right) q_j(x)$$

$$\mathcal{L}_{QCD} = -\frac{1}{2}\mathrm{tr}\left(G_{\mu\nu}G^{\mu\nu}\right) + \sum_{j=u,d,s,\dots} \bar{q}_j(x)\left(i\not{D} - m_j\right)q_j(x)$$

- It is invariant under $q_j(x) \to e^{i\theta_j}q_j(x)$, $\theta_j \neq \theta_j(x)$ $\Longrightarrow$ flavor is conserved
- Clearly, if we put N_f masses $m_j = 0$ then it has a $SU_L(N_f) \otimes SU_R(N_f)$ chiral symmetry, which implements point 3
- Proving points 4 and 5 requires non-trivial QFT calculations
  - Point 5, namely why quarks behave as free particles at high momentum transfer, can be answered within perturbation theory, but you need to learn about loop calculations, renormalization, and quantization of non-abelian gauge theories
  - Point 4 is even more complicated to prove, as it requires non-perturbative calculations (lattice QCD)
- Let us from now on take advantadge of the point 5, and the form of the QCD Lagrangian to learn things on new processes

$R(e^-e^+ \to \text{ hadrons})$

$$R(e^-e^+ \to \text{ hadrons}) \equiv \frac{\sigma(e^-e^+ \to \text{ hadrons})}{\sigma(e^-e^+ \to \mu^+\mu^-)}$$

- Duality hypothesis: $\sigma(e^-e^+ \to \text{ hadrons}) = \sigma(e^-e^+ \to \text{ quarks+gluons})$
- If at high energy the quark behave like free particles $\Longrightarrow$ We can neglect the interaction with gluons $\Longrightarrow$ $\sigma(e^-e^+ \to \text{ hadrons}) \simeq \sum_{j=u,d,s,\ldots} \sigma(e^-e^+ \to q_j\bar{q}_j)$
- Hence, for $\sqrt{s} \gg 2m_j$,

$$R \simeq \sum_{j=u,d,s,\ldots} Q_j^2 N_c \quad , \quad N_c = \text{number of colors}$$

$$R = \begin{cases} \frac{2}{3}N_c \text{ for } 1\text{ GeV} \simeq \sqrt{s} \ll 2m_c \simeq 3\text{ GeV} \\ \frac{10}{9}N_c \text{ for } 2m_c \ll \sqrt{s} \ll 2m_b \simeq 10\text{ GeV} \\ \frac{11}{9}N_c \text{ for } 2m_b \ll \sqrt{s} \ll 2m_t \simeq 350\text{ GeV} \end{cases}$$

- Experiment: 15 GeV $< \sqrt{s} <$ 35 GeV $\Longrightarrow$ $R = 4 \pm 0.5$
- Theory: $R = 11N_c/9$

N_c	1	2	3	4
R	1.2	2.4	3.6	4.8

Heavy quarks

- Since $m_c \sim 1.5$ GeV and $m_b \sim 5$ GeV are larger than 1 GeV and hence much larger than the typical hadronic scales $\Lambda_{QCD} \sim 300 - 400$ MeV:
  - $\alpha_s \equiv g_s^2/4\pi$ at those scales should be small
  - Heavy quarks in hadrons will move slowly $\Longrightarrow$ non-relativistic approximation is reasonable
- Then in analogy to QED, we may write

$$\mathcal{L}_{NRQCD} = \psi^\dagger \left(iD_0 + \frac{\vec{D}^2}{2m_Q} + \vec{\mu}\vec{B} + \cdots \right) \psi \ , \ \vec{\mu} \sim \frac{g_s}{m_Q} \vec{S}$$

  - ψ is a color triplet Schrödinger field describing non-relativistic quarks
  - D_μ are covariant derivatives which contains a 3×3 color matrix with a gluon field
  - $\vec{B}$ is the chromomagnetic field, which also contains a 3×3 color matrix with gluon fields

Heavy-light hadrons

These are hadrons containing a single heavy quark (charm or bottom), $H = (Ql)$, $l = \bar{q}$ or $l = qq$

- The CoM of the hadron roughly coincides with the location of the heavy quark $\Longrightarrow$ the heavy quark is approximately a static source of color field (analogous to the Hidrogen atom, in which the proton is approximately a static source of electric field)

$$\mathcal{L}_{NRQCD} \simeq \psi^\dagger iD_0\psi \equiv \mathcal{L}_{HQET}$$

  - It is flavor independent $\Longrightarrow$ flavor symmetry $\Longrightarrow$ spectrum independent of the heavy quark mass

$$M_H = m_Q + \Lambda'_l + \mathcal{O}\left(\frac{1}{m_Q}\right)$$

  - It is spin independent $\Longrightarrow$ spin symmetry $\Longrightarrow$ approximately degenerated spin doublets

$$M_{H^*} - M_H = \frac{\Lambda_l^2}{m_Q} + \mathcal{O}\left(\frac{1}{m_Q^2}\right)$$

H^* is the spin partner of H

- This implies, for instance,

$$1 \underset{\text{Th}}{\simeq} \frac{M_{\Lambda_b} - M_B}{M_{\Lambda_c} - M_D} \underset{\text{Exp}}{=} 0.82 \quad , \quad 1 \underset{\text{Th}}{\simeq} \frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} \underset{\text{Exp}}{=} 0.88$$

Heavy Quarkonium

These are hadrons containing a heavy quark and a heavy antiquark (charm or bottom), $H = (Q\bar{Q})$, $Q = c, b$

- The heavy quarks move slowly around the CoM $\Longrightarrow$ a flavor independent potential is created (analogous to positronium)

$$\mathcal{L}_{NRQCD} \simeq \psi^\dagger \left(iD_0 + \frac{\vec{\nabla}^2}{2m_Q} \right) \psi$$

  - There is no flavor symmetry anymore, the binding energies depend on m_Q
  - There is an approximate spin symmetry $\Longrightarrow$ spin multiplets (doublets for $L = 0$, quadruplets for $L \neq 0$)
  - The non-relativistic quark model is a reasonable approximation

- The decays to light hadrons are mediated by gluons ($\alpha_s(m_Q)$ small)

$$\Gamma(H \to \text{light hadrons}) \sim \Gamma((Q\bar{Q}) \to \text{light quarks} + \text{gluons}) \simeq \Gamma((Q\bar{Q}) \to \text{gluons})$$

- Gluons have $J^{PC} = 1^{--}$

$$PG^{\mu}(x)P^{-1} = G_{\mu}(\tilde{x}) \quad , \quad CG_{\mu}(x)C^{-1} = -G_{\mu}^{T}(x)$$

You may check that the QCD Lagrangian is invariant under P and C, recall that $Pq(x)P^{-1} = \gamma^0 q(\tilde{x})$ and $Cq(x)C^{-1} = i\gamma^2 q(x)^*$

- Then the states $J^{P\,+}$ decay to two gluons and the states $J^{P\,-}$ to three gluons
- For states in the same spin multiplet we have

$$\frac{\Gamma((Q\bar{Q})[J^{P\,-}] \to g\,g\,g)}{\Gamma((Q\bar{Q})[J^{P\,+}] \to g\,g)} \sim \alpha_s(m_Q) \ll 1$$

- For the ground states, ${}^1S_0$ has $J^{PC} = 0^{-+}$ and ${}^3S_1$ has $J^{PC} = 1^{--}$, hence

$$1 \underset{\text{Th}}{\gg} \frac{\Gamma(J/\psi \to \text{hadrons})}{\Gamma(\eta_c \to \text{hadrons})} \underset{\text{Exp}}{\simeq} 1.9\,10^{-3} \quad , \quad 1 \underset{\text{Th}}{\gg} \frac{\Gamma(\Upsilon \to \text{hadrons})}{\Gamma(\eta_b \to \text{hadrons})} \underset{\text{Exp}}{\sim} 4\,10^{-3}$$

  - The smallness of the experimental values is not only due to $\alpha_s(m_Q)$ being small but also to a numerical accident in the QCD calculation which also occurs in positronium decays for QED