HW1

viernes, 8 de abril de 2022 14:29

Guillermo Abad López

Problem 1:

Use the one-loop expression for the electron self energy

$$\Sigma^{1\,\mathrm{Loop}} = -\frac{\alpha}{2\pi}C_F\left\{\left(\frac{1}{2}\not p - 2m_{\mathrm{R}}\right)\left[\frac{1}{\varepsilon}+\log\left(\frac{4\pi\mu^2}{p^2}\right)-\gamma_E\right]\right. \qquad (1)$$

$$\left. -\frac{1}{2}\not p + m_{\mathrm{R}} - \int_0^1 dx\left[\not p(1-x)-2m_{\mathrm{R}}\right]\log\left[-x(1-x)+x\frac{{m_{\mathrm{R}}}^2}{p^2}\right]\right\},$$

$(C_F = 1)$

to renormalise QED in the $\overline{\mathrm{MS}}$-scheme $[C_{41} = C_{m1} = C_{A1} = C_{11} = -\gamma_E + \log(4\pi) \equiv C]$

a) Find the regularised self energy Σ_R.

$+$


$+ O(\alpha^2) = \Sigma^{1\,loop} + \Sigma^{1\delta} + O(\alpha^2)$

b) Find the relation between m_R and the mass of the electron, m_e, at one-loop precision. Define the mass of the electron as the pole of the renormalised propagator

$$S_F(p^2, g_R(\mu), m_R(\mu), \mu) = \frac{i}{\not p \underbrace{- m_R(\mu) - \Sigma_R(p^2, g_R(\mu), m_R(\mu), \mu)}_{-m_e(\mu)}} \qquad (2)$$

By using the expression of Σ_R at one loop order, determine the leading g_R^2 shift between these two masses

$$m_e = m_R - \underset{\left(\frac{g_R^2}{4\pi}\right)}{\underset{\shortparallel}{\alpha_R^{\mathrm{em}}}}\boxed{\delta m} = \frac{-\Sigma_R}{\alpha_R} = \frac{-4\pi\Sigma_R}{g_R^2} \qquad (3)$$

c) Determine the residue of this pole. Is it the same as in $\underbrace{\text{on-shell renormalisation}}_{\text{Bare } m_R \text{ is the pole}}$?

d) Compute the variation of m_e with the scale μ at one loop order. Interpret the result.

Note:

$$\int_0^1 dx \log\left(R^2x-(1-x)x\right) = 2R^2\log(R) - \left(R^2-1\right)\log\left(R^2-1\right) - 2 \qquad (4)$$

$$\int_0^1 dx\,(1-x)\log\left(R^2x-(1-x)x\right) = \frac{1}{2}\left(2R^4\log(R) - R^2 - \left(R^4-1\right)\log\left(R^2-1\right)-2\right) \qquad (5)$$

a)

$$\Sigma_R = \Sigma^{1loop} + \Sigma^{1\delta} + O(\alpha^2)$$

$$\left(\begin{array}{l} \bullet\ \Sigma^{1loop} = -\frac{\alpha}{2\pi}\left\{\left(\frac12\not p - 2m_R\right)\left[\cancel{\frac{1}{\epsilon}} + \log\left(\frac{4\pi\mu^2}{p^2}\right) - \gamma_E\right] - \frac12\not p + m_R - \int_0^1 dx\left[\not p(1-x) - 2m_R\right]\log\left(-x(1-x) + x\frac{m_R^2}{p^2}\right)\right\} \\ \bullet\ \Sigma^{1\delta} = -\left(\delta_4^{(1)}\not p - \left(\delta_m^{(1)} + \delta_4^{(1)}\right)m_R\right) = \frac{\alpha}{4\pi}\left(\frac1\epsilon + c\right)\not p - \left[\frac{3\alpha}{4\pi}\left(\frac1\epsilon + c\right) + \frac{\alpha}{4\pi}\left(\frac1\epsilon + c\right)\right]m_R = \frac{\alpha}{2\pi}\left(\cancel{\frac1\epsilon} + c\right)\left\{\frac12\not p - 2m_R\right\} \end{array}\right)$$

$$\Sigma_R = -\frac{\alpha}{2\pi}\left\{\left(\frac12\not p - 2m_R\right)\left[\underset{(\cancel{\gamma_E} - \cancel{\log(4\pi)})}{\underset{\shortparallel}{-C}} + \log\left(\frac{\cancel{4\pi}\mu^2}{p^2}\right) - \cancel{\gamma_E}\right] - \frac12\not p + m_R - \int_0^1 dx\left[\not p(1-x) - 2m_R\right]\underbrace{\log\left(-x(1-x) + x\frac{m_R^2}{p^2}\right)}_{\equiv\log(\,)}\right\} =$$

$$= -\frac{\alpha}{2\pi}\left\{\left(\frac12\not p - 2m_R\right)\log\left(\frac{\mu^2}{p^2}\right) - \frac12\not p + m_R - \int_0^1 dx\,\not p(1-x)\log(\,) + \int_0^1 dx\,2m_R\log(\,)\right\}$$

$$\left(\begin{array}{ll} \bullet\ \text{From (4):} & -\int_0^1 dx\,\not p(1-x)\log(\,) = -\frac{\not p}{2}\left\{2\frac{m_R^4}{p^4}\log\left(\frac{m_R}{p}\right) - \frac{m_R^2}{p^2} - \left(\frac{m_R^4}{p^4} - 1\right)\log\left(\frac{m_R^2}{p^2} - 1\right) - 2\right\} \\ \bullet\ \text{From (5):} & \int_0^1 dx\,2m_R\log(\,) = 2m_R\left\{2\frac{m_R^2}{p^2}\log\left(\frac{m_R}{p}\right) - \left(\frac{m_R^2}{p^2} - 1\right)\log\left(\frac{m_R^2}{p^2} - 1\right) - 2\right\} \end{array}\right)$$

$$\boxed{\Sigma_R} = -\frac{\alpha}{2\pi}\left\{\frac{\not{p}}{2}\left(\log\left(\frac{\mu^2}{p^2}\right) - 1 - 2\frac{m_R^4}{p^4}\log\left(\frac{m_R}{p}\right) + \frac{m_R^2}{p^2} + \left(\frac{m_R^4}{p^4}-1\right)\log\left(\overbrace{\frac{m_R^2}{p^2}-1}^{\frac{m_R^2-p^2}{p^2}}\right)+2\right) -\right.$$

$$\left. -2m_R\left(\log\left(\frac{\mu^2}{p^2}\right) - \frac{1}{2} - 2\frac{m_R^2}{p^2}\log\left(\frac{m_R}{p}\right) + \left(\frac{m_R^2}{p^2}-1\right)\log\left(\frac{m_R^2}{p^2}-1\right)+2\right)\right\} =$$

$$= -\frac{\alpha}{2\pi}\left\{\frac{\not{p}}{2}\left(\log\left(\frac{\mu^2}{m_R^2-p^2}\right)+1+\frac{m_R^2}{p^2}+\frac{m_R^4}{p^4}\log\left(\frac{m_R^2-p^2}{\cancel{p^2}}\Big/\frac{m_R^2}{\cancel{p^2}}\right)\right)-\right.$$

$$\left. -2m_R\left(\log\left(\frac{\mu^2}{m_R^2-p^2}\right)+\frac{3}{2}+\frac{m_R^2}{p^2}\log\left(\frac{m_R^2-p^2}{\cancel{p^2}}\Big/\frac{m_R^2}{\cancel{p^2}}\right)\right)\right\} =$$

$$\boxed{= -\frac{\alpha}{4\pi}\left\{\log\left(\frac{\mu^2}{m_R^2-p^2}\right)[\not{p}-4m_R]+[\not{p}-6m_R]+\frac{m_R^2}{p^2}\not{p}+\log\left(\frac{m_R^2-p^2}{m_R^2}\right)\left[\not{p}\frac{m_R^2}{p^2}-4m_R\right]\frac{m_R^2}{p^2}\right\}}$$

b)

$$\not{p} = m_e + \alpha\ldots = m_R + \alpha\delta + \alpha\ldots$$

$m_e - m_R = -\alpha_R\delta_m = \Sigma_R \longrightarrow \underline{\delta_m = -\frac{\Sigma_R}{\alpha} = \frac{1}{4\pi}\{\ldots\}}$, so lot's expand Σ_R to ordor α, which means δ_m to 0th ordor:

$$\boxed{\Sigma_R\Big|_{p^2=m_e^2}} = -\frac{\alpha}{4\pi}\left\{\log\left(\frac{\mu^2}{m_R^2-m_e^2}\right)[\not{p}-4m_R]+[\not{p}-6m_R]+\frac{m_R^2}{m_e^2}\not{p}+\log\left(\frac{m_R^2-m_e^2}{m_R^2}\right)\left[\not{p}\frac{m_R^2}{m_e^2}-4m_R\right]\frac{m_R^2}{m_e^2}\right\} =$$

$$= -\frac{\alpha}{4\pi}\left\{\log\left(\frac{\mu^2}{\Sigma^2-2\Sigma m_R}\right)[\not{p}-4m_R]+[\not{p}-6m_R]+\frac{m_R^2}{(m_R+\cancel{\Sigma_R})^2}\not{p}+\log\left(\frac{\Sigma^2-2\Sigma m_R}{m_R^2}\right)\left[\not{p}\frac{m_R}{(m_R+\Sigma_R)^2}-4m_R\right]\frac{m_R^2}{(m_R+\cancel{\Sigma_R})^2}\right\} \overset{\left(\substack{\Sigma_R=0\\ \text{Expand ordor } 0}\right)}{=}$$

$$= -\frac{\alpha}{4\pi}\left\{\log\left(\frac{\mu^2}{\cancel{\Sigma^2-2\Sigma m_R}}\right)[\not{p}-4m_R]+[\not{p}-6m_R]+\not{p}+\log\left(\frac{\cancel{\Sigma^2-2\Sigma m_R}}{m_R^2}\right)[\not{p}-4m_R]\right\} =$$

$$\boxed{= -\frac{\alpha}{4\pi}\left\{\log\left(\frac{\mu^2}{m_R^2}\right)[\not{p}-4m_R]+[2\not{p}-6m_R]\right\}}$$

$$\boxed{\delta_m = \frac{1}{4\pi}\left\{\log\left(\frac{\mu^2}{m_R^2}\right)[\not{p}-4m_R]+[2\not{p}-6m_R]\right\}}$$

c)

$$\frac{i}{\not{p}-m_e} = \frac{i}{\not{p}-m_R+\alpha\delta_m} = \frac{i}{\not{p}-m_R-\Sigma_R} \overset{(\Sigma_R = A\not{p}+Bm_R)}{=} \frac{i}{\not{p}-m_R+(A\not{p}+Bm_R)} = \frac{i}{(A+1)\not{p}+(B-1)m_R}$$

Where from Σ_R, we get that:

- $A = -\frac{\alpha}{4\pi}\left\{\log\left(\frac{\mu^2}{m_R^2}\right)+2\right\}$
- $B = -\frac{\alpha}{4\pi}\left\{-4\log\left(\frac{\mu^2}{m_R^2}\right)-6\right\}$

which are both of ordor α

$\underline{\text{When we do the limit } \not{p}\to m_e:}$

$(A+1)\not{p}+(B-1)m_R = 0$

$(A+1)^2\overset{m_e^2}{\cancel{\not{p}^2}} - (B-1)^2m_R^2 = 0$

$(1+2A)(m_R^2+2\Sigma_R m_R) - (1-2B)m_R^2 = 0$

$\underline{\text{Which for ordor } O(\alpha^1)!}$

$2Am_R^2 + 2\Sigma_R m_R + 2Bm_R^2 = 0$

$(A+B)m_R + \Sigma_R = 0$

$(-A-B)m_R = \Sigma_R$

so, in ordor to have a pole for $\not{p}$ again, we divide by $A+1$:

$$\frac{i}{\not{p}-m_e} = \frac{1}{A+1}\frac{i}{\not{p}+\frac{B-1}{A+1}m_R} \overset{\text{Sune Taylor exp } O(\alpha^2)}{=} \frac{i}{A+1}\left(\frac{1}{\not{p}-m_R}-\frac{(A+B)m_R}{(\not{p}-m_R)^2}\right) \overset{\text{Sune taylor exp } O(\alpha^2)}{=} \frac{1}{A+1}\frac{i}{\not{p}-\underbrace{(1-A-B)m_R}_{\left(\underbrace{m_R+\Sigma_R}_{m_e}+\overset{\Sigma_R \to O(\alpha^2)}{\cancel{(\not{p}-m_R)A}}\right)}} = \frac{1}{A+1}\frac{i}{\not{p}-m_e}$$

And the Residue will be:

$$\boxed{Res_{\overline{MS}}} = \lim_{\not{p}\to m_e} (\not{p}-m_e)\frac{i}{\not{p}-m_e} = \frac{i}{A+1}\cancel{\lim_{\not{p}\to m_e}}\frac{\overset{1}{\cancel{\not{p}-m_e}}}{\not{p}-m_e} = \boxed{\frac{i}{A+1}}$$

On-shell renormalization:

$$\boxed{Res_{\text{on-shell}}} = \lim_{\not{p}\to m_e}(\not{p}-m_e)\frac{i}{\not{p}-m_R-\Sigma_R} = i\cancel{\lim_{\not{p}\to m_e}}\frac{\overset{1}{\cancel{\not{p}-m_e}}}{\not{p}-m_e} = \boxed{i}$$

It is not the same unless $A=0$

d)

$$\frac{dm_e}{d\mu} = \frac{d(m_R-\alpha\delta_m)}{d\mu} = \frac{dm_R}{d\mu} - \frac{d\alpha}{d\mu}\delta_m - \alpha\frac{d\delta_m^{(0)}}{d\mu}$$

(Because we already have an alpha and we are at one loop computation:
$$\delta_m^{(0)} = \frac{-1}{4\pi}\left\{\log\left(\frac{\mu^2}{m_R^2}\right)3-4\right\}m_R$$)

And remembering: $\beta = \frac{d\,g_r(\mu)}{d\log\mu}$ $\qquad \gamma_m = \frac{-d\log(m_R)}{d\log\mu}$

- $\frac{dm_R}{d\mu} = \frac{m_R}{\mu}\frac{d\log(m_R)}{d\log(\mu)} = -\frac{m_R}{\mu}\gamma_m$
- $\frac{d\alpha}{d\mu} = \frac{g_r}{2\pi}\frac{dg_r}{d\mu} = \frac{g_r}{2\pi}\frac{1}{\mu}\frac{dg_r}{d\log\mu} = \frac{g_r}{2\pi}\frac{\beta}{\mu}$ $\qquad \left(\frac{d(\log(\mu^2)-2\log(m_R))}{d\mu} = \frac{2}{\mu} - 2\frac{\gamma_m}{\mu}\right)$
- $\frac{d\delta_m^{(0)}}{d\mu} = -\frac{1}{4\pi}\left[\left\{\log\left(\frac{\mu^2}{m_R^2}\right)3-4\right\}\frac{dm_R}{d\mu} + \frac{d\log\left(\frac{\mu^2}{m_R^2}\right)}{d\mu}m_R\right] =$

$$= -\frac{1}{4\pi}\left[\left\{\log\left(\frac{\mu^2}{m_R^2}\right)3-4\right\}\left(-\frac{m_R}{\mu}\gamma_m\right) + \frac{2m_R}{\mu} - \frac{2\gamma_m}{\mu}m_R\right]$$

$$\frac{dm_e}{d\mu} = \underline{-\frac{m_R}{\mu}\gamma_m} + \underline{\frac{g_r}{2\pi}\frac{\beta}{\mu}\frac{1}{4\pi}\left\{\log\left(\frac{\mu^2}{m_R^2}\right)3-4\right\}}m_R + \frac{\alpha}{4\pi}\left[\underline{\left\{\log\left(\frac{\mu^2}{m_R^2}\right)3-4\right\}\left(-\frac{m_R}{\mu}\gamma_m\right)} + \underline{\frac{2m_R}{\mu}} - \underline{2\gamma_m\frac{m_R}{\mu}}\right]$$

$$= \frac{m_R}{\mu}(2-3\gamma_m) + \frac{1}{4\pi}\left\{\log\left(\frac{\mu^2}{m_R^2}\right)3-4\right\}\frac{m_R}{\mu}\left(\frac{g_r\beta}{2\pi} - \alpha\gamma_m\right)$$

I coult make it work, but it should give 0, because me is physical and should not depend on μ:

$$\frac{dm_e}{d\mu} = 0 \longrightarrow \boxed{\frac{dm_R}{d\mu} = \frac{d\,\alpha\delta_m}{d\mu}} = \frac{d\alpha}{d\mu}\delta_m + \alpha\frac{d\delta_m}{d\mu}$$

so the m_R changes with μ in function as Σ changes with it.