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PHYSICAL PARTICLES OF THE MASSIVE SCHWINGER MODEL

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The massive Schwinger model is considered in the infinite momentum frame. By assuming its physical particles consist of two fermion bound states, we compute a spectrum. For fermions with large bare masses, the method is reliable. For low-mass fermions, we find we must include states of higher fermion number to adequately describe excited states of the fundamental boson of the theory. We do this for the scalar state in the limit of small bare fermion mass. This representation of the theory provides a unified treatment of both the weak and strong coupling limits, remaining in the fermion representation throughout. We have checked our numerical results with exact calculations wherever possible, and find good agreement.

1. Introduction

The massive Schwinger model [1,2] has come into prominence [3] lately as a gauge theory with many properties one hopes to see in 4-dimensional models of strong interactions [3,4]. The model, 1 + 1 dimensional quantum electrodynamics can be solved exactly only in the case of massless fermions. In this solution, all charged particles are absent as asymptotic states. It has been shown that this feature persists when the fermions are given a bare mass [5,6]. In this paper, we will study an approximation to the model with massive fermions. The Hamiltonian will be written in the infinite momentum frame. Then, we shall use variational techniques to study its spectrum. The spectrum we obtain interpolates smoothly from massless to extremely massive fermions.

We go to the infinite momentum frame to make the vacuum simple. This simplifies the calculation of the lowest lying excitations. In fact, in the massless Schwinger model, the boson is exactly a two-particle state in this frame. To compute the spectrum, we start out restricting our states to the two-particle sector. Then we can write a Schrödinger equation for the state in this approximation and (numerically) solve it. The mass of this state reduces to the exact result as the fermion mass (m) approaches zero. Furthermore, the quantity $\partial M_V/\partial m$ (M_V is the mass of the lowest

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lying (vector) particle) can be calculated exactly by other means and is found to be $e^{\gamma} = 1.78$. We obtain $\sqrt{\frac{1}{3}}\pi = 1.81$. As the fermions get heavier, the mass of the vector particle approaches $2m$, the static limit. We may also trace the wave functions of the states continuously through all values of (e/m). This is not all we see in the two-particle sector, however. For any value of m, there is an infinite family of eigenstates to the Schrödinger equation. This is not a feature of the full Hamiltonian. For example, when $m = 0$, we find a state at $M_V = e/\sqrt{\pi}$, one at $M_S = 2.43\, M_V$, and so on. The fact that $M_S > 2M_V$ causes us to suspect the validity of the two-particle approximation. We then add in four-particle states considered as bound states of two vector particles to compute M_S for small m. This, of course, allows us to recover the second excited state of the massless Schwinger model (which consists of two vector particles, no interaction). We also compute $\partial M_S/\partial m$ to within 2%. For large m, we have many excitations of the two particles with energies less than $2M_V$ which we believe to be stable particles. These are linearly spaced in M_n^2 with a coefficient approximately m-independent. Requiring $M_n^2 < 4m^2 \cong M_0^2$, we obtain a bound $n \leqslant 4m^2/ce^2$ [6]. Finally, there is a region where we may compare our results to calculations done in lattice gauge theory for the Schwinger model. The results agree quite well.

This paper is organized into six sections. In sect. 2 we present the Hamiltonian in the infinite momentum frame. As massless fermions in the 1 + 1 dimensional are notoriously tricky, we explore properties of the free Dirac equation in sect. 3. We then pick up the Schwinger model in sect. 4 and develop a variational scheme for studying the two-particle sector. This is almost identical to 't Hooft's model of a non-Abelian gauge theory in 1 + 1 dimension in the large N limit. In sect. 5 we analyze the eigenvalue equation for states, presenting and discussing solutions. Finally, in sect. 6 we summarize a calculation of the mass of a state coupled to the four-particle sector; the scalar in the massive Schwinger model with $m \ll e$. Some details of the computation are included in the appendix.

2. The Schwinger model in the infinite momentum frame

The vacuum of the Schwinger model is highly non-trivial [4,7]. It is "seized", e.g., it is not an eigenstate of Q_5. This is due to the possibility of creating $q\bar{q}$ pairs with total momentum 0 and $Q_5 = \pm 2$ on the Fock vacuum. The actual vacuum is a coherent superposition of eigenstates of definite Q_5. Our goal of computing the spectrum of the theory by variational techniques would require us to first get a tractable representation of this vacuum. We turn to the infinite momentum frame to avoid this. We take the conventions [8]

$$\tau = \sqrt{\tfrac{1}{2}}(t' + z')\,, \qquad \xi = \sqrt{\tfrac{1}{2}}(t' - z')\,, \qquad x^{\mu} = (\tau, \xi)\,, \tag{2.1}$$

where primed quantities refer to the ordinary reference frame. For γ-matrices we take

$$\gamma_0 = \sqrt{\tfrac{1}{2}}(\gamma_0' - \gamma_1') = \sqrt{2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad \gamma_1 = \sqrt{2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \quad \gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}. \tag{2.2}$$

Note $\gamma_0^2 = \gamma_1^2 = 0$, $\gamma_0\gamma_1 + \gamma_1\gamma_0 = 2I$.

We start with the Lorentz invariant Lagrangian

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu \nabla_\mu - m)\psi - e\overline{\psi}\gamma_\mu \psi A^\mu - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.3}$$

In any formulation of the Schwinger model, there are no physical degrees of freedom associated with A. We pick the gauge $A_1 = A^0 = 0$. To find the fermion degrees of freedom we compute

$$\frac{\delta \mathcal{L}}{\delta(\nabla_0 \psi)} = \psi^\dagger \frac{\gamma_0 + \gamma_1}{\sqrt{2}} i\gamma^0 = i\sqrt{2}\psi^\dagger \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}. \tag{2.4}$$

Here we see an important fact. In the infinite momentum frame, the Dirac field has only one degree of freedom, ψ_1. The other component, ψ_2, is a constrained variable. We therefore impose the commutation relation

$$\{\psi_1(\xi, \tau), \psi_1^\dagger(\xi', \tau)\} = \sqrt{\tfrac{1}{2}}\delta(\xi - \xi'). \tag{2.5}$$

The equation of constraint for ψ_2 is

$$\frac{\partial \psi_2}{\partial \xi} = \frac{-im}{\sqrt{2}}\psi_1, \qquad \text{or} \qquad \psi_2(\xi, \tau) = \frac{-im}{2\sqrt{2}}\int_{-\infty}^{\infty} d\xi' \epsilon(\xi - \xi')\psi_1(\xi', \tau). \tag{2.6}$$

The gauge field may be eliminated in the standard way (remember, $A_1 = 0$)

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\tfrac{1}{2}A_0 \frac{\partial^2}{\partial \xi^2} A_0. \tag{2.7}$$

Maxwell's equation, $\partial_\mu F^{\mu\nu} = j^\nu$ implies

$$j^0(\xi, \tau) = -\frac{\partial^2}{\partial \xi^2} A_0(\xi, \tau), \qquad \text{or}^\star \qquad A_0(\xi, \tau) = -\tfrac{1}{2}\int d\xi' |\xi - \xi'| j_0(\xi', \tau).$$

Finally, the Hamiltonian is

$$H = i\sqrt{2}\psi_1^\dagger \dot{\psi}_1 - \mathcal{L} = m\psi_1^\dagger \psi_2 + \tfrac{1}{2}j^0 A_0 \tag{2.8}$$

$$= \frac{-im^2}{2\sqrt{2}}\int d\xi d\xi' \psi_1^\dagger(\xi)\epsilon(\xi - \xi')\psi_1(\xi') - \tfrac{1}{4}e^2 \int d\xi d\xi' j^0(\xi)|\xi - \xi'| j^0(\xi').$$

⋆ One might also add to this a c-number contribution proportional to ξ, e.g., a background field. This may lead to interesting phenomena such as the "half asymptotic" particle found by Coleman in ref. [6]. We set this background field to zero in what follows.

Now, $j^0 = \overline{\psi}\gamma^0\psi = \sqrt{2}\,\psi_1^\dagger\psi_1$. This, combined with the anticommutation relation eq. (2.5) defines the Hamiltonian form of the theory in these coordinates. Before we go on and analyze it, we shall backtrack and study the free Dirac equation in the 1 + 1 dimensional infinite momentum frame to point out some subtleties.

3. Free fermions

We have seen that in the infinite momentum frame describing a fermion requires only one degree of freedom. This is clear from the form of γ_5. In the ordinary reference frame, γ_5 has eigenvalues ± 1 describing right/left moving particles. In the infinite momentum frame, we have only right movers, so we get only one component. There are some subtleties which arise for massless fermions. We now turn to them.

First, it is useful to have a momentum representation for ψ. We find

$$\psi_1(\xi,\tau) = \frac{1}{2^{1/4}}\int_0^\infty \frac{\mathrm{d}\eta}{\sqrt{2\pi}}\left(b_\eta\,\mathrm{e}^{-i(\eta\xi+m^2\tau/2\eta)} + d_\eta^\dagger\,\mathrm{e}^{i(\eta\xi+m^2\tau/2\eta)}\right), \tag{3.1}$$

with $\{b_\eta^+, b_{\eta'}\} = \delta(\eta-\eta')$, etc. is consistent with eq. (2.5). We also have

$$:H_0: = \tfrac{1}{2}m^2\int_0^\infty \frac{\mathrm{d}\eta}{\eta}\left(b_\eta^\dagger b_\eta + d_\eta^\dagger d_\eta\right), \tag{3.2}$$

$$\psi_2(\xi,\tau) = \frac{m}{2^{3/4}}\int_0^\infty \frac{\mathrm{d}\eta}{\eta\sqrt{2\pi}}\left(b_\eta\,\mathrm{e}^{-i(\eta\xi+m^2\tau/2\eta)} - d_\eta^\dagger\,\mathrm{e}^{i(\eta\xi+m^2\tau/2\eta)}\right). \tag{3.3}$$

From the above, it would seem that when $m = 0$, $\psi_2 = 0$. This, however, is not true. Matrix elements of the operator $\overline{\psi}\psi$, computed with $m \neq 0$, have non-zero limits as m approaches 0, even though the operator itself is explicitly multiplied by m. $\overline{\psi}\psi$ is a Lorentz scalar, so the infinite momentum frame matrix element should equal the non-zero result we recall from the ordinary reference frame. We have $\overline{\psi}\psi = \psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1$. Consider the matrix element

$$\begin{aligned}&\langle 0|:\psi_1^\dagger\psi_2(\xi,\tau)::\psi_2^\dagger\psi_1(0,0):|0\rangle\\ &\quad = \langle 0|\psi_1^\dagger(\xi,\tau)\psi_1(0,0)|0\rangle\langle 0|\psi_2(\xi,\tau)\psi_2^\dagger(0,0)|0\rangle\,.\end{aligned}$$

Momentum expansions yield

$$\langle\psi_1^\dagger(\xi,\tau)\psi_1(0,0)\rangle = \sqrt{\tfrac{1}{2}}\int_0^\infty \frac{\mathrm{d}\eta}{2\pi}\,\mathrm{e}^{-i(\eta\xi+m^2\tau/2\eta)}\,,$$

$$\langle\psi_2(0,0)\psi_2^\dagger(\xi,\tau)\rangle = \frac{m^2}{2\sqrt{2}}\int_0^\infty \frac{\mathrm{d}\eta}{2\pi}\frac{1}{\eta^2}\,\mathrm{e}^{+i(\eta\xi+m^2\tau/2\eta)}\,. \tag{3.4}$$

At $m = 0$, the first contraction is $-i/2\pi\sqrt{2}(\xi - i\epsilon)$. The second integral is non-zero. Letting $x = 1/\xi\eta$,

$$\langle \psi_2(0,0)\psi_2^\dagger(\xi,\tau)\rangle = \frac{m^2\xi}{2\sqrt{2}} \int_0^\infty \frac{\mathrm{d}x}{2\pi} \mathrm{e}^{+i(m^2\xi\tau x/2 + 1/x)} \underset{m\to 0}{\cong} \frac{+i}{2\pi\sqrt{2}\tau} \, .$$

Thus,

$$\langle \overline{\psi}\psi(0,0)\overline{\psi}\psi(\xi,\tau)\rangle = -\frac{1}{4\pi^2\xi\tau} \qquad \text{at } m = 0 \, . \tag{3.5}$$

In the ordinary reference frame we know $\overline{\psi}\psi$ couples right and left moving particles. We suspect that this feature is responsible for the inordinate importance of the region $\eta \simeq m$, and can demonstrate this further by watching what happens to left movers when we go to the infinite momentum frame.

In the ordinary reference frame, parity takes z' to $-z'$. In the infinite momentum frame, this is $\xi \leftrightarrow \tau$. The parity operator implements

$$P\,\psi(\xi,\tau)P^{-1} = A\;\psi(\tau,\xi)\,, \qquad \text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{3.6}$$

It is interesting to see the effect of P on creation operators. We obtain

$$P\,b_\eta^\dagger P^{-1} = \frac{m}{\eta\sqrt{2}} b_{m^2/2\eta}^\dagger \,, \qquad P\,d_\eta^\dagger P^{-1} = -\frac{m}{\eta\sqrt{2}} d_{m^2/2\eta}^\dagger \, . \tag{3.7}$$

The momentum of the particle becomes $m^2/2\eta$; not surprising in view of the fact that η goes to H under P. Thus, the energetic left moves are mapped onto the small η region, and this region is important.

4. Schwinger model

The next thing to do is to understand the anomaly in this frame. We use it then to construct the boson of the massless theory. We have

$$V_\alpha(\tau) = \int_{-\infty}^{\infty} \mathrm{d}\xi\,\mathrm{e}^{i\alpha\xi} :j^0(\xi,\tau): \,, \tag{4.1}$$

$$V_\alpha(\tau) = \int_0^\infty \mathrm{d}\eta [b_\eta^\dagger b_{\eta+\alpha}\theta(\eta+\alpha) - d_{\eta-\alpha}^\dagger d_\eta \theta(\eta-\alpha)$$

$$+ b_\eta^\dagger d_{-\eta-\alpha}^\dagger \theta(-\eta-\alpha) + d_\eta b_{-\eta+\alpha}\theta(-\eta+\alpha)] \, . \tag{4.2}$$

Note that $:V_\alpha:$ annihilates the vacuum for $\alpha > 0$. To quickly get the anomaly, consider $(\alpha; -\beta > 0)$

$$\langle 0 | [V_\alpha, V_\beta] | 0 \rangle = \langle V_\alpha V_\beta \rangle = \int_0^\alpha \mathrm{d}\eta \, \delta(\alpha + \beta) = \alpha \delta(\alpha + \beta) \,. \tag{4.3}$$

As one can explicitly verify, there are no q-number contributions, so

$$[V_\alpha, V_\beta] = \alpha \delta(\alpha + \beta) \,.$$

Note that V in the infinite momentum frame is $j_0 + j_1$ in the ordinary reference frame. This commutator is exactly the (double) Fourier transform of a derivative of a delta function as computed by Schwinger [1].

Then ★

$$:H: = m^2 \int_0^\infty \frac{\mathrm{d}\eta}{2\eta} (b_\eta^\dagger b_\eta + d_\eta^\dagger d_\eta) + \frac{e^2}{4\pi} P \cdot P \int_{-\infty}^\infty \frac{\mathrm{d}\alpha}{\alpha^2} : V_\alpha V_{-\alpha} : \,. \tag{4.4}$$

For $m = 0$, we have

$$-i \frac{\partial}{\partial t} V_\alpha = [H, V_\alpha] = -\frac{e^2}{2\pi} \frac{1}{\alpha} V_\alpha \,,$$

$$\frac{\partial^2}{\partial \xi \partial \tau} j^0(\xi, \tau) = \frac{e^2}{2\pi} j^0(\xi, \tau) \,, \tag{4.5}$$

which is the (massive) Klein-Gordon equation in our coordinates. The mass is entirely due to the anomaly. Thus, we have the usual spectrum for the model; bosons of mass $e/\sqrt{\pi}$ with no interactions. The boson is

$$|P\rangle = \int_0^P \mathrm{d}\eta \, b_\eta^\dagger d_{P-\eta}^\dagger |0\rangle \,.$$

When $m > 0$, we make an ansatz for the vector state

$$|P\rangle = \int_0^P \mathrm{d}\eta \, f_P(\eta) b_\eta^\dagger d_{P-\eta}^\dagger |0\rangle \,. \tag{4.6}$$

Our ansatz is based on the hope that the vector state is mostly two-particle. When $m = 0$ this is exactly true. Also, when $m \gg e$, it is certainly also true; the piece of H

★ Usually we write $H \approx a\,\rho(x) |x - y| \rho(y)$. This neglects a boundary term obtained in going from L to H, and, for example, assigns zero energy to an isolated charge. The Fourier transform of $|x|$ is $(-2\mathrm{d}/\mathrm{d}k) P \cdot P/k \neq 2P \cdot P/k^2$. By using the latter, we recover the missing boundary term and get in infinite energy ($\sim L$) for an isolated charge. This is why we use it.

connecting the two- and four-particle sectors is $O(e^2)$, the kinetic energy term is of order $m^2 \gg e^2$, so the approximation is fine. 't Hooft [9] has studied the two-dimensional Yang-Mills colored quark model in the large N limit. In this limit, graphs connecting the two- and four-particle sectors are suppressed by a factor N^{-1}, and neglected in the order considered. This is essentially the same approximation (indeed, he, too, works in the infinite momentum frame) and we will obtain a similar eigenvalue equation. The difference is that $SU(N)$ has no anomaly.

The eigenvalue equation is obtained by computing the energy of the state ⋆. We quickly find

$$\langle P|H_0|P'\rangle = \delta(P-P')m^2 P \int_0^P \frac{d\eta}{2\eta(P-\eta)} f_P^2(\eta)\,. \tag{4.7}$$

The electric part of $\langle H\rangle$ has a few more terms. We represent them by the diagrams in figs. 1a–c. The first graph, 1a, is a fermion self-energy. It is infrared divergent. Including the scattering diagram 1b, we cancel this divergence. The fact that in the one-particle sector no such cancellation is possible leads to an infinite energy for a single-particle (charged) state.

All together, we find

$$\langle P|H|P'\rangle = \delta(P-P')\left[m^2 \int_0^P \frac{d\eta}{2\eta(P-\eta)} f_P^2(\eta) + \frac{e^2}{2\pi P^2}\left[\int_0^P d\eta\, f_P(\eta)\right]^2 \right.$$

$$\left. + \frac{e^2}{2\pi} P\cdot P \int_0^P d\eta\, f_P(\eta) \int_{-\eta}^{P-\eta} \frac{d\alpha}{\alpha^2}\left[f_P(\eta) - f_P(\eta+\alpha)\right]\right]. \tag{4.8}$$

We also have

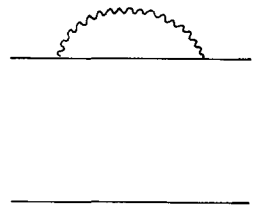
$$\langle P|P'\rangle = \delta(P-P')\int_0^P d\eta\, f_P^2(\eta) = P\,\delta(P-P')\,.$$

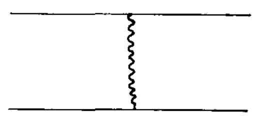
Introducing dimensionless variables $x = (\eta/P)$, $y = (\alpha/P)$, $f_P(\eta) = \psi(\eta/P) = \psi(x)$, we obtain

$$\int_0^1 dx\, \psi^2(x) E[\psi] = \frac{1}{2P}\left[m^2 \int_0^1 \frac{dx}{x(1-x)} \psi^2(x) + \frac{e^2}{\pi}\left(\int_0^1 dx\, \psi(x)\right)\right.$$

$$\left. + \frac{e^2}{\pi} P\cdot P \int_0^1 dx\, \psi(x) \int_{-x}^{1-x} \frac{dy}{y^2}\left[\psi(x) - \psi(x+y)\right]\right]. \tag{4.9}$$

⋆ The theory is super-renormalizable. All divergences are removed by normal ordering H. See ref. [6].


(a)


(b)

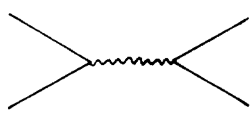

(c)

Fig. 1. Graphs contributing to $\langle H \rangle$ in the two-particle sector.

In our frame, the energy of a particle is $E = M^2/2P$, so we can identify the mass M of the trial state. Now, varying ψ, we obtain

$$2E\,\psi(z) = \frac{1}{2P}\left[m^2 \frac{2\psi(z)}{z(1-z)} + \frac{2e^2}{\pi} \int_0^1 \mathrm{d}x\,\psi(x) \right.$$

$$+ \frac{e^2}{\pi} P \cdot P \int_{-z}^{1-z} \frac{\mathrm{d}y}{y^2}\,[\psi(z) - \psi(z+y)]$$

$$\left. + \frac{e^2}{\pi} P \cdot P \int_0^1 \mathrm{d}x\,\psi(x) \int_{-x}^{1-x} \frac{\mathrm{d}y}{y^2}\,[\delta(x-z) - \delta(x+y-z)] \right] . \qquad (4.10)$$

This simplifies to

$$M^2 \psi(x) = \frac{m^2}{x(1-x)} \psi(x) + \frac{e^2}{\pi} \int_0^1 dx\, \psi(x)$$

$$+ \frac{e^2}{\pi} P \cdot P \int_0^1 \frac{dy}{(x-y)^2} [\psi(x) - \psi(y)] . \tag{4.11}$$

The last term in this equation is regularized *via* a principle part prescription [9]. This comes directly from the Hamiltonian, eq. (4.4). With this regularization, eq. (4.11) is well defined.

Another equation of use to us is obtained by integrating eq. (4.11) over x,

$$\left(M^2 - \frac{e^2}{\pi}\right) \int_0^1 dx\, \psi(x) = m^2 \int_0^1 \frac{dx}{x(1-x)} \psi(x) . \tag{4.12}$$

5. Spectrum in the two-particle sector

The eigenvalue equation for $m = 0$ has a solution $\psi(x) = 1$. This has a mass of $e/\sqrt{\pi}$ and is the "photon" of the Schwinger model. The term in H involving the fermion mass involves m^2. This might cause us to think that to first order, $\delta M^2 \sim m^2$, but this is not right. The most important effect of this term is to force $\psi(0) = \psi(1) = 0$. The last term in H tries to keep ψ as flat as possible consistent with the endpoint values and the normalization. We can actually determine the behavior of ψ near $x = 0\,(1)$, and from this, calculate $\partial M/\partial m|_0$ analytically. We make the ansatz $\psi = x^s(1-x)^s$ for some (small) s depending on m. Eq. (4.11) becomes

$$M^2 x^s(1-x)^s = m^2 x^{s-1}(1-x)^{s-1} + \frac{e^2}{\pi}$$

$$+ \frac{e^2}{\pi} P \cdot P \int_0^1 \frac{dy}{(x-y)^2} [x^s(1-x)^s - y^s(1-y)^s] .$$

The power s is determined by matching the behavior of the right- and left-hand sides of the above equation near $x = 0$. The dominant behavior of the left-hand side is x^s, so the last term on the right-hand side must cancel the $m^2 x^{s-1}$ behavior *. For small x, the integral is

$$\int_0^1 \frac{dy}{(x-y)^2} (x^s(1-x)^s - y^s(1-y)^s) \cong -x^{s-1} + s x^{s-1} \int_0^1 \frac{dz}{1-z} z^{s-1}$$

$$= -x^{s-1}(1 - s\pi \cot \pi s) .$$

* This is the same calculation as 't Hooft's.

Thus, we must have

$$\frac{m^2\pi}{e^2} - 1 + \pi s \cot \pi s = 0 \, .$$

Since we want $s \sim 0$ for $m \simeq 0$, we find

$$m = \sqrt{\tfrac{1}{3}}\pi s \, , \tag{5.1}$$

where from here on we measure m^2 and other masses in units of e^2/π. Substituting this in eq. (4.12), we find

$$(M^2 - 1)(1 + \mathrm{O}(s)) = \frac{2m^2}{s} = 2m\frac{\pi}{\sqrt{3}} \, ,$$

$$M^2 = 1 + 2m\left(\frac{\pi}{\sqrt{3}}\right) . \tag{5.2}$$

This may be compared with the exact result which has $\pi/\sqrt{3}$ replaced by e^γ. Numerically, $\pi/\sqrt{3} = 1.81$, $e^\gamma = 1.78$. Thus, in this region of very light fermions, the approximation is good to 2%.

Another region of interest is the static limit, $m \gg e$. Here, the dominant term in H is the first one, and it causes ψ to be peaked about $x = \frac{1}{2}$. Then, the integral itself has a value near 4, so $M \simeq 2m$. This of course corresponds to two very heavy particles practically unaffected by the interaction.

For intermediate masses, we must do the computation numerically. In fig. 2 we present wave functions computed for various values of m. The expected behavior

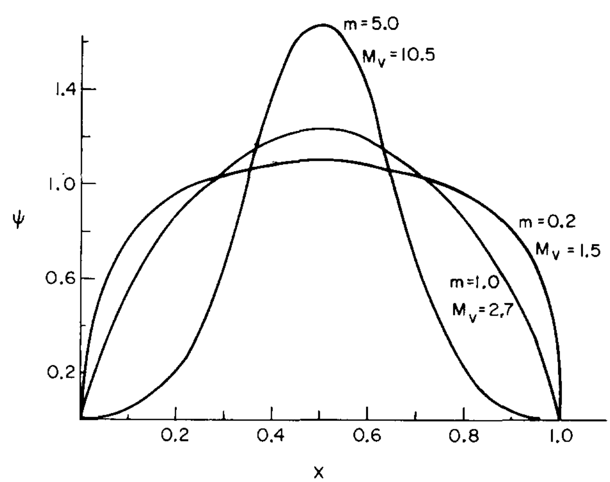


Fig. 2. Two-particle wave functions of the vector particle at a few values of m.

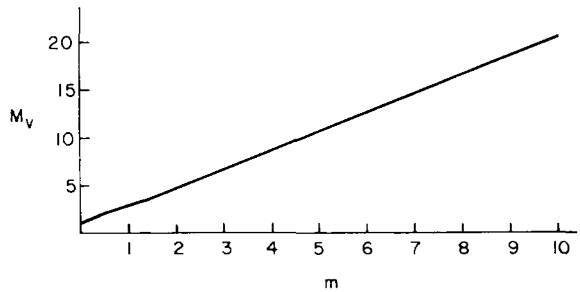


Fig. 3. Mass of the vector particle as a function of m.

for $m \ll e$ and $m \gg e$ is evident. We can also plot the mass of our state from strong to weak coupling. This is shown in fig. 3. Perhaps the most gratifying aspects of the results is the unified description of the physics from strong to weak coupling.

The state above is not the only solution of eq. (4.11). There are infinitely many. These are linearly spaced in mass squared [9] with a spacing of about $10(e^2/\pi)$, approximately independent of m. These states are just two-particle bound states with a potential $V = |x|$ in the ordinary reference frame [6]. This has been exploited by Coleman to get a bound on the number of stable particles in this regime (e.g., the maximum n such that $M_n < 2M_0$). We have numerically computed a few of these and display them in fig. 4.

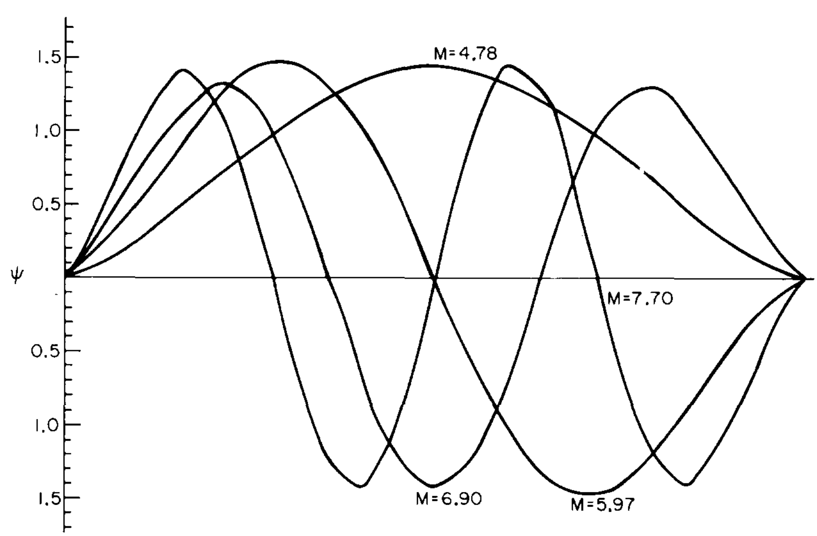


Fig. 4. Excited two-particle states for m = 2.0.

The series of eigenstates of (4.11) is infinite for any m, even for $m = 0$. In fact, at $m = 0$, the next level is at $M_2 = 2.43 > 2M_1$. This means that there is undoubtedly a large coupling between the two- and four-particle states, and a large component of the second level is four-fermion.

6. A description of excited states

As we saw in the last section, the two-particle sector is not sufficient to describe the scalar state of the massless Schwinger model. In general, we expect that the spectrum we computed in the two-particle sector will be unreliable for states whose masses are comparable to twice the mass of the lowest lying state. At that point, we must expand our space of trial states to include four and more fermions. In general, this is extremely difficult. For the nearly massless Schwinger model, however, we can carry out such a computation.

There exists a formulation of the theory in terms of Bose fields [4,6]. The massless case is just a free massive Bose field. The mass term becomes $km \cos(c\phi)$ where k and c are computable constants. When m is small compared to the mass of the Bose field, this term weakly binds two quanta of the ϕ field. This bound state is the scalar state of the model.

Motivated by this, we are led to consider a state consisting of two vector particles bound by H. We start by examining the massless Schwinger model. A state consisting of two vector particles with total momentum P is

$$|P\rangle = \int_0^P \mathrm{d}a \, G(a) V_a^+ V_{P-a}^+ |0\rangle \,.$$

This is an eigenstate of H with energy

$$E = \frac{e^2}{2\pi} \int_0^P \mathrm{d}a \, G^2(a) \left(\frac{1}{a} + \frac{1}{P-a} \right) .$$

Clearly this is minimized by $G^2(a) = \delta(a - \frac{1}{2}P)$ in which case we obtain

$$E = \frac{e^2}{2\pi} \frac{4}{P} = \frac{(4e^2/\pi)}{2P} \,. \tag{6.1}$$

We identify the "mass" of this state as $M_\mathrm{s} = 2M_\mathrm{v}$, thinking of it as the limit of the scalar state for vanishing m. In terms of fermions,

$$V_a^+ V_{P-a}^+ = \int_0^a \mathrm{d}\eta \, b_\eta^\dagger d_{a-\eta}^\dagger \int_0^{P-a} \mathrm{d}\eta' \, b_{\eta'}^\dagger d_{P-a-\eta'}^\dagger$$

$$+ \int_0^P \mathrm{d}\eta [\theta(a + \eta - P) - \theta(a - \eta)] b_\eta^\dagger d_{P-\eta}^\dagger \,. \tag{6.2}$$

Note that this has both two- and four-particle contributions.

We clearly want to replace the four-particle term with two of the vector states we found in sect. 5. The two-particle piece of eq. (6.2) must also be modified. The mass term in H prevents a state from having a zero momentum fraction constituent. Thus, we must give this term a wave function, h, which vanishes like a power at $k = 0$ and $k = P$. Nothing in H demands modifications at $k = a$ or $k = P - a$. The power law for h is determined in the same manner as eq. (5.1). There are extra terms in the equations, but these do not affect the result. We find h must vanish like k^s where $\sqrt{\frac{1}{3}}\pi s = m$. Thus, we take

$$|P\rangle = [\int_0^a \mathrm{d}\eta\, f_a(\eta) b_\eta^\dagger d_{a-\eta}^\dagger \int_0^{P-a} \mathrm{d}\eta'\, f_{P-a}(\eta') b_{\eta'}^\dagger d_{P-a-\eta'}^\dagger$$

$$+ \int_0^P \mathrm{d}\eta\, h(\eta, a) b_\eta^\dagger d_{P-\eta}^\dagger]|0\rangle\,, \tag{6.3}$$

where

$$f_a(\eta) = \eta^s(a-\eta)^s\theta(\eta)\theta(a-\eta)\,,$$

$$h(\eta, a) = [\theta(a - P + \eta) - \theta(a-\eta)]\eta^s(P-\eta)^s\,. \tag{6.4}$$

Note that this state has the proper symmetry under charge conjugation if $G(a) = G(P - a)$. We do not intend to do a straightforward variational calculation on this state. This is because the binding energy is $O(m^2)$. Rather, we will determine the strength of the delta function potential which the two heavy particles feel in the state. This is $O(m)$. It turns out that, in this representation, the strength of the delta function is sensitive to the power s, and we believe that our state is accurate in that aspect.

The first task is to compute the norm of the state. As outlined in the appendix,

$$\langle P|P'\rangle = \delta(P-P')[2\int_0^P \mathrm{d}a\, G^2(a)(\int_0^a \mathrm{d}\eta\, f_a^2(\eta))(\int_0^{P-a} \mathrm{d}\eta'\, f_{P-a}^2(\eta'))$$

$$+ \int \mathrm{d}a\, \mathrm{d}a'\, G(a)G(a')N_1(a, a')\,, \tag{6.5}$$

where

$$N_1(a, a') = -2\int \mathrm{d}\eta\, f_a(\eta) f_{P-a}(a' - a + \eta) f_{a'}(a-\eta) f_{P-a'}(\eta)$$

$$+ \int_0^P \mathrm{d}\eta\, h(\eta, a) h(\eta, a')\,.$$

There are two terms due to the four-particle part. This is due to the exclusion principle. In the massless case, $N_1(a, a') = 0$.

Next, we must compute $\langle H\rangle$. There are many terms. Some contribute to the formation of the two vector particles themselves. Others result from individual constituents in a meson feeling the fields from the constituents of the other meson ("photon exchange"). Our particles are neutral overall, and there can be no "dipole" effects in two dimensions (the electric field of a charge is constant) so these cancel. Net forces do arise from diagrams where two mesons exchange fermions or constituents from two different mesons annihilate. Of course, there is mixing between the two- and four-particle sectors, and terms coming solely from the two-particle sector. The fact that $f_a(k)$ satisfies eq. (4.11) is useful in reducing $\langle H\rangle$. We find (see appendix)

$$\langle P|H|P'\rangle = \delta(P-P')[M_{\mathrm{V}}^2 P \int_0^P \mathrm{d}a\, G^2(a) n_a n_{P-a}$$

$$+ \int \mathrm{d}a\, \mathrm{d}a'\, G(a) G(a') H_1(a, a')] \,, \tag{6.6}$$

where

$$n_a = \frac{1}{a} \int_0^a \mathrm{d}\eta\, f_a^2(\eta) \,,$$

and

$$H_1(a, a') = \frac{e^2}{\pi} \int \mathrm{d}k \frac{\mathrm{d}\alpha}{\alpha^2} [f_a(k) f_{P-a}(a'-a+k) f_{a'}(a-k) f_{P-a'}(k)$$

$$\times \theta(k-\alpha)\theta(a-k+\alpha) - f_a(k-\alpha) f_{P-a}(a'-a+k) f_{P-a'}(k-\alpha) f_{a'}(a-k)$$

$$+ f_a(k) f_{P-a}(a'-a+k) f_{P-a'}(k) f_{a'}(a-k) \theta(P-a'-k+\alpha)\theta(a'-a+k-\alpha)$$

$$- f_a(k) f_{P-a}(a'-a+k-\alpha) f_{P-a'}(k-\alpha) f_{a'}(a-k) - \tfrac{1}{2} h(k, a') h(k+\alpha, a)$$

$$+ \tfrac{1}{2} h(k, a') h(k, a) \theta(k+\alpha)\theta(P-k-\alpha)$$

$$+ f_a(k) f_{P-a}(k+\alpha-a) f_{P-a'}(k+\alpha-a') f_{a'}(k)$$

$$+ f_a(P-k-\alpha) f_{P-a}(k) f_{P-a'}(k+\alpha-a') f_{a'}(k)$$

$$- h(k,a')f_a(k-\alpha)f_{P-a}(k-a) - h(k,a')f_a(P-k)f_{P-a}(k-\alpha)$$

$$- h(k,a)f_{a'}(P-k)f_{P-a'}(k-\alpha) - h(k,a)f_{a'}(k-\alpha)f_{P-a'}(k-a')]$$

$$-\frac{e^2}{\pi}\int \mathrm{d}k\,[w_a f_{P-a}(k)h(k,a') + w_{P-a}f_a(k)h(k,a')$$

$$+ w_{a'}f_{P-a'}(k)h(k,a) + w_{P-a'}f_{a'}(k)h(k,a)]$$

$$- M_\mathrm{V}^2 \int \mathrm{d}k\left[f_a(k)f_{P-a}(a'-a+k)f_{a'}(a-k)f_{P-a'}(k)\left\{\frac{1}{a}+\frac{1}{P-a}+\frac{1}{P-a'}+\frac{1}{a'}\right\}\right]$$

$$+ m^2\int \mathrm{d}k\, f_a(k)f_{P-a}(a'-a+k)f_{P-a'}(k)f_{a'}(a-k)$$

$$\times\left\{\frac{1}{k}+\frac{1}{a-k}+\frac{1}{a'-a+k}+\frac{1}{P-a'-k}\right\}$$

$$+ \tfrac{1}{2}m^2\int \mathrm{d}k\, h(k,a)h(k,a')\left\{\frac{1}{k}+\frac{1}{P-k}\right\},$$

where

$$w_a = \frac{1}{a^2}\int \mathrm{d}k\, f_a(k)\,.$$

It may be verified that this is zero when $m = 0$. The two-particle contributions are crucial for this. Without them, we would find logarithmic divergences in the massless case. We have evaluated this expression for small s. The evaluation is tricky due to the non-analytic nature of the integrands. The techniques used are explained in the appendix.

For small m, we expect G to be peaked strongly at $a = \frac{1}{2}$. We also expect (and find) H_1 and N_1 are $\mathrm{O}(m)$. We therefore make a negligible error by replacing $H_1(a,a')$ by its value at $a = a' = \frac{1}{2}$ in eq. (6.6). This also is true for N_1 in eq. (6.5). Thus, $(P = 1)$

$$E = \frac{M_\mathrm{V}^2\int_0^P \mathrm{d}a\, G^2(a)n_a n_{1-a} + H_1\left[\int \mathrm{d}a\, G(a)\right]^2}{\int_0^P \mathrm{d}a\, 2a(1-a)G^2(a)n_a n_{1-a} + N_1\left[\int \mathrm{d}a\, G(a)\right]^2}\,.$$

Varying with respect to G, we obtain

$$M_\mathrm{V}^2\, G(a)n_a n_{1-a} + H_1\int \mathrm{d}a\, G(a) = E\left[2a(1-a)G(a)n_a n_{1-a} + N_1\int \mathrm{d}a\, G(a)\right].$$

Now, $E = \frac{1}{2}M_s^2$. Rearranging, we find

$$n_a n_{1-a}[M_v^2 - a(1-a)M_s^2]G(a) = [\tfrac{1}{2}M_s^2 N_1 - H_1] \int \mathrm{d}a\, G(a) .$$

Now, n_a is $1 + O(m)$. Since $2N_1 - H_1$ is $O(m)$, we may set $n_a = 1$. Also, we may set $\frac{1}{2}M_s^2 N_1 = 2N_1$. For simplicity, define $2N_1 - H_1 = \alpha$. Then it is straightforward to compute M_s. If $\alpha > 0$, we find

$$M_s = 2M_v - \frac{\pi^2}{4M_v^3}\alpha^2 .$$

One can also verify that G is narrow enough to justify replacing H_1 by a constant.

It is through this last equation that we may obtain the strength of the delta function potential. A bound state of two particles of mass M_v interacting through a potential $-b\delta(x)$ has the energy

$$E = 2M_v - \tfrac{1}{4}b^2 M_v .$$

Then, $\pi\alpha/M_v^2 = b$. A calculation outlined in the appendix shows

$$\alpha = \tfrac{1}{3}\pi^2 s = \sqrt{\tfrac{1}{3}}\pi m e/\sqrt{\pi} . \tag{6.7}$$

Thus,

$$M_s = 2M_v - \frac{\pi^4}{12M_v} m^2 .$$

The exact result [11] ⋆ is $\alpha = e^\gamma m$ which is quite close to our result.

A convenient variable for displaying our results is the binding energy of the scalar state B,

$$B = 1 - \frac{M_s}{2M_v} .$$

For large values of m, this is computed in the two-particle sector. We also know how B behaves at $m \simeq 0$. We match their behaviors in the intermediate region when m is too large to use the derivative information but too small to trust results from the two-particle calculation. B is plotted in fig. 5. This quantity has also been computed *via* lattice gauge theory methods. The results are very close, and the dotted line in fig. 5 is that curve obtained for B in ref. [10].

⋆ A similar calculation appeared in ref. [10]. The authors forgot to use the reduced mass, and forgot a factor of $\frac{1}{2}$ in the Hamiltonian. With these corrections, refs. [10] and [11] agree.

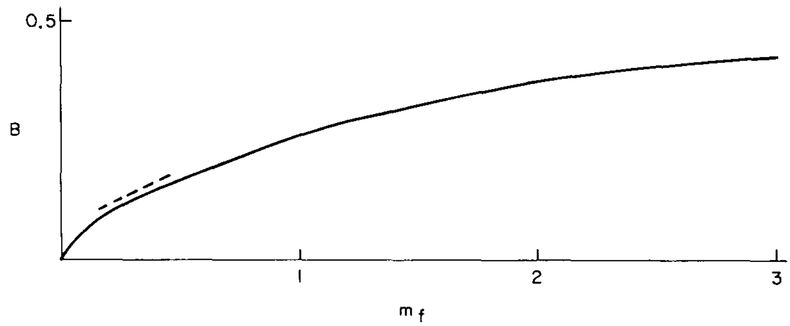


Fig. 5. Binding energy B.

7. Conclusion

We have studied the low lying states of the massive Schwinger model for all values of (e/m). Our approximation remains in the fermion representation of the theory, so we can trace a state's structure from small to large (e/m). It is encouraging that there is no *qualitative* change in, say, the structure of the vector particle from weak to strong coupling. Perhaps the success of strong coupling methods may be traced to this.

These techniques may have useful applications to some other models in field theory.

I wish to thank Profs. J. Kogut and L. Susskind for suggesting this problem and for their continued encouragement. Discussions with M. Peskin are gratefully acknowledged, in particular with regard to the solution of the integral equation. The author also wishes to thank the Department of Physics and Astronomy of Tel-Aviv University, where most of the work was completed, for their warm hospitality.

Appendix

In this appendix, we will outline the steps leading to eqs. (6.6) and (6.7). First, we deal with the norm of the state (6.5). We represent the terms by the diagrams shown in figs. 6a, b. Their contributions are

$$\delta(a-a')\delta(P-P')\int \mathrm{d}k\ \mathrm{d}k'\ f_a^2(k) f_{P-a}^2(k') \qquad \text{from 6a,}$$

$$-\delta(P-P')\int \mathrm{d}k\ f_a(k) f_{P-a}(a'-a+k) f_{P-a'}(k) f_{a'}(a-k) \qquad \text{from 6b.}$$

Letting $a \to P-a$, we will obtain two more identical contributions.

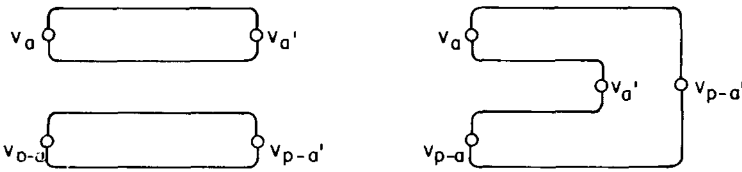


Fig. 6. Graphs contributing to the four-particle norm.

Next, we have a group of diagrams whose sum constitutes a meson. These are shown in fig. 7a. Graphs of this type produce the first term in eq. (6.6).

Now, the long range meson-meson force arising from diagrams of the type shown in fig. 7b vanishes after summing all such diagrams. This is simply, due to the fact that our mesons are electrically neutral. All interactions are due to processes in which two constituents are exchanged, or annihilate and reappear in different final particles. A sample annihilation graph is shown in fig. 7c. This particular one contributes a term

$$\frac{e^2}{\pi}\int \mathrm{d}k \frac{\mathrm{d}\alpha}{\alpha^2} f_a(k) f_{P-a}(k+\alpha-a) f_{P-a'}(k+\alpha-a') f_{a'}(k)\,.$$

Exchange diagrams can be grouped conveniently by utilizing eq. (4.11) in diagrammatic form. This is shown in fig. 7d. When we add all such graphs, we find it convenient to include an extra set of graphs with m^2 vertices and remove them later. This reduces the number of double integrals we have to consider.

Terms coming from the parts of H coupling the two- and four-particle sector typically look like

$$-\frac{e^2}{2\pi}\int \mathrm{d}k \frac{\mathrm{d}\alpha}{\alpha^2} h(k,a) f_a(k-\alpha) f_{P-a}(P-k)\,,$$

or

$$-\frac{e^2}{2\pi a^2}\int \mathrm{d}k' f_a(k') \int \mathrm{d}k\, h(k,a) f_{P-a}(k)\,.$$

Finally, the two-particle part of H has the same form as eq. (4.11) except for the fact that the annihilation term vanishes by (anti)symmetry. These graphs yield eq. (6.6). When f and h are θ functions described in the text (for any G), these terms mediating meson-meson interactions vanish. This involves cancellation of logarithmic divergences. These terms will be evaluated in our trial state. Recall this is defined by

$$f_a(k) = k^s(a-k)^s\,, \qquad h(k,a) = k^s(P-k)^s(\theta(a+k-P) - \theta(a-k))\,.$$

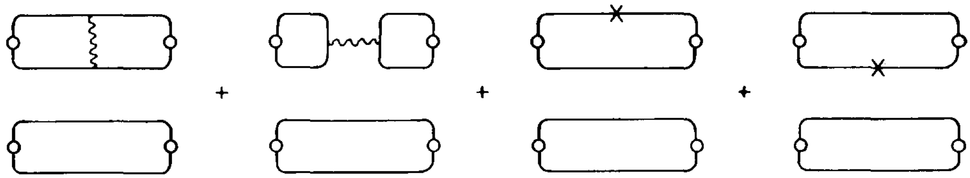


Fig. 7a.

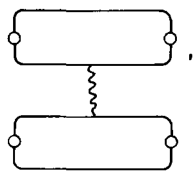


Fig. 7b.

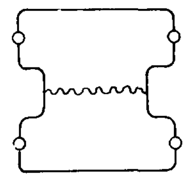


Fig. 7c.

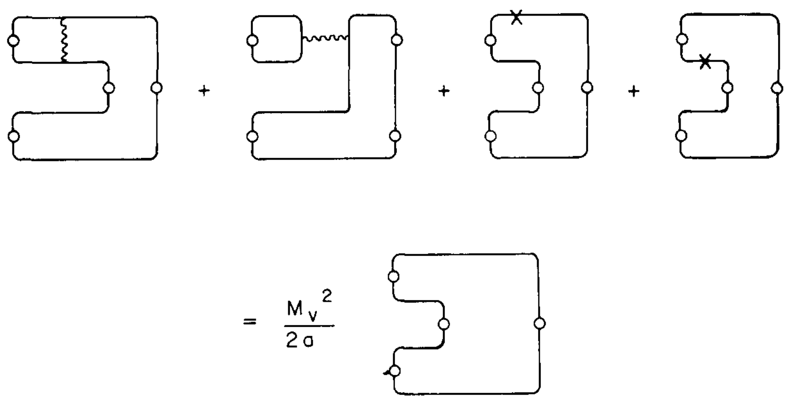


Fig. 7d.

Fig. 7. (a) Graphs constituting individual mesons. (b) Direct interactions between two mesons. (c) An annihilation graph. (d) Graphic form of the two-body Schrödinger equation.

We shall evaluate the integrals for very small s. Since $s = 0$ implies that the sum of the terms vanishes, we shall end up computing the derivative with respect to s of the sum of the terms. In order to familiarize ourselves with the sorts of expressions we shall encounter, we examine two examples:

$$F(s) = \int_0^1 \mathrm{d}x\, x^s(1-x)^s \int_0^1 \frac{\mathrm{d}y}{(x-y)^2} [x^s(1-x)^s - y^s(1-y)^s] ,$$

$$G(s) = \int_0^1 \mathrm{d}x\, x^s(1-x)^s \int_0^1 \frac{\mathrm{d}y}{(x+y)^2} [x^s(1-x)^s - y^s(1-y)^s] ;$$

$F(0) = G(0) = 0$. We might just differentiate the integrands with respect to s and set $s = 0$ to get the result. This procedure misses a significant part of the function, however. That is the contribution due to very small x where the inner integral behaves like $s^2 x^{s-1}$. Naively differentiating, setting $s = 0$, and integrating gives us zero, while we know the x integral gives us a factor $1/s$, so the actual contribution is $O(s)$. We may, however, differentiate with respect to s provided x is bounded away from any value x_E which causes the denominator of the y-integral to vanish at an endpoint of the y integral. Concretely, for F; $x = 0$ or $x = 1$ allows the denominator of the y-integral $1/(x-y)^2$ to vanish at $y = 0$ or $y = 1$. Thus, to apply naive differentiation, we must be in a region $\epsilon < x < 1 - \epsilon$. For G, only $x = 0$ is a trouble spot, so $\epsilon < x < 1$ in it. We may write (for any small ϵ, even $\epsilon = s^a$ where $a > 0$)

$$F(s) = 2 \int_0^{\epsilon} \mathrm{d}x\, x^s(1-x)^s \int_0^1 \frac{\mathrm{d}y}{(x-y)^2} [x^s(1-x)^s - y^s(1-y)^s]$$

$$+ \int_{\epsilon}^{1-\epsilon} \mathrm{d}x\, x^s(1-x)^s \int_0^1 \frac{\mathrm{d}y}{(x-y)^2} [x^s(1-x)^s - y^s(1-y)^s] \,.$$

The first integrals have been evaluated in the text in the analysis that lead to (5.1). We get

$$2 \int_0^{\epsilon} \mathrm{d}x\, x^s(-\tfrac{1}{3}x^{s-1}\pi^2 s^2) \cong -\tfrac{1}{3}\pi^2 s(\epsilon^s) = -\tfrac{1}{3}\pi^2 s$$

as $s \to 0$ (even if $\epsilon = s^a$).
The second integral can be differentiated with respect to s. Then we set $s = 0$ to get

$$\int_{\epsilon}^{1-\epsilon} \mathrm{d}x \int_0^1 \frac{\mathrm{d}x}{(x-y)^2} \ln \frac{x(1-x)}{y(1-y)} = +\tfrac{2}{3}\pi^2 + O(\epsilon) \,.$$

Thus, $F(s) = +\frac{1}{3}\pi^2 s + O(\epsilon s)$ for any given $\epsilon > 0$, so

$$F(s) = \tfrac{1}{3}\pi^2 s \,.$$

In a similar manner, we find $G(s)$ is obtained by a similar procedure. There are some differences, however. The plus sign in the denominator *and* the fact that there is only *one* sensitive endpoint tells us the contribution of $0 < x < \epsilon$ is $-\frac{1}{12}\pi^2 s$. (The plus sign replaces a csc πs with a cot πs.) The contribution from the region $\epsilon < x < 1$ is evaluated as before. As a check on these ideas, we can recover the behavior of the vector particle at $m = 0$ using $F(s)$ computed above. Assume $\psi = x^s(1-x)^s$ in eq. (4.9). We find to lowest order in s,

$$M^2 = \frac{m^2}{s} + 1 + F(s) = 1 + \frac{m^2}{s} + \tfrac{1}{3}\pi^2 s \,.$$

Assuming $m = a_0 s$, we obtain

$$M^2 = 1 + m\left(a_0 + \frac{\pi^2}{3a_0}\right),$$

which is minimized at $a_0 = \sqrt{\frac{1}{3}}\pi$ which is eq. (5.1). In what follows, we will develop each term in (6.6) about $s = 0$ paying careful attention to the endpoint contributions seen above. Another caution; although we are interested in $H(\frac{1}{2}, \frac{1}{2})$, we must consider it as a limit of $H(\frac{1}{2}, a')$ as $a' \to \frac{1}{2}$. Otherwise, we miss some contributions to $H(\frac{1}{2}, \frac{1}{2})$.

As an example, we compute the first (two) terms in (6.6)

$$\left.\frac{\partial v}{\partial s}\right|_{s=0} = -\int\limits_{a-a'}^{a} \mathrm{d}k \int\limits_{k-a}^{k} \frac{\mathrm{d}\alpha}{\alpha^2}\left(2\ln\left(1-\frac{\alpha}{k}\right) + \ln\left(1 - \frac{\alpha}{a-k}\right) + \ln\left(1 + \frac{\alpha}{P-a'-k}\right)\right)$$

The α integrals may be done analytically. Then, we define

$$F_-(A,B) = \int\limits_B^A \frac{\mathrm{d}x}{x}\ln(1-x)\,, \qquad 0 \leqslant (A,B) \leqslant 1\,,$$

$$\tilde{F}_-(A,B) = \int\limits_B^A \frac{\mathrm{d}x}{x}\ln(x-1)\,, \qquad B, A \geqslant 1\,,$$

$$F_+(A,B) = \int\limits_B^A \frac{\mathrm{d}x}{x}\ln(1+x)\,, \qquad B, A \geqslant 0\,.$$

Then,

$$\left.\frac{\partial v}{\partial s}\right|_{s=0} = -\left[3F_-\left(1, \frac{a-a'}{a}\right) + 3F_-\left(\frac{a'}{a}, 0\right) - \tilde{F}_-\left(\frac{P-a}{P-a-a'}, 1\right)\right.$$

$$\left. + F_+\left(\frac{a'}{P-a-a'}, 0\right) + F_-\left(\frac{P-a}{P-a'}, \frac{P-a-a'}{P-a'}\right) + F_-\left(\frac{a}{P-a'}, \frac{a-a'}{P-a'}\right)\right].$$

The region $k \sim a$ contributes, in addition $-\frac{1}{6}\pi^2$. At $a = a' = \frac{1}{2}$, we find this group contributes $\frac{5}{6}\pi^2 s$.

Continuing in this way, we can reduce all terms in eq. (6.6) containing integrals over α. The result for them is $\frac{11}{6}\pi^2 s$.

The fermion mass terms contribute $\frac{1}{2}\pi^2 s$. Another group contributes $(-\frac{8}{3}\pi^2 + 8(1 + 2\ln 2))s$. Now, we find $N_1 = 4(1 + 2\ln 2)s$. Therefore,

$$\alpha = 2N_1 - H_1 = 8(1 + 2\ln 2)s - 8(1 + 2\ln 2)s + (\tfrac{8}{3} - \tfrac{11}{6} - \tfrac{1}{2})\pi^2 s, \qquad \alpha = \tfrac{1}{3}\pi^2 s\,.$$

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