Lecture 5:
Luminosity and angular diameter distances

L: Remember the Friedmann-Robertson-Walker metric, for the standard model of the Universe.

$$-ds^2 = c^2 d\tau^2 = c^2 dt^2 - a^2(t)\left[dr^2 + S_k^2(r)(d\theta^2 + \sin^2\theta d\phi^2)\right] \ , \tag{1}$$

where $S_k(r) = r$ for $k = 0$ (flat case), $S_k(r) = R\sin(r/R)$ for $k = 1$ (closed case), and $S_k(r) = R\sinh(r/R)$ for $k = -1$ (open case). Note that R is the comoving radius of curvature of space here, and is a constant (does not depend on cosmic time t).

Q: If we have a ruler of physical length L, placed along a transverse direction at comoving distance r from us, what is the angle α it subtends on the sky?

$$\alpha = \frac{L}{a_e S_k(r)} = \frac{L}{D_A(r)} \ ; \qquad D_A(r) = \frac{S_k(r)}{1+z} \ . \tag{2}$$

The angular diameter distance is $D_A(r) = a_e S_k(r) = S_k(r)/(1+z)$. The scale factor a_e is that of the ruler when it emits the light we see. The redshift z is the one the observer measures for the ruler, where we assume the scale factor is normalized at the observing epoch to $a_o = 1$ (otherwise we have to multiply $S_k(r)$ by a_o).

Q: If we have a source of luminosity L observed at redshift z, emitting light isotropically at a comoving distance r from us, what is the flux f we receive from it?

The luminosity L is spread over a sphere of physical size $4\pi a_o^2 S_k(r)^2$, where the scale factor a_o is at the time of observation of the flux. However, each emitted photon is redshifted and its energy is reduced by a factor $1+z$. At the same time, the rate at which photons are received is also slowed by another factor $1+z$. So the measured flux is:

$$f = \frac{L}{4\pi a_o^2 S_k(r)^2}\frac{1}{(1+z)^2} \ . \tag{3}$$

The luminosity distance is defined as $f = L/(4\pi D_L^2)$. If the observer is at present, with normalized scale factor $a_o = 1$, then the luminosity distance is

$$f = \frac{L}{4\pi D_L^2} \ ; \qquad D_L = S_k(r)(1+z) \ . \tag{4}$$

L: First evidence for the cosmological constant: supernovae Type Ia. The method is analogous to the one used by Henrietta Leavitt to establish Cepheid variables as standard candles: the measurement of a Cepheid period-luminosity relation in the Magellanic Clouds tells you a method to infer the luminosity once you know the period. The scatter of this relation is further reduced if you measure a correlation with metallicity and correct for any dust absorption. Then you use the inferred luminosity to obtain the distance from the flux.

L: Similarly, observations of Type Ia supernovae (a special type of supernovae that show no hydrogen and are believed to arise from collisions of white dwarfs) show a relation between the duration of their lightcurve and their intrinsic luminosity, which can be tested in nearby galaxies that have independent distance measurements from Cepheid variables. Then we can use this relation to infer luminosity for more distant supernovae. This has led to the measurement of luminosity distance versus redshift up to a high redshift of about one, and only the benchmark model containing $\sim 70\%$ of the critical density as a cosmological constant, and $\sim 30\%$ as matter, matches the observations. This evidence, presented in 1998, culminated other observations (coming from ages of globular clusters and large-scale structure data) that were suggesting the presence of a cosmological constant, and was the first time that a cosmological constant was clearly required by the data. Then, in 2002, observations of the CMB by WMAP confirmed this discovery in a totally independent way.

Q: This luminosity distance tells us how total flux depends on total luminosity, or flux in a given band that is redshifted from the band in which the luminosity is emitted. How does flux per unit frequency vary? Since $d\nu_{\rm obs} = d\nu_{\rm em}/(1+z)$, we have:

$$f_{\nu/(1+z)} = \frac{L_\nu(1+z)}{4\pi D_L^2} \ . \tag{5}$$

Q: How about flux per unit wavelength? Now, a unit of wavelength in which we observe varies as $d\lambda_{\rm obs} = d\lambda{\rm em}\,(1+z)$, so:

$$f_{\lambda(1+z)} = \frac{L_\lambda}{4\pi D_L^2(1+z)} \ . \tag{6}$$