Quantum Field Theory, 2021/2022
Exercise sheet 4: Spin 1 fields & interactions
Hand-in: November 3, 2021

4.1. Consider the free electromagnetic field $A^\mu(x)$. In the Quantum Theory the Lorentz condition is substituted by a condition on the allowed physical states $|\Psi\rangle$

$$\partial_\mu A^{\mu +}(x)|\Psi\rangle = 0 \Rightarrow (a_{(3)\boldsymbol{k}} - a_{(0)\boldsymbol{k}})|\Psi\rangle = 0$$

(a) From the commutation relations among $a_{(\lambda)\boldsymbol{k}}$, $a^\dagger_{(\lambda)\boldsymbol{q}}$, show that:

$$[a_{(3)\boldsymbol{k}} - a_{(0)\boldsymbol{k}}, a^\dagger_{(3)\boldsymbol{q}} - a^\dagger_{(0)\boldsymbol{q}}] = 0$$

(b) Define

$$\alpha_i^\dagger = a^\dagger_{(3)\boldsymbol{k_i}} - a^\dagger_{(0)\boldsymbol{k_i}}$$

show that the most general allowed physical state which contains scalar and longitudinal photons can be written as:

$$|\Psi_{SL}\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots c(n_1, n_2, \ldots)\prod_{i=1}^{\infty}(\alpha_i^\dagger)^{n_i}|0\rangle$$

Where the $c(n_1, n_2, \ldots)$ are complex numbers.

(c) Show that the norm of this state is given by just the coefficient with no photons present:

$$\langle\Psi_{SL}|\Psi_{SL}\rangle = |c(0, 0, \ldots)|^2$$

4.2. Consider a real Klein-Gordon field with a cubic interaction Hamiltonian:

$$\mathcal{H}_{int} = \frac{\lambda}{3!}\phi^3$$

(a) Compute the interaction Feynman rule for the 3-particle interaction vertex.

(b) Compute, **using the Dyson expansion**, the lowest order contribution to the transition matrix element for $p_A, p_B \to k_1, k_2$:

$$\langle k_1, k_2|i\mathcal{T}|p_A, p_B\rangle$$

remember that only fully connected and amputated contributions have to be taken into account.