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Overview on the Anomaly and Schwinger Term in Two Dimensional QED*

C. Adam, R. A. Bertlmann and P. Hofer
Institut für Theoretische Physik
Universität Wien

Abstract

The axial anomaly of two dimensional QED is computed in different ways (perturbative, via dispersion integrals, path integral and index theorem) and their relation is discussed as well as the relation between anomaly, Schwinger term and the Dirac vacuum. Some features of the special case of massless fermions (Schwinger model) and some methods of exactly solving it are demonstrated.

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Contents

1 Introduction

In this article we compute the axial anomaly of QED_2 using different methods (perturbation theory, dispersion relations, exact solution) and discuss the relation between those methods.

The existance of the anomaly can be traced to the fact that n-point-functions are ambiguous when divergent diagrams contribute to them. They may be changed by polynomials of the exterior momenta (which are called "seagulls") where the degree of the polynomial equals the degree of divergence of the most divergent graph. Therefore Ward identities (WI) - the quantum consequence of a classical symmetry - may not be suggested to hold a priori. The addition of a seagull may destroy or restore a WI. When no seagull can be chosen as to fulfill all WI steming from different symmetries the theory is called anomalous. Usually one symmetry is preferred on physical grounds and the anomaly is associated entirely with the other symmetry. In our case the more important symmetry is the vector WI (VWI) and the less important one the axial WI (AWI) (see e.g. [1], [2]).

In the second section we compute the relevant n-point-functions using perturbative and dispersion methods. We evaluate the corresponding Feynman graphs in two ways, using a naive kind of regularization first where the result agrees with a coordinate space calculation (see subsection 2.5; an analogous computation is done in [2]) and with the results of some authors ([7], [14]) but does not fulfill either VWI or AWI. A proper dimensional regularization leads to normal VWI and anomalous AWI.

Next we show the connection with the dispersion relation approach and find that normal VWI and anomalous AWI can be achieved by performing a subtraction or alternatively by carefully choosing the mass dimensions of the invariant amplitudes (for the dispersion approach see [3], [4], [31]).

We, too, compute the imaginary parts of our dispersion integrals from unitarity.

In the third section for the simple case of massless fermions (Schwinger model, [8]) we compute the Schwinger term and the anomaly from the very definition of the Dirac vacuum for fermions.

Further we show how anomalous WI, Schwinger terms and seagulls are related using the wellknown methods of Jackiw and Johnson ([1], [6], see also [7]).

The fourth section reviews some possibilities of exactly solving the massless (Schwinger) model by regularizing the current operator ([8], [9], [11]) or using path integral ([10], [17]) and functional determinant regularization methods.

At last we compare the features of this exact solution with our perturbative results.

We restrict on ordinary QED_2 in this article. Concerning chiral QED_2 which, though suffering from gauge anomalies, can be quantized consistently and is treated on a similar footing like the Schwinger model, we refer to the literature (e.g. [35], [36], [34], [38], [2]).

2 Perturbative approach to the two dimensional anomaly

(Our conventions are as in [12].)

2.1 Perturbative calculations

At the beginning we will compute Feynman diagrams relevant for the axial anomaly in two dimensions. A naively regularized result will not fulfill the correct WI whereas the proper dimensional (or PV) regularized Feynman integral will do so.

The Lagrangian of QED_2 with one fermion is

$$L = \bar{\Psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \qquad \mu,\nu = 0,1. \tag{1}$$

The A-field is treated as an exterior field in the following because this is sufficient for the computation of the anomaly. The currents

$$J_\mu = \bar{\Psi}\gamma_\mu\Psi \tag{2}$$

$$J^5_\mu = \bar{\Psi}\gamma_\mu\gamma_5\Psi \tag{3}$$

$$J^5 = \bar{\Psi}\gamma_5\Psi \tag{4}$$

classically fulfill the identities

$$\partial^\mu J_\mu = 0 \tag{5}$$

and

$$\partial^\mu J^5_\mu = 2imJ^5. \tag{6}$$

The axial anomaly in QED_2 stems from the two point function

$$T^5_{\mu\nu}(x-y) = \langle T(J_\mu(x)J^5_\nu(y))\rangle \tag{7}$$

whose fermion loop is depicted in Fig. 1.

In two dimensions the relation

$$\gamma^\mu\gamma_5 = \epsilon^{\mu\nu}\gamma_\nu \tag{8}$$

holds. Therefore the tensor (7) can be expressed by the vacuum polarization tensor:

$$T^5_{\mu\nu} = \epsilon_{\nu\lambda}T^\lambda_\mu \tag{9}$$

$$T_{\mu\nu}(x-y) = \langle T(J_\mu(x)J_\nu(y))\rangle \tag{10}$$

where the Feynman graph of $T_{\mu\nu}$ is shown in Fig. 2.

According to their mass dimensions the two graphs are (naively) logarithmically divergent.

In addition we need the two point function of the graph of Fig. 3,

$$P^5_\mu(x-y) = \langle T(J_\mu(x)J^5(y))\rangle, \tag{11}$$

to formulate the relevant WI.

Now if the classical symmetries of the theory were valid the following WI should be satisfied (in momentum space)

VWI:

$$q^{\mu}T^5_{\mu\nu}(q) = 0 \tag{12}$$

naive AWI:

$$q^{\nu}T^5_{\mu\nu}(q) = 2mP^5_{\mu}(q). \tag{13}$$

For an explicit calculation of the corresponding Feynman integrals it suffices to compute $T_{\mu\nu}(q)$ because of relation (9).

We have

$$\begin{aligned}
T_{\mu\nu}(q) &= -i\mathrm{tr}\int \frac{d^2p}{(2\pi)^2}\gamma_\mu \frac{i}{\not p - m}\gamma_\nu \frac{i}{\not p - \not q - m} \\
&= i\int \frac{d^2p}{(2\pi)^2}\frac{(p^\alpha p^\beta - p^\alpha q^\beta)\mathrm{tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta + m^2\mathrm{tr}\gamma_\mu\gamma_\nu}{(p^2-m^2)[(p-q)^2-m^2]}
\end{aligned} \tag{14}$$

and, introducing the Feynman parameter integral (see appendix A.6)

$$\begin{aligned}
T_{\mu\nu}(q) &= i\int \frac{d^2p}{(2\pi)^2}\int_0^1 dx \frac{(p^\alpha p^\beta - p^\alpha q^\beta)\mathrm{tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta + m^2\mathrm{tr}\gamma_\mu\gamma_\nu}{\{x(p^2-m^2)+(1-x)[(p-q)^2-m^2]\}^2} \\
&= i\int \frac{d^2p}{(2\pi)^2}\int_0^1 dx \frac{(p^\alpha p^\beta - p^\alpha q^\beta)\mathrm{tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta + m^2\mathrm{tr}\gamma_\mu\gamma_\nu}{[p^2+2pq(x-1)+q^2(1-x)-m^2]^2}.
\end{aligned} \tag{15}$$

If we now perform the momentum integration using the n-dimensional integration formulae of the appendix (A.1)-(A.3) we observe that in our case of two dimensions formula (A.3) seems to contain the singular term $1/2g_{\mu\nu}\frac{M^2-k^2}{1-n/2}$. A naive way of regularizing the integral is done by just omitting this singular term. Doing so we get

$$\begin{aligned}
T_{\mu\nu}(q) &= i\frac{i(-\pi)}{(2\pi)^2}\int_0^1 dx \frac{[q^\alpha q^\beta(x-1)^2 + q^\alpha q^\beta(x-1)]\mathrm{tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta + m^2\mathrm{tr}\gamma_\mu\gamma_\nu}{q^2(1-x)-m^2-q^2(x-1)^2} \\
&= \frac{1}{4\pi}\int_0^1 dx \frac{q^\alpha q^\beta x(x-1)\mathrm{tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta + m^2\mathrm{tr}\gamma_\mu\gamma_\nu}{q^2x(1-x)-m^2}
\end{aligned} \tag{16}$$

and, using the two dimensional identities

$$\mathrm{tr}\gamma_\mu\gamma_\nu = 2g_{\mu\nu}, \tag{17}$$

$$\mathrm{tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta = 2(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\alpha\nu} - g_{\mu\nu}g_{\alpha\beta}) \tag{18}$$

we further have

$$T_{\mu\nu}(q) = \frac{1}{2\pi}\int_0^1 dx \frac{(2q_\mu q_\nu - q^2 g_{\mu\nu})x(x-1) + m^2 g_{\mu\nu}}{q^2x(1-x)-m^2}. \tag{19}$$

The substitution

$$y = 1 - 2x$$

leads to

$$\begin{aligned}
T_{\mu\nu}(q) &= -\frac{1}{2\pi}\int_1^{-1}\frac{dy}{2}\frac{(2q_\mu q_\nu - q^2 g_{\mu\nu})\frac{1}{4}(y^2-1) + m^2 g_{\mu\nu}}{q^2\frac{1}{4}(1-y^2)-m^2} \\
&= \frac{1}{2\pi q^2}\int_0^1 dy \frac{(2q_\mu q_\nu - q^2 g_{\mu\nu})(y^2-1) + 4m^2 g_{\mu\nu}}{(1-\frac{4m^2}{q^2})-y^2}
\end{aligned} \tag{20}$$

where we used the fact that y appears only quadratically. Now we use the integration formulae (A.4), (A.5) and, introducing the short hand notation

$$R := \sqrt{1 - \frac{4m^2}{q^2}} \quad , \quad q^2 > 4m^2 \tag{21}$$

we finally get

$$\begin{aligned} T_{\mu\nu}(q) &= -\frac{1}{2\pi q^2}(2q_\mu q_\nu - q^2 g_{\mu\nu}) + \frac{1}{4\pi q^2}(R - \frac{1}{R})\ln\frac{1+R}{1-R}(2q_\mu q_\nu - q^2 g_{\mu\nu}) \\ &\quad + \frac{1}{4\pi q^2} 4m^2 g_{\mu\nu}\frac{1}{R}\ln\frac{1+R}{1-R} \\ &= \frac{1}{\pi}(\frac{1}{2}g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{1}{4\pi q^2}(2q_\mu q_\nu - q^2 g_{\mu\nu})\frac{1}{R}(-\frac{4m^2}{q^2})\ln\frac{1+R}{1-R} \\ &\quad + \frac{1}{4\pi q^2}(-q^2 g_{\mu\nu})\frac{1}{R}(-\frac{4m^2}{q^2})\ln\frac{1+R}{1-R} \\ &= -\frac{1}{\pi}(\frac{q_\mu q_\nu}{q^2} - \frac{1}{2}g_{\mu\nu}) - \frac{2m^2}{\pi q^2}(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu})\frac{1}{R}\ln\frac{1+R}{1-R}. \end{aligned} \tag{22}$$

This result does not fulfill the requirement of gauge invariance:

$$q^\mu T_{\mu\nu}(q) = -\frac{1}{\pi}(q_\nu - \frac{1}{2}q_\nu) = -\frac{1}{2\pi}q_\nu \neq 0. \tag{23}$$

However, this result coincides with a calculation in coordinate space (see subsection (2.5)), and in addition the integration formula regularized in this naive manner is used by some authors (e.g. [14], [7]).

As a next step we do the same calculation using the proper dimensional regularization instead of the naive one.

Our starting point is expression (15) where now the momentum space dimension is n :

$$T^R_{\mu\nu}(q) = i\int\frac{d^n p}{(2\pi)^n}\int_0^1 dx\frac{(p^\alpha p^\beta - p^\alpha q^\beta)\mathrm{tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta + m^2\mathrm{tr}\gamma_\mu\gamma_\nu}{[p^2 + 2pq(x-1) + q^2(1-x) - m^2]^2}.$$

Keeping the second term of formula (A.3) which we omitted first we get

$$\begin{aligned} T^R_{\mu\nu}(q) &= T_{\mu\nu}(q) + i\frac{i(-\pi)^{\frac{n}{2}}}{(2\pi)^n}\frac{\Gamma(2-\frac{n}{2})\frac{1}{2}g^{\alpha\beta}(M^2 - k^2)}{\Gamma(2)(M^2-k^2)^{2-\frac{n}{2}}(1-\frac{n}{2})}\mathrm{tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta \\ &= T_{\mu\nu}(q) + \frac{1}{4\pi(2-n)}g^{\alpha\beta}\mathrm{tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta \end{aligned} \tag{24}$$

where we have performed the limit $n \to 2$ where it is save. Using now the n-dimensional two-spinor formula

$$g^{\alpha\beta}\mathrm{tr}\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta = 2(2-n)g_{\mu\nu} \tag{25}$$

we observe that the singularity of the additional term is cancelled by a contribution of spin matrix algebra. So in the contrary to the usual case here, when n-dimensional regularization is performed, no singular term is left at all. We get

$$T^R_{\mu\nu}(q) = T_{\mu\nu}(q) + \frac{1}{2\pi} g_{\mu\nu} = -\frac{1}{\pi}(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu})(1 + \frac{2m^2}{q^2}\frac{1}{R}\ln\frac{1+R}{1-R}) =: (q_\mu q_\nu - g_{\mu\nu}q^2)T(q^2) \quad (26)$$

which obviously is gauge invariant.

For later convenience we display the two point functions for the massless case $m = 0$, which, looking at expression (19), at once can be seen to be

$$T_{\mu\nu}(m=0,q) = -\frac{1}{\pi}(\frac{q_\mu q_\nu}{q^2} - \frac{1}{2}g_{\mu\nu}) \quad (27)$$

$$T^R_{\mu\nu}(m=0,q) = -\frac{1}{\pi}(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}). \quad (28)$$

To check the AWI we now need the two point function P^5_μ. Its Feynman integral looks like

$$P^5_\mu(q) = -i\int \frac{d^2p}{(2\pi)^2}\frac{i^2 \mathrm{tr}\gamma_\mu(\not p + m)\gamma_5(\not p - \not q + m)}{(p^2 - m^2)[(p-q)^2 - m^2]} \quad (29)$$

and is convergent. Evaluating the trace we get

$$-\mathrm{tr}\gamma_\mu\gamma_5 m\not q = -2\epsilon_{\mu\nu} m q^\nu. \quad (30)$$

The further steps are in close analogy to the computation above and result in

$$P^5_\mu(q) = -2im\epsilon_{\mu\nu}q^\nu \int\frac{d^2p}{(2\pi)^2}\int_0^1 \frac{dx}{\{x(p^2-m^2) + (1-x)[(p-q)^2 - m^2]\}^2}$$

$$= -\epsilon_{\mu\nu}q^\nu \frac{2m}{\pi}\int_0^1 \frac{dy}{q^2(1-y^2) - 4m^2} \quad (31)$$

$$= -\epsilon_{\mu\nu}q^\nu \frac{m}{\pi q^2}\frac{1}{R}\ln\frac{1+R}{1-R} \quad (32)$$

where R is as in (21).

Now we can check VWI and AWI for the two differently regularized two point functions $T^5_{\mu\nu}$ and $T^{5,R}_{\mu\nu}$. Having

$$T^5_{\mu\nu}(q) = \epsilon_{\nu\lambda}T^\lambda_\mu$$

and using the results (22), (26) and (32) we find that $T^{5,R}_{\mu\nu}$ *fulfills naive* VWI *and anomalous* AWI,

$$q^\mu T^{5,R}_{\mu\nu}(q) = 0 \quad (33)$$

$$q^\nu T^{5,R}_{\mu\nu}(q) = 2mP^5_\mu(q) + \frac{1}{\pi}q^\lambda\epsilon_{\lambda\mu} \quad (34)$$

whereas for the naively regularized $T^5_{\mu\nu}$ the anomaly is equally spread on both WI:

$$q^\mu T^5_{\mu\nu}(q) = \frac{1}{2\pi}q^\lambda\epsilon_{\lambda\nu} \quad (35)$$

$$q^\nu T^5_{\mu\nu}(q) = 2mP^5_\mu(q) + \frac{1}{2\pi}q^\lambda\epsilon_{\lambda\nu}. \tag{36}$$

So we find that the two differently regularized two point functions differ by a constant (times the metric tensor), the only allowed seagull term in this case. Writing $\mathbf{A} = \frac{1}{\pi}$ for the constant (i.e. the anomaly) we have

$$T^{5,R}_{\mu\nu}(q) = T^5_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu}\mathbf{A}$$

but when we add half the anomaly instead of subtracting it from the naively regularized result we get a two point function $\bar{T}_{\mu\nu}$that respects axial symmetry instead of gauge invariance

$$\bar{T}^5_{\mu\nu}(q) = T^5_{\mu\nu}(a) + \frac{1}{2}\epsilon_{\mu\nu}\mathbf{A}$$

$$q^\nu\bar{T}^5_{\mu\nu}(q) = 2mP^5_\mu,$$

and the anomaly entirely occurs in the VWI

$$q^\mu\bar{T}^5_{\mu\nu}(q) = \frac{1}{\pi}q^\lambda\epsilon_{\lambda\nu}.$$

Obviously no seagull can be chosen as to fulfill both VWI and AWI.

2.2 Dispersion relation approach

For the dispersion relation approach to work we need a decomposition of the two point functions into invariants. In addition gauge invariance cannot be suggested to hold a priori, as we have seen in the perturbative calculation, but must be demanded by carefully regularizing (i.e. fixing the subtraction point in dispersion integrals).

Therefore we write the general Lorentz decomposition

$$T_{\mu\nu}(q) = q_\mu q_\nu T_1(q^2) - g_{\mu\nu}T_2(q^2) \tag{37}$$

and

$$P^5_\mu(q) = \epsilon_{\mu\nu}q^\nu P(q^2). \tag{38}$$

It is a wellknown fact that all naive WI must hold for the imaginary parts of the invariant amplitudes (setting $q^2 := t$):

$$\mathrm{Im}tT_1(t) - \mathrm{Im}T_2(t) = 0 \tag{39}$$

$$\mathrm{Im}T_2(t) = 2m\mathrm{Im}P(t). \tag{40}$$

The amplitudes T_1 and P are convergent according to their mass dimensions, so they will be expressed by unsubtracted dispersion relations and we can find their imaginary parts by simply transforming our perturbative results. Taking formula (20) as a starting point we find

$$T_1(q^2) = \frac{1}{\pi q^2}\int_0^1 dy\frac{y^2-1}{1-\frac{4m^2}{q^2}-y^2} = -\frac{1}{\pi}\int_0^1 dy\frac{1-y^2}{(1-y^2)q^2-4m^2}$$

and substituting

$$t = \frac{4m^2}{1-y^2} \quad , \quad dy = \frac{1}{2}(1-\frac{4m^2}{t})^{-1/2}dt \tag{41}$$

results in

$$T_1(q^2) = -\frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\frac{4m^2}{t}}{\frac{4m^2}{t}q^2 - 4m^2}\frac{1}{2}(1-\frac{4m^2}{t})^{-1/2}\frac{4m^2}{t^2}dt$$

$$= \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{dt}{t-q^2}\frac{2m^2}{t^2}(1-\frac{4m^2}{t})^{-1/2}.$$

So we find

$$\mathrm{Im}T_1(t) = \frac{2m^2}{t^2}(1-\frac{4m^2}{t})^{-1/2} \tag{42}$$

and, starting at (31), in a completely analog manner,

$$\mathrm{Im}P(t) = \frac{m}{t}(1-\frac{4m^2}{t})^{-1/2}. \tag{43}$$

Using the WI (39) or (40) we get for $\mathrm{Im}T_2$ the unique result

$$\mathrm{Im}T_2(t) = \frac{2m^2}{t}(1-\frac{4m^2}{t})^{-1/2}. \tag{44}$$

Computing the imaginary parts by Cutkosky cutting rule gives the same results (see subsection 2.4).

What about the real parts? Observe that in the VWI (39) the two amplitudes $\mathrm{Im}T_1, \mathrm{Im}T_2$ differ in mass dimension by a power of the momentum variable t whereas in the AWI (40) $\mathrm{Im}T_2$ and $\mathrm{Im}P$ only differ by a constant (m). This will cause a naive AWI and an anomalous VWI for the real parts if all ampiitudes are taken as unsubtracted dispersion integrals. Indeed we have

$$T_2(q^2) = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{dt}{t-q^2}\frac{2m^2}{t}(1-\frac{4m^2}{t})^{-1/2} = 2mP(q^2) \tag{45}$$

and

$$q^2T_1(q^2) = \frac{q^2}{\pi}\int_{4m^2}^{\infty} \frac{dt}{t-q^2}\frac{2m^2}{t^2}(1-\frac{4m^2}{t})^{-1/2}$$

$$= T_2(q^2) - \frac{1}{\pi}\int_{4m^2}^{\infty} dt\frac{2m^2}{t^2}(1-\frac{4m^2}{t})^{-1/2}$$

$$= T_2(q^2) - \frac{1}{\pi}. \tag{46}$$

Substituting

$$u = \frac{4m^2}{t} \quad , \quad dt = -du\frac{4m^2}{u^2},$$

we get for the integral

$$\int_{4m^2}^{\infty} dt\frac{2m^2}{t^2}(1-\frac{4m^2}{t})^{-1/2} = \int_0^1 \frac{du}{2(1-u)^{1/2}} = 1. \tag{47}$$

But we already know that T_2 is the only term that may be changed by seagulls, according to it's mass dimension of 0. In a most direct way we see this by demanding that all two point functions must be finite for $q \to \infty$: Looking at (37) and (38) this requirement obviously fixes T_1 and P by $T_1(q=\infty) = 0$ and $P(q=\infty) = 0$ whereas for T_2 it must only hold that $T_2(q=\infty) = \mathrm{const}$. In the dispersion integral therefore Cauchy's theorem tells us that we may perform a subtraction in T_2 in order to restore gauge invariance. Subtracting at $q^2 = 0$ we get

$$T_2^R(q^2) = T_2(q^2) - T_2(0) = \frac{q^2}{\pi}\int_{4m^2}^{\infty}\frac{dt}{t(t-q^2)}\mathrm{Im}T_2(t) \tag{48}$$

and using (45) and (47),

$$T_2(0) = \frac{1}{\pi}\int_{4m^2}^{\infty} dt\frac{2m^2}{t^2}(1-\frac{4m^2}{t})^{-1/2} = \frac{1}{\pi},$$

we find that the once subtracted amplitude T_2^R fulfills normal VWI and anomalous AWI:

$$q^2T_1(q^2) = T_2^R(q^2) \tag{49}$$

$$T_2^R(q^2) = 2mP(q^2) - \frac{1}{\pi}. \tag{50}$$

Certainly this result, i.e. using amplitude (48) for the two point function, is exactly the same as the dimensional regularized gauge invariant result of the perturbative calculation as can be seen easily by reversing the transformation from (41) to (42).

We notice that the unsubtracted amplitude T_2 of (45) may be expressed by the trace of the *naively* regularized two point function of the perturbative calculations. Indeed from (20) we compute

$$T_\mu^\mu(q) = \frac{1}{2\pi q^2}\int_0^1 dy\frac{8m^2}{1-\frac{4m^2}{q^2}-y^2} = \frac{1}{\pi}\int_0^1 dy\frac{4m^2}{(1-y^2)q^2-4m^2}$$
$$= -\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{dt}{t-q^2}\frac{2m^2}{t}(1-\frac{4m^2}{t})^{-1/2} = -T_2(q^2)$$

where the substitution was performed as in (41).

Alternatively we could have started with a different decomposition of the two point function $T_{\mu\nu}$:

$$T_{\mu\nu}(q) = q_\mu q_\nu T_1(q^2) - g_{\mu\nu}q^2\tilde{T}_2(q^2). \tag{51}$$

Here the two amplitudes T_1 and $\tilde{T}_2$ have the same mass dimensions so the VWI of the imaginary parts

$$\mathrm{Im}T_1(t) = \mathrm{Im}\tilde{T}_2(t) \tag{52}$$

must hold for the real parts, too,

$$T_1(q^2) = \tilde{T}_2(q^2) := T(q^2)$$

and we may write

$$T_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu}q^2)T(q^2). \tag{53}$$

On the contrary, the naive AWI which certainly holds for the imaginary parts,

$$\mathrm{Im}tT(t) = 2m\mathrm{Im}P(t), \tag{54}$$

is no longer true for the real amplitudes:

$$q^2T(q^2) = \frac{q^2}{\pi}\int_{4m^2}^{\infty}\frac{dt}{t-q^2}\mathrm{Im}T(t)$$

$$= \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{dt}{t-q^2}t\,\mathrm{Im}T(t) - \frac{1}{\pi}\int_{4m^2}^{\infty}dt\mathrm{Im}T(t)$$

$$= 2mP(t) - \frac{1}{\pi} \tag{55}$$

where $\mathrm{Im}T = \mathrm{Im}T_1$ is given in (42).

So choosing the decomposition (51) we get the correct WI without subtractions (which are not allowed in this case because all amplitudes are uniquely fixed).

Precisely this decomposition (51), (53) was used by Dolgov and Zakharov [3] in their original work on the dispersion relation approach (which to our knowledge first was applied by [31] to calculate the anomaly in the pion-nucleon model). For a review and more details on the ABJ anomaly see Hořejší [4].

2.3 Pauli-Villars-regularization

PV-regularization, too, selects the gauge invariant result (26) (as it must be because the invention of an infinite regulator mass respects gauge invariance but does not respect axial symmetry): using the general decomposition

$$T_{\mu\nu}(q) = q_\mu q_\nu T_1(q^2) - g_{\mu\nu}T_2(q^2)$$

we know that because of the logarithmic degree of divergence of $T_{\mu\nu}$ T_1 is uniquely fixed whereas T_2 may be changed by a constant.

The PV-regularized two point function is fixed by

$$T_{\mu\nu}^{PV}(m,q) = T_{\mu\nu}(m,q) - \lim_{M\to\infty} T_{\mu\nu}(M,q) \tag{56}$$

or equivalently by requiring that $T_1^{PV}(m = \infty, q^2) = 0$ and $T_2^{PV}(m = \infty, q^2) = 0$. This requirement contradicts normal AWI. Having $P_\mu^5 = \epsilon_{\mu\nu}q^\nu P$ as in (38), the naive AWI is

$$T_2(q^2) = 2mP(q^2),$$

and, using formula (31), we find

$$\lim_{m\to\infty} 2mP(m,q^2) = \lim_{m\to\infty} -\frac{4m^2}{\pi}\int_0^1 \frac{dy}{q^2(1-y^2)-4m^2} = \frac{1}{\pi}. \tag{57}$$

We conclude that

$$T_2^{PV}(m,q^2) = 2mP(q^2) - \frac{1}{\pi}. \tag{58}$$

T_1 is unique and fulfills the requirement of PV-regularization (as we can see e.g. from (19)). Looking at (22) or (26) for T_1 we see that, as claimed, the PV regularized amplitudes fulfill naive VWI

$$q^2 T_1^{PV}(m,q^2) = T_2^{PV}(m,q^2). \tag{59}$$

2.4 Unitarity relation for the two-dimensional vacuum polarization amplitude

As a consequence of the unitarity for the scattering matrix the following relation for the transition amplitude in 2 dimensions in a general notation can be shown

$$\mathrm{Im} < i \mid T \mid i >= \frac{1}{2} \sum_f (2\pi)^2 \delta^2(P_f - P_i) < i \mid T \mid f >< f \mid T \mid i > . \tag{60}$$

$\sum_f$ means the sum over all possible intermediate states. This equation may be rewritten in a concrete notation for two intermediate states a,b

$$\mathrm{Im} < i \mid T \mid i >= \frac{1}{2} \int \frac{d^2 p_a}{2\pi} \frac{d^2 p_b}{2\pi} \delta_+(p_a^2 - m_a^2) \delta_+(p_b^2 - m_b^2)(2\pi)^2 \delta^2(P_i - p_a - p_b) \mid < a, b \mid T \mid i > \mid^2 . \tag{61}$$

Now we want to derive the imaginary part of the vacuum polarization amplitude. For the graphical representation of the unitarity relation see Figure 4.
Therefore in our notation we achieve for the vacuum polarization amplitude $T_{\mu\nu}$

$$< i \mid T \mid i >= -e^2 < \varepsilon^\mu \varepsilon^\nu > T_{\mu\nu}$$

and by the use of the unitarity relation it is easy to calculate:

$$\mathrm{Im}(< \varepsilon^\mu \varepsilon^\nu > e^2 T_{\mu\nu}) =$$

$$- < \varepsilon^\mu \varepsilon^\nu > \frac{e^2}{2} \int d^2 p_1 d^2 p_2 \delta_+(p_1^2 - m_1^2) \delta_+(p_2^2 - m_2^2) \delta^2(q - p_1 - p_2) \sum_{spins} \bar{u}(p_1) \gamma_\mu v(p_2) \bar{v}(p_2) \gamma_\nu u(p_1).$$

$< \varepsilon^\mu \varepsilon^\nu >$ means the average over all photon polarizations and

$$\sum_{spins} \bar{u}(p_1) \gamma_\mu v(p_2) \bar{v}(p_2) \gamma_\nu u(p_1) = -\mathrm{tr}(\gamma_\mu (\not{p}_2 - m) \gamma_\nu (\not{p}_1 + m)).$$

Integrating over a δ-function and setting $p_1 = p$ we obtain the following expression

$$e^2 < \varepsilon^\mu \varepsilon^\nu > \mathrm{Im}(T_{\mu\nu}) =$$

$$< \varepsilon^\mu \varepsilon^\nu > (\frac{e^2}{2}) \int d^2 p \delta_+(p^2 - m^2) \delta_+((q - p)^2 - m^2) \mathrm{tr}(\gamma_\mu (\not{q} - \not{p} - m) \gamma_\nu (\not{p} + m)).$$

Now we amputate the photons on both sides and arrive at the same result as in the literature [14]

$$\mathrm{Im} T_{\mu\nu} = \frac{1}{2} \int d^2 p \delta_+(p^2 - m^2) \delta_+((q - p)^2 - m^2) \mathrm{tr}(\gamma_\mu (\not{q} - \not{p} - m) \gamma_\nu (\not{p} + m)). \tag{62}$$

Using the vector current conservation

$$T_{\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu}) T(q^2) \quad , \quad g^{\mu\nu} T_{\mu\nu} = -q^2 T(q^2)$$

we obtain for the amplitude

$$\mathrm{Im} q^2 T(q^2) = 2m^2 \int d^2 p \delta_+(p^2 - m^2) \delta_+((q - p)^2 - m^2). \tag{63}$$

There is another way to achieve the same result - by the use of Cutkosky's cutting rule (see [15]): The discontinuity $\mathrm{Disc}(q^2T(q^2))$ is obtained by cutting the propagators

$$\mathrm{Disc}(\frac{1}{q^2 - m^2 + i\epsilon}) = -2\pi i\delta_+(q^2 - m^2).$$

Using the amplitude

$$q^2T(q^2) = -i\int \frac{d^2p}{(2\pi)^2}\frac{4m^2}{(p^2 - m^2)((q-p)^2 - m^2)}$$

and recalling the relation

$$\mathrm{Disc}(q^2T(q^2)) = 2i\mathrm{Im}q^2T(q^2)$$

we reproduce the above imaginary part of the scalar vacuum polarization amplitude. (See result (14) using VWI).
The next job is to solve the phase space integral. The shortest way is to calculate these integrals in a special inertial system ($q = \binom{q_0}{0}$).

$$\mathrm{Im}q^2T(q^2) = 2m^2\int dp_0dp_1\delta_+(p_0^2 - p_1^2 - m^2)\delta_+((q_0 - p_0)^2 - p_1^2 - m^2)$$

For the δ-function we have

$$\delta(p_0^2 - p_1^2 - m^2) = \frac{1}{2\sqrt{p_1^2 + m^2}}(\delta(p_0 + \sqrt{p_1^2 + m^2}) + \delta(p_0 - \sqrt{p_1^2 + m^2}))$$

Integrating over dp_0 gives

$$\mathrm{Im}q^2T(q^2) = \frac{m^2}{2q_0}\int \frac{dp_1}{\sqrt{p_1^2 + m^2}}\delta_+(\frac{q_0}{2} - \sqrt{p_1^2 + m^2}).$$

Using the properties of δ-functions we obtain the following relation for the second δ-function

$$\delta(\frac{q_0}{2} - \sqrt{p_1^2 + m^2}) =$$

$$\frac{q_0}{2}(\frac{q_0^2}{4} - m^2)^{-\frac{1}{2}}(\delta(p_1 - \sqrt{\frac{q_0^2}{4} - m^2}) + \delta(p_1 + \sqrt{\frac{q_0^2}{4} - m^2}))$$

Integrating again provides the final result for the imaginary part of the vacuum polarization amplitude in this special Lorentz frame

$$\mathrm{Im}q^2T(q^2) = \frac{2m^2}{q_0^2}\frac{1}{\sqrt{1 - \frac{4m^2}{q_0^2}}}$$

or written in Lorentz invariant notation

$$\mathrm{Im}q^2T(q^2) = \frac{2m^2}{q^2}\frac{1}{\sqrt{1 - \frac{4m^2}{q^2}}}. \tag{64}$$

Therefore the result holds in every Lorentz frame.

2.5 Two-point-function in coordinate space

For the simple massless case we now compute the Green's function in coordinate space (see e.g. [2]). Using the Wick theorem we get

$$T_{\mu\nu}(x-y) = \langle 0|T(J_\mu(x)J_\nu(y))|0\rangle = \langle 0|T(\bar{\Psi}(x)\gamma_\mu\Psi(x)\bar{\Psi}(y)\gamma_\nu\Psi(y))|0\rangle$$

$$= \mathrm{tr}S(x-y)\gamma_\mu S(x-y)\gamma_\nu \tag{65}$$

where

$$S(z) = \frac{-iz^\mu\gamma_\mu}{2\pi(z^2-i\epsilon)} \tag{66}$$

is the Feynman propagator for massless fermions in two dimensions. The massless scalar propagator is (we do not care about IR regularization because we only need derivatives)

$$D(z) = \frac{-1}{4\pi}\ln(-z^2+i\epsilon) = \frac{-i}{4\pi}\int_0^\infty \frac{ds}{s}e^{-is\frac{z^2+i\epsilon}{4}} + \text{const.} \tag{67}$$

and the relations

$$-i\partial\!\!\!/ D(z) = S(z) \tag{68}$$

$$\partial\!\!\!/ S(z) = -i\partial\!\!\!/\partial\!\!\!/ D(z) = \frac{-i\gamma_\alpha\gamma_\beta}{2\pi}\left(\frac{g^{\alpha\beta}-2z^\alpha z^\beta}{(z^2-i\epsilon)^2}\right) \tag{69}$$

hold. Because of the two dimensional identity

$$g^{\alpha\beta}\mathrm{tr}\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu = 0$$

we can write for the Green's function:

$$T_{\mu\nu}(z) = S^\alpha(z)S^\beta(z)\mathrm{tr}\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu$$

$$= \frac{z^\alpha z^\beta}{(2\pi)^2(z^2-i\epsilon)^2}\mathrm{tr}\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu$$

$$= \frac{-1}{4\pi}\partial^\alpha\partial^\beta D(z)2(g_{\alpha\mu}g_{\beta\nu}+g_{\alpha\nu}g_{\beta\mu}-g_{\alpha\beta}g_{\mu\nu})$$

and finally

$$T_{\mu\nu}(z) = \frac{-1}{2\pi}(2\partial_\mu\partial_\nu D(z) - g_{\mu\nu}\Box D(z)). \tag{70}$$

This is the coordinate space version of the naively regularized result (27): using

$$\Box D(z) = -\delta(z) \tag{71}$$

we get the WI

$$\partial^\mu T^5_{\mu\nu}(z) = \frac{1}{2\pi}\epsilon_{\nu\lambda}\partial^\lambda\delta(z) \tag{72}$$

and

$$\partial^\nu T^5_{\mu\nu}(z) = -\frac{1}{2\pi}\epsilon_{\mu\lambda}\partial^\lambda\delta(z). \tag{73}$$

The dimensionally regularized result is regained by adding a covariant seagull term:

$$T^R_{\mu\nu}(z) = T_{\mu\nu}(z) - \frac{1}{2\pi} g_{\mu\nu}\delta(z) \tag{74}$$

fulfilling the correct WI

$$\partial^\mu T^{5R}_{\mu\nu}(z) = 0 \tag{75}$$

and

$$\partial^\nu T^{5R}_{\mu\nu}(z) = -\frac{1}{\pi}\epsilon_{\mu\lambda}\partial^\lambda\delta(z). \tag{76}$$

3 Schwinger term, anomaly and the Dirac sea

3.1 Computation of the Schwinger term from the Dirac vacuum

The Dirac equation even allows solutions with negative energy. To save the stability of the quantum mechanical system Dirac invented the wellknown "hole theory". For this purpose he redefined the vacuum: in the Dirac vacuum the states having negative energy are completely filled up, and an empty state having negative energy (a hole) is interpreted as an antiparticle having positive energy. This new physical vacuum must be considered in the canonical commutation relations of the second quantization. It can be shown that filling up the infinitely deep Dirac sea causes anomalous Schwinger terms in the commutator of two fermion currents which is done here for two dimensional massless fermions (see [5], [13], [38], [39]).

The two dimensional massless spinors are

$$\Psi_i(x,t) = \frac{1}{\sqrt{2\pi}} \int dk e^{-ikx} a_{i,k}(t) \tag{77}$$

$$\Psi_i^*(x,t) = \frac{1}{\sqrt{2\pi}} \int dk e^{-ikx} a_{i,k}^*(t) \qquad i = 1,2 \tag{78}$$

where the a's fulfill the canonical anti-commutation relations

$$\{a_{i,k}^*(t), a_{j,l}(t')\} \mid_{t=t'} = \delta(k-l)\delta_{ij} \quad , \quad a_k \mid 0\rangle = 0. \tag{79}$$

This leads to an indefinite hamiltonian of the free fermion fields:

$$H_0 = \int_{-\infty}^{\infty} dkk(a_{1,k}^* a_{1,k} - a_{2,k}^* a_{2,k}). \tag{80}$$

To remedy this problem and to have a positive definite hamiltonian all negative energy states are filled up and new creation and annihilation operators are defined:

$$a_{1,k} = b_k \Theta(k) + c_k^*(1 - \Theta(k)) \tag{81}$$

$$a_{2,k} = b_k(1 - \Theta(k)) + c_k^* \Theta(k) \tag{82}$$

$$a_{1,k}^* = b_k^* \Theta(k) + c_k(1 - \Theta(k)) \tag{83}$$

$$a_{2,k}^* = b_k^*(1 - \Theta(k)) + c_k \Theta(k) \tag{84}$$

which fix the Dirac vacuum

$$b_k \mid 0\rangle_D = c_k \mid 0\rangle_D = 0 \tag{85}$$

and fulfill the anti-commutation relations

$$\{b_k^*, b_l\} = \delta(k-l) \quad , \quad \{c_k^*, c_l\} = \delta(k-l) \tag{86}$$

and all other anti-commutators vanish.

The filling up of the infinitely deep Dirac sea now changes the fermi current commutators ([18], [19], [5]): using the representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \gamma^0\gamma^1 = \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{87}$$

the fermion currents of QED_2 read

$$J^{1,5}(x) = \bar{\Psi}(x)\gamma^1\gamma^5\Psi(x) = \Psi_1^*(x)\Psi_1(x) + \Psi_2^*(x)\Psi_2(x) = J^0(x) \tag{88}$$

$$J^{0,5}(x) = \bar{\Psi}(x)\gamma^0\gamma^5\Psi(x) = \Psi_2^*(x)\Psi_2(x) - \Psi_1^*(x)\Psi_1(x) = J^1(x). \tag{89}$$

Introducing the current amplitudes $\rho_i(p)$ in momentum space and normal ordering we have

$$:\Psi_i^*(x)\Psi_i(x) := \frac{1}{2\pi}\int dp e^{-ipx} : \rho_i(p) : \quad , \quad i = 1,2 \tag{90}$$

where

$$\rho_i(p) = \int_{-\infty}^{\infty} dk a_{i,k+p}^* a_{i,k} \tag{91}$$

$$\rho_i(-p) = \int_{-\infty}^{\infty} dk a_{i,k}^* a_{i,k+p} \qquad \text{for} \qquad p > 0 \tag{92}$$

It can be shown ([13]) that after introducing the Dirac vacuum only normal ordered currents can exist.

In the old vacuum the current commutator in momentum space is

$$[\rho_i(p), \rho_i(-p')] = \int_{-\infty}^{\infty} dk dk' [a_{i,k+p}^* a_{i,k}, a_{i,k'}^* a_{i,k'+p'}]$$

$$= \cdots = \int_{-\infty}^{\infty} dk (a_{i,k+p}^* a_{i,k+p'}) - \int_{-\infty}^{\infty} dk (a_{i,k+p-p'}^* a_{i,k}) = 0 \tag{93}$$

as can be shown by shifting the second integration variable by $k \rightarrow k + p'$.

But when the normal ordered current commutator is considered with respect to the new Dirac vacuum annihilation and creation operators the situation changes. To see more precisely what happens we fill the Dirac sea only up to a finite amount of negative energy at the beginning. In this situation old and new operators differ by the following ([13]):

$$a_{1,k} = b_k\Theta_\Lambda(k) + c_k^*(1 - \Theta_\Lambda(k)) \tag{94}$$

$$a_{1,k}^* = b_k^*\Theta_\Lambda(k) + c_k(1 - \Theta_\Lambda(k)) \tag{95}$$

where

$$\Theta_\Lambda(k) = \begin{matrix} 1 & \text{for} & k > 0 \\ 0 & \text{for} & -\Lambda < k < 0 \\ 1 & \text{for} & k < -\Lambda \end{matrix} \quad , \quad \Theta_\Lambda(k) \longrightarrow_{\Lambda\rightarrow\infty} \Theta(k). \tag{96}$$

We get

$$[:\rho_1(p):, :\rho_1(-p'):] =: [\rho_1(p), \rho_1(-p')] : -\delta(p-p')\int_{-\infty}^{\infty} dk(\Theta_\Lambda(k+p) - \Theta_\Lambda(k)) \tag{97}$$

and as before $:[\rho_1(p), \rho_1(-p')]:$ vanishes upon shifting the integration variable.

The second term even vanishes for finite Λ:

$$-\delta(p-p')\left(\int_{k>-p, k<-\Lambda-p} dk - \int_{k>0, k<-\Lambda} dk\right) = 0. \tag{98}$$

But when the Dirac sea is filled up to a infinite depth, i.e. $\Lambda \to \infty$, then

$$- \delta(p-p') \int_{-\infty}^{\infty} dk \Theta(k+p) - \Theta(k) = -p\delta(p-p') \tag{99}$$

and therefore

$$[\lim_{\Lambda\to\infty} : \rho_1(p) :, \lim_{\Lambda\to\infty} : \rho_1(-p') :] = -p\delta(p-p'). \tag{100}$$

In coordinate space this leads to

$$[: \Psi_1^*(x)\Psi_1(x) :, : \Psi_1^*(y)\Psi_1(y) :] = \frac{i}{2\pi}\partial^y\delta(x-y). \tag{101}$$

In the same way can be shown

$$[: \rho_2(p) :, : \rho_2(-p') :] = p\delta(p-p') \tag{102}$$

$$[: \Psi_2^*(x)\Psi_2(x) :, : \Psi_2^*(y)\Psi_2(y) :] = -\frac{i}{2\pi}\partial^y\delta(x-y) \tag{103}$$

and

$$[: \Psi_1^*(x)\Psi_1(x) :, : \Psi_2^*(y)\Psi_2(y) :] = 0. \tag{104}$$

For the commutators of the free fermion currents this just causes the Schwinger anomaly:

$$[J^{0,5}(x,t), J^{1,5}(y,t)] = [: \Psi_2^*(x)\Psi_2(x) :, : \Psi_2^*(y)\Psi_2(y) :] - [: \Psi_1^*(x)\Psi_1(x) :, : \Psi_1^*(y)\Psi_1(y) :] \tag{105}$$

and therefore

$$[J^{0,5}(x,t), J^{1,5}(y,t)] = -\frac{i}{\pi}\partial^y\delta(x-y), \tag{106}$$

$$[J^{0,5}(x,t), J^{0,5}(y,t)] = [J^{1,5}(x,t), J^{1,5}(y,t)] = 0. \tag{107}$$

As a special feature of two dimensions it even holds that

$$[J^1(x,t), J^0(y,t)] = [J^{0,5}(x,t), J^{1,5}(y,t)] = -\frac{i}{\pi}\partial^y\delta(x-y). \tag{108}$$

3.2 Schwinger term and seagull in the BJL-limit

(see [1], [20] for more details)

The canonically defined T-product

$$T_{\mu\nu}(x-y) = \Theta(x_0 - y_0)J_\mu(x)J_\nu(y) + \Theta(y_0 - x_0)J_\nu(y)J_\mu(x) \tag{109}$$

need not be Lorentz covariant and can differ from the Lorentz covariant perturbative result T^* by socalled *seagull terms*:

$$T^*_{\mu\nu}(x-y) = T_{\mu\nu}(x-y) + C_{\mu\nu} + \tilde{C}_{\mu\nu} \tag{110}$$

where

$T_{\mu\nu}$ and $C_{\mu\nu}$ are Lorentz covariant and

$T_{\mu\nu}$ and $C_{\mu\nu}$ are not Lorentz covariant.

In a similar fashion the equal time commutator (ETC) may deviate from the canonical result by gradients of the delta function (socalled Schwinger terms):

$$\delta(x_0 - y_0)[J_\mu(x), J_\nu(y)] = S^1_{\mu\nu}\partial_1\delta(x-y). \tag{111}$$

The non-covariant part $C_{\mu\nu}$ of the seagull is determined by the Schwinger term $S^1_{\mu\nu}$.

A possibility of directly computing the Schwinger and seagull terms once the result from perturbation theory is known is given by the Bjorken-Johnson-Low limit (BJL-limit). In this limit the ETC for the two currents is defined by the relation

$$S^1_{0\nu} = \lim_{q_0\to\infty} q^0 T_{0\nu}(q) \tag{112}$$

Now we will insert our results of the perturbative calculation for the massless case. The Green's functions (27), (28) differ by a multiple of the metric tensor. Therefore we use the general form

$$T^*_{\mu\nu}(q) = \frac{1}{\pi}(ag_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) \tag{113}$$

where a is a parameter. This corresponds to the freedom of redefining a T^*-product by a covariant constant when logarithmic divergent graphs contribute to it. Using T^* instead of T in the above limit produces additional polynomials in q_0 which must be omitted. Besides they are of no importance in an ETC. We get

$$q_0 T^*_{00} = \frac{1}{\pi}q_0(a-1) \quad \to \quad 0 \tag{114}$$

$$q_0 T^*_{01} = -\frac{1}{\pi}q_1 \tag{115}$$

and therefore the ETC

$$\delta(x_0 - y_0)[J^0(x), J^1(y)] = \frac{i}{\pi}\partial^1\delta(x-y) \tag{116}$$

(concerning the signs notice that we are jumping between coordinate and momentum space).

Even the seagull term itself can be determined by a similar limit. The T-product should vanish in the limes $q_0 \to 0$ and the seagulls remain:

$$\lim_{q_0\to 0} T^*_{\mu\nu}(q) = \frac{1}{\pi}ag_{\mu\nu} - \frac{1}{\pi}\delta_{\mu 0}\delta_{\nu 0} \tag{117}$$

and thus

$$\tilde{C}_{\mu\nu} = \frac{1}{\pi}ag_{\mu\nu} \tag{118}$$

and

$$C_{\mu\nu} = -\frac{1}{\pi}\delta_{\mu 0}\delta_{\nu 0} \tag{119}$$

or in a covariant but frame dependent notation

$$C_{\mu\nu} = -\frac{1}{\pi}n_\mu n_\nu \tag{120}$$

where n_μ is a timelike unit vector pointing into an unspecified time direction. In this notation the corresponding Schwinger term is given by (see [20], [1] for more details)

$$S^{\mu\nu\alpha} = \frac{\partial}{\partial n_\alpha} C^{\mu\nu} = -\frac{1}{\pi}(g^{\alpha\mu}n^\nu + g^{\alpha\nu}n^\mu) \tag{121}$$

In our old notation with time direction $n = (1,0)$ we therefore get for the ETC

$$\delta(x_0 - y_0)[J_0(x), J_1(y)] = \frac{i}{\pi}\partial_1\delta(x-y)$$

as above. The space-space component of the ETC obviously vanishes.

The coefficient a of the covariant seagull can only be fixed by imposing special WI on $T^*_{\mu\nu}$. Demanding normal VWI and anomalous AWI we get $a = 1$ and

$$T^*_{\mu\nu}(q) = \frac{1}{\pi}(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) \tag{122}$$

in accordance with (28).

3.3 The anomaly and the Dirac sea

The anomaly - as an effect of QFT - can also be understood in terms of the Dirac sea (see [11], [21], [22], [23], [24] and [37]). We study here the Schwinger model - QED in two dimensions with massless fermions (see section 2) - with x-space compactified on a circle S^1 of length L. The Lagrangian density is

$$L = \bar{\psi} i \not{D} \psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \tag{123}$$

with the covariant derivative

$$D_\mu = \partial_\mu - iA_\mu. \tag{124}$$

The fields are defined on a cylinder of space and time and have to obey the boundary conditions:

$$A_\mu(t, x = -\frac{L}{2}) = A_\mu(t, x = \frac{L}{2}) \tag{125}$$

$$\psi(t, x = -\frac{L}{2}) = \psi(t, x = \frac{L}{2}). \tag{126}$$

We choose a gauge where A_1 is independent of x and A_0 (the coulomb potential) can be neglected. In fact, we treat $A_1(t)$ as an external field which will be switched on adiabatically. Due to the gauge freedom the values $A_1 = 0$ and $A_1 = \frac{2\pi}{L}$ are gauge equivalent and must be identified. So the true configuration space of the gauge potential is itself a circle S_1 of the length $\frac{2\pi}{L}$.

The Lagrangian (123) is invariant under $U(1)$ symmetries leading to the classical conservation laws (we consider here constant phases). The $U(1)$ symmetry

$$\psi \to e^{i\alpha}\psi \tag{127}$$

$$\bar{\psi} \to \bar{\psi}e^{-i\alpha} \tag{128}$$

leads to the conservation of the vector current and the electric charge

$$\partial^\mu j_\mu = 0 \quad \text{and} \quad \dot{Q}(t) = 0$$

with

$$Q(t) = \int dx j_0(t,x) \tag{129}$$

and the axial $U_A(1)$ symmetry (chiral symmmetry)

$$\psi \to e^{i\beta\gamma_5}\psi \tag{130}$$

$$\bar{\psi} \to \bar{\psi}e^{i\beta\gamma_5} \tag{131}$$

causes conservation of the axial current and the axial charge

$$\partial^\mu j^5_\mu = 0 \quad \text{and} \quad \dot{Q}^5(t) = 0$$

with

$$Q^5(t) = \int dx j^5_0(t,x). \tag{132}$$

Introducing the left- and righthanded fermions

$$\psi_L = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} \quad , \quad \psi_R = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}$$

with chirality ± 1

$$\gamma_5 \psi_{L,R} = \pm \psi_{L,R} \tag{133}$$

we get the charges

$$Q_{L,R} = \int dx \bar{\psi}_{L,R}\gamma_0\psi_{L,R} = \int dx \psi^+_{L,R}\psi_{L,R} \tag{134}$$

and in components

$$Q = Q_L + Q_R \tag{135}$$

$$Q^5 = Q_L - Q_R. \tag{136}$$

So the axial charge for L- or R-fermions is

$$Q^5_{L,R} = \int dx \bar{\psi}_{L,R}\gamma_0\gamma_5\psi_{L,R} = \pm \int dx \psi^+_{L,R}\psi_{L,R} = \pm 1 \quad \text{for L,R.} \tag{137}$$

Since Q and Q^5 are conserved the numbers of the L- and R-fermions are conserved separately.

The laws (129) and (132) are valid classically On the quantum level, however, both currents (or charges) (129) and (132) cannot be conserved simultaneously - an anomaly occurs. How can we observe this phenomenon in a theory à la Dirac?

We start from the Dirac equation

$$(i\partial\!\!\!/ + A\!\!\!/)\psi = 0 \tag{138}$$

and we choose the 2-dimensional Dirac matrices in the following way

$$\gamma^0 = \sigma_2 \quad , \quad \gamma^1 = i\sigma_1 \quad , \quad \gamma_5 = \gamma^0\gamma^1 = \sigma_3$$

where σ_i denote the familiar Pauli matric .s. Then the Dirac equation is rewritten by

$$[i\frac{\partial}{\partial t} + \sigma_3(i\frac{\partial}{\partial x} - A_1)]\psi = 0. \tag{139}$$

According to the boundaries (126) the fermion wave function can be expanded into the Fourier series

$$\psi(t,x) = \frac{1}{\sqrt{L}}\sum_k u(k)e^{-iE_k t}e^{i\frac{2\pi}{L}(k+\frac{1}{2})x} \tag{140}$$

which provides the following energy solutions for the L- and R-fermion eigenstates

$$E_k^L = \frac{2\pi}{L}(k+\frac{1}{2}) + A_1, \tag{141}$$

$$E_k^R = -\frac{2\pi}{L}(k+\frac{1}{2}) - A_1, \tag{142}$$

with $k = 0, \pm 1, \pm 2, \ldots$

So comfortably the energy spectrum is discrete because of compactification and it depends linearly on A_1. We have plotted its features on Figure 5. At $A_1 = 0$ the energy levels are degenerate. If we switch on A_1 the levels split; the L-levels increase whereas the R-levels decrease. At $A_1 = \frac{2\pi}{L}$ we reproduce exactly the original level structure as it should be for gauge equivalent values. Precisely this nontrivial restructuring of the infinitely many fermion levels is the source of the anomaly as we shall see.

IR-behaviour:

Now we turn from the one particle case to the multiparticle description of QFT by introducing the Dirac sea. We fill up all negative energy levels and keep all positive levels empty. So at $A_1 = 0$ we have a vacuum - the Dirac sea. However, if we increase A_1 from 0 to $\frac{2\pi}{L}$ we produce - by lifting the L-levels and lowering the R-levels - a left-handed particle and a right-handed hole (see Figure 5).

What does it imply for the charges? The electric charges of the particle and the hole are opposite. So there is no change in the total electric charge and the vector current is conserved, equ. (129). The axial charges, however, are identical for both the L-particle and the R-hole so that the net axial charge changes

$$\Delta Q^5 = 1 + 1 = 2. \tag{143}$$

This change compared with the one in the gauge potential A_1 gives

$$\Delta Q^5 = 2 = \frac{L}{\pi}\Delta A_1, \tag{144}$$

and per time unit we have

$$\frac{\Delta Q^5}{\Delta t} = \frac{L}{\pi}\frac{\Delta A_1}{\Delta t}. \tag{145}$$

Considering the local change in the equation (132)

$$\frac{\partial}{\partial t}\int_0^L dx j_0^5(t,x) = \frac{1}{\pi}\frac{\partial}{\partial t}\int_0^L dx A_1(t) \tag{146}$$

we obtain the corresponding relation for the axial current

$$\partial_0 j_0^5 = \frac{1}{\pi}\partial_0 A_1 \tag{147}$$

or finally written in a Lorentz invariant way we arrive at

$$\partial^\mu j^5_\mu = \frac{1}{\pi}\varepsilon_{\mu\nu}\partial^\mu A^\nu. \tag{148}$$

Equation (148) represents the familiar anomaly result.

UV-behaviour:

This anomaly result we derived by investigating the level crossing at the energy $E = 0$ scale - the IR-region. But alternatively we also can study the behaviour at some UV-cutoff which is more directly related to the perturbative approach in QFT presented before.

In fact, we have to regularize when working with the Dirac sea. The total energy, the total charge of the vacuum, the fermionic wavefunction as superpositions of infinitely many filled states are ill-defined quantities which have to be regularized somehow. A procedure which preserves gauge invariance is the point splitting method of Schwinger [8]. There the gauge invariant regularized currents are

$$j^{reg}_\mu = \lim_{\varepsilon\to 0}\bar\psi(t,x+\varepsilon)\gamma_\mu\psi(t,x)e^{-i\int_x^{x+\varepsilon}dxA_1} \tag{149}$$

$$j^{5,reg}_\mu = \lim_{\varepsilon\to 0}\bar\psi(t,x+\varepsilon)\gamma_\mu\gamma_5\psi(t,x)e^{-i\int_x^{x+\varepsilon}dxA_1}. \tag{150}$$

The rule is that the computations are carried out with fixed ε and afterwards the limit $\varepsilon \to 0$ is taken for the physical quantities.

The regularized charges are defined by

$$Q(t) = \int dx j^{reg}_0(t,x) \tag{151}$$

$$Q^5(t) = \int dx j^{5,reg}_0(t,x) \tag{152}$$

and

$$Q_L = \frac{1}{2}(Q + Q^5) \tag{153}$$

$$Q_R = \frac{1}{2}(Q - Q^5). \tag{154}$$

Calculating the L- and R-charges explicitely we obtain the sum

$$Q_{L,R} = \sum_k e^{-i\varepsilon[\frac{2\pi}{L}(k+\frac{1}{2})+A_1]} \tag{155}$$

with the summation

$$k = -1,-2,-3,\dots \quad \text{for L}$$
$$k = 0,1,2,3,\dots \quad \text{for R}.$$

If we first take the limit $\varepsilon \to 0$ we are back at the unregularized ill-defined quantities

$$Q_{L,R} = \sum_k 1, \tag{156}$$

corresponding to the infinitely many filled levels in the Dirac sea.

But summing first and expanding afterwards in ε provides for the L- and R-sea the result

$$Q_L^{vac} = -\frac{L}{2\pi}\frac{1}{i\varepsilon} + \frac{L}{2\pi}A_1 \tag{157}$$

$$Q_R^{vac} = \frac{L}{2\pi}\frac{1}{i\varepsilon} - \frac{L}{2\pi}A_1. \tag{158}$$

Now we observe the following. The electric charge of the vacuum vanishes

$$Q^{vac} = Q_L^{vac} + Q_R^{vac} = 0. \tag{159}$$

The electric charge and hence the vector current is conserved. The axial charge, on the other hand, contains a divergent constant $\frac{1}{\varepsilon}$ which will be subtracted to define the regularized quantity and leaves a linear dependence on A_1

$$Q^{5,vac} = Q_L^{vac} - Q_R^{vac} = 2\frac{L}{2\pi}A_1. \tag{160}$$

As A_1 increases from 0 to $\frac{2\pi}{L}$ the axial charge changes by 2 as before providing such the same anomaly result.

4 Exact solution of the Schwinger model

We will discuss here those features of the Schwinger model that are intimately related to the anomaly. Other aspects are discussed e.g. in [38]-[40], and recent developments of more mathematical aspects can be found e.g. in [41], [42].

Because the subsequent functional techniques are better fit to Euclidean space we first give a short review of used conventions. In Minkowski space we have

$$g_{\mu\nu} = (1, -1) \quad , \quad \epsilon_{01} = 1 \quad , \quad \gamma_5 = \gamma^0\gamma^1,$$

and the relations

$$\gamma_\mu\gamma_5 = \epsilon_{\mu\nu}\gamma^\nu \quad , \quad \mathrm{tr}\gamma_5\gamma_\mu\gamma_\nu = -2\epsilon_{\mu\nu} \tag{161}$$

hold.

Performing a Wick rotation $x^0 = ix^4$ we arrive at an Euclidean space with negative metric $g_{\mu\nu} = -\delta_{\mu\nu}$ which we call anti-Euclidean (aE). Here the Dirac operator $\not{D} = \not{\partial} + i\not{A}$ is hermitian and $\epsilon_{41} = i\epsilon_{01} = i$.

However, we will prefer Euclidean space ($g_{\mu\nu} = \delta_{\mu\nu}$) where $\epsilon_{41} = -i$ and $\not{D}$ is anti-hermitian. Keeping the matrix γ_5 unchanged,

$$\gamma_5 = \gamma^0\gamma^1 = (i\gamma^4\gamma^1)_{\mathrm{aE}} = (-i\gamma^4\gamma^1)_{\mathrm{E}}, \tag{162}$$

the relations (161) hold in all spaces.

If we instead defined $\bar{\epsilon}_{\mu\nu}$ to be $\bar{\epsilon}_{41} = 1$ in Euclidean space too, then the expression for the anomaly would be identical in all spaces but relations (161) would not be true in Euclidean space and the simple covariant form of the computations would be lost.

For the massless case (Schwinger model) the exact solution shall be computed in several manners. The result, a free theory with massive bosons, is according to the perturbative calculations as we will show subsequently. Our computations in wide parts follow [9]. A first hint on the massivity of the photon - caused by the anomaly - can be seen even on the level of the equations of motion. The Lagrangian

$$L = \bar{\Psi}(i\not{\partial} - e\not{A})\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{163}$$

leads to the equation of motion

$$\partial^\mu F_{\mu\nu} = e\bar{\Psi}\gamma_\nu\Psi = eJ_\nu \tag{164}$$

with the condition

$$\partial^\mu J_\mu = 0 \tag{165}$$

In two dimensions the following relations hold:

$$J^5_\mu = \epsilon_{\mu\nu}J^\nu \tag{166}$$

$$F_{\mu\nu} = -\epsilon_{\mu\nu}\tilde{F} \tag{167}$$

and consequently

$$\partial_\mu\tilde{F} = eJ^5_\mu. \tag{168}$$

Using the anomalous divergence equation for the axial current

$$\partial^\mu J^5_\mu = \frac{e}{\pi}\tilde{F} \tag{169}$$

we get the Klein-Gordon-equation

$$\Box\tilde{F} = \frac{e^2}{\pi}\tilde{F} \tag{170}$$

for a boson with mass $m = \frac{e}{\sqrt{\pi}}$ (remember that we are in Euclidean space). In two dimensions the electric charge has the unit of mass.

4.1 Exact fermion propagator

In a next step we compute the exact fermion propagator. For the free propagator the equation

$$\not\partial_x G_0(x-y) = \delta(x-y) \tag{171}$$

holds. The exact propagator solves the equation

$$(\not\partial_x + ie\not A)G_A(x,y) = \delta(x-y). \tag{172}$$

Now we insert the ansatz

$$G_A(x,y) = e^{-ie(\Phi_A(x)-\Phi_A(y))}G_0(x-y) \tag{173}$$

where Φ is a matrix valued function. We get the equation

$$\not\partial\Phi_A(x) = \not A \tag{174}$$

if the derivation may be performed on the exponent in a naive fashion, i.e. if

$$[\Phi_A, \Phi_A] = 0. \tag{175}$$

This is true just in two dimensions: using (174) we have

$$\Box\Phi_A(x) = \not\partial\not A \tag{176}$$

and

$$\Phi_A(x) = \frac{1}{\Box}\not\partial\not A(x) = -\int dy D(x-y)(\frac{1}{2}\{\gamma_\mu,\gamma_\nu\} + \frac{1}{2}[\gamma_\mu,\gamma_\nu])\partial^\mu A^\nu(y) \tag{177}$$

where $D(x-y)$ is the infra red regularized scalar field propagator. In two dimensions we have

$$\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}I \tag{178}$$

and

$$[\gamma_\mu,\gamma_\nu] = -2\epsilon_{\mu\nu}\gamma_5 \tag{179}$$

which proves our statement. For the function Φ we get

$$\Phi_A(x) = \alpha_A(x)I - \beta_A(x)\gamma_5 \tag{180}$$

where

$$\alpha_A(x) = -\int dy D(x-y)\partial^\mu A_\mu(y) \tag{181}$$

and

$$\beta_A(x) = -\int dy D(x-y)\epsilon_{\mu\nu}\partial^\mu A^\nu(y). \tag{182}$$

To further simplify the computations we sometimes will choose the Lorentz gauge in the sequel. There

$$\partial^\mu A_\mu(x) = 0 \tag{183}$$

and therefore

$$\alpha_A(x) = 0. \tag{184}$$

We get

$$\Phi_A(x) = -\beta_A(x)\gamma_5 \tag{185}$$

and

$$A^\nu(x) = \epsilon^{\mu\nu}\partial_\mu\beta_A(x) \tag{186}$$

in the Lorentz gauge.

Here we can proceed in different manners. One possibility is to compute the exact electromagnetic current depending on the gauge field A_μ via point separation regularization, which shall be done as a next step.

4.2 Current operator and effective Lagrangian

In the Heisenberg picture the field operators propagate with respect to the full Hamiltonian. We compute the electromagnetic current operator where the equal point singularity is regularized by a gauge invariant point splitting technique ([8]). The gauge invariantly regularized current operator is (see [1] for details)

$$\begin{aligned} J_\mu(x) &= \lim_{\epsilon\to 0} \bar\Psi(x)\gamma_\mu\Psi(x+\epsilon)\exp(ie\int_x^{x+\epsilon} dz^\lambda A_\lambda(z) \\ &= \lim_{\epsilon\to 0} \mathrm{tr}\gamma_\mu G_A(x+\epsilon,x)\exp(ie\int_x^{x+\epsilon} dz^\lambda A_\lambda(z) \\ &= \lim_{\epsilon\to 0} \mathrm{tr}\gamma_\mu e^{-ie(\Phi_A(x+\epsilon)-\Phi_A(x))}G_0(\epsilon)\exp(ie\int_x^{x+\epsilon} dz^\lambda A_\lambda(z). \end{aligned} \tag{187}$$

Using

$$G_0(\epsilon) = -i\frac{\gamma_\mu\epsilon^\mu}{2\pi\epsilon^2}$$

and performing the limit symmetrically

$$\lim_{\epsilon\to 0}\frac{\epsilon_\mu\epsilon_\nu}{\epsilon^2} = \frac{g_{\mu\nu}}{2} \tag{188}$$

$$\lim_{\epsilon\to 0}\frac{\epsilon_\nu}{\epsilon^2} = 0 \tag{189}$$

we get with the help of (177)

$$J_\mu(x) = \frac{e}{2\pi}\int dy(\partial_\mu^x D(x-y)\partial_\nu^y + g_{\mu\nu}\delta(x-y))A^\nu(y)$$

$$-\frac{e}{2\pi}\int dy\partial_{\cdot}^{\nu}\mathcal{J}(x-y)\epsilon_{\mu\nu}\epsilon_{\alpha\beta}\partial_{y}^{\alpha}A^{\beta}(y) \tag{190}$$

on which the correct conservation equations hold:

$$\partial^{\mu}J_{\mu}(x)=0 \tag{191}$$

and

$$\partial^{\mu}J_{\mu}^{5}(x)=\partial^{\mu}\epsilon_{\mu\lambda}J^{\lambda}(x)=\frac{e}{\pi}\epsilon_{\mu\nu}\partial^{\mu}A^{\nu}(x)=\frac{e}{\pi}\tilde{F}(x). \tag{192}$$

Inserting the current (190) into the Lagrangian (163) leads to an effective Lagrangian of the photon field:

$$\begin{aligned} L_{\text{int}} &= -J^{\mu}eA_{\mu} = \frac{e^2}{2\pi}\int dyD(x-y)\partial_{\mu}A^{\mu}(x)\partial_{\nu}A^{\nu}(y) \\ &-\frac{e^2}{2\pi}A^{\mu}(x)A_{\mu}(x)+\frac{e^2}{2\pi}\int dyD(x-y)\epsilon_{\mu\nu}\partial^{\mu}A^{\nu}(x)\epsilon_{\alpha\beta}\partial^{\alpha}A^{\beta}(y) \end{aligned} \tag{193}$$

which in the Lorentz gauge contains aside from the photon mass term a further term which however may be transformed away by a redefinition of the photon field. The transformation

$$A^{\mu}(x)\rightarrow A^{\mu}(x)+\epsilon^{\mu\nu}\int dy\partial_{\nu}^{x}D(x-y)\epsilon_{\alpha\beta}\partial_{y}^{\alpha}A^{\beta}(y) \tag{194}$$

leads to the Lagrangian

$$L_{\text{int}}=\frac{e^2}{2\pi}\int dyD(x-y)\partial_{\mu}A^{\mu}(x)\partial_{\nu}A^{\nu}(y)-\frac{e^2}{2\pi}A_{\mu}(x)A^{\mu}(x) \tag{195}$$

and thus in the Lorentz gauge just to the photon mass term.

4.3 Anomaly and photon mass in the path integral formalism

In the following we fix the Lorentz gauge and choose the space-time to be Euclidean! In the Lorentz gauge the photon field can be written as the derivative of a chiral field (186). As a consequence the photon field can be transformed away from the interaction lagrangian by a chiral transformation of the fermion fields:

$$\begin{aligned} L_{\Psi} &= \bar{\Psi}(i\partial\!\!\!/-eA\!\!\!/)\Psi = \bar{\Psi}(i\partial\!\!\!/+e(\partial^{\mu}\beta)\gamma_{\mu}\gamma_5)\Psi \\ &= \bar{\Psi}e^{-ie\beta\gamma_5}i\partial\!\!\!/e^{-ie\beta\gamma_5}\Psi =: \bar{\Psi}'i\partial\!\!\!/\Psi'. \end{aligned} \tag{196}$$

In the corresponding fermionic path integral even the integration measure is changed by this chiral transformation which precisely causes the anomaly. First of all we will compute the anomaly resulting from an infinitesimal chiral transformation using the method of Fujikawa ([17], [10]). The fermionic path integral is

$$Z[A]=\int D\bar{\Psi}D\Psi\exp\left(\int dx\bar{\Psi}(i\partial\!\!\!/-eA\!\!\!/)\Psi\right)=\det(i\partial\!\!\!/-eA\!\!\!/). \tag{197}$$

Under the infinitesimal chiral transformation $\delta\beta$

$$\Psi\rightarrow(1+i\delta\beta\gamma_5)\Psi$$

$$\bar{\Psi} \to \Psi(1 + i\delta\beta\gamma_5)$$

$$L_\Psi \to L_\Psi - (\partial^\mu \delta\beta)\bar{\Psi}\gamma_\mu\gamma_5\Psi \tag{198}$$

the path integral changes according to

$$Z[A] = \int D\bar{\Psi}' D\Psi' e^{\int L_{\Psi'} dx}$$

$$= \int D\bar{\Psi} D\Psi J_\beta[A] e^{\int (L_\Psi + \delta\beta\partial^\mu J^5_\mu) dx} \tag{199}$$

where the functional determinant $J_\beta[A]$ stems from the transformation of the path integral measure.

Expanding the spinors into eigenfunctions of the hermitian operator $i\not{D}$

$$i\not{D}\phi_n := i(\not{\partial} + ie\not{A})\phi_n = \lambda_n\phi_n \tag{200}$$

$$\Psi = \sum_n a_n\phi_n \quad , \quad \bar{\Psi} = \sum_n b_n\phi_n^*, \tag{201}$$

where a_n and b_n are Grassmann numbers, the path integral measure becomes

$$D\bar{\Psi} D\Psi = \prod_n db_n da_n. \tag{202}$$

Under the chiral transformation (198) the Grassmann coefficients are changing according to

$$a_n' = \sum_m a_m(\delta_{mn} + \int dx \phi_n^* i\delta\beta\gamma_5\phi_m) \tag{203}$$

and an identical term for b_n. The transformed measure is

$$\prod_n da_n' db_n' = \prod_n da_n db_n \exp(-2i \int dx \delta\beta \sum_m \phi_m^* \gamma_5 \phi_m). \tag{204}$$

The sum in the exponential is regularized in the following way:

$$\lim_{M\to\infty} \sum_n \phi_n^* \gamma_5 e^{-\frac{\lambda_n^2}{M^2}} \phi_n = \lim_{M\to\infty} \sum_n \phi_n^* \gamma_5 \exp(-\frac{(i\not{D})^2}{M^2})\phi_n. \tag{205}$$

Inserting plane waves, using the completeness relation, evaluating the Dirac operator and substituting

$$k_\mu \to \frac{k_\mu}{M} = k_\mu'$$

leads to

$$\lim_{M\to\infty} M^2 \int \frac{d^2k'}{(2\pi)^2} e^{-ik'xM} e^{k'^2} \mathrm{tr}\gamma_5 \exp\{\frac{1}{M^2}(\partial iA + iA\partial - A^2 + \frac{1}{4}[\gamma_\mu, \gamma_\nu] iF^{\mu\nu})\} e^{ik'xM}$$

$$= -\frac{i}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}. \tag{206}$$

The measure (202) changes according to

$$D\bar{\Psi}'D\Psi' = D\bar{\Psi}D\Psi(1 - \int dx\delta\beta\frac{1}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu})$$

$$= D\bar{\Psi}D\Psi(1 - \frac{1}{\pi}\int dx\delta\beta\tilde{F}) \tag{207}$$

and consequently the anomaly is

$$\partial^\mu J^5_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} = \frac{1}{\pi}\tilde{F} \tag{208}$$

in accordance with our previous calculations.

This result shall now be used to determine the consequences of transformation (196) which turns off the interaction. First we emphasize that the role of transformed and untransformed field are exchanged in our case but also the sign of the transformation parameter is opposite. Further it can be seen that a chiral transformation produces an additional term in the fermionic lagrangian, i.e. every infinitesimally transformed fermion field "sees" an infinitesimally changed effective gauge field. When the infinitesimal transformations are added successively to a finite transformation, this changing of the gauge field certainly must be considered. In (196) the gauge transformation is chosen as to successively turn off the gauge field.

Choosing the parametrized transformation

$$\delta\beta = \delta t\beta \quad , \quad \beta_t = t\beta \quad , \quad t \in [0,1] \tag{209}$$

together with (186), (196) implies

$$A^\nu(t) = \epsilon^{\mu\nu}\partial_\mu\beta(1-t) \tag{210}$$

and therefore for the anomaly belonging to the finite transformation (196)

$$D\bar{\Psi}D\Psi = D\bar{\Psi}'D\Psi'\exp\frac{-1}{\pi}\int dx\int_0^1 \delta t\beta\tilde{F}(1-t)$$

$$= D\bar{\Psi}'D\Psi'\exp\frac{-1}{2\pi}\int dx\beta\tilde{F} \tag{211}$$

as in [26].

Using (186) we further have

$$\tilde{F} = \Box\beta \qquad \text{and} \qquad \partial_\mu\beta = \epsilon_{\mu\nu}A^\nu \tag{212}$$

and therefore

$$-\int dx\beta\tilde{F} = -\int dx\beta\Box\beta = \int dx\partial_\mu\beta\partial^\mu\beta$$

$$= \int dx\epsilon_{\mu\nu}A^\nu\epsilon^{\mu\lambda}A_\lambda = -\int dxA_\mu A^\mu. \tag{213}$$

The path integral therefore becomes

$$Z[A] = \int D\bar{\Psi}D\Psi e^{\int dxL_\Psi} = \int D\bar{\Psi}D\Psi\exp(\int dx(\bar{\Psi}i\partial\!\!\!/\Psi - \frac{e^2}{2\pi}A^\mu A_\mu)) \tag{214}$$

where the up to now suppressed charge has been reinserted. The interaction really vanishes and instead of it the photon requires a mass

$$m_\gamma^2 = \frac{e^2}{\pi}. \tag{215}$$

The computation was done in Lorentz gauge but the gauge invariant generalization of the path integral is obvious (see (195)).

4.4 Heat kernel and zeta function regularization

We now present two more regularizations, in order to compute the anomaly due to an infinitesimal chiral transformation, which are well fit to functional and path integral formalism (see e.g. [32], [33]).

The heat kernel of a hermitian, positive semidefinite operator A is formally

$$G_{A\alpha\beta}(x,y,\tau) = \sum_n e^{-\lambda_n \tau} \Phi_{n\alpha}(x) \Phi^*_{n\beta}(y) \tag{216}$$

where

$$A\Phi_n(x) = \lambda_n \Phi_n(x)$$

and α, β denote all indices which Φ may have. To get rid of the indices we use matrix notation in the sequel.

Because of completeness of the Φ_n the relation

$$G_A(x,y,\tau=0) = \delta(x-y) \tag{217}$$

must hold. Together with this condition the heat kernel fulfills the "heat equation"

$$A_x G_A(x,y,\tau) = -\frac{\partial}{\partial \tau} G_A(x,y,\tau) \tag{218}$$

which may be easily veryfied using the representation (216).

For small τ the expansion

$$G_A(x,y,\tau) = \frac{1}{(4\pi\tau)^{d/2}} \exp(\frac{-(x-y)^2}{4\tau}) \sum_{n=0}^{\infty} a_n(x,y) \tau^n \tag{219}$$

holds where d is the dimension of space time. The coefficients a depend on A and are fixed by the heat equation. a_0 is fixed by condition (217) to be

$$a_0(x,y) = 1. \tag{220}$$

Setting $d = 2$ for our purposes we see at once that the coefficient a_1 of the heat kernel of $-\not{D}^2$ causes the anomaly in Fujikawas approach. The regularization of the sum in (205) is just the trace of the heat kernel:

$$\begin{aligned} &\int dx \delta\beta(x) \lim_{\tau\to 0} \sum_n \Phi_n^*(x) \gamma_5 e^{-\lambda_n \tau} \Phi_n(x) \\ &= \int dx \delta\beta(x) \lim_{\tau \to 0} \mathrm{tr} \gamma_5 G_{-\not{D}^2}(x,x,\tau) = \lim_{\tau\to 0} \mathbf{Tr} \delta\beta \gamma_5 G_{-\not{D}^2} \end{aligned} \tag{221}$$

where Tr means trace with respect to the full operator structure. Inserting the expansion (219) leads to

$$\int dx\delta\beta(x)\mathrm{tr}\gamma_5\frac{1}{4\pi}a_1(x,x). \tag{222}$$

The coefficients of $G_{-\not{D}^2}$ on the diagonal $x = y$ fulfill the relations

$$\not{D}_x^2 a_{n-1}(x,x) = na_n(x,x) \tag{223}$$

steming from the heat equation, and therefore

$$\not{D}^2 a_0(x) = a_1(x) \tag{224}$$

holds, where $a_n(x) := a_n(x,x)$.

Inserting the Dirac operator $\not{D} = \not{\partial} + i\not{A}$ we further get

$$\not{D}^2 1 = i(\not{\partial}\not{A}) - A^2 \tag{225}$$

and for the trace

$$\frac{1}{4\pi}\mathrm{tr}\gamma_5 a_1(x) = -\frac{i}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}(x) \tag{226}$$

which is just half the anomaly in agreement with Fujikawas method (see (206)).

However, the regulator used here and in (205) is an ad hoc modification of an indefinite sum. The zeta function regularization offers a possibility to work with a regularized path integral from the beginning. In this case the chiral Jacobian is the ratio of two regularized determinants and therefore finite:

$$J_\beta[A] = \frac{\det^R \not{D}}{\det^R e^{i\beta\gamma_5}\not{D}e^{i\beta\gamma_5}}. \tag{227}$$

The zeta function of an operator A is

$$\zeta_A(s) = \sum_n \lambda_n^{-s} \tag{228}$$

where λ_n are the eigenvalues of A. The series converges for positive semidefinite A and sufficiently large Re s. To reach smaller s analytic continuation in the complex variable s is understood. Formally we have

$$\frac{d}{ds}\zeta_A(s)|_{s=0} = -\sum_n \ln\lambda_n e^{-s\ln\lambda_n}|_{s=0} = -\ln\prod_n\lambda_n$$

and therefore

$$\det A = \prod_n \lambda_n = \exp(-\frac{d}{ds}\zeta_A(s))|_{s=0}. \tag{229}$$

To proceed further an analytic expression for the zeta function must be found. For this purpose we need the notions of integral kernel and complex power of an operator.

The integral kernel of an operator A fulfills

$$A_x f(x) = \int dy K_A(x,y)f(y).$$

A definition for the complex power of an operator will be given below. Formally we have

$$\zeta_A(s) = \sum_n \lambda_n^{-s} = \mathbf{Tr} A^{-s} \tag{230}$$

which - when defined correctly - even works for non-hermitian, indefinite operators. For an infinitesimally varied operator we may expand

$$\mathbf{Tr}(A+\delta A)^{-s} = \mathbf{Tr} A^{-s} - s\mathbf{Tr} A^{-s-1}\delta A + o(\delta A^2) \tag{231}$$

where the functional trace of an operator means

$$\mathbf{Tr} A = \int dx K_A(x,x) = \sum_n \int dx \lambda_n \Phi_n(x)\Phi_n^*(x)$$

and

$$\mathbf{Tr} A^{-s} = \int dx K_A(x,x,-s) = \sum_n \int dx \lambda_n^{-s} \Phi_n(x)\Phi_n^*(x). \tag{232}$$

One possibility to compute the chiral Jacobian is given by the heat kernel expansion which however needs positive semidefinite operators to work. Therefore we have to compute the change of the squared Dirac operator determinant:

$$J_{\delta\beta}^{-\not{D}^2}[A] = \frac{\det - \not{D}^2}{\det(-\not{D}^2 + \{i\delta\beta\gamma_5, -\not{D}^2\})} = \frac{\det - \not{D}^2}{\det(-\not{D}^2 + \delta(-\not{D}^2))} \tag{233}$$

and

$$\ln J_{\delta\beta}^{-\not{D}^2}[A] = -\frac{d}{ds}|_{s=0}\zeta_{-\not{D}^2}(s) + \frac{d}{ds}|_{s=0}\zeta_{-\not{D}^2+\delta(-\not{D}^2)}(s)$$

$$= -\frac{d}{ds}|_{s=0} s\mathbf{Tr}(-\not{D}^2)^{-s-1}\delta(-\not{D}^2) = -\frac{d}{ds}|_{s=0} 2is\mathbf{Tr}(-\not{D}^2)^{-s}\delta\beta\gamma_5 \tag{234}$$

using the cyclic property of the trace. Inserting the definition of the trace and Mellin transforming to the heat kernel leads to

$$\mathbf{Tr}(-\not{D}^2)^{-s}\delta\beta\gamma_5 = \mathrm{tr}\int dx K_{-\not{D}^2}(x,x,-s)\delta\beta\gamma_5$$

$$= \mathrm{tr}\sum_n \int dx \lambda_n^{-s}\Phi_n(x)\Phi_n^*(x)\delta\beta\gamma_5$$

$$= \mathrm{tr}\frac{1}{\Gamma(s)}\sum_n \int dx \int_0^\infty d\tau \tau^{s-1} e^{-\lambda_n \tau}\Phi_n(x)\Phi_n^*(x)\delta\beta\gamma_5$$

$$= \mathrm{tr}\frac{1}{\Gamma(s)}\int dx \int_0^\infty d\tau \tau^{s-1} G_{-\not{D}^2}(x,x,\tau)\delta\beta\gamma_5. \tag{235}$$

The heat kernel is falling off exponentially for large τ so because $\frac{1}{\Gamma(s)} = s + o(s^2)$ the upper integration boundary may be replaced by a finite number, T (actually in the limit $s \to 0$ only the point $\tau = 0$ will contribute).

Further it is known that the trace of the integral kernel is finite for $s = 0$ [25]. Therefore

$$-\frac{d}{ds}|_{s=0} 2is\mathbf{Tr}(-\not{D}^2)^{-s}\delta\beta\gamma_5 = -2i\mathbf{Tr}(-\not{D}^2)^{-s}\delta\beta\gamma_5|_{s=0}. \tag{236}$$

Inserting now the heat kernel expansion le-.ds to

$$\ln J_{\delta\beta}^{-\not{D}^2} = -2i(s + o(s^2))\mathrm{tr}\int dx \int_0^T d\tau \tau^{s-1}\frac{1}{4\pi\tau}\sum_m a_m(x)\tau^m|_{s=0}\delta\beta\gamma_5$$
$$= -2i\mathrm{tr}\int dx \frac{1}{4\pi} s a_1(x)\frac{1}{s}T^s|_{s=0}\delta\beta\gamma_5 + o(s\frac{1}{s+n})|_{s=0}$$
$$= -\frac{i}{2\pi}\mathrm{tr}\int dx a_1(x)\delta\beta\gamma_5 \qquad (237)$$

which is independent of the integration boundary. We see that only the point $\tau = 0$ contributes, as claimed, in accordance with the previous computation. This also ensures that the upper boundary T may be chosen small enough to make the heat kernel expansion hold. Of course, result (237) is just the anomaly.

Using the neat kernel we had to take the squared Dirac operator because we needed positive semi-definiteness. This does not pose a problem for hermitian operators [26], but in spite of this we want to present a version of the zeta function regularization which does not rely on positivity or hermitizity.

First we need the definition of a complex power of an operator [25]:

$$A^s = \frac{i}{2\pi}\int_\Gamma \lambda^s \frac{1}{A-\lambda}d\lambda \qquad (238)$$

where the curve Γ in the complex plane is chosen as to include the complex powers of all eigenvalues of A and zero. For the chiral Jacobian

$$J_{\delta\beta}^{\not{D}} = \frac{\det \not{D}}{\det(\not{D} + \{i\delta\beta\gamma_5, \not{D}\})}$$

we get

$$\ln J_{\delta\beta}^{\not{D}} = -\frac{d}{ds}|_{s=0}2is\mathrm{Tr}\not{D}^{-s}\delta\beta\gamma_5 = -2i\mathrm{Tr}\not{D}^{-s}\delta\beta\gamma_5|_{s=0} \qquad (239)$$

as in (234), (236).

Inserting now the complex power and inventing plane waves as a complete system to evaluate the trace we get

$$\mathrm{Tr}\not{D}^{-s}\delta\beta\gamma_5 = \frac{i}{2\pi}\int dx \mathrm{tr}\int_\Gamma d\lambda\lambda^{-s}\int\frac{d^2k}{(2\pi)^2}\delta\beta\gamma_5 e^{-ikx}\frac{1}{\not{\partial} + i\not{A} - \lambda}e^{ikx}$$
$$= \frac{i}{2\pi}\int dx\mathrm{tr}\int_\Gamma d\lambda\lambda^{-s}\int\frac{d^2k}{(2\pi)^2}\delta\beta\gamma_5\frac{1}{\not{D} + i\not{k} - \lambda}|_{\partial=0} \qquad (240)$$

where the notation means that derivatives acting to the right give zero. Expanding the denominator we get

$$\mathrm{Tr}\not{D}^{-s}\delta\beta\gamma_5 = \frac{i}{2\pi}\int dx\mathrm{tr}\int_\Gamma d\lambda\lambda^{-s}\int\frac{d^2k}{(2\pi)^2}\delta\beta\gamma_5[\frac{1}{i\not{k}-\lambda} -$$
$$-\frac{1}{i\not{k}-\lambda}\not{D}\frac{1}{i\not{k}-\lambda} + \frac{1}{i\not{k}-\lambda}\not{D}\frac{1}{i\not{k}-\lambda}\not{D}\frac{1}{i\not{k}-\lambda} - + \ldots]|_{\partial=0}$$
$$= \frac{i}{2\pi}\int dx\mathrm{tr}\int d\lambda\lambda^{-s}\int\frac{d^2k'}{(2\pi)^2}\lambda^2\delta\beta\gamma_5[\lambda^{-1}\frac{1}{i\not{k}'-1} -$$

$$- \lambda^{-2}\frac{1}{i\not k' - 1}\not D\frac{1}{i\not k' - 1} + \lambda^{-3}\frac{1}{i\not k' - 1}\not D\frac{1}{i\not k' - 1}\not D\frac{1}{i\not k' - 1} - + \ldots]|_{\partial=0} \tag{241}$$

$$\not k' = \frac{1}{\lambda}\not k.$$

This expansion corresponds to the expansion into Seeley's coefficients [25] for the Dirac operator, but we will choose a different way of evaluating them which is much simpler in this case ([27]).

Setting $s = 0$ only the third term of the expansion will contribute as a residual integral (observe that as claimed the expression is finite at $s = 0$) and using

$$\frac{1}{2\pi i}\int \frac{d\lambda}{\lambda} = 1$$

we get

$$\begin{aligned}
\mathrm{Tr}\not D^{-s}\delta\beta\gamma_5|_{s=0} &= -\int dx \mathrm{tr}\int \frac{d^2k}{(2\pi)^2}\delta\beta\gamma_5\frac{-i\not k - 1}{k^2+1}\not D\frac{-i\not k - 1}{k^2+1}\not D\frac{-i\not k - 1}{k^2+1}|_{\partial=0} \\
&= \int dx \mathrm{tr}\int \frac{d^2k}{(2\pi)^2}\frac{\delta\beta\gamma_5}{(k^2+1)^3}(\not D\not D - \not k\not D\not k\not D - \not k\not D\not D\not k - \not D\not k\not D\not k)|_{\partial=0} \\
&= \int dx \int \frac{d^2k}{(2\pi)^2}\frac{\delta\beta}{(k^2+1)^3}\mathrm{tr}\gamma_5(\not D\not D + k^2\not D\not D)|_{\partial=0} \\
&= -\int dx \int \frac{d^2k}{(2\pi)^2}\frac{\delta\beta}{(k^2+1)^2}2i\epsilon_{\mu\nu}\partial^\mu A^\nu.
\end{aligned} \tag{242}$$

For the chiral Jacobian therefore we get

$$\begin{aligned}
\ln J^{\not D}_{\delta\beta} &= (-2i)(-)\int dx \frac{2i}{(2\pi)^2}\delta\beta\epsilon_{\mu\nu}\partial^\mu A^\nu \int_0^{2\pi} d\phi \int \frac{dk^2}{2}\frac{1}{(k^2+1)^2} \\
&= -\frac{1}{\pi}\int dx\delta\beta\epsilon_{\mu\nu}\partial^\mu A^\nu = -\frac{1}{\pi}\int dx\delta\beta\tilde{F}
\end{aligned} \tag{243}$$

which is the anomaly.

4.5 Anomaly and the index

As noticed by Fujikawa [17] the exponential in the transformation Jacobian - the anomaly - represents an index density of the Dirac operator. This we can understand in the following way:

Let us introduce the positive and negative chirality projectors

$$P_+ = \frac{1}{2}(1+\gamma_5) \quad , \quad P_- = \frac{1}{2}(1-\gamma_5)$$

together with the positive and negative chirality spinors

$$P_+\Phi = \Phi_+ \quad , \quad P_-\Phi = \Phi_-$$

and

$$\gamma_5\Phi_+ = +\Phi_+ \quad , \quad \gamma_5\Phi_- = -\Phi_-.$$

Recall the eigenvalue equation (200)

$$i\not D\Phi_n = \lambda_n\Phi_n$$

then $\gamma_5\Phi_n$ satisfies the equation with negative eigenvalues

$$i\not{D}\gamma_5\Phi_n = -\lambda_n\gamma_5\Phi_n$$

because of

$$\{\gamma_\mu, \gamma_5\} = 0.$$

Since $i\not{D}$ has been chosen hermitian in Euclidean space the two eigenfunctions are orthogonal to each other for $\lambda_n \neq 0$

$$(\Phi_n|\gamma_5\Phi_n) = \int dx\Phi_n^*(x)\gamma_5\Phi_n(x) = 0. \tag{244}$$

For vanishing eigenvalues $\lambda_n = 0$ the zero mode functions Φ_n^0 satisfying

$$i\not{D}\Phi_n^0 = 0 \quad , \quad i\not{D}\gamma_5\Phi_n^0 = 0$$

can be chosen to be γ_5 eigenfunctions

$$\gamma_5\Phi_n^{+0} = +\Phi_n^{+0} \quad , \quad \gamma_5\Phi_n^{-0} = -\Phi_n^{-0} \tag{245}$$

for positive or negative chirality fields.

Considering now the transformation Jacobian (204) for constant β we obtain by virtue of equs. (244) and (245)

$$\int dx \sum_n \Phi_n^*\gamma_5\Phi_n = \int dx \sum_{n,\lambda_n=0} \Phi_n^{*0}\gamma_5\Phi_n^0$$

$$= \sum_{n,\lambda_n=0} \int dx\Phi_n^{*+0}\Phi_n^{+0} - \sum_{n,\lambda_n=0} \int dx\Phi_n^{*-0}\Phi_n^{-0} = n_+ - n_- \tag{246}$$

where n_+ and n_- denote the number of positive and negative chirality zero modes. This difference (246) represents precisely the index of a differential operator.

For defining generally the index of an operator T

$$E \to^T F$$

we need the kernel, range and cokernel

$$\ker T = \{e \in E : Te = 0\}$$

$$\mathrm{range}T = \{Te \in F : e \in E\}$$

$$\mathrm{coker}T = F/\mathrm{range}T \sim \ker T^\dagger$$

with $T^\dagger$ being the adjoint of T.

Then the index of a (Fredholm) operator is defined by

$$\mathrm{index}T = \dim\ker T - \dim\mathrm{coker}T = \dim\ker T - \dim\ker T^\dagger.$$

For our example we choose the Dirac operator restricted to positive chirality spinors - the Weyl operator

$$D_+ = i\not{D}P_+ \tag{247}$$

with its adjoint

$$D_+^\dagger = \not{D} P_- = D_- .$$

Both operators act between the two subspaces of positive and negative chirality spinors

$$S_+ \to^{D_+} S_- \to^{D_-} S_+ .$$

In that case the index of the Weyl operator is

$$\mathrm{index} D_+ = \dim \ker D_+ - \dim \ker D_- = n_+ - n_- .$$

Therefore we obtain the following connection between the anomaly and the index

$$-i \int dx \partial^\mu J_\mu^5 = -\frac{i}{2\pi} \int dx \epsilon_{\mu\nu} F^{\mu\nu} = -\frac{1}{2\pi} \int dx \bar{\epsilon}_{\mu\nu} F^{\mu\nu}$$

$$= 2 \int dx \sum_n \Phi_n^* \gamma_5 \Phi_n = 2(n_+ - n_-) = 2 \mathrm{index} D_+ \qquad (248)$$

where the real $\bar{\epsilon}_{\mu\nu} = i\epsilon_{\mu\nu}$ was introduced for convenience. So Fujikawa's regularization procedure corresponds to a local evaluation of the index.

There is an interesting connection between the index of a differential operator and the heat kernel of its Laplacian [30]. Thus we also can understand Fujikawa's procedure in the light of a heat kernel expansion.

Let D_+ be some elliptic differential operator (and we specialize to the Weyl operator (247) later on) and we construct the two Laplacians

$$\Delta_+ = D_+^\dagger D_+ = D_- D_+$$

$$\Delta_- = D_+ D_+^\dagger = D_-^\dagger D_- = D_+ D_- .$$

Both Laplacians are elliptic, selfadjoint and positive semidefinite. Since we consider here a compact manifold (a sphere in practice) the spectrum of $\Delta_{+(-)}$ is discrete and the degeneracy of each eigenvalue is finite.

The two Laplacians Δ_+ and Δ_- have an identical spectrum for nonvanishing eigenvalues. Suppose the eigenvalue equation

$$\Delta_+ \Phi_\lambda = \lambda \Phi_\lambda$$

then

$$\Psi_\lambda := D_+ \Phi_\lambda$$

is an eigenfunction of Δ_- with the same eigenvalue λ

$$\Delta_- \Psi_\lambda = \Delta_- D_+ \Phi_\lambda = D_+ D_- D_+ \Phi_\lambda = D_+ \Delta_+ \Phi_\lambda = \lambda D_+ \Phi_\lambda = \lambda \Psi_\lambda .$$

So we have

$$\dim E_+(\lambda) = \dim E_-(\lambda)$$

when denoting the spaces of the eigenfunctions by

$$E_+(\lambda) = \{\Phi_\lambda\} \quad , \quad E_-(\lambda) = \{\Psi_\lambda\} .$$

However this is not true for the zero mode, $\lambda = 0$. But the zeros of $\Delta_{+(-)}$ and $D_{+(-)}$ are the same

$$\ker \Delta_{+(-)} = \ker D_{+(-)}.$$

Now we can proof the following relation between the index and the heat kernel (for any $t > 0$)

$$\mathrm{index} D_+ = \mathbf{Tr} e^{-t\Delta_+} - \mathbf{Tr} e^{-t\Delta_-}$$

Proof:

$$\mathbf{Tr}_{E_+} e^{-t\Delta_+} - \mathbf{Tr}_{E_-} e^{-t\Delta_-} = \sum_\lambda (\Phi_\lambda | e^{-t\Delta_+} | \Phi_\lambda) - \sum_\lambda (\Psi_\lambda | e^{-t\Delta_-} | \Psi_\lambda)$$

$$= \sum_\lambda e^{-t\lambda} [\dim E_+(\lambda) - \dim E_-(\lambda)] = e^{-t\cdot 0} [\dim E_+(0) - \dim E_-(0)]$$

$$= \dim \ker \Delta_+ - \dim \ker \Delta_- = \dim \ker D_+ - \dim \ker D_- = \mathrm{index} D_+.$$

Choosing now the Weyl operator from before we get

$$\mathrm{index} D_+ = \mathbf{Tr}_{S_+} e^{-t(i\not{D})^2 P_+} - \mathbf{Tr}_{S_-} e^{-t(i\not{D})^2 P_-} = \mathbf{Tr}_{S_+} e^{-t(i\not{D})^2} P_+ - \mathbf{Tr}_{S_-} e^{-t(i\not{D})^2} P_-$$

$$= \mathbf{Tr}_{S=S_+\oplus S_-} e^{-t(i\not{D})^2} (P_+ - P_-) = \mathbf{Tr}_S \gamma_5 e^{-t(i\not{D})^2}. \tag{249}$$

Formula (249) corresponds to Fujikawa's regularization procedure when expanding the heat kernel, picking up the t-independent part and performing the limit $t \to 0$.

Finally we return to the Atiyah-Singer index theorem [28]. Atiyah and Singer have shown that the above discussed index of an elliptic differential operator can be expressed in terms of topological quantities, socalled characteristic classes. In case of the Dirac operator including Yang-Mills fields the theorem states

$$\mathrm{index} D_+ = \int_M \mathrm{ch}(F)$$

where $\mathrm{ch}(F)$denotes the Chern character

$$\mathrm{ch}(F) = \mathrm{tr} e^{\frac{i}{2\pi}F} = r + \frac{i}{2\pi}\mathrm{tr}F + \frac{1}{2!}(\frac{i}{2\pi})^2 \mathrm{tr}F^2 + \cdots \tag{250}$$

and

$$F = dA + A^2$$

is the curvature two-form including the generators of the gauge group; r is the dimension of the representation of the gauge group. The integral is taken over a compact manifold with dimension $2n$ so that only the n-th term in expansion (250) survives.

In our case we calculate on a two-dimensional sphere S^2 and we consider just the $U(1)$ gauge group. Then the index theorem provides

$$\mathrm{index} D_+ = \frac{i}{2\pi} \int_{S^2} F = \frac{i}{2\pi} \int_{S^2} \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$$

$$= -\frac{i}{4\pi} \int_{S^2} \tilde{F} \epsilon_{\mu\nu} dx^\mu dx^\nu = -\frac{i}{4\pi} \int dx \epsilon^{\mu\nu} F_{\mu\nu} = -\frac{1}{4\pi} \int dx \bar{\epsilon}^{\mu\nu} F_{\mu\nu}$$

where the last expression is in real terms only. This result is in accordance with Fujikawa's calculation of the anomaly, equ. (206). So we can use the Atiyah-Singer index theorem - a topological result - to evaluate the anomaly.

4.6 Comparison with perturbative calculations

The mass of the photon in the Schwinger model can be seen at once in the perturbative result. For the vacuum polarization we found

$$T_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})T(q^2) \tag{251}$$

where in the massless case the amplitude is given by $q^2 T = -\frac{1}{\pi}$. (See result (28)). The photon propagator in the Lorentz gauge is written as

$$D_{\mu\nu}(q) = (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})D(q^2) \tag{252}$$

where for the free propagator

$$D_0(q^2) = -\frac{1}{q^2 - i\epsilon}. \tag{253}$$

For the exact propagator we have

$$\begin{aligned} D(q^2) &= D_0(q^2) - D_0(q^2)e^2q^2T(q^2)D_0(q^2) + D_0(q^2)e^2q^2T(q^2)D_0(q^2)e^2q^2T(q^2)D_0(q^2) + \cdots \\ &= \frac{1}{D_0^{-1}(q^2)(1 + e^2q^2T(q^2)D_0(q^2))} = -\frac{1}{q^2 + e^2q^2T(q^2)} \end{aligned} \tag{254}$$

and in the case of massless fermions we get

$$D_{\mu\nu}(q) = -\frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}}{q^2 - \frac{e^2}{\pi}} \tag{255}$$

which is just a free propagator for a particle with mass $m_\gamma = \frac{e}{\sqrt{\pi}}$.

Examining next a fermion cross section we will also find that the interaction vanishes in the massless limit. Taking the total cross section of $e^+e^- \to e^+e^-$ for definiteness we have, in least order (see Fig. 6)

$$\sigma_{\rm tot}^{e^+e^-\to e^+e^-} = \frac{1}{2w(q^2, m^2, m^2)} \int \frac{dp_1'}{2p_1'^0}\frac{dp_2'}{2p_2'^0}\delta^2(p_1 + p_2 - p_1' - p_2') \sum_{\rm spins} |\langle e^+e^- \mid T \mid e^+e^-\rangle|^2 \tag{256}$$

where m is the electron mass and

$$w(x, y, z) = (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)^{1/2}. \tag{257}$$

Using

$$\langle e^+e^- \mid T \mid e^+e^-\rangle = \frac{e^2}{q^2}\bar v(p_2)\gamma_\mu u(p_1)\bar u(p_1')\gamma^\mu v(p_2') \tag{258}$$

and performing the two dimensional spin trace we get for the unpolarized cross section

$$\begin{aligned} \sigma_{\rm tot}^{e^+e^-\to e^+e^-} &= \frac{1}{2w(q^2, m^2, m^2)} \int \frac{dp_1'}{2p_1'^0}\frac{dp_2'}{2p_2'^0}\delta^2(p_1 + p_2 - p_1' - p_2')\cdot \\ &\cdot \frac{2e^4}{q^4}(p_1p_1'p_2p_2' + p_1p_2'p_2p_1' - p_1p_2p_1'p_2' + m^4). \end{aligned} \tag{259}$$

In two dimensions the following momentum integrals hold:

$$I_2 = \int \frac{dp_1'}{2p_1'^0}\frac{dp_2'}{2p_2'^0}\delta^2(q - p_1' - p_2')f(p_1' \cdot p_2')$$

$$= \frac{1}{q^2}(1 - \frac{4m^2}{q^2})^{-1/2} f(\frac{1}{2}(q^2 - 2m^2)) = w^{-1}(q^2, m^2, m^2) f(\frac{1}{2}(q^2 - 2m^2)) \tag{260}$$

and

$$I_2^{\mu\nu} = \int \frac{dp_1'}{2p_1'^0}\frac{dp_2'}{2p_2'^0}\delta^2(q - p_1' - p_2')f(p_1' \cdot p_2')p_1'^\mu p_2'^\nu$$

$$= w^{-1}(q^2, m^2, m^2) f(\frac{1}{2}(q^2 - 2m^2))[(\frac{q^2}{4} - m^2)g^{\mu\nu} + \frac{m^2}{q^2}q^\mu q^\nu]$$

$$= [(\frac{q^2}{4} - m^2)g^{\mu\nu} + \frac{m^2}{q^2}q^\mu q^\nu]I_2. \tag{261}$$

Using this formulae we get for the cross section

$$\sigma_{\text{tot}}^{e^+e^- \to e^+e^-} = \frac{2e^4m^4}{q^4}w^{-2}(q^2, m^2, m^2)\Theta(q^2 - 4m^2) = \frac{2e^4m^4}{q^8}(1 - \frac{4m^2}{q^2})^{-1}\Theta(q^2 - 4m^2) \tag{262}$$

which vanishes in the limit $m \to 0$.

The total cross section for all orders of perturbation theory can be obtained in a completely analogous manner. We only have to substitute the denominator of the free photon propagator, q^2, by the denominator of the exact photon propagator, $q^2 + e^2q^2T(q^2)$. We get the exact expression

$$\sigma_{\text{tot}}^{e^+e^- \to e^+e^-} = \frac{1}{2w(q^2, m^2, m^2)}\int \frac{dp_1'}{2p_1'^0}\frac{dp_2'}{2p_2'^0}\frac{2e^4}{(q^2 + e^2q^2T(q^2))^2}.$$

$$\cdot \delta^2(p_1 + p_2 - p_1' - p_2')(p_1p_1'p_2p_2' + p_1p_2'p_2p_1' - p_1p_2p_1'p_2' + m^4) \tag{263}$$

for the total cross section and, performing the momentum integrations,

$$\sigma_{\text{tot}}^{e^+e^- \to e^+e^-} = \frac{2e^4m^4}{(q^2 + e^2q^2T(q^2))^2}\frac{1}{q^4}(1 - \frac{4m^2}{q^2})^{-1}\Theta(q^2 - 4m^2) \tag{264}$$

which obviously vanishes in the limit $m \to 0$. Thus we could indeed reproduce the two features of the Schwinger model, vanishing interaction and the photon aquiring a mass $m_\gamma = \frac{e}{\sqrt{\pi}}$, in our perturbative calculations.

5 Summary

In the literature there exist many articles dealing with different treatments of the axial anomaly.

We wanted to review some of those treatments using as a very simple example two dimensional QED and to show how these different treatments are connected to each other.

We showed how the anomaly arises because of the behaviour of some n-point functions in different treatments (perturbative, dispersive, in BJL limit) and how this behaviour can be traced to the very definition of the Dirac vacuum.

Another feature of two dimensional QED is the solvability of the massless case (Schwinger model). This we could use to demonstrate some methods of exactly solving it and to relate the exact solution to the previous results of the article.

So the aim of the article is on one hand to give an overview on two dimensional QED and its anomaly and Schwinger term. On the other hand we wanted to show different possibilities of treating the anomaly and their relations, and to demonstrate some modern techniques using a very simple model. Here all computations are relatively simple and we could use it as a "training model" and show how some modern and advanced computational methods work.

QED in two dimensions and especially the Schwinger model, although it is an abelian model, has interesting topological properties and nonperturbative aspects like condensates and confinement. Thus the Schwinger model is not just of academic interest but serves as a guide for a four dimensional nonabelian field theory (see e.g. [38]-[40] and the references cited there).

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Appendix

Feynman-integrals in n dimensions

$$I_0 = \int \frac{d^n p}{(2\pi)^n} \frac{1}{(p^2 + 2kp + M^2)^\alpha} = \frac{i(-\pi)^{n/2}}{(2\pi)^n} \frac{\Gamma(\alpha - \frac{n}{2})}{\Gamma(\alpha)(M^2 - k^2)^{\alpha - \frac{n}{2}}} \tag{A.1}$$

$$I_\mu = \int \frac{d^n p}{(2\pi)^n} \frac{p_\mu}{(p^2 + 2kp + M^2)^\alpha} = -k_\mu I_0 \tag{A.2}$$

$$I_{\mu\nu} = \int \frac{d^n p}{(2\pi)^n} \frac{p_\mu p_\nu}{(p^2 + 2kp + M^2)^\alpha} = I_0 (k_\mu k_\nu + \frac{1}{2} g_{\mu\nu} \frac{M^2 - k^2}{\alpha - \frac{n}{2} - 1}) \tag{A.3}$$

Needed integrals:

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\frac{1}{2} \ln \mid \frac{a+x}{a-x} \mid + C \tag{A.4}$$

$$\int \frac{dx x^2}{a^2 - x^2} = a(-\frac{x}{a} + \frac{1}{2} \ln \mid \frac{a+x}{a-x} \mid) + C \tag{A.5}$$

Feynman parameter integral:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \tag{A.6}$$

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Figure Captions

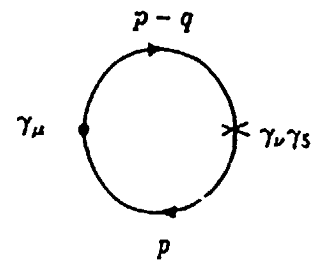

Fig. 1: Fermion loop containing a vector and axial current

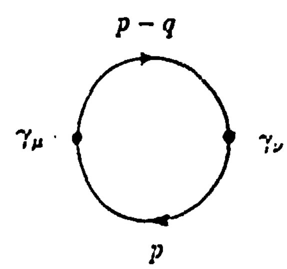

Fig. 2: Fermion loop containing a vector and vector current

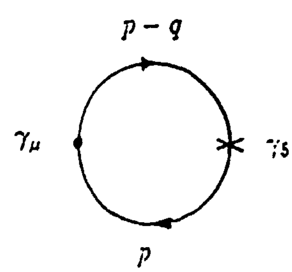

Fig. 3: Fermion loop containing a vector and pseudoscalar current

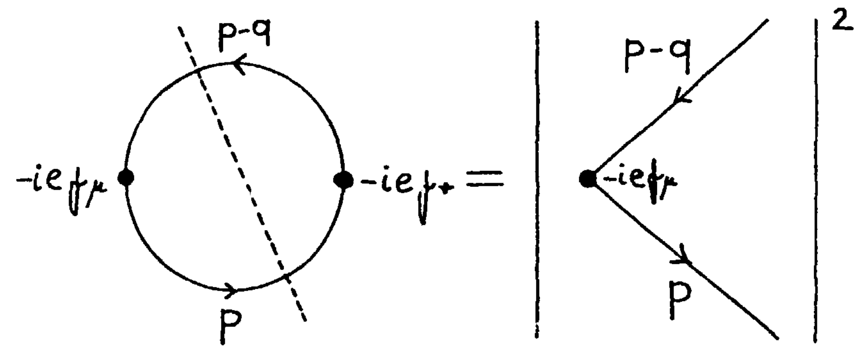

Fig. 4: Graphical representation of the unitarity relation

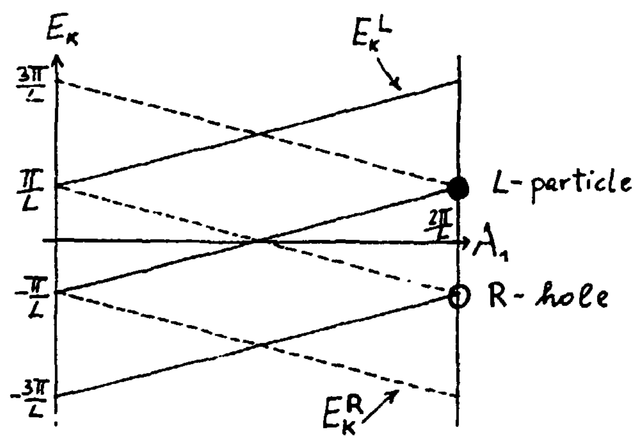

Fig. 5: Crossing of the energy levels

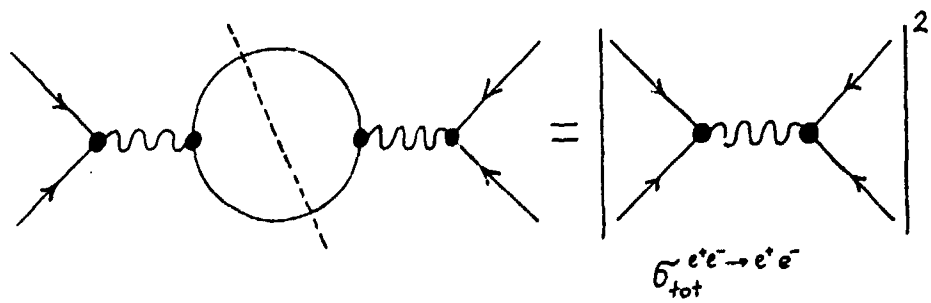

Fig. 6: Derivation of the total cross section