Interactions

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- Up to now: free fields $\Rightarrow$ NO interactions
- Now: description of interacting theory: QED, SQED, ...
- Real theory (QED):
  - Technical problems which obscure the fundamentals
    - spinor, gauge-boson indices, ...

  $\Rightarrow$ Use simplest theory:

Hermitic Klein-Gordon field with quartic self-interaction

$$\mathcal{L} =: \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - m^2\phi\phi\right) - \frac{\lambda}{4!}\phi^4 :$$

Solution program

1. write the Euler-Lagrange equations of motion
2. solve them
3. convert the solutions to operators
4. apply canonical commutation relations between the fields
5. compute matrix-elements

NOT possible!

Several fields (Ψ, A_μ) $\Rightarrow$ coupled partial differential equations

$\Rightarrow$ rely on **perturbation theory**

- Assume that the interaction term is small

$$\lambda \ll 1$$

- make a perturbative expansion around $\lambda = 0$ in a power series

$$A = A_0 + A_1\lambda + A_2\lambda^2 + \ldots$$

- E.g. QED: $\alpha = \frac{e^2}{4\pi} = \frac{1}{137} \ll 1$
- Separate the Lagrangian in a free and an interaction term:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

  - $\mathcal{L}_0$: free-lagrangian $\Rightarrow$ solutions are the free fields
  - $\mathcal{L}_I$: small correction
- To perform this treatment

  $\Rightarrow$ Interaction picture is specially well suited

Interaction Picture

- Schrödinger Picture: States evolve, operators don't evolve:[1]

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}|\psi,t\rangle_S &= H|\psi,t\rangle_S \text{ Schrödinger equation} \\ |\psi,t\rangle_S &= e^{-iH(t-t_0)}|\psi,t_0\rangle_S = U(t,t_0)|\psi,t_0\rangle_S \\ U(t,t_0) &= e^{-iH(t-t_0)} \quad \text{evolution operator} \\ O^S &= \text{constant} \end{aligned} \tag{1}$$

- Heisenberg Picture: States don't evolve, operators evolve:

$$\begin{aligned} |\psi,t\rangle_H &= |\psi,t_0\rangle_H = |\psi,t_0\rangle_S = U^\dagger(t,t_0)|\psi,t\rangle_S \\ O^H(t) &= U^\dagger(t,t_0)O^S U(t,t_0) \\ i\frac{\mathrm{d}}{\mathrm{d}t}O^H(t) &= [O^H(t),H] \end{aligned} \tag{2}$$

$U(t,t_0)$ is unitary $\Rightarrow$ preserves scalar products

[1] Operators in the Schrödinger picture might have an explicit time-dependence, we don't consider this case here.

- Interaction Picture

$$H = \underbrace{H_0}_{\text{H-picture}} + \underbrace{H_{int}}_{\text{S-picture}}$$

$$\begin{aligned} U_0(t,t_0) &= e^{-iH_0(t-t_0)} \\ |\psi,t\rangle_I &= U_0^\dagger(t,t_0)|\psi,t\rangle_S = e^{iH_0(t-t_0)}|\psi,t\rangle_S \\ O^I(t) &= U_0^\dagger(t,t_0)O^S U_0(t,t_0) \end{aligned} \tag{3}$$

$U_0(t,t_0)$ unitary $\Rightarrow$ preserves scalar products.

$$[H_0,H_0] = 0 \Rightarrow H_0^I = H_0^S = H_0$$

both the states and the operators evolve with time:

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}O^I(t) &= [O^I(t),H_0] \Rightarrow \text{Operators evolve with } H_0 \\ i\frac{\mathrm{d}}{\mathrm{d}t}|\psi,t\rangle_I &= H_{int}^I(t)|\psi,t\rangle_I \Rightarrow \text{States evolve with } H_{int}^I \\ H_{int}^I(t) &= U_0^\dagger(t,t_0)H_{int}U_0(t,t_0) = e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)} \end{aligned} \tag{4}$$

If $[H_{int},H_0] \neq 0$, $\Rightarrow$ H_{int} evolves with time.

Interaction Picture

- Exact treatment for any Hamiltonian
- Specially well suited for time-dependent perturbation theory
  - ⇒ We know exact solutions of H_0 and
  - ⇒ H_{int} is a small perturbation
- Defined as a function of the Schrödinger picture (3)
  - ⇒ Quantum Field Theory formulated in Heisenberg picture (2).
  - ⇒ Relation between I-picture and H-picture

Def: Evolution operator in the interaction picture

$$U_I(t,t_0) \equiv U_0^\dagger(t,t_0)U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{5}$$

This operator is called just $U(t,t_0)$ in Peskin-Schroeder and other books.

$$\begin{aligned} |\psi,t\rangle_I &= U_0^\dagger(t,t_0)U(t,t_0)|\psi,t\rangle_H = U_I(t,t_0)|\psi,t\rangle_H \\ &= U_I(t,t_0)|\psi,t_0\rangle_H = U_I(t,t_0)|\psi,t_0\rangle_I \\ O^I(t) &= U_0^\dagger(t,t_0)U(t,t_0)O^H(t)U^\dagger(t,t_0)U_0(t,t_0) \\ &= U_I(t,t_0)O^H(t)U_I^\dagger(t,t_0) \end{aligned} \tag{6}$$

$U_I(t,t_0)$ represents the evolution of the states $|\psi,t\rangle_I$ in the interaction picture.

Differential equation for $U_I(t, t_0)$

$$
\begin{aligned}
i\frac{\mathrm{d}}{\mathrm{d}t}U_I(t,t_0) &= e^{iH_0(t-t_0)}(H - H_0)e^{-iH(t-t_0)} = e^{iH_0(t-t_0)}H_{int}e^{-iH(t-t_0)} \\
&= \underbrace{e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)}}\,\underbrace{e^{iH_0(t-t_0)}e^{-iH(t-t_0)}} \\
&= H^I_{int}(t)U_I(t,t_0) \qquad (7)
\end{aligned}
$$

$$\text{Initial condition: } U_I(t,t) = 1 \qquad (8)$$

Formal solution

$$U_I(t,t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)} \qquad (9)$$

Fulfills differential equation (7) with contour condition (8).

Product of two evolution operators

$$
\begin{aligned}
U_I(t_1,t_2)U_I(t_2,t_3) &= e^{iH_0(t_1-t_0)}e^{-iH(t_1-\cancel{t_2})}\cancel{e^{-iH_0(t_2-t_0)}} \\
&\quad \cancel{e^{iH_0(t_2-t_0)}}e^{-iH(\cancel{t_2}-t_3)}e^{-iH_0(t_3-t_0)} \\
&= e^{iH_0(t_1-t_0)}e^{-iH(t_1-t_3)}e^{-iH_0(t_3-t_0)} \\
&= U_I(t_1,t_3) \\
U_I(t_1,t_2) &= U_I(t_1,t_3)U_I^\dagger(t_2,t_3)
\end{aligned}
\tag{10}
$$

The S-matrix

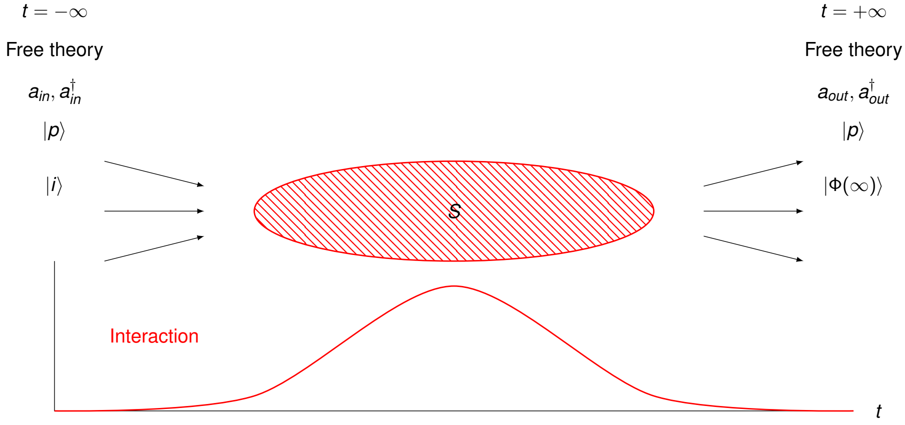
- H_0: Free Hamiltonian
- H_{int}: interaction hamiltonian
- Fields (operators) in the interaction picture
  - $\Rightarrow$ $\phi_I(x)$ are the solutions H_0: the free theory
  - $\Rightarrow$ Admit a description as ladder operators $a^r_{\mathbf{p}}$, $a^{r\dagger}_{\mathbf{p}}$
    - set of states of definite momentum $|p_1 \cdots p_n\rangle$
    - Particle interpretation

- the states $|p_1 \cdots p_n\rangle$ are not eigenstates of the full Hamiltonian H
- they are no longer stationary states $\Rightarrow$ they will evolve with time.
- At t_0 we have a state: set of particles of given momentum:

$$|\psi, t_0\rangle = |p_1 \cdots p_n\rangle$$

- At $t > t_0$: $|\psi, t\rangle = U_I(t, t_0)|\psi, t_0\rangle$
  - $\Rightarrow$ not the same set of particles with the same momentum
  - $\Rightarrow$ maybe: linear combination of different sets of particles

$$|\psi, t\rangle = \sum c_{k_1,\cdots,k_\alpha}|k_1, \cdots, k_\alpha\rangle$$



- At $t_i = -\infty$: prepare input state: $|\Phi(-\infty)\rangle = |i\rangle$
  $\Rightarrow$ given number of particles with definite momenta
- The state evolves, at $t_f = \infty$ it is:
$$|\Phi(\infty)\rangle = S|\Phi(-\infty)\rangle = S|i\rangle$$

S: **scattering operator** or S-matrix for short.

- State $|f\rangle$ (a given set of particles with definite momenta, spin, etc.)
  $\Rightarrow$ probability that $|f\rangle$ is contained in the final state $|\Phi(\infty)\rangle$:

$$|\langle f|\Phi(\infty)\rangle|^2$$

- Probability amplitude:

$$\langle f|\Phi(\infty)\rangle = \langle f|S|i\rangle \equiv S_{fi}$$

  $\Rightarrow$ scattering amplitude matrix.

$$S = U_I(\infty, -\infty)$$

$\Rightarrow$ S is unitary

$$SS^\dagger = 1 \quad \Rightarrow \sum_f |S_{fi}|^2 = \sum_f |\langle f|S|i\rangle|^2 = 1$$

$\Rightarrow$ the probability that anything happens is 1.

Def: Transition matrix $\mathcal{T}$

$$S = 1 + i\mathcal{T}$$

- 1: *nothing happens*
- $\mathcal{T}$: probability amplitude that some interaction took place

$$S^\dagger S = 1 \Rightarrow -i(\mathcal{T} - \mathcal{T}^\dagger) = \mathcal{T}^\dagger \mathcal{T}$$

insert initial-final states: $\langle b|\mathcal{T}|a\rangle = \mathcal{T}_{ba}$

$$-i(\mathcal{T}_{ba} - \mathcal{T}_{ab}^*) = \sum_n \mathcal{T}_{nb}^* \mathcal{T}_{na}$$

for $a = b$

Optical Theorem

$$2\,\mathrm{Im}(\mathcal{T}_{aa}) = \sum_n |\mathcal{T}_{na}|^2$$

$\Rightarrow$ translated to scattering process

the total cross-section equals the imaginary part of the forward scattering amplitude

Perturbative expansion

if H_{int} small perturbation $\Rightarrow$ solve by iterative procedure

Formal solution of eq. (7)

$$U_I(t,t_0) = U_I(t_0,t_0) - i\int_{t_0}^{t} H^I_{int}(t_1)U_I(t_1,t_0)\,\mathrm{d}t_1 \tag{11}$$

- Zeroth order aproximation ($H_{int}=0$):

$$U_I^0(t,t_0) = 1 = U_I(t_0,t_0)$$

- Substitute into (11) and obtain the first order approximation:

$$U_I^1(t,t_0) = 1 - i\int_{t_0}^{t} H^I_{int}(t_1)\,\mathrm{d}t_1$$

- Substitute into (11) to obtain the second order approximation:

$$\begin{aligned} U_I^2(t,t_0) &= 1 - i\int_{t_0}^{t} H^I_{int}(t_1)\,\mathrm{d}t_1 + (-i)^2\int_{t_0}^{t}\mathrm{d}t_1\,H^I_{int}(t_1)\int_{t_0}^{t_1} H^I_{int}(t_2)\,\mathrm{d}t_2 \\ &= 1 - i\int_{t_0}^{t}\mathrm{d}t_1\,H^I_{int}(t_1) + (-i)^2\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\,H^I_{int}(t_1)H^I_{int}(t_2) \end{aligned}$$

Substitute iterativelly into (11):

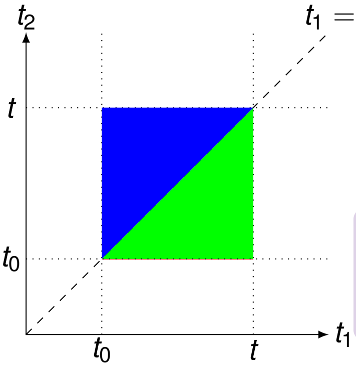
$$\begin{aligned} U_I(t,t_0) &= 1 - i\int_{t_0}^{t} \mathrm{d}t_1\, H^I_{int}(t_1) + (-i)^2 \int_{t_0}^{t}\mathrm{d}t_1 \int_{t_0}^{t_1}\mathrm{d}t_2\, H^I_{int}(t_1)H^I_{int}(t_2) \\ &+ (-i)^3 \int_{t_0}^{t}\mathrm{d}t_1 \int_{t_0}^{t_1}\mathrm{d}t_2 \int_{t_0}^{t_2}\mathrm{d}t_3\, H^I_{int}(t_1)H^I_{int}(t_2)H^I_{int}(t_3) \\ &+ \cdots \\ &+ (-i)^n \int_{t_0}^{t}\mathrm{d}t_1 \int_{t_0}^{t_1}\mathrm{d}t_2 \cdots \int_{t_0}^{t_{n-1}}\mathrm{d}t_n H^I_{int}(t_1)H^I_{int}(t_2)\cdots H^I_{int}(t_n) \\ &+ \cdots \end{aligned} \tag{12}$$

- $t \geq t_1 \geq t_2 \cdots \geq t_n \geq t_0 \Rightarrow$ Hamiltonians are time-ordered
- Introduce time-ordered product

$$\int_{t_0}^{t}\mathrm{d}t_1 \int_{t_0}^{t_1}\mathrm{d}t_2\, H^I_{int}(t_1)H^I_{int}(t_2) = \int_{t_0}^{t}\mathrm{d}t_1 \int_{t_0}^{t_1}\mathrm{d}t_2\, T\{H^I_{int}(t_1)H^I_{int}(t_2)\}$$

$\Rightarrow$ symmetric expression $t_1 \leftrightarrow t_2$.

$$\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \, T\{H^I_{int}(t_1) H^I_{int}(t_2)\}$$



symmetric $t_1 \leftrightarrow t_2$.

$$\int \text{square} = 2 \int \text{triangle}$$

$$\frac{1}{2} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2 \, T\{H^I_{int}(t_1) H^I_{int}(t_2)\}$$

Introduce Hamiltonian density

$$= \frac{1}{2} \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \, T\{\mathcal{H}^I_{int}(x_1) \mathcal{H}^I_{int}(x_2)\}$$

- Same process for other terms
  $\Rightarrow$ $n!$ to compensate extra integrated space

Dyson series

$$
\begin{aligned}
U_I(t,t_0) &= 1 - i\int \mathrm{d}^4x_1\, T\{\mathcal{H}^I_{int}(x_1)\} + \frac{(-i)^2}{2!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\, T\{\mathcal{H}^I_{int}(x_1)\mathcal{H}^I_{int}(x_2)\} \\
&+ \frac{(-i)^3}{3!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\, T\{\mathcal{H}^I_{int}(x_1)\mathcal{H}^I_{int}(x_2)\mathcal{H}^I_{int}(x_3)\} \\
&+ \cdots \\
&+ \frac{(-i)^n}{n!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\cdots\mathrm{d}^4x_n\, T\{\mathcal{H}^I_{int}(x_1)\mathcal{H}^I_{int}(x_2)\cdots\mathcal{H}^I_{int}(x_n)\} \\
&+ \cdots \\
&= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!} T\left\{\int \mathrm{d}^4x_1\mathrm{d}^4x_2\cdots\mathrm{d}^4x_n\, \mathcal{H}^I_{int}(x_1)\mathcal{H}^I_{int}(x_2)\cdots\mathcal{H}^I_{int}(x_n)\right\} \\
&\equiv T\left\{\exp\left[-i\int \mathrm{d}^4x\, \mathcal{H}^I_{int}(x)\right]\right\} \qquad (13)
\end{aligned}
$$

Last line is the exponential definition
Easy reminder of the full expression

$\mathcal{H}^I_{int}(x)$: Interaction hamiltonian density in the interaction picture

$\Rightarrow$ same expression as Heisenberg picture,

$\Rightarrow$ but as a function of the interaction-picture fields

$$
\begin{aligned}
\mathcal{H}^I_{int} &= U_I(t,t_0)\mathcal{H}^H_{int}U_I^\dagger(t,t_0) = \frac{\lambda}{4!}U_I(t,t_0)\phi_H^4 U_I^\dagger(t,t_0) = \\
&= \frac{\lambda}{4!}\underbrace{U_I(t,t_0)\phi_H U_I^\dagger(t,t_0)}\,\underbrace{U_I(t,t_0)\phi_H U_I^\dagger(t,t_0)} \\
&\quad \underbrace{U_I(t,t_0)\phi_H U_I^\dagger(t,t_0)}\,\underbrace{U_I(t,t_0)\phi_H U_I^\dagger(t,t_0)} \\
&= \frac{\lambda}{4!}\phi_I^4
\end{aligned}
$$

$\Rightarrow$ Same with any product of fields

Wick's Theorem

Interactions $\Rightarrow$ time-ordered product. How to compute?
Vaccuum expected value of two fields $\Rightarrow$ Feynman propagator

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\}|0\rangle = \Delta_F(x_1 - x_2)$$

Generalization:

- For any state
- For any number of fields

  $\Rightarrow$ relate it to a normal-ordered product

Notation:

$$\phi_x = \phi_I(x)\ ;\ \phi_y = \phi_I(y)$$

Two fields $\Rightarrow$ separate ϕ^+ and ϕ^-

$$\begin{aligned} T\{\phi_I(x)\phi_I(y)\} &\equiv T\{\phi_x\phi_y\} = T\{(\phi_x^+ + \phi_x^-)(\phi_y^+ + \phi_y^-)\} \\ &= \underbrace{T\{\phi_x^+\phi_y^+\} + T\{\phi_x^-\phi_y^-\}}_{\text{Already normal order}} + T\{\phi_x^+\phi_y^-\} + T\{\phi_x^-\phi_y^+\} \end{aligned}$$

$$
\begin{aligned}
T\{\phi_x\phi_y\} &= \, :\phi_x^+\phi_y^+: + :\phi_x^-\phi_y^-: + \begin{cases} \phi_x^+\phi_y^- + \phi_x^-\phi_y^+ & (x^0 > y^0) \\ \phi_y^-\phi_x^+ + \phi_y^+\phi_x^- & (x^0 < y^0) \end{cases} \\
&= \, :\phi_x^+\phi_y^+: + :\phi_x^-\phi_y^-: + \begin{cases} \phi_y^-\phi_x^+ + \phi_x^-\phi_y^+ + [\phi_x^+, \phi_y^-] & (x^0 > y^0) \\ \phi_y^-\phi_x^+ + \phi_x^-\phi_y^+ + [\phi_y^+, \phi_x^-] & (x^0 < y^0) \end{cases} \\
&= \, :\phi_x\phi_y: + \begin{cases} [\phi_x^+, \phi_y^-] & (x^0 > y^0) \\ [\phi_y^+, \phi_x^-] & (x^0 < y^0) \end{cases} \\
&= \, :\phi_x\phi_y: + \Theta(x^0 - y^0)[\phi_x^+, \phi_y^-] + \Theta(y^0 - x^0)[\phi_y^+, \phi_x^-] \\
&= \, :\phi_x\phi_y: + \Theta(x^0 - y^0)\Delta^+(x - y) + \Theta(y^0 - x^0)\Delta^+(y - x) \\
&= \, :\phi_x\phi_y: + \Delta_F(x - y)
\end{aligned}
$$

$\Rightarrow$ **Wick's theorem** for the time-ordered product of two fields.

New notation: **field contraction**:

$$\phi_x\phi_y \equiv \Delta_F(x-y)$$

if there are a number of fields between the two contracted, we define:

$$\phi_{z_1}\cdots\phi_{z_i}\phi_x\phi_{z_{i+1}}\cdots\phi_{z_k}\phi_y\phi_{z_{k+1}}\cdots\phi_{z_n} \equiv \Delta_F(x-y)\phi_{z_1}\cdots\phi_{z_n} \tag{14}$$

Wick's theorem for two fields:

$$T\{\phi_x\phi_y\} = :\phi_x\phi_y: + \phi_x\phi_y \tag{15}$$

Wick's theorem

$$T\{\phi_{z_1}\cdots\phi_{z_n}\} = :\phi_{z_1}\cdots\phi_{z_n} + \text{(all possible contractions)}: \tag{16}$$

Proof by induction

Example:

$$
\begin{aligned}
T\{\phi_1\phi_2\phi_3\phi_4\} = \; & :\phi_1\phi_2\phi_3\phi_4: \\
& + :\wick{\c{\phi_1}\c{\phi_2}}\phi_3\phi_4: + :\wick{\c{\phi_1}\phi_2\c{\phi_3}}\phi_4: + :\wick{\c{\phi_1}\phi_2\phi_3\c{\phi_4}}: \\
& + :\phi_1\wick{\c{\phi_2}\c{\phi_3}}\phi_4: + :\phi_1\wick{\c{\phi_2}\phi_3\c{\phi_4}}: \\
& + :\phi_1\phi_2\wick{\c{\phi_3}\c{\phi_4}}: \\
& + :\wick{\c1{\phi_1}\c1{\phi_2}\c2{\phi_3}\c2{\phi_4}}: + :\wick{\c1{\phi_1}\c2{\phi_2}\c1{\phi_3}\c2{\phi_4}}: + :\wick{\c1{\phi_1}\c2{\phi_2}\c2{\phi_3}\c1{\phi_4}}:
\end{aligned}
$$

Feynman Diagrams & Feynman Rules

Wick's theorem (16) $\oplus$ Dyson expansion (13)

$\Rightarrow$ compute probability amplitudes.

Example $2 \to 2$ process

$$p_A p_B \to p_1 p_2$$

$$\langle p_1 p_2 | i\mathcal{T} | p_A p_B \rangle$$

Zeroth order: move the a-operators to the right:

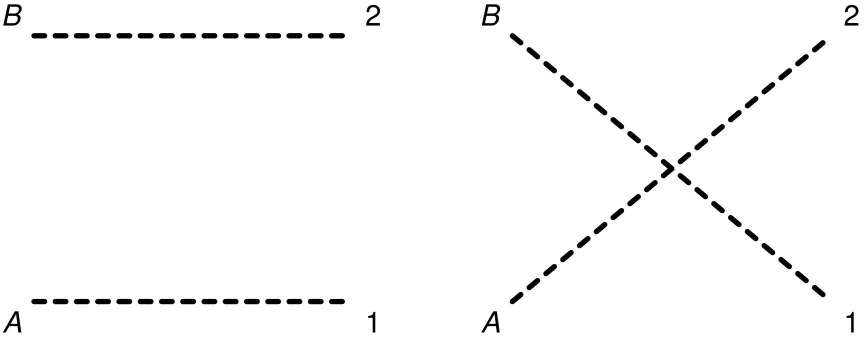
$$
\begin{aligned}
\langle p_1 p_2 | p_A p_B \rangle &= \sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_A}\sqrt{2E_B}\langle 0| a_1 a_2 a_A^\dagger a_B^\dagger |0\rangle \quad , \quad (a_2 \leftrightarrow a_A^\dagger) \\
&= \sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_A}\sqrt{2E_B}\langle 0| a_1 a_A^\dagger a_2 a_B^\dagger + a_1 a_B^\dagger (2\pi)^3 \delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_A) |0\rangle \\
&\quad (a_2 \leftrightarrow a_B^\dagger) \\
&= \sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_A}\sqrt{2E_B}\langle 0| a_1 a_A^\dagger a_B^\dagger a_2 + a_1 a_B^\dagger (2\pi)^3 \delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_A) \\
&\qquad\qquad + a_1 a_A^\dagger (2\pi)^3 \delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_B) |0\rangle \\
&\quad (a_1 \leftrightarrow a_A^\dagger) \quad , \quad (a_1 \leftrightarrow a_B^\dagger) \\
&= \sqrt{2E_1}\sqrt{2E_2}\sqrt{2E_A}\sqrt{2E_B}\langle 0| a_1 a_A^\dagger a_B^\dagger \cancelto{0}{a_2} \\
&\quad + a_B^\dagger \cancelto{0}{a_1} (2\pi)^3 \delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_A) + (2\pi)^6 \delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_B) \delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_A) \\
&\quad + a_A^\dagger \cancelto{0}{a_1} (2\pi)^3 \delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_B) + (2\pi)^6 \delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_A) \delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_B) |0\rangle \\
&= 2E_A 2E_B (2\pi)^6 \left(\delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_A)\delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_B) + \delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_B)\delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_A) \right)
\end{aligned}
$$

$$\langle p_1 p_2 | p_A p_B \rangle = 2E_A 2E_B (2\pi)^6 \left(\delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_A)\delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_B) + \delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_B)\delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_A) \right) \tag{17}$$

δ functions $\Rightarrow$ final state $\equiv$ initial state

Two options $\begin{cases} A = 1 & B = 2 \\ A = 2 & B = 1 \end{cases}$

$\Rightarrow$ 1 in the $S = 1 + i\mathcal{T}$ definition.



(18)

First order term

$$
\begin{aligned}
&\langle p_1 p_2 | T\{-i\frac{\lambda}{4!}\int \mathrm{d}^4x\, \phi_I^4(x)\}|p_A p_B\rangle = \\
&\qquad \langle p_1 p_2| : -i\frac{\lambda}{4!}\int \mathrm{d}^4x\, \phi_I^4(x) + \text{ contractions} : |p_A p_B\rangle
\end{aligned}
\tag{19}
$$

Uncontracted $\phi_I^+ \Rightarrow$ anihilates initial state particle:

$$
\begin{aligned}
\phi_I^+(x)|p\rangle &= \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_k}} a_k e^{-ikx}\sqrt{2E_p}\, a_p^\dagger|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_k}} e^{-ikx}\sqrt{2E_p}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{k})|0\rangle = e^{-ipx}|0\rangle
\end{aligned}
$$

New notation: **Contraction with an external state**

$$
\phi_I(x)|p\rangle = e^{-ipx} \quad ; \quad \langle p|\phi_I(x) = e^{ipx}
$$

Contractions in eq. (19) $\Rightarrow$ 3 kind of terms:

$$
\phi\phi\phi\phi \quad ; \quad \phi\phi\phi\phi \quad ; \quad \phi\phi\phi\phi \quad ; \tag{20}
$$

Uncontracted term

4 uncontracted fields $\Rightarrow$ contracted with external states

$\Rightarrow$ 4! equivalent ways of contracting: $4 \cdot 3 \cdot 2 \cdot 1$

$$4!\,\frac{-i\lambda}{4!}\int \mathrm{d}^4x\,\langle p_1 p_2|\phi(x)\phi(x)\phi(x)\phi(x)|p_A p_B\rangle \tag{21}$$

$$= -i\lambda \int \mathrm{d}^4x\, e^{-i(p_A+p_B-p_1-p_2)x} = -i\lambda(2\pi)^4\delta^4(p_A+p_B-p_1-p_2)$$

- 4! combinatorial: cancels the one in the denominator. (This is the reason it was put there.)
- δ^4: linear momentum conservation: $p_A + p_B = p_1 + p_2$

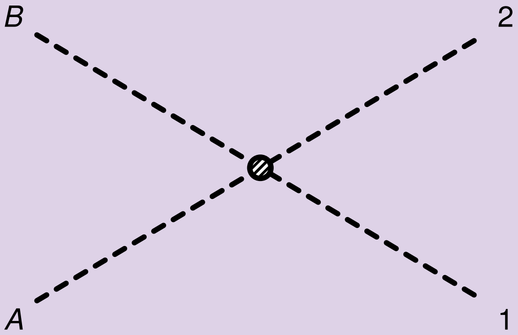
  $\Rightarrow$ needs to appear always $\Rightarrow$ extract it

Def: invariant matrix element $\mathcal{M}$

$$p_1 \cdots p_n \rightarrow k_1 \cdots k_l :\; i\mathcal{T} = i\mathcal{M}\cdot(2\pi)^4\delta^4\left(\sum_i^n p_i - \sum_j^l k_j\right) \tag{22}$$

- diagramatically:
  - two fields are created (at $t = -\infty$) with p_A, p_B,
  - interact at a point x,
  - and emerge with p_1, p_2:

Feyman rule for the 4-point vertex in the $\lambda\phi^4$ theory.



$$i\mathcal{M} = -i\lambda \qquad (23)$$

Fully contracted term

Last term in (20) $\Rightarrow$ 3 equal posibilities

$$3\frac{-i\lambda}{4!}\int \mathrm{d}^4x\langle p_1 p_2|\phi(x)\phi(x)\phi(x)\phi(x)|p_A p_B\rangle = \underbrace{\langle p_1 p_2|p_A p_B\rangle}_{0^{th}\text{ order}}\left(\frac{-i\lambda}{8}\right)\underbrace{\int \mathrm{d}^4x\Delta_F(x-x)\Delta_F(x-x)}_{\text{totally disconnected}} \quad (24)$$

0^{th} order $\Rightarrow$ contributes to the **1** factor of the S-matrix.
Disconnected term

- particle created at point $x \rightarrow$ propagates to the **same point** x
- Second particle: also created at point x, and propagates to the same point
- integrate over all points in space-time x

$\Rightarrow$ **vacuum diagram**

$$= \frac{-i\lambda}{8} \int \mathrm{d}^4x \, \Delta_F(x-x)\Delta_F(x-x) \qquad (25)$$

$\frac{1}{8}$: **symmetry factor** of the diagram: 1

- put the $1/n!$ of the Dyson expansion
- put the $\lambda/4!$ factorial for each vertex
- find the number all equal possible contractions

almost always the different contractions will cancel the $1/n!$, $1/4!$
Alternative computation:

- Don't write the $1/n!$ factorial
- Take all vertices as λ
- Compute a **symmetry factor** S, by counting the number of ways of interchanging components without changing the diagram.

We can build it the following way: start with the vaccum diagram, and assing a label to each line:

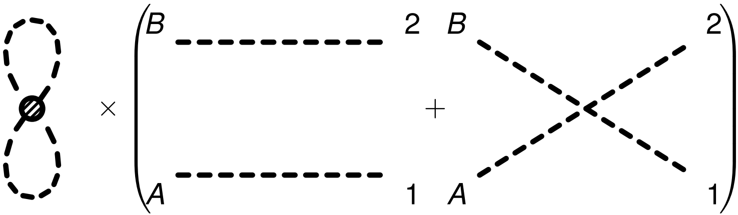
Is symmetric under:

- $a \leftrightarrow b$: a factor 2
- $c \leftrightarrow d$: a factor 2
- upper bubble $\leftrightarrow$ lower bubble: a factor 2

$$S = 2 \times 2 \times 2 = 8$$

$\Rightarrow$ 8 equivalent ways of constructing the same diagram
$\Rightarrow$ Feynman rule (23) we had taken all these as different:
$\Rightarrow$ have to divide by S.

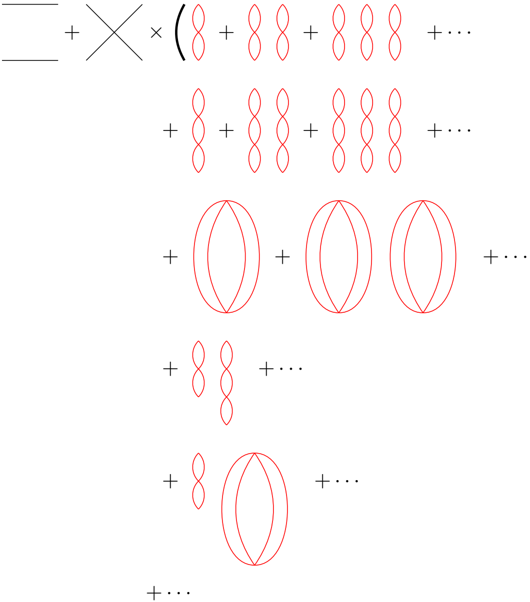
Fully contracted contribution

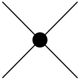


⇒ This is called a **disconnected diagram**

⇒ It is a correction to the non-interacting transition (17), (18)

- Higher order terms, with fully contracted fields,
  - ⇒ also contribute to the same matrix element.
  - ⇒ These terms are known as **vacuum bubbles** or **vacuum diagrams.**
- All other matrix elements will also have contributions from the same kind of disconnected diagrams, and vacuum bubbles.



All other matrix elements:  $\times$ (same)

The same factor appears everywhere

vaccum-vacuum transition:

$$\langle 0|T\left\{\exp\left[-i\int \mathrm{d}^4x\mathcal{H}^I_{int}\right]\right\}|0\rangle \tag{26}$$

which requires all fields to be contracted.

When we add succesive terms to the Dyson expansion of the vaccuum transition (26) we obtain the following kind of contributions

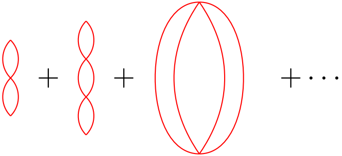
$$\begin{aligned}
8 &= -i\frac{\lambda}{4!}\int \mathrm{d}^4x\,\overline{\phi_x\phi_x}\,\overline{\phi_x\phi_x}\times 3=\frac{3}{4!}V=\frac{1}{8}V\equiv V_i\\
8\,8 &= \frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)\int \mathrm{d}^4x\,\overline{\phi_x\phi_x}\,\overline{\phi_x\phi_x}\int \mathrm{d}^4z\,\overline{\phi_z\phi_z}\,\overline{\phi_z\phi_z}\times 3^2=\frac{1}{2!}V_i^2\\
8\,8\,8 &= \frac{1}{3!}V_i^3
\end{aligned}$$

adding up all diagrams:

$$\sum_n \frac{V_i^n}{n!}=e^{V_i}$$

Other vacuum diagrams have same kind of contributions. define:

$$V_i = \text{ connected vaccum diagram}$$

$V_i =$ 

the vacuum-vacuum transition is:

$$\langle 0| T\left\{ \exp\left[-i \int \mathrm{d}^4 x \mathcal{H}^I_{int} \right] \right\} |0\rangle = \prod_i e^{V_i} = e^{\sum V_i}$$

So, for any transition process we can write:

$$i\mathcal{M} = \left(\sum i\mathcal{M}(\text{connected}) \right) \times e^{\sum V_i}$$

- vacuum transition elements (or vacuum bubbles) appear everywhere
- they are just an overall normalization factor $\Rightarrow$ **discard**[2].

[2] a justification will be given in the LSZ reduction formula.

Partially contracted term

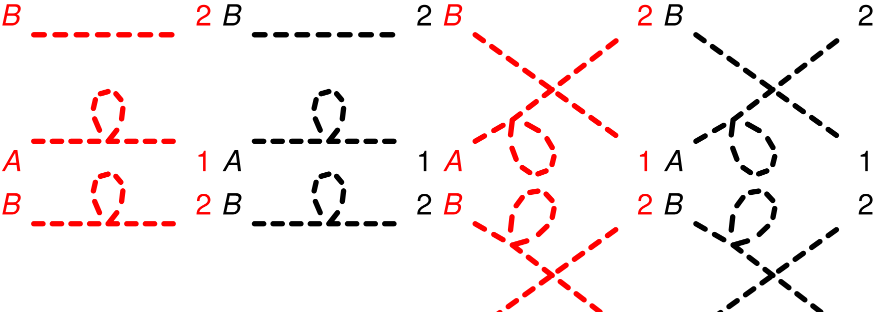
Second term in eq. (20): 2 uncontracted fields: $\phi\phi\phi\phi$ If we contract both in initial or final state:

$$\langle 0|p_A p_B\rangle = 0 \quad \text{or} \quad \langle p_1 p_2|0\rangle = 0$$

$\Rightarrow$ one field to the initial, and the other to the final

$$\frac{-i\lambda}{4!}\int \mathrm{d}^4x\langle p_1 p_2|\phi_x\phi_x\phi_x\phi_x|p_A p_B\rangle + \quad \frac{-i\lambda}{4!}\int \mathrm{d}^4x\langle p_1 p_2|\phi_x\phi_x\phi_x\phi_x|p_A p_B\rangle +$$

$$\frac{-i\lambda}{4!}\int \mathrm{d}^4x\langle p_1 p_2|\phi_x\phi_x\phi_x\phi_x|p_A p_B\rangle + \quad \frac{-i\lambda}{4!}\int \mathrm{d}^4x\langle p_1 p_2|\phi_x\phi_x\phi_x\phi_x|p_A p_B\rangle$$

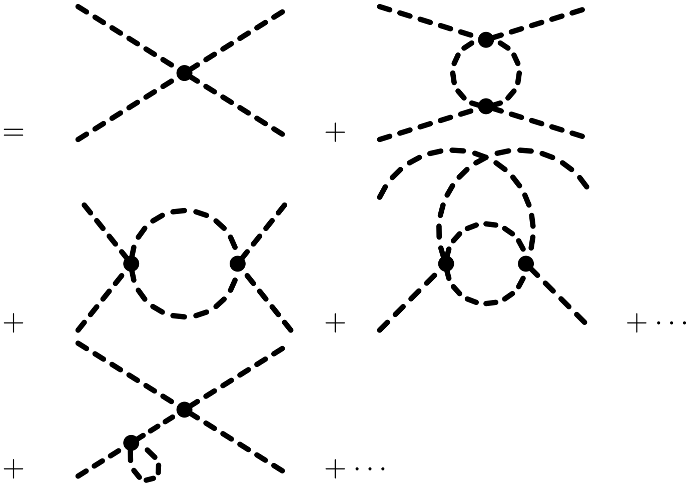


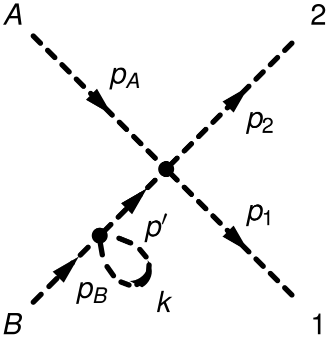
Expression for the transition matrix

So we seem to have found an expression for the transition matrix:

$$\langle p_1 p_2 | i\mathcal{T} | p_A p_B \rangle = \sum \text{ fully connected diagrams}$$

$\langle p_1 p_2 | i\mathcal{T} | p_A p_B \rangle =$



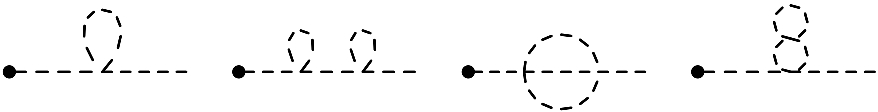


$$
= \frac{1}{2} \int \frac{\mathrm{d}^4 p'}{(2\pi)^4} \frac{i}{p'^2 - m^2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2}
$$
$$
\times (-i\lambda)(2\pi)^4 \delta^4(p_A + p' - p_1 - p_2)
$$
$$
\times (-i\lambda)(2\pi)^4 \delta^4(p_B - p')
$$

$\Rightarrow$ Integrate over p' with the last δ function:

$$
\left.\frac{1}{p'^2 - m^2}\right|_{p'=p_B} = \frac{1}{p_B^2 - m^2} = \frac{1}{0}
$$

$\Rightarrow$ any diagram that has loops in a external leg will have this infinity

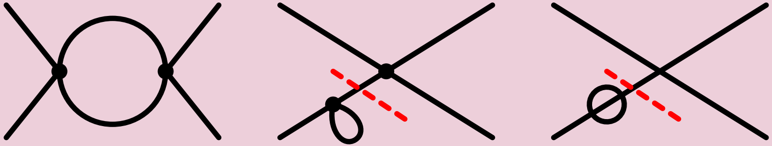


⇒ Similar situation to the vacuum bubbles

⇒ Define the S-matrix to exclude this diagrams[3]

Amputation

- Remove all subdiagrams associated to external legs which can be separated by cutting just one line



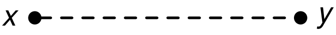
[3]Again: a justification will be given in the LSZ formalism, in which these contributions are the wave-function renormalization constants.

Feynman Rules

To compute a transition matrix element in position space:

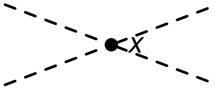
$$\langle p_1 \cdots p_n | i\mathcal{T} | p_A p_B \rangle = i\mathcal{M}(2\pi)^4 \delta^4(p_A + p_B - \sum_i p_i)$$

1. Construct all **fully connected**, **amputated** diagrams whith p_A, p_B incoming, $p_1 \ldots p_n$ outgoing
2. For each internal line (propagator), write a Feynman propagator



$$= \Delta_F(x - y)$$

3. for each vertex:



$$= (-i\lambda) \int \mathrm{d}^4 x$$

4. for each external line

$$= e^{-ipx}$$

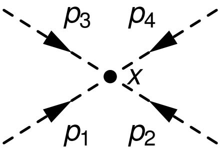
5. Divide by the symmetry factor S

It is usually easier, however, to work out in momentum space

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$$

$\Rightarrow$ each line converging to a vertex will have a e^{-ipx}

$\Rightarrow$ associated either to $\begin{cases} \text{propagator} \\ \text{external line} \end{cases}$

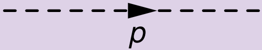


$$\leftrightarrow \int \mathrm{d}^4 x e^{-ip_1 x} e^{-ip_2 x} e^{-ip_3 x} e^{-ip_4 x} = (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)$$

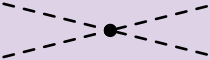
$\Rightarrow$ momentum is conserved at all vertices

Feynman rules in momentum space:

1. Construct all **fully connected**, **amputated** diagrams whith p_A, p_B incoming, $p_1 \dots p_n$ outgoing
2. For each internal line (propagator), write a Feynman propagator



$$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

3. for each vertex:  $= -i\lambda$
4. for each external line  $= 1$
5. Impose momentum conservation at each vertex
6. Integrate over all indetermined momenta: $\displaystyle\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$
7. Divide by the symmetry factor S

Feynman Rules for Fermions & QED
Wick's theorem for fermions & gauge fields

Fermions: extra $-$ sign when interchanging two fermionic fields:

$$T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} = \begin{cases} \psi_\alpha(x)\bar{\psi}_\beta(y) & ; \quad (x^0 > y^0) \\ -\bar{\psi}_\beta(y)\psi_\alpha(x) & ; \quad (x^0 < y^0) \end{cases}$$

$$\wick{\c\psi_x \c{\bar{\psi}}_y} = S_F(x-y) = \begin{cases} \{\psi_x^+, \bar{\psi}_y^-\} & ; \ (x^0 > y^0) \\ -\{\bar{\psi}_y^+, \psi_x^-\} & ; \ (x^0 < y^0) \end{cases}$$

$$\wick{\c\psi_x \c\psi_y} = \wick{\c{\bar{\psi}}_x \c{\bar{\psi}}_y} = 0$$

$$\wick{\c{\bar{\psi}}_x \c\psi_y} \equiv \wick{\c{\bar{\psi}}_{x\alpha} \c\psi_{y\beta}} = -\wick{\c\psi_{y\beta} \c{\bar{\psi}}_{x\alpha}} = -S_F(x-y)_{\beta\alpha}$$

$$:\psi_\alpha^+\psi_\beta^-: = -\psi_\beta^-\psi_\alpha^+$$

Wick's theorem:

- a $-$ sign appears if there is an odd number of fermionic fields between the two contracted fields
- sign taken into account by the definitions

$$:\psi_{1\alpha}\psi_{2\beta}\bar{\psi}_{3\gamma}\bar{\psi}_{4\delta}: = -:\psi_{1\alpha}\psi_{2\beta}\bar{\psi}_{4\delta}\bar{\psi}_{3\gamma}: = -S_F(x_2 - x_4)_{\beta\delta} :\psi_{1\alpha}\bar{\psi}_{3\gamma}:$$

$\Rightarrow$ with these definitions Wick's theorem looks exactly the same for fermions as for bosons (16).

Gauge field

$\Rightarrow$ we only have to take care of the extra index of the fields:

$$A_x^\mu A_y^\nu = D_F^{\mu\nu}(x - y)$$

QED interaction Hamiltonian

$$\mathcal{H}_{int} = -\mathcal{L}_{int} = e\bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x) = e\bar{\psi}(x)\not{A}(x)\psi(x) \tag{27}$$

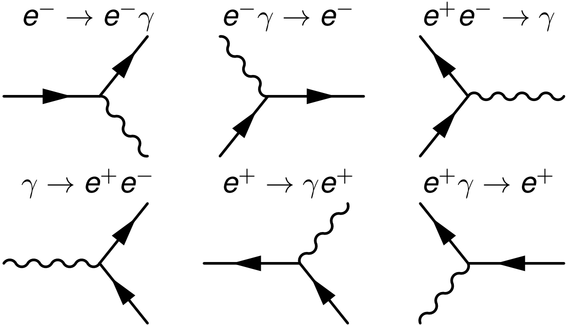
Dyson expansion

- First term: contains only the **1**, contributes only to the non-interacting part
- Second order $\mathcal{O}(e)$:

$$\int \mathrm{d}^4x \, T\{-ie\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x)\} =$$

$$\int \mathrm{d}^4x \; :-ie\bar{\psi}(x)\gamma_\mu A^\mu(x)\psi(x): -ieA^\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x) \quad (28)$$

The first term in (28) can contribute to several 3-particle processes:



…

None will survive momentum conservation

$\mathcal{T} \Rightarrow$ derive Feynman rules

…

Take for definitess: $e^-(p_A) \to e^-(p_1)\gamma(k)$

$\Rightarrow$ we need to specify particle type and polarization

$$\langle f|S|i\rangle = \langle \gamma_\lambda(k); e_r^-(p_1)|S|e_s^-(p_A)\rangle$$

$c_{(\lambda)\boldsymbol{k}}$ the ladder operators for the photon field.

$$\begin{aligned} |i\rangle &= |e_s^-(p_A)\rangle = \sqrt{2E_{p_A}} a_{\boldsymbol{p_A}}^{s\dagger}|0\rangle \quad ; \\ \langle f| &= \langle \gamma_\lambda(k); e_r^-(p_1)| = \sqrt{2E_{p_1}}\sqrt{2E_k}\langle 0|c_{(\lambda)\boldsymbol{k}} a_{\boldsymbol{p_1}}^r \end{aligned}$$

and the transition matrix:

$$\langle f|i\mathcal{T}|i\rangle = \int \mathrm{d}^4x \sqrt{2E_{p_1}}\sqrt{2E_k}\sqrt{2E_{p_A}}\langle 0|c_{(\lambda)\boldsymbol{k}} a_{\boldsymbol{p_1}}^r :-ieA^\mu\bar{\psi}\gamma_\mu\psi: a_{\boldsymbol{p_A}}^{s\dagger}|0\rangle$$

- $b_{\boldsymbol{p}}^w, b_{\boldsymbol{p}}^{w\dagger}$ operators acting on the right or the left vacuum $\Rightarrow 0$
- $c_{(\sigma)\boldsymbol{k}}$ on the right vaccuum $\Rightarrow 0$

$$-ie\sqrt{2E_{p_1}}\sqrt{2E_k}\sqrt{2E_{p_A}}\langle 0|c_{(\lambda)\boldsymbol{k}} a_{\boldsymbol{p_1}}^r A^{\mu-}\bar{\psi}^-\gamma_\mu\psi^+ a_{\boldsymbol{p_A}}^{s\dagger}|0\rangle \qquad (29)$$

$$
\begin{aligned}
\psi^+(x) a^{s\dagger}_{\boldsymbol{p_A}}|0\rangle &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}} a^w_{\boldsymbol{p}} u^w(\boldsymbol{p}) e^{-ipx} a^{s\dagger}_{\boldsymbol{p_A}}|0\rangle \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}} \left(-a^{s\dagger}_{\boldsymbol{p_A}} a^w_{\boldsymbol{p}} + (2\pi)^3 \delta^{sw} \delta^3(\boldsymbol{p}-\boldsymbol{p_A})\right) u^w(\boldsymbol{p}) e^{-ipx}|0\rangle \\
&= \frac{1}{\sqrt{2E_{p_A}}} u^s(\boldsymbol{p_A}) e^{-ip_A x}
\end{aligned}
$$

$\Rightarrow$ Feynman rules for external fermions

$$
\overbracket{\psi(x)|e^-_r}(p)\rangle = \psi^+(x)|e^-_r(p)\rangle = u^r(\boldsymbol{p}) e^{-ipx}
$$

$\Rightarrow$ Different initial-final states have different combinations of $a^r_{\boldsymbol{p}}$, $b^r_{\boldsymbol{p}}$,

$|e^+_r(p)\rangle \sim b^{r\dagger}_{\boldsymbol{p}}|0\rangle$ needs a $\bar{\psi}^+$ $\Rightarrow$ selects a spinor $\bar{v}^r(\boldsymbol{p})$

and they select different kind spinors.

Feynman rules for external spinors

$\contraction{}{\psi}{(x)|}{e}\psi(x)|e_r^-(p)\rangle = u^r(\boldsymbol{p})e^{-ipx}$ Electron in a initial state

$\contraction{\langle}{e}{_r^-(p)|}{\bar\psi}\langle e_r^-(p)|\bar\psi(x) = \bar u^r(\boldsymbol{p})e^{ipx}$ Electron in a final state

$\contraction{}{\bar\psi}{(x)|}{e}\bar\psi(x)|e_r^+(p)\rangle = \bar v^r(\boldsymbol{p})e^{-ipx}$ Positron in a initial state

$\contraction{\langle}{e}{_r^+(p)|}{\psi}\langle e_r^+(p)|\psi(x) = v^r(\boldsymbol{p})e^{ipx}$ Positron in a final state (30)

- arrow indicates the direction of the spinor (or the charge) and **not** of the momentum
- momentum is always incoming in initial states and outgoing in final states.

Photon fields: same as for the real Klein-Gordon field, but now each operator is multiplied by the corresponding polarization vector $\epsilon^{\mu}_{(\lambda)}(\boldsymbol{p})$:

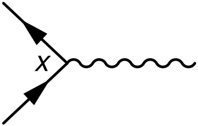
 $A^{\mu}(x)|\gamma_{\lambda}(p)\rangle = \epsilon^{\mu}_{(\lambda)}(\boldsymbol{p})e^{-ipx}$ Photon in a initial state

 $\langle\gamma_{\lambda}(p)|A^{\mu}(x) = \epsilon^{\mu *}_{(\lambda)}(\boldsymbol{p})e^{ipx}$ Photon in a final state(31)

$\mathcal{T}$ for process (29)

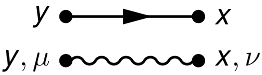
$$\begin{aligned}
\langle f|i\mathcal{T}|i\rangle &= -ie\int \mathrm{d}^4x\,\epsilon^{\mu *}_{(\lambda)}(\boldsymbol{k})\bar{u}^r(\boldsymbol{p_1})\gamma_{\mu}u^s(\boldsymbol{p_A})e^{-i(p_A-p_1-k)x} \\
&= -ie\epsilon^{\mu *}_{(\lambda)}(\boldsymbol{k})\bar{u}^r(\boldsymbol{p_1})\gamma_{\mu}u^s(\boldsymbol{p_A})(2\pi)^4\delta^4(p_A-p_1-k) \\
&= i\mathcal{M}(2\pi)^4\delta^4(p_A-p_1-k) \\
i\mathcal{M} &= -ie\epsilon^{\mu *}_{(\lambda)}(\boldsymbol{k})\bar{u}^r(\boldsymbol{p_1})\gamma_{\mu}u^s(\boldsymbol{p_A})
\end{aligned}$$

Feynman rule in position space for the QED vertex



$$= -ie \int \mathrm{d}^4 x \, \gamma^\mu \tag{32}$$

and complemented with the propagators:



$$\begin{aligned} &= S_F(x-y) \\ &= D_F^{\mu\nu}(x-y) \end{aligned} \tag{33}$$

direction of propagation in fermions is significant,

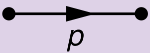
$$S_F(x-y) = -S_F(y-x)$$

.

- substituting the several exponentials
- performing the integrations in the vertices (32)
  - $\Rightarrow$ Feynman rules for QED in momentum space

Feynman rules for QED in momentum space

1. Construct all **fully connected, amputated** diagrams with p_A, p_B incoming, $p_1, \ldots p_n$ outgoing
2. Propagators:

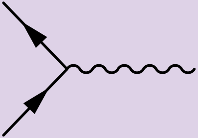


$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} \quad ;$$

$$= \frac{-ig^{\mu\nu}}{p^2 + i\varepsilon} \tag{34}$$

The **direction of propagation of fermions is significant**

3. Vertex:



$$= -ie\,\gamma^{\mu} \tag{35}$$

4. External legs

 $= u^r(\boldsymbol{p})$ Fermion initial

 $= \bar{u}^r(\boldsymbol{p})$ Fermion final

 $= \bar{v}^r(\boldsymbol{p})$ Antifermion initial

 $= v^r(\boldsymbol{p})$ Antifermion final

 $= \epsilon^{\mu}_{(\lambda)}(\boldsymbol{p})$ Photon initial

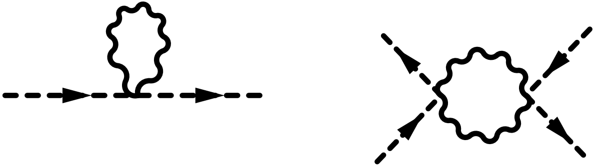
 $= \epsilon^{\mu *}_{(\lambda)}(\boldsymbol{p})$ Photon final

5. Impose momentum conservation at each vertex
6. Integrate over all indetermined momentum: $\displaystyle\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$
7. **Figure out the overall sign of the diagram**

- QED has no symmetry factors, since the three fields in H_{int} are not equivalent.
- A closed photon loop would entail a symmetry factor, however, since each vertex has only one photon, it is not possible to construct a closed photon loop.
- Scalar QED has an additional interaction term with two photons:

$$e^2 A^\mu A_\mu \phi^\dagger \phi \in \mathcal{L}_{int} = -\mathcal{H}_{int}$$

from which we can construct closed photon loops, for example:



which contain symmetry factors.

- Sign: each time we have to commute two fermionic fields, a $-$ sign appears in the expression
- overall sign of the amplitude is irrelevant
- relative sign between different contributions is relevant.

Example: Møller scattering:

$$e^- e^- \to e^- e^-$$

$$|i\rangle = |e_r^-(p_A) e_s^-(p_B)\rangle \quad ; \quad |f\rangle = |e_w^-(k_1) e_t^-(k_2)\rangle$$

- 4 electrons in the external states $\Rightarrow$ 4 fermionic fields $\Rightarrow$ 2^{nd} order in Dyson expansion

$$\langle f| T \left\{ \frac{(-ie)^2}{2!} \int \mathrm{d}^4x \, \mathrm{d}^4y \, \bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y \not{A}_y \psi_y \right\} |i\rangle$$

Only the following contractions contribute

- No photons in external states $\Rightarrow$ two A **must** be contracted
- 4 fermions in external states $\Rightarrow$ 4 ψ fields **can not** be contracted

$$\frac{(-ie)^2}{2!}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\langle f|:\bar{\psi}_x \slashed{A}_x \psi_x \bar{\psi}_y \slashed{A}_y \psi_y :|i\rangle$$

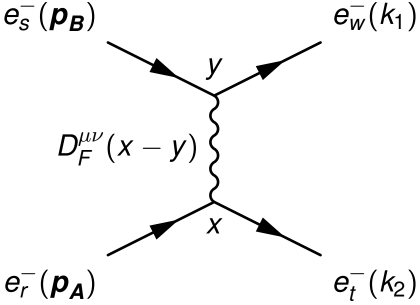
Now we have to contract the fermion fields with the external states:

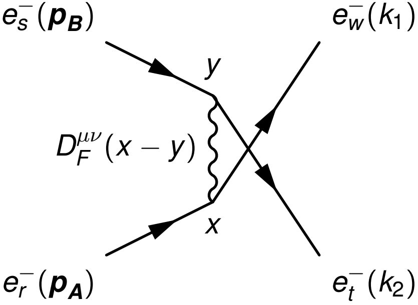
- The $\bar{\psi}$ can only be contracted with $\langle f|$, and the ψ can only be contracted with $|i\rangle$
- In addition, for any contraction of external fields with the fields in x, there is an equivalent contraction with the fields in y, so we count only different kind of contractions and multiply by 2
- two non-equivalent contractions

$$\frac{(-ie)^2}{2!} \times 2 \times \int \mathrm{d}^4x\, \mathrm{d}^4y\; D_F^{\mu\nu}(x-y)\Big($$

$$\langle e_t^-(k_2) e_w^-(k_1)| : \bar{\psi}_x \gamma_\mu \psi_x \bar{\psi}_y \gamma_\nu \psi_y : |e_r^-(p_A) e_s^-(p_B)\rangle$$

$$+ \quad \langle e_t^-(k_2) e_w^-(k_1)| : \bar{\psi}_x \gamma_\mu \psi_x \bar{\psi}_y \gamma_\nu \psi_y : |e_r^-(p_A) e_s^-(p_B)\rangle\Big)$$





How many anticommutations? **1st**: 1 **2nd**: 1 2

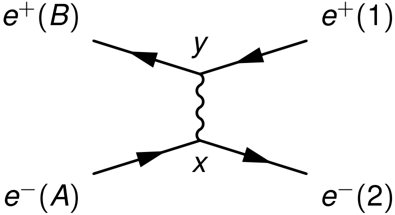
Now let's look at the relative signs, looking at how many times we need to anticommute different fermionic fields to bring them in the order given by the contractions:

- 1st term:
  - ψ_x has to cross over $\bar{\psi}_y$: (-1)
- 2nd term:
  - ψ_x has to cross over $\bar{\psi}_y$: (-1)
  - $\bar{\psi}_y$ has additionaly to cross over $\bar{\psi}_x$: (-1)

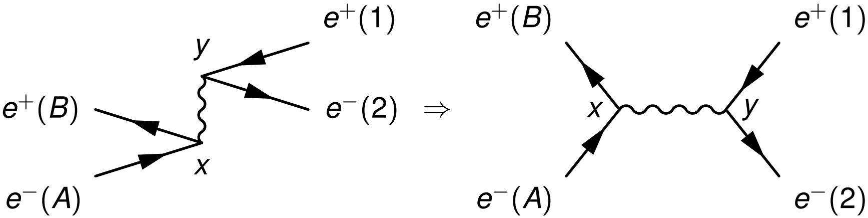
So the second term has an additional $-$ sign. The signs rule is:

7. **If diagram A is obtained from diagram B by the exchange of a fermionic line, they have a relative $-$ sign.**

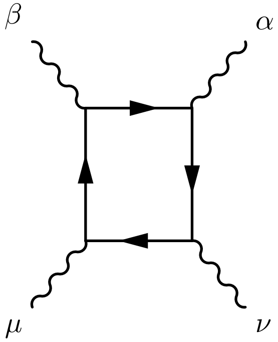
This rule may involve the exchange of initial-final states fermions-antifermions. For example in Bhabha scattering $e^+e^- \to e^+e^-$:



Exchange $B \leftrightarrow 2$:



If there are fermionic loops:



$$
\begin{aligned}
&= \bar\psi_1\gamma^\mu\psi_1\bar\psi_2\gamma^\nu\psi_2\bar\psi_3\gamma^\alpha\psi_3\bar\psi_4\gamma^\beta\psi_4 \\
&= -\mathrm{Tr}(\gamma^\mu\psi_1\bar\psi_2\gamma^\nu\psi_2\bar\psi_3\gamma^\alpha\psi_3\bar\psi_4\gamma^\beta\psi_4\bar\psi_1) \\
&= -\mathrm{Tr}(\gamma^\mu\tilde{S}_F\gamma^\nu\tilde{S}_F\gamma^\alpha\tilde{S}_F\gamma^\beta\tilde{S}_F)
\end{aligned}
$$

8. **Add a $-$ sign for closed fermionic loops.**