4. The quark model and effective theories of hadrons

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4.1 The non-relativistic quark model

So far we have been playing with approximate internal symmetries, but have not specified which is the dynamics of the interaction between quarks. The simplest proposal for this dynamics is the non-relativistic quark model:

- Quarks move slowly in the hadrons $\Longrightarrow$ the main contribution to the hadron mass is the addition of the masses of its quarks
- The Interaction Hamiltonian must depend on the spin in order to explain the mass difference between 0^- and 1^- for mesons and $1/2^+$ and $3/2^+$ for baryons:

$$H_1 = a \sum_{i \neq j} \frac{\vec{S}_i \vec{S}_j}{m_i m_j}$$

- a depends on:
  - quark-quark or quark-antiquark interaction
  - the radial wave function
- For mesons we have

$$M_{q_1 \bar{q}_2} = m_1 + m_2 + \frac{a}{m_1 m_2} \left(\frac{S(S+1)}{2} - \frac{3}{2} \right)$$

$S =$ meson spin

- For baryons we have

$$M_{q_1 q_2 q_3} = m_1 + m_2 + m_3 + a' \sum_{i \neq j}^{3} \frac{\langle \vec{S}_i \vec{S}_j \rangle}{m_i m_j}$$

  - For light baryons (i.e. containing u, d and s only) in the exact $SU(3)$ limit ($m \sim m_u \sim m_d \sim m_s$)

$$M_{q_1 q_2 q_3} = 3m + \frac{a'}{2m^2}\left(S(S+1) - \frac{9}{4}\right)$$

    $S =$ baryon spin
  - For baryons containing u and d quarks only, the formula above already holds in the isospin limit ($m \sim m_u \sim m_d$)
  - We then have

$$J^P = \frac{1}{2}^+ \qquad M = 3m - \frac{3a'}{4m^2}$$
$$J^P = \frac{3}{2}^+ \qquad M' = 3m + \frac{3a'}{4m^2}$$

- Inputing the values of the nucleon and Delta resonance masses we get

$$m \simeq 362\,\text{MeV} \quad , \quad 3a'/2m^2 = M' - M \simeq 292\,\text{MeV}$$

  Note that the value of m and that of $3a'/2m^2$ are similar $\Longrightarrow$ the non-relativistic assumption is not well justified
- For light mesons in the $SU(3)$ (or isospin, if none of the quarks is the strange one) limit we have

$$J^P = 0^- \qquad M = 2m - \frac{3a}{4m^2}$$
$$J^P = 1^- \qquad M' = 2m + \frac{a}{4m^2}$$

- Inputing the values of the pion and rho masses we get

$$m \simeq 302\,\text{MeV} \quad , \quad a/m^2 = M' - M \simeq 630\,\text{MeV}$$

  Note that the value of m is about a half a/m^2 $\Longrightarrow$ the non-relativistic assumption does not hold
- The non-relativistic quark model is inconsistent

4.2 The linear sigma model

- Why is the pion so light ($m_\pi \sim 140$ MeV $\ll m_\rho \sim 770$ MeV)?
- Light pion $\Longrightarrow$ m_u , m_d must be small
- If m_u , m_d are small, how are the rho and the nucleon masses generated?
- Let us assume that $m_u \simeq m_d \simeq 0$
  - Right and left Dirac fields decouple
  - The isospin $SU(2)$ symmetry is enlarged to $SU_L(2) \otimes SU_R(2)$, which is called **chiral symmetry**
  - Since approximate $SU_L(2) \otimes SU_R(2)$ multiplets (parity doublets) are not observed in nature, chiral symmetry must be spontaneously broken (the vacuum is not invariant)
- Let us implement this idea in a meson model

$$\mathcal{L} = \frac{1}{4}\mathrm{tr}\left(\partial_\mu M^\dagger \partial^\mu M\right) - \frac{m^2}{4}\mathrm{tr}\left(M^\dagger M\right) - \lambda\left(\mathrm{tr}\left(M^\dagger M\right)\right)^2$$

$$M(x) = \sigma(x)\mathbb{I}_2 + i\vec{\tau}\vec{\varphi}(x) \quad , \quad \sigma(x)\,,\vec{\varphi}(x) \in \mathbb{R} \quad , \quad m^2\,,\lambda \in \mathbb{R} \quad , \quad \{\vec{\tau}\} = \{\text{Pauli matrices}\}$$

  - $\sigma(x)$ scalar, $\vec{\varphi}(x)$ pseudoscalar $\sim$ pion $\Longrightarrow$ $M(x) \to M^\dagger(\tilde{x})$ under parity

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}\mathrm{tr}\left(\partial_\mu M^\dagger \partial^\mu M\right) - \frac{m^2}{4}\mathrm{tr}\left(M^\dagger M\right) - \lambda\left(\mathrm{tr}\left(M^\dagger M\right)\right)^2 \\ &= \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec{\varphi}\partial^\mu\vec{\varphi} - \frac{m^2}{2}\left(\sigma^2 + \vec{\varphi}^2\right) - 4\lambda\left(\sigma^2 + \vec{\varphi}^2\right)^2\end{aligned}$$

- $\mathcal{L}$ is invariant under $M \to g_L M g_R^\dagger$, $g_L \in SU_L(2)$, $g_R \in SU_R(2)$
- Isospin $SU(2) \subset SU_L(2) \otimes SU_R(2)$ corresponds to $g_L = g_R$
- $\lambda > 0$ for the Hamiltonian to be bounded from below
- If $m^2 > 0$ $\Longrightarrow$ a scalar particle with the same mass as the pseudoscalars must exist, which does not happens in nature
- If $m^2 < 0$, the minimum of the Hamiltonian is not attained at $M = 0$, but at $M \neq 0$
- For constant M, the Hamiltonian reduces to a potential

$$V(\rho) = \frac{m^2}{2}\rho + 4\lambda\rho^2 \quad , \quad \rho = \frac{1}{2}\mathrm{tr}\left(M^\dagger M\right)$$

  - The minimum is attained at

$$0 = \frac{m^2}{2} + 8\lambda\rho \quad , \quad \rho = -\frac{m^2}{16\lambda} \equiv \rho_0 > 0 \quad , \quad V(\rho_0) = -\frac{m^4}{64\lambda} < 0$$

$$V(\rho)=\frac{m^2}{2}\rho+4\lambda\rho^2 \quad , \quad \rho=\frac{1}{2}\mathrm{tr}\left(M^\dagger M\right)$$

- Note that $V(0)=0>V(\rho_0) \Longrightarrow$ we should better expand M about M_0, the ground state configuration $\rho_0=\frac{1}{2}\mathrm{tr}\left(M_0^\dagger M_0\right)$, and quantize the fluctuations
- M_0 must be invariant under isospin, $gM_0g^\dagger=M_0 \Longrightarrow M_0=\sqrt{\rho_0}\,\mathbb{I}_2$
- Note that for configurations $M(x)=M_0U(x)$, $U(x)\in SU(2) \Longrightarrow \rho(x)=\rho_0$ $\Longrightarrow V(\rho(x))=V(\rho_0)$
  - For this configuration

$$\mathcal{L}=\frac{\rho_0}{4}\mathrm{tr}\left(\partial_\mu U^\dagger\partial^\mu U\right)-V(\rho_0)$$

  - $U(x)$ is conventionally written as

$$U(x)=e^{\frac{i\vec{\pi}\vec{\tau}}{f_\pi}} \quad , \quad \vec{\pi}\vec{\tau}=\begin{pmatrix}\pi^0 & \sqrt{2}\pi^+\\ \sqrt{2}\pi^- & -\pi^0\end{pmatrix}$$

$\pi^0, \pi^\pm=(\pi^1\mp i\pi^2)/\sqrt{2}$ are identified with the pion fields

$$\mathcal{L}=\frac{\rho_0}{4}\mathrm{tr}\left(\partial_\mu U^\dagger\partial^\mu U\right)-V(\rho_0)$$

- $V(\rho_0)$ is now an irrelevant constant and will be dropped
- Upon expanding $\mathcal{L}$ up to fourth order in the pion fields, we obtain

$$\mathcal{L}=\frac{\rho_0}{f_\pi^2}\left(\frac{1}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi}+\frac{1}{6f_\pi^2}\left(\left(\partial_\mu\vec{\pi}\vec{\pi}\right)\left(\partial^\mu\vec{\pi}\vec{\pi}\right)-\left(\partial^\mu\vec{\pi}\partial_\mu\vec{\pi}\right)\left(\vec{\pi}\vec{\pi}\right)\right)+\mathcal{O}\left(\frac{1}{f_\pi^4}\right)\right)$$

- In order to have the usual normalization of the kinetic term $f_\pi^2=\rho_0$
- Pions are massless in this limit.
  - This is a particular example of **Goldstone's theorem**: If a relativistic Lagrangian is invariant under a Lie group G, but the ground state is only invariant under a subgroup $H\subset G \Longrightarrow$ $\dim G$-$\dim H$ massless particles arise in the spectrum $\equiv$ **Goldstone bosons**
  - In our case $G=SU_L(2)\otimes SU_R(2)$, $\dim G=3+3=6$, $H=SU(2)$, $\dim H=3 \Longrightarrow$ 3 Goldstone bosons exist, the three pions
  - In nature pions are not massless $\Longrightarrow m_u\neq 0$ or $m_d\neq 0$ or both
  - The interactions between pions are fixed by the symmetries in terms of f_π
  - At small four momenta the interactions are small

- So far we have restricted ourself to configurations such that $V(\rho(x)) = V(\rho_0)$
- For general configurations, we may choose $M(x) = (M_0 + S(x))U(x)$, $S(x) \in \mathbb{R}$, then

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) + \frac{1}{2}\partial_\mu S \partial^\mu S + \left(\frac{f_\pi S}{2} + \frac{S^2}{4}\right)\mathrm{tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) - V((f_\pi + S)^2)$$

$$V((f_\pi + S)^2) = V(f_\pi^2) - m^2 S^2 + 16\lambda f_\pi S^3 + 4\lambda S^4 \quad \Longrightarrow \quad m_S^2 = -2m^2 > 0$$

- What particle does $S(x)$ represent?
  - Do you remember the misterious $f_0(500)$ or σ with $J^{PC} = 0^{++}$ that did not fit in the quark model?
  - Now we have a good candidate for it
- We need to introduce small quark masses
  - $m_u \simeq m_d \equiv m_q$ break $SU_L(2) \otimes SU_R(2)$ but respect isospin
  - We may just introduce a term linear in $m_q \mathbb{I}_2$

$$\delta\mathcal{L} = -c\,\mathrm{tr}\left(M^\dagger m_q \mathbb{I}_2 + m_q \mathbb{I}_2 M\right) = -c\, m_q (f_\pi + S)\mathrm{tr}\left(U^\dagger + U\right) \quad , \quad c \in \mathbb{R}$$

  - Upon expanding up to second order in the pion fields one can identify the pion mass

$$m_\pi^2 = -\frac{4cm_q}{f_\pi} \qquad (\Longrightarrow \quad c < 0)$$

  - Note that $m_\pi \sim \sqrt{m_q}$
  - Rewritting c in terms of m_π

$$\delta\mathcal{L} = \frac{m_\pi^2 f_\pi (f_\pi + S)}{4}\mathrm{tr}\left(U^\dagger + U\right)$$

- Recall that the mass of S, $m_S^2 = -2m^2$ does not depend on the quark mass m_q
- Is this a general feature for the remaining hadrons?
- Let us introduce nucleons in the Lagrangian

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix} \quad , \quad \psi = \psi_L + \psi_R \quad , \quad \psi_R \to g_R \psi_R \quad , \quad \psi_L \to g_L \psi_L$$

  - A mass term does not respect the chiral symmetry
  - But an interaction term with the M field does ($M \to g_L M g_R^\dagger$)

$$\mathcal{L} = \bar\psi_L i\partial\!\!\!/ \psi_L + \bar\psi_R i\partial\!\!\!/ \psi_R - d\,\bar\psi_L M \psi_R - d\,\bar\psi_R M^\dagger \psi_L$$

- The vacuum configuration $M = f_\pi\, \mathbb{I}_2$ induces a mass term for the nucleons

$$m_N = d f_\pi$$

- This mass term is independent of the quark masses m_q
- This is very interesting: we can achieve large hadron masses in spite of having small quark masses if $df_\pi \gg m_q$

- We can understand why the nucleon masses are much larger than the pion masses: $m_N = df_\pi$, $m_\pi = \sqrt{-\frac{4cm_q}{f_\pi}}$, $m_\pi \to 0$ when $m_q \to 0$ but m_N remains finite.
- Let us consider the general configuration $M(x) = (f_\pi + S(x))U(x)$

$$\bar{\psi}_L M \psi_R = \bar{\psi}_L (f_\pi + S) U \psi_R = \bar{\psi}_L (f_\pi + S) u^2 \psi_R = \bar{N}_L (f_\pi + S) N_R$$

$$U \equiv u^2 \quad , \quad N_R \equiv u\psi_R \quad , \quad N_L \equiv u^\dagger \psi_L$$

$$\begin{aligned}\mathcal{L} &= \left(\bar{N}_L u^\dagger\right) i\partial\!\!\!/ \left(uN_L\right) + \left(\bar{N}_R u\right) i\partial\!\!\!/ \left(u^\dagger N_R\right) - d\,\bar{N}_L(f_\pi + S)N_R - d\,\bar{N}_R(f_\pi + S)N_L \\ &= \bar{N}\left(i\left(\partial\!\!\!/ + v\!\!\!/\right) + i a\!\!\!/\gamma^5 - m_N\left(1 + \frac{S}{f_\pi}\right)\right) N\end{aligned}$$

$$N = N_L + N_R \quad , \quad v_\mu \equiv \frac{1}{2}\left(u\partial_\mu u^\dagger + u^\dagger \partial_\mu u\right) \quad , \quad a_\mu \equiv \frac{1}{2}\left(u\partial_\mu u^\dagger - u^\dagger \partial_\mu u\right)$$

- We have now the free Lagrangian well separated from the interactions
- The transformation properties of the new fields under $SU_L(2) \otimes SU_R(2)$ are $(U \to g_L U g_R^\dagger)$

$$u \to g_L u h^\dagger(u) = h(u) u g_R^\dagger \quad , \quad u^\dagger \to g_R u^\dagger h^\dagger(u) = h(u) u^\dagger g_L^\dagger \quad , \quad \Longrightarrow$$

$$N \to h(u)N \ , \ \bar{N} \to \bar{N}h^\dagger(u) \ , \ v_\mu \to h(u)v_\mu h^\dagger(u) + h(u)\partial_\mu h^\dagger(u) \ , \ a_\mu \to h(u)a_\mu h^\dagger(u)$$

- Upon expanding u in terms of the pion fields

$$\begin{aligned} v_\mu &\simeq \frac{1}{8f_\pi^2}\left(\vec{\pi}\vec{\tau}\partial_\mu\vec{\pi}\vec{\tau} - \left(\partial_\mu\vec{\pi}\vec{\tau}\right)\vec{\pi}\vec{\tau}\right) + \mathcal{O}\left(\frac{\partial_\mu \vec{\pi}^4}{f_\pi^4}\right) \\ a_\mu &\simeq \frac{i}{2f_\pi}\partial_\mu\vec{\pi}\vec{\tau} + \mathcal{O}\left(\frac{\partial_\mu \vec{\pi}^3}{f_\pi^3}\right)\end{aligned}$$

  - The interactions of pions with nucleons are given in terms of f_π only
  - v_0 provides the dominant contribution to elastic pion-nucleon scattering at very low energy ($\vec{p} \ll m_\pi$)
  - Note that the pion-nucleon-nucleon vertex is not the one originally proposed by Yukawa ($\sim \bar{N}\vec{\pi}\vec{\tau}N$)
- We should also add to the model an explicit chiral symmetry breaking term due to the quark masses

$$\delta\mathcal{L} = -d'\,\bar{\psi}_L m_q \mathbb{I}_2 \psi_R - d'\,\bar{\psi}_R m_q \mathbb{I}_2 \psi_L = -d' m_q \bar{N}\left(U + U^\dagger\right) N$$

Upon expanding the pion fields one gets a correction to the nucleon mass ($d' \in \mathbb{R}$)

$$m_N = df_\pi + 2d' m_q$$

- Which of these features follow from the spontaneous symmetry breaking pattern $SU_L(2) \otimes SU_R(2) \to SU(2)$ and are independent of the model?

4.3 The non-linear sigma model

- From the linear sigma model we learned that pions with $m_\pi \sim \sqrt{m_q}$ and hadrons with masses $\sim m_N$ independent of m_q can be generated when $m_q \to 0$ through spontaneous chiral symmetry breaking
- Then at momentum $p \ll m_N$ pions can be produced but not the rest of hadrons $\Longrightarrow$ an effective theory can be built with only pions in which p/m_N and m_π/m_N are small parameters (this is called Chiral perturbation theory)
- The building blocks are the field $U(x)$, $U^\dagger U = 1$, containing the pions, and a phantom field $\mathcal{M}(x)$ that is used to implement the explicit symmetry breaking by the quark masses,

$$U(x) \to g_L U(x) g_R^\dagger \quad , \quad \mathcal{M}(x) \to g_L \mathcal{M}(x) g_R^\dagger$$

- $\mathcal{M}(x)$ is eventually set to $\mathcal{M}(x) = m_q \mathbb{I}_2$
- ∂_μ is counted as $p \sim m_\pi$ and $\mathcal{M}(x) \sim m_\pi^2 \sim p^2$
- Then one writes down all possible terms in the Lagrangian at a given order in p, invariant under Lorentz, chiral symmetry, parity and charge conjugation

- Recall that
  - $P: \quad \vec{\pi}(x) \to -\vec{\pi}(\tilde{x}) \quad \Longrightarrow \quad U(x) \to U^\dagger(\tilde{x})$
  - $C: \quad \vec{\tau}\vec{\pi}(x) \to (\vec{\tau}\vec{\pi}(x))^T \quad \Longrightarrow \quad U(x) \to U^T(x)$
- The leading order is $\mathcal{O}(p^2)$, we can only write

$$\mathrm{tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) \quad , \quad \mathrm{tr}\left(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U\right)$$

- These two terms already appeared in the linear sigma model. They are universal $\Longrightarrow$ the low energy physics of pions depends on only two parameters f_π and m_π.
  - $f_\pi \sim 92$ MeV is called the pion decay constant, and it is measured in the $\pi^+ \to \mu^+ \nu_\mu$ decay, as we shall see in the weak interaction section
  - The elastic pion scattering is also given in terms of f_π
  - Multipion production out of pion-pion collisions is also given in terms of f_π
- The next-to-leading order is $\mathcal{O}(p^2)$, we have now terms like

$$\mathrm{tr}\left(\partial_\mu U^\dagger \partial^\nu U\right) \mathrm{tr}\left(\partial_\nu U^\dagger \partial^\mu U\right) \quad , \quad \left(\mathrm{tr}\left(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U\right)\right)^2 \quad , \quad \dots$$

- Each of these terms brings in a new coupling constant

Pion-nucleon Lagrangian

- Nucleon fields cal also be incorporated to the non-linear sigma model, but one must keep in mind that $p \ll m_N$ and hence nucleon pair production must not be considered
- The building blocks are the nucleon fields $N(x)$, $u(x)$ ($v_\mu(x)$, $a_\mu(x)$) and $\mathcal{M}(x)$
- Recall that
  - $P: \quad u(x) \to u^\dagger(\tilde{x}) \quad \Longrightarrow \quad v_\mu(x) \to v^\mu(\tilde{x}) \quad , \quad a_\mu(x) \to -a^\mu(\tilde{x})$
  - $C: \quad u(x) \to u^T(x) \quad \Longrightarrow \quad v_\mu(x) \to -v_\mu^T(x) \quad , \quad a_\mu(x) \to a_\mu^T(x)$
- The leading order Lagrangian is now $\mathcal{O}(p)$

$$\bar{N} i \left(\not{\partial} + \not{v} \right) N \quad , \quad \bar{N} i \not{a} \gamma^5 N$$

  - A mass term of $\mathcal{O}(1) \sim \bar{N}N$ should also be added to the Lagrangian, but it does not contain interactions
  - Note that it does not depend on the quarks masses $\mathcal{M}(x)$
  - The coupling of $\not{v}$ is fixed to 1 by symmetry $\Longrightarrow$ this term already appeared in the linear sigma model, it is universal $\Longrightarrow$ pion-nucleon scattering at very low energies ($p \ll m_\pi$) is given in terms of f_π
  - The coupling of the $\not{a}$ term is called g_A and it is not fixed by the symmetry.
    - ★ In the linear sigma model $g_A = 1$
    - ★ In nature $g_A \simeq 1.25$

- At low energies ($p \ll m_N$), the dynamics of pion-nucleon scattering is given in terms of f_π and g_A
- At next-to-leading order, $\mathcal{O}(p^2)$, terms like the following can be added

$$\bar{N} a_\mu a^\mu N \quad , \quad \bar{N} \left(u \mathcal{M}^\dagger u + u^\dagger \mathcal{M} u^\dagger \right) N \quad , \quad \dots$$

- Each of these terms has an arbitrary constant in front

Final remarks

- The terms containing the S field in the linear sigma model are not universal, and their features are particular to the model. However, it is worth mentioning:
  - We will need a similar pattern of spontaneous symmetry breaking in the context of the electroweak theory
  - The S field here plays a role analogous to Higgs field there
  - In particular the coupling of the Higgs to fermions is proportional to the masses of the fermions, like the couling of the S field to nucleons

- The non-linear sigma model can be easily generalized to flavor $SU(3)$, namely assuming $m_u \simeq m_d \simeq m_s \simeq 0$, we have the chiral symmetry $SU_L(3) \otimes SU_R(3)$ which is spontaneously broken to flavor $SU(3)$
  - $U(x)$ must be replaced by

$$U(x) = e^{\frac{\lambda^a M^a}{f_\pi}} \quad , \quad \lambda^a M^a = M = \sqrt{2} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}$$

  - The kaons and the η are approximate Goldstone bosons like the pion
  - $\mathcal{M}(x)$, in the isospin limit, must be set to

$$\mathcal{M}(x) = \begin{pmatrix} m_q & 0 & 0 \\ 0 & m_q & 0 \\ 0 & 0 & m_s \end{pmatrix}$$