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**Sheet 1: Cosmology:  $\Lambda$ CDM and beyond**  
**Aspects of Dark Matter and Dark Energy**

**Exercise 1.** We may ask ourselves to which extent the present Universe is transparent to light. There are different processes that may stop, absorb or scatter the propagation of light. They depend on the scattering cross-section  $\sigma$  and the number density  $n$  of scattering particles. Estimate the photon mean free path ( $\lambda_\gamma = 1/n\sigma$ ) in each one of the following processes:

i) Thomson scattering of photons on free electrons in the intergalactic medium, with a number density  $n_e = n_p \sim \Omega_B \rho_c / m_p$ ;

ii) Stellar absorption, with a number density  $n_{\text{est}} \sim \Omega_{\text{visible}} \rho_c / M_\odot$  and cross-section  $\sigma_\odot \sim \pi R_\odot^2$ . Check yourself the value of  $\Omega_{\text{visible}}$ <sup>1</sup>;

iii) Absorption by dust in galaxies. Using the information I gave on the blackboard, check in detail that there is roughly an equivalent of  $\sim 0.4$  galaxies like ours (Milky Way) each  $Mpc$ .<sup>2</sup> Use that the average galactic radius is  $R_{\text{gal}} \sim 10 Kpc$ . and that the absorption of light after crossing the galaxy is on average 10% of its geometrical cross-section.

iv) Compare the mean free path with the Hubble radius ( $d_H = 1/H_0 \simeq 3000 h_0^{-1} Mpc$ ) and conclude if the Universe is or is not transparent to light. Which is the dominant effect among the previous three? What is the optical depth of the universe?

v) Translated into time units, the expression  $t_H = 1/H_0$  defines the so-called Hubble time. How many million years correspond to this period? How many photons out of 1000 will undergo an interaction (according to the previous processes) in the next Hubble time?

**Exercise 2.** Let us take the system of equations formed by the Friedmann-Lemaître equation with non-vanishing cosmological constant ( $\Lambda = 8\pi G_N \rho_\Lambda$ ),

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3} (\rho_m + \rho_\Lambda) - \frac{k}{a^2},$$

and the dynamical evolution equation for the scale factor,

$$\ddot{a} = -\frac{4\pi G_N}{3} (\rho_m + 3p_m - 2\rho_\Lambda) a.$$

Here we assume that only one component of matter is present, e.g. non-relativistic, but we could equally understand that both (relativistic and non-relativistic) are around, and then  $\rho_m \rightarrow \rho_m + \rho_r$  and  $p_m \rightarrow p_m + p_r$  (provided of course we are far away from the transition epoch between the two).

<sup>1</sup>Do quote (always) your bibliographical sources every time you need to input data.

<sup>2</sup>Our Galaxy has the equivalent of  $\sim 10^{11}$  stars as our sun.

i) Consider the possibility that the  $\Lambda$ -term is time-dependent:  $\Lambda = \Lambda(t)$ . Show that the equation

$$\dot{\rho}_\Lambda + \dot{\rho}_m + 3H(\rho_m + p_m) = 0$$

is a first integral of the previous system, irrespective of the value of the spatial curvature  $k$ ; that is, a differential equation of smaller order that can replace one of these two equations. For  $\dot{\rho}_\Lambda \neq 0$  there is an exchange of energy between matter and vacuum. Show in detail that any of the previous three equations can be derived from the other two.

ii) Check that  $H^2 + \dot{H} = \ddot{a}/a$  and arrive at the following expression for  $\dot{H}$  in terms of the pressure, matter density and spatial curvature:

$$\dot{H} = -4\pi G(\rho_m + p_m) + \frac{k}{a^2}.$$

What is the physical meaning of the minus sign? Why is this result independent of  $\rho_\Lambda$ ?

iii) Show that the equation for  $\ddot{a}$  above can be replaced by the following expression (if desired):

$$2\dot{H} + 3H^2 = -8\pi G(p_m - \rho_\Lambda) - \frac{k}{a^2}.$$

Assume we are in the matter dominated epoch (MDE), well before the transition from deceleration to acceleration, and suppose that the spatial curvature is  $k = 0$ . Prove that  $H(t) = 2/(3t)$  in that epoch and then check if this result is compatible with the previous expression in the MDE. Repeat the consistency check for the radiation dominated epoch (RDE). What is  $H(t)$  in this case?

**Exercise 3.** Consider the matter density equation indicate in the point i) of the previous exercise.

i) Solve that equation for the case when  $\Lambda = \text{const}$ . Obviously if  $\Lambda$  would be variable, you would need some input on the dynamics of the vacuum energy density. This will be the object of a subsequent exercise later on, but here let us first solve the simplest situation (which you actually know from previous courses). Specifically, assume that the equation of state of matter (EoS) is  $p_m = w_m \rho_m$ , with  $w_m = \text{const}$ . Trading the time variable for the scale factor, solve the mentioned differential equation and show that

$$\rho_m = \rho_m^0 a^{-3(1+w_m)},$$

where  $\rho_m^0$  is the current energy density of matter, assuming that the scale factor is normalized such that  $a = 1$  at present.

ii) Express the general solution in the case that  $w_m$  would be a certain function of the scale factor:  $w_m = w_m(a)$ .

iii) Apply your solution to compute the matter density in the case that  $w_m(a) = a$ . Do you find it a realistic possibility? What would be the matter density in the infinite past? What would be the matter density at the redshift of decoupling of the CMB? And in the infinite future?

**Exercise 4.** i) Using the basic cosmological equations from Exercise 1, show that you can recover Einstein's static solution as a particular case, namely the solution characterized by

$$4\pi G\rho_m = \frac{1}{a^2} = \Lambda.$$

What is the geometry of the Universe satisfying this equation? Why? What is the sign of the vacuum energy for this universe?

ii) Show that the equilibrium solution that you have found in the previous point corresponds to unstable equilibrium and hence corresponds to a runaway solution (as proven originally by Eddington). Specifically, show that if  $a_{\text{ea}}$  is the equilibrium position, a perturbation  $a_{\text{eq}} \rightarrow a_{\text{eq}} + \delta a$  satisfies

$$\delta\ddot{a} = 4\pi G\rho_m\delta a,$$

Why this shows that equilibrium is unstable? Einstein's universe is a “fine-tuned” universe, hence unnatural!

**Exercise 5.** Let us come back to our discussion on energy densities in Exercise 2.

i) Assume that  $w_m = -1$ , What would be the density and pressure? This is a very strange result for matter, isn't it? So let us assume that it actually applies to some conserved fluid  $X$  (maybe some dark energy fluid). To make it more general, assume that it satisfies the EoS  $p_X = w_X\rho_X$ , with constant  $w_X$ . Clearly, if  $w_X = -1$  we are back to the previous situation, which corresponds... to what? And what is the maximum value that  $w_X$  can take such that a dark energy (DE) fluid of this sort can produce late time acceleration for the Universe?

ii) Consider the situation in which  $w_X$  is not constant but follows the evolution law:

$$w_X(a) = w_0 + (1 - a)w_1,$$

where  $w_0$  and  $w_1$  are constants. Notice that this nonconstant parameterization of  $w_X(a)$  looks minimally meaningful since it is well-defined both at present and at any point in the remote past (where the matter density becomes unbounded!). Compute explicitly  $\rho_X(a)$  for such parameterization and provide a physical interpretation of it. Show that

$$\rho_X(a) = \rho_X^0 a^{-3(1+w_0+w_1)} e^{-3w_1(1-a)}.$$

iii) Do you recover from this expression the case when the EoS is constant? Now make a reasonable choice for the range of the parameters  $w_0$  and  $w_1$  such that this fluid can provide a minimally sensible picture of the DE, as we usually conceive it. What is the behavior of the energy density of such DE fluid in the infinite past and in the remote future (for the range of parameters you have chosen)? Do you find it sensible? Argue your position.

**Exercise 6.** Take the cosmological equations you have played with in Exercise 1.

i) Using the equations above, prove that the deceleration parameter of the Universe ( $q \equiv -\ddot{a}a/\dot{a}^2$ ) in the presence of several fluids with *constant* equation of state parameters

$\omega_n$  (for radiation, matter, vacuum energy etc) can be written, at a given cosmological redshift  $z$ , as

$$q(z) = \sum_n (1 + 3\omega_n) \frac{\Omega_n(z)}{2},$$

where  $\Omega_n(z) = \rho_n(z)/\rho_c(z)$  involve the various normalized densities, which are bound to satisfy the sum rule  $\sum_n \Omega_n(z) + \Omega_k(z) = 1$  at any value of the redshift  $z$ , with  $\Omega_k(z) = -(k/H_0^2)(1+z)^2$  the spatial curvature parameter. Evaluate this formula for  $z = 0$  in the  $\Lambda$ CDM model and show that the current value of the acceleration parameter reads

$$q_0 = \frac{\Omega_m^0}{2} - \Omega_\Lambda^0 = \frac{3\Omega_m^0 - 2}{2},$$

where the second equality is valid *only* for a spatially flat universe (why?). What is the value of  $q_0$  according to the present data? Explain the meaning of this numerical value and its sign.

ii) Check that the deceleration parameter can also be computed as follows

$$q(z) = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{1+z}{2H^2} \frac{dH^2(z)}{dz}$$

where  $z$  is the cosmological redshift. Apply this formula to the Hubble function for the  $\Lambda$ CDM and show that the transition redshift from deceleration to acceleration is given by

$$z^* = -1 + \left( \frac{2\Omega_\Lambda^0}{\Omega_m^0} \right)^{1/3}.$$

Does this result assume vanishing  $\Omega_k^0$ ? Evaluate  $z^*$  according to the latest *Planck* satellite data from July 2018 (see arXiv:1807.06209).

**Exercise 7.** Let us make some study on the time evolution and the age of the Universe.

i) Show that the formulae giving the cosmic time as a function of the scale factor is

$$t - t_1 = \int_{a_1}^{a(t)} \frac{da}{a H(a)}$$

where  $H = \dot{a}/a$  is the expansion rate. Here  $t_1$  is some initial time and  $a_1 = a(t_1)$ . What is the formula that gives the age of the Universe,  $t_0$ , in terms of an integral over the cosmological redshift  $z$ ?

ii) In general these integrals cannot be solved by quadrature, but there are some interesting cases that can be worked out easily (see the Table below). Compute  $a = a(t)$  during the matter and radiation dominated epoch for the Einstein-de Sitter's (EdS) Universe (CDM model). Then compute the age of each of the universes in the Table in units of the present Hubble time  $H_0^{-1}$  (**Solution:** a)  $t_0 = 2/(3H_0)$ , b)  $t_0 = 1/H_0$ , c)  $t_0 = \infty$ ). What drives the evolution in Milne's case b)? And in case c)?

(a)	$\Omega_m^0 = 1$	$\Omega_\Lambda^0 = 0$	(Einstein-de Sitter's Universe)
(b)	$\Omega_m^0 \simeq 0$	$\Omega_\Lambda^0 \simeq 0$	(Milne's Universe)
(c)	$\Omega_m^0 = 0$	$\Omega_\Lambda^0 = 1$	(Inflationary Universe)

iii) Using numerical integration, compute the realistic age of our Universe,  $t_0$ , assuming valid the standard  $\Lambda$ CDM model of cosmology, hence  $\Lambda \neq 0$ . To this end, take the latest data released by the *Planck* satellite on the cosmological parameters (assuming the upper bound on  $\Omega_k^0$ , and then 10 times its value). Express the result in Gigayears (Gyr) or (US) billion years; recall that 1 Gyr =  $10^9$  yr. Compare the obtained result with the age value of the EdS Universe. Does it pay to take into account the radiation epoch in the calculation? Why?

**Exercise 8.** We retake now the notion of co-moving “particle horizon”. Consider the FLRW metric expressed in the  $(a(t), \chi, \theta, \phi)$  parametrization (check your notes on cosmology and write down  $ds^2$  explicitly in these coordinates). Of course,  $\chi, \theta, \phi$  for a fundamental observer are comoving coordinates and hence independent of time. The particular one  $\chi$  plays the role of “radial” coordinate (what does it mean?) Let us assume that a light ray propagates fully along the  $\chi$  direction, i.e. with  $\theta = \phi = \text{const.}$  The comoving particle horizon at time  $t$ , which we denote  $\chi_{ph}(t)$ , reads as follows:

$$\chi_{ph}(t) = \int_0^t \frac{cdt'}{a(t')} = \int_0^{a(t)} \frac{cda}{a^2 H(t)} = \int_0^\infty \frac{cdz}{H(z)}$$

i) Explain the origin, meaning and equivalence of these formulas. What is the upper limit of integration of the last integral if we wish to compute the particle horizon at the present time? What is the physical interpretation of  $\chi_{ph}(t)$  for spheric and flat universes? Explain why the proper distance to the particle horizon is  $d_{ph}(t) = a(t) \chi_{ph}$ .

ii) Compute  $d_{ph}(t)$  for a flat radiation-dominated Universe without cosmological constant. Idem for a matter-dominated one at an early epoch (hence with negligible cosmological constant). Why the answer is not just  $ct$ ?

iii) What is the horizon if  $a \propto t^n$ ? Retrieve the previous results as particular cases of it. Show that the Hubble horizon at time  $t$  (i.e. the distance  $d_H(t) = c/H(t)$ ), is related to the particle horizon through  $d_{ph} = nd_H/(1-n)$ . Show that the observable part of the universe expands at a velocity

$$V_{ph} = \frac{c}{1-n} = c + V_r$$

where  $V_r$  is the recession velocity of galaxies at the particle horizon ( $V_r = Hd_{ph}$ ).

iv) Show that the Hubble horizon at time  $t$  is the distance where galaxies recede from us at  $V = c$ , whereas the recession velocity of galaxies at the particle horizon is always  $V_{ph} > c$ . Explain this result and its interpretation.

v) Something strange happens with the above formulae if  $n > 1$ . Why? What it means? Write down the previous results in terms of the deceleration parameter  $q$ . Argue that  $q > n$ . Is it so in the current universe?

vi) Compute the particle horizon at the decoupling time (redshift  $z_d \sim 1000$ ) and compare it to the value of the horizon now, which is  $d_p(t_0) \sim 14$  Gpc. The last number cannot be computed with accuracy using the formulas above. Explain why. Using this result, show that the standard cosmological model predicts that there are  $\sim 10^5$  causally disconnected regions in our universe, in the sense that the CMB that we observe from any

one of these regions is equally homogeneous and isotropic, what is surprising since these hundred thousand regions were never in causal contact!!

**Note:** Assume for simplicity that the universe has been all the time in the matter-dominated epoch up to decoupling. Why this assumption will not alter the result significantly?

**Exercise 9.** Let us now face the age computation of the  $\Lambda$ CDM model analytically. You may assume vanishing spatial curvature, which is realistic (why?).

i) Use the equations of Exercise 1 to show that the Hubble rate  $H = H(t)$  in the matter-dominated epoch (MDE) can be obtained by solving the differential equation

$$\dot{H} + \frac{3}{2}H^2 = 4\pi G \rho_\Lambda = \frac{\Lambda}{2},$$

ii) Solve explicitly this equation for the case of the  $\Lambda$ CDM model with  $\Lambda \neq 0$ , assuming that the Universe is spatially flat. Express your result as

$$H(t) = \sqrt{\Omega_\Lambda^0} H_0 \coth \left( \frac{3H_0 \sqrt{\Omega_\Lambda^0}}{2} t \right),$$

where  $H_0$  is the Hubble rate at present. Check that for  $\Lambda = 0$  you obtain an expected result. Which one? Derive explicitly  $a(t)$  from it.

iii) Consider again the MDE. Verify that the cosmic time  $t$  is related with the cosmological redshift  $z$  as follows:

$$t(z) = \frac{2}{3\sqrt{\Omega_\Lambda^0} H_0} \sinh^{-1} \left( \sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}} (1+z)^{-3/2} \right).$$

What is the limit  $\Omega_\Lambda^0 \rightarrow 0$  of this expression? And what is the limit for  $z \rightarrow \infty$ ? Explain your results.

iv) Let us compute the age of the Universe in the  $\Lambda$ CDM model for the present values of the cosmological parameters. If  $\Omega_k^0 \neq 0$ , the age cannot be given as a simple analytical formula (you then need the numerical procedure. However, for  $\Omega_k^0 = 0$  you can easily use the previous results to obtain an analytical expression for the age of the Universe. Prove the following beautiful formula:

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda^0} H_0} \sinh^{-1} \left( \sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}} \right) = \frac{2}{3H_0} \frac{\tanh^{-1} \sqrt{\Omega_\Lambda^0}}{\sqrt{\Omega_\Lambda^0}}.$$

where the second equality is the most convenient one. Check it!

v) Verify in detail that for  $\Lambda \rightarrow 0$  you recover the age of the Einstein-de Sitter universe given in Exercise 6 above, and also the first correction to this result for small values of  $\Lambda$ . Namely, show that

$$t_0 = \frac{2}{3} H_0^{-1} (1 + \frac{1}{3} \Omega_\Lambda^0 + \dots)$$

Take the ratio of this result with the exact result, and check if it departs significantly from one for the current value of  $\Omega_\Lambda^0$ .

**vi)** In contrast to the exact formula of **iv)** (which is quite nice but also quite opaque) the approximate formula of **v)** is numerically crude, but is qualitatively very useful since it transparently shows that for a non-vanishing and positive  $\Lambda$  the age of the Universe is larger than without it. However, can you feel the intuitive sense of this result beyond the mathematical result? Explain physically why an Universe with  $\Lambda > 0$  is necessarily older than another one with  $\Lambda = 0$  (with the same matter content). Justify your answer, clearly showing that you fully understand the physical reason.

**Exercise 10.** The idea of a strictly constant  $\Lambda$ -term as in the “concordance  $\Lambda$ CDM” is very simple, but it might even be qualified of simpleminded. If we take into account the dynamical character of our expanding Universe, one should rather expect an evolving vacuum energy density  $\rho_\Lambda$  and maybe even a slowly evolving gravitational coupling  $G$ . This is at least suggested on quantum field theory (QFT) grounds. In the Exercise 1 we have admitted in part this possibility, but now we want to exploit it a little further.

**i)** Adopting such more flexible point of view, but still in the context of the FLRW metric, show that the following generalized local conservation law holds:

$$\frac{d}{dt} [G(\rho_m + \rho_\Lambda)] + 3GH(\rho_m + p_m) = 0,$$

of which the equation in Exercise 1i) is a particular case (check it).

**ii)** Write down that equation in expanded form as a differential equation with respect to the cosmological redshift variable  $z$ .

**iii)** Such generalized conservation law can be compatible with an anomalous matter conservation formula of the form

$$\rho_m(z) = \rho_m^0(1+z)^{3(1-\nu)},$$

with  $\nu$  a small parameter  $|\nu| \ll 1$ . Obviously, for  $\nu = 0$  one recovers the standard law. Let us assume that  $G$  is strictly constant for the moment. Integrate the differential equation and show that  $\rho_\Lambda$  can no longer be constant but the following function of the redshift:

$$\rho_\Lambda(z) = \rho_\Lambda^0 + \frac{\nu}{1-\nu} \rho_m^0 [(1+z)^{3(1-\nu)} - 1]$$

where  $\rho_\Lambda^0$  is the current value of  $\rho_\Lambda$ . Interpret the structure of this result.

**iv)** While the above anomalous matter conservation law can be taken as a phenomenological ansatz, there are situations in the context of QFT in curved spacetime where such scenario can be theoretically motivated. For example, the following expression for  $\rho_\Lambda$  as a function of  $H$  has been derived within a particular QFT context<sup>3</sup>:

$$\rho_\Lambda(H) = \rho_\Lambda^0 + \frac{3\nu}{8\pi G}(H^2 - H_0^2).$$

Insert this result into the generalized conservation law found in **ii)** above for  $G = \text{const.}$  and use Friedmann’s equation (with zero spatial curvature) to solve for the density  $\rho_m(z)$ .

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<sup>3</sup>See arXiv:1306.1527 for more theoretical details of the QFT context, if desired, but the simple information provided in the exercise itself is sufficient to solve the specific questions we are addressing. Let us just mention that the small parameter  $\nu$  plays here the role of  $\beta$ -function coefficient for the running of  $\rho_\Lambda$  in the cosmological renormalization group flow.

The result (check it!) is exactly the anomalous conservation law given in section **ii**) above. Subsequently solve for  $\rho_\Lambda(z)$  and find the exact result given in **iii**). Find also the answer for  $\rho_m(z)$  and  $\rho_\Lambda(z)$  when the spatial curvature parameter  $\Omega_k^0 = -k/H_0^2 \neq 0$ .

**Note:** Another way to proceed is to insert the above expression for  $\rho_\Lambda(H)$  in the differential equation of Exercise 8 i) and obtain the Hubble function  $H = H(z)$  (upon first trading the time variable for the redshift) and from here to derive the energy densities  $\rho_\Lambda(z)$  and  $\rho_m(z)$ . Do it in at least one of the possible ways.

**Exercise 11.** Let us consider scalar fields in cosmology. We have derived on the blackboard the explicit expressions for the energy density and pressure, and we found

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi). \quad (0.1)$$

**i)** We had shown that these equations hold good for both the Minkowski and FLRW metrics, where in the last case we used the homogeneity and isotropy of spacetime. In our derivation we compared the perfect fluid form of the energy-momentum tensor with the field theory form of it. Re-derive this result without using at all the fluid form. Use the fact that  $\rho_\phi = T_0^0$  and  $p_\phi = (-1/3)T_i^i$  (sum over  $i$ ). Justify these expressions first.

**ii)** Assume that  $\phi$  does not interact with other components during the cosmological expansion. Rewrite *all* of the equations of Exercise 1 when the incoherent matter system  $(\rho_m, p_m)$  is replaced by the field densities  $(\rho_\phi, p_\phi)$  and  $\Lambda = 0$ . In particular, what is the local conservation law satisfied by  $\rho_\phi$  and  $p_\phi$ ? Re-derive from this law the field equation of motion for  $\phi$ , which we already inferred from the field action.

**iii)** Recompute the field density and pressure in the case when  $\phi$  is *not* homogeneous but still isotropic. Show that in this case the equation of state  $\omega_\phi = p_\phi/\rho_\phi$  reads

$$\omega_\phi = -1 + \frac{\dot{\phi}^2/V(\phi) + \frac{1}{3} (\nabla\phi)^2/V(\phi)}{1 + \frac{1}{2} \dot{\phi}^2/V(\phi) + \frac{1}{2} (\nabla\phi)^2/V(\phi)}$$

Check that the above result boils down to the standard one (which we discussed on the blackboard) for homogeneous scalar fields.

**iv)** Phantom fields are not welcome in QFT, as we have to pay a high price. Replace the kinetic term in the action  $(\nabla\phi)^2/2$  by  $\xi(\nabla\phi)^2/2$ , where  $\xi$  is some coefficient. What would be the expression for  $\omega_\phi$  now? What is the condition on  $\xi$  for the scalar field to be a phantom field? Why we don't like this condition? Check a QFT book, if necessary.

**v)** Explain how space inhomogeneities could make that a time-independent scalar field  $\phi$  to behave as a quintessence field. Could it appear as being phantom-like? Compare it with the standard quintessence case.

**vi)** The *Planck* data show a possible phantom-like behavior of the dark energy (DE). Check and quote the latest observed result by *Planck* on this particular. So somehow we should be ready for a phantom like DE, despite we cannot easily cope with it in QFT (at least using a scalar field  $\phi$ ). If we could mimic the phantom behavior with a non-phantom scalar field, it would be great. Discuss some strategy to make this option possible, at least in principle.