Gauge Excercicos

1) Write the $\mathcal{L}$ in terms of Ψ_L, Ψ_R:

$$\boxed{\mathcal{L}} = \bar{\Psi}^i i \not\partial \Psi^i - m_i \bar{\Psi}^i \Psi^i =$$
$$= \bar{\Psi}^i (P_L + P_R) i \not\partial (P_L + P_R) \Psi^i - m_i \bar{\Psi}^i (P_L + P_R)(P_L + P_R) \Psi^i = \quad (*)$$
$$= \bar{\Psi}^i P_L i \not\partial P_R \Psi^i + \bar{\Psi}^i P_R i \not\partial P_L \Psi^i - m_i (\bar{\Psi}^i P_L^2 \Psi^i + \bar{\Psi}^i P_R^2 \Psi^i) = \quad (**)$$
$$= \boxed{\bar{\Psi}^i_R i \not\partial \Psi^i_R + \bar{\Psi}^i_L i \not\partial \Psi^i_L - m_i (\bar{\Psi}^i_R \Psi^i_L + \bar{\Psi}^i_L \Psi^i_R)}$$

$\left({}^{*} P_L + P_R = \frac{\mathbb{1} - \gamma^5}{2} + \frac{\mathbb{1} + \gamma^5}{2} = \mathbb{1} \right)$

$\left({}^{**} \begin{array}{l} P_L P_R = 0 = P_R P_L \\ P_L \gamma^\mu P_L = P_L P_R \gamma^\mu = 0 = P_R P_L \gamma^\mu = P_R \gamma^\mu P_R \end{array} \right)$

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2)

a) Show that $F^{\mu\nu}$ is gauge invariant:

$$\boxed{F^{\mu\nu}} = \partial^\mu A^\nu - \partial^\nu A^\mu \xrightarrow{(\text{Gauge trans.})} \partial^\mu (A^\nu + \partial^\nu \theta) - \partial^\nu (A^\mu + \partial^\mu \theta) =$$
$$= \partial^\mu A^\nu - \partial^\nu A^\mu + \cancel{\partial^\mu \partial^\nu \theta} - \cancel{\partial^\nu \partial^\mu \theta} = \boxed{F^{\mu\nu}} \quad \text{invariant!!}$$

b) Show that $F^{\mu\nu}$ is not a derivative operator:

$$\boxed{[D_\mu, D_\nu]} = D_\mu D_\nu - D_\nu D_\mu = (\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu) - (\partial_\nu - ieA_\nu)(\partial_\mu - ieA_\mu) =$$
$$= \cancel{\partial_\mu \partial_\nu} - ie\partial_\mu A_\nu - ieA_\mu \partial_\nu - e^2 \cancel{A_\mu A_\nu} - [\cancel{\partial_\nu \partial_\mu} - ie\partial_\nu A_\mu - ieA_\nu \partial_\mu - e^2 \cancel{A_\nu A_\mu}] =$$
$$= -ie[\underbrace{(\partial_\mu A_\nu - A_\nu \partial_\mu)}_{(\partial_\mu A_\nu)} - \underbrace{(\partial_\nu A_\mu - A_\mu \partial_\nu)}_{(\partial_\nu A_\mu)}] = -ie[(\partial_\mu A_\nu) - (\partial_\nu A_\mu)] = \boxed{-ieF^{\mu\nu}}$$

Not a derivative operator!!!

Where the initial $\partial_\mu A_\nu$ also derive the terms following A_ν, but $(\partial_\mu A_\nu)$ does not!

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3)

a)

$$\left.\begin{array}{ll} (1) & [(i\partial)^2 - m^2] A_\mu = 0 \\ (2) & \partial_\mu A^\mu = 0 \end{array}\right\} \overset{?}{\Longleftrightarrow} \partial^\mu F_{\mu\nu} + m^2 A_\nu = 0$$

let's start from (1):

$$[(i\partial)^2 - m^2] A_\nu = 0 \; ; \; \partial^2 A_\nu + m^2 A_\nu = 0 \; ; \; \partial^\mu \partial_\mu A_\nu + m^2 A_\nu = 0$$

(let's add (2) now as: $\partial_\nu \partial^\mu A_\mu = 0$)

$$\partial^\mu \partial_\mu A_\nu + \partial^\mu \partial_\nu A_\mu + m^2 A_\nu = 0 \; ; \; \partial^\mu (\underbrace{\partial_\mu A_\nu + \partial_\nu A_\mu}_{F_{\mu\nu}}) + m^2 A_\nu = 0$$

so finally, we are led to:

$$\boxed{\partial^\mu F_{\mu\nu} + m^2 A_\nu = 0} \quad (3)$$

b) Show that (3) follows from the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^2$$

(for which the eq of motion are:)

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} - \frac{\partial}{\partial x^\beta} \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = 0$$

- $$\frac{\partial \mathcal{L}}{\partial A^\alpha} = \frac{1}{2} m^2 \frac{\partial A^\alpha A_\alpha}{\partial A^\alpha} = m^2 A_\alpha$$
- $$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{4} \frac{\partial F^{\mu\nu} F_{\mu\nu}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{2} F^{\mu\nu} \frac{\partial(\partial^\mu A^\nu - \partial^\nu A^\nu)}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{2} F_{\mu\nu} (g^\mu{}_\beta g^\nu{}_\alpha - g^\nu{}_\beta g^\mu{}_\alpha) =$$
$$= -\frac{1}{2}(F_{\beta\alpha} - F_{\alpha\beta}) = \frac{1}{2}(F_{\alpha\beta} + F_{\alpha\beta}) = F_{\alpha\beta}$$

so finally, we have:

$$m^2 A_\alpha - \partial^\beta F_{\alpha\beta} = 0 \longrightarrow \boxed{\partial^\mu F^{\mu\nu} + m^2 A_\nu = 0}$$

4) If we add a term $A_\mu J^\mu$, the e.o.m. become:

$$\frac{\partial A_\mu J^\mu}{\partial A^\mu} = J_\mu \longrightarrow \partial^\nu F_{\nu\mu} + m^2 A_\mu + J_\mu = 0$$

Which finally can be written:

$$\boxed{\partial^\mu F_{\mu\nu} + m^2 A_\nu = -J_\nu}$$

5) Write the $\mathcal{L}$ showing the Kernel explicitly:

$$\boxed{\mathcal{L}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^2 + A_\mu J^\mu = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2} m^2 A^\mu A_\mu + A_\mu J^\mu =$$

$$= -\frac{1}{4}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A^\mu - \partial^\nu A^\mu \partial_\mu A_\nu + \partial^\nu A^\mu \partial_\nu A_\mu) + \frac{1}{2} A^\mu m^2 A_\mu + A_\mu J^\mu =$$

$$= +\frac{1}{4}(A^\nu \partial^\mu \partial_\mu A_\nu - A^\nu \partial^\mu \partial_\nu A^\mu - A^\mu \partial^\nu \partial_\mu A_\nu + A^\mu \partial^\nu \partial_\nu A_\mu) + \frac{1}{2} A^\mu m^2 g_{\mu\nu} A^\nu + A_\mu J^\mu = \quad ⊛$$

$$= \frac{1}{2}(A^\mu \partial^2 g_{\mu\nu} A^\nu - A^\mu \partial_\nu \partial_\mu A^\nu) + \frac{1}{2} A^\mu m^2 g_{\mu\nu} A^\nu + A_\mu J^\mu =$$

$$= \frac{1}{2} A^\mu (\partial^2 g_{\mu\nu} - \partial_\nu \partial_\mu + m^2 g_{\mu\nu}) A^\nu + A_\mu J^\mu =$$

$$= \boxed{\frac{1}{2} A^\mu \underbrace{((\partial^2 + m^2) g_{\mu\nu} - \partial_\mu \partial_\nu)}_{O_{A\,\mu\nu}} A^\nu + A_\mu J^\mu}$$

(⊛ $\partial^\alpha(A^\beta \partial_\delta A_\epsilon) = (\partial^\alpha A^\beta)(\partial_\delta A_\epsilon) + A^\beta \partial^\alpha \partial_\delta A_\epsilon$ where the total derivative gets cancelled when we integrate $S = \int d^4x \, \mathcal{L}$)

6) From the Kernel find the propagator:

Changing to momentum representation, we have:

$$\tilde{O}_A^{\mu\nu}\, \tilde{D}_{F\,\nu\rho}(K) = i\, \delta^\mu{}_\rho \quad \text{where } \tilde{O} \text{ is } O \text{ in momentum space: } i\partial_\mu \rightarrow K_\mu:$$

$$\left[(-K^2+m^2)\, g^{\mu\nu} + K^\mu K^\nu\right] \tilde{D}_{F\,\nu\rho} = i\, \delta^\mu{}_\rho$$

$$\left[(K^2-m^2)\, g^{\mu\nu} - K^\mu K^\nu\right] \tilde{D}_{F\,\nu\rho} = -i\, \delta^\mu{}_\rho$$

So to find the operator $\tilde{D}_{F\,\nu\rho}$, let's start with a term that will give the delta when contracted with the first $(K^2-m^2)\, g^{\mu\nu}$:

$$\frac{-i}{K^2-m^2+i\varepsilon}\, g_{\nu\rho} \quad \text{will have the correct normalization and give } g^{\mu\nu} g_{\nu\rho} = g^\mu{}_\rho = \delta^\mu{}_\rho$$

And now we need another term $A_{\nu\rho}$ to cancel the remaing term:

$$\left[(K^2-m^2)\, g^{\mu\nu} - K^\mu K^\nu\right] \left(\frac{-i g_{\nu\rho} + A_{\nu\rho}}{K^2-m^2}\right) = -i\, \delta^\mu{}_\rho$$

$$\cancel{-i\,\delta^\mu{}_\rho} + i\,\frac{K^\mu K_\rho}{K^2-m^2} + \left(g^{\mu\nu} - \frac{K^\mu K^\nu}{K^2-m^2}\right) A_{\nu\rho} = \cancel{-i\,\delta^\mu{}_\rho}$$

for which if we see that $A_{\nu\rho}$ has to be dimensionless, and from the two obvious possibilities $i\frac{K_\nu K_\rho}{K^2}$ and $i\frac{K_\nu K_\rho}{m^2}$, the second one fullfills everything:

$$0 = i\,\frac{K^\mu K_\rho}{K^2-m^2} + i\,\frac{K^\mu K_\rho}{m^2} - i\,\frac{K^\mu K^\nu K_\nu K_\rho / m^2}{K^2-m^2} = i\,\frac{K^\mu K_\rho}{K^2-m^2}\left(\cancel{1} + \frac{\cancel{K^2-m^2}}{m^2} - \frac{\cancel{K^2}}{m^2}\right) = 0 \;\checkmark\checkmark$$

So finally:

$$\boxed{\tilde{D}_{F\,\nu\rho} = \frac{-i}{K^2-m^2}\left(g_{\nu\rho} + \frac{K_\nu K_\rho}{m^2}\right)}$$

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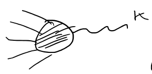
7) Show the completness of the polarizations through the W.I.:



The process with an external photon will have an amplitude:

$$|M|^2 = \sum_a \varepsilon_\mu^{(a)*}(K)\, \varepsilon_\nu^{(a)}(K)\, M^\mu(K)\, M^\nu(K)^* = \sum_a \left|\varepsilon_\mu^{(a)}(K)\, M^\mu(K)\right|^2 \overset{(*)}{=} |M^1|^2 + |M^2|^2 \overset{(**)}{=}$$

$$= |M^1|^2 + |M^2|^2 + |M^3|^2 - |M^0|^2 = -g_{\mu\nu}\, M^\mu(K)\, M^\nu(K)^*$$

So we see that:

$$\boxed{\sum_a \varepsilon_\mu^{(a)}(K)^*\, \varepsilon_\nu^{(a)}(K) = g_{\mu\nu}}$$

((*) We have to take into account only the physical possible processes here, beause it's an external photon, so:
$|\varepsilon_\mu^{(0)} M^\mu|^2 = |\varepsilon_\mu^{(3)} M^\mu|^2 = 0$, and we only get contributions from $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ (transverse))

((**) To get the desired expression, we see from the ward Identity that:
$K_\mu M^\mu = 0$, which for $K^\mu = (|\vec{K}|, 0, 0, |\vec{K}|)$ tells us that $M^3 = M^0 \Rightarrow |M^3|^2 - |M^0|^2 = 0$
and we can add this zero at the previous expression to obtain $g_{\mu\nu}$)

Extra

To see that not physical $\varepsilon^{0,3}$ aren't need to be taken into account in ⊛, let's show it:

- Fixing the gauge $\partial^\mu A_\mu = 0 \Longrightarrow K^\mu \varepsilon^{(a)}_\mu = 0 \xRightarrow{K^\mu=(K,0,0,K)} \varepsilon^{0(a)} = \varepsilon^{3(a)} \Longrightarrow \varepsilon^{(0,3)}_0 = -\varepsilon^{(0,3)}_3$

- Basis completness tells us, that: $g_{ij}\, \varepsilon^{(i)}_\mu \varepsilon^{(j)}_\nu = g_{\mu\nu}$

$$g_{ii}\, \varepsilon^{(i)}_\mu \varepsilon^{(i)}_\mu = g_{\mu\mu} \Longrightarrow$$

(And chosing $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ as transverse to K)

Possible base then:

$$\begin{cases} \varepsilon^{(0)}_\mu = (1/\sqrt{2}, 0, 0, -1/\sqrt{2}) \\ \varepsilon^{(1)}_\mu = (0,1,0,0) \\ \varepsilon^{(2)}_\mu = (0,0,1,0) \\ \varepsilon^{(3)}_\mu = (-1/\sqrt{2}, 0, 0, 1/\sqrt{2}) \end{cases}$$

- Where now it's obvious, that

$$|\varepsilon^{(0)}_\mu M^\mu|^2 = \left|\tfrac{1}{\sqrt{2}} M^0 - \tfrac{1}{\sqrt{2}} M^3\right|^2 = |M^0|^2 \left|\cancel{\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}}\right|^2 = \boxed{0}$$

$$|\varepsilon^{(1)}_\mu M^\mu|^2 = |M^1|^2$$

$$|\varepsilon^{(2)}_\mu M^\mu|^2 = |M^2|^2$$

$$|\varepsilon^{(3)}_\mu M^\mu|^2 = \left|-\tfrac{1}{\sqrt{2}} M^0 + \tfrac{1}{\sqrt{2}} M^3\right|^2 = |M^0|^2 \left|\cancel{-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}}\right|^2 = \boxed{0}$$

($M^3 = M^0$ from Ward Identity)

where we have seen that $\underline{\varepsilon^0 \text{ and } \varepsilon^3 \text{ don't conturbute to the final } M}$.

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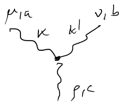
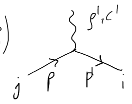
8) Compute $K_\mu \cdot M^{\mu\nu}_{ij}$:

1) 

$$= i g f^{abc} \left(g_{\mu\nu} (K+K')_\rho + g_{\nu\rho}(-K'-q)_\mu + g_{\rho\mu}(q-K)_\nu \right) \delta(K+q-K')\, (2\pi)^4$$

2) 

$$= \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2} g_{\rho\rho'} \delta^c_{c'}$$

3) 

$$= i g\, \bar u(p') \gamma_{\rho'} T^{c'}_{ij} u(p)\, \delta^4(p-q-p')\, (2\pi)^4$$

$$I^{\mu\nu}_{ij}(p,K,p',K') = 1)+2)+3) = i g^2 f^{abc} \int \frac{dq}{\cancel{(2\pi)^4}} \left[\frac{\bar u(p')\gamma^\rho T^c_{ij} u(p) \left(g_{\mu\nu}(K+K')_\rho + g_{\nu\rho}\overbrace{(-K'-q)_\mu}^{-2K'+K} + g_{\rho\mu}\overbrace{(q-K)_\nu}^{-2K+K'} \right)}{q^2} \delta(K+q-K')\delta(p-q-p')\, (2\pi)^{\cancel{8}\,4} \right] =$$

$$= i g^2 f^{abc} \frac{\bar u(p')\gamma^\rho T^c_{ij} u(p)}{(K'-K)^2} \left[\left(g_{\mu\nu}(K+K')_\rho + g_{\nu\rho}(K-2K')_\mu + g_{\rho\mu}(K'-2K)_\nu \right) \delta(p+K-p'-K')\, (2\pi)^4 \right] =$$

$I^{\mu\nu}_{ij} = i M^{\mu\nu}_{ij}\, \delta(\sum_f p_f - \sum_i p_i)\, (2\pi)^4$, so:

$$M^{\mu\nu}_{ij} = g^2 f^{abc} \frac{\bar u(p')\gamma^\rho T^c_{ij} u(p)}{(K'-K)^2} \left(g_{\mu\nu}(K+K')_\rho + g_{\nu\rho}(K-2K')_\mu + g_{\rho\mu}(K'-2K)_\nu \right)$$

And so, then:

$$\boxed{K_\mu \cdot M^{\mu\nu}_{ij}} = g^2 f^{abc} \frac{\bar u(p')\gamma^\rho T^c_{ij} u(p)}{(K'-K)^2} \left(K_\nu (K+K')_\rho + g_{\nu\rho}\, K\cdot(K-2K') + K_\rho (K'-2K)_\nu \right) =$$

$$= g^2 f^{abc} \frac{\bar u(p')\gamma^\rho T^c_{ij} u(p)}{(K'-K)^2} \left(\underline{K_\nu K_\rho} + K_\nu K'_\rho + g_{\nu\rho}(K^2 - 2K\cdot K') + K_\rho K'_\nu - \underline{2K_\rho K_\nu} \right) \overset{⊛}{=}$$

$$= g^2 f^{abc} \frac{\bar u(p')\gamma^\rho T^c_{ij} u(p)}{(K'-K)^2} \left(-\cancel{K_\nu K_\rho} + \cancel{K_\nu K_\rho} + g_{\nu\rho}(K^2 - 2KK') + \overset{0}{\cancel{K'_\rho K'_\nu}} \right) =$$

(this term will have a $\varepsilon_\nu(K')$ multiplied that will make $K'_\nu \cdot \varepsilon_\nu(K') = 0$ ⊛⊛)

$$= g^2 f^{abc} \frac{\bar u(p')\gamma^\rho T^c_{ij} u(p)}{\cancel{(K'-K)^2}} \left(g_{\nu\rho} \frac{(K^2 - 2KK' + K'^2)}{\cancel{(K'-K)^2}} \right) =$$

($K'^2 = 0 = K^2$)

$$= \boxed{g^2 f^{abc}\, \bar u(p')\, \gamma_\nu\, T^c_{ij}\, u(p)}$$

(⊛ $0 = \bar u(p')\, \not{q}\, u(p) = \bar u(p')(\not{K}' - \not{K})\, u(p) \longrightarrow \not{K}' = \not{K}$ effectively $\longrightarrow K'_\rho = K_\rho$)

(⊛⊛ Fixing the gauge $\partial^\mu A_\mu = 0 \longrightarrow K^\mu \varepsilon_\mu(K) = 0$)

9) Show what the product rule of the unitary matrices implies for the algebras:

$$U(t(\theta))\,U(t(\theta'))=U(t(f(\theta,\theta'))) \quad \text{with} \begin{cases} f^a(\theta,\theta')=\theta^a+\theta'^a+g^{abc}\theta^b\theta'^c+\dots \\ U(t(\theta))=\mathbb{1}+it^a\theta^a+\frac{1}{2}t^{ab}\theta^a\theta^b+\dots \end{cases} \quad \text{becomes:}$$

$$\left[1+it^a\theta^a+\frac{1}{2}t^{ab}\theta^a\theta^b\right]\left[1+it^a\theta'^a+\frac{1}{2}t^{ab}\theta'^a\theta'^b\right]=1+it^a(\theta^a+\theta'^a+g^{abc}\theta^b\theta'^c)+\frac{1}{2}t^{ab}(\theta^a+\theta'^a+g^{abc}\theta^b\theta'^c)(\theta^b+\theta'^b+g^{bde}\theta^d\theta'^e)+O(t^3)$$

$$\cancel{1}+\cancel{it^a\theta^a}+\cancel{it^a\theta'^a}-t^a\theta^a t^b\theta'^b+\frac{1}{2}\cancel{t^{ab}\theta^a\theta^b}+\frac{1}{2}\cancel{t^{ab}\theta'^a\theta'^b}=\cancel{1}+\cancel{it^a\theta^a}+\cancel{it^a\theta'^a}+it^a g^{abc}\theta^b\theta'^c+\frac{1}{2}t^{ab}(\cancel{\theta^a\theta^b}+\cancel{\theta'^a\theta'^b}+\theta^a\theta'^b+\theta'^a\theta^b)+\dots;$$

$$-t^a t^b\theta^a\theta'^b=it^a g^{abc}\theta^b\theta'^c+\frac{1}{2}t^{ab}\theta^a\theta'^b+\frac{1}{2}t^{ab}\theta'^a\theta^b;$$

(a↔b, t symmetric)

$$-t^a t^b\theta^a\theta'^b=it^c g^{cab}\theta^a\theta'^b+\frac{1}{2}t^{ab}(\theta^a\theta'^b+\theta'^b\theta^a);$$

$$\boxed{-t^a t^b=it^c g^{cab}+t^{ab}}$$

$$\left(\begin{aligned}&T^a t^b-t^b t^a=-it^c g^{cab}\cancel{+t^{ab}}+it^c g^{cba}\cancel{-t^{ba}}\\ &[t^a,t^b]=-it^c(g^{cab}-g^{cba})=-it^c\underbrace{g^{c[a,b]}}_{f^{cab}}\end{aligned}\right)$$

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10) Compute the f^{abc} of $SU(2)$ directly from the group multiplication rule:

if $U\in SU(2)$ then
$$\begin{cases} UU^+=\mathbb{1} \to (\mathbb{1}+iu+\dots)(\mathbb{1}-iu^++\dots)=\mathbb{1} \to u-u^+=0 \to \underset{\text{hermitic}}{u=u^+} \to u=\begin{pmatrix} a & b^* \\ b & c\end{pmatrix} \quad a,c\in\mathbb{R} \\ \det(U)=1 \to \det(\mathbb{1}+iu+\dots)=1 \to \det\begin{pmatrix}1+ia & b^* \\ b & 1+ic\end{pmatrix}=1 \to \underset{\text{traceless}}{a=-c} \to u=\begin{pmatrix} a & b^* \\ b & -a\end{pmatrix}\end{cases}$$

a) First way:

With commutators of a general basis:

so the elements will the base of hermitian traceless matrixs (dim 3, a, Re(b), iIm(b)), so a basis will be:

$$u_1=\begin{pmatrix} a & 0 \\ 0 & -a\end{pmatrix},\quad u_2=\begin{pmatrix} 0 & Re(b) \\ Re(b) & 0\end{pmatrix},\quad u_3=\begin{pmatrix} 0 & i\,Im(b) \\ -i\,Im(b) & 0\end{pmatrix}\quad \text{for which the commutators are:}$$

- $[u_1,u_2]=\begin{pmatrix}0 & aRe(b)\\ -aRe(b) & 0\end{pmatrix}-\begin{pmatrix}0 & -aRe(b)\\ aRe(b) & 0\end{pmatrix}=\begin{pmatrix}0 & 2aRe(b)\\ -2aRe(b) & 0\end{pmatrix}=\frac{2aRe(b)}{i\,Im(b)}u_3$
- $[u_2,u_3]=\begin{pmatrix}-iRe(b)Im(b) & 0\\ 0 & iRe(b)Im(b)\end{pmatrix}-\begin{pmatrix}iRe(b)Im(b) & 0\\ 0 & -iRe(b)Im(b)\end{pmatrix}=\begin{pmatrix}-i2Re(b)Im(b) & 0\\ 0 & i2Re(b)Im(b)\end{pmatrix}=\frac{-i2Re(b)Im(b)}{a}u_1$
- $[u_3,u_1]=\begin{pmatrix}0 & ia\,Im(b)\\ ia\,Im(b) & 0\end{pmatrix}-\begin{pmatrix}0 & ia\,Im(b)\\ ia\,Im(b) & 0\end{pmatrix}=\begin{pmatrix}0 & -i2a\,Im(b)\\ -i2a\,Im(b) & 0\end{pmatrix}=\frac{-i2a\,Im(b)}{Re(b)}u_2$

$$\circledast \quad \boxed{[u_a,u_b]=-2\epsilon_{abc}\sqrt{\frac{\det(u_a)\det(u_b)}{\det(u_c)}}\,u_c}$$

Which for a basis with the same determinant $\equiv u$ gives: $\boxed{[u_a,u_b]=-2\epsilon_{abc}\sqrt{u}\,u_c}$

$$\left(\circledast\ \det(u_1)=-a^2,\ \det(u_2)=-Re(b)^2,\ \det(u_3)=-Im(b)^2 \longrightarrow \frac{a\cdot Re(b)}{Im(b)}=\frac{\sqrt{-\det(u_1)}\sqrt{-\det(u_2)}}{\sqrt{-\det(u_3)}}=i\sqrt{\frac{\det(u_1)\det(u_2)}{\det(u_3)}}\right)$$

b) Second way:

With the product from the previous excercise, with $t^{ab}=0$ (it is redundant for the structure constants):

$$-t^a t^b=it^c g^{cab}$$

let's compute the products:

$$t_1\cdot t_2=\begin{pmatrix}a&0\\0&-a\end{pmatrix}\begin{pmatrix}0&Re(b)\\Re(b)&0\end{pmatrix}=\begin{pmatrix}0&aRe(b)\\-aRe(b)&0\end{pmatrix}=\frac{aRe(b)}{i\,Im(b)}\begin{pmatrix}0&iIm(b)\\-iIm(b)&0\end{pmatrix}=-\overbrace{i\frac{aRe(b)}{Im(b)}}^{i\,g^{312}}t_3 \longrightarrow g^{312}\overset{\circledast}{=}-i\,\epsilon_{312}\sqrt{\frac{\det(t_1)\det(t_2)}{\det(t_3)}}$$

$$t_2\cdot t_3=\begin{pmatrix}0&Re(b)\\Re(b)&0\end{pmatrix}\begin{pmatrix}0&iIm(b)\\-iIm(b)&0\end{pmatrix}=\begin{pmatrix}-iRe(b)Im(b)&0\\0&iRe(b)Im(b)\end{pmatrix}=\frac{-iRe(b)Im(b)}{a}\begin{pmatrix}a&0\\0&-a\end{pmatrix}=-i\frac{Re(b)Im(b)}{a}t_1 \longrightarrow g^{123}=-i\,\epsilon_{123}\sqrt{\frac{\det(t_2)\det(t_3)}{\det(t_1)}}$$

$$t_3\cdot t_1=\begin{pmatrix}0&iIm(b)\\-iIm(b)&0\end{pmatrix}\begin{pmatrix}a&0\\0&-a\end{pmatrix}=\begin{pmatrix}0&-ia\,Im(b)\\-iaIm(b)&0\end{pmatrix}=\frac{-ia\,Im(b)}{Re(b)}\begin{pmatrix}0&Re(b)\\Re(b)&0\end{pmatrix}=-i\frac{a\,Im(b)}{Re(b)}t_2 \longrightarrow g^{231}=-i\epsilon_{231}\sqrt{\frac{\det(t_3)\det(t_1)}{\det(t_2)}}$$

$$t_2\cdot t_1=\begin{pmatrix}0&Re(b)\\Re(b)&0\end{pmatrix}\begin{pmatrix}a&0\\0&-a\end{pmatrix}=-\begin{pmatrix}0&aRe(b)\\-aRe(b)&0\end{pmatrix}=-\frac{aRe(b)}{i\,Im(b)}\begin{pmatrix}0&iIm(b)\\-iIm(b)&0\end{pmatrix}=i\frac{aRe(b)}{Im(b)}t_3 \longrightarrow g^{321}=-i\,\epsilon_{321}\sqrt{\frac{\det(t_1)\det(t_2)}{\det(t_3)}}$$

$$t_3\cdot t_2=-\begin{pmatrix}0&Re(b)\\Re(b)&0\end{pmatrix}\begin{pmatrix}0&iIm(b)\\-iIm(b)&0\end{pmatrix}=-\begin{pmatrix}-iRe(b)Im(b)&0\\0&iRe(b)Im(b)\end{pmatrix}=\frac{iRe(b)Im(b)}{a}\begin{pmatrix}a&0\\0&-a\end{pmatrix}=i\frac{Re(b)Im(b)}{a}t_1 \longrightarrow g^{132}=-i\,\epsilon_{132}\sqrt{\frac{\det(t_2)\det(t_3)}{\det(t_1)}}$$

$$t_1\cdot t_3=-\begin{pmatrix}0&iIm(b)\\-iIm(b)&0\end{pmatrix}\begin{pmatrix}a&0\\0&-a\end{pmatrix}=-\begin{pmatrix}0&-ia\,Im(b)\\-iaIm(b)&0\end{pmatrix}=\frac{ia\,Im(b)}{Re(b)}\begin{pmatrix}0&Re(b)\\Re(b)&0\end{pmatrix}=i\frac{a\,Im(b)}{Re(b)}t_2 \longrightarrow g^{213}=-i\,\epsilon_{213}\sqrt{\frac{\det(t_3)\det(t_1)}{\det(t_2)}}$$

And we've see that g^{abc} is antysimetric in $b \leftrightarrow c$, so: $\boxed{f^{abc} = 2g^{abc} = -2i\,\epsilon_{abc}\sqrt{\frac{det(t_b)\,det(t_c)}{det(t_a)}}}$

11) Show the Adj rep. fullfills the comutator relation: $\left(\text{Adj. repr}: T^a_{ij} = -if^{aij}\right)$ (Jacobi Identity ⊛)

$$\boxed{[T^a,T^b]_{ck}} = (T^aT^b - T^bT^a)_{ck} = T^a_{cj}T^b_{jk} - T^b_{cj}T^a_{jk} = -f^{acj}f^{bjk} + f^{bcj}f^{ajk} = f^{abj}f^{jck} = \boxed{if^{abj}T^j_{ck}}$$

⊛ Jacobi Identity:

$$[T^c,[T^a,T^b]]_{l,m} + [T^a,[T^b,T^c]]_{l,m} + [T^b[T^c,T^a]]_{l,m} = 0$$

$[T^i,T^j] = f^{ijk}T^k$

$$[T^c, f^{abj}T^j]_{l,m} + [T^a, f^{bcj}T^j]_{l,m} + [T^b, f^{caj}T^j]_{l,m} = 0$$

$$f^{abj}[T^c,T^j]_{l,m} + f^{bcj}[T^a,T^j]_{l,m} + f^{caj}[T^b,T^j]_{l,m} = 0$$

$(T^i,T^j)_{l,m} = f^{ijk}T^k_{l,m}$

$$f^{abj}f^{cjk}\cancel{T^k_{lm}} + f^{bcj}f^{ajk}\cancel{T^k_{lm}} + f^{caj}f^{bjk}\cancel{T^k_{lm}} = 0$$

$$f^{abj}f^{cjk} + f^{bcj}f^{ajk} + f^{caj}f^{bjk} = 0$$

$$-f^{abj}f^{jck} + f^{bcj}f^{ajk} - f^{acj}f^{bjk} = 0$$

12) Using $Tr(t^a_r t^b_r) = C(r)\delta^{ab}$ show that f^{abc} are totally antisymetric:

$$tr\left([t^a_r,t^b_r] = f^{abc}t^c_r\right) \Longrightarrow Tr(t^a_r t^b_r) - tr(t^b_r t^a_r) = Tr(f^{abc}t^c_r)$$

$$C(r)\delta^{ab} - C(r)\delta^{ba} = f^{abc}tr(t^c_r)$$

$$C(r)\,(\cancel{\delta^{ab} - \delta^{ba}})^{0} = f^{abc}tr(T^c_r)$$

$$tr(t^c_r) = 0$$

Going to the adjoint representation:

- $tr(t^a_G) = 0$; $Tr(t^a_{G\,\alpha\beta}) = 0$; $t^a_{G\,\beta\beta} = 0$; $f^a{}_{\beta\beta} = 0$; f antisymetric in last indices
- We already knew that f^{abc} was antisymmetric in the first indices $\left([t^a,t^b] = f^{abc}t^c \longrightarrow [t^a,t^a] = 0 = f^{aac}t^c\right)$

$\boxed{f \text{ totally antisymmetric!}}$

13) Show that if A_μ is hermitic any gaug transformation also is hermitic:

- $A_\mu' = A_\mu + i\theta^a[T^a,A_\mu] + \frac{1}{g}(\partial_\mu\theta^a)T^a$

$(\theta^a, g \in \mathbb{R})$ $\qquad (T^{a\dagger} = T^a)$

- $\boxed{(A_\mu')^\dagger} = A_\mu^\dagger - i\theta^{a*}[T^a,A_\mu]^\dagger + \frac{1}{g^*}(\partial_\mu\theta^{a*})T^{a\dagger} = A_\mu^\dagger - i\theta^a\underbrace{[A_\mu,T^{a\dagger}]}_{(-)} + \frac{1}{g}(\partial_\mu\theta^a)T^{a\dagger} =$

$$= A_\mu^\dagger + i\theta^a[T^a,A_\mu] + \frac{1}{g}(\partial_\mu\theta^a)T^a = \boxed{A_\mu'}$$

$\left(\text{if } A_\mu = A_\mu^\dagger\right)$

$\boxed{\text{if } A_\mu \text{ hermitian then } A_\mu' \text{ hermitian } \forall\theta}$

14) Show that if A_μ is traceless any gaug transformation also is traceless:

- $A_\mu' = A_\mu + i\underbrace{\theta^a [T^a, A_\mu]}_{\substack{\text{traceless} \\ (\circledast [A,B] \text{ traceless})}} + \underbrace{\frac{1}{g}(\partial_\mu \theta^a) T^a}_{\text{traceless } (T^a \text{ traceless})} \Longrightarrow$ $\boxed{\text{if } A_\mu \text{ traceless then } A_\mu' \text{ traceless } \forall \theta}$

$\left(\circledast \; A_{ab} B_{ba} = B_{ba} A_{ab} \overset{(a\leftrightarrow b)}{=} B_{ab} A_{ba} \longrightarrow Tr(AB) = tr(BA) \longrightarrow tr(AB - BA) = 0 \right)$

15)

e) Show $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - ig f^{abc} A_\mu^b A_\nu^c$:

$$\boxed{\frac{i}{g}[D_\mu, D_\nu]_{\underset{(SU(N))}{\alpha\beta}}} = \frac{i}{g}\left[\partial_\mu - ig A_\mu^a T^a, \partial_\nu - ig A_\nu^a T^a\right]_{\alpha\beta} = \frac{i}{g}\left[\partial_\mu \delta_{\alpha x} - ig A_\mu^a T^a_{\alpha x}, \partial_\nu \delta_{x\beta} - ig A_\nu^a T^a_{x\beta}\right]$$

$$= \frac{i}{g}\Big(\overset{0}{[\partial_\mu, \partial_\nu]} - ig \underbrace{[A_\mu^a T^a, \partial_\nu]}_{-\partial_\nu(A_\mu^a T^a)} - ig \underbrace{[\partial_\mu, A_\nu^a T^a]}_{\partial_\mu(A_\nu^a T^a)} - g^2 [A_\mu^a T^a, A_\nu^b T^b]\Big)_{\alpha\beta} \overset{\circledast}{=}$$

$$= \left(\partial_\mu (A_\nu^a T^a) - \partial_\nu (A_\mu^a T^a) - ig A_\mu^a A_\nu^b [T^a, T^b]\right)_{\alpha\beta} =$$

$$= (\partial_\mu A_\nu^a)\, 1\!\!1_{\alpha x} T^a_{x\beta} - (\partial_\nu A_\mu^a)\, 1\!\!1_{\alpha\gamma} T^a_{\gamma\beta} - ig A_\mu^a A_\nu^b \overbrace{[T^a, T^b]_{\alpha\beta}}^{f^{abc} T^c_{\alpha\beta}} =$$

$$= \left((\partial_\mu A_\nu^a) - (\partial_\nu A_\mu^a) - ig f^{abc} A_\mu^b A_\nu^c\right) T^a_{\alpha\beta} \Longrightarrow \boxed{F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - ig f^{abc} A_\mu^b A_\nu^c}$$

a), b) Show we can write $F_{\mu\nu} = F_{\mu\nu}^a T^a$, and that $F_{\mu\nu}^a$ is not a derivative operator:

$\boxed{\text{Both follow from the results in e)}}$.

d) Show that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$:

It also follows from e), if we express the $A_\mu = A_\mu^a T^a$:

$$\boxed{F_{\mu\nu}} \overset{(e)\text{ without } \alpha\beta \text{ indices } \circledast)}{=} \partial_\mu (A_\nu^a T^a) - \partial_\nu (A_\mu^a T^a) - ig[A_\mu^a T^a, A_\nu^b T^b] \overset{(A_\mu = A_\mu^a T^a)}{=} \boxed{\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]}$$

c) Show that $F_{\mu\nu}^a$ transforms in the Adj. rep.:

$$F_{\mu\nu}^a \longrightarrow F_{\mu\nu}^{a'} = \partial_\mu A_\nu^{a'} - \partial_\nu A_\mu^{a'} - ig f^{abc} A_\mu^{b'} A_\nu^{c'} =$$

$$= \underline{\partial_\mu (A_\nu^a} + \cancel{\frac{1}{g}\partial_\nu \theta^a} - i \underline{A_\nu^i T^i_{ab} \theta^b}) -$$

$$- \underline{\partial_\nu (A_\mu^a} + \cancel{\frac{1}{g}\partial_\mu \theta^a} - i \underline{A_\mu^i T^i_{ab} \theta^b}) -$$

$$- \underline{ig f^{abc} (A_\mu^b} + \frac{1}{g}\partial_\mu \theta^b - i A_\mu^\alpha T^\alpha_{b\beta} \theta^\beta)(\underline{A_\nu^c} + \frac{1}{g}\partial_\nu \theta^c - i A_\nu^\gamma T^\gamma_{c\delta} \theta^\delta) =$$

$$= F_{\mu\nu}^a - i \boxed{\partial_\mu (A_\nu^i T^i_{ab} \theta^b) - \partial_\nu (A_\mu^i T^i_{ab} \theta^b)} - ig f^{abc}\left(\frac{1}{g}\partial_\mu \theta^b - i A_\mu^\alpha T^\alpha_{b\beta} \theta^\beta\right)\left(\frac{1}{g}\partial_\nu \theta^c - i A_\nu^\gamma T^\gamma_{c\delta} \theta^\delta\right) =$$

$$= \underline{F_{\mu\nu}^a} - i\, \underline{F_{\mu\nu}^i T^i_{ab} \theta^b} - i \underbrace{\left[A_\nu^i T^i_{ab} (\partial_\mu \theta^b) - A_\mu^i T^i_{ab} (\partial_\nu \theta^b)\right]}_{①} - ig \underbrace{f^{abc} (\dots)(\dots)}_{②}$$

so, to show that it transforms in the Adj. rep. we need to show that the last two terms cancel each other when T^i_{ab} is in the adjoint ① = - ② :

$$② = -ig f^{abc}(\ldots)(\ldots) = -ig f^{abc}\Big[\underbrace{\tfrac{1}{g^2}\partial_\mu\theta^b\partial_\nu\theta^c + A_\mu^\alpha A_\nu^\gamma T^\alpha{}_{b\beta}T^\gamma{}_{cd}\,\theta^\beta\theta^\delta}_{=0 \text{ by symmetry}} - \underbrace{\tfrac{i}{g}\left(A_\mu^\alpha T^\alpha_{b\beta}\theta^\beta(\partial_\nu\theta^c) + A_\nu^\gamma T^\gamma_{cd}\theta^\delta(\partial_\mu\theta^b)\right)}\Big] =$$

$$= -i A_\mu^\alpha \underbrace{f^{abc} f^{\alpha b\beta}}_{f^{\alpha ac}}\theta^\beta(\partial_\nu\theta^c) - i A_\nu^\gamma \underbrace{f^{abc} f^{\gamma c\delta}}_{f^{\gamma ab}}\theta^\delta(\partial_\mu\theta^b)$$

$$= -i A^i_{[\nu} f^{iab}\partial_{\mu]}\theta^b$$ which cancel with ①

so finally we see that:

$$\boxed{F_{\mu\nu}^a \longrightarrow F_{\mu\nu}^a - ig\theta^b f^{bai} F^i_{\mu\nu}}$$ which is the adjoint!!!

---

16) Express the Yang-Mills Lagrangian in terms of A_μ:

$$\boxed{\mathcal{L}_{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \quad \left(\text{with } F^a_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc}A_\mu^b A_\nu^c\right)$$

$$= -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc}A_\mu^b A_\nu^c\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + g f^{abc}A^{\mu b}A^{\nu c}\right) =$$

$$= -\frac{1}{4}\Big\{\underbrace{(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})}_{\text{Original non-abelian } \mathcal{L}} + \underbrace{2(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)\, g f^{abc}A^{\mu b}A^{\nu c}}_{4\partial_\mu A_\nu^a\, g f^{abc}A^{\mu b}A^{\nu c}} + g^2 f^{abc}f^{ade}A_\mu^b A_\nu^c A^{\mu d}A^{\nu e}\Big\} =$$

$$= \boxed{-\frac{1}{2}A_\mu^a\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu^a - g f^{abc}(\partial_\mu A_\nu^a)A^{\mu b}A^{\nu c} - g^2 f^{abc}f^{ade}A_\mu^b A_\nu^c A^{\mu d}A^{\nu e}}$$

(Excercice 5))

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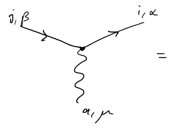
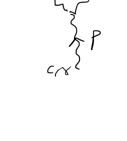
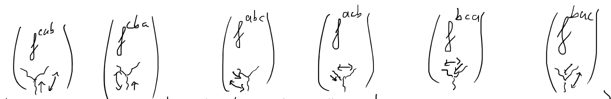
17) Derive the Feynman Rules of the YM + fermion theory:

• $g\bar{\psi}_{i\alpha}A_\mu^a \gamma^\mu_{\alpha\beta} t^a_{ij}\psi_{j\beta} \longrightarrow$ 

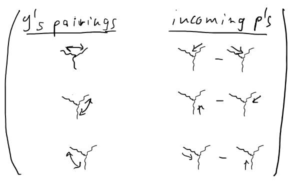
$= \boxed{ig\gamma^\mu_{\alpha\beta}t^a_{ij}}$

• $g f^{eab}(\partial_\mu A_\nu^c)A^{\mu a}A^{\nu b} \longrightarrow$ 



$$= -ig f^{cab}\left\{(p_c^\mu g^{\nu\gamma} - p_c^\nu g^{\mu\gamma}) + (p_a^\nu g^{\mu\gamma} - p_a^\gamma g^{\mu\nu}) + (p_b^\gamma g^{\mu\nu} - p_b^\mu g^{\gamma\nu})\right\} =$$

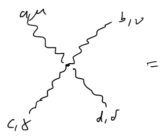
$$= \boxed{-ig f^{cab}\left\{g^{\mu\nu}(p_b - p_a)^\gamma + g^{\nu\gamma}(p_c - p_b)^\mu + g^{\gamma\mu}(p_a - p_c)^\nu\right\}}$$



(◦ Each term is given by $-g f^{cab}(\partial_\mu A_\gamma^c)A^{\mu a}A^{\nu b}g^{\gamma\nu}$ which translates into one of the incoming momentas p, pairing it then with one of the other legs, "$-ig f^{cab} p_c g^{\gamma\nu}$" for example, and then subtracting the other pairing because of f^{cab} antisymmetry. $f^{\square\square\square} \longrightarrow 3! = 6$ combinations)

$$\left(f^{xab} f^{xcd} = f^{xba} f^{xdc} = f^{xcd} f^{xab} = f^{xdc} f^{xba}\right) \quad \left(f^{xab} f^{xdc} = f^{xba} f^{xcd} = f^{xdc} f^{xab} = f^{xcd} f^{xba}\right)$$

• $\frac{g^2}{4} f^{xab} f^{xcd} A^a_\mu A^b_\nu A^{\mu c} A^{\nu d} \longrightarrow$



$$\boxed{= -ig^2\Big\{ f^{xab} f^{xcd}\left(g^{\mu\gamma} g^{\nu\delta} - g^{\mu\delta} g^{\nu\gamma}\right) + f^{xad} f^{xbc}\left(g^{\mu\nu} g^{\gamma\delta} - g^{\mu\gamma} g^{\nu\delta}\right) + f^{xac} f^{xdb}\left(g^{\mu\delta} g^{\nu\gamma} - g^{\mu\nu} g^{\gamma\delta}\right)\Big\}}$$

f's pairing	g's pairing

$$\left(f^{xab} f^{xcd} = f^{xba} f^{xdc} = f^{xcd} f^{xab} = f^{xdc} f^{xba} \longrightarrow \right.$$ factor 4, and for each of these, only switching $f^{x\square\square} f^{x\square\square}$ gives the − term in each term)

- Each term is given by $\frac{g^2}{4} f^{xab} f^{xcd} A^a_\mu A^b_\nu A^c_\gamma A^d_\delta g^{\gamma\mu} g^{\delta\nu}$, which translates into pairing each of the last two indices with one of the first, such as: $\frac{g^2}{4} f^{xab} f^{xcd} g^{\gamma\mu} g^{\delta\nu}$ or $\frac{g^2}{4} f^{xab} f^{xcd} g^{\gamma\nu} g^{\delta\mu}$ for each $f^{x\square\square} f^{x\square\square}$, which translates to: if we have an f's pairing then the g's pairing are the other two.
- In the previous case because we had explicitly an incoming $\vec{p}$ the symmetry was broken, and we didn't got a factor of 3, instead we got the 3 different momentas (p_a, p_b, p_c), with the pairings for each case.
- This time because we didn't had explicit momentas we can just get a factor ~~4~~ cancelling $\frac{g^2}{\cancel{4}}$, and simply not interchange the legs, only do all the possible different pairings for f's and g's.
- $f^{x\square\square} f^{x\square\square} \longrightarrow 4! = 24$ combinations, which groups of 4 are equivalent $\longrightarrow$ 6 diff terms $\cdot \cancel{4}\left(\frac{g^2}{\cancel{4}}\right)$

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18) Add a scalar field in the adjoint representation to $\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi$:

a) Find the extra terms in the Lagrangian:

First we have to add the kinematic term for ϕ, which transforms as:

$$\mathrm{tr}\{m\phi^2\} \longrightarrow \mathrm{Tr}\{m(U\phi U^\dagger)(U\phi U^\dagger)\} = \mathrm{tr}\{m\phi^2\} \quad \text{invariant} \checkmark$$

$$\mathrm{tr}\{(\partial_\mu\phi)^\dagger(\partial^\mu\phi)\} \longrightarrow \mathrm{tr}\{\partial_\mu(U\phi U^\dagger)\,\partial^\mu(U\phi U^\dagger)\} =$$

$$= \mathrm{Tr}\{[(\partial_\mu U)\phi U^\dagger + U(\partial_\mu\phi)U^\dagger + U\phi(\partial_\mu U^\dagger)]\cdot[(\partial^\mu U)\phi U^\dagger + U(\partial^\mu\phi)U^\dagger + U\phi(\partial^\mu U^\dagger)]\} =$$

$$= \mathrm{Tr}\{(\partial_\mu\phi)^\dagger(\partial^\mu\phi) + f(\partial_\mu U(x))\ldots\}$$ we need a covariant derivative that cancels the $\partial_\mu U$ terms!!!

If we check the form of the transformation, we get, which will be useful in a moment:

$$\begin{cases} U = e^{-igT^a\theta^a(x)}, \quad \alpha(x) = T^a\theta^a(x) \\ U^\dagger = e^{+igT^a\theta^a(x)} \end{cases} \longrightarrow \begin{cases} (\partial_\mu U)U^\dagger = -igT^a\partial_\mu\theta^a = -ig(\partial_\mu\alpha) \\ (\partial_\mu U^\dagger)U = +igT^a\partial_\mu\theta^a = +ig(\partial_\mu\alpha) \end{cases}$$

The covariant derivative we will need has the form:

$$D_\mu\phi_a = \partial_\mu\phi_a + g f_{abc} A^b_\mu \phi^c \quad \text{or} \quad D_\mu\phi = \partial_\mu\phi - ig[A_\mu, \phi]$$

as we will see now doing it's transformation:

$$\left(A_\mu' = U A_\mu U^+ - i\frac{1}{g}\underbrace{(\partial_\mu U)U^+}_{\substack{\text{(I'll use this later:)}\\ -(\partial_\mu\alpha) \circledast}} \right)$$

- $D_\mu'\phi' = \partial_\mu\phi' - ig[A_\mu',\phi'] = \partial_\mu(U\phi U^+) - ig[A_\mu', U\phi U^+] =$

$$= \underline{\partial_\mu(U\phi U^+)} - igU[A_\mu,\phi]U^+ - (\partial_\mu U)U^+(U\phi U^+) + (U\phi U^+)(\partial_\mu U)U^+ =$$

$$= \underline{U(\partial_\mu\phi)U^+ + U\phi(\partial_\mu U^+)} - igU[A_\mu,\phi]U^+ + (U\phi U^+)(\partial_\mu U)U^+$$

- $(D'^\mu\phi')^+ = U(\partial^\mu\phi)U^+ + (\partial^\mu U)\phi U^+ + \underbrace{igU[\phi,A^\mu]U^+}_{-igU[A_\mu,\phi]U^+} + U(\partial^\mu U^+)(U\phi U^+)$

then the lagrangian will transform as:

$$tr\{(D^\mu\phi)^+(D_\mu\phi)\} \longrightarrow tr\left\{(\partial^\mu\phi' + ig[A^\mu,\phi]^{+'})(\partial_\mu\phi' - ig[A_\mu,\phi]')\right\} = tr\{(D'^\mu\phi')^+(D_\mu'\phi')\} =$$

$$= tr\Big\{(\partial_\mu\phi)^+(\partial^\mu\phi) + \cancel{U(\partial_\mu\phi)\phi\overset{(-\partial_\mu\alpha)}{(\partial_\mu U^+)}} - ig(\partial_\mu\phi)[A_\mu,\phi] + \cancel{(\partial_\mu\phi)\phi\overset{(\partial_\mu\alpha)}{U^+(\partial_\mu U)}} +$$

$$+ \cancel{\overset{(\partial_\mu\alpha)}{(\partial_\mu U)}\phi(\partial_\mu\phi)U^+} + \cancel{\overset{-(\partial\alpha)^2}{(\partial_\mu U)\phi^2(\partial_\mu U^+)}} - ig\cancel{\overset{(\partial_\mu\alpha)}{(\partial_\mu U)}\phi[A_\mu,\phi]U^+} + \cancel{(\partial_\mu U)\phi^2\overset{(\partial\alpha)^2}{U^+(\partial_\mu U)}U^+} +$$

$$- ig[A_\mu,\phi](\partial_\mu\phi) - ig\cancel{U[A_\mu,\phi]\phi\overset{(-\partial\alpha)}{(\partial_\mu U^+)}} - g^2[A_\mu,\phi]^2 - ig\cancel{[A_\mu,\phi]\phi\overset{(\partial\alpha)}{U^+(\partial_\mu U)}}$$

$$+ \cancel{\underset{(-\partial\alpha)}{(\partial_\mu U^+)}U\phi(\partial_\mu\phi)} + \cancel{U\underset{(\partial\alpha)^2}{(\partial_\mu U^+)}U\phi^2(\partial_\mu U^+)} - ig\cancel{\underset{(-\partial\alpha)}{(\partial_\mu U^+)}U\phi[A_\mu,\phi]} + \cancel{(\partial_\mu U^+)U\phi^2\underset{-(\partial\alpha)^2}{U^+(\partial_\mu U)}}\Big\} =$$

$$= tr\{(\partial_\mu\phi)^+(\partial^\mu\phi) - ig(\partial_\mu\phi)[A^\mu,\phi] - ig[A_\mu,\phi](\partial_\mu\phi) - g^2[A_\mu,\phi]^2\} =$$

$$= tr\{(\partial_\mu\phi - ig[A^\mu,\phi])^+(\partial^\mu\phi - ig[A_\mu,\phi])\} \checkmark \quad \underline{\text{invariant!!!}}$$

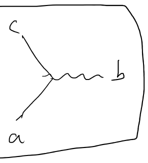
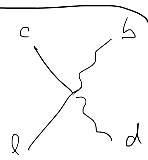
that means we need to add: $D_\mu^{ij} = \delta^{ij}\partial_\mu + g f^{ijk} A_\mu^k$ (adjoint)

Which in new terms for the Lagrangian means:

$$(D_\mu\phi^a)^+(D^\mu\phi^a) = (\partial_\mu\phi^a)(\partial^\mu\phi^a) + \underline{2(\partial^\mu\phi^a)f^{abc}A_\mu^b\phi^c} + \underline{g^2 f^{abc}f^{ade}A_\mu^b A^{\mu d}\phi^c\phi^e}$$

Adjoint form

So the new terms added are:

$\boxed{2gp_a f^{abc} \text{ for }}$  and $\boxed{g^2 f^{abc} f^{ade} \text{ for }}$ 

b) Can this couple to the gauge field with a different coupling $g' \neq g$?

For non-Abelian cases the couplings to a same SU(N) transformation, must be the same $\forall$ representations! Whereas in the Abelian case, we can have different g's for each representation.

(Of course for different $SU(N)$ transformations this wouldn't be the case, because directly we are coupling to different Gauge fields, so for example $SU(3)\times SU(2)$ will have g_3 and g_2 respectively with $g_3 \neq g_2$ and $A_{1,2,3}$, $B_{1,2}$ different gauge fields.)

So in this case $g' = g$, because ψ and ϕ are transforming under the same $SU(N)$ symmetry, in a <u>non-Abelian case</u>, so <u>all representations have the same coupling g</u>.

---

19) Show step (I), from:

$$\left(\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \frac{1}{\cancel{Z_1[0]}}\left(\frac{-i\delta}{\delta J(x)}\right)\left(\frac{-i\delta}{\delta J(y)}\right)\left\{\cancel{Z_1[0]}\, e^{-\frac{1}{2}JD_FJ}\right\}\Bigg|_{J=0} = \quad \left(JD_FJ = \int d^4x' d^4y'\, J(x')\, D_F(x'-y')\, J(y')\right) \right.$$

$$\left. = -\frac{\delta}{\delta J(x)}\left\{-\frac{1}{2}e^{-\frac{1}{2}JD_FJ}\left[\int d^4y'\left(D_F(y-y')+D_F(y'-y)\right)J(y')\right]\right\}\Bigg|_{J=0} \overset{\mathrm{I}}{=} \dots = D_F(x-y) \right)$$

$$\frac{1}{2}\frac{\delta}{\delta J(x)}\left\{e^{-\frac{1}{2}JD_FJ}\left[\overbrace{\int d^4y'\left(D_F(y-y')+D_F(y'-y)\right)J(y')}^{I(y,J)}\right]\right\}\Bigg|_{J=0} =$$

$$= \frac{1}{2}e^{-\frac{1}{2}JD_FJ}\left\{\left[D_F(y-x)+D_F(x-y)\right]-\frac{1}{2}I(y,J)\,I(x,J)\right\}\Bigg|_{J=0} =$$

$$= \frac{1}{2}e^{-0}\left\{\left[D_F(y-x)+D_F(x-y)\right]-\left(\int d^4y'\, 0\right)^2\right\} = \frac{1}{2}\left(D_F(y-x)+D_F(x-y)\right) = \boxed{D_F(x-y)}$$

---

20) Show that for N Grassman variables ψ^i and a matrix B^{ij}:

$$\int \prod_i d\bar\psi^i d\psi^i\, e^{-\bar\psi_i B_{ij}\psi_j} \propto \det B \qquad \left(N=2 \text{ case } \int d\bar\psi_1 d\psi_1 d\bar\psi_2 d\psi_2\, e^{-\bar\psi_1 B_{11}\psi_1 - \bar\psi_1 B_{12}\psi_2 - \bar\psi_2 B_{21}\psi_1 - \bar\psi_2 B_{22}\psi_2}\right)$$

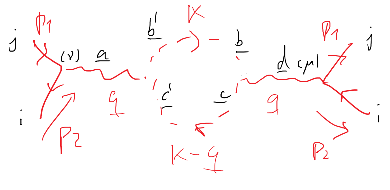
$$\boxed{\int \prod_i d\bar\psi^i d\psi^i\, e^{-\bar\psi_i B_{ij}\psi_j}} \overset{\circledast}{=} \det(B)\int \prod_i d\bar\psi^i d\psi'^i\, e^{-\bar\psi_i\psi'_i} = \det(B)\int \prod_i d\bar\psi^i d\psi'^i\left(1-\bar\psi_i\psi'_i\right) =$$

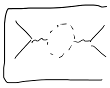
$$= \det(B)\left(\int \prod_i \cancelto{0}{d\bar\psi^i} d\psi'^i - \int \prod_i d\bar\psi^i \cancelto{-1}{d\psi'^i}\, \bar\psi_i\psi'_i\right) = \det(B)\,(0+1) = \boxed{\det(B)}$$

$$\left(\overset{\circledast}{} \int \prod_i d\bar\psi^i d\psi^i \underset{(B \text{ is the Jacobian matrix})}{\overset{(\text{if } \psi'^j = B_{ji}\psi^i)}{=}} \det(B)\int \prod_i d\bar\psi^i d\psi'^i\right)$$

---

21) Check explicitly the imaginary part of:





$$= \int \frac{d^D K}{(2\pi)^D} \left(ig\,\bar{\Psi}_i(p_2)\gamma^\mu T^d_{ij}\Psi_j(p_1)\right)\left(-g f^{cdb}(K-q)_\mu\right)\left(\frac{i\delta^{cc'}}{(K-q)^2}\right)\left(\frac{i\delta^{bb'}}{K^2}\right)\left(-g f^{b'ac'}K_\nu\right)\left(ig\,\Psi_j(p_1)\gamma^\nu T^a_{ji}\bar{\Psi}_i(p_2)\right) =$$

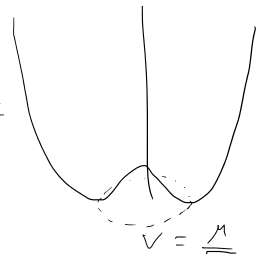
$$= g^4 \int \frac{d^D K}{(2\pi)^D} \frac{\bar{\Psi}_i(p_2)\,(\not K-\not q)\,T^d_{ij}\Psi_j(p_1)\overbrace{f^{cdb}f^{bac}}^{-N\delta^{ad}}\Psi_j(p_1)\not K\,T^a_{ji}\bar{\Psi}_i(p_2)}{(K-q)^2\,K^2} = -Ng^4\int\frac{d^DK}{(2\pi)^D}\frac{\bar{\Psi}(\not K-\not q)T^a\Psi\,\Psi\not K T^a\bar{\Psi}}{(K-q)^2\,K^2} =$$

$$= -Ng^4\,\overbrace{\bar{\Psi}\gamma_\mu T^a\Psi}^{J^a_\mu}\,\overbrace{\Psi\gamma_\nu T^a\bar{\Psi}}^{J^{+a}_\nu}\underbrace{\int\frac{d^DK}{(2\pi)^D}\frac{(K-q)^\mu K^\nu}{(K-q)^2K^2}}_{\frac{g^{\mu\nu}}{96\pi q^2}} = -Ng^4 J^a_\mu J^{+a}_\nu \frac{g^{\mu\nu}}{96\pi q^2} = \boxed{\frac{-Ng^4 J^a_\mu J^{+\mu a}}{96\pi q^2}}$$

---

22) Consider the linear sigma model Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a)(\partial^\mu\phi^a) + \underbrace{\frac{1}{2}\mu^2\phi^a\phi^a - \frac{\lambda}{4}(\phi^a\phi^a)^2}_{U(\phi^a)}$$

$U(\phi) =$ 

$v = \frac{\mu}{\sqrt{\lambda}}$

a) Find $v^2 = \phi_0^2$:

$$\partial_{|\phi|^2}\left(\frac{1}{2}\mu^2\phi^a\phi^a - \frac{\lambda}{4}(\phi^a\phi^a)^2\right) = 0 \longrightarrow \boxed{\phi_0^2 = \frac{\mu^2}{\lambda}}$$

$$\left(\frac{\partial(ax^2 - bx^4)}{\partial x^2} = a - 2bx^2 = 0 \longrightarrow x^2 = \frac{a}{2b}\right)$$

b) Take $\phi_0 = (0,0,\dots,0,v)$, write the Lagrangian in terms of the new components and find the masses:

$$\phi_0 = \left(0,\dots,0,\frac{\mu}{\sqrt{\lambda}}\right) \quad \text{with} \quad \hat{\phi}(x) = \Big(\overbrace{\phi_1,\dots,\phi_{n-1}}^{\pi_i},\ \overbrace{\phi_n - \frac{\mu}{\sqrt{\lambda}}}^{\sigma(x)}\Big)$$

- $\phi^a\phi^a = \pi^2 + \left(\sigma + \frac{\mu}{\sqrt{\lambda}}\right)^2 = \pi^2 + \sigma^2 + \frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma$
- $(\phi^a\phi^a)^2 = \pi^4 + \left(\sigma + \frac{\mu}{\sqrt{\lambda}}\right)^4 + 2\pi^2\left(\sigma + \frac{\mu}{\sqrt{\lambda}}\right)^2 =$

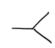
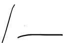
$$= \pi^4 + \left(\sigma^4 + 4\sigma^3\frac{\mu}{\sqrt{\lambda}} + 6\sigma^2\frac{\mu^2}{\lambda} + 4\sigma\frac{\mu^3}{\lambda\sqrt{\lambda}} + \frac{\mu^4}{\lambda^2}\right) + 2\pi^2\sigma^2 + 4\pi^2\sigma\frac{\mu}{\sqrt{\lambda}} + 2\pi^2\frac{\mu^2}{\lambda}$$

$$U = \frac{\mu^2}{2}\phi^a\phi^a - \frac{\lambda}{4}(\phi^a\phi^a)^2 =$$

$$= -\frac{\lambda}{4}(\pi^4+\sigma^4) - \mu\sqrt{\lambda}\,\sigma^3 + \left(\frac{\mu^2}{2} - \frac{3\mu^2}{2}\right)\sigma^2 + \cancelto{0}{\left(\frac{\mu^3}{\sqrt{\lambda}} - \frac{\mu^3}{\sqrt{\lambda}}\right)}\sigma + \left(\frac{\mu^4}{2\lambda} - \frac{\mu^4}{4\lambda}\right) - \frac{\lambda}{2}\pi^2\sigma^2 - \mu\sqrt{\lambda}\,\pi^2\sigma + \cancelto{0}{\left(\frac{\mu^2}{2} - \frac{\mu^2}{2}\right)}\pi^2 =$$

$$= -\frac{\lambda}{4}(\pi^4 + \sigma^4 + 2\pi^2\sigma^2) - \mu\sqrt{\lambda}(\sigma^3 + \pi^2\sigma) - \mu^2\sigma^2 + \frac{\mu^4}{4\lambda}$$

(— = σ ; - - - = π)

 $= \frac{-i\lambda}{4}$ /  $= -\frac{i\lambda}{4}$ /  $= -\frac{i\lambda}{2}$ /  $= i\mu\sqrt{\lambda}$ /  $= i\mu\sqrt{\lambda}$ /  $= \mu^2$

So finally the Lagrangian will be:

$$\mathcal{L}(\pi^i,\sigma) = \frac{1}{2}\left\{(\partial_\mu\pi^i)(\partial^\mu\pi^i) + (\partial_\mu\sigma)(\partial^\mu\sigma)\right\} - \mu^2\sigma^2 - \mu\sqrt{\lambda}(\sigma^3 + \pi^2\sigma) - \frac{\lambda}{4}(\pi^4 + \sigma^4 + 2\pi^2\sigma^2)$$

where the masses are: $\boxed{m_a = \begin{cases} 0 & \text{for } \pi^i \\ \sqrt{2}\mu & \text{for } \sigma \end{cases}}$ $\left(-\frac{1}{2}m^2 = -\mu^2\ ;\ m^2 = 2\mu^2\right)$

23) Work out the masses of the 3 Gauge bosons, and the remaining symmetries after SSB of:

a) Scalar field in the fundamental of SU(2), with $\Phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix}$ [spinor representation]:

If ϕ is in the fundamental the covariant derivative will be:

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha - ig A_\mu^a T^a_{fund\,\alpha\kappa} \phi_\kappa = \partial_\mu \phi_\alpha - i\frac{g}{2} A_\mu^a \sigma^a_{\alpha\kappa} \phi_\kappa$$

When we add the VEV ($\hat{\phi} = \phi - \phi_0$), the coupling terms becomes:

$$|D_\mu \phi|^2 = |D_\mu \hat{\phi}|^2 + (D_\mu \phi_0)^\dagger (D^\mu \hat{\phi}) + (D_\mu \hat{\phi})^\dagger (D^\mu \phi_0) + \underline{|D_\mu \phi_0|^2}$$

Where the only term that contains only 2 gauge bosons (future mass term) is the last one:

(Symmetrized due to $A_\mu^a A^{\mu b}$) $\qquad (\{T^a, T^b\} = \frac{1}{2}\delta^{ab})$

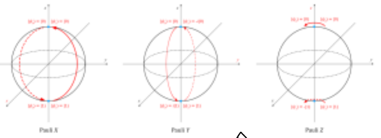
$$|D_\mu \phi_0|^2 = \frac{1}{2} g^2 (0\ v)\, T^a T^b \begin{pmatrix}0\\v\end{pmatrix} A_\mu^a A^{\mu b} = \frac{1}{2} g^2 (0\ v) \frac{T^a T^b + T^b T^a}{2} \begin{pmatrix}0\\v\end{pmatrix} A_\mu^a A^{\mu b} =$$

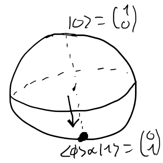
$$= \frac{1}{8} g^2 (0\ v)\, \delta^{ab} \begin{pmatrix}0\\v\end{pmatrix} A_\mu^a A^{\mu b} = \frac{g^2}{8} v^2 (A_\mu^a)^2$$

Which compared with a typical mass term ($\frac{1}{2} m_{ab}^2 A_\mu^a A^{\mu b}$):

$$m_a^2 = 2\left(\frac{g^2}{8} v^2\right) \longrightarrow \boxed{m_a = \frac{gv}{2} \quad \forall a}$$

So all Gauge boson adquire a mass as we expected.





$$\begin{cases} \frac{\sigma_x}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \\ \frac{\sigma_y}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-i\\0\end{pmatrix} \\ \frac{\sigma_z}{2}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix} \end{cases}$$

No generator leaves $\langle\phi\rangle$ invariant!

All 3 generators are SSB!

- Actually $\forall \phi_0 \neq 0$ all 3 generators would break since neither has an eigen value $= 1$. (Only $T^a \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$)
- If we don't substitute ϕ_0 in $|D_\mu \phi_0|^2 = \frac{1}{8} g^2 \phi_0^\dagger \phi_0 (A_\mu^a)^2 = \frac{g^2}{8} |\phi_0|^2 (A_\mu^a)^2$, so $m_a = 0$ only if $\phi_0 = 0$

b) Scalar field in the adjoint with $\Phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\v\end{pmatrix}$ [vector representation]:

If ϕ is in the adjoint the covariant derivative will be:

$(T_{adj}{}^a{}_{\alpha\kappa} = -i f^{a\alpha\kappa} = -i\,\epsilon_{a\alpha\kappa})$

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha - ig A_\mu^a T_{adj\,\alpha\kappa} \phi_\kappa = \partial_\mu \phi_\alpha - g\,\epsilon_{a\alpha\kappa} A_\mu^a \phi_\kappa$$

When we add the VEV ($\hat{\phi} = \phi - \phi_0$), the coupling terms becomes:

$$|D_\mu \phi|^2 = |D_\mu \hat{\phi}|^2 + (D_\mu \phi_0)^\dagger (D^\mu \hat{\phi}) + (D_\mu \hat{\phi})^\dagger (D^\mu \phi_0) + \underline{|D_\mu \phi_0|^2}$$

Where the only term that contains only 2 gauge bosons (future mass term) is the last one:

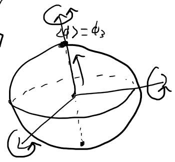
$$|D_\mu \phi_0|^2 = \frac{1}{2} g^2 (0\,0\,v)\, T^a T^b \begin{pmatrix}0\\0\\v\end{pmatrix} A_\mu^a A^{\mu b} = \frac{g^2}{2} (0\,0\,v)_\kappa \overbrace{\epsilon_{a\alpha\kappa}\epsilon_{b\alpha\ell}}^{\delta_{ab}\delta_{\kappa\ell} - \delta_{a\ell}\delta_{\kappa b}} \begin{pmatrix}0\\0\\v\end{pmatrix}_\ell A_\mu^a A^{\mu b} =$$

$$= \frac{g^2}{2}\left(\delta_{ab} v^2 - (0\,0\,v)_b \begin{pmatrix}0\\0\\v\end{pmatrix}_a\right) A_\mu^a A^{\mu b} = \frac{g^2}{2} v^2 \left((A_\mu^1)^2 + (A_\mu^2)^2\right)$$

Which compared with a typical mass term ($\frac{1}{2} m_{ab}^2 A_\mu^a A^{\mu b}$):

$$m_1^2 = m_2^2 = g^2 v^2 \longrightarrow \boxed{\begin{cases} m_{1,2} = gv \\ m_3 = 0 \end{cases}}$$

So all Gauge boson except the direction of the VEV gained mass, as expected.



$$\begin{cases} R_1\begin{pmatrix}0\\0\\v\end{pmatrix} \neq \begin{pmatrix}0\\0\\v\end{pmatrix} \\ R_2\begin{pmatrix}0\\0\\v\end{pmatrix} \neq \begin{pmatrix}0\\0\\v\end{pmatrix} \\ R_3\begin{pmatrix}0\\0\\v\end{pmatrix} = \begin{pmatrix}0\\0\\v\end{pmatrix}\end{cases}$$

Only the rotation in the same axis than $\langle\phi\rangle$ leave it invariant!

Only one unbroken and 2 broken

- Actually $\forall \phi_0 \neq 0$, we will break all the generators except the one in the direction we move the vacua!
- If we don't substitute ϕ_0 we get:

$$|D_\mu \phi_0|^2 = g^2(\delta_{ab}|\phi_0|^2 - \phi_b\phi_a)\, A_\mu^a A^{\mu b} = \frac{g^2}{2}(v^2 - \phi_{0i}^2)(A_\mu^i)^2$$

which means $m_i = 0$ only in the direction ϕ_{0i}.

---

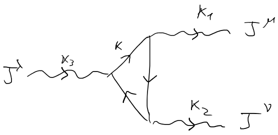
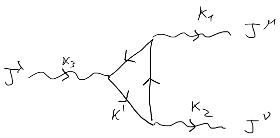
24) Show if the correlation function of 3 vector$^{a)}$ / axial$^{b)}$ currents have an anomaly or not:

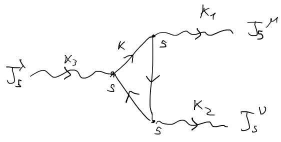
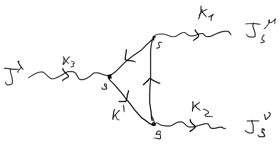
$$\begin{cases}\Delta_V^{\mu\nu\lambda} \equiv \langle 0|\, T\{J^\mu J^\nu J^\lambda\}\,|0\rangle & \text{WI} \checkmark_{a)} \\ \Delta_A^{\mu\nu\lambda} \equiv \langle 0|\, T\{J_5^\mu J_5^\nu J_5^\lambda\}\,|0\rangle & \text{anomaly !!}_{b)}\end{cases}$$

which in momenta space are:

$$\tilde{\Delta}_X^{\mu\nu\lambda} = \int d^4x_1\, d^4x_2\, d^4x_3\; e^{-i(k_1x_1 + k_2x_2 + k_3x_3)}\; \Delta_X^{\mu\nu\lambda} \qquad \text{(to momentum space)}$$

and represent the following diagrams (at lowest order):

$\tilde{\Delta}_V^{\mu\nu\lambda} =$  $+$  $= \tilde{\Delta}_{V_1}^{\mu\nu\lambda} + \tilde{\Delta}_{V_2}^{\mu\nu\lambda}$

$\tilde{\Delta}_A^{\mu\nu\lambda} =$  $+$  $= \tilde{\Delta}_{A_1}^{\mu\nu\lambda} + \tilde{\Delta}_{A_2}^{\mu\nu\lambda}$

b) Let's start by the 3 axial currents:

$$\tilde{\Delta}_{A_1}^{\mu\nu\lambda} = i\int dk\, \mathrm{Tr}\left[\gamma^\lambda\gamma_5 \frac{1}{\not{k} - \not{k}_3}\gamma^\nu \gamma_5 \frac{1}{\not{k} - \not{k}_1}\gamma^\mu\gamma_5 \frac{1}{\not{k}}\right]$$

new!

$$= -i\int dk\, \mathrm{Tr}\left[\gamma^\lambda\gamma_5 \frac{1}{\not{k} - \not{k}_3}\gamma^\nu \gamma_5 \frac{1}{\not{k} - \not{k}_1}\gamma_5\gamma^\mu \frac{1}{\not{k}}\right] \qquad (\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu)$$

$$\tilde{\Delta}_{A_2}^{\mu\nu\lambda} = i\int dk\, \mathrm{Tr}\left[\gamma^\lambda\gamma_5 \frac{1}{\not{k}' - \not{k}_3}\gamma^\mu \gamma_5 \frac{1}{\not{k}' - \not{k}_1}\gamma^\nu\gamma_5 \frac{1}{\not{k}}\right]$$

$$= -i\int dk\, \mathrm{Tr}\left[\gamma^\lambda\gamma_5 \frac{1}{\not{k}' - \not{k}_3}\gamma^\mu \gamma_5 \frac{1}{\not{k}' - \not{k}_1}\gamma_5\gamma^\nu \frac{1}{\not{k}}\right] \qquad (\gamma^\nu\gamma_5 = -\gamma_5\gamma^\nu)$$

so if I can show that:

$$\gamma_5 \frac{1}{\not{a}}\gamma_5 \propto -\frac{1}{\not{a}}$$

we will have the same as in class, thus an anomaly. Let's show it then:

$$\boxed{\gamma_5 \frac{1}{\not{a}}\gamma_5} = \frac{1}{a^\mu}\left(\gamma_5 \frac{1}{\gamma_\mu}\gamma_5\right) \overset{\circledast}{=} \frac{1}{a^\mu}\left(\gamma^5\gamma^\mu\gamma_5\right) =$$

$$= \frac{-1}{a^\mu}\left(\cancel{\gamma^5\gamma^5}\,\gamma^\mu\right) = -\frac{1}{a^\mu}\gamma^\mu = -\frac{1}{a^\mu\gamma_\mu} = \boxed{-\frac{1}{\not{a}}}$$

$$\circledast \quad \left.\begin{aligned} \frac{1}{\not{a}} &= \frac{1}{a_\mu\gamma^\mu} \\ \frac{1}{\not{a}} &= \left(\frac{1}{a}\right)_\mu \gamma^\mu = \left(\frac{1}{a^\mu}\right)\gamma^\mu \end{aligned}\right\} \frac{1}{\gamma^\mu} = \gamma_\mu$$

Summarazing:

$$\mathrm{tr}\left(\gamma^\lambda\gamma_5\,\gamma^\alpha\gamma^\nu\gamma^5\gamma^\beta\gamma^\mu\gamma^5\gamma^\delta\right) = \mathrm{tr}\left(\gamma^\lambda\gamma_5\,\gamma^\alpha\gamma^\nu\underbrace{\cancel{\gamma^5}\cancel{\gamma^5}}_{\mathbb{1}}\gamma^\beta\gamma^\mu\gamma^\delta\right) = \mathrm{tr}\left(\gamma^\lambda\gamma_5\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\gamma^\delta\right)$$

(2 new γ^5 compared with class) (commute 2 times)

same as in class $\Longleftrightarrow$ anomally!!

a) And now let's do the 3 vector currents:

Furry's theorem

From Wikipedia, the free encyclopedia

In quantum electrodynamics, **Furry's theorem** states that if a Feynman diagram consists of a closed loop of fermion lines connected to an odd number of vertices, its contribution vanishes. As a corollary, no single photon can arise or be destroyed from the vacuum state.[1] The theorem was first derived by Wendell H. Furry in 1937,[2] as a direct consequence of the conservation of energy and charge invariance (C-symmetry).

Furry's theorem is based on the invariance of the vacuum under charge conjugation and the symmetry of the photon-fermion vertex under such. It is therefore not valid for non-Abelian gauge theories in which C-odd contributions also occur. For example, a scattering of three real gluons is not forbidden in quantum chromodynamics, but is instead proportional to the structure constant of the associated Lie algebra.[3]

(a) (b)

Figure 1: Lowest-order Feynman diagrams of $Z \to \gamma\gamma$ transition.

where $\tilde{T}^{\mu\nu\lambda}(k, k')$ is the tensor corresponding to diagram 1a.

Z boson couples with fermions via both vector and axial currents of the form

$$(g_V\bar{\psi}\gamma^\mu\psi + g_A\bar{\psi}\gamma^\mu\gamma^5\psi)Z_\mu. \qquad (2.3)$$

where g_V and g_A are coupling constants. It follows from the Furry's theorem [6, §79], that proportional to the vector coupling part of the amplitude equals zero. Thus, we need to consider only axial coupling. The most general representation of the latter is [7][1]

So what all of this is telling us, is that the odd-C diagrams will cancel each other when we take the two contributions, in our case the odd-C diagrams will be:

$$J_V^\mu = \bar{\psi}\gamma^\mu\psi \xrightarrow{C} -J_V^\mu \quad \text{odd under charge conjugation}$$

$$J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \longrightarrow +J_A^\mu \quad \text{even under charge conjugation}$$

Example: $\psi \propto \begin{pmatrix}1\\0\end{pmatrix}$ (2D)

$$\to J_V^\mu \propto (0\;1)\gamma^\mu\begin{pmatrix}1\\0\end{pmatrix} = -(1\;0)\gamma^\mu\begin{pmatrix}0\\1\end{pmatrix} = -C J_V^\mu$$

$$\to J_A^\mu \propto (0\;1)\gamma^\mu\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = (1\;0)\gamma^\mu\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = +C J_A^\mu$$

$$\Downarrow$$

a) $\Delta_V^{\mu\nu\lambda} \equiv \langle 0|T\{J^\mu J^\nu J^\lambda\}|0\rangle$ odd under C $\longrightarrow$  = −  $\longrightarrow$ $\boxed{\text{Vanishes } \Delta_V = \Delta_{V_1} + \Delta_{V_2} = 0}$

$(\Delta_{V_1} = -\Delta_{V_2})$

b) $\Delta_A^{\mu\nu\lambda} \equiv \langle 0|T\{J_5^\mu J_5^\nu J_5^\lambda\}|0\rangle$ even under C $\longrightarrow$ $\Delta_{A_1} = \Delta_{A_2} \longrightarrow \Delta_A = \Delta_{A_1} + \Delta_{A_2} = 2\Delta_{A_1} \neq 0$

Not need to vanish

So we have seen that $\Delta_V^{\mu\nu\lambda} = 0$