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Week 6:

1. Calculate the decay width of the $f_0(500)$ to two pions (both charged and neutral) in the linear sigma model.

$$\mathcal{L}_{\text{int}} = \frac{1}{f_\pi} S \partial_\mu \vec{\pi} \partial^\mu \vec{\pi} - \frac{m_\pi^2}{2 f_\pi} S \vec{\pi}^2$$

Take $f_\pi = 92$ MeV and $m_\pi = 140$ MeV. Discuss which values of m_{f_0} in the allowed PDG range provide values of the decay width compatible with the range given in the PDG, if any.

Linear model:

$$\underset{\{\text{Real K.G.}\}}{S \longrightarrow \pi^0 \pi^0} \quad ; \quad \underset{\{\text{Complex K.G.}\}}{S \longrightarrow \pi^+ \pi^-}$$

With fields:

$$\vec{\pi} = (\pi^1, \pi^2, \pi^3) \quad ; \quad \begin{cases} \pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \mp i\pi^2) & (\pi^- = (\pi^+)^+) \\ \pi^0 = \pi^3 \end{cases} \quad ; \quad f_0(500) = S$$

The decay width are given by:

$$\begin{cases} \Gamma(S \to \pi^0\pi^0) = \dfrac{|\mathcal{M}_0|^2}{16\pi\, m_{f_0}} \sqrt{1 - \dfrac{4 m_\pi^2}{m_{f_0}^2}} \\ \Gamma(S \to \pi^+\pi^-) = \dfrac{|\mathcal{M}_\pm|^2}{16\pi\, m_{f_0}} \sqrt{1 - \dfrac{4 m_\pi^2}{m_{f_0}^2}} \end{cases}$$

so we need to compute the amplitudes $|\mathcal{M}_0|^2, |\mathcal{M}_\pm|^2$, starting by first order:

$$i\mathcal{M}_0^{(1)} = \langle \underbrace{\vec{p}_{\pi^0} \vec{k}_{\pi^0}} | \, i\mathcal{L}_{int}(\vec{0}) \, | \underbrace{\vec{p}\,'_S} \rangle = \qquad (\vec{p}\,' = \vec{p} + \vec{k})$$

$$= i \overbrace{\langle 00 | a_{\pi^0}(\vec{p})\, a_{\pi^0}(\vec{k})\, 2\sqrt{E_p E_k}} \left(\frac{1}{f_\pi} S (\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi}) - \frac{m_\pi^2}{2 f_\pi} S \vec{\pi}^2 \right)\Bigg|_{\vec{0}} \overbrace{\sqrt{2E_{p'}}\, a_S^\dagger(\vec{p}\,') | 0 \rangle}$$

$$= \frac{2}{f_\pi} \sqrt{2 E_p E_k E_{p'}} \Big\{ \langle 00 | a_{\pi^0}(\vec{p})\, a_{\pi^0}(\vec{k})\, S (\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi})\, a_S^\dagger(\vec{p}\,') | 0 \rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})\, S\,\vec{\pi}\vec{\pi}\, a_s^+(\vec{p}\,')|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$= \frac{2}{f_\pi}\sqrt{2E_pE_kE_{p'}}\Big\{\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})\, S\, a_s^+(\vec{p}\,')(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi})|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})\, S\, a_s^+(\vec{p}\,')\,\vec{\pi}\vec{\pi}|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$= \frac{2}{f_\pi}\sqrt{2E_pE_kE_{p'}}\Big\{\cancelto{0}{\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})a_s^+(\vec{p}\,')} S(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi})|0\rangle +$$

$$+\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})[S, a_s^+(\vec{p}\,')](\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi})|0\rangle -$$

$$-\frac{m_\pi^2}{2}\cancelto{0}{\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})a_s^+(\vec{p}\,')} S\,\vec{\pi}\vec{\pi}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})[S, a_s^+(\vec{p}\,')]\,\vec{\pi}\vec{\pi}|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$= \frac{2}{f_\pi}\sqrt{2E_pE_kE_{p'}}\Big\{\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})[S, a_s^+(\vec{p}\,')](\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi})|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})[S, a_s^+(\vec{p}\,')]\,\vec{\pi}\vec{\pi}|0\rangle\Big\}\Big|_{\vec{0}}$$

So now we need to compute $[S, a_s^+(\vec{p}\,')]$:

$$[S, a_s^+(\vec{p}\,')] = \int\frac{d^3\ell}{\cancel{(2\pi)^3}\sqrt{2E_\ell}}\Big(\underbrace{[a_s(\vec{\ell}), a_s^+(\vec{p}\,')]}_{\cancel{(2\pi)^3}\delta^3(\vec{\ell}-\vec{p}\,')} e^{-i\ell x} + \cancelto{0}{[a_s^+(\vec{\ell}), a_s^+(\vec{p}\,')]} e^{i\ell x}\Big) = \frac{e^{-ip'x}}{\sqrt{2E_{p'}}}$$

Let's continue computing M_0:

$$M_0^{(1)} = \frac{2}{f_\pi}\sqrt{2E_pE_kE_{p'}}\Big\{\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})\frac{e^{-ip'x}}{\sqrt{2E_{p'}}}(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi})|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})\frac{e^{-ip'x}}{\sqrt{2E_{p'}}}\,\vec{\pi}\vec{\pi}|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$= \frac{2}{f_\pi}\sqrt{E_pE_k}\,\underbrace{e^{-ip'\vec{0}}}_{1}\Big\{\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi})|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|a_{\pi^0}(\vec{p})a_{\pi^0}(\vec{k})\,\vec{\pi}\vec{\pi}|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$= \frac{2}{f_\pi}\sqrt{E_p E_K}\Big\{\langle 00|\, a_{\pi^0}(\vec{p})\,(\partial_\mu\vec{\pi})\, a_{\pi^0}(\vec{K})\,(\partial^\mu\vec{\pi})|0\rangle +$$

$$+\langle 00|\, a_{\pi^0}(\vec{p})\,[a_{\pi^0}(\vec{K}), \partial_\mu\vec{\pi}]\,(\partial^\mu\vec{\pi})|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\, a_{\pi^0}(\vec{p})\,\vec{\pi}\, a_{\pi^0}(\vec{K})\,\vec{\pi}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\, a_{\pi^0}(\vec{p})\,[a_{\pi^0}(\vec{K}),\vec{\pi}]\,\vec{\pi}|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$= \frac{2}{f_\pi}\sqrt{E_p E_K}\Big\{\langle 00|\, a_{\pi^0}(\vec{p})\,\cancel{(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi})\, a_{\pi^0}(\vec{K})|0\rangle}^{\,0} +$$

$$+\langle 00|\, a_{\pi^0}(\vec{p})\,(\partial_\mu\vec{\pi})\,[a_{\pi^0}(\vec{K}), (\partial^\mu\vec{\pi})]|0\rangle +$$

$$+\langle 00|\, \cancel{[a_{\pi^0}(\vec{K}), \partial_\mu\vec{\pi}]\,(\partial^\mu\vec{\pi})\, a_{\pi^0}(\vec{p})|0\rangle}^{\,0} +$$

$$+\langle 00|\, [a_{\pi^0}(\vec{K}), \partial_\mu\vec{\pi}]\,[a_{\pi^0}(\vec{p}), (\partial^\mu\vec{\pi})]|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\, \cancel{a_{\pi^0}(\vec{p})\,\vec{\pi}\,\vec{\pi}\, a_{\pi^0}(\vec{K})|0\rangle}^{\,0} -$$

$$-\frac{m_\pi^2}{2}\langle 00|\, a_{\pi^0}(\vec{p})\,\vec{\pi}\,[a_{\pi^0}(\vec{K}),\vec{\pi}]|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\, \cancel{[a_{\pi^0}(\vec{K}),\vec{\pi}]\,\vec{\pi}\, a_{\pi^0}(\vec{p})|0\rangle}^{\,0} -$$

$$-\frac{m_\pi^2}{2}\langle 00|\, [a_{\pi^0}(\vec{K}),\vec{\pi}]\,[a_{\pi^0}(\vec{p}),\vec{\pi}]|0\rangle\Big\}\Big|_{\vec{0}} =$$

Because $[a_{\pi^0}(\vec{K}), (\partial^\mu\vec{\pi})] = \partial^\mu[a_{\pi^0}(\vec{K}),\vec{\pi}]$, we only need to compute one commutator, which gives 0 for the first and second components of $\vec{\pi}$ and the same as the previously computed for the third component (both S and π^0 are Real K.G. fields):

$$[a_{\pi^0}(\vec{K}),\vec{\pi}] = \begin{pmatrix} 0 \\ 0 \\ \frac{e^{iKx}}{\sqrt{2E_K}} \end{pmatrix} \quad ; \quad [a_{\pi^0}(\vec{K}), (\partial^\mu\vec{\pi})] = iK^\mu \begin{pmatrix} 0 \\ 0 \\ \frac{e^{iKx}}{\sqrt{2E_K}} \end{pmatrix}$$

So we end up with:

$$\mathcal{M}_a^{(1)} = \frac{2}{f_\pi}\sqrt{E_p E_K}\Big\{\langle 00|\, a_{\pi^0}(\vec{p})\,(\partial_\mu\vec{\pi})\, iK^\mu \begin{pmatrix} 0 \\ 0 \\ \frac{e^{iKx}}{\sqrt{2E_K}} \end{pmatrix}|0\rangle +$$

$$+\langle 00|\,(iK_\mu)(ip^\mu)\begin{pmatrix} 0 \\ 0 \\ \frac{e^{iKx}}{\sqrt{2E_K}} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \frac{e^{ipx}}{\sqrt{2E_p}} \end{pmatrix}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\, a_{\pi^0}(\vec{p})\,\vec{\pi}\begin{pmatrix}0\\0\\ \frac{e^{ikx}}{\sqrt{2E_K}}\end{pmatrix}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\begin{pmatrix}0\\0\\ \frac{e^{ikx}}{\sqrt{2E_K}}\end{pmatrix}\begin{pmatrix}0\\0\\ \frac{e^{ipx}}{\sqrt{2E_p}}\end{pmatrix}|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$=\frac{2}{f_\pi}\sqrt{E_pE_K}\Big\{\langle 00|\big[a_{\pi^0}(\vec{p}),(\partial_\mu\pi^3)\big]^{\circledast}(iK^\mu)\frac{e^{ikx}}{\sqrt{2E_K}}|0\rangle +$$

($\circledast$ As we have seen $a|0\rangle = 0$)

$$+\langle 00|(iK_\mu)(ip^\mu)\frac{e^{ikx}}{\sqrt{2E_K}}\frac{e^{ipx}}{\sqrt{2E_p}}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\big[a_{\pi^0}(\vec{p}),\pi^3\big]^{\circledast}\frac{e^{ikx}}{\sqrt{2E_K}}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\frac{e^{ikx}}{\sqrt{2E_K}}\frac{e^{ipx}}{\sqrt{2E_p}}|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$=\frac{2}{f_\pi}\sqrt{E_pE_K}\Big\{\langle 00|(ip_\mu)\frac{e^{ipx}}{\sqrt{2E_p}}(iK^\mu)\frac{e^{ikx}}{\sqrt{2E_K}}|0\rangle +$$

$$+\langle 00|(iK_\mu)(ip^\mu)\frac{e^{ikx}}{\sqrt{2E_K}}\frac{e^{ipx}}{\sqrt{2E_p}}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\frac{e^{ipx}}{\sqrt{2E_p}}\frac{e^{ikx}}{\sqrt{2E_K}}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\frac{e^{ikx}}{\sqrt{2E_K}}\frac{e^{ipx}}{\sqrt{2E_p}}|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$=\frac{\cancel{2}}{f_\pi}\cancel{\sqrt{E_pE_K}}\,\frac{\overset{1}{\cancel{e^{ik\vec{0}}}}}{\cancel{\sqrt{2E_K}}}\frac{\overset{1}{\cancel{e^{ip\vec{0}}}}}{\cancel{\sqrt{2E_p}}}\Big\{2(ip_\mu)(iK^\mu)-2\frac{m_\pi^2}{2}\Big\}\Big|_{\vec{0}} = \frac{-2}{f_\pi}\Big\{pK+\frac{m_\pi^2}{2}\Big\}$$

And in the C.M. frame $\vec{p}' = 0$ and $\vec{K} = -\vec{p}$, so $K = \bar{p}$ and:

$(|\vec{p}|^2 = E_p^2 - m_\pi^2)$ $(2E_p = m_{f_0})$

$$pK = p\bar{p} = E_p^2 + |\vec{p}|^2 = -m_\pi^2 + 2E_p^2 = -m_\pi^2 + \frac{m_{f_0}^2}{2}$$

So finally:

$$\boxed{M_0^{(1)} = \frac{-2}{f_\pi}\Big\{-m_\pi^2 + \frac{m_{f_0}^2}{2} + \frac{m_\pi^2}{2}\Big\} = \frac{1}{f_\pi}\Big\{m_\pi^2 - m_{f_0}^2\Big\}}$$

Now we want to do the same for $S \longrightarrow \pi^+\pi^-$:

$$\mathcal{M}_{+/-}^{(1)} = \langle \underbrace{\vec{p}_{\pi^+} \vec{K}_{\pi^-}} | \, i \mathcal{L}_{int}(\vec{0}) \, | \underbrace{\vec{p}_S'} \rangle = \qquad (\vec{p}' = \vec{p} + \vec{K})$$

$$= i \langle 00 | a_{\pi^+}(\vec{p}) \, b_{\pi^-}(\vec{K}) \, 2\sqrt{E_p E_K} \left(\frac{1}{f_\pi} S (\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi}) - \frac{m_\pi^2}{2 f_\pi} S \vec{\pi}^2 \right) \Big|_{\vec{0}} \sqrt{2E_{p'}} \, a_S^+(\vec{p}') |0\rangle$$

$$= \frac{2}{f_\pi} \sqrt{2 E_p E_K E_{p'}} \Big\{ \langle 00 | a_{\pi^+}(\vec{p}) \, b_{\pi^-}(\vec{K}) \, S (\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi}) a_S^+(\vec{p}') |0\rangle -$$

$$- \frac{m_\pi^2}{2} \langle 00 | a_{\pi^+}(\vec{p}) \, b_{\pi^-}(\vec{K}) \, S \, \vec{\pi}\vec{\pi} \, a_S^+(\vec{p}') |0\rangle \Big\} \Big|_{\vec{0}} =$$

$$= \frac{2}{f_\pi} \sqrt{2 E_p E_K E_{p'}} \Big\{ \langle 00 | a_{\pi^+}(\vec{p}) \, b_{\pi^-}(\vec{K}) \, S \, \underline{a_S^+(\vec{p}')} \, \underline{(\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi})} |0\rangle -$$

$$- \frac{m_\pi^2}{2} \langle 00 | a_{\pi^+}(\vec{p}) \, b_{\pi^-}(\vec{K}) \, S \, \underline{a_S^+(\vec{p}')} \, \underline{\vec{\pi}\vec{\pi}} |0\rangle \Big\} \Big|_{\vec{0}} =$$

$$= \frac{2}{f_\pi} \sqrt{2 E_p E_K E_{p'}} \Big\{ \langle 00 | a_{\pi^+}(\vec{p}) \, b_{\pi^-}(\vec{K}) \, [S, a_S^+(\vec{p}')] (\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi}) |0\rangle -$$

$$- \frac{m_\pi^2}{2} \langle 00 | a_{\pi^+}(\vec{p}) \, b_{\pi^-}(\vec{K}) \, [S, a_S^+(\vec{p}')] \, \vec{\pi}\vec{\pi} |0\rangle \Big\} \Big|_{\vec{0}}$$

$$= \frac{2}{f_\pi} \sqrt{E_p E_K} \Big\{ \langle 00 | a_{\pi^+}(\vec{p}) \, b_{\pi^-}(\vec{K}) \, (\partial_\mu \vec{\pi})(\partial^\mu \vec{\pi} |0\rangle -$$

$$- \frac{m_\pi^2}{2} \langle 00 | a_{\pi^+}(\vec{p}) \, b_{\pi^-}(\vec{K}) \, \vec{\pi}\vec{\pi} |0\rangle \Big\} \Big|_{\vec{0}} =$$

$$= \frac{2}{f_\pi} \sqrt{E_p E_K} \Big\{ \langle 00 | [a_{\pi^+}(\vec{p}), \partial_\mu \vec{\pi}] [b_{\pi^-}(\vec{K}), \partial^\mu \vec{\pi}] |0\rangle +$$

$$+ \langle 00 | [b_{\pi^-}(\vec{K}), \partial_\mu \vec{\pi}] [a_{\pi^+}(\vec{p}), \partial^\mu \vec{\pi}] |0\rangle -$$

$$- \frac{m_\pi^2}{2} \langle 00 | [a_{\pi^+}(\vec{p}), \vec{\pi}] [b_{\pi^-}(\vec{K}), \vec{\pi}] |0\rangle -$$

$$- \frac{m_\pi^2}{2} \langle 00 | [b_{\pi^-}(\vec{K}), \vec{\pi}] [a_{\pi^+}(\vec{p}), \vec{\pi}] |0\rangle \Big\} \Big|_{\vec{0}} =$$

So now we have to commute the commutators:

$$[b_{\pi^-}(\vec{K}), \vec{\pi}] = \begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix} \frac{e^{iKx}}{\sqrt{2E_K}}$$

$$[b_{\pi^-}(\vec{K}), \partial^\mu \vec{\pi}] = (iK^\mu) \begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix} \frac{e^{iKx}}{\sqrt{2E_K}} \qquad \left(\begin{cases} \pi^1 = \dfrac{\pi^+ + \pi^-}{2} \\ \pi^2 = i \dfrac{\pi^+ - \pi^-}{2} \end{cases} \right)$$

$$[a_{\pi^+}(\vec{p}), \vec{\pi}] = \begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix} \frac{e^{ipx}}{\sqrt{2E_p}}$$

$$[a_{\pi^+}(\vec{p}), \partial^\mu \vec{\pi}] = (ip^\mu)\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix} \frac{e^{ipx}}{\sqrt{2E_p}}$$

Substituting these values, we find:

$$\mathcal{M}_\pm^{(1)} = \frac{2}{f_\pi}\sqrt{E_p E_K}\Big\{\langle 00|\,(ip^\mu)\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix}\frac{e^{ipx}}{\sqrt{2E_p}}(iK^\mu)\begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix}\frac{e^{ikx}}{\sqrt{2E_K}}|0\rangle +$$

$$+\langle 00|(iK^\mu)\begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix}\frac{e^{ikx}}{\sqrt{2E_r}}(ip^\mu)\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix}\frac{e^{ipx}}{\sqrt{2E_p}}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix}\frac{e^{ipx}}{\sqrt{2E_p}}\begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix}\frac{e^{ikx}}{\sqrt{2E_K}}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix}\frac{e^{ikx}}{\sqrt{2E_K}}\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix}\frac{e^{ipx}}{\sqrt{2E_p}}|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$= \frac{1}{f_\pi}\Big\{\langle 00|\,(ip^\mu)\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix}(iK^\mu)\begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix}|0\rangle +$$

$$+\langle 00|(iK^\mu)\begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix}(ip^\mu)\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix}\begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix}|0\rangle -$$

$$-\frac{m_\pi^2}{2}\langle 00|\begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix}\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix}|0\rangle\Big\}\Big|_{\vec{0}} =$$

$$= \frac{-2}{f_\pi}\left\{pK\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix}\begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix} + \frac{m_\pi^2}{2}\begin{pmatrix} 1/2 \\ i/2 \\ 0 \end{pmatrix}\begin{pmatrix} 1/2 \\ -i/2 \\ 0 \end{pmatrix}\right\} =$$

$$= \frac{-2}{f_\pi}\left\{pK\left(\frac{1}{4}+\frac{1}{4}\right) + \frac{m_\pi^2}{2}\left(\frac{1}{4}+\frac{1}{4}\right)\right\} =$$

$$= \frac{-1}{f_\pi}\left\{pK + \frac{m_\pi^2}{2}\right\} = \frac{-1}{f_\pi}\left\{E_p^2 + |\vec{p}|^2 + \frac{m_\pi^2}{2}\right\} = \frac{-1}{f_\pi}\left\{-m_\pi^2 + \underbrace{2E_p^2}_{\frac{m_{\rho_0}^2}{2}} + \frac{m_\pi^2}{2}\right\}$$

So finally, we again obtain:

$$\mathcal{M}_\pm^{(1)} = \frac{1}{f_\pi}\left\{\frac{m_\pi^2}{2} - \frac{m_{\rho_0}^2}{2}\right\} = \frac{1}{2}\mathcal{M}_0^{(1)}$$

Which means that:

$$\left.\begin{aligned} \Gamma(S\to\pi_0\pi_0) &= \frac{|M_0^{(1)}|^2}{16\pi m_{f_0}}\sqrt{1-\frac{4m_\pi^2}{m_{f_0}^2}} \\ \Gamma(S\to\pi_+\pi_-) &= \frac{|M_\pm^{(1)}|^2}{16\pi m_{f_0}}\sqrt{1-\frac{4m_\pi^2}{m_{f_0}^2}} = \frac{\frac{1}{4}|M_0^{(1)}|^2}{16\pi m_{f_0}}\sqrt{1-\frac{4m_\pi^2}{m_{f_0}^2}} \end{aligned}\right\} \Gamma_{Total} = \frac{5}{4}\frac{|M_0^{(1)}|^2}{16\pi m_{f_0}}\sqrt{1-\frac{4m_\pi^2}{m_{f_0}^2}}$$

Which substituting $M_0^{(1)}$ gives:

$$\boxed{\Gamma_{tot} = \frac{5}{4}\frac{(m_\pi^2 - m_{f_0}^2)^2}{16\pi m_{f_0} f_\pi^2}\sqrt{1-\frac{4m_\pi^2}{m_{f_0}^2}}}$$

From PDG we obtain a m_{f_0} range that goes from 400 to 800 MeV, so:

$$\Gamma_{tot} = \frac{5}{4}\frac{(140^2 - (800\sim 400)^2)^2}{16\pi(800\sim 400)\, 92^2}\sqrt{1-\frac{4\cdot 140^2}{(800\sim 400)^2}} = (103{,}4 \sim 1324{,}2)\ MeV$$

The PDG give a total width for all decays of $(100\sim 800)\ MeV$, so our Γ_{tot} must be smaller than those values, and this is fullfilled for values of m_{f_0} around:

$$\boxed{m_{f_0} \approx (400\sim 700)\ MeV} \longrightarrow \left(\Gamma \approx (103{,}4 \sim 850)\ MeV\right)$$