5. QED for leptons

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5.1 QED for elementary spin 1/2 particles

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad D_\mu = \partial_\mu + iqA_\mu$$

- The Lagrangian above is for a single elementary particle of electric charge q (and its antiparticle)
- Recall that the concept of elementary depends on the energy scale we are (for instance, for $E \lesssim 1$ GeV, hadrons look like elementary particles)
- If we have n elementary particles of charges q_j then

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_j^n \bar{\psi}_j(i\gamma^\mu D^j_\mu - m_j)\psi_j \quad , \quad D^j_\mu = \partial_\mu + iq_jA_\mu$$

  - The Lagrangian above is invariant under $\psi_j \to e^{i\theta_j}\psi_j$, $\theta_j \neq \theta_j(x)$, $\forall j = 1, \dots, n$ $\Longrightarrow$ there are n conserved charges $\Longrightarrow$ each flavor j is conserved
  - The interaction Lagrangian reads

$$\mathcal{L}_I = \sum_j^n -q_j\bar{\psi}_j\gamma^\mu A_\mu\psi_j \equiv -\sum_j^n j^\mu_j A_\mu \equiv -j^\mu A_\mu$$

- The simplest physical processes requires second order in $\mathcal{L}_I$
- We shall focuse on two that were relevant to the discovery of QCD
  - $e^- \, \mu^- \rightarrow e^- \, \mu^-$
  - $e^+ \, e^- \rightarrow \mu^+ \, \mu^-$
  - They are related by crossing
- The amplitude is given by

$$\begin{aligned} i\mathcal{M} &= \frac{i^2}{2!}\int d^4x \, \langle f| \mathrm{T}\left\{\mathcal{L}_I(0)\mathcal{L}_I(x)\right\}|i\rangle = i^2\int d^4x \, \langle f|\mathrm{T}\left\{j_e^\mu(0)A_\mu(0)j_m^\nu(x)A_\nu(x)\right\}|i\rangle \\ &= i^2\int d^4x \, \langle f|\mathrm{T}\left\{A_\mu(0)A_\nu(x)\right\}\mathrm{T}\left\{j_e^\mu(0)j_m^\nu(x)\right\}|i\rangle \end{aligned}$$

- The initial and final states may be written as

$$|i\rangle = |0\rangle_\gamma \, |i\rangle_e \, |i\rangle_\mu \quad , \quad |f\rangle = |0\rangle_\gamma \, |f\rangle_e \, |f\rangle_\mu$$

- Then

$$i\mathcal{M} = i^2\int d^4x \; {}_\gamma\langle 0|\,\mathrm{T}\left\{A_\mu(0)A_\nu(x)\right\}|0\rangle_\gamma \; {}_e\langle f|\, j_e^\mu(0)\,|i\rangle_e \; {}_\mu\langle f|\, j^\nu{}_m(x)\,|i\rangle_\mu$$

- Consider $e^- \, \mu^- \rightarrow e^- \, \mu^-$, then

$$|i\rangle_e = |\vec{p}_A\lambda_A\,;\rangle_e \quad , \quad |i\rangle_\mu = |\vec{p}_B\lambda_B\,;\rangle_\mu \quad , \quad |f\rangle_e = |\vec{p}_1\lambda_1\,;\rangle_e \quad , \quad |f\rangle_\mu = |\vec{p}_2\lambda_2\,;\rangle_\mu$$

- Hence,

$${}_\gamma\langle 0|T\{A_\mu(0)A_\nu(x)\}|0\rangle_\gamma = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(0-x)}}{k^2+i\eta}(-g_{\mu\nu}) \quad , \quad u(l) \equiv u_{\lambda_l}(\vec{p}_l)$$

$${}_e\langle f|\, j_e^\mu(0)\,|i\rangle_e = {}_e\langle f|\, q_e\bar{\psi}_e(0)\gamma^\mu\psi_e(0)\,|i\rangle_e = q_e\,\bar{u}(1)\gamma^\mu u(A)$$

$${}_\mu\langle f|\, j^\nu{}_m(x)\,|i\rangle_\mu = {}_\mu\langle f|\, q_m\bar{\psi}_\mu(x)\gamma^\nu\psi_\mu(x)\,|i\rangle_\mu = q_m\,\bar{u}(2)\gamma^\nu u(B)e^{-ix.(p_B-p_2)}$$

- Then ($q_e = q_m = -e$)

$$i\mathcal{M} = \frac{(-ig_{\mu\nu})}{(p_B-p_2)^2}\bar{u}(1)\left(-iq_e\gamma^\mu\right)u(A)\bar{u}(2)\left(-iq_m\gamma^\nu\right)u(B) = \frac{ie^2\bar{u}(1)\gamma^\mu u(A)\bar{u}(2)\gamma_\mu u(B)}{(p_B-p_2)^2}$$

- In order to calculate cross-sections only $|\mathcal{M}|^2$ is relevant so global phases (and signs) can be dropped
- There is a shortcut to calculate the amplitudes: the Feynman rules, see for instance pg. 15-17 of this link or pg. 228-233 of Griffiths' e-book

- In order to calculate cross-sections only $|\mathcal{M}|^2$ is relevant so global phases (and signs) can be dropped
- $|\mathcal{M}|^2$ depends on the three momentum and third component of the spin (or helicity) of each of the particles in the initial and final state
- If the beam is not polarized $\Longrightarrow$ average over the initial spin states (polarizations)
- If spin states (polarizations) are not measured $\Longrightarrow$ sum over all final spin states (polarizations)
- Then

$$|\mathcal{M}|^2 \to |\overline{\mathcal{M}}|^2 \equiv \frac{1}{2}\sum_{\lambda_A=+,-}\frac{1}{2}\sum_{\lambda_B=+,-}\sum_{\lambda_1=+,-}\sum_{\lambda_2=+,-}|\mathcal{M}|^2$$

- Since

$$|\mathcal{M}|^2 = \frac{e^4\bar{u}(1)\gamma^\mu u(A)\bar{u}(2)\gamma_\mu u(B)\left(\bar{u}(1)\gamma^\nu u(A)\bar{u}(2)\gamma_\nu u(B)\right)^*}{(p_B-p_2)^4}$$

- Consider

$$\sum_{\lambda_A=+,-}\sum_{\lambda_1=+,-}\bar{u}(1)\gamma^\mu u(A)\left(\bar{u}(1)\gamma^\nu u(A)\right)^* = \sum_{\lambda_A=+,-}\sum_{\lambda_1=+,-}\bar{u}(1)\gamma^\mu u(A)\left(\bar{u}(1)\gamma^\nu u(A)\right)^\dagger =$$

$$\sum_{\lambda_A=+,-}\sum_{\lambda_1=+,-}\bar{u}(1)\gamma^\mu u(A)\left(\bar{u}(A)\gamma^0\gamma^{\nu\,\dagger}\gamma^0 u(1)\right) = \sum_{\lambda_A=+,-}\sum_{\lambda_1=+,-}\mathrm{tr}\left(\gamma^\mu u(A)\bar{u}(A)\gamma^\nu u(1)\bar{u}(1)\right) =$$

$$\mathrm{tr}(\gamma^\mu \sum_{\lambda_A=+,-}u(A)\bar{u}(A)\ \gamma^\nu \sum_{\lambda_1=+,-}u(1)\bar{u}(1)\ ) = \mathrm{tr}\left(\gamma^\mu\left(\not{p}_A+m_e\right)\gamma^\nu\left(\not{p}_1+m_e\right)\right)$$

- Then

$$\mathrm{tr}\left(\gamma^\mu\left(\not{p}_A+m_e\right)\gamma^\nu\left(\not{p}_1+m_e\right)\right) = \mathrm{tr}\left(\gamma^\mu\not{p}_A\gamma^\nu\not{p}_1\right) + m_e^2\mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) =$$
$$4\left(p_A^\mu p_1^\nu - g^{\mu\nu}p_A.p_1 + p_A^\nu p_1^\mu\right) + 4m_e^2 g^{\mu\nu} \equiv 2L_e^{\mu\nu}$$

- Analogously

$$\sum_{\lambda_B=+,-}\sum_{\lambda_2=+,-}\bar{u}(2)\gamma^\mu u(B)\left(\bar{u}(2)\gamma^\nu u(B)\right)^* = 4\left(p_B^\mu p_2^\nu - g^{\mu\nu}p_B.p_2 + p_B^\nu p_2^\mu\right) + 4m_\mu^2 g^{\mu\nu} \equiv 2L_m^{\mu\nu}$$

- Then

$$|\overline{\mathcal{M}}|^2 = \frac{e^4 L_e^{\mu\nu}L_{\mu\nu\, m}}{(p_B-p_2)^4}$$

$$\begin{aligned} L_e^{\mu\nu}L_{\mu\nu\, m} &= 8\left((p_A p_B)(p_1 p_2) + (p_A p_2)(p_1 p_B) - m_e^2(p_B p_2) - m_\mu^2(p_A p_1) + 2m_e^2 m_\mu^2\right) \\ &= 4\left((s-m_e^2-m_\mu^2)^2 + \frac{t^2}{2} + st\right) \end{aligned}$$

- We have used the Mandelstam variables:

$$\begin{aligned} s &\equiv (p_A+p_B)^2 = (p_1+p_2)^2 = m_e^2+m_\mu^2+2(p_A p_B) = m_e^2+m_\mu^2+2(p_1p_2) \\ t &\equiv (p_A-p_1)^2 = (p_2-p_B)^2 = 2m_e^2 - 2(p_A p_1) = 2m_\mu^2 - 2(p_B p_2) \\ u &\equiv (p_A-p_2)^2 = (p_1-p_B)^2 = m_e^2+m_\mu^2-2(p_A p_2) = m_e^2+m_\mu^2-2(p_1 p_B) \\ & s+t+u = 2m_e^2+2m_\mu^2 \end{aligned}$$

- Finally (recall $m_\mu \gg m_e$)

$$|\overline{\mathcal{M}}|^2 = \frac{4e^4}{t^2}\left((s-m_e^2-m_\mu^2)^2+\frac{t^2}{2}+st\right) \simeq \frac{4e^4}{t^2}\left((s-m_\mu^2)^2+\frac{t^2}{2}+st\right)$$

- In the high energy limit $s \gg m_\mu^2$

$$|\overline{\mathcal{M}}|^2 \simeq \frac{2e^4}{t^2}\left(s^2+u^2\right)$$

- In the CoM frame, the kinematics of a $A\,B \to 1\,2$ collision is fixed except for the angle between the incoming and outgoing direction. If we keep this angle unintegrated, we have

$$\sigma = \frac{1}{4\sqrt{(p_A\cdot p_B)^2-m_A^2m_B^2}}\int\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\int\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}|\mathcal{M}|^2(2\pi)^4\delta(p_A+p_B-p_1-p_2)$$

$$\Longrightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{|\mathcal{M}|^2 p_f}{64\pi^2 s\, p_i} \equiv \text{ Differential cross section}$$

$$p_i \equiv |\vec{p}_A| = |-\vec{p}_B| = \frac{\sqrt{(s-(m_A+m_B)^2)(s-(m_A-m_B)^2)}}{2\sqrt{s}}$$

$$p_f \equiv |\vec{p}_1| = |-\vec{p}_2| = \frac{\sqrt{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}}{2\sqrt{s}}$$

- For $e^-\mu^- \to e^-\mu^-$, $p_i = p_f$
- In the high energy limit ($p_i \gg m_\mu$)
  - $s=(p_A+p_B)^2 \simeq (|\vec{p}_A|+|\vec{p}_B|)^2 = 4p_i^2$
  - $t=(p_A-p_1)^2 \simeq -(\vec{p}_A-\vec{p}_1)^2 = -4p_i^2\sin^2\frac{\theta}{2}$
  - $u=(p_A-p_2)^2 \simeq -(\vec{p}_A-\vec{p}_2)^2 = -4p_i^2\cos^2\frac{\theta}{2}$
- Then

$$|\overline{\mathcal{M}}|^2 \simeq 2e^4\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}$$

  - Independent of p_i
  - Differential cross section forward enhanced (in fact it blows up !)
- In the low energy limit ($p_i \ll m_e$)
  - $s=(p_A+p_B)^2 \simeq (m_e+m_\mu)^2 \simeq m_\mu^2+2m_em_\mu$
  - $t=(p_A-p_1)^2 \simeq -(\vec{p}_A-\vec{p}_1)^2 = -4p_i^2\sin^2\frac{\theta}{2}$
  - $u=(p_A-p_2)^2 \simeq (m_e-m_\mu)^2 \simeq m_\mu^2-2m_em_\mu$
- Then

$$|\overline{\mathcal{M}}|^2 \simeq \frac{16e^4m_e^2m_\mu^2}{(\vec{p}_A-\vec{p}_1)^2} = \frac{e^4m_e^2m_\mu^2}{p_i^4\sin^4\frac{\theta}{2}}$$

- This is nothing but Coulomb scattering with relativistic normalization
- The differential cross section (which is independent of the normalization) reads

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \simeq \frac{|\mathcal{M}|^2}{64\pi^2(m_e+m_\mu)^2} \simeq \frac{e^4 m_e^2}{64\pi^2 p_i^4 \sin^4\frac{\theta}{2}} = \frac{\alpha^2 m_e^2}{4p_i^4 \sin^4\frac{\theta}{2}}$$

which agrees with Rutherford's formula

The LAB frame

- This is the suitable frame to describe fixed target experiments

$$p_A = (E_A\,,\vec{p}_A) \quad , \quad p_B = (m_B\,,0) \quad , \quad p_1 = (E_1\,,\vec{p}_1) \quad , \quad p_2 = (E_2\,,\vec{p}_2)$$
$$E_A + m_B = E_1 + E_2 \quad , \quad \vec{p}_A = \vec{p}_1 + \vec{p}_2$$

- The cross section may be written as

$$\begin{aligned}\sigma &= \frac{1}{4\sqrt{(p_A\cdot p_B)^2 - m_A^2 m_B^2}}\int\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\int\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}|\mathcal{M}|^2(2\pi)^4\delta(p_A+p_B-p_1-p_2)\\ &= \frac{1}{4m_B|\vec{p}_A|}\int\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\int d^4p_2\,\theta(p_2^0)\,\delta(p_2^2-m_2^2)\,|\mathcal{M}|^2(2\pi)\delta(p_A+p_B-p_1-p_2)\\ &= \frac{1}{4m_B|\vec{p}_A|}\int\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\theta(p_A^0+p_B^0-p_1^0)\delta\left((p_A+p_B-p_1)^2-m_2^2\right)|\mathcal{M}|^2(2\pi)\end{aligned}$$

- Let us introduce

$$q \equiv p_A - p_1 \quad , \quad \nu \equiv \frac{q.p_B}{m_B} = p_A^0 - p_1^0 = E_A - E_1 = q^0$$
$$(p_A+p_B-p_1)^2 - m_2^2 = (q+p_B)^2 - m_2^2 = q^2 + 2q.p_B + m_B^2 - m_2^2$$

- Let us particularize to $e^-\mu^- \to e^-\mu^-$ $\Longrightarrow$ $m_A = m_1 = m_e$, $m_B = m_2 = m_\mu$, and assume that $E_A \gg m_A$

$$q^2 + 2q.p_B + m_B^2 - m_2^2 = q^2 + 2m_B\nu \quad , \quad q^2 \simeq -4E_A E_1 \sin^2\frac{\theta}{2}$$

$$\begin{aligned}\sigma &= \frac{1}{4m_B|\vec{p}_A|}\int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\theta(p_A^0+p_B^0-p_1^0)\delta\left((p_A+p_B-p_1)^2-m_2^2\right)|\mathcal{M}|^2 \\ &\simeq \frac{1}{4m_B E_A}\int d\Omega \int \frac{dE_1\, E_1}{(2\pi)^2\, 2}\theta(q^0+m_B)\frac{\delta\left(\frac{q^2}{2m_B}+\nu\right)}{2m_B}|\mathcal{M}|^2\end{aligned}$$

- Hence

$$\left(\frac{d\sigma}{dE_1 d\Omega}\right)_{LAB} = \frac{|\mathcal{M}|^2 E_1}{64\pi^2 E_A m_B^2}\theta(\nu+m_B)\delta\left(\frac{q^2}{2m_B}+\nu\right)$$

- The delta function allows to carry out the integral over E_1

$$\delta\left(\frac{q^2}{2m_B}+\nu\right) \simeq \delta\left(-\frac{2E_A E_1}{m_B}\sin^2\frac{\theta}{2}+E_A-E_1\right) = \frac{1}{1+\frac{2E_A}{m_B}\sin^2\frac{\theta}{2}}\delta\left(E_1-\frac{E_A}{1+\frac{2E_A}{m_B}\sin^2\frac{\theta}{2}}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \frac{|\mathcal{M}|^2}{64\pi^2 m_B^2}\frac{1}{\left(1+\frac{2E_A}{m_B}\sin^2\frac{\theta}{2}\right)^2} = \frac{|\mathcal{M}|^2 E_1^2}{64\pi^2 m_B^2 E_A^2} \quad , \quad E_1 = \frac{E_A}{1+\frac{2E_A}{m_B}\sin^2\frac{\theta}{2}}$$

- Let us write in a convenient form for later use ($E_A \gg m_e$)

$$\overline{|\mathcal{M}|^2} \simeq \frac{4e^4}{t^2}\left((s-m_\mu^2)^2+\frac{t^2}{2}+st\right)$$

$$t=q^2 \simeq -4E_A E_1\sin^2\frac{\theta}{2} \quad , \quad s\simeq m_\mu^2+2E_A m_\mu \quad , \quad E_1 = \frac{E_A}{1+\frac{2E_A}{m_\mu}\sin^2\frac{\theta}{2}}$$

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{4e^4}{q^4}\left(4E_A^2 m_\mu^2+\frac{q^2}{2}\left(-4E_A E_1\sin^2\frac{\theta}{2}\right)+\left(m_\mu^2+2E_A m_\mu\right)\left(-4E_A E_1\sin^2\frac{\theta}{2}\right)\right) \\ &= \frac{4e^4}{q^4}\left(4E_A E_1\left(1+\frac{2E_A}{m_\mu}\sin^2\frac{\theta}{2}\right)m_\mu^2+\frac{q^2}{2}\left(-4E_A E_1\sin^2\frac{\theta}{2}\right)\right. \\ &\quad \left.+\left(m_\mu^2+2E_A m_\mu\right)\left(-4E_A E_1\sin^2\frac{\theta}{2}\right)\right) \\ &= \frac{16e^4 m_\mu^2 E_A E_1}{q^4}\left(\cos^2\frac{\theta}{2}-\frac{q^2}{2m_\mu^2}\sin^2\frac{\theta}{2}\right)\end{aligned}$$

- The differential cross sections read

$$\left(\frac{d\sigma}{dE_1 d\Omega}\right)_{LAB} = \frac{4\alpha^2 E_1^2}{q^4}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m_\mu^2}\sin^2\frac{\theta}{2}\right)\delta\left(\frac{q^2}{2m_\mu}+\nu\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \frac{\alpha^2 E_1}{4E_A^3\sin^4\frac{\theta}{2}}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m_\mu^2}\sin^2\frac{\theta}{2}\right)$$

  - This is an arbitrary (and redundant) way of writing the differential cross sections, which is convenient to experimentalist
  - Remember that the independent variables are E_A and θ
- In the limit $m_e \ll E_A \ll m_\mu \implies E_1 \simeq E_A$, Mott's formula is recovered

$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB} \simeq \frac{\alpha^2\cos^2\frac{\theta}{2}}{4E_A^2\sin^4\frac{\theta}{2}}$$

5.2 Pair creation

$e^- e^+ \to \mu^- \mu^+$

- This process is related by crossing to $e^- \mu^- \to e^- \mu^-$
- Assume the beams are unpolarized and polarizations are not measured
- Then $|\overline{\mathcal{M}}|^2(e^- e^+ \to \mu^+\mu^-)$ is related to $|\overline{\mathcal{M}}|^2(e^- \mu^- \to e^- \mu^-)$

$$\begin{array}{cccccc} e^- & \mu^- & \to e^- & \mu^- & , & e^- \; e^+ \to \mu^+ \; \mu^- \\ p_A & p_B & p_1 & p_2 & & p'_A \; p'_B \quad p'_1 \; p'_2 \end{array}$$

$$p'_A = p_A \quad , \quad p'_B = -p_1 \quad , \quad p'_1 = -p_B \quad , \quad p'_2 = p_2$$

$$\begin{aligned} s' &= (p'_A + p'_B)^2 = (p_A - p_1)^2 = t \\ t' &= (p'_A - p'_1)^2 = (p_A + p_B)^2 = s \\ u' &= (p'_A - p'_2)^2 = (p_A - p_2)^2 = u \end{aligned}$$

- Hence $|\overline{\mathcal{M}}|^2(\underset{p_A}{e^-}\ \underset{p_B}{e^+} \to \underset{p_1}{\mu^+}\underset{p_2}{\mu^-})$ can be obtained from $|\overline{\mathcal{M}}|^2(\underset{p_A}{e^-}\ \underset{p_B}{\mu^-} \to \underset{p_1}{e^-}\ \underset{p_2}{\mu^-})$ by

$$s \to t \quad , \quad t \to s \quad , \quad u \to u$$

- Then

$$|\overline{\mathcal{M}}|^2 = \frac{4e^4}{s^2}\left((t-m_e^2-m_\mu^2)^2+\frac{s^2}{2}+st\right) \simeq \frac{4e^4}{s^2}\left((t-m_\mu^2)^2+\frac{s^2}{2}+st\right)$$

- In the high energy limit $t \gg m_\mu^2$

$$|\overline{\mathcal{M}}|^2 \simeq \frac{2e^4}{s^2}\left(t^2+u^2\right)$$

  - In the *CoM* frame

$$s \simeq 4E_A^2 \quad , \quad t \simeq -2E_A^2(1-\cos\theta) \quad , \quad u \simeq -2E_A^2(1+\cos\theta)$$

$$|\overline{\mathcal{M}}|^2_{CoM} \simeq e^4\left(1+\cos^2\theta\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} \simeq \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \quad \Longrightarrow \quad \sigma = \frac{4\pi\alpha^2}{3s}$$

  - Note that the maximum of pair production is attained at $\theta = 0,\ \pi$

- If we allow energies close to the muon mass ($t \gtrsim m_\mu^2$)

$$p_i = |\vec{p}_A| = |-\vec{p}_B| = \frac{\sqrt{s-4m_e^2}}{2} \simeq \frac{\sqrt{s}}{2} \simeq E_A \simeq E_1$$

$$p_f = |\vec{p}_1| = |-\vec{p}_2| = \frac{\sqrt{s-4m_\mu^2}}{2} = \frac{\sqrt{s}}{2}\sqrt{1-\frac{4m_\mu^2}{s}} \equiv \frac{\sqrt{s}}{2}\beta$$

$$t \simeq m_\mu^2 - 2(p_Ap_1) = m_\mu^2 - 2(E_AE_1 - |\vec{p}_A||\vec{p}_1|\cos\theta) = m_\mu^2 - \frac{s}{2}(1-\beta\cos\theta)$$

$$|\overline{\mathcal{M}}|^2_{CoM} \simeq e^4\left(\left(1+\cos^2\theta\right)+\frac{4m_\mu^2}{s}\sin^2\theta\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} \simeq \frac{\alpha^2\beta}{4s}\left(\left(1+\cos^2\theta\right)+\frac{4m_\mu^2}{s}\sin^2\theta\right) \quad \Longrightarrow \quad \sigma = \frac{2\pi\alpha^2\beta}{3s}(1+\beta)$$

$e^- \, e^+ \to \pi^- \pi^+$

- Let us assume that $E_A \ll 1$ GeV so that the pions may be considered point-like particles
- The relevant interaction Lagrangians are the one of QED for the electron and the one of SQED for the pion

$$\mathcal{L}_I = -q_e \bar{\psi}\gamma^\mu A_\mu \psi + i q_\pi A^\mu \left(\partial_\mu \phi^* \phi - \phi^* \partial_\mu \phi\right) + q_\pi^2 A^\mu A_\mu \phi^* \phi$$

- The last term is quadratic in q_π and hence it does not contribute at leading order
- We may use the same formula as for electron-muon scattering

$$i\mathcal{M} \;=\; i^2 \int d^4x \; {}_\gamma\langle 0|\,\mathrm{T}\left\{A_\mu(0)A_\nu(x)\right\}|0\rangle_\gamma \; {}_e\langle f|\, j_e^\mu(0)\,|i\rangle_e \; {}_\pi\langle f|\, j^\nu{}_\pi(x)\,|i\rangle_\pi$$

$$j_\pi^\nu = -i q_\pi \left(\partial^\nu \phi^* \phi - \phi^* \partial^\nu \phi\right)$$

$$|i\rangle_e = |\vec{p}_A \lambda_A \,;\, \vec{p}_B \lambda_b\rangle_e \quad , \quad |i\rangle_\pi = |0\rangle_\pi \quad , \quad |f\rangle_e = |0\rangle_e \quad , \quad |f\rangle_\pi = |\vec{p}_1 \,;\, \vec{p}_2\rangle_\pi$$

- For unpolarized beams and $E_A \gg m_e$ one eventually obtains (this is the exercise for this week !)

$$\overline{|\mathcal{M}|^2} \simeq \frac{e^4}{s^2}\left(-\frac{(u-t)^2}{2} + \frac{s^2}{2} - 2m_\pi^2 s\right) \;,\; \overline{|\mathcal{M}|^2_{CoM}} \simeq \frac{e^4 \beta^2 \sin^2\theta}{2} \;,\; \beta \equiv \sqrt{1 - \frac{4m_\pi^2}{s}}$$

- Which leads to

$$\left(\frac{d\sigma}{d\Omega}\right)_{CoM} \simeq \frac{\alpha^2 \beta^3}{8s}\sin^2\theta \quad \Longrightarrow \quad \sigma = \frac{\pi \alpha^2 \beta^3}{3s}$$

  - The maximum occurs at $\theta = \pi/2$, versus $\theta = 0, \pi$ for muons $\Longrightarrow$ measuring the angular distribution in pair production allows to tell apart the spin of the produced particles
  - For $s \gg m_\pi^2$, $\sigma \simeq \pi\alpha^2/3s$, a fourth of the muon's one
- The following processes are related by crossing to pion pair production
  - $\pi^- \pi^+ \to e^- \, e^+$
  - $e^- \, \pi^- \to e^- \, \pi^-$
  - $e^+ \, \pi^+ \to e^+ \, \pi^+$

5.3 Other elementary QED processes

- Møller scattering $e^- \, e^- \to e^- \, e^-$
  - There are two contributions to the amplitude with a relative minus sign
  - The cross section must be divided by 2! because the final state has two identical particles
  - It is related by crossing to Bhabha scattering and to $e^+ \, e^+ \to e^+ \, e^+$
- Bhabha scattering $e^- \, e^+ \to e^- \, e^+$
  - There are two contributions to the amplitude with the same sign
  - It is related by crossing to Møller scattering
- Compton scattering $e^- \, \gamma \to e^- \, \gamma$
  - There are two contributions to the amplitude with the same sign
  - The photon polarizations enter in the amplitude
  - The Dirac propagators enters in the amplitude
- Electron-positron annihilation $e^- \, e^+ \to \gamma \, \gamma$
  - There are two contributions to the amplitude with the same sign
  - The photon polarizations enter in the amplitude
  - The Dirac propagators enters in the amplitude
  - It is related by crossing to Compton scattering and to $\gamma \, \gamma \to e^- \, e^+$
  - Important for positronium ($e^- \, e^+$ bound state) decay