Quantum Field Theory, 2021/2022
Exercise sheet 3 part b: Dirac Field
Hand-in: October 27, 2021

3.2. Using the chiral representation for the 4-component Dirac spinors, or otherwise,

(a) prove the following relations:

$$\bar{u}^r(\boldsymbol{p})v^s(\boldsymbol{p}) = 0 \quad ; \quad u^{r\dagger}(-\boldsymbol{p})v^s(\boldsymbol{p}) = 0 \quad ;$$

(b) Given the fermion field expansion in normal modes

$$\psi(x) = \sum_{s=1,2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} \left(a^s_{\boldsymbol{p}} u^s(\boldsymbol{p}) e^{-ipx} + b^{s\dagger}_{\boldsymbol{p}} v^s(\boldsymbol{p}) e^{ipx} \right) \tag{1}$$

and the equal-time-anti-commutation relations

$$\begin{aligned} \{\psi_\alpha(t,\boldsymbol{x}), \psi^\dagger_\beta(t,\boldsymbol{y})\} &= \delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\ \{\psi_\alpha(t,\boldsymbol{x}), \psi_\beta(t,\boldsymbol{y})\} &= 0 \end{aligned} \tag{2}$$

compute the anti-commutation relations among the $a^s_{\boldsymbol{p}}$, $a^{s\dagger}_{\boldsymbol{p}}$, $b^r_{\boldsymbol{p}}$ and $b^{r\dagger}_{\boldsymbol{p}}$ operators.

[3 points]

3.3. Using that under Lorentz transformations the spinor field changes as

$$\psi(x) \to \psi'(x') = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi(x)$$

and the properties of the Pauli σ matrices,

(a) find the spin part of the associated conserved current.

(b) Show that the generators of the rotational part of the transformations can be written as:

$$\frac{1}{2}\Sigma^k = \frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

(c) show that the associated conserved charge is

$$S^k = \int \mathrm{d}^3 x : \psi^\dagger(x) \frac{\Sigma^k}{2} \psi(x) :$$

(d) Find S^k as a function of $a^s_{\boldsymbol{p}}$, $a^{s\dagger}_{\boldsymbol{p}}$, $b^r_{\boldsymbol{p}}$ and $b^{r\dagger}_{\boldsymbol{p}}$ operators.

(e) Consider a state of one particle at rest ($\boldsymbol{p} = 0$). Show that it is an eigenstate of S^z, and find the spin-z eigenvalues for each kind of particle ($a^{r\dagger}_{\boldsymbol{0}}$, $b^{r\dagger}_{\boldsymbol{0}}$, $r = 1, 2$). Hint: You can use that $J^z|0\rangle = 0$.

[4 points]