EXCERCICES COSMO

lunes, 2 de mayo de 2022 17:57

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Sheet 1: Cosmology: ΛCDM and beyond
Aspects of Dark Matter and Dark Energy

Exercise 1. We may ask ourselves to which extent the present Universe is transparent to light. There are different processes that may stop, absorb or scatter the propagation of light. They depend on the scattering cross-section σ and the number density n of scattering particles. Estimate the photon mean free path ($\lambda_\gamma = 1/n\,\sigma$) in each one of the following processes:

i) Thomson scattering of photons on free electrons in the intergalactic medium, with a number density $n_e = n_p \sim \Omega_B \rho_c/m_p$;

ii) Stellar absorption, with a number density $n_{\rm est} \sim \Omega_{\rm visible}\ \rho_c/M_\odot$ and cross-section $\sigma_\odot \sim \pi R_\odot^2$. Check yourself the value of $\Omega_{\rm visible}$[1];

iii) Absorption by dust in galaxies. Using the information I gave on the blackboard, check in detail that there is roughly an equivalent of ~ 0.4 galaxies like ours (Milky Way) each $Mpc.^2$. Use that the average galactic radius is $R_{\rm gal} \sim 10Kpc.$ and that the absorption of light after crossing the galaxy is on average 10% of its geometrical cross-section.

iv) Compare the mean free path with the Hubble radius ($d_H = 1/H_0 \simeq 3000\,h_0^{-1}Mpc$) and conclude if the Universe is or is not transparent to light. Which is the dominant effect among the previous three? What is the optical depth of the universe?

v) Translated into time units, the expression $t_H = 1/H_0$ defines the so-called Hubbe time. How many million years correspond to this period? How many photons out of 1000 will undergo an interaction (according to the previous processes) in the next Hubble time?

i)

$$\lambda_{\rm Thomson} = \frac{1}{n_e\,\sigma_{\rm Thomson}} = \frac{m_p}{\Omega_B\,\rho_c\,\sigma_{\rm Thomson}} \quad \text{with} \quad \begin{cases} \sigma_{\rm Thomson} = \frac{8\pi}{3}\left(\frac{\alpha\hbar c}{mc^2}\right)^2 = 6{,}6524587\cdot 10^{-29}\ m^2 \quad \text{(wiki)} \\ \Omega_B = 0{,}05 \quad \text{(class)} \qquad h=0{,}7 \\ \rho_c = \left(10{,}54\ \frac{GeV}{m^3}\right) h^2 \simeq 5{,}1646\ \frac{GeV}{m^3} \quad \text{(class)} \\ m_p = 0{,}938272\ GeV \quad \text{(Wiki)} \end{cases}$$

$$\boxed{\lambda_{\rm Thomson}} = \frac{m_p}{\Omega_B\,\rho_c\,\sigma_{\rm Thomson}} = \frac{0{,}938272\ \cancel{GeV}}{0{,}05\cdot 5{,}1646\ \frac{\cancel{GeV}}{m^{\cancel{3}}}\ 6{,}652458\cdot 10^{-29}\ \cancel{m^2}} = \boxed{5{,}4618\cdot 10^{28}\ m}$$

ii)

$$\lambda_{\rm stel.abs.} = \frac{1}{n_{\rm est}\,\sigma_\odot} = \frac{M_\odot}{\Omega_{\rm vis}\,\rho_c\,\pi R_\odot^2} \quad \text{with} \quad \begin{cases} \rho_c = 2{,}8\,h^2\,10^{11}\ \frac{M_\odot}{Mpc^3} \overset{h=0{,}7}{=} 1{,}37\ 10^{11}\ \frac{M_\odot}{Mpc^3} \quad \text{(class)} \\ R_\odot = 6{,}957\cdot 10^5\ Km \quad \text{(Wiki)} \\ 1\,Mpc = 3{,}086\cdot 10^{19}\ Km \quad \text{(wiki)} \\ \Omega_{vis} = f_*\cdot \Omega_m = 0{,}01\cdot 0{,}3 = 0{,}003 \quad \text{(class)} \end{cases}$$

$$\boxed{\lambda_{\rm stel.abs.}} = \frac{\cancel{M_\odot}}{\Omega_{vis}\ 1{,}37\cdot 10^{11}\ \frac{\cancel{M_\odot}}{Mpc^3}\ \pi\ 6{,}957^2\cdot 10^{5^2}} = \frac{3{,}086^3\cdot 10^{57}\ Km^{\cancel{3}}}{0{,}003\cdot 1{,}37\cdot 10^{11}\cdot \pi\cdot 6{,}957^2\ 10^{10}\ \cancel{Km^2}} = \boxed{4{,}7\cdot 10^{40}\ m}$$

iii)

From class we know that to compute this is better to use Ω_m rather than Ω_{vis}, so we then got:

$$\lambda_{galaxy\ dust} = \frac{1}{n_{galaxies} \cdot \sigma_{galaxies}} \quad \text{where} \quad \begin{cases} n_{galaxies} = \frac{\Omega_m \cdot \rho_c}{M_{galaxy}} = \frac{\Omega_m \cdot \rho_c}{M_\odot \cdot 10^{11}} \\ \sigma_{galaxies} = \frac{1}{10} A_{galaxy} = \frac{1}{10} \pi R^2_{galaxy} \\ R_{galaxy} = 0{,}01\ Mpc \quad (\text{class}) \end{cases}$$

$$\boxed{\lambda_{galaxy\ dust}} = \frac{M_\odot \cdot 10^{11} \cdot 10}{\Omega_m \cdot \rho_c \cdot \pi\ 0{,}01^2 Mpc^2} = \frac{\cancel{M_\odot} \cdot \cancel{10^{11}} \cdot 10}{0{,}3 \cdot 1{,}37 \cdot \cancel{10^{11}} \frac{\cancel{M_\odot}}{Mpc^3} \cdot \pi \cdot 0{,}01^2 \cancel{Mpc^2}} = 7{,}74 \cdot 10^4 Mpc = \boxed{2{,}39 \cdot 10^{27}\ m}$$

iv)

$$\boxed{d_H} = \frac{1}{H_0} \overset{(\text{class})}{\approx} \frac{3000}{h_0}\ Mpc \overset{(h_0 = 0{,}7)}{=} 4285{,}71\ Mpc = 4285{,}71 \cdot 3{,}086 \cdot 10^{22}\ m = \boxed{1{,}3226 \cdot 10^{26}\ m}$$

So, we see that the universe is pretty transparent due to $\lambda_{galaxy} > d_H \cdot 10$. Which is the dominant effect of the previous 3. $\lambda_{stel. abs.} > \lambda_{thomson} > \lambda_{galaxy\ dust} > d_H$

The optical depth of the universe is then:

$$\boxed{\tau = \sigma N = \lambda^{-1} = \lambda^{-1}_{dust.\ gal.} = 4{,}18 \cdot 10^{-28}\ m^{-1}}$$

(we only consider the most important of the three, the others are negligible)

v)

The Hubble time is around:

$$\boxed{t_H} = \frac{1}{H_0} \overset{(\text{class})}{=} \left(70\ \frac{Km/sec}{Mpc}\right)^{-1} = \left(70\ \frac{\cancel{Km} \Big/ \frac{years}{365 \cdot 24 \cdot 60^2}}{3{,}086 \cdot 10^{19} \cancel{Km}}\right)^{-1} = \left(7{,}1533 \cdot 10^{-11}\ years^{-1}\right)^{-1} = 1{,}3979 \cdot 10^{10}\ years$$

$$= \boxed{13979\ M_{years}}$$

The remaining photons without interaction will be of the order:

$$\boxed{N = N_0 \cdot e^{-d_H/\lambda}} = 1000 \cdot e^{-1{,}32 \cdot 10^{26} / 2{,}39 \cdot 10^{27}} = 1000 \cdot \underbrace{e^{-0{,}05523}}_{0{,}946} \simeq \boxed{946{,}27}$$

So only 54 out of 1000 photons will be absorbed in a Hubble time! Showing that the universe is pretty much transparent as we said previously!

Exercise 2. Let us take the system of equations formed by the Friedmann-Lemaître equation with non-vanishing cosmological constant ($\Lambda = 8\pi\, G_N\, \rho_\Lambda$),

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi\, G_N}{3}\,(\rho_m + \rho_\Lambda) - \frac{k}{a^2}\,, \qquad (1)$$

and the dynamical evolution equation for the scale factor,

$$\ddot a = -\frac{4\pi\, G_N}{3}\,(\rho_m + 3\,p_m - 2\,\rho_\Lambda)\,a\,. \qquad (2)$$

Here we assume that only one component of matter is present, e.g. non-relativistic, but we could equally understand that both (relativistic and non-relativistic) are around, and then $\rho_m \to \rho_m + \rho_r$ and $p_m \to p_m + p_r$ (provided of course we are far away from the transition epoch between the two).

[1] Do quote (always) your bibliographical sources every time you need to input data.
[2] Our Galaxy has the equivalent of $\sim 10^{11}$ stars as our sun.

i) Consider the possibility that the Λ-term is time-dependent: $\Lambda = \Lambda(t)$. Show that the equation

$$\dot\rho_\Lambda + \dot\rho_m + 3\,H\,(\rho_m + p_m) = 0 \qquad (3)$$

is a first integral of the previous system, irrespective of the value of the spatial curvature k; that is, a differential equation of smaller order that can replace one of these two equations. For $\dot\rho_\Lambda \neq 0$ there is an exchange of energy between matter and vacuum. Show in detail that any of the previous three equations can be derived from the other two.

ii) Check that $H^2 + \dot H = \ddot a/a$ and arrive at the following expression for $\dot H$ in terms of the pressure, matter density and spatial curvature:

$$\dot H = -4\pi\, G\,(\rho_m + \overset{w_m \rho_m}{\overset{\shortparallel}{p_m}}) + \frac{k}{a^2}\,.$$

What is the physical meaning of the minus sign ? Why is this result independent of ρ_Λ?

iii) Show that the equation for $\ddot a$ above can be replaced by the following expression (if desired):

$$2\dot H + 3\,H^2 = -8\pi\, G\,(p_m - \rho_\Lambda) - \frac{k}{a^2}\,.$$

Assume we are in the matter dominated epoch (MDE), well before the transition from deceleration to acceleration, and suppose that the spatial curvature is $k = 0$. Prove that $H(t) = 2/(3t)$ in that epoch and then check if this result is compatible with the previous expression in the MDE. Repeat the consistency check for the radiation dominated epoch (RDE). What is $H(t)$ in this case?

i) Derivating $\dot a^2$ from respect to time, we got:

$$\frac{d\dot a^2}{dt} = 2\dot a \ddot a \;; \qquad \frac{d\,(1)_r a^2}{dt} = 2\dot a\,(2)_r \;; \qquad \frac{d(1)_r}{dt} a^2 + (1)_r 2a\dot a = 2\dot a\,(2)_r$$

(where $(X)_r$ means the right part of eq. (X))

$$\underbrace{\left\{\frac{8\pi \cancel{G_N}}{\cancel{3}}(\dot\rho_m + \dot\rho_\Lambda) + \frac{2\cancel{k}}{\cancel{a^3}}\dot a\right\}}_{d(1)_r/dt} a^2 + 2\underbrace{\left\{\frac{8\pi \cancel{G_N}}{\cancel{3}}(\rho_m + \rho_\Lambda) - \frac{\cancel{k}}{\cancel{a^2}}\right\}}_{(1)_r} a\dot a = \cancel{2}\dot a \underbrace{\left(-\frac{4\pi \cancel{G_N}}{\cancel{3}}(\rho_m + 3p_m - 2\rho_\Lambda)\,a\right)}_{(2)_r}$$

$$(\dot\rho_m + \dot\rho_\Lambda)\,a^2 + 2(\rho_m + \cancel{\rho_\Lambda})\,a\dot a = -(\rho_m + 3p_m - 2\cancel{\rho_\Lambda})\,a\dot a$$

$$(\dot\rho_m + \dot\rho_\Lambda)\,a^2 + 3(\rho_m + p_m)\,a\dot a = 0 \;; \quad \boxed{\dot\rho_m + \dot\rho_\Lambda + 3H(\rho_m + p_m) = 0} \qquad (3)$$

So we have shown that (1) and (2) $\longrightarrow$ (3), and from $\frac{d(1)}{dt}a^2 + (1)\,2a\dot a = 2\dot a(2)$ it's obvious that:

$$(2)_r = \frac{1}{2}\frac{d(1)_r}{dt}\frac{a^2}{\dot a} + (1)_r a = \left\{\frac{4\pi G_N}{3}(\rho_m+\rho_\Lambda) + \frac{\cancel{K}}{a^3}\dot a\right\}\frac{a^2}{\dot a} + \left\{\frac{8\pi G_N}{3}(\rho_m+\rho_\Lambda) - \frac{\cancel{K}}{a^2}\right\} a \Bigg\}$$

$$\boxed{\ddot a = -\frac{4\pi G_N}{3}(\rho_m + 3p_m - 2\rho_\Lambda)\, a}$$

$$(2)_\ell = \frac{1}{2}\frac{d(1)_\ell}{dt}\frac{a^2}{\dot a} + (1)_\ell a = \frac{\cancel{2}}{\cancel{a}}\,\frac{\ddot a a - \cancel{\dot a^2}}{a^2}\,\frac{a^2}{\cancel{\dot a}} + \frac{\cancel{\dot a^2}}{a^2} a = \ddot a$$

So we have now also shown that (1) and (3) $\longrightarrow$ (2), and we are left with (2) and (3) $\longrightarrow$ (1) which we obtain from:

$$\frac{d(1)}{dt}a^2 + (1)\,2a\dot a = \frac{d(1)a^2}{dt} = 2\dot a(2) \longrightarrow (1)\,a^2 = \int 2\dot a\,(2)\,dt\,, \quad \text{so:}$$

$$(1)_r = \frac{1}{a^2}\int 2\dot a\left(-\frac{4\pi G_N}{3}(\rho_m + 3\cancel{p_m}^{\,0} - 2\rho_\Lambda)\,a\right)dt = -\frac{8\pi G_N}{3a^2}\int \overbrace{\dot a a}^{\frac{1}{2}\frac{da^2}{dt}}(\rho_m - 2\rho_\Lambda)\,dt = -\frac{8\pi G_N}{3\cancel{a^2}}\frac{1}{2}\left\{\cancel{a^2}(\rho_m - 2\rho_v - \rho_m + 2\rho_\Lambda)\right\} + C(t)$$

$$= \frac{8\pi G_N}{3}(\rho_m + \rho_\Lambda) + C(t)$$

$$(1)_\ell = \frac{1}{a^2}\int 2\dot a\ddot a\,dt = \frac{\dot a^2}{a^2} = H^2$$

So we obtain:

$$\boxed{H^2 = \frac{8\pi G_N}{3}(\rho_m+\rho_\Lambda) + C(t)} \quad (1)$$

, and we have checked the 3 possibilities!

ii)

Let's derive H respect time and check:

$$\boxed{H^2 + \dot H = \left(\frac{\dot a}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^{\cdot} = \cancel{\left(\frac{\dot a}{a}\right)^2} + \frac{\ddot a a - \cancel{\dot a^2}}{a^2} = \frac{\ddot a}{a}} \Longrightarrow \dot H = \frac{\ddot a}{a} - H^2$$

And substituting (2) and (1) we get:

$$\boxed{\dot H = \overbrace{-\frac{4\pi G_N}{3}(\rho_m + 3p_m - 2\cancel{\rho_\Lambda})\frac{\cancel{a}}{\cancel{a}}}^{\ddot a/a} - \overbrace{\left(\frac{8\pi G_N}{3}(\rho_m + \cancel{\rho_\Lambda}) - \frac{K}{a^2}\right)}^{H^2} = -4\pi G_N(\rho_m + p_m) + \frac{K}{a^2}}$$

Where the minus sign shows that matters tends to stop the expansion of the universe!

And ρ_Λ doesn't appear because it produces a constant H, not an increase in it!

iii)

If we are in MDE $\rho_m \gg \rho_v$ or ρ_Λ and $K=0$, we then have:

$$2\dot H + 3H^2 = -8\pi G\,\rho_m \cancel{w_m}^{\,0} = 0\ ; \quad 2\dot H = -3H^2\ ; \quad \frac{dH}{dt} = -\frac{3}{2}H^2\ ; \quad \int\frac{dH}{H^2} = -\frac{3}{2}\int dt\ ;$$

$$-\frac{1}{H(t)} = -\frac{3}{2}t\ ; \quad \boxed{H(t) = \frac{2}{3t}}$$

Now let's check this with the previous expression:

$$\boxed{2\dot H + 3H^2} = -2\frac{2}{3t^2} + 3\left(\frac{2}{3t}\right)^2 = -\frac{4}{3t^2} + \frac{4}{3t^2} = \boxed{0}$$

works ✓

If we are now in RDE, it would be the same, but changing $w_m = 0 \longrightarrow w_r = \frac{1}{3}$, so we get:

$$2\dot H + 3H^2 = -8\pi G\rho_r\frac{1}{3} = H^2 \longrightarrow 2\dot H + 2H^2 = 0\ ; \quad \dot H + H^2 = 0 \longrightarrow \boxed{H(t) = \frac{1}{t}}$$

And finally, let's also check this result:

$$\boxed{2\dot H + 3H^2} = -2\frac{1}{t^2} + 3\frac{1}{t^2} = \frac{1}{t^2} = H^2 = \boxed{\frac{-8\pi G_N}{3}\rho_r}$$

works ✓

Exercise 3. Consider the matter density equation indicate in the point i) of the previous exercise.

i) Solve that equation for the case when Λ =const. Obviously if Λ would be variable, you would need some input on the dynamics of the vacuum energy density. This will be the object of a subsequent exercise later on, but here let us first solve the simplest situation (which you actually know from previous courses). Specifically, assume that the equation of state of matter (EoS) is $p_m = w_m \rho_m$, with w_m =const. Trading the time variable for the scale factor, solve the mentioned differential equation and show that

$$\rho_m = \rho_m^0 a^{-3(1+w_m)},$$

where ρ_m^0 is the current energy density of matter, assuming that the scale factor is normalized such that $a = 1$ at present.

ii) Express the general solution in the case that w_m would be a certain function of the scale factor: $w_m = w_m(a)$.

iii) Apply your solution to compute the matter density in the case that $w_m(a) = a$. Do you find it a realistic possibility? What would be the matter density in the infinite past? Waht would be the matter density at the redshift of decoupling of the CMB? And in the infinite future?

i)

So, starting from an "easy" (3), we get to:

(Could also be done with: $\int_1^a \frac{da'}{a'}$)

$$\overset{0\,(\Lambda\,cte)}{\cancel{\dot\rho_\Lambda}} + \dot\rho_m + 3H(\rho_m + \overset{w_m \rho_m}{p_m}) = 0 \; ; \quad \frac{d\rho_m}{dt} + 3(1+w_m)\frac{\dot a}{a}\rho_m = 0 \; ; \quad \int_{\rho_m^0}^{\rho_m} \frac{d\rho_m}{\rho_m} = -3(1+w_m)\int_0^t \frac{\dot a}{a}\,dt \; ;$$

$$\ln\left(\frac{\rho_m}{\rho_m^0}\right) = -3(1+w_m)\ln\left(\frac{a}{\cancel{a(0)}}\right) \; ; \quad \frac{\rho_m}{\rho_m^0} = e^{-3(1+w_m)\ln(a)} \; ; \quad \boxed{\rho_m = \rho_m^0 \cdot a^{-3(1+w_m)}} \overset{*}{\equiv} \rho_{m\,\text{"simple"}}$$

ii)

If $w_m = w_m(a)$, we then have:

$$\ln\left(\frac{\rho_m}{\rho_m^0}\right) = -3\ln\left(\frac{a}{a(0)}\right) - 3\int_1^a \frac{w_m(a')}{a'}\,da' = -3\ln(a) - 3\left(w_m(0)\ln(a') + w_m'(0)\,a' + O(a'^2)\right)\Big|_1^a =$$

where $w_m(0)$ is w_m at $t=-\infty$, so we finally get:

$$\boxed{\rho_m = \rho_m^0\, e^{-3(1+w_m(0))\ln(a) - 3w_m'(0)(a-1)} = \rho_m^0\, a^{-3(1+w_m(0))}\, e^{-3w_m'(0)(a-1)}}$$

iii)

If $w_m(a) = a$, from our solution we got:

$$\boxed{\rho_m = \rho_m^0\, a^{-3}\, e^{-3(a-1)}}$$

If we now do again the previous steps with $w_m(a) = a$ from the start, we get:

$$\boxed{\rho_m = \rho_m^0\, e^{-3\ln(a) - 3\int_1^a \frac{\cancel{a}}{\cancel{a}}da} = \rho_m^0\, a^{-3}\, e^{-3(a-1)}}$$

<u>same!</u>

This answer would give the same result on ρ_m for the present time with:

$$\rho_m = \rho_m^0\, 1^{-3}\, \cancel{e^{0}}^{\,1} = \underline{\rho_m^0} \longleftrightarrow \rho_{m_{\text{"simple"}}}^{\circledast} = \rho_m^0\, 1^{-3(1+w_m)} = \underline{\rho_m^0} \qquad \underline{\text{same!}}$$

In the infinite past we would have:

$$\rho_m = \rho_m^0\, 0^{-3}\, e^{-3(0-1)} \longrightarrow \underline{\infty} \longleftrightarrow \rho_{m_{\text{"simple"}}} = \rho_m^0\, 0^{-3(1+w_m)} = \begin{cases} \infty & \text{for } w_m > -1 \quad \underline{\text{same to this case}} \\ \rho_m^0 & \text{for } w_m = -1 \\ 0 & \text{for } w_m < -1 \end{cases}$$

And in the infinite future we would have then:

$$\rho_m = \rho_m^0\, \infty^{-3}\, e^{-3(\infty-1)} = \underline{0} \longleftrightarrow \rho_{m_{\text{"simple"}}} = \rho_m^0\, \infty^{-3(1+w_m)} = \begin{cases} 0 & \text{for } w_m > -1 \quad \underline{\text{same to this case}} \\ \rho_m^0 & \text{for } w_m = -1 \\ \infty & \text{for } w_m < -1 \end{cases}$$

So all the limits behave similar to when $w_m = cte$ and $w_m > -1$, so it's seems kind of realistic.

And the last limit at the decoupling of CMB ($z = 1000$), we have:

$$\rho_m = \rho_m^0\, (10^{-3})^{-3}\, e^{-3(10^{-3})} \simeq \underline{\rho_m^0\, 10^9} \longleftrightarrow \rho_{m_{\text{"simple"}}} = \rho_m^0\, (10^{-3})^{-3(1+w_m)} = \rho_m^0 \cdot 10^{9(1+w_m)} = \begin{cases} \simeq \underline{\rho_m^0 \cdot 10^9} & \text{for } w_m > -1 \quad \underline{\text{same for this case}} \\ = \rho_m^0 & \text{for } w_m = -1 \\ \simeq \rho_m^0 \cdot 10^{-9} & \text{for } w_m < -1 \end{cases}$$

Exercise 4. i) Using the basic cosmological equations from Exercise 1, show that you can recover Einstein's static solution as a particular case, namely the solution characterized by

$$4\pi G\rho_m = \frac{1}{a^2} = \Lambda \,. = 8\pi G_N \rho_\Lambda$$

What is the geometry of the Universe satisfying this equation? Why? What is the sign of the vacuum energy for this universe?

ii) Show that the equilibrium solution that you have found in the previous point corresponds to unstable equilibrium and hence corresponds to a runaway solution (as proven originally by Eddington). Specifically, show that if a_{ea} is the equilibrium position, a perturbation $a_{eq} \to a_{eq} + \delta a$ satisfies

$$\delta\ddot{a} = 4\pi G\rho_m \delta a \,,$$

Why this shows that equilibrium is unstable? Einstein's universe is a "fine-tuned" universe, hence unnatural!

i)

First we see that we will need $\underline{\ddot a = 0}$ in order to have:

$$\ddot a = -\frac{4\pi G_N}{3}(\rho_m + 3p_m - 2\rho_\Lambda)\,a \longrightarrow \boxed{4\pi G_N \rho_m = 8\pi G_N \rho_\Lambda} \;;$$

The next things we need for this solutions are that $\begin{cases} \dot a = 1 \ (a = t) & \text{and } k = 0 ; \\ \dot a = 0 \ (a = cst) & \text{and } k = 1 : \end{cases}$ We'll go with this one, since it is the typical static solution

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G_N}{\cancel{3}}\left(\rho_m + \overbrace{\rho_\Lambda}^{3\rho_\Lambda}\right) - \frac{k}{a^2} \longrightarrow \boxed{\frac{1}{a^2} = 8\pi G_N \rho_\Lambda = 4\pi G_N \rho_m}$$

We then see this is the solution for a closed ($k > 0$) static universe ($a = constant$, $\dot a = 0$).

Finally, the sign for the vacuum energy is positive since: $\boxed{\Lambda = \frac{1}{a^2} = \left(\frac{1}{a}\right)^2 \geq 0}$

ii)

From $4\pi G \rho_m = \frac{1}{a^2} = \Lambda = 8\pi G \rho_\Lambda \longrightarrow \rho_m = 2\rho_\Lambda \longrightarrow \ddot{a} = -\frac{4\pi G}{\cancel{3}}(\cancel{\rho_m} + \cancel{3} p_m - \cancel{2\rho_\Lambda})a = -4\pi G p_m a$

And from conservation equation (with $\dot{\rho}_m + \dot{\rho}_\Lambda = 0$):

$$3H(\rho_m + p_m) = 0 \longrightarrow \rho_m = -p_m \longrightarrow \ddot{a} = 4\pi G \rho_m a$$

Which with the perturbation beomes:

$$\cancel{\ddot{a}} + \delta\ddot{a} = 4\pi G \rho_m (\cancel{a} + \delta a) \longrightarrow \boxed{\delta\ddot{a} = 4\pi G \rho_m \delta a}$$

$\delta\ddot{a} \propto \delta a$, means that if we separate $a_{eq} \to a_{eq} + \delta a$ from the equilibrium, in the next step the separation increases, and then increases even more, going away from the equilibrium exponientally:

$$\ddot{x} = c x \longrightarrow x = e^{cT}, \text{ so: } \boxed{\delta a = e^{4\pi G_N T}}$$

Fine-tuned universe!

Exercise 5. Let us come back to our dicsussion on energy densities in Exercise 2.

H

i) Assume that $w_m = -1$, What would be the density and pressure? This is a very strange result for matter, isn't it? So let us assume that it actually applies to some conserved fluid X (maybe some dark energy fluid). To make it more general, assume that it satisfies the EoS $p_X = w_X \rho_X$, with constant w_X. Clearly, if $w_X = -1$ we are back to the previous situation, which corresponds... to what? And what is the maximum value that w_X can take such that a dark energy (DE) fluid of this sort can produce late time acceleration for the Universe?

ii) Consider the situation in which w_X is not constant but follows the evolution law:

$$w_X(a) = w_0 + (1-a)w_1 \,,$$

where w_0 and w_1 are constants. Notice that this nonconstant parameterization of $w_X(a)$ looks minimally meaningful since it is well-defined both at present and at any point in the remote past (where the matter density becomes unbounded!). Compute explicitly $\rho_X(a)$ for such parameterization and provide a physical interpretation of it. Show that

$$\rho_X(a) = \rho_X^0 a^{-3(1+w_0+w_1)} e^{-3w_1(1-a)} \,.$$

iii) Do you recover from this expression the case when the EoS is constant? Now make a reasonable choice for the range of the parameters w_0 and w_1 such that this fluid can provide a minimally sensible picture of the DE, as we usually conceive it. What is the behavior of the energy density of such DE fluid in the infinite past and in the remote future (for the range of parameters you have chosen)? Do you find it sensible? Argue your position.

i)

We start from:

$$\begin{cases} \rho_m = \rho_m^0 \, a^{-3(1+w_m)} \\ p_m = w_m \rho_m = w_m \rho_m^0 \, a^{-3(1+w_m)} \end{cases} \xrightarrow{w_m = -1} \begin{cases} \rho_m = \rho_m^0 \\ \boxed{p_m = -\rho_m^0 = -\rho_m} \end{cases}$$

which corresponds to the situation in the previous excercice with:

$p_m = -\rho_m = \text{constant}$

Einstein static solution!

The maximum w_m value for having a later expansion is:

$$\ddot{a} = -4\pi G_N p_m = -4\pi G_N w_m \rho_m, \text{ so for } \ddot{a} > 0 \text{ (later expansion), we need } \boxed{w_m < 0}$$

ii)

If we now have $w_x(a) = w_0 + (1-a)\,w_1$, the solution for ρ_m will be:

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0 \xrightarrow{\text{Ex 3)}} \boxed{\rho_m} = \rho_m^0\, a^{-3(1+w_m(0))}\, e^{-3w_m'(0)(a-1)} \overset{(w_x'(a) = -w_1)}{=}$$

$$= \boxed{\rho_m^0\, a^{-3(1+w_0+w_1)}\, e^{-w_1(a-1)}} \checkmark\checkmark$$

To get it's physical interpretation let's compute its limits first:

$$\begin{cases} w_0+w_1 > -1 \\ \rho_m(a=0) \longrightarrow \infty \quad , \; \rho_m(a=\infty) \longrightarrow 0 \quad , \; \rho_m(a=1) = \rho_m^0 & \underline{\infty \longrightarrow \rho_m^0 \longrightarrow 0} \\ w_0+w_1 = -1 \\ \rho_m(a=0) = \rho_m^0 e^{w_1} \quad , \; \rho_m(a=\infty) \longrightarrow 0 \quad , \; \rho_m(a=1) = \rho_m^0 & \underline{\rho_m^0 e^{w_1} \longrightarrow \rho_m^0 \longrightarrow 0} \\ w_0+w_1 < -1 \\ \rho_m(a=0) \longrightarrow 0 \quad , \; \rho_m(a=\infty) \longrightarrow 0 \quad , \; \rho_m(a=1) = \rho_m^0 & \underline{0 \longrightarrow \rho_m^0 \longrightarrow 0} \end{cases}$$

The pressure from matter has a constant component $w_0 + w_1$ and a linear component $-w_1 a$, so when the universe expands the pressure starts reducing. We see from the limits that the initial conditions change depending on the constant part, but <u>the ending is always $\rho_m = 0$.</u>

iii)

For the constant case we only need to set $w_1 = 0$, which then gives:

$$\boxed{\rho_m = \rho_m^0\, a^{-3(1+w_0)}}$$ which is <u>the expression we found in Ex 3), where EoS is constant.</u>

Now we need to think of sensible choices for w_0 and w_1. The first thing one can think of, is that which when $a=1$, we expect $w_x = -1 \longrightarrow w_x(a=1) = \underline{w_0 = -1}$. Another good asumption would be that when and if a varies w_x doesn't vary by alot, so that $\underline{|w_1| < 1}$. From here we identify the two cases we have seen in class:

- $(1-a)\,w_1 < 0$: phantom DE $(w_x < -1)$
- $(1-a)\,w_1 > 0$: quintessace $(w_x > 1)$

so, we see that <u>if we had quintessence in the past $w_1 > 0$, in the future we will have phantom DE</u> and viceversa.

So we have two possibilities

		past (a=0)	present (a=1)	future ($a\to\infty$)	
$\bullet\; w_1 < 0 \longrightarrow w_1 + w_0 < -1$ so:		phantom, $\rho_m \to 0$	$\rho_m = \rho_m^0$	quintessence, $\rho_m \to 0$	✓
$\bullet\; w_1 > 0 \longrightarrow w_1 + w_0 > -1$ so:		quintessace, $\rho_m \to \infty$	$\rho_m = \rho_m^0$	phantom, $\rho_m \to 0$	✗

It would be sensible to have $\rho: 0 \longrightarrow \rho_m^0 \longrightarrow 0$, because we would have a start and finish where matter didn't dominate, and a present where it could be dominating, as our best theories sugest. So the reasonable choices to me are:

$$\boxed{w_0 = -1} \quad , \quad \boxed{w_1 < 0 \quad \text{with } |w_1| < 1} \qquad (w_0 + w_1 < -1)$$

Exercise 6. Take the cosmological equations you have played with in Exercise 1.

i) Using the equations above, prove that the deceleration parameter of the Universe ($q \equiv -\ddot{a}\,a/\dot{a}^2$) in the presence of several fluids with *constant* equation of state parameters ω_n (for radiation, matter, vacuum energy etc) can be written, at a given cosmological redshift z, as

$$q(z) = \sum_n (1 + 3\omega_n)\,\frac{\Omega_n(z)}{2}\,,$$

where $\Omega_n(z) = \rho_n(z)/\rho_c(z)$ involve the various normalized densities, which are bound to satisfy the sum rule $\sum_n \Omega_n(z) + \Omega_k(z) = 1$ at any value of the redshift z, with $\Omega_k(z) = -(k/H_0^2)(1+z)^2$ the spatial curvature parameter. Evaluate this formula for $z = 0$ in the ΛCDM model and show that the current value of the acceleration parameter reads

$$q_0 = \frac{\Omega_m^0}{2} - \Omega_\Lambda^0 = \frac{3\Omega_m^0 - 2}{2}\,,$$

where the second equality is valid *only* for a spatially flat universe (why?). What is the value of q_0 according to the present data? Explain the meaning of this numerical value and its sign.

ii) Check that the deceleration parameter can also be computed as follows

$$q(z) = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{1+z}{2H^2}\frac{dH^2(z)}{dz}$$

where z is the cosmological redshift. Apply this formula to the Hubble function for the ΛCDM and show that the transition redshift from deceleration to acceleration is given by

$$z^* = -1 + \left(\frac{2\Omega_\Lambda^0}{\Omega_m^0}\right)^{1/3}.$$

Does this result assume vanishing Ω_k^0? Evaluate z^* according to the latest *Planck* satellite data from July 2018 (see arXiv:1807.06209).

i)

Let's start by writing the decelerator parameter as:

From Ex 2) $\qquad P_\Lambda = -\rho_\Lambda$; $-2\rho_\Lambda = \rho_\Lambda + 3P_\Lambda$

$$-q = \frac{\ddot{a}\,a}{\dot{a}^2} = \frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = \frac{1}{H^2}\frac{\ddot{a}}{a} = \frac{1}{H^2}\left(-\frac{4\pi G_N}{3}(\rho_m + 3P_m - 2\rho_\Lambda)\right) = -\frac{4\pi G_N}{3H^2}\sum_n (1+3w_n)\rho_n$$

In the other hand we have:

$$(1+z)^2$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}(\rho_m + \rho_\Lambda) - \frac{K}{a^2} = \frac{8\pi G_N}{3}(\rho_m + \rho_\Lambda) + H_0^2\,\Omega_K(z) = \frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda + \rho_c\,\Omega_K(z)\right) =$$

$$= \frac{8\pi G_N}{3}\rho_c\,(\underbrace{\Omega_m + \Omega_\Lambda + \Omega_K(z)}_{1}) = \frac{8\pi G_N}{3}\rho_c$$

So finally we have that:

$$\boxed{q} = \frac{4\cancel{\pi}\cancel{G_N}}{\cancel{3}\,\frac{8\cancel{\pi}\cancel{G_N}}{\cancel{3}}\rho_c}\sum_n (1+3w_n)\rho_n = \boxed{\sum_n (1+3w_n)\frac{\Omega_n}{2}}$$

Now we have to substitute $z=0$, giving:

$$\boxed{q_0} = (1+3\overset{0}{\cancel{w_m}})\frac{\Omega_{m0}}{2} + (1+3\overset{-1/3}{\cancel{w_\Lambda}})\frac{\Omega_{\Lambda 0}}{2} = \boxed{\frac{\Omega_{m0}}{2} - \Omega_{\Lambda 0}}$$

And in a flat model because $\sum_n \Omega_n + \overset{0}{\cancel{\Omega_K}} = 1 \longrightarrow \Omega_\Lambda + \Omega_m = 1 \longrightarrow \Omega_\Lambda = 1 - \Omega_m$, so:

$$\boxed{q_0} = \frac{\Omega_{m0}}{2} - (1 - \Omega_m) = \frac{3\Omega_{m0}}{2} - 1 = \boxed{\frac{3\Omega_{m0} - 2}{2}}$$

ii)

Since $w_m = 0$, $w_\Lambda = -1$ $\Longrightarrow$ $\rho_m = \rho_m^0\, a^{-3} = \rho_m^0 (1+z)^3$ and $\rho_\Lambda = \rho_\Lambda^0$, so:

$$H^2(z) = \frac{8\pi G_N}{3}(\rho_m + \rho_\Lambda) = \underbrace{\frac{8\pi G_N}{3}}_{H_0^2/\rho_c}\left(\rho_m^0 (1+z)^3 + \rho_\Lambda^0\right) = H_0^2\left(\Omega_m^0 (1+z)^3 + \Omega_\Lambda^0\right)$$

And from:

$$H = \frac{\dot a}{a} \; ; \; \dot H = \frac{\ddot a\, a - \dot a^2}{a^2} \; ; \; \dot H = -q H^2 - H^2 \; ; \; \frac{\dot H}{H^2} = -q - 1 \; ; \; \boxed{q = -\frac{\dot H}{H^2} - 1}$$

which with the previously computed $H(z) = H_0\sqrt{\Omega_m^0 (1+z)^3 + \Omega_\Lambda^0}$, gives:

$$\boxed{q(z)} \overset{(*)}{=} -1 - \frac{\dot z}{2H^3}\frac{dH^2}{dz} \overset{(**)}{=} \boxed{-1 - \frac{(1+z)}{2H^2}\frac{dH^2}{dz}}$$

$$\left(\left.\begin{aligned} (*)\quad \dot H &= \frac{H_0^2}{2H}\, 3(1+z)^2\, \dot z\, \cancel{\Omega_m} \\ \frac{dH^2}{dz} &= H_0^2\, 3(1+z)^2\, \cancel{\Omega_m^0} \end{aligned}\right\} \quad \begin{aligned} \dot H &= \frac{\dot z}{2H}\frac{dH^2}{dz} \\ \frac{\dot H}{H^2} &= \frac{\dot z}{2H^3}\frac{dH^2}{dz} \end{aligned} \right) \quad \left((**)\; \dot H(z) = \frac{\dot a}{a} = \frac{-\dot z}{\cancel{(z+1)^2}}\,\frac{\cancel{z+1}}{1} = -\frac{\dot z}{(z+1)} \right)$$

And to end we need to find the transition redshift (z^*) from deceleration to acceleration:

$$\ddot a = 0 \longrightarrow q = 0 = -1 - \left.\frac{(1+z)}{2H^2}\frac{dH^2}{dz}\right|_{z^*} = -1 + \frac{1+z^*}{2\cancel{H_0^2}\left(\Omega_m^0 (1+z^*)^3 + \Omega_\Lambda^0\right)}\cancel{H_0^2}\,\Omega_m^0\, 3(1+z^*)^2$$

which means that our z^* fullfills:

$$0 = -1 + \frac{\Omega_m^0\, 3\,(1+z^*)^3}{2\left(\Omega_m^0 (1+z^*)^3 + \Omega_\Lambda^0\right)} \; ; \; 2\left(\Omega_m^0 (1+z)^3 + \Omega_\Lambda^0\right) = \Omega_m^0\, 3(1+z^*)^3$$

$$\Omega_m^0 (1+z^*)^3 = 2\,\Omega_\Lambda^0 \; ; \; (1+z^*) = \sqrt[3]{\frac{2\Omega_\Lambda^0}{\Omega_m^0}} \; ; \; \boxed{z^* = -1 + \left(\frac{2\,\Omega_\Lambda^0}{\Omega_m^0}\right)^{1/3}}$$

where this result asumed Ω_k^0 vanishing since $k=0$.

And finally, we are going to evaluate z^* according to the last measures of Plank satellite on July 2018:

$$\begin{cases} \Omega_m^0 = 0{,}315 \pm 0{,}007 \\ \Omega_k^0 = 0{,}0010 \pm 0{,}0002 \end{cases} \quad \left(\Sigma\, \Omega = 1\right) \longrightarrow \Omega_\Lambda = 0{,}684 \longrightarrow \boxed{z^* = -1 + \left(\frac{2\cdot 0{,}684}{0{,}315}\right)^{1/3} = 0{,}633}$$

Exercise 7. Let us make some study on the time evolution and the age of the Universe.
i) Show that the formulae giving the cosmic time as a function of the scale factor is

$$t - t_1 = \int_{a_1}^{a(t)} \frac{da}{a\,H(a)}$$

where $H = \dot{a}/a$ is the expansion rate. Here t_1 is some initial time and $a_1 = a(t_1)$. What is the formula that gives the age of the Universe, t_0, in terms of an integral over the cosmological redshift z?

ii) In general these integrals cannot be solved by quadrature, but there are some interesting cases that can be worked out easily (see the Table below). Compute $a = a(t)$ during the matter and radiation dominated epoch for the Einstein-de Sitter's (EdS) Universe (CDM model). Then compute the age of each of the universes in the Table in units of the present Hubble time H_0^{-1} (**Solution**: a) $t_0 = 2/(3H_0)$, b) $t_0 = 1/H_0$, c) $t_0 = \infty$.). What drives the evolution in Milne's case b)? And in case c)?

(a)	$\Omega_m^0 = 1$	$\Omega_\Lambda^0 = 0$	(Einstein-de Sitter's Universe)
(b)	$\Omega_m^0 \simeq 0$	$\Omega_\Lambda^0 \simeq 0$	(Milne's Universe)
(c)	$\Omega_m^0 = 0$	$\Omega_\Lambda^0 = 1$	(Inflationary Universe)

iii) Using numerical integration, compute the realistic age of our Universe, t_0, assuming valid the standard ΛCDM model of cosmology, hence $\Lambda \neq 0$. To this end, take the latest data released by the *Planck* satellite on the cosmological parameters (assuming the upper bound on Ω_k^0, and then 10 times its value). Express the result in Gigayears (Gyr) or (US) billion years; recall that 1 Gyr $= 10^9$ yr. Compare the obtained result with the age value of the EdS Universe. Does it pay to take into account the radiation epoch in the calculation? Why?

i)

It is very easy to proof this, we will start from:

$$H = \frac{\dot a}{a} \;;\quad Ha = \frac{da}{dt} \;;\quad dt = \frac{da}{H(a)\,a} \;;\quad \int_{t_1}^{t} dt = \int_{a(t_1)}^{a(t)} \frac{da}{H(a)\,a} \;;\quad \boxed{t - t_1 = \int_{a_1}^{a(t)} \frac{da}{H(a)\,a}}$$

To compute the age of the universe we only need to asign $\begin{cases} a(t) = a(\text{present}) = 1 \\ a_1 = a(\text{origin}) = 0 \end{cases}$, which gives:

$$\text{present} - \text{origin} = \boxed{t_0 = \int_0^1 \frac{da}{H(a)\,a}}$$

And now we need to express $a = \frac{1}{z+1}$, $da = \frac{-dz}{(z+1)^2}$, so:

$$\boxed{t_0 = \int_{\substack{a=0\\ z=\infty}}^{\substack{z=0\\ a=1}} \frac{-dz\;(z+1)}{H(z)\;(z+1)^2} = \int_0^\infty \frac{dz}{H(z)(z+1)}} \quad \text{where} \quad \boxed{H(z) = \frac{\dot a}{a} = \frac{-\dot z}{(z+1)^{\cancel{2}}}\,\frac{\cancel{z+1}}{1} = -\frac{\dot z}{(z+1)}}$$

$$\left(\underline{\text{Check:}} \quad \int_{t_1}^{t} dt = \int_{z_1}^{z(t)} \frac{-dz}{H(z)(z+1)} \Rightarrow t_0 = t - t_1 = \int_0^\infty \frac{dz}{H(z)(z+1)} \right)$$

ii)

Now we want to compute $a(t)$, so:

$$\int_0^t dt = \int_0^{a(t)} \frac{da}{a' H(a)} \quad \text{with} \quad H^2 \overset{(EdS)}{=} H_0^2\left(\Omega_m^0 a^{-3} + \Omega_r^0 a^{-4}\right) = \begin{cases} \text{matter era: } H = H_0 a^{-3/2} \\ \text{radiation era: } H = H_0 a^{-2} \end{cases}$$

So:

$$T = \int_0^{a(t)} \frac{da}{a H(a)} \begin{cases} \dfrac{1}{H_0}\displaystyle\int_0^{a(t)} \frac{da}{a^{-3/2}} = \frac{1}{H_0}\frac{2}{3} a^{3/2}\Big|_0^{a(t)} \\ \dfrac{1}{H_0}\displaystyle\int_0^{a(t)} \frac{da}{a^{-1}} = \frac{1}{H_0}\frac{1/2}{a^{-2}}\Big|_0^{a(t)} \end{cases} \Longrightarrow \begin{cases} t = \dfrac{2}{3H_0} a(t)^{3/2} \; ; \; \boxed{a(t) = \left(\frac{3}{2} H_0 t\right)^{2/3} \quad \text{matter era}} \\ t = \dfrac{1}{2H_0} a(t)^2 \; ; \; \boxed{a(t) = \sqrt{2 H_0 t} \quad \text{radiation era}} \end{cases}$$

And now we will compute t_0 for each of the universes in the Table:

a) $\Omega_m^0 = 1$, $\Omega_\Lambda^0 = 0$, EdS universe:

Using our previous results, $\overset{1}{\cancel{a(t_0)}} = \left(\frac{3}{2} H_0 t_0\right)^{2/3}$; $\boxed{t_0 = \frac{2}{3} H_0}$

b) $\Omega_m^0 = 0$, $\Omega_\Lambda^0 = 0$, Milne's Universe:

$$H = H_0 a^{-1} \Longrightarrow t_0 = \frac{1}{H_0}\int_0^{a(t_0)} \frac{da}{a^0} = \frac{1}{H_0} a \Big|_0^{a(t)} \Longrightarrow t_0 = \frac{1}{H_0} \overset{1}{\cancel{a(t_0)}} \; ; \; \boxed{t_0 = \frac{1}{H_0}}$$

c) $\Omega_m^0 = 0$, $\Omega_\Lambda^0 = 1$, Inflationary Universe:

$$H = H_0 a^0 \Longrightarrow t_0 = \frac{1}{H_0}\int_0^{a(t_0)} \frac{da}{a} = \frac{1}{H_0} \ln(a) \Big|_0^{a(t_0)} \Longrightarrow t_0 = \frac{1}{H_0}\left(\overset{0}{\cancel{\ln(1)}} - \overset{-\infty}{\cancel{\ln(0)}}\right) \; ; \; \boxed{t_0 = \infty}$$

iii)

For the ΛCDM model we will have:

$$H^2(z) = H_0^2\left(\Omega_m^0 a^{-3} + \Omega_k^0 a^{-2} + \Omega_\Lambda^0 a^{-3(1+w_\Lambda)}\right)$$

Where from Plank 2018, we have:

- $H_0 = 67{,}4 \; Km \cdot s^{-1} \cdot Mpc^{-1}$
- $\Omega_m^0 = 0{,}315$
- $w_\Lambda = -1{,}03$
- $\Omega_k^0 = 0{,}001 \pm 0{,}002 \xrightarrow{\times 10 \text{ upper bound}} \Omega_k^0 = 0{,}03$
- $\Omega_\Lambda^0 = 1 - \Omega_m^0 - \Omega_k^0 = 0{,}655$

With this values and using Mathematica we get:

And so, the age of the universe will be:

$(1 \; Mpc = 3{,}086 \cdot 10^{19} \; Km)$

And so, the age of the universe will be:

$(1\,\mathrm{Mpc} = 3{,}086 \cdot 10^{19}\,\mathrm{Km})$

$$\boxed{t_0 = \int_0^{a(t_0)=1} \frac{da}{a\,H(a)} = \frac{1}{H_0}\int_0^1 \frac{da}{\left(\Omega_m^0\, a^{-3} + \Omega_k^0\, a^{-2} + \Omega_\Lambda^0\, a^{-3(1+w_\Lambda)}\right)^{1/2} a} = \frac{1}{H_0}\, 0{,}945 = 14{,}03 \cdot 10^{-3}\ \frac{\mathrm{s\,Mpc}}{\mathrm{Km}} = 13{,}73\ \mathrm{Gyrs}}$$

So comparing this result with t_0 for EdS, which is around $t_0 \simeq 9{,}68$ Gyr, we can see that the age of the universe in the ΛCDM model is bigger than in EdS, due to Λ!

The reason we have not included the radiation terms is because they go with a^{-4}, and they go to 0 very fast, further more the time where radiation dominated was pretty short, so it wouldn't contribute much to the computation of the age of the universe.

Exercise 8. We retake now the notion of co-moving "particle horizon". Consider the FLRW metric expressed in the $(a(t), \chi, \theta, \phi)$ parametrization (check your notes on cosmology and write down ds^2 explicitly in these coordinates). Of course, χ, θ, ϕ for a fundamental observer are comoving coordinates and hence independent of time. The particular one χ plays the role of "radial" coordinate (what does it mean?) Let us assume that a light ray propagates fully along the χ direction , i.e.with $\theta = \phi =$const. The comoving particle horizon at time t, which we denote $\chi_{ph}(t)$, reads as follows:

$$\chi_{ph}(t) = \int_0^t \frac{cdt'}{a(t')} = \int_0^{a(t)} \frac{cda}{a^2 H(t)} = \int_0^\infty \frac{cdz}{H(z)}$$

i) Explain the origin, meaning and equivalence of these formulas. What is the upper limit of integration of the last integral if we wish to compute the particle horizon at the present time? What is the physical interpretation of $\chi_{ph}(t)$ for spheric and flat universes? Explain why the proper distance to the particle horizon is $d_{ph}(t) = a(t)\,\chi_{ph}$.

ii) Compute $d_{ph}(t)$ for a flat radiation-dominated Universe without cosmological constant. Idem for a matter-dominated one at an early epoch (hence with negligible cosmological constant). Why the answer is not just ct?

iii) What is the horizon if $a \propto t^n$? Retrieve the previous results as particular cases of it. Show that the Hubble horizon at time t (i.e. the distance $d_H(t) = c/H(t)$), is related to the particle horizon through $d_{ph} = n d_H/(1-n)$. Show that the observable part of the universe expands at a velocity

$$V_{ph} = \frac{c}{1-n} = c + V_r$$

where V_r is the recession velocity of galaxies at the particle horizon ($V_r = H d_{ph}$).

iv) Show that the Hubble horizon at time t is the distance where galaxies recede from us at $V = c$, whereas the recession velocity of galaxies at the particle horizon is always $V_{ph} > c$. Explain this result and its interpretation.

v) Something strange happens with the above formulae if $n > 1$. Why? What it means? Write down the previous results in terms of the deceleration parameter q .

vi) Compute the particle horizon at the decoupling time (redshift $z_d \sim 1000$) and compare it to the value of the horizon now, which is $d_p(t_0) \sim 14$ Gpc. The last number cannot be computed with accuracy using the formulas above. Explain why. Using this result, show that the standard cosmological model predicts that there are $\sim 10^5$ causally disconnected regions in our universe, in the sense that the CMB that we observe from any one of these regions is equally homogeneous and isotropic, what is surprising since these hundred thousand regions were never in causal contact!!

Note: Assume for simplicity that the universe has been all the time in the matter-dominated epoch up to decoupling. Why this assumption will not alter the result significantly?

$$ds^2 = -c^2 dt^2 + a^2(t)\left\{d\chi^2 + r(\chi)\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right\} \quad \text{where} \quad r(\chi) = \begin{cases} \sinh\chi & \text{for } K=-1 \\ \chi & \text{for } K=0 \\ \sin\chi & \text{for } K=+1 \end{cases}$$

i)

For a light ray propagating in χ:

$$d\theta = d\phi = 0 \longrightarrow 0 = ds^2 = -c^2dt^2 + a^2(t)\,d\chi^2 \longrightarrow d\chi = \frac{c\,dt}{a(t)} \;;\; \boxed{\chi_{ph}(t) = \int_0^t \frac{c\,dt'}{a(t')}}$$

Where we have find the first formula for the comoving particle horizon, to get the next ones:

$$H = \frac{\dot a}{a} \longrightarrow \frac{da}{dt} = aH \longrightarrow dt = \frac{da}{a\,H} \longrightarrow \boxed{\chi_{ph}(t) = \int_0^{a(t)} \frac{c\,da}{a^2 H(a)}}$$

And to get the final expression:

$$1+z = \frac{1}{a} \longrightarrow da = -\frac{dz}{(1+z)^2} \longrightarrow \boxed{\chi_{ph}(t) = -\int_\infty^{z(t)} \frac{dz}{H(z)} \overset{t=t_0}{=} \int_0^\infty \frac{dz}{H(z)}}$$

Where χ_{ph} is the "radius" of the observable universe. But if we actually wanted to compute the particle horizon at present time, the upper limit then should start at the decoupling epoch $\begin{cases} z_{dec} \sim 10^3 \\ a_{dec} \sim 10^{-3} \end{cases}$.

Finally say that $\chi_{ph}(t)$ does not change with the expansion of the universe, in order to do so and to obtain the proper distance, we need to multiply by the scale factor: $\boxed{d_{ph}(t) = a(t)\,\chi_{ph}(t)}$

ii)

$\Lambda = 0$, flat radiation dominated universe:

$$a \propto t^{1/2} \longrightarrow d_{ph}(t) = a\int_0^t \frac{c\,dt'}{a} = t^{1/2}\int_0^t \frac{c\,dt'}{t'^{1/2}} = 2c\,t^{1/2}\cdot t^{1/2} \longrightarrow \boxed{d_{ph}(t) = 2ct}$$

$\Lambda = 0$, flat matter dominated universe:

$$a \propto t^{2/3} \longrightarrow d_{ph}(t) = a\int_0^t \frac{c\,dt'}{a} = t^{2/3}\int_0^t \frac{c\,dt'}{t'^{2/3}} = 3\,c\,t^{2/3}\,t^{1/3} \longrightarrow \boxed{d_{ph}(t) = 3ct}$$

The answer ct would be for a total static universe, in both our cases we have expansion, and then the horizon grows faster.

$$\int \frac{dx}{x^n} = \int x^{-n}\,dx \qquad x^{-(n-1)} = \frac{1}{x^{n-1}}$$

iii)

If we had $a \propto t^n$ then we would have:

$$d_{ph}(t) = a\int_0^t \frac{c\,dt'}{a} = t^n \int_0^T \frac{c\,dt'}{t'^n} = \begin{cases} t^n \dfrac{c}{t^{n-1}}\,\dfrac{1}{1-n} = \dfrac{1}{1-n}\,ct & \text{for } n<1 \text{ or } n>1 \\ ct\,(\ln(t) - \ln(0)) = \infty & \text{for } n=1 \end{cases}$$

Which fullfills both the previous cases for $n = \frac{1}{2}$ and $n = \frac{2}{3}$.

Now let's find the relation of the Hubble horizon d_H with the particle horizon d_{ph}:

$$\left.\begin{aligned} d_H = \frac{c}{H} \text{ with } H = \frac{\dot a}{a} = \frac{n t^{n-1}}{t^n} = \frac{n}{t} \longrightarrow d_H = \frac{ct}{n} \end{aligned}\right\} \quad \boxed{d_H = \frac{1-n}{n}\,d_{pH} \;;\; d_{pH} = \frac{n}{1-n}\,d_H}$$

$$d_H = \frac{c}{H} \quad \text{with} \quad H = \frac{\dot a}{a} = \frac{n t^{n-1}}{t^n} = \frac{n}{t} \longrightarrow d_H = \frac{ct}{n}$$
$$d_{ph} = \frac{1}{1-n} ct$$
$$\boxed{d_H = \frac{1-n}{n} d_{pH} \quad ; \quad d_{pH} = \frac{n}{1-n} d_H}$$

If we define the recession velocity of galaxies as: $V_r = H d_{ph} = \frac{cn}{1-n}$

The velocity will then be: $\boxed{V_{ph} = \frac{d_{ph}}{t} = \frac{c}{1-n} = \frac{c(1-n+n)}{1-n} = c + \frac{cn}{1-n} = c + V_r}$

iv)

The recession velocity of a galaxy at the Hubble horizon, $d_H = \frac{c}{H}$ is:

$$\boxed{V_r = H d_H = H \frac{c}{H} = c}$$

$V_r = \frac{cn}{1-n}$

And because $V_{ph} = c + V_r$
$$\begin{cases} V_{ph} > c & \text{if} \quad V_r > 0 \quad n < 1 \longrightarrow \underline{\text{expanding universe}} \\ V_{ph} < c & \text{if} \quad V_r < 0 \quad n > 1 \longrightarrow \underline{\text{contracting universe}} \end{cases}$$

v)

If $n > 1$ then $V_r < 0 \longrightarrow V_{ph} < c \longrightarrow$ contraction

To see this better we will compute the deceleration parameter:

$$\boxed{q = -\frac{\ddot a a}{\dot a^2} = -\frac{\cancel{k}(n-1)\cancel{t^{n-2}} \cdot \cancel{t^{n}}}{\cancel{k t^{n-1}} \cdot n \cancel{t^{n-1}}} = -\frac{n-1}{n} = \frac{1-n}{n}}$$

$n < 1 \qquad q > 0 \qquad \underline{\text{expanding universe}}$

$n > 1 \qquad q < 0 \qquad \underline{\text{contracting universe}}$

So the precious results now are:

$$V_r = c/q \quad , \quad V_{ph} = c + c/q = c\,\frac{1+q}{q}$$

vi)

As before, we are not going to consider radiation, because it's briefness, so:

$$H^2 = H_0^2 a^{-3} \longrightarrow \chi_d = \int_0^{a_d} \frac{c\,da}{a^2 H(a)} = \int_0^{a_d} \frac{c\,da}{H_0 a^{1/2}} = \frac{1}{H_0} 2c\, a^{1/2}\Big|_0^{a_d} = \frac{2c}{H_0}\sqrt{a_d}$$

And then the proper particle horizon at decoupling (d_d), will be:

$$d_d = a_d \chi_d = \frac{2c}{H_0} a_d^{3/2} \longrightarrow \boxed{d_d = 0{,}28 \text{ Mpc}}$$

Plank 2018 ($H_0 = 67{,}4$ Km/s Mpc)

which is 10^4 or 10^5 less orders of magnitude, <u>telling us we have around those 10^5 disconnected regions, which in the past were much closer, explaing it's homogeneities!</u>

Exercise 9. Let us now face the age computation of the ΛCDM model analytically. You may assume vanishing spatial curvature, which is realistic (why?).

i) Use the equations $(k=0)$ of Exercise 1 to show that the Hubble rate $H = H(t)$ in the matter-dominated epoch (MDE) can be obtained by solving the differential equation

$$\dot{H} + \frac{3}{2}H^2 = 4\pi\, G\, \rho_\Lambda = \frac{\Lambda}{2}\,,$$

ii) Solve explicitly this equation for the case of the ΛCDM model with $\Lambda \neq 0$, assuming that the Universe is spatially flat. Express your result as

$$H(t) = \sqrt{\Omega_\Lambda^0}\; H_0 \coth\left(\frac{3H_0\sqrt{\Omega_\Lambda^0}}{2}\, t\right)\,,$$

where H_0 is the Hubble rate at present. Check that for $\Lambda = 0$ you obtain an expected result. Which one? Derive explicitly $a(t)$ from it.

iii) Consider again the MDE. Verify that the cosmic time t is related with the cosmological redshift z as follows:

$$t(z) = \frac{2}{3\sqrt{\Omega_\Lambda^0}H_0}\sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}}\,(1+z)^{-3/2}\right)\,.$$

What is the limit $\Omega_\Lambda^0 \to 0$ of this expression? And what is the limit for $z \to \infty$? Explain your results.

iv) Let us compute the age of the Universe in the ΛCDM model for the present values of the cosmological parameters. If $\Omega_k^0 \neq 0$, the age cannot be given as a simple analytical formula (you then need the numerical procedure. However, for $\Omega_k^0 = 0$ you can easily use the previous results to obtain an analytical expression for the age of the Universe. Prove the following beautiful formula:

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda^0}H_0}\sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}}\right) = \frac{2}{3\,H_0}\,\frac{\tanh^{-1}\sqrt{\Omega_\Lambda^0}}{\sqrt{\Omega_\Lambda^0}}\,.$$

where the second equality is the most convenient one. Check it!

v) Verify in detail that for $\Lambda \to 0$ you recover the age of the Einstein-de Sitter universe given in Exercise 6 above, and also the first correction to this result for small values of Λ. Namely, show that

$$t_0 = \frac{2}{3}\,H_0^{-1}(1 + \frac{1}{3}\,\Omega_\Lambda^0 + ...)$$

Take the ratio of this result with the exact result, and check if it departs significantly from one for the current value of Ω_Λ^0.

vi) In contrast to the exact formula of **iv)** (which is quite nice but also quite opaque) the approximate formula of **v)** is numerically crude, but is qualitatively very useful since

it transparently shows that for a non-vanishing and positive Λ the age of the Universe is larger than without it. However, can you feel the intuitive sense of this result beyond the mathematical result? Explain physically why an Universe with $\Lambda > 0$ is necessarily older than another one with $\Lambda = 0$ (with the same matter content). Justify your answer, clearly showing that you fully understand the physical reason.

Using the results from Plank 2018, it is good enough to assume $\Lambda = 0$

i)

Now using exercise 2), we have:

$(\Lambda = 8\pi G_N \rho_\Lambda)$

$$\left.\begin{aligned} H^2 &= \frac{8\pi G_N}{3}(\rho_m + \rho_\Lambda) \\ \dot{H} &= -4\pi G_N(\rho_m + p_m) \end{aligned}\right\} \boxed{\dot{H} + \frac{3}{2}H^2} = 4\pi G_N(-\cancel{\rho_m} - p_m + \cancel{\rho_m} + \rho_\Lambda) = 4\pi G_N(\rho_\Lambda - \overset{0}{\cancel{p_m}}) = 4\pi G_N\rho_\Lambda = \boxed{\frac{\Lambda}{2}}$$

ii)

Let's now solve this:

$$\frac{dH}{dt} + \frac{3}{2}H^2 = \frac{\Lambda}{2} \longrightarrow \frac{dH}{dt} = \frac{\Lambda}{2} - \frac{3}{2}H^2 \longrightarrow \int_{H(0)}^{H(t)} \frac{dH}{\frac{\Lambda}{2} - \frac{3}{2}H^2} = \int_0^t dt$$

Which using that $\int \frac{dx}{a - bx^2} = \tanh^{-1}\left(\sqrt{\frac{b}{a}}\,x\right)/\sqrt{ab}$, we have:

$(\Lambda = 3H_0^2\Omega_\Lambda^0)$

Which using that $\int \frac{dx}{a-bx^2} = \tanh^{-1}\left(\sqrt{\frac{b}{a}}\, x\right)/\sqrt{ab}$, we have:

$$\left.\frac{\tanh^{-1}\left(\sqrt{\frac{2}{\Lambda}}\sqrt{\frac{3}{2}}\, H\right)}{\sqrt{\frac{\Lambda}{2}\frac{3}{2}}}\right|_{H(0)}^{H(t)} = t - 0 \longrightarrow t = \frac{2}{\sqrt{3}\sqrt{\Lambda}}\tanh^{-1}\left(\sqrt{\frac{3}{\Lambda}}\, H\right) \overset{(\Lambda = 3H_0^2\Omega_\Lambda^0)}{=} \frac{2}{3H_0\sqrt{\Omega_\Lambda^0}}\tanh^{-1}\left(\frac{H}{H_0\sqrt{\Omega_\Lambda^0}}\right)$$

so finally: $\boxed{H(t) = \sqrt{\Omega_\Lambda^0}\, H_0 \tanh\left(\frac{3H_0\sqrt{\Omega_\Lambda^0}}{2}\, t\right)}$ which is different to the solution you provide!

(Check from Mathematica:

```
DSolve[H'[t] + a*H[t]^2 == b, H[t], t]
{{H[t] → √b Tanh[√a √b t + √a √b c1] / √a}}
```
)

From my solution:

And taking the limit $\Lambda \to 0$ $\left(\tanh(x) \simeq x - \frac{x^3}{3} + \frac{2x^5}{15}\dots\right)$, we get:

$$\boxed{H(t)} = \overbrace{\frac{3}{2}\Omega_\Lambda^0 H_0^2}^{\Lambda}\, t = \boxed{\frac{\Lambda}{2} t}$$

For which $a(t)$ will be:

$$H = \frac{\dot a}{a} = \frac{\Lambda}{2} t \longrightarrow \frac{da}{a} = \frac{\Lambda}{2} t\, dt \longrightarrow \ln(a) = \frac{\Lambda}{4} t^2 + C \longrightarrow \boxed{a(t) = C' \cdot e^{\frac{\Lambda}{4} t^2}}$$

So, we see that $a(t) \xrightarrow[\Lambda \to 0]{} \boxed{a(t) = \text{constant}, \quad \dot a(t) = 0}$ which is the static Einstein solution!

From the solution you provide:

And taking the limit $\Lambda \to 0$ $\left(\tanh(x) \simeq x - \frac{x^3}{3} + \frac{2x^5}{15}\dots\right)$, we get:

$$\boxed{H(t)} = \cancel{\sqrt{\Omega_\Lambda^0}}\, \cancel{H_0}\, \frac{2}{3\cancel{H_0}\cancel{\sqrt{\Omega_\Lambda^0}}\, t} = \boxed{\frac{2}{3t}}$$

For which $a(t)$ will be:

$$H = \frac{\dot a}{a} = \frac{2t}{3} \longrightarrow \frac{da}{a} = \frac{2\, dt}{3t} \longrightarrow \ln(a) = \frac{2}{3}\ln(t) + C \longrightarrow \boxed{a(t) = C \cdot t^{2/3}}$$

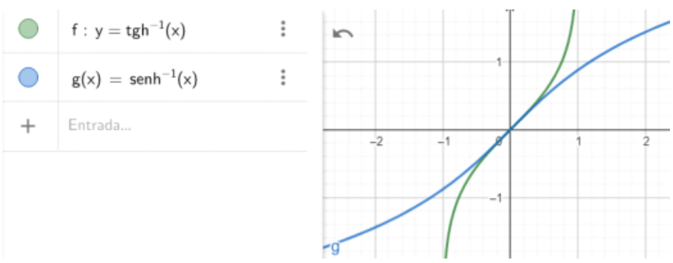
Which is the solution in a $\Lambda = 0$, flat matter dominated universe found in ex 7) ii)

iii)

Now, from a MDE, we will have: $H^2 = H_0^2\left(\Omega_m^0 a^{-3} + \Omega_\Lambda^0\right)$, for which:

$$t = \int_0^{a(t)} \frac{da}{a H(a)} = \frac{1}{H_0}\int_0^{a(t)} \frac{da}{\sqrt{\Omega_m^0 a^{-1} + \Omega_\Lambda^0 a^2}} = \frac{1}{H_0}\, \frac{2\cancel{\sqrt{a}}\,\cancel{\sqrt{\Omega_m^0 a^{-1} + \Omega_\Lambda^0 a^2}}\, \tanh^{-1}\left(\frac{\sqrt{\Omega_\Lambda^0}\, a^{3/2}}{\sqrt{\Omega_m^0 + \Omega_\Lambda^0 a^3}}\right)}{3\sqrt{\Omega_\Lambda^0}\,\cancel{\sqrt{\Omega_m^0 + \Omega_\Lambda^0 a^3}}} = \frac{2\tanh^{-1}\left(\frac{a^{3/2}}{\sqrt{\Omega_m^0/\Omega_\Lambda^0 + a^3}}\right)}{3H_0\sqrt{\Omega_\Lambda^0}}$$

But we can recall that: $\tanh^{-1}(x) \simeq \sinh^{-1}(x)$ when $x < \frac{1}{2}$:



So, considering that $\Omega_m^0/\Omega_\Lambda^0 \overset{(MDE)}{\gg} 1 \longrightarrow \boxed{t \simeq \frac{2\sinh^{-1}\left(\frac{a^{3/2}}{\sqrt{\Omega_m^0/\Omega_\Lambda^0 + a^3}}\right)}{3H_0\sqrt{\Omega_\Lambda^0}}}$ $\left(\text{for } a^3 \ll \frac{\Omega_m^0}{\Omega_\Lambda^0}\right)$

The limits for this expression are:

- $\Omega_\Lambda^0 \to 0$: $\boxed{t = \frac{2a^{3/2}}{3H_0\sqrt{\Omega_m^0}} - O(\Omega_\Lambda^0) = \frac{2a^{3/2}}{3H_0\sqrt{\Omega_m^0}}}$ (Taylor around $\Omega_\Lambda^0=0$)

  Where we see that $a \propto t^{2/3}$ as one would expect from a MDE.

- $z \to \infty$ $(a \to 0)$: $\boxed{t = \frac{2a^{3/2}}{3H_0\sqrt{\Omega_m^0}} - O(a^{5/2}) \longrightarrow 0}$ (Taylor around $a=0$)

  Which obviously is right, since $z \to \infty$ is the infinite past when $t=0$

iv) To compute the present time we only need to set $a=1$ $(z=0)$, so:

$$t_0 = \frac{2\,\mathrm{Sinh}^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}}\right)}{3H_0\sqrt{\Omega_\Lambda^0}}$$

now, to get the second equality, we consider that because $\Omega_k = 0 \longrightarrow 1 = \Omega_\Lambda^0 + \Omega_m^0$, so

$$t_0 = \frac{2\,\mathrm{Sinh}^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{1-\Omega_\Lambda^0}}\right)}{3H_0\sqrt{\Omega_\Lambda^0}} = \frac{2\tanh^{-1}\left(\sqrt{\Omega_\Lambda^0}\right)}{3H_0\sqrt{\Omega_\Lambda^0}}$$

"Numerical proof"

(for an analitic proof we would need to consider that $\mathrm{Sinh}^{-1}(x) = \ln\left(x+\sqrt{x^2+1}\right)$)

Where one is over the other:

They have same Taylor expansion around 0 up to order 50...

In[21]:= Series[ArcSinh[√(x / (1 − x))], {x, 0, 50}]

$$\text{Out[21]=} \sqrt{x} + \frac{x^{3/2}}{3} + \frac{x^{5/2}}{5} + \frac{x^{7/2}}{7} + \frac{x^{9/2}}{9} + \frac{x^{11/2}}{11} + \frac{x^{13/2}}{13} + \frac{x^{15/2}}{15} + \frac{x^{17/2}}{17} + \frac{x^{19/2}}{19} + \frac{x^{21/2}}{21} + \frac{x^{23/2}}{23} + \frac{x^{25/2}}{25} + \frac{x^{27/2}}{27} + \frac{x^{29/2}}{29} + \frac{x^{31/2}}{31} + \frac{x^{33/2}}{33} + \frac{x^{35/2}}{35} + \frac{x^{37/2}}{37} + \frac{x^{39/2}}{39} + \frac{x^{41/2}}{41} + \frac{x^{43/2}}{43} + \frac{x^{45/2}}{45} + \frac{x^{47/2}}{47} + \frac{x^{49/2}}{49} + \frac{x^{51/2}}{51} + \frac{x^{53/2}}{53} + \frac{x^{55/2}}{55} + \frac{x^{57/2}}{57} + \frac{x^{59/2}}{59} + \frac{x^{61/2}}{61} + \frac{x^{63/2}}{63} + \frac{x^{65/2}}{65} + \frac{x^{67/2}}{67} + \frac{x^{69/2}}{69} + \frac{x^{71/2}}{71} + \frac{x^{73/2}}{73} + \frac{x^{75/2}}{75} + \frac{x^{77/2}}{77} + \frac{x^{79/2}}{79} + \frac{x^{81/2}}{81} + \frac{x^{83/2}}{83} + \frac{x^{85/2}}{85} + \frac{x^{87/2}}{87} + \frac{x^{89/2}}{89} + \frac{x^{91/2}}{91} + \frac{x^{93/2}}{93} + \frac{x^{95/2}}{95} + \frac{x^{97/2}}{97} + \frac{x^{99/2}}{99} + O[x]^{101/2}$$

In[19]:= Series[ArcTanh[√x], {x, 0, 50}]

$$\text{Out[19]=} \sqrt{x} + \frac{x^{3/2}}{3} + \frac{x^{5/2}}{5} + \frac{x^{7/2}}{7} + \frac{x^{9/2}}{9} + \frac{x^{11/2}}{11} + \frac{x^{13/2}}{13} + \frac{x^{15/2}}{15} + \frac{x^{17/2}}{17} + \frac{x^{19/2}}{19} + \frac{x^{21/2}}{21} + \frac{x^{23/2}}{23} + \frac{x^{25/2}}{25} + \frac{x^{27/2}}{27} + \frac{x^{29/2}}{29} + \frac{x^{31/2}}{31} + \frac{x^{33/2}}{33} + \frac{x^{35/2}}{35} + \frac{x^{37/2}}{37} + \frac{x^{39/2}}{39} + \frac{x^{41/2}}{41} + \frac{x^{43/2}}{43} + \frac{x^{45/2}}{45} + \frac{x^{47/2}}{47} + \frac{x^{49/2}}{49} + \frac{x^{51/2}}{51} + \frac{x^{53/2}}{53} + \frac{x^{55/2}}{55} + \frac{x^{57/2}}{57} + \frac{x^{59/2}}{59} + \frac{x^{61/2}}{61} + \frac{x^{63/2}}{63} + \frac{x^{65/2}}{65} + \frac{x^{67/2}}{67} + \frac{x^{69/2}}{69} + \frac{x^{71/2}}{71} + \frac{x^{73/2}}{73} + \frac{x^{75/2}}{75} + \frac{x^{77/2}}{77} + \frac{x^{79/2}}{79} + \frac{x^{81/2}}{81} + \frac{x^{83/2}}{83} + \frac{x^{85/2}}{85} + \frac{x^{87/2}}{87} + \frac{x^{89/2}}{89} + \frac{x^{91/2}}{91} + \frac{x^{93/2}}{93} + \frac{x^{95/2}}{95} + \frac{x^{97/2}}{97} + \frac{x^{99/2}}{99} + O[x]^{101/2}$$

And same expansion around a, up to order 5:

Series[ArcSinh[√((x + a) / (1 − (x + a)))], {x, 0, 5}]

$$\mathrm{ArcSinh}\left[\sqrt{\frac{a}{1-a}}\right] - \frac{\sqrt{-\frac{a}{-1+a}}\,x}{2\left(\sqrt{-\frac{1}{-1+a}}\,(-1+a)\,a\right)} + \frac{\sqrt{-\frac{a}{-1+a}}\,(-1+3a)\,x^2}{8\sqrt{-\frac{1}{-1+a}}\,(-1+a)^2 a^2} + \frac{\left(-3\sqrt{-\frac{a}{-1+a}} + 10a\sqrt{-\frac{a}{-1+a}} - 15a^2\sqrt{-\frac{a}{-1+a}}\right)x^3}{48\sqrt{-\frac{1}{-1+a}}\,(-1+a)^3 a^3} + \frac{\left(-5\sqrt{-\frac{a}{-1+a}} + 21a\sqrt{-\frac{a}{-1+a}} - 35a^2\sqrt{-\frac{a}{-1+a}} + 35a^3\sqrt{-\frac{a}{-1+a}}\right)x^4}{128\sqrt{-\frac{1}{-1+a}}\,(-1+a)^4 a^4} + \frac{\left(-35\sqrt{-\frac{a}{-1+a}} + 180a\sqrt{-\frac{a}{-1+a}} - 378a^2\sqrt{-\frac{a}{-1+a}} + 420a^3\sqrt{-\frac{a}{-1+a}} - 315a^4\sqrt{-\frac{a}{-1+a}}\right)x^5}{1280\sqrt{-\frac{1}{-1+a}}\,(-1+a)^5 a^5} + O[x]^6$$

truncate higher-order terms | coefficient list | nth coefficient... | inverse series | Pade approximant

Series[ArcTanh[√(x + a)], {x, 0, 5}]

$$\mathrm{ArcTanh}\left[\sqrt{a}\right] - \frac{x}{2\left((-1+a)\sqrt{a}\right)} + \frac{(-1+3a)\,x^2}{8(-1+a)^2 a^{3/2}} + \frac{(-3+10a-15a^2)\,x^3}{48(-1+a)^3 a^{5/2}} + \frac{(-5+21a-35a^2+35a^3)\,x^4}{128(-1+a)^4 a^{7/2}} + \frac{(-35+180a-378a^2+420a^3-315a^4)\,x^5}{1280(-1+a)^5 a^{9/2}} + O[x]^6$$

v) For checking t_0 in $\Lambda \to \infty$, we are going to Taylor expand it:

(⊛ Series[2 ArcTanh[√x] / (3 * c * √x), {x, 0, 5}]

$$\frac{2}{3c} + \frac{2x}{9c} + \frac{2x^2}{15c} + \frac{2x^3}{21c} + \frac{2x^4}{27c} + \frac{2x^5}{33c} + O[x]^{11/2}$$)

$$\boxed{t_0' \overset{⊛}{\approx} \frac{2}{3H_0} + \frac{2\Omega_\Lambda^0}{9H_0} + \dots} = \frac{2}{3H_0}\left(1 + \frac{\Omega_\Lambda^0}{3} + \frac{\Omega_\Lambda^{0\,2}}{5} + \frac{\Omega_\Lambda^{0\,3}}{7} + \dots\right) = \frac{2}{3H_0}\sum_{n=0} \frac{\Omega_\Lambda^{0\,n}}{1+2n}$$

And now doing the ratio to the exact result, we get

$$\boxed{\frac{t_0'}{t_0}} = \frac{\cancel{2}}{\cancel{3H_0}}\left(1 + \frac{\Omega_\Lambda^0}{3} + \dots\right) \Big/ \frac{\cancel{2}\tanh^{-1}\left(\sqrt{\Omega_\Lambda^0}\right)}{\cancel{3H_0}\sqrt{\Omega_\Lambda^0}} = \boxed{\frac{\sqrt{\Omega_\Lambda^0}}{\tanh^{-1}\left(\sqrt{\Omega_\Lambda^0}\right)}\left(1 + \frac{\Omega_\Lambda^0}{3} + \dots\right)}$$

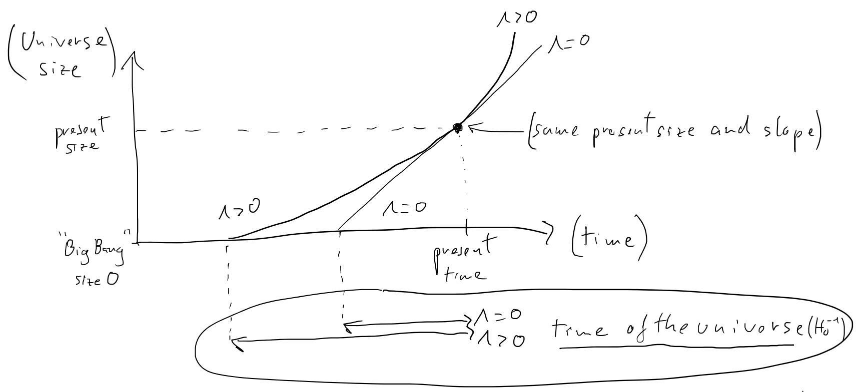
which from the Plank 2018 results is:

$$\frac{t_0'}{t_0} = 0{,}87 \quad \text{with an error of } 13\%$$

t_0

vi)

If in the present time we are at the same size of universe, and with the same first derivative (slope = current expansion rate), we can see the answer with an easy graphic considering that $\Lambda > 0$ will have positive 2nd derivative, and $\Lambda = 0$ will have 2nd derivative ≈ 0 (considering only the Λ contribution, because the others are equal:



From where we see that it's obvious that a $\Lambda > 0$ universe is older than a $\Lambda = 0$, due to the "2nd derivative".

Exercise 10. The idea of a strictly constant Λ-term as in the "concordance ΛCDM" is very simple, but it might even be qualified of simpleminded. (by expansion up to order S^0, so...) If we take into account the dynamical character of our expanding Universe, one should rather expect an evolving vacuum energy density ρ_Λ and maybe even a slowly evolving gravitational coupling G. This is at least suggested on quantum field theory (QFT) grounds. In the Exercise 1 we have admitted in part this possibility, but now we want to exploit it a little further.

i) Adopting such more flexible point of view, but still in the context of the FLRW metric, show that the following generalized local conservation law holds:

$$\frac{d}{dt}\left[G(\rho_m + \rho_\Lambda)\right] + 3GH(\rho_m + p_m) = 0\,,$$

of which the equation in Exercise 1i) is a particular case (check it).

ii) Write down that equation in expanded form as a differential equation with respect to the cosmological redshift variable z.

iii) Such generalized conservation law can be compatible with an anomalous matter conservation formula of the form

$$\rho_m(z) = \rho_m^0 (1+z)^{3(1-\nu)}\,,$$

with ν a small parameter $|\nu| \ll 1$. Obviously, for $\nu = 0$ one recovers the standard law. Let us assume that G is strictly constant for the moment. Integrate the differential equation and show that ρ_Λ can no longer be constant but the following function of the redshift:

$$\rho_\Lambda(z) = \rho_\Lambda^0 + \frac{\nu}{1-\nu}\,\rho_m^0\left[(1+z)^{3(1-\nu)} - 1\right]$$

where ρ_Λ^0 is the current value of ρ_Λ. Interpret the structure of this result.

iv) While the above anomalous matter conservation law can be taken as a phenomenological ansatz, there are situations in the context of QFT in curved spacetime where such scenario can be theoretically motivated. For example, the following expression for ρ_Λ as a function of H has been derived within a particular QFT context [3]:

$$\rho_\Lambda(H) = \rho_\Lambda^0 + \frac{3\nu}{8\pi\,G}(H^2 - H_0^2)\,.$$

Insert this result into the generalized conservation law found in **ii)** above for G =const. and use Friedmann's equation (with zero spatial curvature) to solve for the density $\rho_m(z)$. The result (check it!) is exactly the anomalous conservation law given in section **ii)** above.

Subsequently solve for $\rho_\Lambda(z)$ and find the exact result given in **iii)**. Find also the answer

Subsequently solve for $\rho_\Lambda(z)$ and find the exact result given in **iii)**. Find also the answer for $\rho_m(z)$ and $\rho_\Lambda(z)$ when the spatial curvature parameter $\Omega_k^0 = -k/H_0^2 \neq 0$.

Note: Another way to proceed is to insert the above expression for $\rho_\Lambda(H)$ in the differential equation of Exercise 8 i) and obtain the Hubble function $H = H(z)$ (upon first trading the time variable for the redshift) and from here to derive the energy densities $\rho_\Lambda(z)$ and $\rho_m(z)$. Do it in at least one of the possible ways.

i)

Starting from $H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G_N}{3}(\rho_m + \rho_\Lambda)$, and deriving it:

$$2\frac{\dot a}{a}\frac{\ddot a a - \dot a \dot a}{a^2} = \frac{8\pi}{3}\frac{d}{dt}\left[G_N(\rho_m+\rho_\Lambda)\right] \quad ; \quad \frac{\ddot a}{a} = \frac{4\pi}{3}\frac{d}{dt}\left[G_N(\rho_m+\rho_\Lambda)\right]\frac{1}{H} + H^2$$

which from $\ddot a = -\frac{4\pi G_N}{3}(\rho_m + 3p_m - 2\rho_\Lambda)\,a$, we got:

$$-G_N(\rho_m + 3p_m - 2\rho_\Lambda) = \frac{d}{dt}\left[G_N(\rho_m+\rho_\Lambda)\right]\frac{1}{H} + 2G_N(\rho_m+\rho_\Lambda)\,;$$

$$\frac{d}{dt}\left[G_N(\rho_m+\rho_\Lambda)\right] = HG_N\left(-\rho_m - 3p_m + \cancel{2\rho_\Lambda} - 2\rho_m - \cancel{2\rho_\Lambda}\right)$$

$$\boxed{\frac{d}{dt}\left[G_N(\rho_m+\rho_\Lambda)\right] + 3G_N H(\rho_m + p_m)}$$

ii)

Now let's write this as a differential equation for z:

$$\frac{d}{dt} = \frac{dz}{dt}\frac{d}{dz} = \underbrace{\frac{dz}{da}}_{=-\frac{1}{a^2}}\underbrace{\frac{da}{dt}}_{=aH}\frac{d}{dz} = -\frac{H}{a}\frac{d}{dz} = -H(1+z)\frac{d}{dz}$$

so we will have:

$$\boxed{(1+z)\frac{d}{dz}\left[G_N(\rho_m+\rho_\Lambda)\right] = 3G_N(\rho_m + p_m)}$$

iii)

If G_N is constant and we have $\rho_m = \rho_m^0(1+z)^{3(1-\nu)}$, the previous equation becomes:

$$(1+z)\left(\rho_m^0\,3(1-\nu)(1+z)^{3(1-\nu)-1} + \frac{d\rho_\Lambda}{dz}\right) = 3(\rho_m + \underbrace{\cancel{p_m}}_{0\,(\text{since } |\nu|\ll 0)})$$

$$3(\cancel{1}-\nu)\rho_m + (1+z)\frac{d\rho_\Lambda}{dz} = 3\cancel{\rho_m} \quad ; \quad 3\nu\rho_m = (1+z)\frac{d\rho_\Lambda}{dz}$$

$$\int_{\rho_\Lambda^0}^{\rho_\Lambda} d\rho_\Lambda = \int_0^z \frac{3\nu\rho_m}{1+z}dz \quad ; \quad \rho_\Lambda = \rho_\Lambda^0 + 3\nu\rho_m^0\int_0^z (1+z)^{3(1-\nu)-1}dz = \rho_\Lambda^0 + \cancel{3}\nu\rho_m^0 \left.\frac{(1+z)^{3(1-\nu)}}{\cancel{3}(1-\nu)}\right|_0^z$$

Which finally gives:

$$\boxed{\rho_\Lambda = \rho_\Lambda^0 + \frac{\nu}{1-\nu}\rho_m^0\left((1+z)^{3(1-\nu)} - 1\right)}$$

Where obviously for $\nu = 0$, we have $\boxed{\rho_\Lambda = \rho_\Lambda^0 = cte}$

And also, obviously, when $z=0$, we have: $\boxed{\rho_\Lambda = \rho_\Lambda^0}$ which is the actual value for ρ_Λ.

Exercise 11. Let us consider scalar fields in cosmology. We have derived on the blackboard the explicit expressions for the energy density and pressure, and we found

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{0.1}$$

i) We had shown that these equations hold good for both the Minkowski and FLRW metrics, where in the last case we used the homogeneity and isotropy of spacetime. In our derivation we compared the perfect fluid form of the energy-momentum tensor with the field theory form of it. Re-derive this result without using at all the fluid form. Use the fact that $\rho_\phi = T^0_0$ and $p_\phi = (-1/3)T^i_i$ (sum over i). Justify these expressions first.

ii) Assume that ϕ does not interact with other components during the cosmological expansion. Rewrite *all* of the equations of Exercise 1 when the incoherent matter system (ρ_m, p_m) is replaced by the field densities (ρ_ϕ, p_ϕ) and $\Lambda = 0$. In particular, what is the local conservation law satisfied by ρ_ϕ and p_ϕ? Re-derive from this law the field equation of motion for ϕ, which we already inferred from the field action.

iii) Recompute the field density and pressure in the case when ϕ is *not* homogeneous but still isotropic. Show that in this case the equation of state $\omega_\phi = p_\phi/\rho_\phi$ reads

$$\omega_\phi = -1 + \frac{\dot{\phi}^2/V(\phi) + \frac{1}{3}\left(\nabla\phi\right)^2/V(\phi)}{1 + \frac{1}{2}\dot{\phi}^2/V(\phi) + \frac{1}{2}\left(\nabla\phi\right)^2/V(\phi)}$$

Check that the above result boils down to the standard one (which we discussed on the blackboard) for homogeneous scalar fields.

iv) Phantom fields are not welcome in QFT, as we have to pay a high price. Replace the kinetic term in the action $(\nabla\phi)^2/2$ by $\xi(\nabla\phi)^2/2$, where ξ is some coefficient. What would be the expression for ω_ϕ now? What is the condition on ξ for the scalar field to be a phantom field? Why we don't like this condition? Check a QFT book, if necessary.

v) Explain how space inhomogeneities could make that a time-independent scalar field ϕ to behave as a quintessence field. Could it appear as being phantom-like? Compare it with the standard quintessence case.

vi) The *Planck* data show a possible phantom-like behavior of the dark energy (DE). Check and quote the latest observed result by *Planck* on this particular. So somehow we should be ready for a phantom like DE, despite we cannot easily cope with it in QFT (at least using a scalar field ϕ). If we could mimic the phantom behavior with a non-phantom scalar field, it would be great. Discuss some strategy to make this option possible, at least in principle.

i) Let's start from the action of a scalar field:

$$S_m = -\int d^4x\,\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right] = -\int d^4x\,\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$$

And then we know that the energy-momentum tensor is:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = +\frac{2}{\sqrt{-g}}\left\{\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\left[\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - V(\phi)\right] + \sqrt{-g}\left[\frac{1}{2}\frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}}\partial_\alpha\phi\,\partial_\beta\phi\right]\right\} =$$

$$\left(\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}\right)$$

$$= g_{\mu\nu}\left[-\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi + V(\phi)\right] - 2\left[-\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\right] =$$

$$= -\frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\,\partial^\alpha\phi + g_{\mu\nu}V(\phi) + \partial_\mu\phi\,\partial_\nu\phi =$$

$$= \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - V(\phi)\right)$$

So finally using $\rho_\phi = T_0^0$ and $p_\phi = -\frac{1}{3}T_i{}^i$, we get: $\left(\text{using } g = \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}\right)$

$$T_0^0 = \partial_0\phi\,\partial^0\phi - \overset{1}{\delta^0_0}\left(\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - V(\phi)\right) =$$

So finally ... we get: (using $g = \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}$)

$$\bullet\ \boxed{\rho_\phi} = \partial_0\phi\,\partial^0\phi - \cancel{g^0_0}^{\,1}\left(\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - V(\phi)\right) =$$

$$= \dot\phi^2 - \left(\frac{\dot\phi^2}{2} - \frac{(\vec\nabla\phi)^2}{2} - V(\phi)\right) = \frac{\dot\phi^2}{2} + \frac{(\vec\nabla\phi)^2}{2} + V(\phi) = \boxed{\frac{\dot\phi^2}{2} + V(\phi)}$$

(Homogeneous and Isotropic scalar fields $\vec\nabla\phi = 0$)

$$\bullet\ \boxed{P_\phi} = -\frac{1}{3}T_i^{\ i} = -\frac{1}{3}\left\{\partial_i\phi\,\partial^i\phi - \cancel{g_i^{\ i}}^{\,3}\left(\frac{\dot\phi^2}{2} - \frac{(\vec\nabla\phi)^2}{2} - V(\phi)\right)\right\} =$$

$$= +\frac{1}{3}(\vec\nabla\phi)^2 + \left(\frac{\dot\phi^2}{2} - \frac{(\vec\nabla\phi)^2}{2} - V(\phi)\right) = \frac{\dot\phi^2}{2} - \frac{(\vec\nabla\phi)^2}{6} - V(\phi) \overset{\vee}{=} \boxed{\frac{\dot\phi^2}{2} - V(\phi)}$$

ii)

Let's use then $\rho_m = \rho_\phi$ and $P_m = P_\phi$, with $\Lambda = 0$:

$$\boxed{H^2} = \frac{8\pi G_N}{3}\left(\rho_\phi + \cancel{\rho_\Lambda}^{\,0}\right) - \frac{K}{a^2} = \boxed{\frac{8\pi G_N}{3}\left(\frac{\dot\phi^2}{2} + V(x)\right) - \frac{K}{a^2}}$$

$$\boxed{\ddot a} = -\frac{4\pi G_N}{3}(\rho_\phi + 3p_\phi)\,a = -\frac{4\pi G_N}{3}\left(\frac{\dot\phi^2}{2} + V(x) + \frac{3}{2}\dot\phi^2 - 3V(x)\right)a = \boxed{-\frac{8\pi G_N}{3}\left(\dot\phi^2 - V(x)\right)a}$$

$$\boxed{\dot H} = \frac{\ddot a a - \dot a^2}{a^2} = \frac{\ddot a}{a} - H^2 = -\frac{8\pi G_N}{3}\left[\left(\dot\phi^2 - \cancel{V(x)}\right) + \left(\frac{\dot\phi^2}{2} + \cancel{V(x)}\right)\right] + \frac{K}{a^2} = \boxed{-4\pi G_N\,\dot\phi^2 + \frac{K}{a^2}}$$

$$\boxed{2\dot H + 3H^2} = 8\pi G_N\left(-\dot\phi^2 + \frac{\dot\phi^2}{2} + V(x)\right) + 2\frac{K}{a^2} - 3\frac{K}{a^2} = \boxed{-8\pi G_N\left(\frac{\dot\phi^2}{2} - V(x)\right) - \frac{K}{a^2}}$$

And the equations of motions:

$$\dot\rho_\phi + 3H(\rho_\phi + P_\phi) = 0 \longrightarrow \frac{d}{dt}\left(\frac{\dot\phi^2}{2} + V(x)\right) + 3H\left(\frac{\dot\phi^2}{2} + \cancel{V(x)} + \frac{\dot\phi^2}{2} - \cancel{V(x)}\right) = 0$$

$$\cancel{\dot\phi}\ddot\phi + V'(x) + 3H\dot\phi^{\cancel{2}} = 0 \longrightarrow \boxed{\ddot\phi + 3H\dot\phi + V'(x) = 0}$$

iii)

Now we are going to consider ϕ being isotropic but no homogeneous $(\vec\nabla\phi \neq 0)$, and show the form of w_ϕ:

$$\boxed{w_\phi} = \frac{P_\phi}{\rho_\phi} = \frac{\frac{\dot\phi^2}{2} - \frac{(\vec\nabla\phi)^2}{6} - V(\phi)}{\frac{\dot\phi^2}{2} + \frac{(\vec\nabla\phi)^2}{2} + V(\phi)} = \frac{\frac{\dot\phi^2}{2V} - \frac{(\vec\nabla\phi)^2}{6V} - 1}{\frac{\dot\phi^2}{2V} + \frac{(\vec\nabla\phi)^2}{2V} + 1} =$$

$$= -1 + \frac{\cancel{1} + \frac{\dot\phi^2}{2V} + \frac{(\vec\nabla\phi)^2}{2V} + \frac{\dot\phi^2}{2V} - \frac{(\vec\nabla\phi)^2}{6V} - \cancel{1}}{1 + \frac{\dot\phi^2}{2V} + \frac{(\vec\nabla\phi)^2}{2V}} = \boxed{-1 + \frac{\frac{\dot\phi^2}{V} + \frac{(\vec\nabla\phi)^2}{3V}}{1 + \frac{\dot\phi^2}{2V} + \frac{(\vec\nabla\phi)^2}{2V}}}$$

Which for the homogeneous case gets reduced to:

$$\boxed{w_\phi} = -1 + \frac{\dot\phi^2/V}{1 + \frac{\dot\phi^2}{2V}} = \frac{-1 - \frac{\dot\phi^2}{2V} + \frac{\dot\phi^2}{V}}{1 + \frac{\dot\phi^2}{2V}} = \boxed{\frac{\frac{\dot\phi^2}{2} - V}{\frac{\dot\phi^2}{2} + V}}$$

iv)

Let's change $(\vec{\nabla}\phi)^2 \longrightarrow \xi(\vec{\nabla}\phi)^2$, so:

$$w_\phi = -1 + \underbrace{\frac{\dot{\phi}^2/V + \xi(\vec{\nabla}\phi)^2/3V}{1+\dot{\phi}^2/2V + \xi(\vec{\nabla}\phi)^2/2V}}_{\delta w} = -1 + \delta w$$

where for the scalar field to be phantom-like, we need $w_\phi < -1$, so $\delta w < 0$.

Looking at δw, we see that we need that the numerator is <0, but the denominator still positive:

$$\begin{cases} \xi < -\dfrac{\dot{\phi}^2/\cancel{V}}{(\vec{\nabla}\phi)^2/3\cancel{V}} = -3\dfrac{\dot{\phi}^2}{(\vec{\nabla}\phi)^2} \\ \xi > -\dfrac{1+\dot{\phi}^2/\cancel{2V}}{(\vec{\nabla}\phi)^2/\cancel{2V}} = -\dfrac{2V+\dot{\phi}^2}{(\vec{\nabla}\phi)^2} \end{cases} \qquad \boxed{-\frac{2V+\dot{\phi}^2}{(\vec{\nabla}\phi)^2} < \xi < -3\frac{\dot{\phi}^2}{(\vec{\nabla}\phi)^2}}$$

so ξ is negative at least

This is not acceptable because the kinetic term would be negative, making states with infinite negative energy.

v)

Having a time independent field means $\dot{\phi} = 0$, so:

$$w_\phi = -1 + \frac{\frac{(\vec{\nabla}\phi)^2}{3V}}{1+\frac{(\vec{\nabla}\phi)^2}{2V}} = -1 + \frac{(\vec{\nabla}\phi)^2/3}{V+(\vec{\nabla}\phi)^2/2}$$

And doing the same we did previously:

$$w_\phi = -1 + \frac{(\vec{\nabla}\phi)^2/3}{V+(\vec{\nabla}\phi)^2/2} \longrightarrow w_\phi = -1 + \underbrace{\frac{\xi(\vec{\nabla}\phi)^2/3}{V+\xi(\vec{\nabla}\phi)^2/2}}_{\delta w}$$

But this time to have quintaessence, we need $w_\phi > -1$ and $w_\phi < -\frac{1}{3}$, which if $\xi > 0$, is fullfilled if $\vec{\nabla}\phi \neq 0$. So inhomogeinities make it behave as quintaessence!

In this case because the field is time independent, it doesn't matter if $\xi < 0$, so we can have phantom fields ϕ, without the problem we had before!

vi)

A phantom DE behaviour without the need of phantom scalar can be achived through the ΛXCDM model, where DE is a mixture of Λ and X:

$$\begin{cases} \Omega^0_{DE} = \Omega^0_\Lambda + \Omega^0_X = 0,7 \\ w_{DE} = \dfrac{P_{DE}}{\rho_{DE}} = \dfrac{P_\Lambda + P_X}{\rho_\Lambda + \rho_X} = 1 + (1+w_X)\dfrac{\rho_X}{\rho_{DE}} = -1 + (1+w_X)\dfrac{\Omega_X}{\Omega_{DE}} \end{cases}$$

for which to have a phantom behaviour of DE, we need $w_{DE} < -1$, so:

$$(1+w_X)\frac{\Omega_X}{\Omega_{DE}} < 0$$

but $\boxed{w_X > -1}$ since we don't want neither Λ or X to be phantom, the only ~~o~~ the possibility then is: $\boxed{\Omega_X < 0}$

which is allowed.

A few exercicios from class:

1) $R = 12H^2 + 6\dot{H} \overset{\text{show}}{=} \dfrac{6\gamma(2\gamma+1)}{(t_* - t)^2}$ if $a(t) = \dfrac{A}{(t_* - t)^\gamma}$:

To show this, we will first compute H and $\dot{H}$:

- $H = \dfrac{\dot{a}}{a} = \dfrac{-\cancel{A}\gamma/(t_* - t)^{\gamma+1}}{\cancel{A}/(t_* - t)^\gamma} = \dfrac{-\gamma}{t_* - t}$

- $\dot{H} = \dfrac{\ddot{a}}{a} - \dfrac{\dot{a}^2}{a^2} = \dfrac{\ddot{a}}{a} - H^2 = \dfrac{+\cancel{A}\gamma(\gamma+1)/(t_* - t)^{\gamma+2}}{\cancel{A}/(t_* - t)^\gamma} - H^2 = \dfrac{\gamma(\gamma+1)}{(t_* - t)^2} - H^2$

so then we have:

$$\boxed{R = \frac{(12-6)\gamma^2}{(t_* - t)^2} + \frac{6\gamma(\gamma+1)}{(t_* - t)^2} = \frac{12\gamma^2 + 6\gamma}{(t_* - t)^2} = \frac{6\gamma(2\gamma+1)}{(t_* - t)^2}}$$

2) $\Delta t = t_* - t_0$ if $w_0 = -1{,}05$, $\gamma^{-1} = \frac{3}{2}(|w_0| - 1) = 0{,}075$:

So, let's start from: $\Delta t = t_* - t_0 = \dfrac{1}{H_0}\dfrac{\gamma}{\sqrt{1-\Omega_m^0}}$

which using the values from Plank 2018, goes to:

$$\boxed{\Delta t} = 0{,}234\,s \cdot \frac{Mpc}{Km} \cdot \frac{3{,}0857 \cdot 10^{19}\,Km}{1\,Mpc} \cdot \frac{1h}{3600\,s} \cdot \frac{1\,day}{24h} \cdot \frac{1\,year}{365\,day} \cdot \frac{1\,Gyear}{10^9\,years} = \boxed{233{,}85\ Gyears}$$

3) Find ρ_ϕ and P_ϕ of a non-homogeneous scalar field ϕ:

$\boxed{\text{Done in ex 11) ;) !}}$