DYNAMICS AND SPECTRUM OF THE SCHWINGER MODEL

Guillermo Abad López

Master's thesis dissertation
Sep 02, 2022

Universitat de Barcelona

Facultat de Física

Outline

Table of Contents

Motivation and introduction

The Schwinger model, named after Julian Schwinger, describes 1+1 d QED, which posses some really interesting properties such as:

- confinement
- mass gap
- θ parameter
- anomaly
- charge shielding
- phase transition

First we will review the model at the classical level, then at the quantum level to finally end with the massive case.

Table of Contents

Introduction to massless classical electrodynamics

The QED lagrangian reads:

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi \tag{2.1}$$

For which the quations of motion are:

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = 0 \quad \longrightarrow \quad \partial_\nu F^{\nu\mu} = e\bar{\psi}\gamma^\mu\psi \tag{2.2}$$

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \psi)} = 0 \quad \longrightarrow \quad \bar{\psi}(i\not{D} + m) = 0 \tag{2.3}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \bar{\psi})} = 0 \quad \longrightarrow \quad (i\not{D} - m)\psi = 0 \tag{2.4}$$

from where we can see a conserved current:

$$j_V^\mu = \frac{1}{e}\partial_\nu F^{\nu\mu} = \bar{\psi}\gamma^\mu\psi \tag{2.5}$$

More generally the theory has a local U(1) gauge symmetry:

$$\begin{cases} \psi(x) \;\rightarrow\; e^{ie\alpha(x)}\psi(x) \\ A^\mu(x) \rightarrow\; A^\mu(x) - \partial^\mu\alpha(x) \end{cases} \tag{2.6}$$

Chiral symmetry for classical electrodynamics in even d

For even dimensions it's useful to work with the corresponding chiral states:

$$\begin{cases} \psi_R = P_R\psi = \frac{1+\gamma^5}{2}\psi & \text{right-handed chirality fermions} \\ \psi_L = P_L\psi = \frac{1-\gamma^5}{2}\psi & \text{left-handed chirality fermions} \end{cases} \tag{2.7}$$

for which the lagrangian becomes:

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\not{D}\psi_R - m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R) \tag{2.8}$$

where only the mass term mixes the right and left terms! Meaning that for the massless case we should have two independent conserved currents:

$$\begin{cases} j^\mu_V = j^\mu_R + j^\mu_L = \bar{\psi}_R\gamma^\mu\psi_R + \bar{\psi}_L\gamma^\mu\psi_L = \bar{\psi}\gamma^\mu\psi \\ j^\mu_A = j^\mu_R - j^\mu_L = \bar{\psi}_R\gamma^\mu\psi_R - \bar{\psi}_L\gamma^\mu\psi_L = \bar{\psi}\gamma^\mu\gamma^5\psi \end{cases} \tag{2.9}$$

And finally going to the massless 1+1 d case, we finally arrive to the Schwinger model, which conserves both of the above currents:

$$\mathcal{L}_{QED_{m=0}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{\partial}\psi - ej^\mu_V A_\mu \tag{2.10}$$

Gauge part of the Lagrangian (1)

Focusing in the gauge part of the Lagrangian:

$$\mathcal{L}_g = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - ej_V^{\mu}A_{\mu} = \frac{1}{2}F_{01}^2 - e(j_V^0 A_0 + j_V^1 A_1) \text{ with } F_{\mu\nu} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \quad (2.11)$$

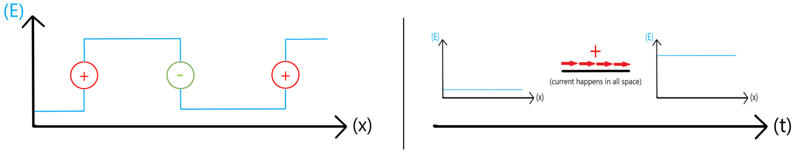
The equations of motion are then:

$$\partial_\nu F^{\nu\mu} = ej_V^{\mu} \longrightarrow \begin{cases} \partial_1 E(t,x) = ej_V^0(t,x) \equiv e\rho_V(t,x) \\ \partial_0 E(t,x) = -ej_V^1(t,x) \equiv -e\mathbf{j}_V(t,x) \end{cases} \quad (2.12)$$

which give the green funtions:

$$\begin{cases} E_\rho(t,x) = e\int_{-\infty}^{x} \rho_V \delta(x')dx' + F(t) = e\rho_V H(x) + F(t) \\ E_j(,x) = -e\int_{-\infty}^{t} \mathbf{j}_V \delta(t')dt' + G(x) = -e\mathbf{j}_V H(t) + G(x) \end{cases} \quad (2.13)$$

Graphical representation to the electric field contribution from point charges and from instant currents (Green's functions):



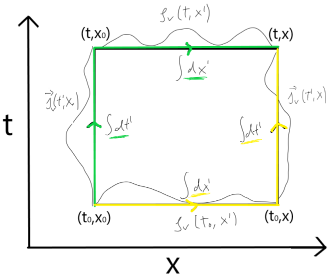
Gauge part of the Lagrangian (2)

So each step function will contribute as:

$$E(t,x) = e\int_{-\infty}^{x}\rho_V(t,x')dx' + F(t) = E(t,x_0) + e\int_{x_0}^{x}\rho_V(t,x')dx'$$

$$E(t,x) = -e\int_{-\infty}^{t}\mathbf{j}_V(t',x)dt' + G(x) = E(t_0,x) - e\int_{t_0}^{t}\mathbf{j}_V(t',x)dt'$$

And considering all of them together we get two equivalent integration paths:



$$\underline{E(t,x)} = E(t_0,x_0) + e\int_{x_0}^{x}\rho_V(t,x')dx' - \int_{t_0}^{t}\vec{j}_V(t',x_0)dt'$$

$$\underline{E(t,x)} = E(t_0,x_0) - e\int_{t_0}^{t}\vec{j}_V(t',x)dt' + e\int_{x_0}^{x}\rho_V(t_0,x')dx'$$

Infinite energy and confinement of charges

The energy contained in the electric field

$$\mathcal{E} = \int dx \frac{1}{2} F_{01}^2 \tag{2.14}$$

so, neutrally charged states will be the only one with finite energy.
So now let's consider the simplest neutrally charged state, which would be a charge q at position $x = L/2$ and a charge q at position $x = +L/2$. From the equations of motion we get

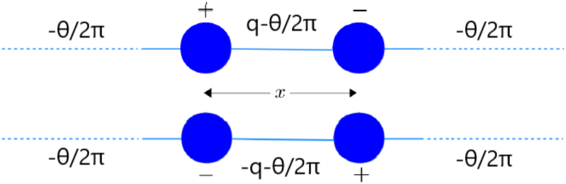
$$\partial_1 F^{01} = eq[\delta(-L/2) - \delta(L/2)] \longrightarrow F^{01} = \begin{cases} eq \text{ between the charges} \\ 0 \text{ outside} \end{cases} \tag{2.15}$$

so for this case we get and energy:

$$\mathcal{E} = \frac{e^2 q^2}{2} L \tag{2.16}$$

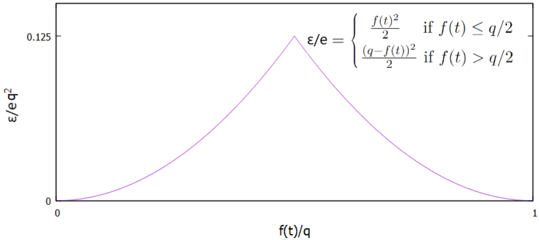
The θ parameter on the line (1)

Two distinct configurations for a pair of positive-negative charges in a background field (we have set e=1 for clarity):



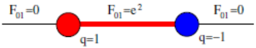
The θ parameter on the line (2)

Plot of energy density divided by the coupling and the charge squared ε/eq^2, as a function of the background field (which we normalized to be between 1 and 0 as: $f(t) = \frac{\theta}{2\pi}$) divided by the charge $f(t)/q$:

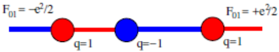


Confinement of massive charges in function of the θ angle

Flux tubes between positive-negative charges, and their effect as θ varies [in a) the flux is confining, and in b) it is not, because the θ external field has equated half the flux tubes]



a) When $\theta = 0$, there is a confining string between particles and anti-particles



b) When $\theta = \pi$, the string tensions cancel on either side and alternating particles/anti-particles feel no long-distance force.

Fermionic part of the Lagrangian

The complete classical Schwinger model

The complete equations of motion then are:

$$\begin{cases} \partial_1 F^{10} = e j_V^0 = e\bar{\psi}\gamma^0\psi \;\rightarrow\; \partial_1 E = e(|\psi_R|^2 + |\psi_L|^2) \\ \partial_0 F^{01} = e j_V^1 = e\bar{\psi}\gamma^1\psi \;\rightarrow\; -\partial_0 E = e(|\psi_R|^2 - |\psi_L|^2) \\ i(\partial_0 + \partial_1)\psi_R = e(A^0 - A^1)\psi_R \;\text{ and }\; i(\partial_0 + \partial_1)\psi_R^* = -e(A^0 - A^1)\psi_R^* \\ i(\partial_0 - \partial_1)\psi_L = e(A^0 + A^1)\psi_L \;\text{ and }\; i(\partial_0 - \partial_1)\psi_L^* = -e(A^0 + A^1)\psi_L^* \end{cases} \tag{2.17}$$

Which give the solutions:

- $$E(t,x) = E(t,x_0) + \int_{x_0}^{x} e\left(|\psi_R(x',t)|^2 + |\psi_L(x',t)|^2\right) dx' \tag{2.18}$$

- $$E(t,x) = E(t_0,x) - \int_{t_0}^{t} e\left(|\psi_R(x,t')|^2 - |\psi_L(x,t')|^2\right) dt' \tag{2.19}$$

- $$\begin{cases} \psi_R = e^{-ieK(A^0 - A^1)} G_R(t-x) \\ \psi_L = e^{-ieL(A^0 + A^1)} G_L(t+x) \end{cases} \;\text{ and }\; \begin{cases} \psi_R^* = e^{ieK(A^0 - A^1)} G_R(t-x)^* \\ \psi_L^* = e^{ieL(A^0 + A^1)} G_L(t+x)^* \end{cases} \tag{2.20}$$

Solution to the classical Schwinger model (1)

And canceling the phases, we are left only with the right/left moving functions G's:

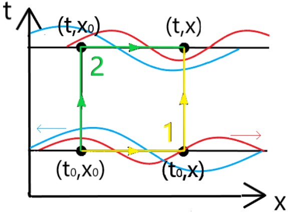
$$\begin{cases} \partial_1 E = e(|G_R(t-x)|^2 + |G_L(t+x)|^2) \\ \partial_0 E = -e(|G_R(t-x)|^2 - |G_L(t+x)|^2) \end{cases} \tag{2.21}$$

Which can again take two path for the integration:

$$\Delta_E^1(t,x,t_0,x_0) = e\left(\int_{x_0}^{x} \left(|\psi_R(t_0,x')|^2 + |\psi_L(t_0,x')|^2\right) dx' - \int_{t_0}^{t} \left(|\psi_R(t',x)|^2 - |\psi_L(t',x)|\right.\right.$$

$$\Delta_E^2(t,x,t_0,x_0) = e\left(-\int_{t_0}^{t} \left(|\psi_R(t',x_0)|^2 - |\psi_L(t',x_0)|^2\right) dt' + \int_{x_0}^{x} \left(|\psi_R(t,x')|^2 + |\psi_L(t,x'\right.\right.$$

which we can schematically see represented here:



Solution to the classical Schwinger model (2)

Explain how you rotate the two integrals.

Final discussion fo the classical model

Grassman

Table of Contents

Photons develop a mass in 1+1 d

ABJ, Chiral or Axial anomaly in 1+1 d

Hamiltonian formalism of the Schwinger model

Spectrum of the Schwinger model in a circle

The irrelevance of the θ parameter in the massless model

Explicit canonical quantization of the Schwinger model (1)

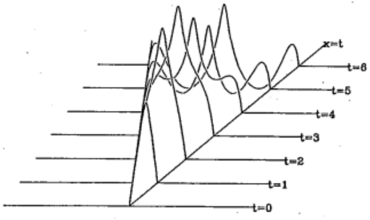
Explicit canonical quantization of the Schwinger model (2)

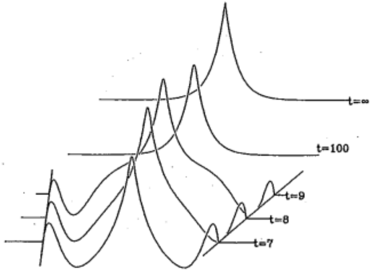
Bosonization of the massless Schwinger model (1)

Bosonization of the massless Schwinger model (2)

Screening of external charges

Induced current density for different times when an external charge is located at the center. **?**





Revisit ABJ anomaly with Bosonization

Table of Contents

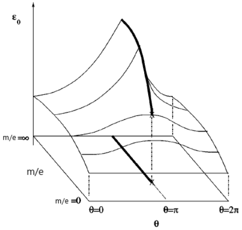
Introduction to the massive Schwinger model

Bosonization of the massive Schinger model

The two regimes of the massive Schwinger model

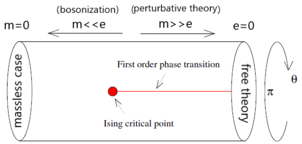
The relevance of the θ parameter in the massive model

Schematic plot of the vacuum energy density as a function of m/e and θ. The heavy line marks the first-order transition line, where the energy density has a cusp, terminating at the second order critical point $(m/e)_c$, where the slope no longer has a discontinuity.



Critical point $(m/e)_c$ for the massive Schwinger model

Phase diagram of the Schwinger model, based on the phase diagram of 1+1 d scalar theory of David Tong.



The weak coupling regime (massive, $m >> e$)

Table of Contents

Discussion and outlook

Thank you for your time