Guillermo Abad López

Quantum Field Theory, Sheet 3 b)

3.2)

a)

- $\boxed{\bar{u}^r(\vec{p})\, v^s(\vec{p})} = \left(\xi^{r\dagger} \sqrt{p\sigma}^{\dagger}, \xi^{r\dagger} \sqrt{p\bar{\sigma}}^{\dagger}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\sigma}\, \eta^s \\ -\sqrt{p\bar{\sigma}}\, \eta^s \end{pmatrix} =$

$$= \xi^{r\dagger} \left(\underbrace{\sqrt{p\bar{\sigma}}\sqrt{p\sigma}}_{1 m} - \underbrace{\sqrt{p\sigma}\sqrt{p\bar{\sigma}}}_{1 m}\right) \eta^s = \xi^{r\dagger} (\cancel{m - m})^{0} \eta^s = \boxed{0}$$

- $\boxed{u^{\dagger r}(\vec{p})\, v^s(\vec{p})} = \left(\xi^{r\dagger} \sqrt{p\sigma}, \xi^{r\dagger} \sqrt{p\bar{\sigma}}\right) \begin{pmatrix} \sqrt{p\sigma}\, \eta^s \\ -\sqrt{p\bar{\sigma}}\, \eta^s \end{pmatrix} =$

$$= \xi^{r\dagger} \left(\sqrt{p\sigma}\sqrt{p\sigma} - \sqrt{p\bar{\sigma}}\sqrt{p\bar{\sigma}}\right) \eta^s =$$

$$= \xi^{r\dagger} \left(\cancel{\sqrt{(p_0 + \vec{p}\vec{\sigma})(p_0 - \vec{p}\vec{\sigma})}} - \cancel{\sqrt{(p_0 - \vec{p}\vec{\sigma})(p_0 + \vec{p}\vec{\sigma})}}\right) \eta^s = \boxed{0}$$

(We have used that $\left.\begin{array}{l} \sqrt{p\sigma} = e^{\frac{1}{2}\eta \vec{p}\vec{\sigma}} \\ \sqrt{p\bar{\sigma}} = e^{-\frac{1}{2}\eta \vec{p}\vec{\sigma}} \end{array}\right\}$ so $\sqrt{p\sigma}\sqrt{p\bar{\sigma}} = m\, e^{\frac{1}{2}\eta(\vec{p}\vec{\sigma} - \vec{p}\vec{\sigma})} = m\sqrt{e^{\eta(\vec{p}\vec{\sigma}-\vec{p}\vec{\sigma})}} = m\sqrt{(p\sigma)(p\bar{\sigma})}$ We can unite the square roots! (if they commute)

$= m\sqrt{e^0} = m\,\mathbb{1}$

$[\vec{p}\vec{\sigma}, \vec{p}\vec{\sigma}] = 0$

$\left(e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]\dots}\right)$)

~~b)~~

$$\cancel{\psi(x) = \sum_{s=1,2} \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \left(a_{\vec{p}}^s\, u^s(\vec{p})\, e^{-ipx} + b_{\vec{p}}^{s\dagger}\, v^s(\vec{p})\, e^{ipx}\right)}$$

~~with:~~

$$\cancel{\begin{cases} \{\psi_\alpha(t,\vec{x}), \psi_\beta^\dagger(t,\vec{y})\} = \delta_{\alpha\beta}\, \delta^3(\vec{x}-\vec{y}) \\ \{\psi_\alpha(t,\vec{x}), \psi_\beta(t,\vec{y})\} = 0 \end{cases}}$$

~~Using now that:~~

$$\cancel{\mathbb{1} = \frac{1}{2m} \sum_s \left(u^s(\vec{p})\bar{u}^s(\vec{p}) - v^s(\vec{p})\bar{v}^s(\vec{p})\right) \longrightarrow \delta_{\alpha\beta} = \frac{1}{2m} \sum_s \left(u_\alpha^s(\vec{p})\bar{u}_\beta^s(\vec{p}) - v^s(\vec{p})\bar{v}^s(\vec{p})\right)}$$

$$\cancel{\mathbb{1}\cdot\gamma^0 = \frac{1}{2m} \sum_s \left(u^s(\vec{p})u^{s\dagger}(\vec{p}) - v^s(\vec{p})v^{s\dagger}(\vec{p})\right) \longrightarrow \delta_{\alpha\beta'}\gamma^{0\beta'}{}_{\beta} = \frac{1}{2m} \sum_s \left(u_\alpha^s(\vec{p})u_\beta^\dagger(\vec{p}) - v^s(\vec{p})v^{s\dagger}(\vec{p})\right)}$$

b)

$$\hat{\psi}(x) = \sum_{s=1,2} \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \left(\hat{a}_{\vec{p}}^{\,s}\, u^s(\vec{p})\, e^{-ipx} + \hat{b}_{\vec{p}}^{\,s\dagger}\, v^s(\vec{p})\, e^{ipx} \right) \qquad (1)$$

with:

$$\begin{cases} \{\hat{\psi}_\alpha(t,\vec{x}), \hat{\psi}_\beta^\dagger(t,\vec{y})\} = \delta_{\alpha\beta}\, \delta^3(\vec{x}-\vec{y}) \\ \{\hat{\psi}_\alpha(t,\vec{x}), \hat{\psi}_\beta(t,\vec{y})\} = 0 \end{cases} \qquad (2)$$

And we want to compute $\{\hat{a},\hat{a}^\dagger, \hat{b},\hat{b}^\dagger\}$, so first let's write this ladder operators in terms of the fields, so when we anti-commute them we will be left with (2):

- First we see that $\hat{a}$ is in the $\hat{\psi}(x)^+$ with $u^s(\vec{p})$ and in $\hat{\psi}(x)^-$ we have $v^s(\vec{p})$, with diferent signs in the exponentials, so let's try to get something like:

$$\triangleright \; u^\dagger(\vec{p}')\, u(\vec{p}) + u^\dagger(\vec{p}')\, v(\vec{p}) = u^\dagger(\vec{p}')\, u(\vec{p}) \underbrace{\delta(\vec{p}'-\vec{p})}_{e^{i(\vec{p}-\vec{p}')x}} + u^\dagger(\vec{p}')\, v(\vec{p}) \underbrace{\delta(\vec{p}'+\vec{p})}_{e^{i(\vec{p}'+\vec{p})x}}$$

(What we needed!)

- So the first term gives us our $\hat{a}$, and the second gives 0:

$$\triangleright \int d^3x\, u^{s'}(\vec{p}')^\dagger\, \hat{\psi}(x)\, e^{ip'x} = \int d^3x \left(\sum_{s=1,2} \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left[\hat{a}_{\vec{p}}^{\,s}\, u^{s'\dagger}(\vec{p}')\, u^s(\vec{p})\, e^{-i(p-p')x} + \hat{b}_{\vec{p}}^{\,s\dagger}\, u^{s'\dagger}(\vec{p}')\, v^s(\vec{p})\, e^{i(p+p')x} \right] \right) =$$

$$= \sum_{s=1,2} \int \frac{d^3p}{\sqrt{2E_p}} \left[\hat{a}_{\vec{p}}^{\,s}\, u^{s'\dagger}(\vec{p}')\, u^s(\vec{p})\, \delta(p-p') + \hat{b}_{\vec{p}}^{\,s\dagger}\, u^{s'\dagger}(\vec{p}')\, v^s(\vec{p})\, \delta(p+p')\, e^{2iE_pt} \right]$$

$$= \sum_{s=1,2} \frac{1}{\sqrt{2E_{p'}}} \left[\hat{a}_{\vec{p}'}^{\,s}\, \underbrace{u^{s'\dagger}(\vec{p}')\, u^s(\vec{p})}_{2E_p\delta^{s's}} + \hat{b}_{\vec{p}'}^{\,s\dagger}\, \underbrace{u^{s'\dagger}(-\vec{p}')\, v^s(\vec{p})}_{0}\, e^{2iE_pt} \right] =$$

$$= \sqrt{2E_{p'}}\, \hat{a}_{\vec{p}'}^{\,s'}$$

- And finally we see that:

$$\boxed{\hat{a}_{\vec{p}}^{\,s} = \int \frac{d^3x}{\sqrt{2E_p}}\, u^s(\vec{p})^\dagger\, \hat{\psi}(x)\, e^{ipx}} \quad \text{and so} \quad \boxed{\hat{a}_{\vec{p}}^{\,s\dagger} = \int \frac{d^3x}{\sqrt{2E_p}}\, \hat{\psi}^\dagger(x)\, u^s(\vec{p})\, e^{-ipx}}$$

(3)

- Analogically we can easy see that:

$$\boxed{\hat{b}_{\vec{p}}^{\,s\dagger} = \int \frac{d^3x}{\sqrt{2E_p}}\, v^s(\vec{p})^\dagger\, \hat{\psi}(x)\, e^{-ipx}} \quad \text{and so} \quad \boxed{\hat{b}_{\vec{p}}^{\,s} = \int \frac{d^3x}{\sqrt{2E_p}}\, \hat{\psi}^\dagger(x)\, v^s(\vec{p})\, e^{ipx}}$$

So from (3) we see that

- $\boxed{\{\hat{a}^s_{\vec{p}}, \hat{b}^{r\dagger}_{\vec{k}}\}} \propto \{\hat{\psi}(x), \hat{\psi}(y)\} \boxed{= 0}$
- $\boxed{\{\hat{a}^{s\dagger}_{\vec{p}}, \hat{b}^r_{\vec{k}}\}} \propto \{\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(y)\} \boxed{= 0}$

And we are left with checking:

- $$\boxed{\{\hat{a}^s_{\vec{p}}, \hat{b}^r_{\vec{k}}\}} = \iint \frac{d^3x\, d^3y}{2\sqrt{E_p E_k}} \{u^s_\alpha(\vec{p})^\dagger \hat{\psi}_\alpha(x), \hat{\psi}_\beta(y)^\dagger v^r_\beta(\vec{k})\} e^{i(px+ky)} =$$
$$= \iint \frac{d^3x\, d^3y}{2\sqrt{E_p E_k}} u^s_\alpha(\vec{p})^\dagger v^r_\beta(\vec{k}) \underbrace{\{\hat{\psi}_\alpha(x), \hat{\psi}^\dagger_\beta(y)\}}_{\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})} e^{i(px+ky)} =$$
$$= \int \frac{d^3x}{2\sqrt{E_p E_k}} u^s_\alpha(\vec{p})^\dagger v^r_\alpha(\vec{k}) e^{i(p+k)x} =$$
$$= \frac{u^s(\vec{p})^\dagger v^r(\vec{k})}{2\sqrt{E_p E_k}} (2\pi)^3 \delta^3(\vec{p}+\vec{k}) e^{-i(E_p+E_k)t} \approx \frac{(2\pi)^3}{2E_p} \cancelto{0}{u^s(\vec{p})^\dagger v^r(-\vec{p})} \delta^3(\vec{p}+\vec{k}) e^{2iE_pt} \boxed{= 0}$$

- $\boxed{\{\hat{a}^{s\dagger}_{\vec{p}}, \hat{b}^{r\dagger}_{\vec{k}}\}} = \{\hat{b}^r_{\vec{k}}, \hat{a}^s_{\vec{p}}\}^\dagger = \{\hat{a}^s_{\vec{p}}, \hat{b}^r_{\vec{k}}\}^\dagger = 0^\dagger \boxed{= 0}$

And finally the important ones:

- $$\boxed{\{\hat{a}^s_{\vec{p}}, \hat{a}^{r\dagger}_{\vec{k}}\}} = \iint \frac{d^3x\, d^3y}{2\sqrt{E_p E_k}} \{u^s_\alpha(\vec{p})^\dagger \hat{\psi}_\alpha(x), \hat{\psi}^\dagger_\beta(y) u^r_\beta(\vec{k})\} e^{i(px-ky)} =$$
$$= \iint \frac{d^3x\, d^3y}{2\sqrt{E_p E_k}} u^s_\alpha(\vec{p})^\dagger u^r_\beta(\vec{k}) \delta_{\alpha\beta} \delta^3(\vec{x}-\vec{y}) e^{i(px-ky)} =$$
$$= \int \frac{d^3x}{2\sqrt{E_p E_k}} u^s(\vec{p})^\dagger u^r(\vec{k}) e^{i(p-k)x} = \frac{\overbrace{u^s(\vec{p})^\dagger u^r(\vec{k})}^{2E_p\delta^{sr}}}{2\cancel{\sqrt{E_p E_k}}} (2\pi)^3 \delta^3(\vec{p}-\vec{k}) \cancelto{=1}{e^{i(E_p-E_k)t}} \approx \boxed{(2\pi)^3 \delta^{sr} \delta^3(\vec{p}-\vec{k})}$$

- $$\boxed{\{\hat{b}^s_{\vec{p}}, \hat{b}^{r\dagger}_{\vec{k}}\}} = \iint \frac{d^3x\, d^3y}{2\sqrt{E_p E_k}} \{\hat{\psi}^\dagger_\alpha(x) v^s_\alpha(\vec{p}), v^{r\dagger}_\beta(\vec{k}) \hat{\psi}_\beta(y)\} e^{i(px-ky)} =$$
$$(\{\psi, \psi^\dagger\} = \{\psi^\dagger, \psi\})$$
$$= \iint \frac{d^3x\, d^3y}{2\sqrt{E_p E_k}} v^s_\alpha(\vec{p}) v^{r\dagger}_\beta(\vec{k}) \delta_{\alpha\beta} \delta^3(\vec{x}-\vec{y}) e^{i(px-ky)} =$$
$$= \int \frac{d^3x}{2\sqrt{E_p E_k}} v^s(\vec{p}) \cdot v^r(\vec{k})^\dagger e^{i(p-k)x} = \frac{\overbrace{v^s(\vec{p}) v^r(\vec{k})^\dagger}^{(2E_p\delta^{sr})^T}}{2\cancel{\sqrt{E_p E_k}}} (2\pi)^3 \delta^3(\vec{p}-\vec{k}) \cancelto{1}{e^{i(E_p-E_k)t}} \approx \boxed{(2\pi)^3 \delta^{sr} \delta^3(\vec{p}-\vec{k})}$$

3.3)

$$\begin{cases}\Psi^\alpha(x)\longrightarrow \Psi'^\alpha(x') = \left[e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\right]^\alpha_\beta \Psi^\beta(x)\\ x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu\end{cases} \Longrightarrow \begin{cases}\delta\Psi^\alpha = -\frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})^\alpha_\beta\Psi^\beta(x)\\ \delta x^\mu = \omega^\mu{}_\nu x^\nu\end{cases}$$

a)

• $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^\alpha)}\left(\delta\Psi^\alpha - \delta x^\nu(\partial_\nu\Psi^\alpha)\right) + \delta x^\mu \cancelto{0}{\mathcal{L}} =$

(but in Dirac: $\mathcal{L} = \bar\Psi(i\partial\!\!\!/ - m)\Psi$ with eq.o.m $(i\partial\!\!\!/-m)\Psi=0$ so $\mathcal{L}$ in shell $=0$)

$$= \bar\Psi_\alpha i\gamma^\mu\left(-\frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})^\alpha_\beta\Psi^\beta - \omega^\nu{}_\kappa x^\kappa(\partial_\nu\Psi^\alpha)\right)=$$

$$= \omega^{\rho\sigma}\left[i\bar\Psi_\alpha\gamma^\mu\left(-x_\sigma(\partial_\rho\Psi^\alpha)\right) - \frac{i}{4}(\sigma_{\rho\sigma})^\alpha_\beta\Psi^\beta\right]$$

But if we want the conserved current for any $\omega^{\rho\sigma}$ we get:

• $j^{\mu\rho\sigma} = i\bar\Psi_\alpha\gamma^\mu\left(-x^\sigma(\partial^\rho\Psi^\alpha) - \frac{i}{4}\underbrace{(\sigma^{\rho\sigma})^\alpha_\beta\Psi^\beta}_{\text{spin part}}\right)$

(from the first term we would get $L^{\rho\sigma}$, the orbital momentum part $\to J^{\rho\sigma} = L^{\rho\sigma}+S^{\rho\sigma}$ [the first term is like the one for K.G fields])

$$\boxed{j^{\mu\rho\sigma}_{spin} = \frac{1}{4}\bar\Psi_\alpha\gamma^\mu(\sigma^{\rho\sigma})^\alpha_\beta\Psi^\beta} \quad ; \quad \boxed{j^\mu_{spin} = \frac{\omega_{\rho\sigma}}{4}\bar\Psi_\alpha\gamma^\mu(\sigma^{\rho\sigma})^\alpha_\beta\Psi^\beta}$$

b)

$\sigma^{\rho\sigma}$ is contracted with $\omega_{\rho\sigma}$ always, and for rotations $\omega_{\rho\sigma}=0$ except for the  which are $\theta^k = \frac{1}{2}\epsilon^{klm}\omega_{lm}$, so the terms of $\sigma_{\rho\sigma}$ that contribute will be the same , which tells us we will need only $\epsilon^{kij}\sigma^{ij}$ terms. Let's check if $\frac{\Sigma^k}{2}$ fullfills that:

• $\sigma^{\rho\sigma} = \frac{i}{2}[\gamma^\rho,\gamma^\sigma] = \frac{i}{2}\left[\begin{pmatrix}-\sigma^\rho\sigma^\sigma & 0\\ 0 & -\sigma^\rho\sigma^\sigma\end{pmatrix} - \begin{pmatrix}-\sigma^\sigma\sigma^\rho & 0\\ 0 & -\sigma^\sigma\sigma^\rho\end{pmatrix}\right] = (\sigma^\sigma\sigma^\rho - \sigma^\rho\sigma^\sigma)\frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

then:

• $\epsilon^{\alpha\rho\sigma}\sigma^{\rho\sigma} = -i\epsilon^{\alpha\rho\sigma}\sigma^\rho\sigma^\sigma\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = -i\underbrace{\epsilon^{\alpha\rho\sigma}}_{\text{antsim}}\left(\underbrace{\cancelto{0}{\frac{\sigma^\rho\sigma^\sigma}{2}+\frac{\sigma^\sigma\sigma^\rho}{2}}}_{\text{symetric}} + i\epsilon^{\rho\sigma k}\sigma^k\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$(\epsilon^{\alpha\rho\sigma}\sigma^\sigma\sigma^\rho = -\epsilon^{\alpha\sigma\rho}\sigma^\sigma\sigma^\rho)$

$$= +\overbrace{\epsilon^{\alpha\rho\sigma}\epsilon^{k\rho\sigma}}^{2\delta^\alpha_k}\sigma^k\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = +2\sigma^\alpha\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 2\begin{pmatrix}\sigma^\alpha & 0\\ 0 & \sigma^\alpha\end{pmatrix}$$

so we see that:

$\boxed{\frac{1}{2}\Sigma^k} = \frac{1}{2}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} = \boxed{\frac{1}{4}\epsilon^{kij}\sigma^{ij}}$ which is exactly what we anticipated, and the term that generates the rotational part of the transformations.

Another way of checking this is if we know $\theta^{K} = \frac{1}{2}\epsilon^{Kij} w_{ij}$ generate rotations, taking those terms and the rest 0, let's check $\frac{i}{4} w_{\alpha'\beta'}\sigma^{\alpha'\beta'} = \frac{1}{4}\theta_K \Sigma^K$:

- $\boxed{w_{\alpha'\beta'}\sigma^{\alpha'\beta'}} = \frac{1}{2}\left(w_{\alpha'\beta'}\sigma^{\alpha'\beta'} - w_{\beta'\alpha'}\sigma^{\alpha'\beta'}\right) = \frac{1}{2}\left(\delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'} - \delta^{\alpha}_{\beta'}\delta^{\beta}_{\alpha'}\right) w_{\alpha\beta}\sigma^{\alpha'\beta'} = \frac{1}{2}\epsilon^{K\alpha\beta}\epsilon_{K\alpha'\beta'} w_{\alpha\beta}\sigma^{\alpha'\beta'} =$

$$= \frac{1}{2}\left(\epsilon^{K\alpha\beta} w_{\alpha\beta}\right)\left(\epsilon_{K\alpha'\beta'}\sigma^{\alpha'\beta'}\right) = \frac{1}{2}\left(2\theta^K\right)\left(2\Sigma_K\right) = \boxed{2\theta^K \Sigma_K}$$

so, yes, $\frac{1}{2}\Sigma^K = \frac{1}{2}\begin{pmatrix}\sigma^K & 0 \\ 0 & \sigma^K\end{pmatrix}$ are the generators of the rotational part of the transformation.

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c)

Our current for only the rotational part will be then:

- $j^{\mu}_{\text{spin rotational}} = \theta_K \bar{\psi}_\alpha \gamma^\mu (\Sigma^K)^\alpha_{\ \beta}\psi^\beta$ (I've taken out some constant factors)

- $j^{\mu K}_{\text{spin rotational}} = \bar{\psi}_\alpha \gamma^\mu (\Sigma^K)^\alpha_{\ \beta}\psi^\beta$

(for any θ_K)

And so the conserved current will be:

$$\boxed{S^K} = \int d^3x\, j^{0K} = \int d^3x\, \bar{\psi}_\alpha \gamma^0 (\Sigma^K)^\alpha_{\ \beta}\psi^\beta = \boxed{\int d^3x\, \psi^\dagger \Sigma^K \psi}$$

(a factor 2 left, doesn't change the conservation)

or normally ordered:

$$:S^K: \equiv \boxed{S^K} = \int d^3x\, :j^{0K}: = \boxed{\int d^3x\, :\psi^\dagger \Sigma^K \psi:}$$

(only a cte from not normal ordered, so it's conserved too)

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d)

d) $$S^K = \int d^3x \left[\sum_{s=1,2}\int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(\hat{a}_{\vec p}^{s\dagger} u_\alpha^{s\dagger}(\vec p) e^{ipx} + \hat{b}_{\vec p}^{s} v_\alpha^{s\dagger}(\vec p) e^{-ipx}\right)\right]\frac{(\Sigma^K)^\alpha_\beta}{2}$$

$$\left[\sum_{s'=1,2}\int\frac{d^3p'}{(2\pi)^3\sqrt{2E_{p'}}}\left(\hat{a}_{\vec p'}^{s'} u^{\beta s'}(\vec p') e^{-ip'x} + \hat{b}_{\vec p'}^{s'\dagger} v^{\beta s'}(\vec p') e^{ip'x}\right)\right] =$$

$$= \iiint\frac{d^3x\, d^3p\, d^3p'}{(2\pi)^6\, 2\sqrt{E_p E_{p'}}}\sum_{s,s'=1,2}\left[\hat{a}_{\vec p}^{\dagger s} u_\alpha^{s\dagger}(\vec p)\frac{(\Sigma^K)^\alpha_\beta}{2}\left(\hat{a}_{\vec p'}^{s'} u^{\beta s'}(\vec p') e^{i(p-p')x} + \hat{b}_{\vec p'}^{s'\dagger} v^{\beta s'}(\vec p') e^{i(p+p')x}\right) + \right.$$

$$\left. + \hat{b}_{\vec p}^{s} v_\alpha^{s\dagger}(\vec p)\frac{(\Sigma^K)^\alpha_\beta}{2}\left(\hat{a}_{\vec p'}^{s'} u^{\beta s'}(\vec p') e^{-i(p+p')x} + \hat{b}_{\vec p'}^{s'\dagger} v^{\beta s'}(\vec p') e^{-i(p-p')x}\right)\right] =$$

$$= \int\frac{d^3p}{(2\pi)^3\, 2E_p}\sum_{s,s'=1,2}\left[\hat{a}_{\vec p}^{\dagger} u_\alpha^{s\dagger}(\vec p)\frac{(\Sigma^K)^\alpha_\beta}{2}\left(\hat{a}_{\vec p}^{s'} u^{\beta s'}(\vec p) + \hat{b}_{-\vec p}^{s'\dagger} v^{\beta s'}(-\vec p)\, e^{2iE_pt}\right) + \right.$$

$$\left. + \hat{b}_{\vec p}^{s} v_\alpha^{s}(\vec p)^\dagger\frac{(\Sigma^K)^\alpha_\beta}{2}\left(\hat{a}_{-\vec p}^{s'} u^{\beta s'}(-\vec p) e^{-2iE_pt} + \hat{b}_{\vec p}^{s'\dagger} v^{\beta s'}(\vec p)\right)\right] =$$

$$= \int\frac{d^3p}{(2\pi)^3\, 2E_p}\sum_{s,s'=1,2}\left[\hat{a}_{\vec p}^{s\dagger}\left(\sqrt{p\sigma}\,\xi^{s\dagger}\sigma^K, \sqrt{p\bar\sigma}\,\xi^{s\dagger}\sigma^K\right)\left(\hat{a}_{\vec p}^{s'}\begin{pmatrix}\sqrt{p\sigma}\,\xi^{s'}\\ \sqrt{p\bar\sigma}\,\xi^{s'}\end{pmatrix} + \hat{b}_{-\vec p}^{s'\dagger}\begin{pmatrix}\sqrt{-p\sigma}\,\eta^{s'}\\ -\sqrt{-p\bar\sigma}\,\eta^{s'}\end{pmatrix} e^{2iE_pt}\right) + \right.$$

$$\left. + \hat{b}_{\vec p}^{s}\left(\sqrt{p\sigma}\,\eta^{s\dagger}\sigma^K, -\sqrt{p\bar\sigma}\,\eta^{s\dagger}\sigma^K\right)\left(\hat{a}_{-\vec p}^{s'}\begin{pmatrix}\sqrt{-p\sigma}\,\xi^{s'}\\ \sqrt{-p\bar\sigma}\,\xi^{s'}\end{pmatrix} e^{-2iE_pt} + \hat{b}_{\vec p}^{s'\dagger}\begin{pmatrix}\sqrt{p\sigma}\,\eta^{s'}\\ -\sqrt{p\bar\sigma}\,\eta^{s'}\end{pmatrix}\right)\right] =$$

$$\boxed{\begin{aligned} = \int\frac{d^3p}{(2\pi)^3\, 2E_p}\sum_{s,s'=1,2}\Big[& \hat{a}_{\vec p}^{s\dagger}\hat{a}_{\vec p}^{s'}\,\xi^{s\dagger}\left(\sqrt{p\sigma}\,\sigma^K\sqrt{p\sigma} + \sqrt{p\bar\sigma}\,\sigma^K\sqrt{p\bar\sigma}\right)\xi^{s'} + \\ & + \hat{a}_{\vec p}^{s\dagger}\hat{b}_{-\vec p}^{s'\dagger}\,\xi^{s\dagger}\left(\sqrt{p\sigma}\,\sigma^K\underbrace{\sqrt{-p\sigma}}_{\sqrt{p\bar\sigma}} - \sqrt{p\bar\sigma}\,\sigma^K\underbrace{\sqrt{-p\bar\sigma}}_{\sqrt{p\sigma}}\right)\eta^{s'} e^{2iE_pt} + \\ & + \hat{b}_{\vec p}^{s}\hat{a}_{-\vec p}^{s'}\,\eta^{s\dagger}\left(\sqrt{p\sigma}\,\sigma^K\sqrt{-p\sigma} - \sqrt{p\bar\sigma}\,\sigma^K\sqrt{-p\bar\sigma}\right)\xi^{s'} e^{-2iE_pt} + \\ & + \hat{b}_{\vec p}^{s}\hat{b}_{\vec p}^{s'\dagger}\,\eta^{s\dagger}\left(\sqrt{p\sigma}\,\sigma^K\sqrt{p\sigma} + \sqrt{p\bar\sigma}\,\sigma^K\sqrt{p\bar\sigma}\right)\eta^{s'}\Big] = \end{aligned}}$$

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e)

• $\boxed{S^z|\vec{0}^r_a;\,{}_b\rangle} = S^z\left(\sqrt{2m}\, a_{\vec 0}^{r\dagger}|0\rangle\right) = \sqrt{2m}\left([S^z, a_{\vec 0}^{r\dagger}] + \cancel{a_{\vec 0}^{r\dagger} \overbrace{S^z}^{S^z|0\rangle=0}}\right)|0\rangle =$

$$= \sqrt{2m}\int \frac{d^3p}{(2\pi)^3\, 2E_p}\sum_{s,s'}\Big[a_{\vec p}^{s\dagger}\overbrace{[a_{\vec p}^{s'}, a_0^{r\dagger}]}^{(2\pi)^3\delta^{s'r}\delta^3(\vec p-\vec 0)}\xi^{s\dagger}\left(\sqrt{p\sigma}\,\sigma^z\sqrt{p\sigma}+\sqrt{p\bar\sigma}\,\sigma^z\sqrt{p\bar\sigma}\right)\xi^{s'} +$$

$([X,a^\dagger]\neq 0$ for $X=a$ only and $[a^\dagger a, a^\dagger] = +a^\dagger[a,a^\dagger])$

$$+ b_{\vec p}^{s}\overbrace{[a_{\vec p}^{s'}, a_0^{r\dagger}]}^{(2\pi)^3\delta^{s'r}\delta^3(\vec p-\vec 0)}\eta^{s\dagger}\left(\sqrt{p\sigma}\,\sigma^z\sqrt{p\bar\sigma}-\sqrt{p\bar\sigma}\,\sigma^z\sqrt{p\sigma}\right)\xi^{s'}e^{-2iE_pt}\Big]|0\rangle =$$

$$=\frac{\sqrt{2m}}{2m}\sum_s\Big[a_{\vec 0}^{s\dagger}\xi^{s\dagger}\left(\sqrt m\,\sigma^z\sqrt m+\sqrt m\,\sigma^z\sqrt m\right)\xi^r + b_{\vec 0}^s\eta^{s\dagger}\left(\cancelto{0}{\sqrt m\,\sigma^z\sqrt m-\sqrt m\,\sigma^z\sqrt m}\right)\xi^{s'}e^{-2iE_pt}\Big]|0\rangle =$$

$$=\frac{2m}{2m}\sqrt{2m}\sum_s\left[a_{\vec 0}^{s\dagger}\xi^{s\dagger}\sigma^z\xi^r|0\rangle\right] = \sqrt{2m}\Big(a_{\vec 0}^{1\dagger}\underbrace{\xi^{1\dagger}\sigma^z}_{+\xi^{1\dagger}}\xi^r + a_{\vec 0}^{2\dagger}\underbrace{\xi^{2\dagger}\sigma^z}_{-\xi^{2\dagger}}\xi^r\Big)|0\rangle =$$

$$=\sqrt{2m}\Big(a_{\vec 0}^{1\dagger}\underbrace{\xi^{1\dagger}\xi^r}_{\delta^{1r}} - a_{\vec 0}^{2\dagger}\underbrace{\xi^{2\dagger}\xi^r}_{\delta^{2r}}\Big)|0\rangle = \begin{cases}\sqrt{2m}\,a_{\vec 0}^{1\dagger}|0\rangle = \boxed{+|\vec 0^1_a;\,{}_b\rangle \text{ if } r=1}\\ -\sqrt{2m}\,a_{\vec 0}^{2\dagger}|0\rangle = \boxed{-|\vec 0^2_a;\,{}_b\rangle \text{ if } r=2}\end{cases}$$

• $\boxed{S^z|\,;\vec 0^r\rangle} = S^z\left(\sqrt{2m}\, b_0^{r\dagger}|0\rangle\right) = \sqrt{2m}\left([S^z, b_0^{r\dagger}] + \cancel{b_0^{r\dagger} \overbrace{S^z}^{S^z|0\rangle=0}}\right)|0\rangle =$

$$=\sqrt{2m}\int\frac{d^3p}{(2\pi)^3\, 2E_p}\sum_{s,s'}\Big[-\overbrace{[b_p^s, b_0^{r\dagger}]}^{(2\pi)^3\delta^{sr}\delta^3(\vec p-0)}a_{\vec p}^{s'}\eta^{s\dagger}\left(\sqrt{p\sigma}\,\sigma^z\sqrt{p\bar\sigma}-\sqrt{p\bar\sigma}\,\sigma^z\sqrt{p\sigma}\right)\xi^{s'}e^{-2iE_pt} -$$

$([X,b^\dagger]\neq 0$ for $X=b$ only and $[bb^\dagger, b^\dagger] = -[b,b^\dagger]b^\dagger)$

⊛ In normal ordering this would have been: $[b^\dagger b, b^\dagger] = +b^\dagger[b,b^\dagger]$ positive!

$$-\overbrace{[b_p^s, b_0^{r\dagger}]}^{(2\pi)^3\delta^{sr}\delta^3(\vec p-0)}b_p^{s'\dagger}\eta^{s\dagger}\left(\sqrt{p\sigma}\,\sigma^z\sqrt{p\sigma}+\sqrt{p\bar\sigma}\,\sigma^z\sqrt{p\bar\sigma}\right)\eta^{s'}\Big]|0\rangle =$$

$$=\frac{\sqrt{2m}}{2m}\sum_{s'}\Big[-a_{\vec 0}^{s'}\eta^{r\dagger}\left(\cancelto{0}{\sqrt m\,\sigma^z\sqrt m-\sqrt m\,\sigma^z\sqrt m}\right)\xi^{s'}e^{-2iE_pt} - b_{\vec 0}^{s'\dagger}\eta^{r\dagger}\left(\sqrt m\,\sigma^z\sqrt m+\sqrt m\,\sigma^z\sqrt m\right)\eta^{s'}\Big]|0\rangle =$$

$$=\frac{2m}{2m}\sqrt{2m}\sum_{s'}\left[-b_{\vec 0}^{s'\dagger}\eta^{r\dagger}\sigma^z\eta^{s'}\right]|0\rangle = \sqrt{2m}\Big(-b_{\vec 0}^{1\dagger}\eta^{r\dagger}\underbrace{\sigma^z\eta^1}_{+\eta^1} - b_{\vec 0}^{2\dagger}\eta^{r\dagger}\underbrace{\sigma^z\eta^2}_{-\eta^2}\Big)|0\rangle =$$

$$=\sqrt{2m}\Big(-b_{\vec 0}^{1\dagger}\underbrace{\eta^{r\dagger}\eta^1}_{\delta^{r1}} + b_{\vec 0}^{2\dagger}\underbrace{\eta^{r\dagger}\eta^2}_{\delta^{r2}}\Big)|0\rangle = \begin{cases}-\sqrt{2m}\,b_{\vec 0}^{1\dagger}|0\rangle = \boxed{-|{}_a;\vec 0^1_b\rangle \text{ if } r=1}\\ +\sqrt{2m}\,b_{\vec 0}^{2\dagger}|0\rangle = \boxed{+|{}_a;\vec 0^2_b\rangle \text{ if } r=2}\end{cases}$$

So, we see that particle^r and antiparticle^r have oposite spin z-component,
This is because we have worked with S^k without normal ordering, with normal order, we would have obtained the same for a's and b's ⊛

particles $\begin{cases}1\to\oplus\\2\to\ominus\end{cases}$

antiparticles $\begin{cases}1\to\ominus\\2\to\oplus\end{cases}$