Lie Groups

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Definition: Group G:

$\{g_i\} \in G$ with an operation "$\cdot$" such that:

- $g_i \cdot g_j \in G$ (internal operation)
- $g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k$ (associative)
- $\exists e \mid g_i \cdot e = g_i \quad \forall g_i \in G$ (neutral element)
- $\forall g \quad \exists g^{-1} \mid g \cdot g^{-1} = e$ (inverse element)

It can be proved:

- $e \cdot g = g$
- $g^{-1} \cdot g = e$
- e is unique
- g^{-1} is unique
- $(g^{-1})^{-1} = g$
- $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$
- $a \cdot b = a \cdot c \leftrightarrow b = c \leftrightarrow b \cdot a = c \cdot a$
- $a \cdot a = a \leftrightarrow a = e$

Definition: Lie Group: continuous group:

- $\theta \equiv \theta_a \in \mathbb{R}, \quad a = 1, \ldots N$ parameters, $N \equiv$ group dimension
- $g = g(\theta_1, \ldots, \theta_N)$ continuous, differentiable
- $g(0) = e$
- $g(-\theta) = g^{-1}(\theta)$

Definition: Subgroup:

$H \subset G$ such that H is group

Definition: Invariant subgroup:

$H \subset G$ is a subgroup such that:

$$\forall h \in H \; \& \; \forall g \in G: \quad g \cdot h \cdot g^{-1} \in H$$

Definition: Simple group:

has no proper invariant subgroups

Example: $SU(N)$ is an invariant subgroup of $U(N)$

$U(N)$: $\{\text{matrix } U, N \times N | U^\dagger = U^{-1}\}$
$SU(N)$: $\{\text{matrix } U, N \times N | U^\dagger = U^{-1}, \det(U) = 1\}$

$$SU(N) \subset U(N)$$

$$\begin{aligned} & A \in U(N) \quad , \quad S \in SU(N) \\ B &= A\,S\,A^{-1} \in U(N) \\ \det(B) &= \det(A\,S\,A^{-1}) = \det(A)\det(S)\det(A)^{-1} = \det S = 1 \\ &\Rightarrow B \in SU(N) \end{aligned}$$

$\Rightarrow$ $SU(N)$ is invariant

Definition: Representation R:

Each element $g \in G$ is assigned a linear operator in a vector space:

$V \equiv$ vector space , $D_R : V \xrightarrow{\text{linear}} V$

$R : g \to D_R(g)$ such that:

- $D_R(e) = \mathbb{1}$
- $D_R(g_i \cdot g_j) = D_R(g_i) \cdot D_R(g_j)$

If $\dim(V) = n$ finite $\to D_R$ are $n \times n$ matrices

Example: rotation group in 3-D: $SO(3)$

Defining object: $\begin{pmatrix} x \\ y \\ x \end{pmatrix} \to M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- $\Rightarrow$ Fundamental representation: F
- $\Rightarrow$ Vector space of dimension: 3. $\dim(F) = 3$
- $\Rightarrow$ 3 parameters in the group (e.g.: rotations around x, y and z angles, or 3 Euler angles)
  - $\Rightarrow$ Lie group of dimension 3
  - $\Rightarrow$ 3 parameters = 3 generators

Some representations:

- Spinors $\begin{pmatrix} a \\ b \end{pmatrix}$ $\Rightarrow$ Vector space of dimension 2.
  (but dim(group)=3: 3 generators: 3 Pauli matrices 2×2)
- Trivial: $D_R(g) = 1 \ \ \forall g \in SO(3)$
  - $\Rightarrow$ Scalar particle $\phi \to \phi$
  - $\Rightarrow$ Vector space of dimension 1.

Definition: Equivalent representations:

R and R' are equivalent if:

$$\exists\, S \quad | \quad D_R(g) = S^{-1} D_{R'}(g) S \quad \text{(basis change)}$$

Definition: Reducible representation:

Leaves invariant a non-trivial subspace:

- V: defining vector space
- $V' \subsetneq V$ a non-trivial subspace ($V' \neq \{0\}$)
- if $\forall v' \in V' \;\; \forall g \in G \to D_R(g) v' \in V'$

e.g.: has a zero-diagonal block:

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & i & j \\ 0 & 0 & k & l \end{pmatrix} \forall D_R(g) \text{ then } \begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix} \text{ is an invariant subspace}$$

$\Rightarrow$ is a reducible representation

Definition: Irreducible representation (*irrep*):

has no invariant subspaces

Definition: Completely reducible representation:

$\exists$ a basis in which D_R is block-diagonal

$$\Rightarrow D_R = D_1 \oplus D_2 \oplus D_3 \cdots \text{ (direct sum)} \left(\begin{array}{c|c|c|c|c} & 0 & \cdots & 0 & 0 \\ \hline 0 & & 0 & \cdots & 0 \\ \hline 0 & 0 & & 0 & \vdots \\ \hline \vdots & \vdots & 0 & \ddots & 0 \\ \hline 0 & 0 & \cdots & 0 & \end{array}\right)$$

- If g is close to the identity:

$$\theta_i = \delta\theta_i \ll 1$$

$$D_R(g(\delta\theta)) = \mathbb{1} - i\delta\theta_a T_R^a$$

Group generators in representation R

$$T_R^a = i\frac{\partial D_R}{\partial \theta_a} \quad , \quad a = 1, \cdots, N \text{ (group dimension)}$$

$$D_R = e^{-i\theta_a T_R^a} \quad \text{(near the identity)}$$

If D_R is unitary ($D_R^\dagger D_R = \mathbb{1}$) $\Rightarrow$ T_R^a are hermitic.

Definition: Structure constants: f^{abc}

$[T_R^a, T_R^b] = if^{abc}T_R^c$ independent of representation!

Group's generator algebra: describes the local structure of the group near the neutral element.

Two groups with the same generator algebra are locally isomorf in the vicinity of the neutral element.

- Abelian group:

$$[T^a_R, T^b_R] = 0$$

$$e^{-i\alpha_a T^a} e^{-i\beta_b T^b} = e^{-i(\alpha_a T^a + \beta_b T^b)} = e^{-i(\alpha_c + \beta_c) T^c}$$

$\Rightarrow$ 1-Dimensional representations

Definition: Casimir operators: C

- Commutes with all group elements
- $C = \lambda \mathbb{1}$: λ label the possible irreducible representations

e.g.: Rotation in 3-dim $SO(3)$ (locally equivalent to $SU(2)$):

- Generators: J^k, $k = 1, 2, 3$ spin matrices
  Group dimension: 3. (3 parameters $\equiv$ 3 generators)
- $[J^k, J^l] = i\varepsilon^{klm} J^m$: structure constants ε^{klm}, $\varepsilon^{123} = 1$
- $C = \vec{J}^2 = (J^1)^2 + (J^2)^2 + (J^3)^2 = \lambda \mathbb{1}$
  $\lambda = j(j+1) \quad , \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ Spin states
- Dimension of representation: $2j + 1$

$j = 0$	$\lambda = 0$	Scalar trivial $D_R(g) = 1$
$j = \frac{1}{2}$	$\lambda = \frac{3}{4}$	Dim=2. 2-spinors. Spin 1/2 $\begin{pmatrix} a \\ b \end{pmatrix} \qquad J^i = \frac{\sigma^i}{2}$ Pauli matrices
$j = 1$	$\lambda = 2$	Dim=3. Vectors. Spin 1 $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdots$
$\vdots$	$\vdots$	$\vdots$

Important irreducible representations:

- Fundamental (defining representation)
- Adjoint: Jacobi identity:

$$\begin{aligned} &[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0 \\ &-f^{ade}f^{bcd}T^e-f^{bde}f^{cad}T^e-f^{cde}f^{abd}T^e=0 \\ &f^{abd}f^{cde}+f^{bcd}f^{ade}+f^{cad}f^{bde}=0 \end{aligned} \qquad (1)$$

Definition: adjoint representation:

$$\left(T^a_{\mathrm{adj}}\right)^{bc}=-if^{abc} \qquad (2)$$

$$[T^a_{\mathrm{adj}},T^b_{\mathrm{adj}}]^{cd} \;=\; -\left(f^{ace}f^{bed}-f^{bce}f^{aed}\right)=-\left(f^{ace}f^{bed}+f^{aed}f^{cbe}\right)$$

Jacobi identity (1) with: $b\to c$, $d\to e$, $c\to b$, $e\to d$

$$[T^a_{\mathrm{adj}},T^b_{\mathrm{adj}}]^{cd}=+f^{bae}f^{ced}=-f^{abe}f^{ced}=f^{abe}f^{ecd}=if^{abe}(-if^{ecd})=if^{abe}\left(T^e_{\mathrm{adj}}\right)^{cd}$$

Definition (2) fulfills the generator algebra

$\Rightarrow$ is a representation of the group generators:

$$[T^a_{\mathrm{adj}},T^b_{\mathrm{adj}}]^{cd}=if^{abe}\left(T^e_{\mathrm{adj}}\right)^{cd}$$