QUANTUM FIELD THEORY

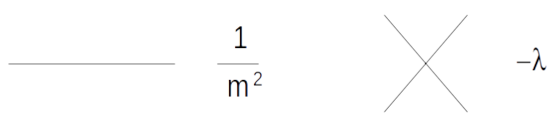
Fall 2021

A QFT toy model and asymptotic series

We are going to study a toy model of a quantum field theory that despite its simplicity, captures some important features of realistic quantum field theories. We can think of this toy model as the limit of the Euclidean $\lambda\phi^4$ theory when ϕ is restricted to be independent of spacetime coordinates (in this sense this toy model is a 0+0 QFT). In this limit, ϕ is just an ordinary variable and the path integral boils down to an ordinary integral. Since ϕ does not depend on space-time coordinates, there is no kinetic term,

$$Z(\lambda, j) = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\left(\frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right) + j\phi}$$

$Z(\lambda, 0)$ can be computed exactly (ask `Mathematica` !). Notice that there is no i in the exponent, this is the toy version of an Euclidean QFT. The Feynman rules are (again, there is no i in the Feynman rule for the vertex),



1) *Free Theory.* Set $\lambda = 0$ (so there are no vertices). You are going to compute the n-point functions $< 0|\phi^n|0 >$ of the free theory in three different ways. Of course, you should obtain the same result with all three methods.

i) Compute the n-point function by carrying out the integrals **(1 point)**

$$< 0|\phi^n|0 >= \frac{\int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} \ \phi^n \ e^{-\frac{m^2}{2}\phi^2}}{\int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} \ e^{-\frac{m^2}{2}\phi^2}}$$

ii) Compute the generating function $Z(j)_{\lambda=0}$ directly, and then obtain the n-point functions by taking derivatives of this generating function. **(1 point)**

iii) Give a purely combinatorial interpretation of $< 0|\phi^n|0 >$ and use it to evaluate these n-point functions. **(1 point)**

2) *Interacting theory.* Let's now consider $\lambda \neq 0$. It is easy to argue that $Z(j=0)$ is convergent for $\lambda \geq 0$ and divergent for $\lambda < 0$. Nevertheless, let's pretend that we can write $Z(j=0)$ as a power series in $\tilde{\lambda} = \frac{\lambda}{m^4}$. To do so, expand $e^{-\frac{\lambda}{4!}\phi^4}$ in a power series, exchange $\sum$ and $\int$, and perform the integrals. The answer is of the form,

$$Z_{pert}(j=0) = \frac{1}{m}\sum_{k=0}^{\infty} c_k(-\tilde{\lambda})^k \tag{1}$$

- Provide an explicit expression of the coefficients c_k. **(1 point)**
- Since each vacuum diagram evaluates to just its (inverse) symmetry factor and the appropriate power of λ, the coefficients c_k give the sum of the (inverse) symmetry factors of the vacuum diagrams of the $\lambda\phi^4$ theory at order k. Draw and evaluate the relevant vacuum diagrams, and check this up to order λ^2. **(1 point)**
- Draw and evaluate all the connected vacuum diagrams up to order λ^3. Check up to this order that $W(\tilde{\lambda}) = \log\frac{Z(\tilde{\lambda})}{Z(0)}$ is their generating function. **(1 point)**

3) n-point functions in the interacting theory **(1+1+1 points)**

- Compute $W(\lambda, j) = \log Z(\lambda, j)/Z(\lambda, 0)$ up to order λ^2, and derive from it $<\Omega|\phi^2|\Omega>_c$ and $<\Omega|\phi^4|\Omega>_c$ to this order.
- Reproduce these results by drawing and evaluating the relevant connected diagrams.
- Compute $<\Omega|\phi^2|\Omega>_{1\text{PI}}$ to order λ^3 by evaluating the relevant diagrams. Verify to this order that $<\Omega|\phi^2|\Omega>_c = 1/(m^2 - <\Omega|\phi^2|\Omega>_{1\text{PI}})$.

4) The power series (1) is easily seen to be *divergent* for any non-zero value of λ. This is not surprising, since the integral diverges for $\lambda < 0$, and therefore the power series can't have any non-zero radius of convergence. In fact, in the complex λ plane, $Z(\lambda)$ has a branch-cut along the negative real axis. This is not as bad as it might seem: it can be shown that (setting $m = 1$)

$$\left| Z(\lambda)_{j=0} - \sum_{k=0}^{n} c_k(-\lambda)^k \right| < |c_{n+1}\lambda^{n+1}| \tag{2}$$

so for fixed n, taking λ small enough the perturbative series gives an arbitrarily good estimate of the full answer. By definition, this is an *asymptotic series*. This feature is shared with many Quantum Field Theories: there is an old argument by Dyson[1] showing that QED should be unstable for $\alpha < 0$ so the radius of convergence of perturbative QED is zero.

- Plot $Z(\lambda)_{j=0}$ in the range $\lambda \in [0, 1]$. On the same graph, plot $Z_{pert}(\lambda)_{j=0}$ truncated at $n = 5$ and $n = 10$. Is the truncated series with more terms always a better approximation to the full result? **(1 point)**

[1]F. J. Dyson, "Divergence of Perturbation Theory in Quantum Electrodynamics", Phys. Rev. 85, 631 (1952).