Spin 1/2

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2020-2021

Classical theory

- Objective: build a Lorentz-invariant Lagrangian for fermions, with a derivative term
- Define sets of Pauli matrices

$$\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3) \;\; ; \;\; \sigma^\mu = (1, \boldsymbol{\sigma}) \;\; ; \;\; \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$$

  these objects are **not** 4-vectors
- Define the bilineals:

$$\psi_R^\dagger \sigma^\mu \psi_R \quad ; \quad \psi_L^\dagger \bar{\sigma}^\mu \psi_L \tag{1}$$

  these bilineals **do transform** as 4-vectors.

Transformation of $\psi_R^\dagger\sigma^\mu\psi_R$ by an infinitesimal Lorentz transformation

$$\begin{aligned}\psi_R &\to e^{(-i\boldsymbol{\theta}+\boldsymbol{\eta})\cdot\boldsymbol{\sigma}/2}\psi_R \simeq (1+\frac{1}{2}(-i\boldsymbol{\theta}+\boldsymbol{\eta})\cdot\boldsymbol{\sigma}))\psi_R\\ \psi_R^\dagger &\to \psi_R^\dagger e^{(i\boldsymbol{\theta}+\boldsymbol{\eta})\cdot\boldsymbol{\sigma}/2} \simeq \psi_R^\dagger(1+\frac{1}{2}(i\boldsymbol{\theta}+\boldsymbol{\eta})\cdot\boldsymbol{\sigma}))\end{aligned}$$

$$\begin{aligned}\psi_R^\dagger\sigma^\mu\psi_R &\to \psi_R^\dagger(\sigma^\mu+(i\boldsymbol{\theta}+\boldsymbol{\eta})\cdot\frac{\boldsymbol{\sigma}}{2}\sigma^\mu+\sigma^\mu(-i\boldsymbol{\theta}+\boldsymbol{\eta})\cdot\frac{\boldsymbol{\sigma}}{2})\psi_R+\mathcal{O}(\theta^2,\eta^2)\\ &= \psi_R^\dagger(\sigma^\mu+\frac{i\theta^i}{2}(\sigma^i\sigma^\mu-\sigma^\mu\sigma^i)+\frac{\eta^i}{2}(\sigma^i\sigma^\mu+\sigma^\mu\sigma^i))\psi_R\end{aligned}$$

properties of Pauli matrices:

$[\sigma^i,\sigma^j]=2i\epsilon^{ijk}\sigma^k$; $[\sigma^i,\sigma^0]=0$; $\sigma^i\sigma^j+\sigma^j\sigma^i=\{\sigma^i,\sigma^j\}=2\delta^{ij}$; $\{\sigma^i,\sigma^0\}=2\sigma^i$

$$\psi_R^\dagger\sigma^\mu\psi_R \to \psi_R^\dagger(\sigma^\mu+\frac{i\theta^i}{2}(\sigma^i\sigma^\mu-\sigma^\mu\sigma^i)+\frac{\eta^i}{2}(\sigma^i\sigma^\mu+\sigma^\mu\sigma^i))\psi_R$$

$[\sigma^i,\sigma^j]=2i\epsilon^{ijk}\sigma^k$; $[\sigma^i,\sigma^0]=0$; $\sigma^i\sigma^j+\sigma^j\sigma^i=\{\sigma^i,\sigma^j\}=2\delta^{ij}$; $\{\sigma^i,\sigma^0\}=2\sigma^i$

- $\mu=0$

$$\psi_R^\dagger\sigma^0\psi_R \to \psi_R^\dagger(\sigma^0+0+\eta^i\sigma^i))\psi_R=\psi_R^\dagger\sigma^0\psi_R+\eta^i\psi_R^\dagger\sigma^i\psi_R \quad (2)$$

- $\mu=j$

$$\begin{aligned}\psi_R^\dagger\sigma^j\psi_R &\to \psi_R^\dagger(\sigma^j+\frac{i\theta^i}{2}(2i\varepsilon^{ijk})\sigma^k+\eta^i\delta^{ij})\psi_R=\psi_R^\dagger\sigma^j\psi_R-\theta^i\varepsilon^{ijk}\psi_R^\dagger\sigma^k\psi_R+\eta^j\psi_R^\dagger\psi_R\\ &= \psi_R^\dagger\sigma^j\psi_R-\theta^i\varepsilon^{ijk}\psi_R^\dagger\sigma^k\psi_R+\eta^j\psi_R^\dagger\sigma^0\psi_R \qquad (3)\end{aligned}$$

- eq. (2), and the first and 3rd term of eq. (3):
  4-vector infinitesimal Lorentz transformations

$$x'^0=x^0+\eta^ix^i \quad ; \quad x'^j=x^j+\eta^jx^0$$

- the second term in eq. (3) is an infinitesimal rotation.

$\Rightarrow$ $\psi_R^\dagger\sigma^\mu\psi_R$ transforms as a 4-vector.

$\Rightarrow$ The same can be computed for $\psi_L^\dagger\bar{\sigma}^\mu\psi_L$.

- construct an invariant Lagrangian: contract with 4-vector $p_\mu = i\partial_\mu$

$$\mathcal{L}_L = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L \quad ; \quad \mathcal{L}_R = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R \quad ; \qquad (4)$$

- ψ and $\psi^\dagger$ as independent fields,

equations of motion

$$\begin{aligned} \psi_L &: \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_L} - \frac{\partial \mathcal{L}}{\partial \psi_L} = 0 \Rightarrow i\partial_\mu \psi_L^\dagger \bar{\sigma}^\mu = 0 \\ \psi_L^\dagger &\Rightarrow \quad -i\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \end{aligned}$$

equivalent equations

- Equivalent computation for ψ_R:

Weyl-equations:

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \quad , \quad i\sigma^\mu \partial_\mu \psi_R = 0 \quad (5)$$

Left-handed field

- ψ_L fulfills a Klein-Gordon equation:

$$\begin{aligned} (\partial_0 - \sigma^i \partial_i)\psi_L &= 0 \\ \partial_0 \psi_L &= \sigma^i \partial_i \psi_L \\ \partial_0^2 \psi_L &= \sigma^i \partial_i \partial_0 \psi_L = \sigma^i \sigma^j \partial_i \partial_j \psi_L \\ &= \frac{1}{2}(\sigma^i \sigma^j + \sigma^j \sigma^i)\partial_i \partial_j \psi_L \quad [\partial_i \partial_j \text{ is symmetric } i \leftrightarrow j] \\ &= \delta^{ij} \partial_i \partial_j \psi_L \\ (\partial_0^2 - \partial_i^2)\psi_L = 0 \quad &\Rightarrow \quad \boxed{\partial_\mu \partial^\mu \psi_L = 0} \end{aligned}$$

$\Rightarrow$ Klein-Gordon equation for a **massless** field

- Separate the solutions in positive & negative energy fields:

$$\psi_L^+ = u_L e^{-ipx} \quad ; \quad \psi_L^- = u_L e^{ipx} \tag{6}$$
$$p^\mu = (E, \boldsymbol{p}) \quad ; \quad E^2 - \boldsymbol{p}^2 = 0$$

- spin is $\boldsymbol{S} = \boldsymbol{\sigma}/2$
  $\Rightarrow$ compute the **helicity** (projection of spin the momentum direction) of the states in (6),

helicity:

$$h = \hat{\boldsymbol{p}} \cdot \boldsymbol{S} = \frac{1}{2} \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}$$

$\hat{\boldsymbol{p}} \equiv$ unitary vector in the $\boldsymbol{p}$ direction
Weyl equation (5):

$$\bar{\sigma}^\mu \partial_\mu \psi_L = (\partial_0 - \sigma^i \partial_i) u_L e^{\mp ipx} = \mp i (E + \boldsymbol{\sigma} \cdot \boldsymbol{p}) u_L e^{\mp ipx} = 0$$
$$\Rightarrow \quad \boldsymbol{\sigma} \cdot \boldsymbol{p}\, u_L = -E u_L \Rightarrow \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}\, u_L = -u_L \Rightarrow \boxed{h = -\frac{1}{2}}$$

Energy-momentum tensor

- $\mathcal{L}$ does not depend on $\partial_\mu \psi_L^\dagger$
- due to the eq. of motion: $\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \Rightarrow \mathcal{L} = 0$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_L} \partial^\nu \psi_L - g^{\mu\nu} \mathcal{L} = i \psi_L^\dagger \bar{\sigma}^\mu \partial^\nu \psi_L$$

Canonical momenta

$$\Pi_{\psi_L} = i \psi_L^\dagger \bar{\sigma}^0 = i \psi_L^\dagger \quad ; \quad \Pi_{\psi_L^\dagger} = 0$$

$U(1)$ phase symmetry:

the Lagrangian is invariant $\psi_L \to e^{-i\alpha}\psi_L$

$\Rightarrow$ conserved current:

$$\begin{aligned}
\alpha j^\mu &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_L}\delta\psi_L = i\psi_L^\dagger \bar{\sigma}^\mu(-i\alpha\psi_L) = \alpha\psi_L^\dagger\bar{\sigma}^\mu\psi_L \\
j^\mu &= \psi_L^\dagger\bar{\sigma}^\mu\psi_L \;; \\
\partial_\mu j^\mu &= 0 \;;\quad Q = \int \mathrm{d}^3x\,\psi_L^\dagger\psi_L
\end{aligned}$$

these will be the electromagnetic current and electromagnetic charge.

Right-handed field

Same process for the ψ_R field, and find:

$$\begin{aligned}
h &= \frac{1}{2} \;;\quad T^{\mu\nu} = i\psi_R^\dagger\sigma^\mu\partial^\nu\psi_R \;;\quad \Pi_{\psi_R} = i\psi_R^\dagger \;;\quad \Pi_{\psi_R^\dagger} = 0 \;; \\
j^\mu &= \psi_R^\dagger\sigma^\mu\psi_R \;;\quad Q = \int \mathrm{d}^3x\,\psi_R^\dagger\psi_R
\end{aligned}$$

ψ_R and ψ_L describe zero mass $s = 1/2$ particles with positive & negative helicity.

- ψ_L and ψ_R are **not** parity invariant:
  under parity $\boldsymbol{p} \to -\boldsymbol{p}$, $\boldsymbol{s} \to \boldsymbol{s} \Rightarrow h \to -h$

$$\begin{aligned}
\psi_L &\to \psi_L' \quad \text{transforms as } \psi_R \\
\psi_R &\to \psi_R' \quad \text{transforms as } \psi_L
\end{aligned}$$

$\Rightarrow$ for parity-invariant theory (QED, QCD), we must combine $\psi_L \oplus \psi_R$ in a Dirac spinor: $(1/2, 0) \oplus (0, 1/2)$.

Mass term

- a bilinear term of the fields
- we can not add: $\psi_L^\dagger \psi_L \;\; ; \;\; \psi_R^\dagger \psi_R$
  zero components of the 4-vectors (1) $\Rightarrow$ not Lorentz invariant

Under Lorentz transformations:

$$\psi_L \to \Lambda_L \psi_L \;\; ; \;\; \psi_R \to \Lambda_R \psi_R$$
$$\Lambda_L^\dagger \Lambda_R = \mathbb{1} = \Lambda_R^\dagger \Lambda_L$$

$\Rightarrow$ The combinations $\psi_L^\dagger \psi_R$, $\psi_R^\dagger \psi_L$ are **Lorentz scalars**

$$\begin{array}{rl} \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L & \text{scalar (+1 under parity)} \\ i(\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L) & \text{pseudo-scalar (-1 under parity)} \end{array}$$

$\Rightarrow$ Lorentz & parity invariant Lagrangian:

$$\mathcal{L}_D = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$

$\Rightarrow$ equations of motion:

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = m\psi_R \;\; ; \;\; i\sigma^\mu \partial_\mu \psi_R = m\psi_L \;\; ;$$

$\Rightarrow$ which are the **Dirac** equations for the Weyl spinors.

- Apply same procedure as for the massless field:

$$\partial_\mu \partial^\mu \psi_L + m^2 \psi_L = 0 \;\; ; \;\; \partial_\mu \partial^\mu \psi_R + m^2 \psi_R = 0$$

$\Rightarrow$ Klein-Gordon eqs. for a field of mass m.

4-component Dirac fields

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

4×4 Dirac matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad ; \quad \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

- ⇒ Dirac equation $i\gamma^\mu \partial_\mu \psi_D = m\psi_D$
- ⇒ mass-term

$$\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L = \psi_D^\dagger \gamma^0 \psi_D$$

- ⇒ Define: $\bar{\psi}_D = \psi_D^\dagger \gamma^0$

Dirac Lagrangian can be written as:

$$\mathcal{L}_D = \bar{\psi}_D (i\gamma^\mu \partial_\mu - m)\psi_D = \bar{\psi}_D (i\partial\!\!\!/ - m)\psi_D$$

Definition: for a 4-vector a^μ: $a\!\!\!/ = a_\mu \gamma^\mu$

Some more definitions:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

and the chirality-projection operators:

$$P_L = \frac{1}{2}(\mathbb{1} - \gamma^5) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad P_R = \frac{1}{2}(\mathbb{1} + \gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$P_L \psi_D = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \quad ; \quad P_R \psi_D = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \quad ;$$

- ⇒ **chiral representation** of the Dirac matrices
- ⇒ Any set of matrices that obey the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

  are a valid representation of the Dirac matrices.
- ⇒ Different representations are related by a unitary basis change:

$$\psi_D' = U\psi_D \quad ; \quad \gamma'^\mu = U\gamma^\mu U^\dagger$$

standard or Dirac representation

basis change

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}$$

$$\psi'_D = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad ; \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$P_L = \frac{1}{2} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \quad ; \quad P_R = \frac{1}{2} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \quad ;$$

Chirality

in the presence of mass,

$\Rightarrow$ ψ_L and ψ_R do not longer represent helicity states,
$\Rightarrow$ they are called **chiral** states.
$\Rightarrow$ When $m = 0$ chirality = helicity.

- Chiral representation is useful for:
  - Zero mass
  - high-energy ($m/E \to 0$)
  - Chiral theories: SM ψ_L has different interactions than ψ_R.

  at $m \to 0$ chirality is conserved, so a change $\psi_L \leftrightarrow \psi_R$ involves a suppression factor m/E.
- Dirac representation useful for:
  - low energies (or non-relativistic), when mass is important. The Dirac representation separates particle/anti-particle (positive & negative energy components)

But most of time we use γ^μ properties, and **not** specific representations

$$\begin{aligned}\sigma^{\mu\nu} &= \frac{i}{2}[\gamma^\mu,\gamma^\nu]\ \text{(Definition)}\\ \{\gamma^\mu,\gamma^\nu\} &= 2g^{\mu\nu}\\ \{\gamma^\mu,\gamma^5\} &= 0\\ \gamma^0\gamma^{\mu\dagger}\gamma^0 &= \gamma^\mu\end{aligned}$$

$$\begin{aligned}\gamma^5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\\ \gamma^5\sigma^{\mu\nu} &= \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}\\ \Sigma^i &\equiv \gamma^5\gamma^0\gamma^i = \frac{1}{2}\epsilon_{ijk}\sigma^{jk}\end{aligned}$$

Contracting indices:

$$\begin{aligned}\not{a}\not{b} &= ab - i\sigma_{\mu\nu}a^\mu b^\nu\\ \gamma^\mu\gamma_\mu &= 4\\ \gamma^\mu\not{a}\gamma_\mu &= -2\not{a}\\ \gamma^\mu\not{a}\not{b}\gamma_\mu &= 4ab\\ \gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu &= -2\not{c}\not{b}\not{a}\end{aligned}$$

See: V.I. Borodulin, R.N. Rogalyov, S.R. Slabospitsky, *CORE: Compendium of relations*, hep-ph/9507456, but **careful with conventions!!!!**[1]

The 16 matrix-set:

$$\mathbb{1},\ \gamma^\mu,\ \gamma^5,\ \gamma^\mu\gamma^5,\ \sigma^{\mu\nu}$$

are linearly independent, and form a basis of the 4×4 matrix space.

[1] It defines the σ^μ and $\bar{\sigma}^\mu$ opposite to us, and the Dirac 4-spinor also opposite to us!, but generic properties of Dirac γ matrices are OK.

Symmetries of the Dirac Lagrangian

Energy-momentum:

$$T^{\mu\nu} = \bar{\psi} i \gamma^\mu \partial^\nu \psi$$

The canonical momenta are:

$$\Pi_\psi = i\bar{\psi}\gamma^0 = i\psi^\dagger \ \ ; \ \ \Pi_{\bar{\psi}} = 0$$

Charge: $\psi \to e^{-i\alpha}\psi$:

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

Chiral symmetry: if $m = 0$:

$$\psi \to e^{-i\alpha\gamma^5}\psi \ \ ; \ \ \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \to \begin{pmatrix} e^{i\alpha}\psi_L \\ e^{-i\alpha}\psi_R \end{pmatrix}$$

$$j^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \quad \text{axial current}$$

under a Lorentz transformation:

$$\psi \to e^{\frac{-i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi$$

So the generators are:

$$S^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}$$

and the spin operators are:

$$S^{ij} = \frac{1}{2}\sigma^{ij}$$

the following bilinears transform as:

$$\begin{array}{rcl} \bar{\psi}\psi \text{ scalar} & ; & \bar{\psi}\gamma^5\psi \text{ pseudo-scalar, P=-1} \\ \bar{\psi}\gamma^\mu\psi \text{ vector} & ; & \bar{\psi}\gamma^\mu\gamma^5\psi \text{ axial vector} \\ \bar{\psi}\sigma^{\mu\nu}\psi \text{ tensor} & & \end{array}$$

Majorana mass

Majorana spinor $\Rightarrow$ massive Dirac equation:

$$(i\partial\!\!\!/ - m)\psi_M = 0$$

try to build a mass-term for the Lagrangian:

$$\bar{\psi}_M\psi_M = \left(\psi_L^\dagger, -i\xi^*\psi_L^T\sigma^2\right)\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}\psi_L\\ i\xi\sigma^2\psi_L^*\end{pmatrix} = i\xi\psi_L^\dagger\sigma^2\psi_L^* - i\xi^*\psi_L^T\sigma^2\psi_L$$

$$i\psi_L^T\sigma^2\psi_L = i\begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \psi_2\psi_1 - \psi_1\psi_2$$

$\Rightarrow$ So the mass term is zero **unless** $\psi_1\psi_2$ do not **commute**.

$\Rightarrow$ Classically Majorana fermions can not have mass, we need quantum!

$\Rightarrow$ Also, Majorana fermions can not have $U(1)$ symmetries:

$$\psi \to e^{-i\alpha}\psi \Rightarrow \begin{cases}\psi_L \to e^{-i\alpha}\psi_L\\ \psi_R \to e^{-i\alpha}\psi_R\end{cases}$$

BUT $\psi_L \sim \psi_R^*$: can not have $U(1)$ charges (electromagnetism, etc.)

Explicit solution to the Dirac equation

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}}\left(u(\boldsymbol{p})e^{-ipx} + v(\boldsymbol{p})e^{ipx}\right) \tag{7}$$

u, v are 4-component spinors:

$$(i\partial\!\!\!/ - m)\psi = 0$$

- Positive energy ($E > 0$):

$$\psi^+ \simeq u(\boldsymbol{p})e^{-ipx} \Rightarrow (p\!\!\!/ - m)u(\boldsymbol{p}) = 0$$

- Negative energy ($E < 0$):

$$\psi^- \simeq v(\boldsymbol{p})e^{ipx} \Rightarrow (-p\!\!\!/ - m)v(\boldsymbol{p}) = 0$$

To find and explicit solution,

1. go to an *easy* frame,
2. find the solution,
3. make a Lorentz transformation to the original frame.

$m \neq 0$

Go to the proper reference frame $\boldsymbol{p} = \boldsymbol{0}$, $E = m$:

$$(\not p - m)u = 0 \quad \Rightarrow \quad (\gamma^0 - \mathbb{1})u = 0 \Rightarrow \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0 \Rightarrow u_L = u_R$$

$$(-\not p - m)v = 0 \quad \Rightarrow \quad (\gamma^0 + \mathbb{1})v = 0 \Rightarrow \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} v_L \\ v_R \end{pmatrix} = 0 \Rightarrow v_L = -v_R$$

$\Rightarrow$ only two independent solutions for u and two for v. We choose:[2]

$$u_L^s(\boldsymbol{0}) = u_R^s(\boldsymbol{0}) = \sqrt{m}\xi^s \quad ; \quad s = 1, 2 \tag{8}$$

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9}$$

$$u^1(\boldsymbol{0}) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad ; \quad u^2(\boldsymbol{0}) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad ; \tag{10}$$

[2] Careful with different normalizations in different books! this is the same as the one

Make a boost in the $\hat{\boldsymbol{p}} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}$ direction:

$$u^s(\boldsymbol{p}) = \begin{pmatrix} e^{-\frac{1}{2}\eta \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}} u_L^s(\boldsymbol{0}) \\ e^{\frac{1}{2}\eta \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}} u_R^s(\boldsymbol{0}) \end{pmatrix} \quad ; \quad e^{\pm\eta \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}} = \cosh\eta \pm \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\sinh\eta$$

$$\cosh\eta = \gamma = \frac{E}{m} \quad ; \quad \sinh\eta = \gamma\beta = \frac{|\boldsymbol{p}|}{m}$$

$$e^{\pm\eta \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}} = \frac{1}{m}(E \pm \boldsymbol{p}\cdot\boldsymbol{\sigma})$$

$$p^\mu\sigma_\mu = p\sigma = E - \boldsymbol{p}\cdot\boldsymbol{\sigma} \quad ; \quad p^\mu\bar{\sigma}_\mu = p\bar{\sigma} = E + \boldsymbol{p}\cdot\boldsymbol{\sigma} \quad ; \quad e^{\alpha/2} = \sqrt{e^\alpha}$$

$$u^s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi^s \end{pmatrix} \tag{11}$$

$\Rightarrow$ $\sqrt{\ }$ is taken in the matrix sense.

- It's easy to see that the solution (11) fulfills the Dirac equation.

Ultra-relativistic limit $E \gg m$

with momentum in the z-direction $p^\mu = (E, 0, 0, E)$

$$u^s(\boldsymbol{p}) = \sqrt{E}\begin{pmatrix}\sqrt{1-\sigma^3}\xi^s \\ \sqrt{1+\sigma^3}\xi^s\end{pmatrix} = \sqrt{E}\begin{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}^{1/2}\xi^s \\ \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}^{1/2}\xi^s\end{pmatrix} = \sqrt{2E}\begin{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\xi^s \\ \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\xi^s\end{pmatrix}$$

$$u^1(\boldsymbol{p}) = \sqrt{2E}\begin{pmatrix}0 \\ \xi^1\end{pmatrix} \quad ; \quad u^2(\boldsymbol{p}) = \sqrt{2E}\begin{pmatrix}\xi^2 \\ 0\end{pmatrix}$$

Fields normalization

$$\begin{aligned}
u^{r\dagger}(\boldsymbol{p})u^s(\boldsymbol{p}) &= \begin{pmatrix}\xi^{r\dagger}\sqrt{p\sigma}^\dagger & \xi^{r\dagger}\sqrt{p\bar{\sigma}}^\dagger\end{pmatrix}\begin{pmatrix}\sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi^s\end{pmatrix} = \xi^{r\dagger}(p\sigma + p\bar{\sigma})\xi^s \\
&= \xi^{r\dagger}(p^0 - \boldsymbol{p}\cdot\boldsymbol{\sigma} + p^0 + \boldsymbol{p}\cdot\boldsymbol{\sigma})\xi^s = 2p^0\xi^{r\dagger}\xi^s \\
&= 2E\delta^{rs}
\end{aligned}$$

Additional relations

$$\begin{aligned}
\bar{u}^r(\boldsymbol{p})u^s(\boldsymbol{p}) &= \begin{pmatrix}\xi^{r\dagger}\sqrt{p\bar{\sigma}}^\dagger & \xi^{r\dagger}\sqrt{p\sigma}^\dagger\end{pmatrix}\begin{pmatrix}\sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi^s\end{pmatrix} = \xi^{r\dagger}(\sqrt{p\bar{\sigma}}\sqrt{p\sigma} + \sqrt{p\sigma}\sqrt{p\bar{\sigma}})\xi^s \\
&= \xi^{r\dagger}2m\xi^s = 2m\delta^{rs}
\end{aligned}$$

where we have used:

$$p\sigma\, p\bar{\sigma} = (p^0 - \boldsymbol{p}\cdot\boldsymbol{\sigma})(p^0 + \boldsymbol{p}\cdot\boldsymbol{\sigma}) = (p^0)^2 - (\boldsymbol{p}\cdot\boldsymbol{\sigma})^2 = (p^0)^2 - \boldsymbol{p}^2 = m^2$$

Completeness relations:

$$\begin{aligned}
\sum_{s=1,2} u^s(\boldsymbol{p})\bar{u}^s(\boldsymbol{p}) &= \sum_s \begin{pmatrix}\sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi^s\end{pmatrix}\begin{pmatrix}\xi^{s\dagger}\sqrt{p\bar{\sigma}} & \xi^{s\dagger}\sqrt{p\sigma}\end{pmatrix} = \begin{pmatrix}\sqrt{p\sigma}\sqrt{p\bar{\sigma}} & \sqrt{p\sigma}\sqrt{p\sigma} \\ \sqrt{p\bar{\sigma}}\sqrt{p\bar{\sigma}} & \sqrt{p\bar{\sigma}}\sqrt{p\sigma}\end{pmatrix} \\
&= \begin{pmatrix}m & p\sigma \\ p\bar{\sigma} & m\end{pmatrix} = \not{p} + m \qquad (12)
\end{aligned}$$

where we have used : $\sum_s \xi^s\xi^{s\dagger} = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \mathbb{1}$

Due to the Dirac equation $(\not{p} - m)u^s = 0$:

$$0 = \sum_{s=1,2} (\not{p} - m)u^s(\boldsymbol{p})\bar{u}^s(\boldsymbol{p}) = (\not{p} - m)(\not{p} + m) = p^2 - m^2 = 0$$

For the negative energy spinors $v^s(\boldsymbol{p})$

$$v_l^s(\mathbf{0}) = \sqrt{m}\eta^s \;\; ; \;\; \eta^{s\dagger}\eta^r = \delta^{rs} \;\; ; \;\; v^s(\mathbf{0}) = \sqrt{m}\begin{pmatrix} \eta^s \\ -\eta^s \end{pmatrix}$$

it is convenient to define the η^s as the charge-conjugates of ξ^s:

$$\eta^s = -i\sigma^2\xi^{s*} \;\; ; \;\; \eta^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\; ; \;\; \eta^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$v^1(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \;\; ; \;\; v^2(\mathbf{0}) = \sqrt{m}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

the spinor in any reference frame is

$$v^s(\boldsymbol{p}) = \begin{pmatrix} \sqrt{p\sigma}\eta^s \\ -\sqrt{p\bar{\sigma}}\eta^s \end{pmatrix}$$

in the ultra-relativistic limit $E \gg m$, with $\boldsymbol{p}$ in the z-direction

$$v^1(\boldsymbol{p}) = \sqrt{2E}\begin{pmatrix} \eta^1 \\ 0 \end{pmatrix} \;\; ; \;\; v^2(\boldsymbol{p}) = -\sqrt{2E}\begin{pmatrix} 0 \\ \eta^2 \end{pmatrix}$$

the normalizations are:

$$v^{r\dagger}(\boldsymbol{p})v^s(\boldsymbol{p}) = 2E\delta^{rs} \;\;;\;\; \bar{v}^r(\boldsymbol{p})v^s(\boldsymbol{p}) = -2m\delta^{rs}$$

complemented by:

$$\begin{aligned} \bar{u}^r(\boldsymbol{p})v^s(\boldsymbol{p}) = \bar{v}^r(\boldsymbol{p})u^s(\boldsymbol{p}) = 0 \;\;; \\ u^{r\dagger}(-\boldsymbol{p})v^s(\boldsymbol{p}) = v^{r\dagger}(-\boldsymbol{p})u^s(\boldsymbol{p}) = 0 \;\;; \end{aligned} \tag{13}$$

and the completeness relation:

$$\sum_s v^s(\boldsymbol{p})\bar{v}^s(\boldsymbol{p}) = \not{p} - m \tag{14}$$

the completeness relations (12) and (14) lead to:

$$\sum_s u^s(\boldsymbol{p})\bar{u}^s(\boldsymbol{p}) - v^s(\boldsymbol{p})\bar{v}^s(\boldsymbol{p}) = 2m\mathbb{1}$$

⇒ Definition: of the *positive* and *negative* energy projection operators:

$$\Lambda^{\pm}(p) = \frac{\pm\not{p} + m}{2m} \tag{15}$$

$$\Lambda^+(p) = \frac{1}{2m}\sum_s u^s\bar{u}^s \;\;;\;\; \Lambda^-(p) = -\frac{1}{2m}\sum_s v^s\bar{v}^s \;\;;\;\; \Lambda^+(p) + \Lambda^-(p) = \mathbb{1}$$

$$\Lambda^+(p)u^r(\boldsymbol{p}) = u^r(\boldsymbol{p}) \;\;;\;\; \Lambda^+(p)v^r(\boldsymbol{p}) = 0 \;\;;$$

$$\Lambda^-(p)u^r(\boldsymbol{p}) = 0 \;\;;\;\; \Lambda^-(p)v^r(\boldsymbol{p}) = v^r(\boldsymbol{p})$$

Quantization: first attempt

- Convert the field $\psi \Rightarrow$ operator.
- Add an explicit operator to each component of the classical solution (7),

$$\psi(x) = \sum_{s=1,2} \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a^s_{\boldsymbol{p}} u^s(\boldsymbol{p}) e^{-ipx} + b^s_{\boldsymbol{p}} v^s(\boldsymbol{p}) e^{ipx} \right) \tag{16}$$

- a and b: quantum operators.
- Impose the canonical equal-time-commutation relations
  $\Rightarrow$ restitute the spinor indices

$$\begin{aligned}
[\psi_\alpha(t,\boldsymbol{x}),\Pi_\beta(t,\boldsymbol{y})] &= [\psi_\alpha(t,\boldsymbol{x}), i\psi^\dagger_\beta(t,\boldsymbol{y})] = i\delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \Rightarrow \\
[\psi_\alpha(t,\boldsymbol{x}),\psi^\dagger_\beta(t,\boldsymbol{y})] &= \delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\
[\psi_\alpha(t,\boldsymbol{x}),\psi_\beta(t,\boldsymbol{y})] &= 0 \\
[\Pi_\alpha(t,\boldsymbol{x}),\Pi_\beta(t,\boldsymbol{y})] &= [i\psi^\dagger_\alpha(t,\boldsymbol{x}), i\psi^\dagger_\beta(t,\boldsymbol{y})] = 0
\end{aligned} \tag{17}$$

$\Rightarrow$ Canonical harmonic oscillator commutation relations

$$\begin{aligned}
&[a^r_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{q}}] = (2\pi)^3\delta^{rs}\delta^3(\boldsymbol{p}-\boldsymbol{q}) \;\; ; \;\; [b^r_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}}] = (2\pi)^3\delta^{rs}\delta^3(\boldsymbol{p}-\boldsymbol{q}) \;\; ; \\
&[a^r_{\boldsymbol{p}}, a^s_{\boldsymbol{q}}] = [b^r_{\boldsymbol{p}}, b^s_{\boldsymbol{q}}] = [a^r_{\boldsymbol{p}}, b^s_{\boldsymbol{q}}] = [a^r_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}}] = 0
\end{aligned} \tag{18}$$

Hamiltonian as a function of a and b operators:

$$H = \int d^3x\, \psi^\dagger(x) i\partial_0 \psi(x) = \int \frac{d^3p}{(2\pi)^3} E_p \sum_{r=1,2} (a^{r\dagger}_{\boldsymbol{p}} a^r_{\boldsymbol{p}} - b^{r\dagger}_{\boldsymbol{p}} b^r_{\boldsymbol{p}}) \tag{19}$$

$\Rightarrow$ **the b-type particles count as negative energy!!**

$$[a^r_{\boldsymbol{k}}, H] = E_k a^r_{\boldsymbol{k}} \;\; ; \;\; [b^r_{\boldsymbol{k}}, H] = -E_k b^r_{\boldsymbol{k}} \;\; ; \tag{20}$$

$\Rightarrow$ $a^{r\dagger}_{\boldsymbol{p}}$-operator **adds** an energy E_k to the system

$\Rightarrow$ but the $b^{r\dagger}_{\boldsymbol{p}}$-operator **removes** an energy E_k from the system

- We could define another operator: $d^s_{\boldsymbol{p}} = b^{s\dagger}_{\boldsymbol{p}}$

$\Rightarrow$ correct commutation relations with H in (20),

$\Rightarrow$ but $[d, d^\dagger]$ commutators in (18) would have the wrong sign,

$\Rightarrow$ no harmonic-oscillator and rising-lowering states interpretation

$\Rightarrow$ Hamiltonian (19) (after applying normal-ordering), would anyway have a negative term.

- whatever definition one makes for the *b*-operators:
  - $\Rightarrow$ it ruins either commutation relations
  - $\Rightarrow$ leaves the Hamiltonian unbounded from below,
- **unless**, somewhat, one could define an operator such that

$$dd^\dagger = -d^\dagger d$$

and change the sign of the second term in the Hamiltonian (19).

Quantization: second attempt

- HINT: instead of the commutation relations (18) one could define **anti-commutation** relations:

$$\{A, B\} \equiv AB + BA$$

$$\begin{aligned}
\{\psi_\alpha(t,\boldsymbol{x}), \Pi_\beta(t,\boldsymbol{y})\} &= \{\psi_\alpha(t,\boldsymbol{x}), i\psi^\dagger_\beta(t,\boldsymbol{y})\} = i\delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \Rightarrow \\
\{\psi_\alpha(t,\boldsymbol{x}), \psi^\dagger_\beta(t,\boldsymbol{y})\} &= \delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\
\{\psi_\alpha(t,\boldsymbol{x}), \psi_\beta(t,\boldsymbol{y})\} &= 0 \\
\{\Pi_\alpha(t,\boldsymbol{x}), \Pi_\beta(t,\boldsymbol{y})\} &= \{i\psi^\dagger_\alpha(t,\boldsymbol{x}), i\psi^\dagger_\beta(t,\boldsymbol{y})\} = 0 \qquad (21)
\end{aligned}$$

which translate to:

$$\begin{aligned}
&\{a^r_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{q}}\} = (2\pi)^3\delta^{rs}\delta^3(\boldsymbol{p}-\boldsymbol{q}) \;\; ; \;\; \{b^r_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}}\} = (2\pi)^3\delta^{rs}\delta^3(\boldsymbol{p}-\boldsymbol{q}) \;\; ; \\
&\{a^r_{\boldsymbol{p}}, a^s_{\boldsymbol{q}}\} = \{b^r_{\boldsymbol{p}}, b^s_{\boldsymbol{q}}\} = \{a^r_{\boldsymbol{p}}, b^s_{\boldsymbol{q}}\} = \{a^r_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}}\} = 0 \qquad (22)
\end{aligned}$$

Define consistently Wick/normal ordering for spin 1/2 operators:

$$: a^r_{\boldsymbol{p}} a^{s\dagger}_{\boldsymbol{q}} : \equiv -a^{s\dagger}_{\boldsymbol{q}} a^r_{\boldsymbol{p}} \;\; ; \;\; : b^r_{\boldsymbol{p}} b^{s\dagger}_{\boldsymbol{q}} : \equiv -b^{s\dagger}_{\boldsymbol{q}} b^r_{\boldsymbol{p}} \;\; ; \qquad (23)$$

- Anti-commutation relations (22) are symmetric ($b \leftrightarrow b^\dagger$),
  $\Rightarrow$ A renaming does not ruin them.
- Under this renaming: the second term in the Hamiltonian becomes positive:

$$: H : \to - : b^r_{\boldsymbol{p}} b^{r\dagger}_{\boldsymbol{p}} := b^{r\dagger}_{\boldsymbol{p}} b^r_{\boldsymbol{p}}$$

In summary: define

$$\psi(x) = \sum_{s=1,2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} \left(a^s_{\boldsymbol{p}} u^s(\boldsymbol{p}) e^{-ipx} + b^{s\dagger}_{\boldsymbol{p}} v^s(\boldsymbol{p}) e^{ipx} \right) \tag{24}$$

with the equal-time-anti-commutation relations (21), which lead to (22) and the normal ordering (23), with a Hamiltonian:

$$H = \int \mathrm{d}^3 x \; : \psi^\dagger(x) i\partial_0 \psi(x) := \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p (a^{r\dagger}_{\boldsymbol{p}} a^r_{\boldsymbol{p}} + b^{r\dagger}_{\boldsymbol{p}} b^r_{\boldsymbol{p}}) \tag{25}$$

where a sum over repeated indices is understood.

Operators anti-commute

$$|1^r_{\boldsymbol{p}}, 1^s_{\boldsymbol{k}}\rangle = \sqrt{2E_p}\sqrt{2E_k} a^{r\dagger}_{\boldsymbol{p}} a^{s\dagger}_{\boldsymbol{k}} |0\rangle = -\sqrt{2E_p}\sqrt{2E_k} a^{s\dagger}_{\boldsymbol{k}} a^{r\dagger}_{\boldsymbol{p}} |0\rangle = -|1^s_{\boldsymbol{k}}, 1^r_{\boldsymbol{p}}\rangle$$

$\Rightarrow$ states are **anti-symmetric** under particle exchange,
$\Rightarrow$ they are **fermions**
$\Rightarrow$ there can be only one particle in a given state

Consistent with

$$\{a^{r\dagger}_{\boldsymbol{p}}, a^{r\dagger}_{\boldsymbol{p}}\} = 2 a^{r\dagger}_{\boldsymbol{p}} a^{r\dagger}_{\boldsymbol{p}} = 0$$

$\Rightarrow$ the would-be two particle state:

$$a^{r\dagger}_{\boldsymbol{p}} |1^r_{\boldsymbol{p}}\rangle = a^{r\dagger}_{\boldsymbol{p}} a^{r\dagger}_{\boldsymbol{p}} |0\rangle = 0$$

- using the anti-commutation relations (22)

$$\begin{aligned}
a^{r\dagger}_{\mathbf{k}} n^{as}_{\mathbf{p}} &= a^{r\dagger}_{\mathbf{k}} a^{s\dagger}_{\mathbf{p}} a^{s}_{\mathbf{p}} = -a^{s\dagger}_{\mathbf{p}} a^{r\dagger}_{\mathbf{k}} a^{s}_{\mathbf{p}} = -a^{s\dagger}_{\mathbf{p}}(-a^{s}_{\mathbf{p}} a^{r\dagger}_{\mathbf{k}} + \{a^{s}_{\mathbf{p}}, a^{r\dagger}_{\mathbf{k}}\}) \\
&= a^{s\dagger}_{\mathbf{p}} a^{s}_{\mathbf{p}} a^{r\dagger}_{\mathbf{k}} - a^{s\dagger}_{\mathbf{p}} (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{k})\delta^{rs} \\
a^{r\dagger}_{\mathbf{k}} n^{as}_{\mathbf{p}} - n^{as}_{\mathbf{p}} a^{r\dagger}_{\mathbf{k}} &= [a^{r\dagger}_{\mathbf{k}}, n^{as}_{\mathbf{p}}] = -a^{s\dagger}_{\mathbf{p}} (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{k})\delta^{rs}
\end{aligned}$$

⇒ the same relation as for the Klein-Gordon operators

⇒ equivalent expression is found for the b operators

$$\begin{aligned}
[a^{r\dagger}_{\mathbf{k}}, H] &= -E_k a^{r\dagger}_{\mathbf{k}} \;\; ; \;\; [a^{r}_{\mathbf{k}}, H] = E_k a^{r}_{\mathbf{k}} \\
[b^{r\dagger}_{\mathbf{k}}, H] &= -E_k b^{r\dagger}_{\mathbf{k}} \;\; ; \;\; [b^{r}_{\mathbf{k}}, H] = E_k b^{r}_{\mathbf{k}}
\end{aligned}$$

⇒ same relations as for the Klein-Gordon operators

⇒ $a^{r\dagger}_{\mathbf{k}}$ and $b^{r\dagger}_{\mathbf{k}}$ operators create particles with energy E_k

⇒ $a^{r}_{\mathbf{k}}$ and $b^{r}_{\mathbf{k}}$ operators remove particles with energy E_k.

We could define new operators:

$$c = a^\dagger \;\; ; \;\; d = b^\dagger$$

such that (schematically):

$$[c^\dagger, c^\dagger c] = [a, aa^\dagger] = -[a, a^\dagger a + \delta] = -[a, a^\dagger a] = -a\delta = -c^\dagger\delta$$

⇒ c, and d follow the same commutation relations as the usual a and b.

The Hamiltonian is:

$$H = -\int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p (c^{r\dagger}_{\mathbf{p}} c^{r}_{\mathbf{p}} + d^{r\dagger}_{\mathbf{p}} d^{r}_{\mathbf{p}})$$

⇒ We have negative energies.

- The *vacuum* of the c & d operators is:

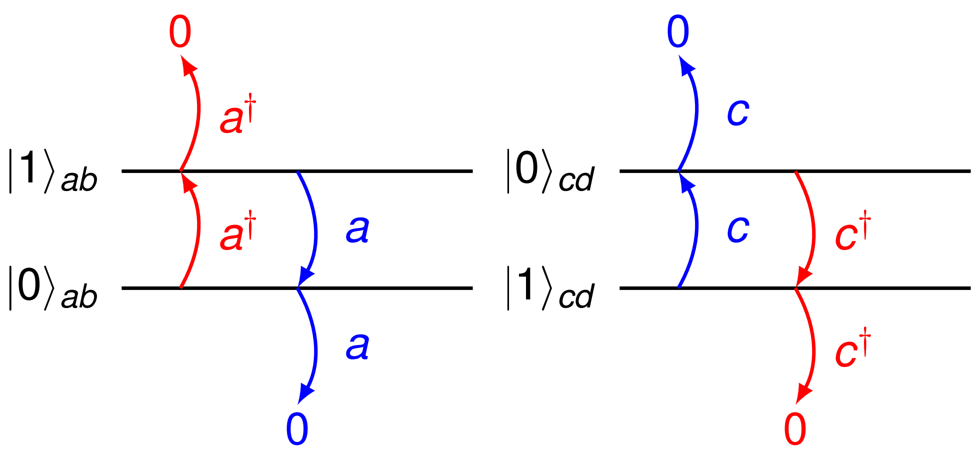
$$|0\rangle_{cd} = |\psi\rangle \text{ with all } a \text{ \& } b \text{ states filled}$$

such that

$$c^r_{\boldsymbol{p}}|0\rangle_{cd} = a^{r\dagger}_{\boldsymbol{p}}|0\rangle_{cd} = 0$$

since $|0\rangle_{cd}$ contains the factor $a^{r\dagger}_{\boldsymbol{p}}$.

- The d operator is our original b operator in eq. (16), and this is the reason why we could the change $b \leftrightarrow b^\dagger$.



Feynman propagator

Definition: fermion Feynman propagator

$$S_F(x-y) = \langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle \tag{26}$$

fermion indices have to be understood:

$$S_F(x-y)_{\alpha\beta} = \langle 0|T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}|0\rangle$$

Dirac's equation inhomogeneous Green's function, with the $+i\varepsilon$ prescription.

$$(i\partial\!\!\!/_x - m)S_F(x-y) = i\delta^4(x-y)$$

Fourier Transform: $S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \tilde{S}_F(p)$

$$\int \frac{d^4p}{(2\pi)^4} (\not{p} - m) e^{-ip(x-y)} \tilde{S}_F(p) = i\delta^4(x-y)$$

$$\Rightarrow (\not{p} - m)\tilde{S}_F(p) = i \Rightarrow \tilde{S}_F(p) = \frac{i}{\not{p} - m} = \frac{i(\not{p}+m)}{p^2 - m^2}$$

- we have used that $\not{p}\not{p} = p^2$
- Add the Feynman prescription

$$\boxed{\tilde{S}_F(p) = \frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon} = \frac{i}{\not{p} - m + i\varepsilon}} \tag{27}$$

and

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$$

- Another way of writing the fermion propagator is:

$$S_F(x-y) = (i\not{\partial}_x + m)\Delta_F(x-y)$$

$\Rightarrow$ fermion propagators cancel in the same space-time regions as the Klein-Gordon propagator.

- Numerator of (27) contains the completeness relation (12).

$$\sum_{s=1,2} u^s(\mathbf{p})\bar{u}^s(\mathbf{p}) = \not{p} + m$$

$\Rightarrow$ General feature of Green's functions

if several states $\varphi_\ell(p)$ have the same momentum p, the Green's function will be:

$$i\frac{\sum_\ell \varphi_\ell(p)\varphi_\ell^*(p)}{p^2 - m^2 + i\varepsilon}$$

If we compute the propagator (26) as anti-commutators of *a* and *b* operators,

- separating the *positive* and *negative* energy components,
- during the computation one encounters the following expressions:

$$
\begin{aligned}
\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle &= \{\psi^+_\alpha(x), \bar{\psi}^-_\beta(y)\} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)} \sum_r u^r_\alpha \bar{u}^r_\beta = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)} (\not{p}+m)_{\alpha\beta} \\
&= (i\not{\partial}_x + m)_{\alpha\beta} \Delta^+(x-y) \\
\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle &= \{\bar{\psi}^+_\beta(y), \psi^-_\alpha(x)\} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} e^{-ip(y-x)} \sum_r v^r_\alpha \bar{v}^r_\beta = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} e^{ip(x-y)} (\not{p}-m)_{\alpha\beta} \\
&= (i\not{\partial}_x + m)_{\alpha\beta} \Delta^-(x-y) = -(i\not{\partial}_x + m)_{\alpha\beta} \Delta^+(y-x)
\end{aligned}
$$

$\Rightarrow$ Explicit appeareance of the sum over all states

- With these expressions we can write the propagators corresponding to the Klein-Gordon $\Delta(x-y)$ and the retarded $\Delta_R(x-y)$ propagators.

Electric charge

The $U(1)$ charge current

$$j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$$

is conserved, and the (electric) charge is:

$$Q = \int \mathrm{d}^3x\, j^0 = \int \mathrm{d}^3x\, \psi^\dagger(x)\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_s (a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} - b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}})$$

so the *a*- and *b*-particles have opposite charge.