Condensed Matter —Assignment 2 (Quantum Gases)

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Weakly interacting Bose gas

We consider a gas with N bosons at zero temperature constrained within a cubic region of volume $V = L^3$.

Part (a)

The Hamiltonian is defined by:

$$H = \underbrace{\sum_{\boldsymbol{k}} \epsilon^0_k a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}}_{\text{kinetic term}} + \underbrace{\frac{g}{2}\int_V d^3x\, d^3y\, a^\dagger(\boldsymbol{x})\, a^\dagger(\boldsymbol{y})\, \delta^3(\boldsymbol{x}-\boldsymbol{y})\, a(\boldsymbol{y})\, a(\boldsymbol{x})}_{\text{“contact” interaction}}$$

where $a^\dagger_{\boldsymbol{k}}$ creates a particle with momentum $\boldsymbol{k}$ and kinetic energy $\epsilon^0_k = k^2/2m$, and $a^\dagger(\boldsymbol{x})$ creates a particle at position $\boldsymbol{x}$.

We substitute $a(\boldsymbol{x}) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} a_{\boldsymbol{k}}$ to obtain:

$$\begin{aligned}
H &= \sum_{\boldsymbol{k}} \epsilon^0_k a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{g}{2V^2}\sum_{\boldsymbol{pqrs}}\int_V d^3x\, d^3y\, \delta^3(\boldsymbol{x}-\boldsymbol{y})\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} e^{-i\boldsymbol{q}\cdot\boldsymbol{y}}\, e^{i\boldsymbol{r}\cdot\boldsymbol{y}} e^{i\boldsymbol{s}\cdot\boldsymbol{x}}\, a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{r}} a_{\boldsymbol{s}} \\
&= \sum_{\boldsymbol{k}} \epsilon^0_k a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{g}{2V^2}\sum_{\boldsymbol{pqrs}}\int_V d^3x\, e^{-i(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}}\, e^{i(\boldsymbol{r}+\boldsymbol{s})\cdot\boldsymbol{x}}\, a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{r}} a_{\boldsymbol{s}} \\
&= \sum_{\boldsymbol{k}} \epsilon^0_k a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{g}{2V^2}\sum_{\boldsymbol{mnij}}\int_0^L\int_0^L\int_0^L d^3x\, e^{-2\pi i(\boldsymbol{m}+\boldsymbol{n})\cdot\boldsymbol{x}/L}\, e^{2\pi i(\boldsymbol{i}+\boldsymbol{j})\cdot\boldsymbol{x}/L}\, a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{r}} a_{\boldsymbol{s}} \\
&= \sum_{\boldsymbol{k}} \epsilon^0_k a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{g}{2V^2}\sum_{\boldsymbol{mnij}} V\delta_{\boldsymbol{m}+\boldsymbol{n},\boldsymbol{i}+\boldsymbol{j}}\, a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{r}} a_{\boldsymbol{s}} \\
&= \underbrace{\sum_{\boldsymbol{k}} \epsilon^0_k a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}}_{\text{kinetic term}} + \underbrace{\frac{g}{2V}\sum_{\boldsymbol{kk'q}} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}\, a^\dagger_{-\boldsymbol{k}+\boldsymbol{q}/2}\, a_{\boldsymbol{k'}+\boldsymbol{q}/2}\, a_{-\boldsymbol{k'}+\boldsymbol{q}/2}}_{\text{interaction term}}
\end{aligned}$$

where at the third equality we used that momentum is quantized according to $\boldsymbol{k} = 2\pi\boldsymbol{n}/L$, where $\boldsymbol{n}$ is a vector of integers, at the fourth equality we used the formula for the inner product of two L-periodic Fourier basis vectors, and at the last equality we have used $\boldsymbol{i}+\boldsymbol{j} = \boldsymbol{m}+\boldsymbol{n} \Rightarrow \boldsymbol{p}+\boldsymbol{q} = \boldsymbol{r}+\boldsymbol{s}$ to define:

$$\boldsymbol{q} \equiv \boldsymbol{p}+\boldsymbol{q} = \boldsymbol{r}+\boldsymbol{s} \qquad \boldsymbol{k} \equiv \frac{\boldsymbol{p}-\boldsymbol{q}}{2} \qquad \boldsymbol{k'} \equiv \frac{\boldsymbol{r}-\boldsymbol{s}}{2}$$

Examining the second term, we notice that it is an interaction between two excitations where total momentum $\boldsymbol{q}$ is conserved, since:

$$\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) + \left(-\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) - \left(\boldsymbol{k'}+\frac{\boldsymbol{q}}{2}\right) - \left(-\boldsymbol{k'}+\frac{\boldsymbol{q}}{2}\right) = 0$$

There is an exchange of momentum $\boldsymbol{k}-\boldsymbol{k'}$ between the excitations.

Note that in this form the Hamiltonian is not diagonal, and so we cannot infer properties of the ground state of the whole gas.

Part (b)

We now expand the sum in the interaction term, writing:

$$
\begin{aligned}
&\sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}\, a^\dagger_{-\boldsymbol{k}+\boldsymbol{q}/2}\, a_{\boldsymbol{k}'+\boldsymbol{q}/2}\, a_{-\boldsymbol{k}'+\boldsymbol{q}/2} \\
&\quad = \sum_{\boldsymbol{k}=0,\boldsymbol{k}'=0,\boldsymbol{q}=0} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}\, a^\dagger_{-\boldsymbol{k}+\boldsymbol{q}/2}\, a_{\boldsymbol{k}'+\boldsymbol{q}/2}\, a_{-\boldsymbol{k}'+\boldsymbol{q}/2} \\
&\qquad + \sum_{\boldsymbol{k}\neq 0,\boldsymbol{k}'=0,\boldsymbol{q}=0} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}\, a^\dagger_{-\boldsymbol{k}+\boldsymbol{q}/2}\, a_{\boldsymbol{k}'+\boldsymbol{q}/2}\, a_{-\boldsymbol{k}'+\boldsymbol{q}/2} \\
&\qquad + \sum_{\boldsymbol{k}=0,\boldsymbol{k}'\neq 0,\boldsymbol{q}=0} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}\, a^\dagger_{-\boldsymbol{k}+\boldsymbol{q}/2}\, a_{\boldsymbol{k}'+\boldsymbol{q}/2}\, a_{-\boldsymbol{k}'+\boldsymbol{q}/2} \\
&\qquad + \sum_{\boldsymbol{k}=\boldsymbol{q}/2,\boldsymbol{k}'=\boldsymbol{q}/2,\boldsymbol{q}\neq 0} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}\, a^\dagger_{-\boldsymbol{k}+\boldsymbol{q}/2}\, a_{\boldsymbol{k}'+\boldsymbol{q}/2}\, a_{-\boldsymbol{k}'+\boldsymbol{q}/2} \\
&\qquad + \sum_{\boldsymbol{k}=-\boldsymbol{q}/2,\boldsymbol{k}'=-\boldsymbol{q}/2,\boldsymbol{q}\neq 0} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}\, a^\dagger_{-\boldsymbol{k}+\boldsymbol{q}/2}\, a_{\boldsymbol{k}'+\boldsymbol{q}/2}\, a_{-\boldsymbol{k}'+\boldsymbol{q}/2} \\
&\qquad + \sum_{\boldsymbol{k}=-\boldsymbol{q}/2,\boldsymbol{k}'=\boldsymbol{q}/2,\boldsymbol{q}\neq 0} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}\, a^\dagger_{-\boldsymbol{k}+\boldsymbol{q}/2}\, a_{\boldsymbol{k}'+\boldsymbol{q}/2}\, a_{-\boldsymbol{k}'+\boldsymbol{q}/2} \\
&\qquad + \sum_{\boldsymbol{k}=\boldsymbol{q}/2,\boldsymbol{k}'=-\boldsymbol{q}/2,\boldsymbol{q}\neq 0} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}\, a^\dagger_{-\boldsymbol{k}+\boldsymbol{q}/2}\, a_{\boldsymbol{k}'+\boldsymbol{q}/2}\, a_{-\boldsymbol{k}'+\boldsymbol{q}/2} \\
&\qquad + \ldots
\end{aligned}
$$

where the neglected terms are neither linear nor quadratic in $a_0^\dagger a_0$. This approximation is justified when there are few excitations, and so most interactions involve particles belonging to the condensate.

We may simplify this expansion as follows:

$$
\begin{aligned}
&\sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} a^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}\, a^\dagger_{-\boldsymbol{k}+\boldsymbol{q}/2}\, a_{\boldsymbol{k}'+\boldsymbol{q}/2}\, a_{-\boldsymbol{k}'+\boldsymbol{q}/2} \\
&\quad = a_0^\dagger\, a_0^\dagger\, a_0\, a_0 + \sum_{\boldsymbol{k}\neq 0} a^\dagger_{\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}}\, a_0\, a_0 + \sum_{\boldsymbol{k}'\neq 0} a_0^\dagger\, a_0^\dagger\, a_{\boldsymbol{k}'}\, a_{-\boldsymbol{k}'} \\
&\qquad\quad + \sum_{\boldsymbol{q}\neq 0} a^\dagger_{\boldsymbol{q}}\, a_0^\dagger\, a_{\boldsymbol{q}}\, a_0 + \sum_{\boldsymbol{q}\neq 0} a_0^\dagger\, a^\dagger_{\boldsymbol{q}}\, a_0\, a_{\boldsymbol{q}} + \sum_{\boldsymbol{q}\neq 0} a_0^\dagger\, a^\dagger_{\boldsymbol{q}}\, a_{\boldsymbol{q}}\, a_0 + \sum_{\boldsymbol{q}\neq 0} a^\dagger_{\boldsymbol{q}}\, a_0^\dagger\, a_0\, a_{\boldsymbol{q}} \\
&\quad = N_0^2 + N_0 \sum_{\boldsymbol{k}\neq 0} a^\dagger_{\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}} + N_0 \sum_{\boldsymbol{k}'\neq 0} a_{\boldsymbol{k}'}\, a_{-\boldsymbol{k}'} + 4N_0 \sum_{\boldsymbol{q}\neq 0} a^\dagger_{\boldsymbol{q}}\, a_{\boldsymbol{q}} \\
&\quad = N_0^2 + N_0 \sum_{\boldsymbol{k}\neq 0} \left(a^\dagger_{\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}} + a_{\boldsymbol{k}}\, a_{-\boldsymbol{k}} + 4 a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}} \right)
\end{aligned}
$$

where we defined $N_0 \equiv \langle s|\, a_0^\dagger a_0\, |s\rangle$, the number of particles in the condensate—that is, the occupation number of the single-particle ground state. Now:

$$
\begin{aligned}
H &\simeq \sum_{\boldsymbol{k}} \epsilon_k^0 a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{g}{2V} N_0^2 + \frac{g}{2V} N_0 \sum_{\boldsymbol{k}\neq 0} \left(4 a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}} + a_{\boldsymbol{k}}\, a_{-\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}} \right) \\
&= \sum_{\boldsymbol{k}} \epsilon_k^0 a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{g N_0^2}{2V} + \frac{g N_0}{V} \sum_{\boldsymbol{k}\neq 0} \left[a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}\, a_{-\boldsymbol{k}} + \frac{1}{2} \left(a_{-\boldsymbol{k}}\, a_{\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}} \right) \right] \\
&= \underbrace{\sum_{\boldsymbol{k}} \epsilon_k^0 a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}}_{\text{kinetic energy}} + \underbrace{\frac{g N_0^2}{2V}}_{\text{interaction energy of ground-state particles}} + \underbrace{\frac{g N_0}{V} \sum_{\boldsymbol{k}\neq 0} \left(a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}\, a_{-\boldsymbol{k}} \right)}_{\text{interaction energy of excitation with ground state}} + \underbrace{\frac{g N_0}{2V} \sum_{\boldsymbol{k}\neq 0} \left(a_{-\boldsymbol{k}}\, a_{\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}} \right)}_{\text{pair production/annihilation}}
\end{aligned}
$$

This first summed term arises from the potential, and is proportional to $N_0(N - N_0)$, that is, it represents the interaction energy between $N - N_0$ excited particles and the N_0 particles in the condensate. The second summed term is a true interaction, which allows pair production or annihilation of excitations with opposite momenta—corresponding to pairs of particles scattering into or out of the condensate.

The Hamiltonian is still not in a diagonal form.

Part (c)

We now make use of the number operator $N = N_0 + \sum_{\boldsymbol{k}\neq 0} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}$ and discard higher-order terms in a and $a^\dagger$:

$$
\begin{aligned}
H = &\sum_{\boldsymbol{k}} \epsilon^0_k a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{g}{2V}\Big(N - \sum_{\boldsymbol{p}\neq 0} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}\Big)\Big(N - \sum_{\boldsymbol{q}\neq 0} a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{q}}\Big) \\
&\quad + \frac{g}{V}\Big(N - \sum_{\boldsymbol{p}\neq 0} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}\Big) \sum_{\boldsymbol{k}\neq 0}\Big[a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}\, a_{-\boldsymbol{k}} + \frac{1}{2}\Big(a_{-\boldsymbol{k}}\, a_{\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}}\Big)\Big] \\
\simeq &\sum_{\boldsymbol{k}} \epsilon^0_k a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{g}{2V}\Big(N^2 - 2N\sum_{\boldsymbol{k}\neq 0} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}\Big) + \frac{g}{V} N \sum_{\boldsymbol{k}\neq 0}\Big[a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}} + a^\dagger_{-\boldsymbol{k}}\, a_{-\boldsymbol{k}} + \frac{1}{2}\Big(a_{-\boldsymbol{k}}\, a_{\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}}\Big)\Big] \\
= &\frac{g}{2V}N^2 + \sum_{\boldsymbol{k}\neq 0}\Big(\epsilon^0_k - \frac{g}{2V}2N + 2\frac{g}{V}N\Big) a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{g}{2V}N\sum_{\boldsymbol{k}\neq 0}\Big(a_{-\boldsymbol{k}}\, a_{\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}}\Big) \\
= &\frac{gnN}{2} + \sum_{\boldsymbol{k}\neq 0}\Big[(\epsilon^0_k + gn) a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{gn}{2}\Big(a_{-\boldsymbol{k}}\, a_{\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}}\, a^\dagger_{-\boldsymbol{k}}\Big)\Big]
\end{aligned}
$$

where $n \equiv N/V$ is the density of particles, and we have entered the term with ϵ^0_k inside the sum over $\boldsymbol{k} \neq 0$ since $\epsilon^0_0 = 0$.

The first term inside the sum aggregates the kinetic energy of an excitation with the energy of its interactions with particles in the ground state.

Part (d)

The Bogoliubov transformation is defined by:

$$
\begin{bmatrix} a_{\boldsymbol{k}} \\ a^\dagger_{-\boldsymbol{k}} \end{bmatrix} = \begin{bmatrix} u_k & -v_k \\ -v_k & u_k \end{bmatrix} \begin{bmatrix} \alpha_{\boldsymbol{k}} \\ \alpha^\dagger_{-\boldsymbol{k}} \end{bmatrix}
$$

where $u_k^2 - v_k^2 = 1$ and u_k, v_k depend only on the magnitude of the vector k.

We may easily invert this relationship, to obtain:

$$
\begin{bmatrix} \alpha_{\boldsymbol{k}} \\ \alpha^\dagger_{-\boldsymbol{k}} \end{bmatrix} = \begin{bmatrix} u_k & v_k \\ v_k & u_k \end{bmatrix} \begin{bmatrix} a_{\boldsymbol{k}} \\ a^\dagger_{-\boldsymbol{k}} \end{bmatrix}
$$

Or simply:

$$
\alpha_{\boldsymbol{k}} = u_k a_{\boldsymbol{k}} + v_k a^\dagger_{-\boldsymbol{k}} \qquad\qquad \alpha^\dagger_{\boldsymbol{k}} = v_k a_{-\boldsymbol{k}} + u_k a^\dagger_{\boldsymbol{k}}
$$

We now calculate the commutator:

$$
\begin{aligned}
[\alpha_{\boldsymbol{k}}, \alpha^\dagger_{\boldsymbol{k}}] &= [u_k a_{\boldsymbol{k}} + v_k a^\dagger_{-\boldsymbol{k}}, v_k a_{-\boldsymbol{k}} + u_k a^\dagger_{\boldsymbol{k}}] \\
&= [u_k a_{\boldsymbol{k}}, v_k a_{-\boldsymbol{k}}] + [u_k a_{\boldsymbol{k}}, u_k a^\dagger_{\boldsymbol{k}}] + [v_k a^\dagger_{-\boldsymbol{k}}, v_k a_{-\boldsymbol{k}}] + [v_k a^\dagger_{-\boldsymbol{k}}, u_k a^\dagger_{\boldsymbol{k}}] \\
&= u_k v_k [a_{\boldsymbol{k}}, a_{-\boldsymbol{k}}] + u_k u_k [a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}}] + v_k v_k [a^\dagger_{-\boldsymbol{k}}, a_{-\boldsymbol{k}}] + v_k u_k [a^\dagger_{-\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}}] \\
&= u_k u_k - v_k v_k \\
&= 1
\end{aligned}
$$

where at the second-last equality we used the commutation relation $[a^\dagger_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = \delta_{\boldsymbol{p},\boldsymbol{q}}$.

The remaining commutation relations follow similarly, and $\alpha_{\boldsymbol{k}}, \alpha^\dagger_{\boldsymbol{k}}$ satisfy the usual bosonic algebra.

We observe that creation of a pseudo-particle by the operator $\alpha^\dagger_{\boldsymbol{k}}$ corresponds to a linear combination of destruction a particle with momentum $-\boldsymbol{k}$ with creation of a particle of momentum $\boldsymbol{k}$.

Part (e)

The Hamiltonian may now finally be expressed in the diagonal form:

$$H = \frac{gnN}{2} - \frac{1}{2}\sum_{k\neq 0}\left(\epsilon_k^0 + gn - \varepsilon(k)\right) + \sum_{\boldsymbol{k}\neq 0}\varepsilon(k)\,\alpha_{\boldsymbol{k}}^{\dagger}\alpha_{\boldsymbol{k}}$$

The first two terms represent the energy of the ground state of the whole system. Notice that there is a contribution from every momentum mode $\boldsymbol{k}$, and so the ground state is a state containing $a_{\boldsymbol{k}}^{\dagger}$ excitations with every allowed momentum—the ground state of the gas is completely distinct from the state where every particle is in the single-particle ground state!

The third term has the form of a kinetic energy associated with an $\alpha_{\boldsymbol{k}}^{\dagger}$ excitation from the ground state of the gas. Thus, the Bogoliubov dispersion relation:

$$\varepsilon(k) = \sqrt{\left(\epsilon_k^0\right)^2 + 2gn\epsilon_k^0}$$

gives the energy associated with an $\alpha_{\boldsymbol{k}}^{\dagger}$ excitation of momentum $\boldsymbol{k}$.

Part (f)

We now consider the number operator:

$$\begin{aligned}
N &= \sum_{\boldsymbol{k}} a_{\boldsymbol{k}}^{\dagger}a_{\boldsymbol{k}} = N_0 + \sum_{\boldsymbol{k}\neq 0} a_{\boldsymbol{k}}^{\dagger}a_{\boldsymbol{k}} \\
&= N_0 + \sum_{\boldsymbol{k}\neq 0}\left(-v_k\,\alpha_{-\boldsymbol{k}} + u_k\,\alpha_{\boldsymbol{k}}^{\dagger}\right)\left(u_k\,\alpha_{\boldsymbol{k}} - v_k\,\alpha_{-\boldsymbol{k}}^{\dagger}\right) \\
&= N_0 + \sum_{\boldsymbol{k}\neq 0}\left(-u_kv_k\,\alpha_{-\boldsymbol{k}}\alpha_{\boldsymbol{k}} + v_k^2\,\alpha_{-\boldsymbol{k}}\alpha_{-\boldsymbol{k}}^{\dagger} + u_k^2\,\alpha_{\boldsymbol{k}}^{\dagger}\alpha_{\boldsymbol{k}} - u_kv_k\,\alpha_{\boldsymbol{k}}^{\dagger}\alpha_{-\boldsymbol{k}}^{\dagger}\right) \\
&= N_0 + \sum_{\boldsymbol{k}\neq 0}\left(v_k^2 - u_kv_k\,\alpha_{-\boldsymbol{k}}\alpha_{\boldsymbol{k}} + v_k^2\,\alpha_{-\boldsymbol{k}}^{\dagger}\alpha_{-\boldsymbol{k}} + u_k^2\,\alpha_{\boldsymbol{k}}^{\dagger}\alpha_{\boldsymbol{k}} - u_kv_k\,\alpha_{\boldsymbol{k}}^{\dagger}\alpha_{-\boldsymbol{k}}^{\dagger}\right) \\
&= N_0 + \sum_{\boldsymbol{k}\neq 0}\left[v_k^2 + \left(v_k^2 + u_k^2\right)\alpha_{\boldsymbol{k}}^{\dagger}\alpha_{\boldsymbol{k}} - u_kv_k\left(\alpha_{-\boldsymbol{k}}\alpha_{\boldsymbol{k}} + \alpha_{\boldsymbol{k}}^{\dagger}\alpha_{-\boldsymbol{k}}^{\dagger}\right)\right]
\end{aligned}$$

Notice that N is not diagonal in the basis where H is diagonal.

In an eigenstate of the Hamiltonian, the last term in the sum does not contribute, and we're left with:

$$N = N_0 + \sum_{k\neq 0}\left[v_k^2 + \left(v_k^2 + u_k^2\right)\alpha_{\boldsymbol{k}}^{\dagger}\alpha_{\boldsymbol{k}}\right]$$

The first term is the number of particles in the single-particle ground state. The second term inside the sum over $\boldsymbol{k}$ is proportional to the number of excitations with momentum $\boldsymbol{k}$ above the ground state of the gas. The first term inside the sum is proportional to the number of excitations with momentum $\boldsymbol{k}$ when the gas is in its ground state.

In the ground state of the gas, the condensate is depleted by $N - N_0$ particles:

$$
\begin{aligned}
N - N_0 = \sum_{\boldsymbol{k}\neq 0} v_k^2 \approx \frac{V}{(2\pi\hbar)^3} \int_{\text{all space}} d^3k v_k^2 \\
&= \frac{V}{(2\pi\hbar)^3} \int_0^\infty 4\pi k^2 dk v_k^2 \\
&= \frac{V}{2\pi^2\hbar^3} \int_0^\infty dk k^2 \frac{1}{2} \left(\frac{\epsilon_k^0 + gn}{\sqrt{\left(\epsilon_k^0\right)^2 + 2gn\epsilon_k^0}} - 1 \right) \\
&= \frac{V}{4\pi^2\hbar^3} m\sqrt{2m} \int_0^\infty d\epsilon \sqrt{\epsilon} \left(\frac{\epsilon + gn}{\sqrt{\epsilon(\epsilon + 2gn)}} - 1 \right) \\
&= \frac{V}{\pi^2\hbar^3} \left(\frac{mgn}{2} \right)^{\frac{3}{2}} \underbrace{\int_0^\infty dx \sqrt{x} \left(\frac{x+1}{\sqrt{x(x+2)}} - 1 \right)}_{= \frac{2^{3/2}}{3}} \\
&= \frac{V}{3\pi^2\hbar^3} \left(\frac{m 4\pi\hbar^2 an}{m} \right)^{\frac{3}{2}} \\
&= \frac{8}{3\sqrt{\pi}} N \sqrt{na^3}
\end{aligned}
$$

where at the approximate equality we have introduced the sum-integral correspondence in reciprocal space given by $\sum_{\boldsymbol{k}} \approx V/(2\pi\hbar)^3 \int d^3k$, at the second equality we have changed to spherical coordinates due to the symmetric dependence of v_k on $\boldsymbol{k}$, at the third equality we have explicitly introduced said dependence, at the fourth equality we have defined the change of variables given by $\epsilon = \epsilon_k^0 = k^2/2m$, $d\epsilon = kdk/m$, at the fifth equality we have performed yet again another change of variables towards $x = \epsilon/gn$, $dx = d\epsilon/gn$, at the sixth equality we have introduced the strength of the interaction $g = 4\pi\hbar^2 a/m$, with a being the s-wave scattering length, and in the final equality we have used $N = nV$.