Solution of Probs. 6.2 and 6.3 of Blundell

Uniaxial ferromagnet

Consider a spin system where each site contains a spin S particle. The corresponding Hamiltonian is

$$\hat{H} = -\sum_{i,j} \left[J_{i,j} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j + K_{i,j} \hat{S}_i^z \hat{S}_j^z \right]. \tag{0.1}$$

(a) Rewriting the Hamiltonian as

$$\hat{H} = -\sum_{i,j} \left[(J_{i,j} + K_{i,j}) \hat{S}_i^z \hat{S}_j^z + \frac{J_{i,j}}{2} (\hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+) \right], \tag{0.2}$$

it is clear that the state where spins are aligned fully-up is an eigenstate (because the off-diagonal elements contain $\hat{S}^+$ operators, which give zero on such a state).

(b) At low enough temperature (or equivalently for large enough spin), the so-called *Holstein-Primakoff relations* give $S_i^+ \approx \sqrt{2S} a_i$, $S_i^- \approx \sqrt{2S} a_i^\dagger$ and $S_i^z = S - a_i^\dagger a_i$, where $a_i^\dagger$ is the operator creating a (bosonic) spin excitation at position $\mathbf{r}_i$ (for a detailed derivation of these relations, see Chap. 17.4 in the book by Grosso and Pastori Parravicini). With the aid of such transformation, one finds

$$\hat{H} = -\sum_{i,j} \left[(J_{i,j} + K_{i,j})(S - \hat{a}_i^\dagger a_i)(S - \hat{a}_j^\dagger a_j) + 2S J_{i,j} (a_i^\dagger a_j + a_i a_j^\dagger) \right]. \tag{0.3}$$

Neglecting quartic terms (which is again allowed at low enough temperature) one finds

$$\hat{H} \approx -\sum_{i,j} \left[(J_{i,j} + K_{i,j})(S^2 - S\hat{a}_i^\dagger a_i - S\hat{a}_j^\dagger a_j) + 2S J_{i,j} (a_i^\dagger a_j + a_i a_j^\dagger) \right] \tag{0.4}$$

$$= E_0 + 2S \sum_i \left[(\hat{a}_i^\dagger a_i) \sum_j (J_{i,j} + K_{i,j}) - \sum_j J_{i,j} (a_i^\dagger a_j + a_i a_j^\dagger) \right] \tag{0.5}$$

$$= E_0 + 2S \sum_{\mathbf{q}} [J(\mathbf{0}) + K(\mathbf{0}) - J(\mathbf{q})] \hat{a}_{\mathbf{q}}^\dagger a_{\mathbf{q}}, \tag{0.6}$$

where we have introduced the Fourier transforms $J(\mathbf{q}) = \sum_{\mathbf{q}} J_{i,j}\, e^{i\mathbf{q}\cdot(\mathbf{r}_i - \mathbf{r}_j)}$. This shows that the spin-wave dispersion is $\hbar\omega(\mathbf{q}) = 2S[J(\mathbf{0}) + K(\mathbf{0}) - J(\mathbf{q})]$.

(c) Let us now restrict the couplings J and K to nearest-neighbors J_0 and K_0.

1. In a 1D chain $J(\mathbf{q}) = J_0(e^{iqa} + e^{-iqa}) = 2J_0 \cos qa$, so that $\hbar\omega(\mathbf{q}) = 4S[J_0(1 - \cos qa) + K_0]$.
2. In a 2D square lattice $J(\mathbf{q}) = 2J_0(\cos q_x a + \cos q_y a)$, so that $\hbar\omega(\mathbf{q}) = 8S \left[J_0 \left(1 - \frac{\cos q_x a + \cos q_y a}{2} \right) + K_0 \right]$.
3. In a 3D bcc crystal $J(\mathbf{q}) = J_0 \sum_{\pm} e^{i(\pm q_x \pm q_y \pm q_z)a/2} = 8J_0 \cos\frac{q_x a}{2} \cos\frac{q_y a}{2} \cos\frac{q_z a}{2}$, so that $\hbar\omega(\mathbf{q}) = 16S \left[J_0 \left(1 - \cos\frac{q_x a}{2} \cos\frac{q_y a}{2} \cos\frac{q_z a}{2} \right) + K_0 \right]$.

(d) At small momenta, these expressions become

1. $\hbar\omega(\mathbf{q}) = 2S[J_0 q^2 a^2 + 2K_0]$.
2. $\hbar\omega(\mathbf{q}) = 2S[J_0 q^2 a^2 + 4K_0]$.
3. $\hbar\omega(\mathbf{q}) = 2S[J_0 q^2 a^2 + 8K_0]$.

(e) In the Ising case ($J_0 = 0$) the excitation energy is independent of $\mathbf{q}$: $\hbar\omega = zSK_0$ (where z is the number of nearest neighbors: $z = 2, 4, 8$ in the above cases). The number of spin waves is proportional to $e^{-\hbar\omega/k_BT}$, their energy to $\hbar\omega e^{-\hbar\omega/k_BT}$, and the heat capacity per spin is

$$C = \frac{1}{N}\frac{\partial E}{\partial T} = \frac{(\hbar\omega)^2}{k_BT^2}e^{-\hbar\omega/k_BT}, \tag{0.7}$$

in agreement with the high-temperature limit of the result found in Prob. 6.1 (with J there equal to $\hbar\omega$ here).

(f) In the Heisenberg case ($K_0 = 0$) the excitation energy is $\hbar\omega_q = Dq^2$, where $D = 2SJ_0a^2$ is the *spin stiffness*. The number of spin waves is

$$N_s = \int \frac{\mathrm{d}^d q}{e^{Dq^2/k_BT} - 1}. \tag{0.8}$$

This integral converges in 3 and higher spatial dimensions, but it diverges (in the infrared, i.e., for small q) in 1d and 2d. This shows that the isotropic Heisenberg model doesn't feature long-range order in 1d and 2d, due to the proliferation of long wavelength excitations.