Universitat de Barcelona
Facultat de Física

Advanced Quantum Mechanics
Year 2022-23
Exam

Instructions

Write **your name** or **NIUB** in all pages, preferably at the top end.

You have **3 hours** to finish the exam.

Answer **ALL** questions in **Section A**.

Answer **TWO** of THREE questions only in **Section B**. If you attempt more than TWO questions, the best TWO solutions will be taken into account.

The mark carried by an individual part of a question is indicated in square brackets **[ ]**.

Section A - answer all questions

A1. For a certain type of atom, the observable $\hat{A}$ has eigenvalues ± 1, with corresponding eigenfunctions $|a_+\rangle$ and $|a_-\rangle$. Another observable $\hat{B}$ also has eigenvalues of ± 1, but corresponding eigenfunctions $|b_+\rangle$ and $|b_-\rangle$. The eigenfunctions are related by

$$|b_+\rangle = \frac{|a_+\rangle + |a_-\rangle}{\sqrt{2}}, \qquad |b_-\rangle = \frac{|a_+\rangle - |a_-\rangle}{\sqrt{2}}.$$

(a) Show that $\hat{C} = \hat{A} + \hat{B}$ is an observable.

[3 marks]

(b) Find the possible results of a measurement of $\hat{C}$.

[3 marks]

(c) Find the probability of obtaining each result when a measurement of $\hat{C}$ is performed on an atom in the state $|a_+\rangle$.

[2 marks]

(d) Find the corresponding state of the atom immediately after the measurement in terms of $|a_+\rangle$ and $|a_-\rangle$.

[2 marks]

A2. Consider the overlap between a coherent state $|z\rangle$ and another coherent state $|z'\rangle$, $\langle z|z'\rangle$.

(a) Starting from the Fock expansion, prove that the overlap can be expressed as

$$\langle z|z'\rangle = e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|z'|^2 + z^* z'}$$

[4 marks]

(b) Assume $z = \sqrt{2}x_0$ and $z' = i\sqrt{2}p_0$, with x_0 and p_0 real constants. Find the modulus of $\langle z|z'\rangle$. Can this modulus be zero?

[3 marks]

(c) What is the phase of $\langle z|z'\rangle$? What condition must x_0 and p_0 satisfy for $\langle z|z'\rangle$ to be purely imaginary?

[3 marks]

A3. (a) The angular momentum operator is defined as $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$. What is the result of the commutator $\left[\hat{L}_x, \hat{L}_y\right]$? Give the full mathematical derivation for maximum punctuation.

[7 marks]

(b) Describe one physical consequence of this result.

[3 marks]

A4. The non-relativistic kinetic energy operator is $\hat{K} = \frac{\hat{p}^2}{2m}$. The lowest order relativistic correction to this operator is $\delta\hat{K} = -\frac{1}{8}\frac{\hat{p}^4}{m^3c^2}$.

(a) Express $\delta\hat{K}$ in terms of powers of the lowering, $\hat{a}$, and raising, $\hat{a}^\dagger$, operators of the harmonic oscillator, using any relevant physical constants.

[4 marks]

(b) Find the first-order correction to the ground-state energy of a 1D harmonic oscillator due to the effect of $\delta\hat{K}$. Express your results in standard $\hbar\omega$ units.

[6 marks]

A5. Consider the elastic scattering of a projectile interacting with a target through the potential $V(r) = V_0\frac{e^{-\frac{r}{R}}}{r^2}$.

(a) Find the form factor $V(q)$ for this potential, where q is the momentum transfer.

[4 marks]

(b) Compute the corresponding Born approximation expression for the differential cross section.

[4 marks]

(c) How does this expression simplify when $q \ll R^{-1}$?

[2 marks]

You may use the integral result $\int_0^\infty dx \frac{e^{-ax}\sin(bx)}{x} = \arctan\frac{b}{a}$.

Section B - answer TWO of THREE questions

B1. Consider a bosonic annihilation operator $\hat{a}$, and the corresponding creation operator $\hat{a}^\dagger$.

(a) Show that the Boguilubov transformation

$$\hat{b} = u\hat{a} + v\hat{a}^\dagger \,,$$
$$\hat{b}^\dagger = u\hat{a}^\dagger + v\hat{a} \,.$$

preserves the canonical commutation relations if u and v are real and $u^2 - v^2 = 1$.

[6 marks]

(b) Show that the transformed Hamiltonian $\hat{H} = \bar{\omega}\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)$ is equivalent to the untransformed Hamiltonian,

$$\hat{H} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \frac{\Delta}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right) .$$

[6 marks]

(c) Find $\bar{\omega}$, u and v as a function of ω and Δ.

[6 marks]

(d) Show that the following commutator relation holds:

$$\left[\hat{a}, e^{-z\hat{a}^\dagger\hat{a}^\dagger}\right] = -2z\hat{a}^\dagger e^{-z\hat{a}^\dagger\hat{a}^\dagger} .$$

[5 marks]

(e) $|\bar{0}\rangle$ is the vacuum for the b operators, i.e. $\hat{b}\,|\bar{0}\rangle = 0$. Show that $|\bar{0}\rangle$ can be written as a coherent condensate of paired bosons on this vacuum,

$$|\bar{0}\rangle = e^{-z\hat{a}^\dagger\hat{a}^\dagger}\,|0\rangle \,,$$

by finding the value of z that guarantees that b annihilates the vacuum $|\bar{0}\rangle$.

[2 marks]

You may use Hadamard's lemma

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \frac{1}{3!}\left[\hat{A}, \left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right]\right] + \cdots$$

for $\hat{A}$ and $\hat{B}$ two arbitrary operators.

B2. Consider a one dimensional harmonic oscillator with angular frequency ω.

(a) The oscillator is acted upon by a force with the following time profile:

$$\hat{F}(t) = \frac{F_0 \tau^2}{t^2 + \tau^2}\,.$$

What is the corresponding perturbing hamiltonian acting on the oscillator?

[2 marks]

(b) At $t = -\infty$, the oscillator is known to be in the ground state. Using time-dependent perturbation theory at first order, calculate the probability that the oscillator is in the first excited state at $t = \infty$.

[10 marks]

(c) If the oscillator was originally on a different state $|n\rangle$, what states could it transition into? Find the corresponding transition probabilities.

[8 marks]

(d) Explain physically the dependence of the previous results on the parameter τ.

[5 marks]

You may use the integral $\int_{-\infty}^{\infty} dx \frac{e^{i\lambda x}}{x^2+a^2} = \frac{\pi}{a} e^{-|\lambda| a}$

B3. Consider a localized potential in one-dimension subject to a beam of particles incident from the left. Outside the region of the potential, we know that the wavefunction of the particles is a plane wave, $e^{\pm ikx}$ of wavevector $k = \sqrt{2mE}$, so that asymptotically we have $\psi = e^{ikx} + re^{-ikx}$ at $x \to -\infty$ and $\psi = te^{ikx}$ at $x \to \infty$.

(a) Prove, by conservation of the probability current or else, that $|t|^2 + |r|^2 = 1$.

[8 marks]

(b) Consider the scattering matrix S for a system like this. Knowing that S is unitary, prove that its eigenvalues, $s_{\pm}$, have unit magnitudes $|s_{\pm}| = 1$.

[5 marks]

(c) Consider a one-dimensional potential wall of height V_0 and total length L,

$$V(x) = \begin{cases} V_0, & -L/2 < x < L/2, \\ 0, & \text{otherwise} \end{cases}\,.$$

Set up the system of equations that would allow you to find the coefficients r and t as a function of k and $\gamma = \frac{2mV_0}{\hbar^2}$. Briefly describe how you would solve such a system.

[7 marks]

(d) Find r and t in the limit $V_0 \to \infty$. What is the corresponding phase-shift?

[5 marks]