Quantum Information
week 3

Last week...

- We learned how to describe states of composite systems via the tensor product
- We learned how to factor pure states of composite systems and discovered that there are states that cannot be factored
- We learned about the Schmidt decomposition of bipartite quantum states
- We introduced assemblages of quantum states and how to effectively describe them via the density operator
- We learned about the partial trace operation and how to use it to describe the states of parts of a pure entangled state
- We proved that arbitrary quantum states cannot be perfectly cloned.

Quantum Information week 3

This week…

- We will learn how to describe the evolution of open quantum systems
- We will introduce the most general description for evolving arbitrary states of quantum systems
- We will learn about Completely Positive and Trace Preserving (CPTP) maps
- We will see how to use CPTP maps to describe quantum channels
- You will become members of "the church of the larger Hilbert space"
- Do some fundamental quantum protocols (metrology and discrimination)

Closed quantum systems

Postulate 2: The evolution of a closed quantum system is described by a time-dependent unitary operator $U(t) : \mathbb{H} \to \mathbb{H}$ such that

$$\rho(t) = U(t)\rho U^{\dagger}(t)$$

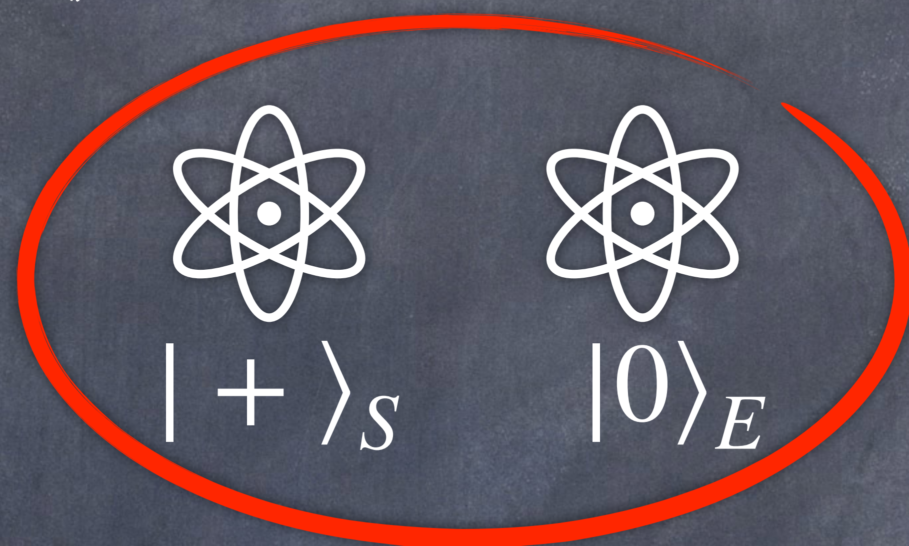
What do we mean when we say that a quantum system is closed?

Unitary dynamics describes the evolution of isolated systems—systems that do not interact with anything else.

Isolated systems are an idealization. In reality no systems is perfectly isolated$\Rightarrow$ Unitary dynamics is not the correct description.

Open quantum systems

Consider the following situation



$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The system and environment are in the bipartite product state

$$|\psi\rangle_{SE} = |+\rangle_S \otimes |0\rangle_E \in \mathbb{H}_2^{\otimes 2}$$

System-plus-environment are a closed system $\Rightarrow$ Evolve unitarily

$$|\phi\rangle_{SE} = U|\psi\rangle_{SE} \in \mathbb{H}_2^{\otimes 2}$$

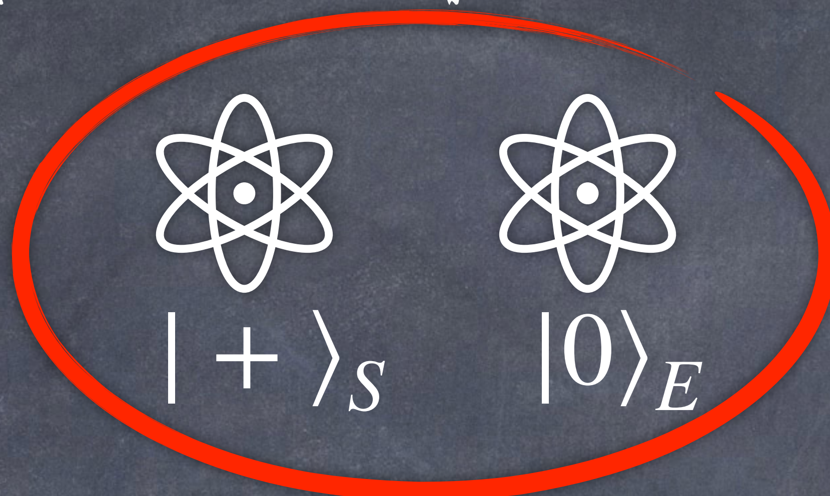
Open quantum systems

$$|+\rangle_S \qquad |0\rangle_E$$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We are only interested to know what happens to the system qubit. How should we describe it's state?

Open quantum systems



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We are only interested to know what happens to the system qubit. How should we describe it's state?

$$\begin{aligned}
\mathrm{tr}_E\left(|\phi\rangle_{SE}\langle\phi|\right) &= \mathrm{tr}_E\left(U|\psi\rangle_{SE}\langle\psi|U^\dagger\right) \\
&= \sum_{k=0}^{1} \mathbb{I}_S \otimes \langle k|\left(U|\psi\rangle_{SE}\langle\psi|U^\dagger\right)\mathbb{I}_S \otimes |k\rangle_E \\
&= \sum_{k=0}^{1} {}_E\langle k|U|0\rangle_E\left(|+\rangle_S\langle +|\right){}_E\langle 0|U|k\rangle_E \\
&= \sum_{k=0}^{1} E_k\left(|+\rangle_S\langle +|\right)E_k^\dagger \equiv \mathcal{E}\left(|+\rangle_S\langle +|\right)
\end{aligned}$$

Open quantum systems

$|+\rangle_S \qquad |0\rangle_E$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We are only interested to know what happens to the system qubit. How should we describe it's state?

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&= \sum_{k=0}^{1} {}_E\langle k|U|0\rangle_E\left(|+\rangle_S\langle+|\right){}_E\langle 0|U|k\rangle_E \\
&= \sum_{k=0}^{1} E_k\left(|+\rangle_S\langle+|\right)E_k^\dagger \equiv \mathscr{E}\left(|+\rangle_S\langle+|\right) \\
&= \frac{\mathbb{I}}{2}
\end{aligned}$$

$$E_0 = |0\rangle_S\langle 0|$$
$$E_1 = |1\rangle_S\langle 1|$$

Quantum Operations

Definition 28: The most general quantum operation is described by a map $\mathscr{E} : \mathscr{B}(\mathbb{H}_{\text{in}}) \to \mathscr{B}(\mathbb{H}_{\text{out}})$ satisfying

1. **Trace non-increasing:** $0 \le \operatorname{tr}\left(\mathscr{E}(\rho)\right) \le 1$
2. **Linearity:** $\mathscr{E}\left(\sum_{i=1}^{N} p_i \rho_i\right) = \sum_{i=1}^{N} p_i \mathscr{E}(\rho_i)$
3. **Complete Positivity**: Let $\rho_{SE} \in \mathscr{B}\left(\mathbb{H}_S \otimes \mathbb{H}_E\right)$, $\rho_{SE} \ge 0$. Then $\mathscr{E} \otimes \mathbb{I}\left(\rho_{SE}\right) \ge 0$

Remarks: The first property simply says that quantum operations must map valid density operators to valid density operators. The second property is linearity.

Complete Positivity

Suppose that Alice and Bob share the entangled state

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00 + |11\right)$$

and let Alice perform an arbitrary operation on her part of the composite system.

Complete positivity states that it does not matter with what other systems our system of interest is related, acting locally on it should not affect the remaining systems.

If complete positivity failed then we could signal faster than light!

Quantum Operations

Theorem 8: A map $\mathscr{E} : \mathscr{B}(\mathbb{H}_{\text{in}}) \rightarrow \mathscr{B}(\mathbb{H}_{\text{out}})$ satisfies

1. **Trace non-increasing:** $0 \leq \text{tr}\left(\mathscr{E}(\rho)\right) \leq 1$
2. **Linearity:** $\mathscr{E}\left(\sum_{i=1}^{N} p_i \rho_i\right) = \sum_{i=1}^{N} p_i \mathscr{E}(\rho_i)$
3. **Complete Positivity:** Let $\rho_{SE} \in \mathscr{B}\left(\mathbb{H}_S \otimes \mathbb{H}_E\right)$, $\rho_{SE} \geq 0$. Then $\mathscr{E} \otimes \mathbb{I}\left(\rho_{SE}\right) \geq 0$

if and only if

$$\mathscr{E}(\rho) = \sum_{i=1}^{N} E_i \rho E_i^{\dagger}$$

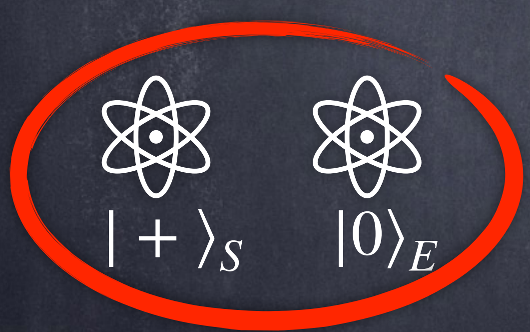
where $\{E_i : \mathbb{H}_{\text{in}} \rightarrow \mathbb{H}_{\text{out}}\}_{i=1}^{N}$, $E_i \geq 0, \forall i$ and $\rho \in \mathscr{B}(\mathbb{H}_{\text{in}})$

Quantum Operations

Remarks: The decomposition of any quantum operation $\mathcal{E} : \mathcal{B}(\mathbb{H}_{\text{in}}) \to \mathcal{B}(\mathbb{H}_{\text{out}})$ as

$$\mathcal{E}(\rho) = \sum_{i=1}^{N} E_i \, \rho E_i^\dagger$$

is known as its Kraus representation and ~~the positive operators~~ $\{E_i : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}\}_{i=1}^{N}$ are known as its Kraus operators



$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathrm{tr}_E(U|\psi\rangle_{SE}\langle\psi|U^\dagger) = \sum_{k=0}^{1} {}_E\langle k|U|0\rangle_E (|+\rangle_S\langle +|) \, {}_E\langle 0|U|k\rangle_E$$

Remark: Note that the first condition implies $\sum_i E_i^\dagger E_i \leq \mathbb{I}$. If equality holds we say the quantum operation is trace-preserving

Examples of Quantum Operations

1. Unitary Evolution

$$\rho \to U\rho U^\dagger$$

This is a quantum operation with a single Kraus operator.

2. Quantum Measurements

$$\rho \to \sum_k M_K \rho M_k^\dagger$$

This is a quantum operation whose Kraus operators are the measurement operators M_k

The Postulates of Quantum Theory

in the most general terms possible...

Postulate 1: Associated to any physical system is a density operator $\rho \in \mathscr{B}(\mathbb{H})$, $\rho > 0$, $\mathrm{tr}(\rho) = 1$. If the system is known to be in state ρ_i with probability p_i then $\rho = \sum_i p_i \rho_i$.

Postulate 2: The evolution of a quantum system is described by a completely positive, (generally time-dependent) trace non-increasing map $\mathscr{E} : \mathscr{B}(\mathbb{H}_{\text{in}}) \rightarrow \mathscr{B}(\mathbb{H}_{\text{out}})$ such that

$$\rho(t) = \mathscr{E}_t\left(\rho\right)$$

Postulate 3: The state of a composite quantum system is described by a density operator $\rho \in \mathscr{B}\left(\bigotimes_{i=1}^{N} \mathbb{H}_i\right)$. If the state of each constituent system is given by ρ_i then the state of the composite system is $\rho = \bigotimes_{i=1}^{N} \rho_i$

Examples of Quantum Operations

1. The bit-flip channel

Consider a quantum operation which, with probability p does nothing to the state and with probability 1-p flips $|0\rangle \leftrightarrow |1\rangle$

$$\mathscr{E}(\rho) = p\,\rho + (1-p)\sigma_x \rho \sigma_x$$

What are the Kraus operators?

Examples of Quantum Operations

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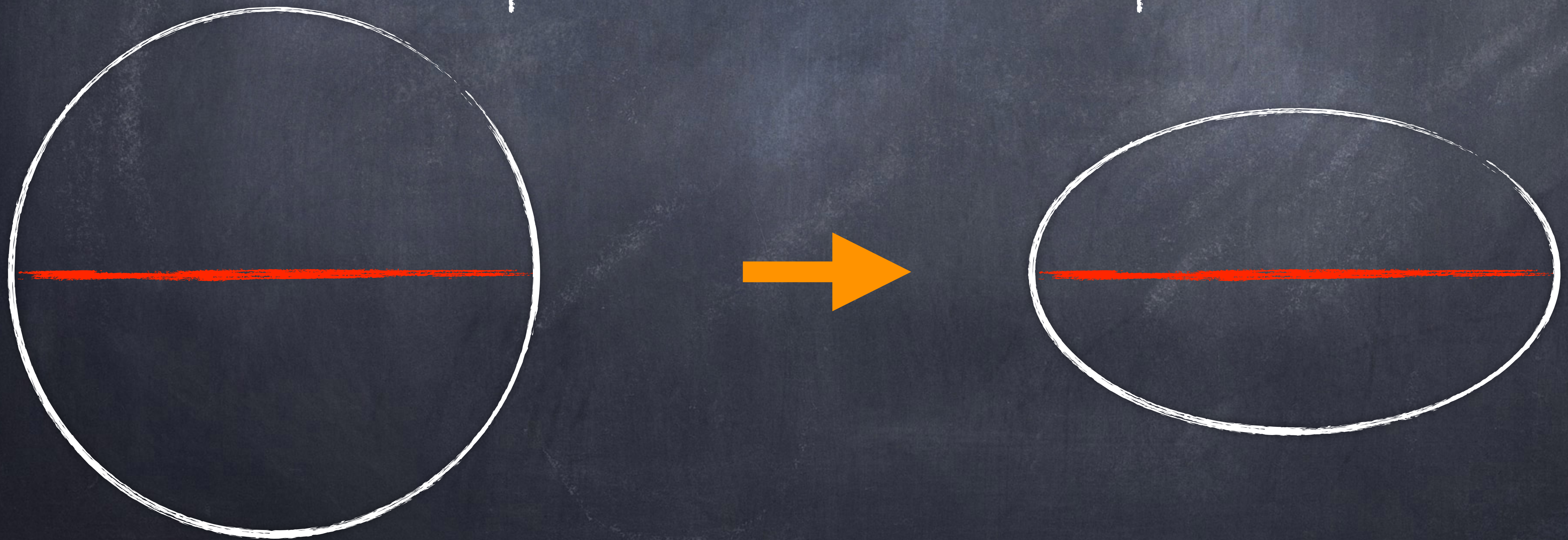
What does such an operation do to the Bloch Sphere?



Examples of Quantum Operations

2. The phase-flip channel

Consider a quantum operation which, with probability p does nothing to the state and with probability 1-p flips $|+\rangle \leftrightarrow |-\rangle$

$$\mathscr{E}(\rho) = p\,\rho + (1-p)\sigma_z\rho\sigma_z$$

What are the Kraus operators?

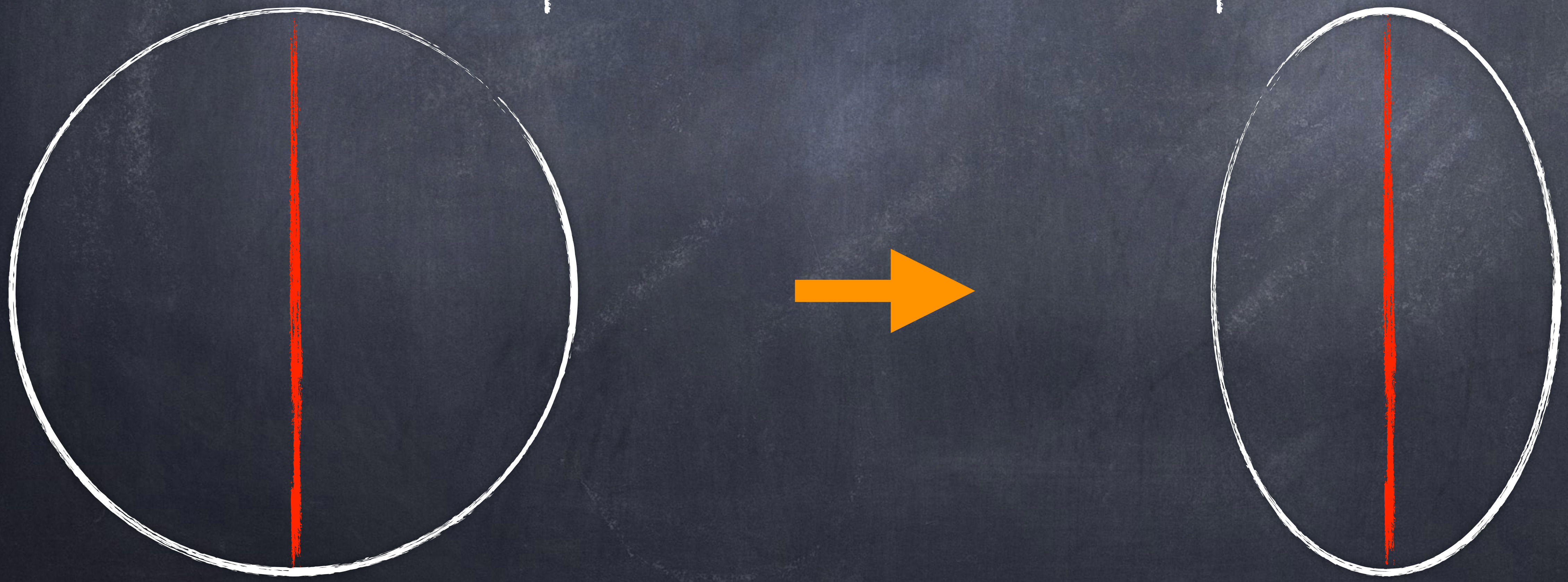
Examples of Quantum Operations

2. The phase-flip channel

Consider a quantum operation which, with probability p does nothing to the state and with probability 1-p flips $|0\rangle \leftrightarrow |1\rangle$

$$\mathcal{E}(\rho) = p\,\rho + (1-p)\sigma_z \rho \sigma_z$$

What does such an operation do to the Bloch Sphere?



Examples of Quantum Operations

3. <u>The bit phase-flip channel</u>

Consider a quantum operation which, with probability p does nothing to the state and with probability 1-p flips $|+i\rangle \leftrightarrow |-i\rangle$

$$\mathcal{E}(\rho) = p\,\rho + (1-p)\sigma_y \rho \sigma_y$$

What are the Kraus operators?

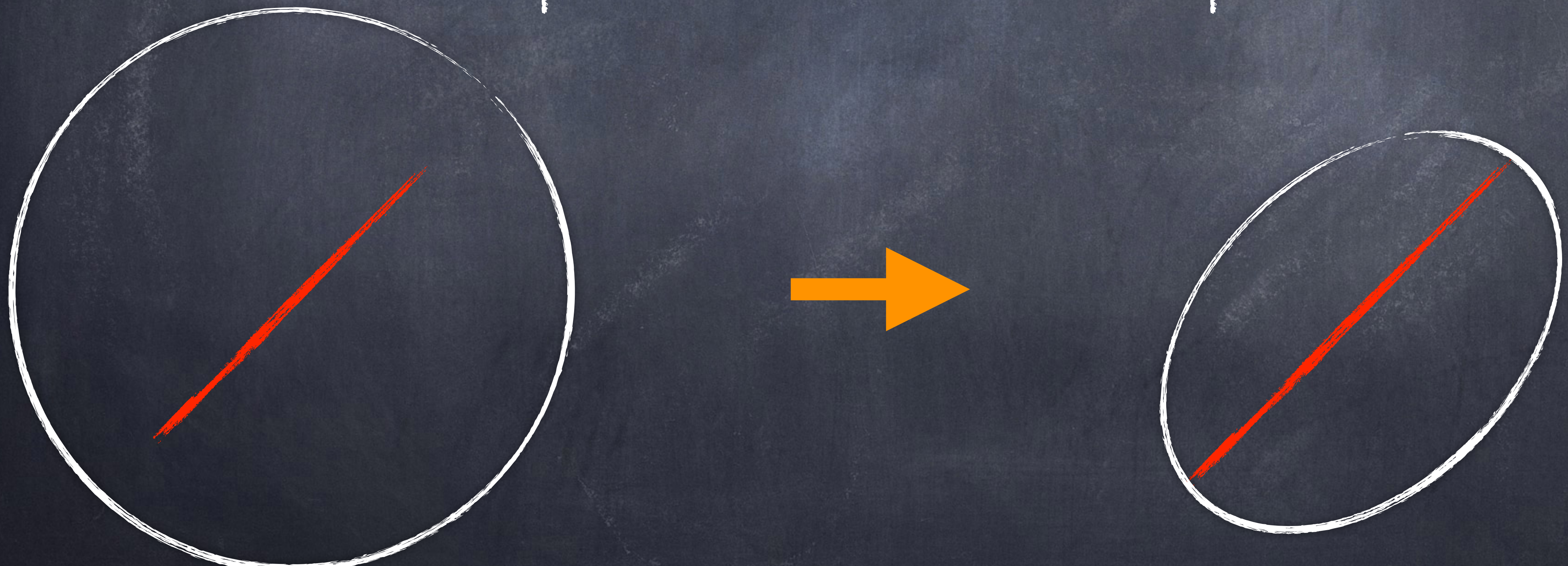
Examples of Quantum Operations

3. <u>The bit phase-flip channel</u>

Consider a quantum operation which, with probability p does nothing to the state and with probability 1-p flips $|0\rangle \leftrightarrow |1\rangle$

$$\mathscr{E}(\rho) = p\,\rho + (1-p)\sigma_y\rho\sigma_y$$

What does such an operation do to the Bloch Sphere?



Examples of Quantum Operations

4. The depolarizing channel

Consider a quantum operation which, with probability p does nothing to the state and with probability 1-p replaces the state with the completely mixed state $\frac{\mathbb{I}}{2}$

$$\mathcal{E}(\rho) = p\,\rho + (1-p)\frac{\mathbb{I}}{2}$$

What are the Kraus operators?

Examples of Quantum Operations

4. The depolarizing channel

Consider a quantum operation which, with probability p does nothing to the state and with probability 1-p replaces the state with the completely mixed state $\frac{\mathbb{1}}{2}$

$$\mathscr{E}(\rho) = p\,\rho + (1-p)\frac{\mathbb{1}}{2}$$

What does such an operation do to the Bloch Sphere?

Examples of Quantum Operations

5. Spontaneous emission.

Spontaneous emission is a naturally occurring noise process whereby an atom, initially in its excited state spontaneously decays to its ground state by emitting a photon. The CPTP map describing such a process has Kraus operators given by

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \sqrt{\gamma} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Verify that this is a trace -preserving quantum operation and compute its action on an arbitrary state $\rho \in \mathscr{B}(\mathbb{H}_2)$.

Draw what happens to the Bloch sphere under such an operation.

Examples of Quantum Operations

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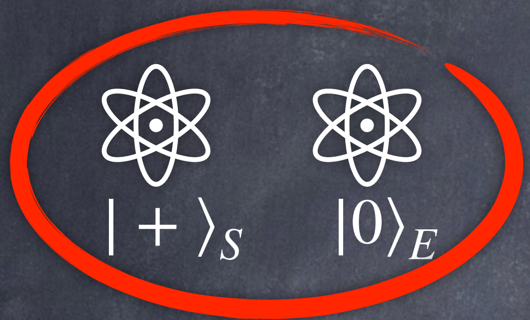
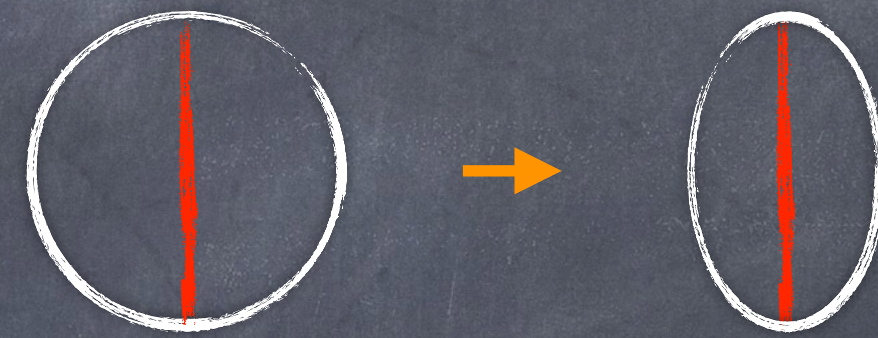
The Kraus representation

Let's have another look at some of the quantum operations we have seen.



$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



$$E_0 = |0\rangle\langle 0|$$

$$E_1 = |1\rangle\langle 1|$$

$$F_0 = \frac{1}{\sqrt{2}}\mathbb{I}$$

$$F_1 = \frac{1}{\sqrt{2}}\sigma_z$$

Let $\rho \in \mathscr{B}(\mathbb{H}_2)$. Compute the action of these two operations on ρ

The Kraus representation

Theorem 9: [Freedom of Kraus Representation] Let $\{E_i : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}\}_{i=1}^{N}$ and $\{F_i : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}\}_{i=1}^{M}$ be the Kraus operators of the quantum operations $\mathscr{E} : \mathscr{B}(\mathbb{H}_{\text{in}}) \to \mathscr{B}(\mathbb{H}_{\text{out}})$ and $\mathscr{F} : \mathscr{B}(\mathbb{H}_{\text{in}}) \to \mathscr{B}(\mathbb{H}_{\text{out}})$ respectively. Then $\mathscr{E} = \mathscr{F}$ if and only if

$$E_i = \sum_j U_{ij} F_j$$

where U is a unitary matrix.

I will give you some time to think about how to prove this.

Last time...

- We discussed how practically no quantum system is ever really "closed"
- We showed how to describe the evolution of quantum systems that are "open", i.e. interact with their environment
- We introduced the concept of Completely Positive, Trace-Preserving (CPTP) maps and showed how these objects describe the most general evolution permissible in quantum theory
- We showed how to represent CPTP maps in terms of their Kraus representation
- We saw a few examples of CPTP maps and saw how they can be understood as "quantum channels"

Today...

- We will revisit some of the arguments so far from a more mathematical point of view
- I will try to enrol you all in the "church of the larger Hilbert space"
- We will see how to physically realise POVMs
- Discuss about quantum estimation theory and in particular phase estimation
- Wrap up things

Representations of Quantum Operations

How do we specify the action of a CPTP map?

1. We give a description of what it does to a state

"... with probability p does nothing to the state and with probability 1-p flips $|0\rangle \leftrightarrow |1\rangle$"

2. Specify it's Kraus operators.

$$\{E_0 = \sqrt{p}\,\mathbb{1}, \quad E_1 = \sqrt{1-p}\,\sigma_x\}$$

Method (1) is not mathematically very handy whilst often times the Kraus operators are not known ahead of time, and are not unique (as we saw last time).

Representations of Quantum Operations

1. The Matrix representation

Recall that CPTP maps are linear and that $\mathscr{B}(\mathbb{H})$ is a vector space under the Hilbert-Schmidt inner product.

Define an orthonormal basis for $\mathscr{B}(\mathbb{H})$, $\{|i\rangle\langle j|\}$. Then

Theorem 11: Let $\mathscr{E} : \mathscr{B}(\mathbb{H}_{\text{in}}) \to \mathscr{B}(\mathbb{H}_{\text{out}})$, and let $\{|i\rangle\langle j|\}$ be an orthonormal basis of $\mathscr{B}(\mathbb{H}_{\text{in}})$. Then the matrix representation of $\mathscr{E}$, $\mathbb{M}(\mathscr{E})$ is a $\cancel{|\mathbb{H}_{\text{in}}| \times |\mathbb{H}_{\text{out}}|}$ matrix with coefficients

$\mathcal{H}_{in}$

$$\mathbb{M}_{ij,kl}(\mathscr{E}) = \operatorname{tr}\Big(\underbrace{|l\rangle\langle k|}_{H_{out}}\,\mathscr{E}(|i\rangle\langle j|)\Big)$$

Representations of Quantum Operations

What is the matrix representation of the bit-flip channel?

Verbal description: with probability p does nothing to the state and with probability 1-p flips $|0\rangle \leftrightarrow |1\rangle$

$$\mathscr{E}(\rho) = p\,\rho + (1-p)\sigma_x \rho \sigma_x$$

Representations of Quantum Operations

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$$\mathcal{E}(\rho) = p\,\rho + (1-p)\sigma_x \rho \sigma_x$$

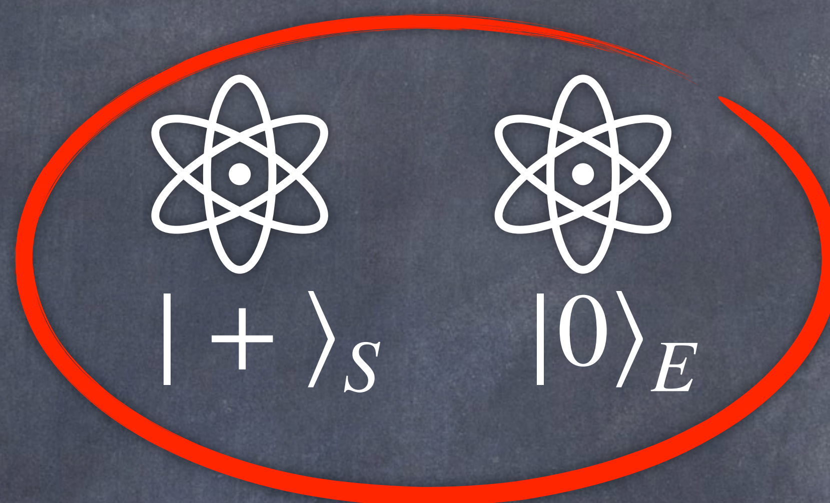
The matrix representation of this map is

$$\mathbb{M}(\mathcal{E}) = \begin{pmatrix} p & 0 & 0 & 1-p \\ 0 & p & 1-p & 0 \\ 0 & 1-p & p & 0 \\ 1-p & 0 & 0 & p \end{pmatrix}$$

Representations of Quantum Operations

2. (Stinespring's) Dilation and the Church of the larger Hilbert space

Let's revisit the argument that led to us introducing CPTP maps



Our reasoning was:

1. Every system interacts with the rest of the "universe"

2. The entire universe is a closed system $\Rightarrow$ Unitary evolution

3. The dynamics of the system are obtained by "tracing out" the rest of the universe

Representations of Quantum Operations

Theorem 12: Any CPTP map $\mathscr{E} : \mathscr{B}(\mathbb{H}) \to \mathscr{B}(\mathbb{H})$ can be obtained by

$$\mathscr{E}(\rho) = \mathrm{tr}_2\left(U\rho U^\dagger\right)$$

for a suitable choice of $U : \mathbb{H}^{\otimes 2} \to \mathbb{H}^{\otimes 2}$

Remark: Theorem 12 is more general than how it is stated here. It not only holds for CPTP maps but it holds for every linear map

Remark: It suffices to take the dimension of the auxiliary system (the environment) to be the same as that of the system for the case of CPTP maps.

Remark: It suffices to take the initial state of the system-plus-environment as separable.

Representations of Quantum Operations

3. The Choi-Jamiołkowski Isomorphism

Recall that the Kraus representation of a CPTP map is not unique. In fact you may of noticed that the various Kraus representations of a CPTP map are related in much the same way as the different assemblages of quantum states.

Is this a coincidence?

Theorem 13: Let $\mathscr{E} : \mathscr{B}(\mathbb{H}_{\text{in}}) \rightarrow \mathscr{B}(\mathbb{H}_{\text{out}})$ be a CPTP map. Then the Choi-Jamiołkowski representation of the map is given by

$$J(\mathscr{E}) = \left(\mathbb{I} \otimes \mathscr{E}\right) |\Phi^+\rangle\langle\Phi^+|$$

where $|\Phi^+\rangle = \dfrac{1}{\sqrt{d}} \displaystyle\sum_{i=1}^{d} |i\rangle \otimes |i\rangle \ \in \mathbb{H}^{\otimes 2}$ with $d = |\mathbb{H}|$

Representations of Quantum Operations

Remark: Recall the definition of complete positivity. As $|\Phi^+\rangle\langle\Phi^+| \in \mathscr{B}(\mathbb{H}_{\text{in}})$ it follows that $J(\mathscr{E}) \in \mathscr{B}(\mathbb{H}_{\text{out}})$.

Remark: Notice the connection with the matrix representation of CPTP maps

$$|\Phi^+\rangle\langle\Phi^+| = \frac{1}{d}\sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j|$$

Given the Choi-Jamiołkowski state, one can retrieve the action of the CPTP map on any state $\rho \in \mathscr{B}(\mathbb{H}_{\text{in}})$ by

$$\mathscr{E}(\rho) = d\,\mathrm{tr}_1\left((\rho^T \otimes \mathbb{I})J(\mathscr{E})\right)$$

Representations of Quantum Operations

What is the Choi-Jamiołkowski representation of the bit-flip channel?

Verbal description: with probability p does nothing to the state and with probability 1-p flips $|0\rangle \leftrightarrow |1\rangle$

$$\mathcal{E}(\rho) = p\,\rho + (1-p)\sigma_x \rho \sigma_x$$

Representations of Quantum Operations

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Verbal description: with probability p does nothing to the state and with probability 1-p flips $|0\rangle \leftrightarrow |1\rangle$

$$\mathscr{E}(\rho) = p\,\rho + (1-p)\sigma_x \rho \sigma_x$$

$$J(\mathscr{E}) = \frac{1}{2}\begin{pmatrix} p & 0 & 0 & p \\ 0 & 1-p & 1-p & 0 \\ 0 & 1-p & 1-p & 0 \\ p & 0 & 0 & p \end{pmatrix}$$

Physically implementing POVMs

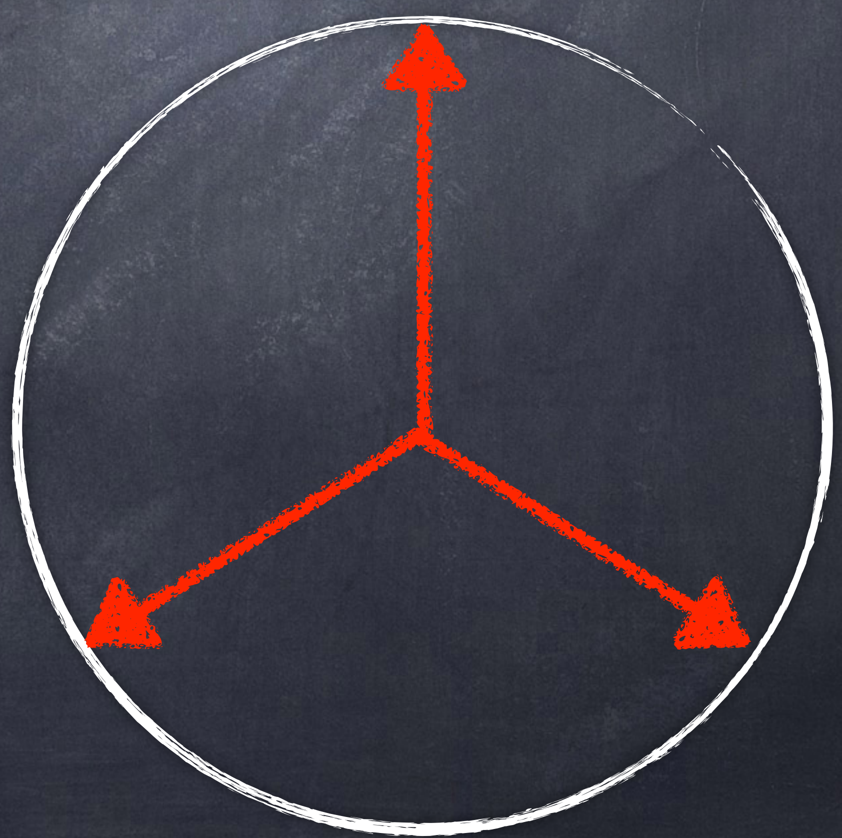
Definition 18: The set of positive operators $\left\{ E_m : \mathbb{H} \to \mathbb{H}, E_m \geq 0, \forall m | \sum_m E_m = \mathbb{I} \right\}$ form the elements of a Positive Operator Value Measure (or POVM for short)

We can obtain the requisite measurement operator M_m of the corresponding POVM element E_m by noting that

$$M_m = E_m^{\frac{1}{2}}$$

$$E_0 = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} \frac{1}{3} & \sqrt{\frac{1}{6}} \\ \sqrt{\frac{1}{6}} & \frac{1}{2} \end{pmatrix}, \quad E_2 = \begin{pmatrix} \frac{1}{3} & -\sqrt{\frac{1}{6}} \\ -\sqrt{\frac{1}{6}} & \frac{1}{2} \end{pmatrix},$$

$$M_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad M_1 = \begin{pmatrix} \sqrt{\frac{2}{15}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \sqrt{\frac{3}{10}} \end{pmatrix} \quad M_2 = \begin{pmatrix} \sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \sqrt{\frac{3}{10}} \end{pmatrix}$$



Physically implementing POVMs

Theorem 14 [Naimark Dilation]: Let $\{E_i \geq 0, E_i : \mathbb{H} \rightarrow \mathbb{H}, | \sum_{i=1}^{n} E_i = \mathbb{I}\}$ be a POVM. Then there exists an isometry $V : \mathbb{H} \rightarrow \mathbb{H} \otimes \mathbb{C}_n$ such that

$$E_i = V^\dagger \left(\mathbb{I} \otimes |i\rangle\langle i| \right) V$$

Remark: Indeed, it is easy to show that

$$V = \sum_{i=1}^{n} E_i^{\frac{1}{2}} \otimes |i\rangle$$

Physically implementing POVMs

Lemma 4: Let $\{E_i \geq 0, E_i : \mathbb{H} \to \mathbb{H}, | \sum_{i=1}^{n} E_i = \mathbb{I}\}$ be a POVM. Then there exists a projective measurement $P_i : \mathbb{H} \otimes \mathbb{C}_n \to \mathbb{H} \otimes \mathbb{C}_n$ and a fiducial state $|0\rangle \in \mathbb{C}_n$ such that

$$\mathrm{tr}\left(E_i \rho\right) = \mathrm{tr}\left(P_i \left(\rho \otimes |0\rangle\langle 0|\right)\right)$$

Moreover the projectors P_i are given by $P_i = U^\dagger \left(\mathbb{I} \otimes |i\rangle\langle i|\right) U$ with $U(\mathbb{I} \otimes |0\rangle) = V$