Quantum Information Theory. Assignment 1

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Q1. Let $\rho \in \mathcal{B}(\mathcal{H})$ be a density operator.

(i) Show that $Tr(\rho^2) \leq 1$ with equality if and only if $\rho = |\psi\rangle\langle\psi|$. (1 point)

(ii) For the case $dim\mathcal{H} = 2$, show that any ρ can be written as (1 point)

$$\rho = \frac{\mathbb{1} + \vec{r}\vec{\sigma}}{2}$$

where $\vec{r} \in \mathbb{R}^3$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector form by the Pauli matrices and $||\vec{r}|| \leq 1$.

(iii) Show that ρ corresponds to a pure state if and only if $||\vec{r}|| = 1$. (1 point)

(iv) Show that any orthogonal rotation in the Bloch sphere (with angles θ, φ) corresponds to a unitary transformation of the density matrix: (1 point)

$$\rho \to \rho' = U_{\vec{n}\theta} \rho U^{\dagger}_{\vec{n}\theta}$$

and thus $SO(3) \simeq SU(2)/(-\mathbb{1}, \mathbb{1})$.

(v) Which rotations one should do to a qubit to achieve the following transformation? (1 point)

$$|\psi\rangle = |0\rangle \Longrightarrow |\psi'\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Q2. *Maximally entangled states*

(i) Show that a maximally entangled state of a bipartite system is of the form: (1 point)

$$|\Psi\rangle = (\mathbb{1} \otimes U)\,|\Omega\rangle$$

where $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$ and $\{|i\rangle\}$ denotes an orthonormal basis.

(ii) Given a pure bipartite system $|\Psi\rangle_{AB} \in \mathcal{C}^d \otimes \mathcal{C}^{d'}$, show for which cases the corresponding reduced density matrices of each subsytem are identical and its dependence on the dimensions d and d'. (1 point)

Q3. *Quantum teleportation*

(i) Let $|\psi\rangle = \alpha\,|0\rangle_1 + \beta\,|1\rangle_1 \in \mathcal{H}_1$ be a single qubit state ($\alpha, \beta \in \mathbb{C}$) and let $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_2\,|0\rangle_3 + |1\rangle_2\,|1\rangle_3) \in \mathcal{H}_2 \otimes \mathcal{H}_3$ be a bipartite entangled state. Write down, in the compact notatio,n the composite state $|\Psi\rangle \equiv |\psi\rangle \otimes |\Phi^+\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ (1 point)

(ii) Assume that Alice is in possession of systems 1 and 2, whilst Bob is in possession of system 3. If Alice measures her two systems in the *Bell basis*,

$$\left|\Phi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right)$$
$$\left|\Psi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right),$$

determine the probabilities of obtaining each of the four possible outcomes as well as the corresponding post measurement state. (2 points)

(iii) Assume that Bob knows what measurement Alice has performed but he does not know Alice's measurement outcome. Write down the density matrix describing Bob's state of system 3. (1 point)

(iv) Given each of Alice's measurement outcomes determine the operation Bob must perform in order that his system is in the state $|\psi\rangle$ in each case. (1 point)

(v) How many classical bits does Alice need to send to Bob in order to communicate her measurement outcome? How does this compare with the amount of information (in bits) that Alice would have to send Bob in order to completely specify the state $|\psi\rangle$. (1 point)

(vi) Draw the quantum circuit corresponding to the teleportation protocol. Your protocol must be such that it accounts for all possible measurement outcomes. Note that classical information, such as that obtained from a quantum measurement, can be represented by a double line. Thus, a single line denotes a quantum state, a box will be a measurement and the classical information going out of the box will be a double line (2 points)

Q4. *Reduced density matrices.*

Let Alice and Bob share the entangled state

$$\left|\Phi^{+}\right\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B\right) \in \mathcal{H}_A \otimes \mathcal{H}_B$$

.

(i) Suppose that Alice chooses to perform $\sigma_x^i \sigma_y^j$, $i, j \in (0,1)$, on her part of the entangled state. Write down the state of the composite system for each of the possible choices of Alice. (1 point)

(ii) Describe the reduced density matrix of Alice and Bob for the above cases. Write two different purifications for Alice and Bob reduced density matrices. (1 point)

Q5. In quantum physics, a measurement with two distinct outcomes can be described by a couple of orthogonal projectors that sum up to the identity:

$$\{\Pi_1, \Pi_2\} \text{ s.t. } \Pi_i \Pi_j = \Pi_j \Pi_i = \delta_{i,j} \Pi_i, \quad \Pi_1 + \Pi_2 = \mathbb{1}.$$

Note that any pair of orthogonal projectors summing up to the identity can be turned onto an observable $A = a_1 \Pi_1 + a_2 \Pi_2$, with $a_1 \neq a_2$ and, conversely, to every observable one can associate a corresponding set of orthogonal projectors that add up to the identity.

(i) Consider the operator Π, whose expression in the canonical basis,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

is

$$\Pi = \frac{1}{5}\begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Show that $\Pi = |\phi\rangle\langle\phi|$ where $|\phi\rangle \in \mathbb{C}_3$.

(1 point)

(ii) Consider the operators B and C, whose matrix representations in the canonical basis is

$$B = \frac{1}{2}\begin{pmatrix} -1 & 0 & -3\mathrm{i} \\ 0 & 2 & 0 \\ 3\mathrm{i} & 0 & -1 \end{pmatrix}, \qquad C = \frac{1}{6}\begin{pmatrix} 1 & 2 & \mathrm{i} \\ 2 & 4 & 2\mathrm{i} \\ \mathrm{i} & -2\mathrm{i} & 1 \end{pmatrix}.$$

Which of B, C, correspond to an observable? Compute the eigenvalues and corresponding eigenvectors for that observable.

(1 point)

(iii) Write down the projectors, $\{\Theta_1, \Theta_2\}$, corresponding to the observable in (**ii**).

(1 point)

(iv) Consider now the measurements determined by the two previous observables: $\mathcal{M} = \{\Pi_1 = \Pi, \Pi_2 = \mathbb{1} - \Pi\}$ and $\mathcal{N} = \{\Theta_1, \Theta_2\}$. Suppose that a quantum system is initially in the state $|\psi\rangle = \frac{1}{\sqrt{6}}(|0\rangle + 2|1\rangle + \mathrm{i}|2\rangle)$. Compute

(a) the probability $p(\theta_1, \pi_2)$ of measuring $\mathcal{N}$ and then $\mathcal{M}$ and obtaining the outcomes corresponding to Θ_1 and Π_2 respectively;

(1 point)

(b) the probability $p(\pi_2, \theta_1)$ of measuring $\mathcal{M}$ and then $\mathcal{N}$ and obtaining the outcomes corresponding to Π_2 and Θ_1, respectively.

(1 point)

Q6. An $n \times n$ circulant matrix is, C, is any matrix of the form

$$C = \begin{pmatrix} c_1 & c_2 & \dots & c_n \\ c_2 & c_3 & \dots & c_1 \\ \vdots & \ddots & \vdots & \\ c_n & c_{n-1} & \dots & c_1 \end{pmatrix}$$

(i) Let $f(x) = \sum_{x=1}^{n} c_n x^{n-1}$. Show that $C = f(P)$ where

$$P := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(1 point)

(ii) Show that C is a *normal matrix*, i.e., $C^\dagger C = CC^\dagger$

(1 point)

(iii) Determine the eigenvalues and eigenvectors of C

(2 point)

Q7. Let $x_0, x_1, y_0, y_1 \in \{0, 1\}$, where Alice has x_0, x_1 and Bob has y_0, y_1. Alice and Bob wish to calculate the following Boolean function

$$g(x_0, x_1, y_0, y_1) := x_1 \oplus y_1 \oplus (x_0 \cdot y_0)$$

where $\oplus$ denotes the XOR operation and $\cdot$ the AND operation. To do this Alice and Bob share the maximally entangled state

$$|\Phi_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\,.$$

Alice applies the unitary matrix $U_A\left(-\frac{\pi}{16} + x_0\frac{\pi}{4}\right)$ on her part of the maximally entangled state whilst Bob applies $U_B\left(-\frac{\pi}{16} + y_0\frac{\pi}{4}\right)$ on his part, where

$$U(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Let Alice and Bob measure their systems in the computational basis, $\{|0\rangle, |1\rangle\}$, and denote the outcomes of their measurement $a, b \in \{0, 1\}$ respectively. Determine the probability that

$$a \oplus b = x_0 \cdot y_0 \,. \tag{1}$$

(2 s)

Q8. Consider the vector

$$\mathbf{v} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^6 \,.$$

Determine the Schmidt decomposition of $\mathbf{v}$ over $\mathbb{C}^6 = \mathbb{C}^2 \otimes \mathbb{C}^3$ and over $\mathbb{C}^6 = \mathbb{C}^3 \otimes \mathbb{C}^2$

(2 points)