Assignment 2 (Quantum Gases)

1. Weakly interacting Bose gas

Consider a system of N bosons at temperature $T = 0$ in a volume V. If the bosons are non-interacting, they are all condensed in the single-particle ground state, i.e. $|\text{g.s.}\rangle = \frac{1}{\sqrt{N!}}(a_0^\dagger)^N |0\rangle$, where $|0\rangle$ is the vacuum state. The aim of this exercise is to investigate the effect of weak interactions.

(a) Consider the Hamiltonian

$$H = \sum_{\mathbf{k}} \epsilon_k^0 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}\int d^3x\, d^3x' a^\dagger(\mathbf{x}) a^\dagger(\mathbf{x}') U(\mathbf{x}-\mathbf{x}') a(\mathbf{x}') a(\mathbf{x}), \tag{1.1}$$

where $U(\mathbf{x}) = g\delta(\mathbf{x})$ is the interaction and $\epsilon_k^0 = \frac{k^2}{2m}$ is the single-particle kinetic energy ($\hbar = 1$). By going to Fourier space, defined by $a(\mathbf{x}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}}$ and $U(\mathbf{k}) = \int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}} U(\mathbf{x})$, show that

$$H = \sum_{\mathbf{k}} \epsilon_k^0 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{g}{2V}\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} a_{\mathbf{k}+\mathbf{q}/2}^\dagger a_{-\mathbf{k}+\mathbf{q}/2}^\dagger a_{\mathbf{k}'+\mathbf{q}/2} a_{-\mathbf{k}'+\mathbf{q}/2}. \tag{1.2}$$

▷ Interpret the momenta in the last term (a diagram would be useful).

(b) Expand the Hamiltonian in powers of N_0, keeping for the interaction part only terms that are linear or quadratic in N_0, to find the approximate Hamiltonian

$$H = \sum_{\mathbf{k}} \epsilon_k^0 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{g}{2V}N_0^2 + \frac{gN_0}{V}\sum_{\mathbf{k}\neq\mathbf{0}}\left[a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} + \frac{1}{2}(a_{-\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger)\right] \tag{1.3}$$

Interpret, for instance diagrammatically, the terms in the last expression.

(c) Use the relation between the total number of particles, and the condensed particles, $N = N_0 + \sum_{\mathbf{k}\neq 0} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$, to replace all N_0 in the expression, and neglect terms which have more than two creation or annihilation operators. You should find

$$H = \frac{gnN}{2} + \sum_{\mathbf{k}\neq\mathbf{0}}\left[(\epsilon_k^0 + gn) a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{gn}{2}(a_{-\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger)\right] \tag{1.4}$$

with $n = N/V$ the density.

(d) The Bogoliubov transformation consists in defining new operators

$$\begin{pmatrix} a_{\mathbf{k}} \\ a_{-\mathbf{k}}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix}\begin{pmatrix} \alpha_{\mathbf{k}} \\ \alpha_{-\mathbf{k}}^\dagger \end{pmatrix} \tag{1.5}$$

with $u_k = \sqrt{\frac{1}{2}\left(\frac{\epsilon_k^0 + gn}{\epsilon(k)} + 1\right)}$ and $v_k = \sqrt{\frac{1}{2}\left(\frac{\epsilon_k^0 + gn}{\epsilon(k)} - 1\right)}$, so that $u_k^2 - v_k^2 = 1$. We have also introduced the Bogoliubov dispersion $\epsilon(k) = \sqrt{(\epsilon_k^0 + gn)^2 - g^2n^2} = \sqrt{(\epsilon_k^0)^2 + 2gn\epsilon_k^0}$.

▷ Show that the Bogoliubov transformation preserves the bosonic commutation relations. (if you wish, do this only for $[\alpha_{\mathbf{k}}, \alpha_{\mathbf{k}}^\dagger]$)

(e) Following the Bogoliubov transformation, the Hamiltonian takes the form

$$H = \frac{gnN}{2} - \frac{1}{2}\sum_{\mathbf{k}\neq\mathbf{0}}[\epsilon_k^0 + ng - \epsilon(k)] + \sum_{\mathbf{k}\neq\mathbf{0}} \epsilon(k)\alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}}. \tag{1.6}$$

▷ Discuss the physical meaning of the various terms.

(f) Show that the number operator $\hat{N}$ evaluated in an eigenstate of the Hamiltonian (1.6) takes the form

$$\hat{N} = N_0 + \sum_{\mathbf{k}\neq\mathbf{0}} v_k^2 + \sum_{\mathbf{k}\neq\mathbf{0}}(u_k^2 + v_k^2)\alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}}. \tag{1.7}$$

▷ Discuss the meaning of the various terms.
▷ Show that the depletion of the condensate in the ground state is $N - N_0 = \frac{8}{3\sqrt{\pi}}(na^3)^{1/2}N$.