

Quantum Information Theory — Assignment 2

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QUESTION 1

In this question we're presented with sets of matrices $\{M_m\}$ and asked if each set is a legal generalized measurement, if it represents a projective measurement, or if it represents a POVM.

We recall the following definitions:

Measurement	Matrices	Definition
Generalized measurement	$\{M_m\}$	$\sum_m M_m^\dagger M_m = \mathbb{1}$
Projective measurement	$\{P_m\}$	$\sum_m P_m = \mathbb{1}, \quad P_m P_j = \delta_{ij} P_j$
POVM	$\{E_m\}$	$\sum_m E_m = \mathbb{1}, \quad E_m \geq 0$

PART (I)

First, we're given this set of matrices:

By inspection, these matrices sum to the identity, but they are not projection matrices, since they manifestly do not equal their own squares. However, each of the matrices is positive semidefinite, as may be quickly verified using software.¹ Therefore, this set of matrices forms a POVM.

On the other hand, $\sum_m M_m^\dagger M_m \neq \mathbb{1}$, and so they are not a generalized measurement.²

PART (II)

Next, we're presented with:

Again, the matrices sum to the identity. Again, they are not projection matrices. But this time, the first two matrices are not positive semidefinite, and so this set of matrices is neither a projective measurement, nor a POVM.

Worse, these matrices do not even form a legal generalized measurement, as is easy to verify using software.

PART (III)

This time we have four matrices:

¹That is, they have non-negative eigenvalues.

²On the other hand, since every positive semidefinite matrix can be factorized as $E = M^\dagger M$, we may easily construct a set of measurement operators $\{M_m\}$ from these $\{E_m\}$.

These matrices do not sum to the identity, a fact which may be established by staring at them. Thus, they are neither a projective measurement, nor a POVM.

But $\sum_m M_m^\dagger M_m = \mathbb{1}$, and so they are a generalized measurement.

PART (IV)

Finally, we're asked about these four suspicious-looking characters:

These matrices do sum to the identity, though one must stare pretty hard to see it. The first matrix is quite manifestly not a projection matrix, so they are not a projective measurement. But they are all positive semidefinite, and therefore constitute a POVM.

On the other hand, $\sum_m M_m^\dagger M_m = \frac{1}{2}\mathbb{1}$, and so they are not a generalized measurement.³

QUESTION 2

Here we consider the following Choi operator for a map Λ :

$$J(\Lambda) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \sqrt{1-\gamma} & 0 & 0 & 1-\gamma \end{pmatrix}$$

PART (I)

We diagonalize this matrix, and obtain the eigenvalues $\lambda_1 = \gamma/2$, $\lambda_2 = 1 - \gamma/2$, $\lambda_{3,4} = 0$. The non-zero eigenvalues have eigenvectors:

$$|\lambda_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\lambda_2\rangle = \frac{1}{\sqrt{2-\lambda}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \sqrt{1-\lambda} \end{pmatrix}$$

The Kraus operators are obtained by rearranging the elements of the vectors $\sqrt{\lambda_i} |\lambda_i\rangle$ into square matrices.

They are:

$$K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} = \sqrt{\gamma} |0\rangle\langle 1| \quad K_2 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} = |0\rangle\langle 0| + \sqrt{1-\gamma} |1\rangle\langle 1|$$

These matrices are both manifestly positive semi-definite, and it's easy to verify by direct evaluation that $K_1^\dagger K_1 + K_2^\dagger K_2 = \mathbb{1}$. Finally, we recognize that these are the Kraus operators of the *amplitude damping channel*.⁴

PART (II)

The matrix representation may be found according to the following formula:

$$M_{ij}^{k\ell} = \text{Tr}\{|\ell\rangle\langle k| \Lambda(|i\rangle\langle j|)\} = \langle k| \Lambda(|i\rangle\langle j|) |\ell\rangle$$

where the matrices $\Lambda(|i\rangle\langle j|)$ are the four 2×2 blocks of $J(\Lambda)$.

³We may obtain a generalized measurement just by scaling these matrices, that is, by taking the set $\{\sqrt{2}M_m\}$.

⁴See Wilde, 2017, section 4.7.

So the matrix representation of Λ may be written:

$$\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ \gamma & 0 & 0 & 1-\gamma \end{pmatrix}$$

QUESTION 3

PART (I)

We consider the action of a unitary operator $U \otimes U$ on the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

We begin by writing a general U as:

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \det(U) = |\alpha|^2 + |\beta|^2 = e^{i\theta}$$

where $e^{i\theta}$ is just a phase, with modulus 1.

Then:

$$\begin{aligned} \sqrt{2}(U \otimes U)|\Psi^-\rangle &= (\alpha|0\rangle - \beta^*|1\rangle) \otimes (\beta|0\rangle + \alpha^*|1\rangle) - (\beta|0\rangle + \alpha^*|1\rangle) \otimes (\alpha|0\rangle - \beta^*|1\rangle) \\ &= \alpha\beta|00\rangle + \alpha\alpha^*|01\rangle - \beta^*\beta|10\rangle - \beta^*\alpha^*|11\rangle - \alpha\beta|00\rangle + \beta\beta^*|01\rangle - \alpha\alpha^*|10\rangle + \beta^*\alpha^*|11\rangle \\ &= (|\alpha|^2 + |\beta|^2)(|01\rangle - |10\rangle) \\ &= \det(U)(|01\rangle - |10\rangle) \\ &= e^{i\theta}\sqrt{2}|\Psi^-\rangle \end{aligned}$$

And so, $|\Psi^-\rangle$ is invariant under transformations of this form (apart from the phase of U).

PART (II)

We now consider the action of $U \otimes U \otimes U$ on the tripartite GHZ and W states.

$$\begin{aligned} \sqrt{2}(U \otimes U \otimes U)(|000\rangle + |111\rangle) &= (\alpha|0\rangle - \beta^*|1\rangle) \otimes (\alpha|0\rangle - \beta^*|1\rangle) \otimes (\alpha|0\rangle - \beta^*|1\rangle) \\ &\quad + (\beta|0\rangle + \alpha^*|1\rangle) \otimes (\beta|0\rangle + \alpha^*|1\rangle) \otimes (\beta|0\rangle + \alpha^*|1\rangle) \\ &= (\alpha^3 + \beta^3)|000\rangle + (-\beta^*)^3 + (\alpha^*)^3|111\rangle \\ &\quad + (-\alpha^2\beta^* + \beta^2\alpha^*)(|001\rangle + |010\rangle + |100\rangle) \\ &\quad + (\alpha(\beta^*)^2 + \beta(\alpha^*)^2)(|011\rangle + |101\rangle + |110\rangle) \end{aligned}$$

and:

$$\begin{aligned} \sqrt{3}(U \otimes U \otimes U)(|001\rangle + |010\rangle + |100\rangle) &= (\alpha|0\rangle - \beta^*|1\rangle) \otimes (\alpha|0\rangle - \beta^*|1\rangle) \otimes (\beta|0\rangle + \alpha^*|1\rangle) \\ &\quad + (\alpha|0\rangle - \beta^*|1\rangle) \otimes (\beta|0\rangle + \alpha^*|1\rangle) \otimes (\alpha|0\rangle - \beta^*|1\rangle) \\ &\quad + (\beta|0\rangle + \alpha^*|1\rangle) \otimes (\alpha|0\rangle - \beta^*|1\rangle) \otimes (\alpha|0\rangle - \beta^*|1\rangle) \\ &= 3\alpha^2\beta|000\rangle - 3(\beta^*)^2\alpha^*|111\rangle \\ &\quad + (\alpha^2\alpha^* - 2\alpha\beta^*\beta)(|001\rangle + |010\rangle + |100\rangle) \\ &\quad + ((\beta^*)^2\beta - 2\alpha\beta^*\alpha^*)(|011\rangle + |101\rangle + |110\rangle) \end{aligned}$$

Neither of these states is invariant under this transformation.

PART (III)

We now ask whether such a transformation could transform the GHZ state into a W state. From the equation above, we are looking for a U such that:

$$\alpha^3 + \beta^3 = 0 \quad -(\beta^*)^3 + (\alpha^*)^3 = 0 \quad -\alpha^2\beta^* + \beta^2\alpha^* = 1 \quad \alpha(\beta^*)^2 + \beta(\alpha^*)^2 = 0$$

From the first and second of these constraints, we obtain:

$$\beta = -\omega\alpha \quad \alpha^* = \lambda\beta^* \quad \omega^3 = \lambda^3 = 1 \quad \omega = -1/\lambda^*$$

Then from the third and fourth constraints we have:

$$-(1 - \omega^2\lambda)\alpha^2\beta^* = 1 \quad (1 - \omega\lambda^2)\alpha(\beta^*)^2 = 0$$

From the first of these, neither α nor β^* may be zero, and so from the second it must be that $\omega\lambda^2 = 1$. But, writing $\omega = \exp(\frac{2}{3}\pi in)$ and $\lambda = \exp(\frac{2}{3}\pi im)$, we obtain:

$$1 = \exp(2i\pi k) = \exp(\frac{2}{3}\pi i(n + m)) \quad \Rightarrow \quad k = \frac{1}{3}(n + m)$$

which has solutions for integer k, n, m only where n and m are both integer multiples of 3. That is, when $\lambda = \omega = 1$. But for this choice of λ, ω , which is required to satisfy the fourth constraint, the third constraint cannot be satisfied.

We have shown that no unitary transformation of the given form can take the GHZ state to the W state. In fact, it's well-known that no LOCC operations can do this, and that the GHZ state and the W state belong to distinct classes of entanglement.

QUESTION 4

Here we consider the family of isotropic states:

$$\rho_{\text{iso}}(\alpha) \equiv \alpha |\Phi_{00}\rangle\langle\Phi_{00}| + (1 - \alpha) \frac{\mathbb{1}}{d^2} \quad |\Phi_{00}\rangle \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$$

which for any real α is a linear combination of a maximally entangled state and a maximally mixed state.

PART (I) AND PART (II)

For a state $\rho_{\text{iso}}(\alpha)$ to be a well-defined quantum state, it must have unit trace, and be positive semidefinite.

Since both $|\Phi_{00}\rangle\langle\Phi_{00}|$ and $\frac{\mathbb{1}}{d^2}$ have unit trace, the first condition is satisfied for every value of α .

It's possible to find the bounds on α just by evaluating the sandwich of ρ_{iso} with an arbitrary vector, but since we've also been asked to find the eigenvalues, it's quicker to just do that first.

To find the eigenvalues of ρ_{iso} we must solve the eigenvalue equation:

$$\rho_{\text{iso}}(\alpha) |\lambda\rangle = \left[\alpha |\Phi_{00}\rangle\langle\Phi_{00}| + (1 - \alpha) \frac{\mathbb{1}}{d^2} \right] |\lambda\rangle = \lambda |\lambda\rangle$$

which may be rearranged to:

$$|\Phi_{00}\rangle\langle\Phi_{00}| |\lambda\rangle = \frac{1}{\alpha} \left(\lambda - \frac{1 - \alpha}{d^2} \right) |\lambda\rangle$$

and this is just the eigenvalue equation of the projection matrix $|\Phi_{00}\rangle\langle\Phi_{00}|$.

Therefore, we have eigenvalues λ_0, λ_1 satisfying:

$$\frac{1}{\alpha} \left(\lambda_0 - \frac{1-\alpha}{d^2} \right) = 0 \qquad \frac{1}{\alpha} \left(\lambda_1 - \frac{1-\alpha}{d^2} \right) = 1$$

Solving, we find:

$$\lambda_0 = \frac{1-\alpha}{d^2} \qquad \lambda_1 = \alpha + \frac{1-\alpha}{d^2}$$

A matrix is positive definite when its eigenvalues are non-negative. Thus, from the expressions for the eigenvalues, we obtain the following bounds on α :

$$\frac{-1}{d^2 - 1} \leq \alpha \leq 1$$

PART (III)

Here we wish to find the general form of density matrices for bipartite systems which are invariant under transformations of type $U \otimes U^*$. We consider a basis where U is diagonal, that is:

$$U = \sum_k \lambda_k |k\rangle\langle k| \qquad |\lambda_k| = 1$$

We begin by writing such an arbitrary density matrix ρ in this basis:

$$\rho = \sum_{ijkl} \rho_{ijkl} |ij\rangle\langle k\ell| \tag{1}$$

We now operate on ρ with the transformation:

$$(U \otimes U^*)\rho(U \otimes U^*)^\dagger = \sum_{ijkl} \lambda_i \lambda_j^* \lambda_k^* \lambda_\ell \rho_{ijkl} |ij\rangle\langle k\ell|$$

Now, if $\rho = (U \otimes U^*)\rho(U \otimes U^*)^\dagger$, the matrix elements in this basis must match:

$$\rho_{ijkl} = \lambda_i \lambda_j^* \lambda_k^* \lambda_\ell \rho_{ijkl}$$

These condition must be satisfied for arbitrary U —that is, for any set of eigenvalues λ_i —which can only happen if either:

- $i = j$ and $k = \ell$,
- $i = k$ and $j = \ell$, or
- $\rho_{ijkl} = 0$.

Thus, we can split the sum (1) into three terms, one with $i = j, k = \ell$, one with $i = k, j = \ell$, and a third with terms for which $i = j = k = \ell$:

$$\rho = \sum_{i \neq k} \rho_{iikk} |ii\rangle\langle kk| + \sum_{i \neq j} \rho_{ijij} |ij\rangle\langle ij| + \sum_i \rho_{iiii} |ii\rangle\langle ii|$$

Now, the matrix elements of ρ must be invariant under any unitary transformation, and in particular, under any permutation of the basis vectors. Thus, the elements ρ_{iikk} cannot depend on i, k , the elements ρ_{ijij} cannot depend on i, j , and the elements ρ_{iiii} cannot depend on i .

If you're inclined to disbelieve us here, please just consider the action of $X \otimes X$ on matrices of this form:

$$(X \otimes X)\rho(X \otimes X)^\dagger = (X \otimes X) \begin{pmatrix} C1 & 0 & 0 & A1 \\ 0 & B2 & 0 & 0 \\ 0 & 0 & B3 & 0 \\ A4 & 0 & 0 & C4 \end{pmatrix} (X \otimes X)^\dagger = \begin{pmatrix} C4 & 0 & 0 & A4 \\ 0 & B3 & 0 & 0 \\ 0 & 0 & B2 & 0 \\ A1 & 0 & 0 & C1 \end{pmatrix}$$

Dear reader, your skepticism was unfounded.

So ρ must take the form:

$$\rho = A \sum_{i \neq k} |ii\rangle\langle kk| + B \sum_{i \neq j} |ij\rangle\langle ij| + C \sum_i |ii\rangle\langle ii| = \begin{pmatrix} C & 0 & 0 & A \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ A & 0 & 0 & C \end{pmatrix}$$

We now consider the action of the Hadamard $H \otimes H$ on this state:

$$(H \otimes H)\rho(H \otimes H)^\dagger = \frac{1}{2} \begin{pmatrix} A+B+C & 0 & 0 & A-B+C \\ 0 & -A+B+C & -A-B+C & 0 \\ 0 & -A-B+C & -A+B+C & 0 \\ A-B+C & 0 & 0 & A+B+C \end{pmatrix}$$

and we see that we must have $C = A + B$. Finally:

$$\rho = A \sum_{ik} |ii\rangle\langle kk| + B \sum_{ij} |ij\rangle\langle ij| = A d |\Phi_{00}\rangle\langle\Phi_{00}| + B \mathbb{1}$$

and, since ρ must have unit trace, we see that it must be of form $\rho = \rho_{\text{iso}}(\alpha)$ for some α .

Finally, the identity matrix is invariant under any unitary transformation, and it's trivial to verify that $|\Phi_{00}\rangle$ is invariant under transformations of form $U \otimes U^*$. So every isotropic state is invariant under these transformations.

Thus, a density matrix ρ is invariant under such transformations if and only if it is an isotropic state.

PART (IV)

A bipartite state ρ_{AB} is entangled if $\rho_A \otimes \mathbb{1}_B \not\geq \rho_{AB}$.

We begin by calculating the reduced density matrix of our state ρ_{iso} .

$$\text{Tr}_B \rho_{\text{iso}} = \text{Tr}_B \left[\alpha \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle\langle jj| + (1-\alpha) \frac{\mathbb{1}}{d^2} \right] = \alpha \frac{1}{d} \mathbb{1}_A + \frac{1-\alpha}{d^2} d \mathbb{1}_A = \frac{\mathbb{1}_A}{d}$$

The eigenvalue equation for the operator $\rho_A \otimes \mathbb{1}_B - \rho_{AB}$ becomes:

$$\left[\frac{1}{d} \mathbb{1}_A \otimes \mathbb{1}_B - \alpha |\Phi_{00}\rangle\langle\Phi_{00}| - \frac{1-\alpha}{d^2} \mathbb{1}_A \otimes \mathbb{1}_B \right] |\lambda\rangle = \lambda |\lambda\rangle$$

which, following the same procedure as before, we may rearrange to obtain:

$$|\Phi_{00}\rangle\langle\Phi_{00}| |\lambda\rangle = \frac{-1}{\alpha} \left(\lambda - \frac{1}{d} + \frac{1-\alpha}{d^2} \right) |\lambda\rangle$$

so our eigenvalues are found by solving:

$$\frac{-1}{\alpha} \left(\lambda_0 - \frac{1}{d} + \frac{1-\alpha}{d^2} \right) = 0 \qquad \frac{-1}{\alpha} \left(\lambda_1 - \frac{1}{d} + \frac{1-\alpha}{d^2} \right) = 1$$

which produces:

$$\lambda_0 = \frac{1}{d} - \frac{1-\alpha}{d^2} \qquad \lambda_1 = \frac{1}{d} - \frac{1-\alpha}{d^2} - \alpha$$

Now, ρ_{iso} is entangled either of these eigenvalues is negative. But $\lambda_0 < 0$ when $\alpha < 1-d$, which never occurs for $d \geq 2$, and $\lambda_1 < 0$ when $\alpha > 1/(1+d)$. So we have entanglement when:

$$\alpha > \frac{1}{1+d}$$

PART (V)

A bipartite state ρ_{AB} is entangled if $\rho_{AB}^{T_A} \not\geq 0$.

We must calculate the partial transpose of our density matrix:

$$\rho_{\text{iso}}^{T_A} = \left[\alpha \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle\langle jj| + (1-\alpha) \frac{\mathbb{1}}{d^2} \right]^{T_A} = \alpha \frac{1}{d} \sum_{i,j=0}^{d-1} |ji\rangle\langle ij| + (1-\alpha) \frac{\mathbb{1}}{d^2}$$

we again rearrange the eigenvalue equation to obtain:

$$\sum_{i,j=0}^{d-1} |ji\rangle\langle ij| |\lambda\rangle = \frac{d}{\alpha} \left(\lambda - \frac{1-\alpha}{d^2} \right) |\lambda\rangle$$

We now notice that the matrix on the left hand side squares to the identity, and so its eigenvalues are ± 1 , and the eigenvalues we're looking for are:

$$\lambda_{\pm 1} = \frac{1-\alpha}{d^2} \pm \frac{\alpha}{d}$$

Our state ρ_{iso} is entangled either of these eigenvalues is negative. But $\lambda_{+1} < 0$ when $\alpha < \frac{1}{1+d}$, which never occurs for $d \geq 2$, and $\lambda_{-1} < 0$ when $\alpha > 1/(1+d)$, exactly as we found in the previous part.

PART (VI)

We now narrow our attention to $d = 2$, where ρ_{iso} is entangled for $\frac{1}{3} < \alpha \leq 1$, and we are asked whether the operator $W \equiv |\Phi_{00}\rangle\langle\Phi_{00}|^{T_A}$ is an entanglement witness for ρ_{iso} . We must evaluate:

$$\begin{aligned} \text{Tr}[W\rho_{\text{iso}}] &= \text{Tr}\left[|\Phi_{00}\rangle\langle\Phi_{00}|^{T_A} \left(\alpha |\Phi_{00}\rangle\langle\Phi_{00}| + (1-\alpha) \frac{\mathbb{1}}{d^2} \right)\right] \\ &= \alpha \text{Tr}\left[|\Phi_{00}\rangle\langle\Phi_{00}|^{T_A} |\Phi_{00}\rangle\langle\Phi_{00}|\right] + \frac{1-\alpha}{d^2} \text{Tr}\left[|\Phi_{00}\rangle\langle\Phi_{00}|^{T_A}\right] \\ &= \alpha \cdot \frac{1}{2} + \frac{1-\alpha}{4} \cdot 1 = \frac{\alpha+1}{4} \end{aligned}$$

This quantity is negative only when $\alpha < -1$, which is disallowed, so this is not a good entanglement witness.

On the other hand, the operator $W \equiv \frac{1}{2} - |\Phi_{00}\rangle\langle\Phi_{00}|$ is a good entanglement witness for this family of states, since:

$$\begin{aligned} \text{Tr}[W\rho_{\text{iso}}] &= \text{Tr}\left[\left(\frac{1}{2} - |\Phi_{00}\rangle\langle\Phi_{00}|\right) \left(\alpha |\Phi_{00}\rangle\langle\Phi_{00}| + (1-\alpha) \frac{\mathbb{1}}{d^2} \right)\right] \\ &= \text{Tr}\left[\frac{1}{2} \alpha |\Phi_{00}\rangle\langle\Phi_{00}| - |\Phi_{00}\rangle\langle\Phi_{00}| \alpha |\Phi_{00}\rangle\langle\Phi_{00}| + \frac{1}{2} \frac{1-\alpha}{4} \mathbb{1} - |\Phi_{00}\rangle\langle\Phi_{00}| \frac{1-\alpha}{4}\right] \\ &= \frac{\alpha}{2} - \alpha + \frac{1}{2} \frac{1-\alpha}{4} \cdot 4 - \frac{1-\alpha}{4} = \frac{1-3\alpha}{4} \end{aligned}$$

This quantity is negative when $\alpha > \frac{1}{3}$, as required.

Furthermore, for any separable state $|\psi\rangle$:

$$\text{Tr}[W|\psi\rangle\langle\psi|] = \text{Tr}\left[\left(\frac{1}{2} - |\Phi_{00}\rangle\langle\Phi_{00}|\right) |\psi\rangle\langle\psi|\right] = \frac{1}{2} \text{Tr}[|\psi\rangle\langle\psi|] - \text{Tr}[|\Phi_{00}\rangle\langle\Phi_{00}| |\psi\rangle\langle\psi|] = \frac{1}{2} - |\langle\Phi_{00}|\psi\rangle|^2 \geq 0$$

since the maximum overlap between a separable state and the maximally entangled state is $\frac{1}{\sqrt{2}}$.

Next, we're asked for an entanglement witness for the family of states:

$$\rho(\alpha) \equiv \alpha |\psi\rangle\langle\psi| + (1-\alpha) \frac{\mathbb{1}}{d^2} \qquad |\psi\rangle = a|01\rangle + b|10\rangle$$

Since $\frac{1}{2} - |\Phi_{00}\rangle\langle\Phi_{00}|$ was a good entanglement witness for the states $\rho_{\text{iso}}(\alpha)$, we're motivated to try the operator $W \equiv \frac{1}{2} - |\Psi_+\rangle\langle\Psi_+|$ for these states. Since this operator has almost the same structure as the previous operator, the proof that $\text{Tr}[W |\psi\rangle\langle\psi|] > 0$ for all separable states $|\psi\rangle$ is identical. It remains to show that $\text{Tr}[W\rho(\alpha)] < 0$ for entangled states from the family $\rho(\alpha)$.

TODO TODO

QUESTION 5

The completely dephasing channel is:

$$\Lambda[\rho] \equiv \text{Tr}_2 \left[U_{\text{CNOT}}(\rho \otimes |0\rangle\langle 0|) U_{\text{CNOT}}^\dagger \right] \quad U_{\text{CNOT}} \equiv |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$$

It's straightforward to expand and simplify this definition:

$$\begin{aligned} \Lambda[\rho] &= \text{Tr}_2 [(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X)(\rho \otimes |0\rangle\langle 0|)(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X)] \\ &= \text{Tr}_2 [(|0\rangle\langle 0| \rho |0\rangle\langle 0|) \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \rho |1\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \rho |0\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \rho |1\rangle\langle 1| \otimes |1\rangle\langle 1|] \\ &= |0\rangle\langle 0| \rho |0\rangle\langle 0| + |1\rangle\langle 1| \rho |1\rangle\langle 1| \end{aligned}$$

Luckily, $|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$, and these projectors are positive semidefinite, and so the Kraus operators are just:

$$K_0 = \Pi_0 = |0\rangle\langle 0|, \quad K_2 = \Pi_1 = |1\rangle\langle 1|$$

QUESTION 6

We're asked to show that partial transpose of any density matrix representing a bipartite system of two cubits has at most one negative eigenvalue. Our proof follows the outline given in Sanpera 1998.⁵

CONDITION FOR A STATE TO BE A PRODUCT STATE

An arbitrary product state may be written:

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = ac|00\rangle + bd|11\rangle + ad|01\rangle + bc|10\rangle$$

and by considering this expression, we obtain a condition for a vector to be a product state:

$$WZ = XY \quad \Leftrightarrow \quad W|00\rangle + Z|11\rangle + X|01\rangle + Y|10\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

That is, the above state is a product state, and may be factorized by choosing a, b, c, d such that $ac = W, bd = Z, ad = X, bc = Y$, if and only if $WZ = XY$.

EVERY TWO-DIMENSIONAL SUBSPACE CONTAINS A PRODUCT STATE

We now ask if it's possible for the two-dimensional subspace spanned by two eigenvectors, $|\lambda_1\rangle$ and $|\lambda_2\rangle$, of a Hermitian matrix to contain only entangled states. To investigate this question, we choose a basis $\{|i\rangle \otimes |j\rangle\}$ of $\mathcal{H}_A \otimes \mathcal{H}_B$, with $\{|i\rangle\}$ a basis of \mathcal{H}_A and $\{|j\rangle\}$ a basis of \mathcal{H}_B , so that $|\lambda_1\rangle$ takes the simple form:

$$|\lambda_1\rangle = \alpha|00\rangle + \beta|11\rangle$$

and then, since the eigenvectors are orthogonal, we must be able to write $|\lambda_2\rangle$ in the form:

$$|\lambda_2\rangle = \epsilon(\beta|00\rangle + \gamma|01\rangle + \delta|10\rangle - \alpha|11\rangle)$$

⁵Sanpera, Tarrach, and Vidal, Physical Review A, 58, 1998.

We will not insist that $|\lambda_2\rangle$ be normalized, and so, without loss of generality, we may take $\epsilon = 1$, leaving:

$$|\lambda_2\rangle = \beta^* |00\rangle + \gamma |01\rangle + \delta |10\rangle - \alpha^* |11\rangle$$

Consider an arbitrary element of the two-dimensional subspace, which must be a linear combination of $|\lambda_1\rangle$ and $|\lambda_2\rangle$:

$$\begin{aligned} A|\lambda_1\rangle + B|\lambda_2\rangle &= A(\alpha |00\rangle + \beta |11\rangle) + B(\beta^* |00\rangle + \gamma |01\rangle + \delta |10\rangle - \alpha^* |11\rangle) \\ &= (A\alpha + B\beta^*) |00\rangle + (A\beta - B\alpha^*) |11\rangle + B\gamma |01\rangle + B\delta |10\rangle \end{aligned}$$

According to the condition established above, this is a product state if:

$$(A\alpha + B\beta^*) \cdot (A\beta - B\alpha^*) = B\gamma \cdot B\delta$$

Now, our subspace contains only entangled states if there is no choice of A, B which satisfies this equation. But this is just a single equation, quadratic in the variables A and B , and may always be solved over the complex numbers.

So the subspace contains at least one product state.

THE PARTIAL TRANSPOSE CANNOT HAVE TWO NEGATIVE EIGENVALUES

The remaining part of the proof is by contradiction.

Suppose we have a density matrix $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ representing the bipartite state of two cubits, and suppose that ρ^{TA} has two negative eigenvalues $\lambda_1, \lambda_2 < 0$ for corresponding eigenvectors $|\lambda_1\rangle, |\lambda_2\rangle$, which are necessarily orthogonal, since ρ^{TA} is Hermitian.

For some coefficients A, B we have a linear combination which is a product state:

$$A|\lambda_1\rangle + B|\lambda_2\rangle = |a\rangle \otimes |b\rangle$$

It's clear that the following sandwich is strictly negative:

$$\begin{aligned} \langle a| \langle b| \rho^{TA} |a\rangle |b\rangle &= (A^* \langle \lambda_1| + B^* \langle \lambda_2|) \rho^{TA} (A |\lambda_1\rangle + B |\lambda_2\rangle) \\ &= (A^* \langle \lambda_1| + B^* \langle \lambda_2|) (A \lambda_1 |\lambda_1\rangle + B \lambda_2 |\lambda_2\rangle) \\ &= |A|^2 \lambda_1 + |B|^2 \lambda_2 < 0 \end{aligned}$$

But, writing $\rho = \sum_{ijkl} \rho_{ij,kl} |ij\rangle\langle kl|$, we may expand the meat of this sandwich as follows:

$$\begin{aligned} 0 > \langle a| \langle b| \rho^{TA} |a\rangle |b\rangle &= \langle a| \langle b| \left[\sum_{ijkl} \rho_{kj,kl} |ij\rangle\langle il| \right] |a\rangle |b\rangle \\ &= \sum_{ijkl} \rho_{kj,kl} \langle a|k\rangle \langle b|j\rangle \langle i|a\rangle \langle \ell|b\rangle \\ &= \sum_{ijkl} \rho_{kj,kl} \langle a|i\rangle^* \langle b|j\rangle \langle k|a\rangle^* \langle \ell|b\rangle \\ &= \langle a^*| \langle b| \rho |a^*\rangle |b\rangle \end{aligned}$$

And so ρ is not positive semidefinite, contradicting the assumption that it is a density matrix.

We conclude that ρ^{TA} has at most one negative eigenvalue.

QUESTION 7

We're presented with the family of maps parameterized by $0 \leq \alpha \leq 1$ and defined by:

$$\Lambda_\alpha(\rho) = \frac{1}{2}\mathbb{1} + \alpha(X\rho Z + Z\rho X)$$

This is manifestly not a linear function of 2×2 matrices, nor trace-preserving, nor is it completely positive, so we need to turn it into something that is all of these things.⁶

Fortunately, we see that the maximally-mixed state $\frac{1}{2}\mathbb{1}$ can be written as a map acting on a unit trace density matrix ρ :

$$\frac{1}{2}\mathbb{1} = \pi(\rho) \quad \text{where} \quad \pi(\rho) \equiv \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z) = \frac{1}{2} \text{Tr}\{\rho\} \mathbb{1}$$

So we can replace the definition of Λ_α with the following redefinition:

$$\Lambda_\alpha(\rho) = \pi(\rho) + \alpha(X\rho Z + Z\rho X) \quad (2)$$

whenever we need a proper linear map.

PART (I)

We may express the eigenvalue equation for the matrix $\Lambda_\alpha(\rho)$ as:

$$(X\rho Z + Z\rho X) |\lambda\rangle = \frac{1}{\alpha} \left(\lambda - \frac{1}{2} \right) |\lambda\rangle$$

To solve this equation, we will express the density matrix ρ in the Pauli basis $\rho = \frac{1}{2}(\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma})$ where $\|\mathbf{r}\| \leq 1$. Then:

$$\begin{aligned} X\rho Z &= X \frac{1}{2}(r_0\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma})Z = \frac{1}{2}(ir_2\mathbb{1} + r_3X - iY + r_1Z) \\ Z\rho X &= (X\rho Z)^\dagger = \frac{1}{2}(-ir_2\mathbb{1} + r_3X + iY + r_1Z) \end{aligned}$$

and so:

$$X\rho Z + Z\rho X = r_3X + r_1Z$$

The eigenvalues of this matrix are easily found to be $\pm\sqrt{r_1^2 + r_3^2}$, and therefore:

$$\lambda = \frac{1}{2} \pm \alpha\sqrt{r_1^2 + r_3^2}$$

The maximum possible value of $\sqrt{r_1^2 + r_3^2}$ is 1, and so both eigenvalues are non-negative for every ρ when $\alpha \leq \frac{1}{2}$.

The resulting density matrix is:

$$\Lambda_\alpha(\rho) = \frac{1}{2}(\mathbb{1} + 2\alpha r_3X + 2\alpha r_1Z)$$

in the Pauli basis, and so we see that the effect of the map was to:

- project away the Y component of ρ ,
- swap and—depending on α —rescale the X and Z components.

When $\alpha > \frac{1}{2}$, the rescaling is an unphysical expansion of the Bloch sphere; if $\alpha < \frac{1}{2}$, it is a contraction of the Bloch sphere, and the resulting state is more mixed than the original state.

The final state lies in the $X - Z$ plane of the Bloch sphere.

PART (II)

To find the Choi matrix for these maps, we work from the definition:

$$J(\Lambda_\alpha) \equiv (\mathbb{1} \otimes \Lambda_\alpha)(|\Omega\rangle\langle\Omega|) = \sum_{i,j=0}^{d-1} |i\rangle\langle j| \otimes \Lambda_\alpha(|i\rangle\langle j|)$$

⁶A quantum map must be linear; it does not make sense to talk about a “linear” map that operates only on unit trace matrices, because the set of unit trace matrices is not closed under vector addition, nor under scalar multiplication.

so all we need to do is apply Λ_α as redefined in (2) to each $|i\rangle\langle j|$, resulting in:

$$J(\Lambda_\alpha) = \frac{1}{2} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} + \alpha \begin{pmatrix} X & Z \\ Z & -X \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

This is manifestly the Choi state for a trace-preserving map, and the eigenvalues of this matrix are:

$$\frac{1}{2}, \frac{1}{2}(1 \pm 4\alpha)$$

which are all non-negative when $\alpha \leq \frac{1}{4}$. Thus, Λ_α is a completely positive map when $\alpha \leq \frac{1}{4}$.

PART (III)

For $\alpha = \frac{1}{4}$, the eigenvalues and eigenvectors of the Choi matrix are:

$$\frac{1}{2}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \frac{1}{2}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad 0, \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad 1, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

and we may rearrange the elements of these vectors to obtain the Kraus operators:

$$K_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\mathbb{1}}{2} \quad K_2 = \frac{i}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{Y}{2} \quad K_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{H}{\sqrt{2}}$$

(The eigenvector with eigenvalue 0 does not give rise to a Kraus operator.)

It's trivial to verify that $\sum K_i^\dagger K_i = \mathbb{1}$ as befits a list of Kraus operators and that each $K_i^\dagger K_i$ is positive definite. Therefore:

$$\Lambda_{\frac{1}{4}}(\rho) = \frac{1}{4}\rho + \frac{1}{4}Y\rho Y + \frac{1}{2}H\rho H$$