MASTER IN QUANTUM SCIENCE AND TECHNOLOGY

Quantum Information Theory
Second Assignment

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1 For each of the following set of matrices determine whether they are admissible quantum measurements. Determine also whether they are projective measurements or POVMs.

(i)

$$E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad E_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ \frac{-1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sum_m E_m = E_1 + E_2 + E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{I}$$

The E_m operators satisfy the completeness relation. Furthermore, the operators E_m are positive because all their eigenvalues are positive:

$$\det(E_1 - \lambda I) = 0 \Longrightarrow \lambda_1 = 1; \quad \lambda_2 = \frac{1}{3}; \quad \lambda_3 = 0; \quad \lambda_4 = 0$$

$$\det(E_2 - \lambda I) = 0 \Rightarrow \lambda_1 = 1; \quad \lambda_2 = \frac{1}{3}; \quad \lambda_3 = 0; \quad \lambda_4 = 0$$

$$\det(E_3 - \lambda I) = 0 \Rightarrow \lambda_1 = \frac{2}{3}; \quad \lambda_2 = \frac{2}{3}; \quad \lambda_3 = 0; \quad \lambda_4 = 0$$

they are clearly positive operators, so they are admisible quantum measurements. We also have that $E_1^\dagger = E_1, E_2^\dagger = E_2$ and $E_3^\dagger = E_3$. Now let's check if they are idempotent

$$E_1 \cdot E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \neq E_1$$

We see that E_1 is not idempotent, so is not a projective measurement $\Rightarrow E_1, E_2$ and E_3 describe a POVMs.

(ii)

$$E_1 = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} \end{pmatrix} \quad E_2 = \begin{pmatrix} \frac{1}{3} & 0 & \frac{-1}{2} \\ 0 & 0 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{3} \end{pmatrix} \quad E_3 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\sum_m E_m = E_1 + E_2 + E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}$$

To check whether they are positive let's compute their eigenvalues:

$$\det\left(E_1 - \lambda I\right) = 0 \Rightarrow \lambda_1 = \frac{5}{6}; \quad \lambda_2 = \frac{-1}{6}; \quad \lambda_3 = 0$$

Given that one of the eigenvalues of E_1 is negative, E_1 is non positive and it cannot represent a valid measurement operator $\Rightarrow$ The set $\{E_1, E_2, E_3\}$ do not represent a valid measurement.

(iii)

$$E_1 = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad E_2 = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

$$E_3 = \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \quad E_4 = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\sum_\infty E_m = E_1 + E_2 + E_2 + E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The operators do not satisfy the completeness relation $\Rightarrow$ they are not a valid measurement.

(iv)

$$E_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \quad E_2 = \frac{1}{2}\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3}e^{-i\frac{2\pi}{3}} \\ \frac{\sqrt{2}}{3}e^{i\frac{2\pi}{3}} & \frac{2}{3} \end{pmatrix}$$

$$E_3 = \frac{1}{2}\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3}e^{-i\frac{4\pi}{3}} \\ \frac{\sqrt{2}}{3}e^{i\frac{4\pi}{3}} & \frac{2}{3} \end{pmatrix}; E_4 = \frac{1}{2}\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix}$$

$$\sum_m E_m = E_1 + E_2 + E_3 + E_4 =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I} \rightarrow \text{ Completeness relation}$$

Let's check whether they are positive or not:

$$\det\left(E_1-\lambda I\right)=0\Rightarrow\lambda_1=0;\quad\lambda_2=\frac{1}{2}$$
$$\det\left(E_2-\lambda I\right)=0\Rightarrow\lambda_1=0;\quad\lambda_2=\frac{1}{2}$$
$$\det\left(E_3-\lambda I\right)=0\Rightarrow\lambda_1=0;\quad\lambda_2=\frac{1}{2}$$
$$\det\left(E_4-\lambda\mathbb{I}\right)=0\Rightarrow\lambda_1=0;\quad\lambda_2=\frac{1}{2}$$

The operators E_m are also positive, so they are a valid measurement.
Let's see if they are projective measurements:

$$E_1\cdot E_1=\begin{pmatrix}\frac{1}{4}&0\\0&0\end{pmatrix}\neq E_1$$

Then E_1 is not a projective measurement $\Rightarrow$The operators E_m are POVMs.

2 Consider the following Choi-Jamiolkowski state of a map:

$$J(\Lambda)=\frac{1}{2}\begin{pmatrix}1&0&0&\sqrt{1-\gamma}\\0&0&0&0\\0&0&\gamma&0\\\sqrt{1-\gamma}&0&0&1-\gamma\end{pmatrix}$$

(i) Determine the Kraus operators of the map Λ

$$J(\Lambda)=\frac{1}{2}\begin{pmatrix}1&0&0&\sqrt{1-\gamma}\\0&0&0&0\\0&0&\gamma&0\\\sqrt{1-\gamma}&0&0&1-\gamma\end{pmatrix}$$

$$2J(\Lambda)=\begin{pmatrix}1&0&0&\sqrt{1-\gamma}\\0&0&0&0\\0&0&\gamma&0\\\sqrt{1-\gamma}&0&0&1-\gamma\end{pmatrix}$$

We diagonalise $2J(\Lambda)$ for simplicity

$$\begin{vmatrix}1-\lambda&0&0&\sqrt{1-\gamma}\\0&-\lambda&0&0\\0&0&\gamma-\lambda&0\\\sqrt{1-\gamma}&0&0&1-\gamma-\lambda\end{vmatrix}=0$$

$$\begin{vmatrix}-\lambda&0\\0&\gamma-\lambda\end{vmatrix}=0\qquad\lambda_1=0\ \ \lambda_2=\gamma$$

$$\begin{vmatrix} 1-\lambda & \sqrt{1-\gamma} \\ \sqrt{1-\gamma} & 1-\gamma-\lambda \end{vmatrix} = 0 \qquad \lambda_3 = 0 \quad \lambda_4 = 2-\gamma$$

The only non-zero eigenvalues are $\lambda = \gamma$ and $\lambda = 2-\gamma$

The eigenvector associated to $\lambda = \gamma$ is, trivially, $|v_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$

The eigenvector associated to $\lambda = 2-\gamma$ is:

$$\begin{pmatrix} 1-2+\gamma & 0 & 0 & \sqrt{1-\gamma} \\ 0 & -2+\gamma & 0 & 0 \\ 0 & 0 & -2+2\gamma & 0 \\ \sqrt{1-\gamma} & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0 \Rightarrow b = 0, c = 0$$

$$\Rightarrow \sqrt{1-\gamma}a - d = 0 \quad \Big|v_2\Big\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \sqrt{1-\gamma} \end{pmatrix} C$$

where C is the normalization constant whose value is $C = \frac{1}{\sqrt{2-\gamma}} \Rightarrow |v_2\rangle = \begin{pmatrix} 1/\sqrt{2-\gamma} \\ 0 \\ \\ \sqrt{\frac{1-\gamma}{2-\gamma}} \end{pmatrix}$

Then $2J(\Lambda) = \sum_i \lambda_i \left|v_i\right\rangle \left\langle v_i\right|$

$$\begin{aligned} 2J(\Lambda) = \gamma|10\rangle\langle 10| + (2-\gamma)\left(\frac{1}{\sqrt{2-\gamma}}|00\rangle + \sqrt{\frac{1-\gamma}{2-\gamma}}|11\rangle\right) \\ \times \left(\frac{1}{\sqrt{2-\gamma}}\langle 00| + \sqrt{\frac{1-\gamma}{2-\gamma}}\langle 11|\right) \end{aligned}$$

Then, we can obtain the kraus from $\sqrt{\lambda_i}\left|v_i\right\rangle$ expressed as a matrix and transposing that matrix.

- For $\lambda = \gamma \quad \sqrt{\lambda_i}|v_i\rangle = \begin{pmatrix} 0 \\ 0 \\ \sqrt{\gamma} \\ 0 \end{pmatrix} \quad K_0 = \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}^\top = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$

- For $\lambda = 2-\gamma \quad \sqrt{\lambda_i}|v_i\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \sqrt{1-\gamma} \end{pmatrix} \quad K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$

Finally

$$K_0 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} = \sqrt{\gamma}|0\rangle\langle 1|$$

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} = 1|0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$$

(ii) $M_{ij,lk} = \text{Tr}[|k\rangle\langle l|\Lambda(|i\rangle\langle j|)]$

$$M_{i,j,lk} = \text{Tr}\left[|k\rangle\langle l|\Big(\gamma|0\rangle\langle 1|\ i\rangle\langle j|1\rangle\langle 0|+(|0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|)|i\rangle\langle j| \times(|0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|\Big)\right]$$

$$= \text{Tr}\left[|k\rangle\langle l|\left(\gamma|0\rangle\langle 0|\delta_{i1}\delta_{j1} + (|0\rangle\langle 0|i\rangle\langle j| + \sqrt{1-\gamma}|1\rangle\langle 1|i\rangle\langle j|)(|0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|)\right)\right]$$

$$M_{ij,lk} = \text{Tr}\left[|k\rangle\langle l|\left(\gamma|0\rangle\langle 0\left|\delta_{i1}\delta_{j1}+\right|0\rangle\langle 0|\delta_{i0}\delta_{j0} + \sqrt{1-\gamma}|0\rangle\langle 1|\delta_{i0}\delta_{j1} + \sqrt{1-\gamma}|1\rangle\langle 0|\delta_{i1}\delta_{j0} + (1-\gamma)|1\rangle\langle 1|\delta_{i1}\delta_{j1}\right]$$

With this simplified expression

- $i_j = 00$

$$M_{00,lk} = \text{Tr}\left(|k\rangle\langle 0|\delta_{l0}\right) = \delta_{l0}\delta_{k0}$$

- $ij = 01$

$$M_{01,lk} = \text{Tr}(\sqrt{1-\gamma}|k\rangle\langle e|0\rangle\langle 1|) = \sqrt{1-\gamma}\delta_{l0}\delta k1$$

- $ij = 10$

$$M_{10,lk} = \text{Tr}(\sqrt{1-\gamma}|k\rangle\langle l|1\rangle\rangle 0| = \sqrt{1-\gamma}\delta_{l1}\delta_{k0}$$

- $ij = 11$

$$M_{11,lk} = \text{Tr}(\gamma|k\rangle\langle l|0\rangle\langle 0| + (1-\gamma)|1\rangle\langle l|1\rangle\langle k| = \gamma\delta_{l0}\delta_{k0} + (1-\gamma)\delta_{l1}\delta_{k1}$$

- Finally:

$$M(\Lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ \gamma & 0 & 0 & (1-\gamma) \end{pmatrix}$$

3 Exercise 3

i) The state $|\psi^-\rangle_{AB}$ in vectorial form is

$$|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] =$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

A general unitary matrix U is

$$U = \begin{pmatrix} g & y \\ -y^* & g^* \end{pmatrix}$$

with the condition

$$gg^* + yy^* = 1$$

Let's apply $U \otimes U$ on each term of $|\psi^-\rangle_{AB}$

$$\begin{aligned} U \otimes U|0\rangle_A|1\rangle_B &= (U|0\rangle_A) \otimes (U|\nabla\rangle_B) = \\ &= \begin{pmatrix} g & y \\ -y^* & g^* \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} g & y \\ -y^* & g^* \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \\ &= \begin{pmatrix} g \\ -y^* \end{pmatrix} \otimes \begin{pmatrix} y \\ g^* \end{pmatrix} = \begin{pmatrix} gy \\ gg^* \\ -yy^* \\ -y^*g^* \end{pmatrix} \end{aligned}$$

$$\begin{aligned} U \otimes U|1\rangle_A|0\rangle_B &= (U|1\rangle_A) \otimes (U10\rangle_B) = \\ &= \begin{pmatrix} y \\ g^* \end{pmatrix} \otimes \begin{pmatrix} g \\ -y^* \end{pmatrix} = \begin{pmatrix} yg \\ -yy^* \\ +g^*g \\ -g^*y^* \end{pmatrix} \end{aligned}$$

$$U \otimes U \left|\psi^-\right\rangle_{AB} = \begin{pmatrix} gy - yg \\ gg^* + yy^* \\ -\left(gg^* + yy^*\right) \\ -y^*g^* + g^*y^* \end{pmatrix} =$$

Finally we obtain

$$= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = |\psi-\rangle_{AB}$$

which is the same state. Thus $|\psi-\rangle_{AB}$ is invariant under unitaries of the form $U \otimes U$.

ii) The two entangled states are

$$|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|W\rangle_{ABC} = \tfrac{1}{\sqrt{3}} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] = \tfrac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Remenbering the result of applying our general unitary U given before to the states of the computational basis $U|0\rangle = \begin{pmatrix} g \\ -y^* \end{pmatrix} \quad U|1\rangle = \begin{pmatrix} y \\ g^* \end{pmatrix}$

we have that

$U \otimes U \otimes U|GHZ\rangle_{ABC} = \tfrac{1}{\sqrt{2}}[(U|0\rangle) \otimes (U|0\rangle) \otimes (U|0\rangle)$

$+(U|1\rangle) \otimes (U|1\rangle) \otimes (U|1\rangle)] =$

$$= \tfrac{1}{\sqrt{2}} \left[\begin{pmatrix} g \\ -y^* \end{pmatrix} \otimes \begin{pmatrix} g \\ -y^* \end{pmatrix} \otimes \begin{pmatrix} g \\ -y^* \end{pmatrix} + \begin{pmatrix} y \\ g^* \end{pmatrix} \otimes \begin{pmatrix} y \\ g^* \end{pmatrix} \otimes \begin{pmatrix} y \\ g^* \end{pmatrix} \right] =$$

$$= \tfrac{1}{\sqrt{2}} \left[\begin{pmatrix} g^2 \\ -gy^* \\ -gy^* \\ (y^*)^2 \end{pmatrix} \otimes \begin{pmatrix} g \\ -y^* \end{pmatrix} + \begin{pmatrix} y^2 \\ yg^* \\ yg^* \\ (g)^* \end{pmatrix} \otimes \begin{pmatrix} y \\ g^* \end{pmatrix} \right] =$$

$$= \frac{1}{\sqrt{2}} \left[\begin{pmatrix} g^3 \\ -g^2y^* \\ -g^2y^* \\ g\left(y^*\right)^2 \\ -g^2y^* \\ g\left(y^*\right)^2 \\ g\left(y^*\right)^2 \\ -\left(y^*\right)^3 \end{pmatrix} + \begin{pmatrix} y^3 \\ y^2g^* \\ y^2g^* \\ y\left(g^*\right)^2 \\ y^2g^* \\ y\left(g^*\right)^2 \\ y\left(g^*\right)^2 \\ \left(g^*\right)^3 \end{pmatrix} \right] =$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} g^3 + y^3 \\ -g^2y^* + y^2g^* \\ -g^2y^* + y^2g^* \\ g\left(y^*\right)^2 + y\left(g^*\right)^2 \\ -g^2y^* + y^2g^* \\ g\left(x^*\right)^2 + y\left(g^*\right)^2 \\ g\left(y^*\right)^2 + y\left(g^*\right)^2 \\ -\left(y^*\right)^3 + \left(g^*\right)^3 \end{pmatrix}$$

and the state $|W\rangle_{ABC}$ transforms as

$$U \otimes U \otimes U|W\rangle_{ABC} = \frac{1}{\sqrt{3}}[(U|0\rangle)(U|0\rangle)(U|1\rangle) + (U|0\rangle)(U|1\rangle)(U|0\rangle) + (U|1\rangle)(U|0\rangle)(U|0\rangle)] =$$

$$= \tfrac{1}{\sqrt{3}} \left[\begin{pmatrix} g \\ -y^* \end{pmatrix} \otimes \begin{pmatrix} g \\ -y^* \end{pmatrix} \otimes \begin{pmatrix} y \\ g^* \end{pmatrix} + \begin{pmatrix} g \\ -y^* \end{pmatrix} \otimes \begin{pmatrix} y \\ g^* \end{pmatrix} \otimes \begin{pmatrix} g \\ -y^* \end{pmatrix} \right.$$

$$+\begin{pmatrix} y \\ g^* \end{pmatrix} \otimes \begin{pmatrix} g \\ -y^* \end{pmatrix} \otimes \begin{pmatrix} g \\ -y^* \end{pmatrix}\Bigg]$$

$$= \tfrac{1}{\sqrt{3}}\left[\begin{pmatrix} g^2 \\ -y^*g \\ -y^*g \\ (y^*)^2 \end{pmatrix} \otimes \begin{pmatrix} y \\ g^* \end{pmatrix} + \begin{pmatrix} gy \\ gg^* \\ -yy^* \\ -y^*g^* \end{pmatrix} \otimes \begin{pmatrix} g \\ -y^* \end{pmatrix} + \begin{pmatrix} yg \\ -yy^* \\ g^*g \\ -g^*y^* \end{pmatrix} \otimes \begin{pmatrix} g \\ -y^* \end{pmatrix}\right]$$

$$= \tfrac{1}{\sqrt{3}} \cdot \left[\begin{pmatrix} g^2y \\ g^2g^* \\ -y^*y_g \\ -y^*g^*g \\ -y^*yg \\ -y^*g^*g \\ (y^*)^2 y \\ (y^*)^2 g^* \end{pmatrix} + \begin{pmatrix} y^2y \\ -gy^* \\ g^2y^* \\ -y^*g^* \\ -yy^*g \\ y \cdot (y^*)^2 \\ -y^*g^*g \\ (y^*)^2 g^* \end{pmatrix} + \begin{pmatrix} yg^2 \\ -yy^*g \\ -yy^*g \\ y\,(y^*)^2 \\ g^*g^2 \\ -y^*g^*g \\ -g^*y^*g \\ g^*\,(y^*)^2 \end{pmatrix}\right] = \tfrac{1}{\sqrt{3}}\begin{pmatrix} 3g^2y \\ g^2g^* - 2gyy^* \\ g^2g^* - 2yy^*g \\ -2y^*gg^* + y\,(y^*)^2 \\ -2yy^*g + g^*g^2 \\ -2y^*g^*g + y\,(y^*)^2 \\ (y^*)^2\, y - 2y^*g^*g \\ 3\,(y^*)^2\, g^* \end{pmatrix}$$

iii) We are going to prove that such transformation is not possible proving that it does not exist unitary U_A for such transformation. First we have the next reduced density matrix of the A system in the $|W\rangle\langle$ state

$$\rho_w = |W\rangle\langle W| = \frac{1}{3}\left(|001\rangle + |010\rangle + |100\rangle\right) \cdot \left(\langle 001| + \langle 010| + \langle 100|\right)$$

$$\rho_w^A = \mathrm{tr}_{BC}\left(\rho_w\right) = \frac{1}{3}(|0\rangle\langle 0| + |0\rangle\langle 0| + |1\rangle\langle 1|) = = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

The reduced density matrix of the A system in the $|GHZ\rangle$ state after aplying the transformation is

$$\mathrm{tr}_{BC}\left(U_A \otimes U_B \otimes U_C |GHZ\rangle\langle GHZ| U_A^\dagger \otimes U_B^\dagger \times U_C^\dagger\right) =$$
$$= t_{VBC}\left[U_A \otimes U_B \otimes U_C \left(\frac{1}{2}(|000\rangle\langle 000| + |000\rangle\langle 111| + \right.\right.$$
$$\left.+|111\rangle\langle 000| + |111\rangle\langle 111|) U_A^\dagger \otimes U_B^\dagger \otimes U_C^\dagger\right] =$$
$$= tr_B\left(\frac{1}{2}\left[U_A \otimes U_B |00\rangle\langle 00| U_A^\dagger \otimes U_B^\dagger + 0 + 0 + \right.\right.$$
$$\left.\left.+U_A \otimes U_B \langle 11\rangle\langle 11| U_A^\dagger \otimes U_B^\dagger\right]\right) = \frac{1}{2}\left(U_A|0\rangle\langle 0|U_A^\dagger\right.$$
$$\left.+U_A|1\rangle\langle 1|U_A^\dagger\right) = U_A \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} U_A^\dagger = U_A \cdot \rho_{GHZ}^A \cdot U_A^\dagger$$

Given that the matrix ρ_w^A and ρ_{GHZ}^A do not have the same eigenvalues, they cannot be unitarily related, what implies that U_A cannot exist. Therefore, it does not exist an unitary $U_A \otimes U_B \otimes U_C$ which perform the transformation from $|GHZ\rangle$ to $|W\rangle$.

4 Exercise 4

The isotropic states are a family of bipartite systems defined as:

$$\rho_{iso} = \alpha \left|\Phi_{00}\right\rangle \left\langle\Phi_{00}\right| + \frac{1-\alpha}{d^2} \left(\mathbb{I}_A \otimes \mathbb{I}_B\right).$$

i)

A proper quantum state must satisfy $\rho \geq 0$ and $Tr\left(\rho\right) = 1$. Imposing these properties to ρ_{iso}, we can find the values of α to become a proper quantum state.

First, we impose $Tr\left(\rho_{iso}\right) = 1$:

$$Tr\left(\rho_{iso}\right) = Tr\left(\alpha \left|\Phi_{00}\right\rangle \left\langle\Phi_{00}\right| + \frac{1-\alpha}{d^2} \left(\mathbb{I}_A \otimes \mathbb{I}_B\right)\right) =$$

$$= \alpha Tr\left(\left|\Phi_{00}\right\rangle \left\langle\Phi_{00}\right|\right) + \frac{1-\alpha}{d^2} Tr\left(\mathbb{I}_A \otimes \mathbb{I}_B\right) = \alpha + (1-\alpha)\frac{d^2}{d^2} = 1,$$

where in the last equation we have used the cyclic property of trace $Tr\left(\left|\Phi_{00}\right\rangle \left\langle\Phi_{00}\right|\right) = Tr\left(\left\langle\Phi_{00}|\Phi_{00}\right\rangle\right) = Tr(1) = 1$ and $Tr\left(\mathbb{I}_A \otimes \mathbb{I}_B\right) = Tr\left(\mathbb{I}_{AB}\right) = d^2$. As it can be seen, $Tr\left(\rho_{iso}\right) = 1$ is always fulfilled independently of the value α.

The next step consists of imposing that quantum states are semi-positive. The easiest way to impose positivity on a matrix is on a diagonal basis, then semi-positivity is achieved when all the diagonal elements are greater or equal to 0. Specifically for ρ_{iso}, it is enough to work on a basis where the first element is the vector $\left|\Psi_{00}\right\rangle$ and the rest are orthonormal to each other. In this basis, ρ_{iso} is written in the matrix form

$$\rho_{iso} = \begin{bmatrix} \alpha & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} + \frac{1-\alpha}{d^2} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \frac{1}{d^2} \begin{bmatrix} \left(d^2-1\right)\alpha + 1 & & & \\ & 1-\alpha & & \\ & & \ddots & \\ & & & 1-\alpha \end{bmatrix}$$

As result, ρ_{iso} is semi-positive when

$$\left(d^2-1\right)\alpha + 1 \geq 0 \Rightarrow \alpha \geq -\frac{1}{d^2-1},$$

$$1-\alpha \geq 0 \Rightarrow 1 \geq \alpha.$$

Therefore, ρ_{iso} is a proper quantum state if

$$1 \geq \alpha \geq -\frac{1}{d^2-1}$$

ii)

Using the basis defined in section i), the diagonal form of ρ_{iso} is

$$\rho_{iso} = \frac{1}{d^2} \begin{bmatrix} \left(d^2-1\right)\alpha + 1 & & & \\ & 1-\alpha & & \\ & & \ddots & \\ & & & 1-\alpha, \end{bmatrix}$$

where we have the eigenvalue $\lambda_1 = \frac{\left(d^2-1\right)\alpha+1}{d^2}$ with degeneracy 1 and $\lambda_2 = \frac{1-\alpha}{d^2}$ with degeneracy d^2-1.
iii)

In this section, we have to demonstrate that a matrix ρ is invariant under the transformations $U \otimes U^* \rho \left(U \otimes U^*\right)^\dagger$ for any unitary matrix U if and only if $\rho = \rho_{iso}$.

The first step is to find restrictions for the coefficient values of our matrix. We then take a general matrix written on some orthonormal basis in the form

$$\rho = \sum_{a,b,c,d=1}^{d} \rho_{abcd} \left|a\right\rangle \left\langle b\right| \otimes \left|c\right\rangle \left\langle d\right|$$

Now consider a unitary transformation U that interchanges $|a\rangle$ by $|a'\rangle$, where $a \neq a'$ and leaves the rest unchanged. Notice that in this case, all the elements of U are real, hence $U^* = U$. The invariance implies that $\rho_{abcd} = \rho_{a'b'c'd'}$. Because our ρ must be invariant under all these kinds of transformations, this state must have all the coefficients of the matrix equals. Now, we know that our matrix has to take the form

$$\rho = C \sum_{a,b,c,d=1}^{d} \left|a\right\rangle \left\langle b\right| \otimes \left|c\right\rangle \left\langle d\right|.$$

The next step is to find invariance under the general matrix transformation. Here we decompose the unitary transformation U in a general orthonormal basis $U \otimes U^* = \sum_{i,j,k,l} U_{ij} U^*_{kl} \left|i\right\rangle \left\langle j\right| \otimes \left|k\right\rangle \left\langle l\right|$. Then,

$$\left(U \otimes U^*\right) \rho = C \left(\sum_{i,j,k,l} U_{ij} U^*_{kl} \left|i\right\rangle \left\langle j\right| \otimes \left|k\right\rangle \left\langle l\right|\right) \left(\sum_{a,b,c,d} \left|a\right\rangle \left\langle b\right| \otimes \left|c\right\rangle \left\langle d\right|\right) =$$

$$= C \sum_{i,k} \sum_{a,b,c,d} U_{ia} U^*_{kc} \left|i\right\rangle \left\langle b\right| \otimes \left|k\right\rangle \left\langle d\right|$$

Finally, we apply the adjoint operator behind the ρ. Then,

$$\left(U \otimes U^*\right) \rho \left(U \otimes U^*\right)^\dagger = C \sum_{i,k} \sum_{a,b,c,d} U_{ia} U^*_{kc} \left|i\right\rangle \left\langle b\right| \otimes \left|k\right\rangle \left\langle d\right| \sum_{v,w,t,z} U^\dagger_{v,w} \left(U^*_{t,z}\right)^\dagger \left|v\right\rangle \left\langle w\right| \otimes \left|t\right\rangle \left\langle z\right| =$$

$$= C \sum_{w,z} \sum_{i,k} \sum_{a,b,c,d} U_{ia} U^*_{kc} U^\dagger_{bw} \left(U^*_{dz}\right)^\dagger \left|i\right\rangle \left\langle w|k\right\rangle \left\langle z\right|$$

In order to be invariant under a general transformation U we need to impose conditions into our states rather than impose conditions into the matrix elements. The only possible way that a state is invariant under any unitary transformation is that the contributions of the different operators cancel out. Here the two possible options are:

$$\sum_{a,b} U_{ia} U^\dagger_{bw} = \mathbb{I},$$

$$\sum_{a,c} U_{ia} U^*_{kc} = \sum_{a,c} U_{ia} \left(U^*_{ck}\right)^T = \sum_{a,c} U_{ia} U^\dagger_{ck} = \mathbb{I}.$$

Now, it is easy to find the conditions that the matrix ρ has to fulfill. The first option requires that $a = b$ and $c = d$, which means that $\rho_1 = C \sum_{a,c} \left|a\right\rangle \left\langle a\right| \otimes \left|c\right\rangle \left\langle c\right| = C \left(\mathbb{I}_A \otimes \mathbb{I}_B\right)$. If we substitute this conditions in the equation $(U \otimes U^*)\,\rho\,(U \otimes U^*)^\dagger$ we obtain

$$C\sum_{w,z}\sum_{i,k}\sum_{a,c} U_{ia}U^*_{kc}U^\dagger_{aw}\left(U^*_{cz}\right)^\dagger |i\rangle\langle w| \otimes |k\rangle\langle z| = C\sum_{w,z}\sum_{i,k}\delta_{iw}\delta_{kz}|i\rangle\langle w| \otimes |k\rangle\langle z| =$$

$$= C\sum_{i,k}|i\rangle\langle i| \otimes |k\rangle\langle k| = C\left(\mathbb{I}_A \otimes \mathbb{I}_B\right) = \rho_1.$$

For the second option, the requirement is now $a = c$ and $b = d$, which means $\rho_2 = C\sum_{a,b}|a\rangle\langle b| \otimes |a\rangle\langle b| = C\,|\Phi_{00}\rangle\langle\Phi_{00}|$. If we substitute this conditions in the equation $(U \otimes U^*)\,\rho\,(U \otimes U^*)^\dagger$ we obtain

$$C\sum_{w,z}\sum_{i,k}\sum_{a,b} U_{ia}U^*_{ka}U^\dagger_{bw}\left(U^*_{bz}\right)^\dagger |i\rangle\langle w| \otimes |k\rangle\langle z| = C\sum_{w,z}\sum_{i,k}\delta_{ik}\delta_{wz}|i\rangle\langle w| \otimes |k\rangle\langle z| =$$

$$= C\sum_{i,w}|i\rangle\langle w| \otimes |i\rangle\langle w| = C\,|\Phi_{00}\rangle\langle\Phi_{00}| = \rho_2.$$

Additionally, we have to impose the additional condition $Tr\,(\rho) = 1$ due to ρ being a density matrix. This yields that $\rho_1 = \frac{1}{d^2}\left(\mathbb{I}_A \otimes \mathbb{I}_B\right)$ and $\rho_2 = \frac{1}{d}\sum_{a,b}|a\rangle\langle b| \otimes |a\rangle\langle b| = |\Phi_{00}\rangle\langle\Phi_{00}|$.

Finally, the most general state invariant under $(U \otimes U^*)$ has to be a linear combination of ρ_1 and ρ_2. This kind of state corresponds exactly with the isotropic states ρ_{iso}. We can conclude then that a ρ state is invariant under the transformations $(U \otimes U^*)$ if and only if it is an isotropic state.

iv)

To find the values of α for which ρ_{iso} is entangled, we use the entanglement reduction criterion. It states that if $\rho_A \otimes \mathbb{I}_B - \rho_{AB} \ngeq 0$, or, $\rho_B \otimes \mathbb{I}_A - \rho_{AB} \ngeq 0$, then ρ_{AB} is entangled.

For our specific case, we have computed ρ_A as

$$\rho_A = Tr_B\left(\rho_{iso}\right) = \sum_{k=0}^{d-1} {}_B\langle k|\rho_{iso}\,|k\rangle_B = \sum_{k=0}^{d-1} {}_B\langle k|\left(\frac{\alpha}{d}\sum_{i,j=0}^{d-1}|ii\rangle\langle jj| + \frac{1-\alpha}{d^2}\left(\mathbb{I}_A \otimes \mathbb{I}_B\right)\right)|k\rangle_B =$$

$$= \frac{\alpha}{d}\sum_{i=0}^{d-1}|i\rangle_A\langle i| + \frac{(1-\alpha)d}{d^2}\mathbb{I}_A = \frac{1}{d}\mathbb{I}_A$$

In the basis where ρ_{iso} is diagonal, the entanglement reduction criterion is also directly computed. The matrix $\rho_A \otimes \mathbb{I}_B - \rho_{AB}$ correspond to

$$\frac{1}{d^2}\begin{bmatrix} -d^2\alpha + \alpha + d - 1 & & & \\ & d-1+\alpha & & \\ & & \ddots & \\ & & & d-1+\alpha \end{bmatrix}.$$

Now, the values of α, for which ρ_{iso} is entangled, are obtained imposing

$$(1-d^2)\alpha + d - 1 < 0 \Rightarrow \alpha > \frac{1}{d+1},$$

$$d - 1 + \alpha < 0 \Rightarrow \alpha < 1 - d.$$

Here, the last inequality does not fulfill the range of values for α to be a proper quantum state. Hence, ρ_{iso} is entangled for $1 \geq \alpha > \frac{1}{d+1}$.

v)

In this subsection, we aim to recover again the values of α for which ρ_{iso} is entangled. However, this time using the PPT criterion. The PPT criterion states that, if ρ_{iso} is separable, then $\rho_{iso}^{T_A} \geq 0$. Equivalently, if $\rho_{iso}^{T_A} < 0$, the state is entangled.

For our specific case,

$$\rho_{iso}^{T_A} = \left(\frac{\alpha}{d}\sum_{i,j=0}^{d-1} |ii\rangle\langle jj| + \frac{1-\alpha}{d^2}(\mathbb{I}_A \otimes \mathbb{I}_B)\right)^{T_A} = \frac{\alpha}{d}\sum_{i,j=0}^{d-1} |ji\rangle\langle ij| + \frac{1-\alpha}{d^2}(\mathbb{I}_A \otimes \mathbb{I}_B)$$

Due to the identity $\mathbb{I}_A \otimes \mathbb{I}_B$ is diagonal in all the bases, the only matrix we need to diagonalize now is $Q = \sum_{i,j=0}^{d-1} |ji\rangle\langle ij|$. To compute the eigenvalues for this matrix we used the property that $Q^2 = \mathbb{I}$, then the eigenvectors v of this matrix have to fulfill

$$Q^2\vec{v} - \vec{v} = 0 \Rightarrow (\lambda^2 - 1) = 0,$$

which yields as the possible eigenvalues are $\lambda_\pm = \pm 1$.

As result, the eigenvalues of $\rho_{iso}^{T_A}$ are:

$$\frac{\alpha}{d} + \frac{1-\alpha}{d^2} \quad \text{for} \quad \lambda_+ = 1,$$

$$-\frac{\alpha}{d} + \frac{1-\alpha}{d^2} \quad \text{for} \quad \lambda_- = -1.$$

To find the values of α for which ρ_{iso} is entangled we have to impose that the eigenvalues obtained are none positive. As a result, we obtained:

$$d\alpha + 1 - \alpha < 0 \Rightarrow \alpha < -\frac{1}{d-1},$$

$$-d\alpha + 1 - \alpha < 0 \Rightarrow \alpha > \frac{1}{d+1},$$

where the first inequation gives an α value outside of the possible range. From the second equation, we again recover the result obtained in the previous section.

vi)

The entanglement witness $W = (|\Phi_{00}\rangle\langle\Phi_{00}|)^{T_A}$ for ρ_{iso} has to fulfill

$$Tr\left(W\rho_{iso}\right) < 0 \quad \text{and} \quad Tr\left(W\sigma\right) \geq 0,$$

where σ are all the possible separable states.

First, let's compute W. Using the computational basis, W can be expressed as

$$W = \frac{1}{d}\sum_{i,j=0}^{d-1} |ji\rangle\langle ij|$$

The next step consists in computing $W\rho_{iso}$

$$W\rho_{iso} = \frac{\alpha}{d^2}\left(\sum_{i,j=0}^{d-1} |ji\rangle\langle ij|\right)\left(\sum_{l,m=0}^{d-1} |ll\rangle\langle mm|\right) + \frac{1-\alpha}{d^2}W\left(\mathbb{I}_A \otimes \mathbb{I}_B\right) =$$

$$= \frac{\alpha}{d^2}\left(\sum_{l,m=0}^{d-1} |ll\rangle\langle mm|\right) + \frac{1-\alpha}{d^2}W = \frac{\alpha}{d}|\Phi_{00}\rangle\langle\Phi_{00}| + \frac{1-\alpha}{d^2}W$$

Before calculating directly $Tr(W\rho_{iso})$, it is worth to keep in mind that $|\Phi_{00}\rangle\langle\Phi_{00}|$ is the density matrix of a pure state, so $Tr(|\Phi_{00}\rangle\langle\Phi_{00}|) = 1$.

Then, the only element left is to compute the $Tr(W)$

$$Tr(W) = \sum_{k=0}^{d-1}\langle kk|\left(\frac{1}{d}\sum_{i,j=0}^{d-1}|ji\rangle\langle ij|\right)|kk\rangle = \frac{1}{d}\sum_{k=0}^{d-1}1 = \frac{d}{d} = 1$$

Consequently, $Tr(W\rho_{iso})$ is

$$Tr(W\rho_{iso}) = \frac{\alpha}{d}Tr\left(|\Phi_{00}\rangle\langle\Phi_{00}|\right) + \frac{1-\alpha}{d^2}Tr(W) =$$

$$\frac{\alpha}{d} + \frac{1-\alpha}{d^2} = \frac{1}{d^2}\left((d-1)\alpha + 1\right)$$

Imposing now the condition $Tr\left((W\rho_{iso}\right) < 0$, we obtain the inequation $\alpha < \frac{-1}{d-1}$. This is out of the possible values of α, therefore, this operator is not a good entanglement witness operator for ρ_{iso}.

In general, the Bell stater are good entanglement witness. For the stare $\rho_\omega = \alpha|\psi\rangle\langle\psi| + \frac{(1-\alpha)}{4}\,\mathbb{I}\otimes\mathbb{I}$ we will use the operator $V = \left(|\phi^-\rangle\langle\phi^-|\right)^\top$.

$$\left|\phi^-\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \text{ and } |\psi\rangle = a|01\rangle + b|10\rangle$$

In order to work easily with $|\psi\rangle$, we will write in the general form for quits:

$$(\psi) = \cos\left(\frac{\theta}{2}\right)|01\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|10\rangle$$

To check if is a good entanglement witness, this operator must fulfill $Tr\left(V\rho_w\right) < 0$ for ρ_w entangled, and $Tr(V\sigma) > 0$ for any $\sigma = \sum\sigma_{ijkl}|ij\rangle\langle kl|$

First,

$$\mathrm{Tr}\left(V\rho_\omega\right) = \mathrm{Tr}\left(\alpha V|\psi\rangle\langle\psi| + \frac{(1-\alpha)}{4}V(\mathbb{I}\otimes\mathbb{I})\right) =$$

$$= \alpha\,\mathrm{Tr}(\langle\psi|V|\psi\rangle) + \frac{(1-\alpha)}{4}\,\mathrm{Tr}(V)$$

$$\mathrm{Tr}(V) = \mathrm{Tr}\left(\left(\left|\phi^-\right\rangle\left\langle\phi^-\right|\right)^{T_A}\right) = \mathrm{Tr}\left(\left|\phi^-\right\rangle\left\langle\phi^-\right|\right) = 1$$

$$V=\left(\frac{1}{2}|00\rangle\langle 00|-|00\rangle\langle 11|-|11\rangle\langle 00|+|11\rangle\langle 11|\right)^{T_A}=$$
$$=\left(\frac{1}{2}(00\rangle\langle 00|+|11\rangle\langle 11|-|10\rangle\langle 01|-|01\rangle\langle 10|\right)$$
$$\langle\psi|U|\psi\rangle=\langle\psi|\left[-\cos\left(\frac{\theta}{2}\right)|10\rangle-\sin\left(\frac{\theta}{2}\right)e^{i\varphi}(01\rangle\right]=$$
$$=-e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\langle 10|10\rangle-e^{i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cdot$$
$$\langle 01|01\rangle=-\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos(\varphi)=-\frac{1}{2}\sin(\theta)\cos(\varphi)$$
$$\operatorname{Tr}\left(V\rho_\omega\right)=-\frac{\alpha}{2}\cos(\varphi)\sin(\theta)+\frac{(1-\alpha)}{4}=$$
$$=\frac{1-(1+2\sin\theta\cos(\varphi))}{4}<0$$

Thus we have that

$$\alpha>\frac{1}{1+2\sin\theta\cos\varphi}$$

Knowing that for $\sin\theta\cos(\varphi)=1\quad\alpha>\frac{1}{3}$ as the PPT cuiterion dictates.

$$Tr(V\omega)=Tr\left(\left(\mathbb{I}\otimes\sigma_2\right)W\left(\not{\Vdash}\otimes\sigma_2\right)(\rho\otimes\sigma)\right)$$

Where we have descrived $|\phi^-\rangle=(I\otimes\sigma_z)\,|\phi_{00}\rangle$

$$\begin{aligned}V&=\left(\left|\phi^-\right\rangle\left\langle\phi^-\right|\right)^{T_4}=(11\otimes\sigma_z)\left(|\phi_{00}\rangle\langle\phi_{00}|\right)^{T_4}(1\mid\otimes\sigma_z)=\\&=(11\otimes\sigma_z)\,W\,(11\otimes\sigma_z)\end{aligned}$$

ω is separable $\rightarrow\omega=\rho\otimes\sigma$

$$\begin{aligned}&T_V(V\omega)=T_r\left(W\left(\rho\otimes\sigma_z\sigma\sigma_z\right)\right)=\\&=\operatorname{Tr}\left[\frac{1}{d}\sum_{ij}|ji\rangle\langle ij|\right)\left(\sum_{k,l,n,m}\rho_{kl}\sigma_{nm}|kn\rangle\langle lm|\right)\bigg]\\&=Tr\left[\frac{1}{d}\sum_{i,j,l,m}\rho_{il}\sigma_{jm}|ji\rangle\langle lm|\right]=\frac{i}{d}\sum_{i,j,l,m}\rho_{il}\sigma_{jm}\cdot\\&\cdot\langle l\mid j\rangle\langle m\mid i\rangle=\frac{1}{d}\sum_{ij}\rho_{ij}\sigma_{ij}=\frac{1}{d}\sum_i(\rho\sigma)_{ii}=\\&=\frac{1}{d}Tr(\rho\sigma)\geq 0\end{aligned}$$

Then we can conclude V is a good entanglement witness

5 Find out the Kraus operators for the complete dephasing channel of a single qubit, which can be expressed as:

$$\Lambda[\rho] = \mathrm{Tr}_2\left[U_{CNOT}(\rho \otimes |0\rangle\langle 0|)U^{+}_{CNOT}\right]$$
$$\Lambda[\rho] = \mathrm{Tr}_2\left[(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X)(\rho \otimes |0\rangle\langle 0|)(|0\rangle\langle 0| \otimes I + |1\rangle\left\langle 1\right| \otimes X)\right]$$
$$\Lambda[\rho] = Tr_2[(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes x)\left(\rho|0\rangle\langle 0| \otimes |0\rangle\left\langle 0\right| + \rho|1\rangle\langle 1| \otimes |0\rangle\left\langle 1|\right)\right]$$
$$\Lambda[\rho] = Tr_2[|0\rangle\langle 0|\rho|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0|\rho|1\rangle\langle 1| \otimes |0\rangle\langle 1|$$
$$+ |1\rangle\langle 1|\rho|0\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1| \otimes |1\rangle\langle 1|]$$
$$\Lambda[\rho] = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$$
$$\Lambda[\rho] = k_0\rho k_0^{+} + k_1\rho k_1^{\top} \Rightarrow \quad \begin{array}{c} k_0 = |0\rangle\langle 0| \\ k_1 = |1\rangle\langle 1| \end{array}$$

$\sum k_i^{+}k_i = I \rightarrow$ Trace preserving map

6 Exercise 6

Let's suppose that ρ^{T_B} has two negative eigenvalues, and let's denote their eigenvectors as $|v_1\rangle$ and $|v_2\rangle$. These two eigenvectors generate a plane P in $\mathbb{C}^2 \otimes \mathbb{C}^2$. Each vector belonging to the plate can be expressed as

$$|w\rangle = \alpha |v_1\rangle + \beta |v_2\rangle$$

where

$$\alpha, \beta \in \mathbb{C}$$

We are going to use a theorem which is proven in the paper *Local description of quantum inseparability* by Anna Sanpera, Rolf Tarrach and Euifre Vidal. This theorem states the following

Theorem: Any plane P in $\mathbb{C}^2 \otimes \mathbb{C}^2$ contains at least one product vector. Some planes contain only one.

Then it exists a product vector in the plane P such that

$$|e, f\rangle = \alpha |v_1\rangle + \beta |v_2\rangle$$

Therefore we have

$$\rho^{T_B} |e, f\rangle = -\lambda_1 \cdot \alpha |v_1\rangle - \lambda_2 \cdot \beta |v_2\rangle \tag{1}$$

where λ_1 and λ_2 are the corresponding eigenvalues associated with $|v_1\rangle$ and $|v_2\rangle$ respectively.

Applying the bra $\langle e, f|$ on the left in (1), we finally have

$$\left\langle e, f \left| \rho^{T_B} \right| e, f \right\rangle = -\lambda_1 |\alpha|^2 - \lambda_2 |\beta|^2 < 0$$

The last inequality is due to the fact that $\lambda_1, \lambda_2 > 0$. Thus, taking the conjugate we also have

$$\langle e^*, f^* | \rho | e^*, f^* \rangle < 0$$

Which is impossible since ρ should be greater than or equal to zero. Thus we conclude that ρ^{T_B} can have at most one negative eigenvalue.

7 Kraus, Choi and CPPT maps. Consider the following family of maps:

$$\boldsymbol{\Lambda_\alpha(\rho) = \frac{1}{2}\mathbb{I} + \alpha\left(X\rho Z + Z\rho X\right), \qquad 0 \leqslant \alpha \leqslant 1,} \tag{2}$$

where as usual X, Y, Z denote the Pauli matrices.

(i) Determine the values of α for which these maps are positive. Use the Bloch representation to see the effect of the map and indicate how the Bloch sphere changes.

A map Λ is positive ($\Lambda \geqslant 0$) if all of its eigenvalues are non-negative. In this exercise we are given the following family of maps:

$$\Lambda_\alpha(\rho) = \frac{1}{2}\mathbb{I} + \alpha\left(X\rho Z + Z\rho X\right) = \frac{1}{2}\mathbb{I} + \alpha\left(\sigma_x\rho\sigma_z + \sigma_z\rho\sigma_x\right) \tag{3}$$

For density matrices of dimension $d = 2$ we can express any ρ in terms of the identity and the Pauli matrices as:

$$\rho = \frac{\mathbb{I} + \vec{r}\cdot\vec{\sigma}}{2}$$

where $\vec{r} = (r_x, r_y, r_z)$ with $\sqrt{r_x^2 + r_y^2 + r_z^2} \leqslant 1$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Therefore, any ρ can be expressed as:

$$\rho = \frac{\mathbb{I} + r_x\cdot\sigma_x + r_y\cdot\sigma_y + r_z\cdot\sigma_z}{2} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{r_x}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{r_y}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \frac{r_z}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\Rightarrow \boxed{\rho = \frac{1}{2}\begin{pmatrix}1 + r_z & r_x - ir_y\\ r_x + ir_y & 1 - r_z\end{pmatrix}} \tag{4}$$

If we now compute the family of maps Λ_α in matricial form using expression (4) for ρ, we have:

$$\begin{aligned}
\Lambda_\alpha(\rho) &= \frac{1}{2}\mathbb{I} + \alpha\left(\sigma_x\rho\sigma_z + \sigma_z\rho\sigma_x\right) = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
&+ \alpha\left[\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 + r_z & r_x - ir_y\\ r_x + ir_y & 1 - r_z\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 + r_z & r_x - ir_y\\ r_x + ir_y & 1 - r_z\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\\
&= \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \alpha\left[\frac{1}{2}\begin{pmatrix}r_x + ir_y & r_z - 1\\ r_z + 1 & -r_x + ir_y\end{pmatrix} + \frac{1}{2}\begin{pmatrix}r_x - ir_y & r_z + 1\\ r_z - 1 & -r_x - ir_y\end{pmatrix}\right]\\
&= \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\alpha}{2}\begin{pmatrix}2r_x & 2r_z\\ 2r_z & -2r_x\end{pmatrix} = \begin{pmatrix}\frac{1}{2} + \alpha r_x & \alpha r_z\\ \alpha r_z & \frac{1}{2} - \alpha r_x\end{pmatrix}
\end{aligned}$$

$$\Rightarrow \boxed{\Lambda_\alpha(\rho) = \begin{pmatrix}\frac{1}{2} + \alpha r_x & \alpha r_z\\ \alpha r_z & \frac{1}{2} - \alpha r_x\end{pmatrix}} \tag{5}$$

The eigenvalues $\{\lambda\}$ of this family of maps will therefore be:

$$\det(\Lambda_\alpha - \lambda\mathbb{I}) = 0$$

$$\begin{vmatrix}\frac{1}{2} + \alpha r_x - \lambda & \alpha r_z\\ \alpha r_z & \frac{1}{2} - \alpha r_x - \lambda\end{vmatrix} = 0 \Rightarrow \lambda^2 - \lambda + \frac{1}{4} - \alpha^2(r_x^2 + r_z^2) = 0$$

The two possible solutions of this second order equation are:

$$\Rightarrow \boxed{\lambda_\pm = \frac{1}{2} \pm \alpha\sqrt{r_x^2 + r_z^2}} \tag{6}$$

At this point it is important to remember the main objective of this exercise, which is to determine the values of α for which the maps Λ_α are positive. Therefore, we have to determine which values of α make the eigenvalues given by (6) non-negative.

- The term $\sqrt{r_x^2 + r_z^2}$ will always be positive, and therefore $\lambda_+ = \frac{1}{2} + \alpha\sqrt{r_x^2 + r_z^2}$ will indeed be positive for all possible values of r_x, r_z, and α (as $0 \leqslant \alpha \leqslant 1$).
- On the other hand, the other eigenvalue, $\lambda_- = \frac{1}{2} - \alpha\sqrt{r_x^2 + r_z^2}$, will not be necessarily positive for all cases. To obtain the most general condition for α (the one that fulfills for all possible ρs), we can take into account that the highest value that $\sqrt{r_x^2 + r_z^2}$ can achieve is 1. As a result, we have:

$$\frac{1}{2} \geqslant \alpha\sqrt{r_x^2 + r_z^2} \Rightarrow \frac{1}{2} \geqslant \alpha$$

As a result, the family of maps given by $\underline{\Lambda_\alpha(\rho)}$ will be positive (non-negative) if:

$$\boxed{0 \leqslant \alpha \leqslant \frac{1}{2}} \tag{7}$$

At this point, we can use the Bloch representation to indicate how the Bloch sphere changes. If we consider how the components of $\vec{r}$ of an initial state given by expression (4) change when compared with the state after applying the map Λ_α we see that:

$$\begin{pmatrix} \frac{1}{2} + \alpha r_x & \alpha r_z \\ \alpha r_z & \frac{1}{2} - \alpha r_x \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + r'_z & r'_x - ir'_y \\ r'_x + ir'_y & 1 - r'_z \end{pmatrix} \Rightarrow \begin{cases} r'_z = 2\alpha r_x \\ r'_x - ir'_y = 2\alpha r_z \\ r'_x + ir'_y = 2\alpha r_z \\ r'_z = 2\alpha r_x \end{cases} \Rightarrow \begin{cases} r'_z = 2\alpha r_x \\ r'_x = 2\alpha r_z \\ r'_y = 0 \end{cases}$$

where $\alpha \leqslant \frac{1}{2}$, as we have seen earlier. From this result, we can see that the Bloch sphere will become completely flat in the y direction (due to r'_y being 0 for any random initial state) and that the x and z axes will become flatter (the magnitude of flattening will depend on the value of α).

(ii) Which is the corresponding Choi state of the $\Lambda_\alpha(\rho)$? For which values of α is the map a CPTP map?

The Choi state of $\Lambda_\alpha(\rho)$ can be computed by considering:

$$J(\Lambda) = (\mathbb{I} \otimes \Lambda)|\Omega\rangle\langle\Omega| \,, \qquad |\Omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$$

where as has been indicated, $|\Omega\rangle$ is a maximally entangled state. In the case of this exercise, we can rewrite this as:

$$J(\Lambda_\alpha) = (\mathbb{I} \otimes \Lambda_\alpha)|\Omega\rangle\langle\Omega| \,, \qquad |\Omega\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Thus, we have:

$$\begin{aligned} J(\Lambda_\alpha) = (\mathbb{I} \otimes \Lambda_\alpha)|\Omega\rangle\langle\Omega| = \frac{1}{2}(&|0\rangle\langle 0| \otimes \Lambda_\alpha(|0\rangle\langle 0|) + |0\rangle\langle 1| \otimes \Lambda_\alpha(|0\rangle\langle 1|) \\ &+ |1\rangle\langle 0| \otimes \Lambda_\alpha(|1\rangle\langle 0|) + |1\rangle\langle 1| \otimes \Lambda_\alpha(|1\rangle\langle 1|)) \end{aligned}$$

By using that:

$$\mathbb{I} = \frac{1}{2}(\mathbb{I}\rho\mathbb{I} + X\rho X + Y\rho Y + Z\rho Z)$$

where X, Y, Z denote the corresponding Pauli matrices, we can write the family of maps $\Lambda_\alpha(\rho)$ as:

$$\Lambda_\alpha(\rho) = \frac{1}{4}(\mathbb{I}\rho\mathbb{I} + X\rho X + Y\rho Y + Z\rho Z) + \alpha(X\rho Z + Z\rho X) \tag{8}$$

Using this expression of Λ_α, we can obtain:

$$\Lambda_\alpha(|0\rangle\langle 0|) = \begin{pmatrix} 1/2 & \alpha \\ \alpha & 1/2 \end{pmatrix} \qquad \Lambda_\alpha(|0\rangle\langle 1|) = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$$

$$\Lambda_\alpha(|1\rangle\langle 0|) = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \qquad \Lambda_\alpha(|1\rangle\langle 1|) = \begin{pmatrix} 1/2 & -\alpha \\ -\alpha & 1/2 \end{pmatrix}$$

We can then use this results in the expression of $J(\Lambda_\alpha)$, obtaining:

$$\begin{aligned} J(\Lambda_\alpha) &= (\mathbb{I} \otimes \Lambda_\alpha)|\Omega\rangle\langle\Omega| \\ &= \frac{1}{2}(|0\rangle\langle 0| \otimes \Lambda_\alpha(|0\rangle\langle 0|) + |0\rangle\langle 1| \otimes \Lambda_\alpha(|0\rangle\langle 1|) \quad + |1\rangle\langle 0| \otimes \Lambda_\alpha(|1\rangle\langle 0|) + |1\rangle\langle 1| \otimes \Lambda_\alpha(|1\rangle\langle 1|)) \\ &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1/2 & \alpha \\ \alpha & 1/2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}\right. \\ &\quad \left. + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1/2 & -\alpha \\ -\alpha & 1/2 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 1/2 & \alpha & \alpha & 0 \\ \alpha & 1/2 & 0 & -\alpha \\ \alpha & 0 & 1/2 & -\alpha \\ 0 & -\alpha & -\alpha & 1/2 \end{pmatrix} \end{aligned}$$

As a result, we have that the corresponding Choi state of the $\Lambda_\alpha(\rho)$ is:

$$\Rightarrow \boxed{J(\Lambda_\alpha) = \frac{1}{2}\begin{pmatrix} 1/2 & \alpha & \alpha & 0 \\ \alpha & 1/2 & 0 & -\alpha \\ \alpha & 0 & 1/2 & -\alpha \\ 0 & -\alpha & -\alpha & 1/2 \end{pmatrix}} \tag{9}$$

A CPTP map must be linear, positive, trace-preserving, and completely positive. In our case, we only need to check if $J(\Lambda_\alpha)$ is trace-preserving and completely positive (the previous conditions have already been proved). Regarding the trace, we can observe that Tr($J(\Lambda_\alpha)$)=1 = Tr(ρ). Therefore, the corresponding map is trace preserving. The only thing left to check is if it can be completely positive. For this, we first need to find the eigenvalues of the Choi state associated to this family of maps, and check that all of them are non-negative. Thus,

$$\det(J(\Lambda_\alpha) - \lambda\mathbb{I}) = 0 \Rightarrow \left(\frac{1}{4} - \lambda\right)^4 - 4\left(\frac{\alpha}{2}\right)^2\left(\frac{1}{4} - \lambda\right)^2 = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = 1/4, \qquad \lambda_3 = \frac{1}{4} + \alpha, \qquad \lambda_4 = \frac{1}{4} - \alpha$$

If we now impose that all eigenvalues must be positive (or zero), and we take into account that α is originally bounded between 0 and 1, we find that:

$$\frac{1}{4} - \alpha \geqslant 0 \Rightarrow \alpha \leqslant \frac{1}{4} \Rightarrow \boxed{0 \leqslant \alpha \leqslant \frac{1}{4}}$$

As a result, the map will be a CPTP map for $0 \leqslant \alpha \leqslant 1/4$.

(iii) Find the Kraus representation of the map for α = 1/4.

For $\alpha = 1/4$ we have the following map:

$$\Lambda_{1/4}(\rho) = \frac{1}{2}\mathbb{I} + \frac{1}{4}\left(X\rho Z + Z\rho X\right)$$

which we can rewrite as:

$$\begin{aligned}
\Lambda_{1/4}(\rho) &= \frac{1}{4}(\mathbb{I}\rho\mathbb{I} + X\rho X + Y\rho Y + Z\rho Z) + \frac{1}{4}(X\rho Z + Z\rho X) \\
&= \frac{1}{4}(\mathbb{I}\rho\mathbb{I} + X\rho X + Y\rho Y + Z\rho Z + X\rho Z + Z\rho X) \\
&= \frac{1}{4}(\mathbb{I}\rho\mathbb{I} + Y\rho Y + (X + Z)\rho(X + Z)) \\
&= \frac{\mathbb{I}}{2}\rho\frac{\mathbb{I}}{2} + \frac{Y}{2}\rho\frac{Y}{2} + \frac{X+Z}{2}\rho\frac{X+Z}{2}
\end{aligned}$$

where we can clearly see that the Kraus operators for this map are:

$$\boxed{K_1 = \frac{1}{2}\mathbb{I} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}} \quad \boxed{K_2 = \frac{1}{2}Y = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}} \quad \boxed{K_3 = \frac{1}{2}(X + Z) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

We can additionally see that:

$$\sum_i K_i K_i^T = K_1 K_1^T + K_2 K_2^T + K_3 K_3^T = \mathbb{I}$$

which is the expected result, coherent with what has been obtained in the previous section ($\sum_i K_i K_i^T = \mathbb{I}$ for a CPTP map).