Quantum Statistical Inference - Homework 1

Team 1: Aatif Kaisar Khan, Guillermo Abad López and Guillermo García Pérez

10 January 2025

Question 1

We are given two quantum systems independently prepared, each of them described by the ensemble $\{1/2, |\phi_1\rangle\,; 1/2, |\phi_2\rangle\}$, such that they are qubit states, the overlap between them is $c = \langle\phi_1|\phi_2\rangle \in \mathbb{R}$ (so $c = \langle\phi_1|\phi_2\rangle = \langle\phi_2|\phi_1\rangle \in [-1, 1]$), and each state is normalized, $\langle\phi_1|\phi_1\rangle = \langle\phi_2|\phi_2\rangle = 1$. Obeying these conditions, we can take:

$$|\phi_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\phi_2\rangle = \begin{bmatrix} c \\ \sqrt{1-c^2} \end{bmatrix} \tag{1}$$

Thus, the density matrix of each system is:

$$\rho_{sys} = \frac{1}{2}\,|\phi_1\rangle\langle\phi_1| + \frac{1}{2}\,|\phi_2\rangle\langle\phi_2| = \frac{1}{2}\big(\,|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|\,\big) \tag{2}$$

As each system is prepared independently, the global state of both systems is given by the tensor product of both states:

$$\begin{aligned}\rho = \rho_{sys_A} \otimes \rho_{sys_B} = \frac{1}{2}\big(\,|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|\,\big) \otimes \frac{1}{2}\big(\,|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|\,\big) = \\ = \frac{1}{4}\big(\,|\phi_1\phi_1\rangle\langle\phi_1\phi_1| + |\phi_1\phi_2\rangle\langle\phi_1\phi_2| + |\phi_2\phi_1\rangle\langle\phi_2\phi_1| + |\phi_2\phi_2\rangle\langle\phi_2\phi_2|\,\big)\end{aligned} \tag{3}$$

where $|\phi_i\phi_j\rangle\langle\phi_i\phi_j| \equiv |\phi_i\rangle\langle\phi_i| \otimes |\phi_j\rangle\langle\phi_j|$ for $i, j = 1, 2$.

Now, if the systems are in the same state, then they can be described by the pure state $|\phi_i\phi_i\rangle\,,\ i = 1, 2$, and, equivalently, if they are in different states, then they can be described by $|\phi_i\phi_j\rangle\,,\ i \neq j,\ i, j = 1, 2$. Therefore, if $\rho_=$ represents a mixed state that describes the case in which both systems are in the same state, it can be written as:

$$\rho_= = \frac{1}{2}\big(\,|\phi_1\phi_1\rangle\langle\phi_1\phi_1| + |\phi_2\phi_2\rangle\langle\phi_2\phi_2|\,\big) \tag{4}$$

and if $\rho_{\neq}$ represents the mixed state which describes the case in which the systems are in different states, we have:

$$\rho_{\neq} = \frac{1}{2}\big(\,|\phi_1\phi_2\rangle\langle\phi_1\phi_2| + |\phi_2\phi_1\rangle\langle\phi_2\phi_1|\,\big) \tag{5}$$

Thus, comparing with the above global state, it is evident that: $\rho = \frac{1}{2}(\rho_= + \rho_{\neq})$ which corresponds to a mixed state given by the ensemble $\{\eta_=, \rho_=; \eta_{\neq}, \rho_{\neq}\}$ with $\eta_= = \eta_{\neq} = 1/2$, as one would have expected when starting the exercise.

Now, we use the minimum error approach to maximize the probability of a successful identification, i.e.,

$$P_{\mathrm{s}} = \frac{1}{2} + \frac{1}{2}||\eta_=\rho_= - \eta_{\neq}\rho_{\neq}||_1 = \frac{1}{2}\left(1 + \frac{1}{2}||\rho_= - \rho_{\neq}||_1\right) \tag{6}$$

where $||\rho_= - \rho_{\neq}||_1 = \mathrm{Tr}\Big(\sqrt{(\rho_= - \rho_{\neq})(\rho_= - \rho_{\neq})^\dagger}\Big)$. Now using the above details, we find:

$$\rho_= - \rho_{\neq} = \frac{d^2}{2}\begin{bmatrix} d^2 & -cd & -cd & c^2 \\ -cd & -d^2 & c^2 & cd \\ -cd & c^2 & -d^2 & cd \\ c^2 & cd & cd & d^2 \end{bmatrix} \quad \text{with} \quad d = \sqrt{1-c^2} = \langle\phi_1|\phi_2\rangle \tag{7}$$

Thus,

$$\text{Eigenvalues of } (\rho_= - \rho_{\neq})(\rho_= - \rho_{\neq})^\dagger = \left\{\frac{1}{4}d^4,\ \frac{1}{4}d^4,\ \frac{1}{4}d^4,\ \frac{1}{4}d^4\right\} \tag{8}$$

Therefore,

$$||\rho_= - \rho_{\neq}||_1 = \mathrm{Tr}\left(\sqrt{(\rho_= - \rho_{\neq})(\rho_= - \rho_{\neq})^\dagger}\right) = \sum_{k=1}^{4}\left|\frac{d^2}{2}\right| = \sum_{k=1}^{4}\left|\frac{1-c^2}{2}\right| = 2(1-c^2) \tag{9}$$

Finally, using in 9 in 6, we get:

$$\boxed{P_{\mathrm{s,global}} = 1 - \frac{c^2}{2}} \tag{10}$$

Question 2

Here, the strategy involves measuring each individual system independently to identify which state each system is in. Given that each system is prepared in either $|\phi_1\rangle$ or $|\phi_2\rangle$, the optimal success probability for discriminating between the two states for a single system is:

$$P_{\text{s,local}}^{(1)} = \frac{1}{2}\left(1 + \sqrt{1 - |\langle\phi_1|\phi_2\rangle|^2}\right) = \frac{1}{2}\left(1 + \sqrt{1 - c^2}\right) \tag{11}$$

Now for two systems, the total success probability is the product of the success probabilities of identifying each system independently: $P_{\text{s,local}} = P_{\text{s,local}}^{(1)} \cdot P_{\text{s,local}}^{(1)}$. And substituting the single-system success probability, we obtain:

$$\boxed{P_{\text{s,local}} = \left[\frac{1}{2}\left(1 + \sqrt{1 - c^2}\right)\right]^2 = \frac{1}{4}\left(1 + \sqrt{1 - c^2}\right)^2} \tag{12}$$

which with some manipulations and expanding in Taylor the square root, it can be expressed like:

$$P_{\text{s,local}} = \frac{1}{2}\left(\underbrace{1 - \frac{c^2}{2}}_{P_{\text{s,global}}} + \underbrace{\sqrt{1 - c^2}}_{\text{Unit-Circle}}\right) = \frac{1}{2}\left(1 - \frac{c^2}{2} + \underbrace{1 - \frac{c^2}{2} - O(c^4)}_{\sqrt{1-c^2}}\right) = P_{\text{s,global}} - \sum_{N>1}\underbrace{\frac{(2N-2)!!}{2^N N!}c^{2N}}_{>0\ (\forall N)} \tag{13}$$

The first equality of Eq. 13, being the average between $P_{\text{s,global}}$ and the top half of a Unit Circle, together with the last result, which is $P_{\text{s,global}}$ minus terms strictly positive, tells us that the result for the local probability of success should be similar to the global one but with the extremes more curved (strictly) down, towards 0, like in a unit circle. If we explicitly plot both, we exactly see these behaviours:

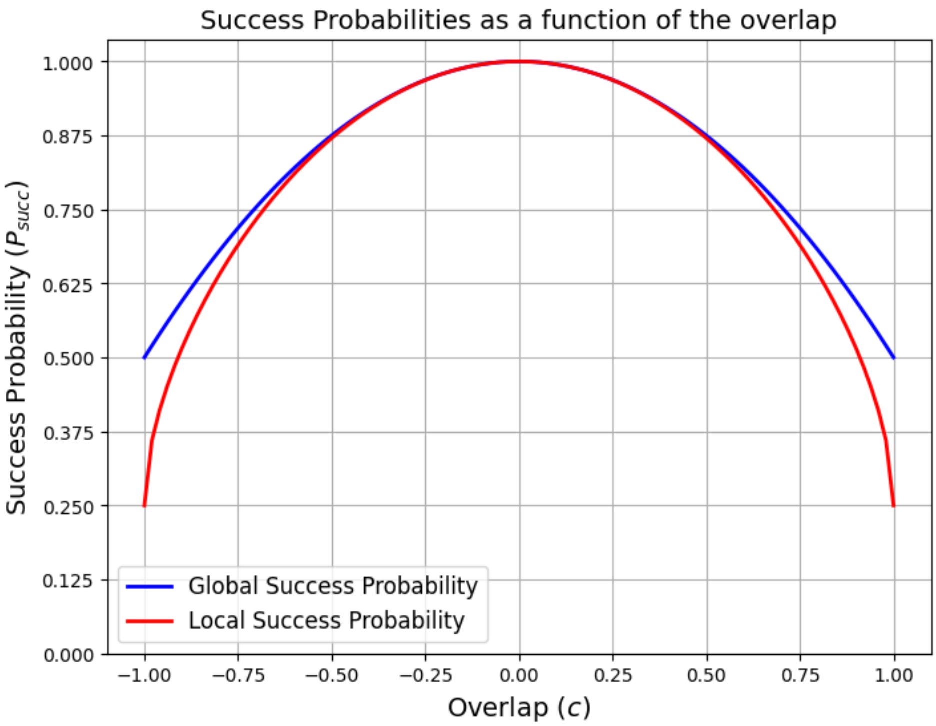

Figure 1: Comparison between global and local measurements, for distinguishing equal or different states in 2 systems.

It is evident from Eq. 13 and Fig. 1 that global measurements always outperform local measurements in discriminating between quantum states, particularly better when the overlap c between the states is large. This advantage should be obvious since local measurements are a subset of global ones, meaning global measurements should have all the capabilities of local ones, plus additionally, they also can exploit quantum correlations between the two systems.

For orthogonal states ($c = 0$), both global and local measurements achieve perfect discrimination with $P_s = 1$, as the states are fully distinguishable. However, as the overlap increases ($|c| \to 1$), the success probability decreases for both strategies, but more so for local measurements. In the extreme case of identical states ($|c| = 1$), global measurements yield $P_{s,\text{global}} = 0.5$, while local measurements drop to $P_{s,\text{local}} = 0.25$, showcasing the inherent limitations of local strategies that treat the systems independently.

Question 3

In this case, we are asked to unambiguously identify states $\rho_=$ and $\rho_{\neq}$, first allowing global measurements and then only local measurements.

Section a

First, we are asked to express the $\rho_=$ and $\rho_{\neq}$ states with the following new set of unnormalized (denoted with $'$) states, which might make the distinguishing task easier:

$$\begin{aligned} |\varphi_1'\rangle &= |\phi_1\phi_1\rangle + |\phi_2\phi_2\rangle\,, \quad & |\varphi_2'\rangle &= |\phi_1\phi_2\rangle + |\phi_2\phi_1\rangle \\ |\varphi_3'\rangle &= |\phi_1\phi_1\rangle - |\phi_2\phi_2\rangle\,, \quad & |\varphi_4'\rangle &= |\phi_1\phi_2\rangle - |\phi_2\phi_1\rangle \end{aligned} \tag{14}$$

We shall start by normalizing these states:

$$\begin{aligned} \langle\varphi_1'|\varphi_1'\rangle &= 2 + 2c^2 \;\to\; |\varphi_1\rangle \equiv \frac{|\phi_1\phi_1\rangle + |\phi_2\phi_2\rangle}{\sqrt{2}\sqrt{1+c^2}}, \quad & \langle\varphi_2'|\varphi_2'\rangle &= 2 + 2c^2 \;\to\; |\varphi_2\rangle \equiv \frac{|\phi_1\phi_2\rangle + |\phi_2\phi_1\rangle}{\sqrt{2}\sqrt{1+c^2}} \\ \langle\varphi_3'|\varphi_3'\rangle &= 2 - 2c^2 \;\to\; |\varphi_3\rangle \equiv \frac{|\phi_1\phi_1\rangle - |\phi_2\phi_2\rangle}{\sqrt{2}\sqrt{1-c^2}}, \quad & \langle\varphi_4'|\varphi_4'\rangle &= 2 - 2c^2 \;\to\; |\varphi_4\rangle \equiv \frac{|\phi_1\phi_2\rangle - |\phi_2\phi_1\rangle}{\sqrt{2}\sqrt{1-c^2}} \end{aligned} \tag{15}$$

The states $|\varphi_1\rangle$, $|\varphi_2\rangle$ and $|\varphi_3\rangle$ are totally symmetric under the exchange of the labels of both systems, while $|\varphi_4\rangle$ is totally antisymmetric.

We can realize that the global states where both systems are in the same state can be given by a linear combination of $|\varphi_1\rangle$ and $|\varphi_3\rangle$ and the global states where both systems are in different states can be given in terms of $|\varphi_2\rangle$ and $|\varphi_4\rangle$:

$$\begin{aligned} |\phi_1\phi_1\rangle &= \frac{\sqrt{1+c^2}\,|\varphi_1\rangle + \sqrt{1-c^2}\,|\varphi_3\rangle}{\sqrt{2}}, \quad & |\phi_1\phi_2\rangle &= \frac{\sqrt{1+c^2}\,|\varphi_2\rangle + \sqrt{1-c^2}\,|\varphi_4\rangle}{\sqrt{2}} \\ |\phi_2\phi_1\rangle &= \frac{\sqrt{1+c^2}\,|\varphi_2\rangle - \sqrt{1-c^2}\,|\varphi_4\rangle}{\sqrt{2}}, \quad & |\phi_2\phi_2\rangle &= \frac{\sqrt{1+c^2}\,|\varphi_1\rangle - \sqrt{1-c^2}\,|\varphi_3\rangle}{\sqrt{2}} \end{aligned} \tag{16}$$

If we express these states in terms of density matrices:

$$\begin{aligned} |\phi_1\phi_1\rangle\langle\phi_1\phi_1| &= \frac{1}{2}\Big((1+c^2)\,|\varphi_1\rangle\langle\varphi_1| + \sqrt{1-c^4}\,|\varphi_1\rangle\langle\varphi_3| + \sqrt{1-c^4}\,|\varphi_3\rangle\langle\varphi_1| + (1-c^2)\,|\varphi_3\rangle\langle\varphi_3|\Big) \\ |\phi_1\phi_2\rangle\langle\phi_1\phi_2| &= \frac{1}{2}\Big((1+c^2)\,|\varphi_2\rangle\langle\varphi_2| + \sqrt{1-c^4}\,|\varphi_2\rangle\langle\varphi_4| + \sqrt{1-c^4}\,|\varphi_4\rangle\langle\varphi_2| + (1-c^2)\,|\varphi_4\rangle\langle\varphi_4|\Big) \\ |\phi_2\phi_1\rangle\langle\phi_2\phi_1| &= \frac{1}{2}\Big((1+c^2)\,|\varphi_2\rangle\langle\varphi_2| - \sqrt{1-c^4}\,|\varphi_2\rangle\langle\varphi_4| - \sqrt{1-c^4}\,|\varphi_4\rangle\langle\varphi_2| + (1-c^2)\,|\varphi_4\rangle\langle\varphi_4|\Big) \\ |\phi_2\phi_2\rangle\langle\phi_2\phi_2| &= \frac{1}{2}\Big((1+c^2)\,|\varphi_1\rangle\langle\varphi_1| - \sqrt{1-c^4}\,|\varphi_1\rangle\langle\varphi_3| - \sqrt{1-c^4}\,|\varphi_3\rangle\langle\varphi_1| + (1-c^2)\,|\varphi_3\rangle\langle\varphi_3|\Big) \end{aligned} \tag{17}$$

Then, expressing $\rho_=$ and $\rho_{\neq}$ in terms of these states, the off-diagonal terms vanish and we are left with:

$$\boxed{\rho_= = \frac{1}{2}\Big((1+c^2)\,|\varphi_1\rangle\langle\varphi_1| + (1-c^2)\,|\varphi_3\rangle\langle\varphi_3|\Big)} \quad \boxed{\rho_{\neq} = \frac{1}{2}\Big((1+c^2)\,|\varphi_2\rangle\langle\varphi_2| + (1-c^2)\,|\varphi_4\rangle\langle\varphi_4|\Big)} \tag{18}$$

*These expressions for the asked unnormalized case would be trivial:

$$\boxed{\rho_= = \frac{1}{4}\Big(\,|\varphi_1'\rangle\langle\varphi_1'| + |\varphi_3'\rangle\langle\varphi_3'|\,\Big)} \quad \boxed{\rho_{\neq} = \frac{1}{4}\Big(\,|\varphi_2'\rangle\langle\varphi_2'| + |\varphi_4'\rangle\langle\varphi_4'|\,\Big)} \tag{19}$$

Section b

The orthogonality relations between the states $\{|\varphi_i\rangle\}_{i=1}^4$ are:

$$\begin{aligned}\langle\varphi_1|\varphi_2\rangle = \langle\varphi_2|\varphi_1\rangle = \frac{2c}{1+c^2} \\ \langle\varphi_1|\varphi_3\rangle = \langle\varphi_1|\varphi_4\rangle = \langle\varphi_2|\varphi_3\rangle = \langle\varphi_2|\varphi_4\rangle = \langle\varphi_3|\varphi_4\rangle = 0\end{aligned} \tag{20}$$

Therefore, all states are orthogonal between them apart from $|\varphi_1\rangle$ and $|\varphi_2\rangle$. But, we can define the orthogonal states to $|\varphi_1\rangle$ and $|\varphi_2\rangle$, such that:

$$\begin{aligned}|\varphi_1^\perp\rangle: \quad \langle\varphi_1|\varphi_1^\perp\rangle = \langle\varphi_3|\varphi_1^\perp\rangle = \langle\varphi_4|\varphi_1^\perp\rangle = 0 \\ |\varphi_2^\perp\rangle: \quad \langle\varphi_2|\varphi_2^\perp\rangle = \langle\varphi_3|\varphi_2^\perp\rangle = \langle\varphi_4|\varphi_2^\perp\rangle = 0\end{aligned} \tag{21}$$

so our whole Hilbert space can be spanned by the basis $\{|\varphi_1\rangle, |\varphi_1^\perp\rangle, |\varphi_3\rangle, |\varphi_4\rangle\}$ or the basis $\{|\varphi_2\rangle, |\varphi_2^\perp\rangle, |\varphi_3\rangle, |\varphi_4\rangle\}$. With the new states, our states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ can be written as:

$$\begin{aligned}|\varphi_2\rangle = \frac{2c}{1+c^2}|\varphi_1\rangle + \sqrt{1-\left(\frac{2c}{1+c^2}\right)^2}|\varphi_1^\perp\rangle = \frac{2c}{1+c^2}|\varphi_1\rangle + \frac{1-c^2}{1+c^2}|\varphi_1^\perp\rangle \\ |\varphi_1\rangle = \frac{2c}{1+c^2}|\varphi_2\rangle + \sqrt{1-\left(\frac{2c}{1+c^2}\right)^2}|\varphi_2^\perp\rangle = \frac{2c}{1+c^2}|\varphi_2\rangle + \frac{1-c^2}{1+c^2}|\varphi_2^\perp\rangle\end{aligned} \tag{22}$$

We may think about using a POVM defined by the projectors of the elements of any of the defined basis to distinguish between $\rho_=$ and $\rho_{\neq}$. If the outcome of our measurement was $|\varphi_1^\perp\rangle$, as this state is orthogonal to the two states that form $\rho_=$ but it is present in $\rho_{\neq}$, we would know with complete certainty that our state was $\rho_{\neq}$. And following a similar argument, if we measured outcome $|\varphi_2^\perp\rangle$, we would be able to know with certainty that the state was $\rho_=$.

Notice that two other possible outcomes also allow us to distinguish our states without ambiguity are $|\varphi_3\rangle$ and $|\varphi_4\rangle$. If we obtained $|\varphi_3\rangle$, as it is only present in $\rho_=$ and is orthogonal to the states that form $\rho_{\neq}$, we would be sure that the state measured was $\rho_=$. Similarly, if we obtained $|\varphi_4\rangle$, we would know without doubt that the state was $\rho_{\neq}$, as it is the only state where this outcome is present.

Thus, $E_1 \propto |\varphi_2^\perp\rangle\langle\varphi_2^\perp|$, $E_2 \propto |\varphi_1^\perp\rangle\langle\varphi_1^\perp|$, $E_3 \propto |\varphi_3\rangle\langle\varphi_3|$ and $E_4 \propto |\varphi_4\rangle\langle\varphi_4|$ yield measurement outcomes that allow us to discriminate our states unambiguously. This statement can be expressed as:

$$P(E_1|\rho_{\neq}) = P(E_2|\rho_=) = P(E_3|\rho_{\neq}) = P(E_4|\rho_=) = 0 \tag{23}$$

Therefore, the POVM that should be used for an unambiguous discrimination is:

$$E_1 = \alpha_1 |\varphi_2^\perp\rangle\langle\varphi_2^\perp|, \; E_2 = \alpha_2 |\varphi_1^\perp\rangle\langle\varphi_1^\perp|, \; E_3 = \alpha_3 |\varphi_3\rangle\langle\varphi_3|, \; E_4 = \alpha_4 |\varphi_4\rangle\langle\varphi_4|, \; E_Q = \mathbb{1} - \sum_{i=1}^4 E_i \tag{24}$$

such that all the elements of the POVM are positive semidefinite. At this point, we shall ask ourselves why we are using a POVM with 5 outcomes if the only overlap comes from $|\varphi_1\rangle$ and $|\varphi_2\rangle$, and a POVM with only 3 would also allow us to discriminate these states (E_1, E_2 and E_Q). This is because we want to achieve the best unambiguous discrimination allowed. Therefore, we want to reduce the probability of measuring E_Q as much as possible. If, for example, we only used E_1 and E_2 for our task, the outcomes E_3 and E_4 would be inside E_Q, and we would be minimizing over outcomes that actually help us in our task. Meaning, that having a bigger POVM is better for this task than a simpler one.

The probability for unambiguous discrimination, in general, will then be:

$$\begin{aligned}P_{unambiguous} &= P(E_1|\rho_=)\eta_= + P(E_2|\rho_{\neq})\eta_{\neq} + P(E_3|\rho_=)\eta_= + P(E_4|\rho_{\neq})\eta_{\neq} = \\ &= \frac{1}{2}\Big(P(E_1|\rho_=) + P(E_2|\rho_{\neq}) + P(E_3|\rho_=) + P(E_4|\rho_{\neq})\Big)\end{aligned} \tag{25}$$

The probability of success is then, a maximization over all the POVMs parameters $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$:

$$P_s = \max_E P_{unambiguous} = \frac{1}{2}\max_E \Big(P(E_1|\rho_=) + P(E_2|\rho_{\neq}) + P(E_3|\rho_=) + P(E_4|\rho_{\neq})\Big) \tag{26}$$

such that $E_1, E_2, E_3, E_4, E_Q \geq 0$ and $\mathbb{1} = E_Q + E_1 + E_2 + E_3 + E_4$.

Now, to obtain the probability of success, we need to maximize the coefficients $\alpha_{1,2,3,4}$ under the conditions for a valid POVM. Imposing the first, positivity of $E_{1,2,3,4}$, is easier, and gives:

$$\alpha_1 \geq 0, \quad \alpha_2 \geq 0, \quad \alpha_3 \geq 0, \quad \alpha_4 \geq 0 \tag{27}$$

To obtain the constraints given by the positivity of E_Q, notice that we will get an upper block with non-diagonal terms and a lower block with only elements in the diagonal terms $|\varphi_3\rangle\langle\varphi_3|$ and $|\varphi_4\rangle\langle\varphi_4|$:

$$E_Q = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \text{ such that } B_1 \text{ is non-diagonal and } B_2 = \begin{bmatrix} 1-\alpha_3 & 0 \\ 0 & 1-\alpha_4 \end{bmatrix} \tag{28}$$

Therefore, if we diagonalize E_Q and impose that its eigenvalues are non-negative, the last two eigenvalues give the following constraints:

$$0 \leq \alpha_3 \leq 1, \quad 0 \leq \alpha_4 \leq 1 \tag{29}$$

As these coefficients are independent from α_1 and α_2, their maximization over the permitted values yields $\boxed{\alpha_3 = \alpha_4 = 1}$.

Solving the maximization for $\alpha_{1,2}$ is not so trivial, we can see that from computing each term of P_s individually:

$$\begin{aligned}
P(E_1|\rho_=) &= \mathrm{Tr}\{E_1\rho_=\} = \frac{1+c^2}{2}\mathrm{Tr}\{E_1\,|\varphi_1\rangle\langle\varphi_1|\} + \frac{1-c^2}{2}\cancelto{0}{\mathrm{Tr}\{E_1\,|\varphi_3\rangle\langle\varphi_3|\}} = \frac{1+c^2}{2}\alpha_1\left|\langle\varphi_2^\perp|\varphi_1\rangle\right|^2 \\
P(E_2|\rho_{\neq}) &= \mathrm{Tr}\{E_2\rho_{\neq}\} = \frac{1+c^2}{2}\mathrm{Tr}\{E_2\,|\varphi_2\rangle\langle\varphi_2|\} + \frac{1-c^2}{2}\cancelto{0}{\mathrm{Tr}\{E_2\,|\varphi_4\rangle\langle\varphi_4|\}} = \frac{1+c^2}{2}\alpha_2\left|\langle\varphi_1^\perp|\varphi_2\rangle\right|^2 \\
P(E_3|\rho_=) &= \mathrm{Tr}\{E_3\rho_=\} = \frac{1+c^2}{2}\cancelto{0}{\mathrm{Tr}\{E_3\,|\varphi_1\rangle\langle\varphi_1|\}} + \frac{1-c^2}{2}\mathrm{Tr}\{E_3\,|\varphi_3\rangle\langle\varphi_3|\} = \frac{1-c^2}{2}\cancelto{1}{\alpha_3}\cancelto{1}{|\langle\varphi_3|\varphi_3\rangle|^2} \\
P(E_4|\rho_{\neq}) &= \mathrm{Tr}\{E_4\rho_{\neq}\} = \frac{1+c^2}{2}\cancelto{0}{\mathrm{Tr}\{E_4\,|\varphi_2\rangle\langle\varphi_2|\}} + \frac{1-c^2}{2}\mathrm{Tr}\{E_4\,|\varphi_4\rangle\langle\varphi_4|\} = \frac{1-c^2}{2}\cancelto{1}{\alpha_4}\cancelto{1}{|\langle\varphi_4|\varphi_4\rangle|^2}
\end{aligned} \tag{30}$$

where we see, that the overlap of the last two terms, corresponding to $|\varphi_3\rangle$ and $|\varphi_4\rangle$ disappears, while for the first two, corresponding to $|\varphi_1\rangle$ and $|\varphi_2\rangle$, doesn't. This leaves our probability of success as:

$$P_s = \frac{1}{2}\max_{\alpha_{1,2,3,4}}\left(\frac{1+c^2}{2}\Big(P(E_{1(\alpha_1)}|\varphi_1) + P(E_{2(\alpha_2)}|\varphi_2)\Big) + \frac{1-c^2}{2}(\cancelto{1}{\alpha_3} + \cancelto{1}{\alpha_4})\right) = P_s^{1,2} + \frac{1-c^2}{2} \tag{31}$$

where the first half contains the probability of success (P_s) of discriminating between $|\varphi_1\rangle$ and $|\varphi_2\rangle$, and the other simpler half contains the P_S from when the measurements fall into the subspaces without overlaps, $|\varphi_3\rangle$ and $|\varphi_4\rangle$.

The complicated first part seems to tell us, that we should solve a simpler problem of distinguishing between $|\varphi_1\rangle$ and $|\varphi_2\rangle$ and then extrapolate to our original problem. To do so, we can check the symmetry of our $\rho_=$ and $\rho_{\neq}$ states respect $|\varphi_1\rangle$ and $|\varphi_2\rangle$ in 18, and define a new ensemble composed by two pure states with equal priors too:

$$\{1/2, |\varphi_1\rangle\,;\ 1/2, |\varphi_2\rangle\} \tag{32}$$

such that we may define two new effective states where the priors are those from the original states, $\eta_=^{eff} = \eta_{\neq}^{eff} = \eta_= = \eta_{\neq} = 1/2$ (but where the total contribution will be missing a $\frac{1+c^2}{2}$ factor), as:

$$\rho_=^{eff} \equiv |\varphi_1\rangle\langle\varphi_1|\,, \quad \rho_{\neq}^{eff} \equiv |\varphi_2\rangle\langle\varphi_2| \tag{33}$$

For the unambiguous discrimination of these new *effective* states, the probability of success is:

$$P_s^{eff} = \max_E\left(\eta_=^{eff}P(E_1|\rho_=^{eff}) + \eta_{\neq}^{eff}P(E_2|\rho_{\neq}^{eff})\right) = \frac{1}{2}\max_E\left(P(E_1|\rho_=^{eff}) + P(E_2|\rho_{\neq}^{eff})\right) \tag{34}$$

where the POVM is the same as in the original problem but with only three elements, $\{E_1, E_2, E_Q\}$, and for the reduced Hilbert space spanned by $\{|\varphi_1\rangle, |\varphi_1^\perp\rangle\}$ (or, similarly, by $\{|\varphi_2\rangle, |\varphi_2^\perp\rangle\}$).

Then, the problem of unambiguously discriminating the effective states is equivalent to the part of our original problem regarding $|\varphi_1\rangle$ and $|\varphi_2\rangle$, as we can see that:

$$\begin{aligned} P(E_1|\rho_=) &= \frac{1+c^2}{2}\,\mathrm{Tr}\left\{E_1\rho_=^{eff}\right\} = \frac{1+c^2}{2}P(E_1|\rho_=^{eff}) \\ P(E_2|\rho_{\neq}) &= \frac{1+c^2}{2}\,\mathrm{Tr}\left\{E_2\rho_{\neq}^{eff}\right\} = \frac{1+c^2}{2}P(E_2|\rho_{\neq}^{eff}) \end{aligned} \tag{35}$$

So the success probabilities for E_1 and E_2 can be related through a prefactor as:

$$P_s^{1,2} = \eta_= P_s(E_1|\rho_=) + \eta_{\neq}P_s(E_2|\rho_{\neq}) = \eta_=\frac{1+c^2}{2}P_s(E_1|\rho_=^{eff}) + \eta_{\neq}\frac{1+c^2}{2}P_s(E_2|\rho_=^{eff}) = \frac{1+c^2}{2}P_s^{eff} \tag{36}$$

and the problem now reduces to find the success probability of the effective ensemble.

Following the development done in the Lecture Notes (Section 1.6) for the unambiguous discrimination between two qubit pure states with equal priors (these states are not qubit states but, as stated in the notes, *two pure states can always be considered to belong to a Hilbert space of dimension 2 with a conveniently chosen basis*), the probability of success for unambiguous discrimination when these two states have an over lap Γ is:

$$P_s^{eff} = 1 - |\Gamma| \tag{37}$$

In this case, the overlap between the states under study, as shown in Eq. 20, is $\Gamma = \langle\varphi_1|\varphi_2\rangle = \frac{2c}{1+c^2}$, so the probability of success for outcomes E_1 and E_2 is:

$$P_s^{1,2} = \frac{1+c^2}{2}(1-|\Gamma|) = \frac{1+c^2}{2}\left(1-\left|\frac{2c}{1+c^2}\right|\right) = \frac{1+c^2}{2}\frac{1+c^2-2|c|}{1+c^2} = \frac{(1-|c|)^2}{2} \tag{38}$$

Finally, the complete probability for unambiguous discrimination between our two states $\rho_=$ and $\rho_{\neq}$, we follow Eq. 31, getting:

$$P_s = P_s^{1,2} + \frac{1-c^2}{2} = \frac{(1-|c|)^2}{2} + \frac{1-c^2}{2} = 1-|c| \quad \rightarrow \quad \boxed{P_{s,global} = 1-|c|} \tag{39}$$

Surprisingly, this expression is identical to the probability of success for the unambiguous identification of only one system in a state given by the ensemble $\{1/2, |\phi_1\rangle\,; 1/2, |\phi_2\rangle\}$.

Section c

If now we are restricted to local measurements, then we need to perform an unambiguous identification in each system independently. That means that for each system we need to clearly determine if the state is $|\phi_1\rangle$ or $|\phi_2\rangle$. Thus, we will consider that an identification has been successful if the states of both systems are unambiguously identified.
For each individual identification, each system has the POVM $\{E_1, E_2, E_Q\}$, such that $E_1 \propto |\phi_2^{\perp}\rangle\langle\phi_2^{\perp}|$ and $E_2 \propto |\phi_1^{\perp}\rangle\langle\phi_1^{\perp}|$. Then, if we call the systems A and B, the possible outcomes we can measure in each system are:

$$\{E_1^A, E_2^A, E_Q^A\} \text{ and } \{E_1^B, E_2^B, E_Q^B\} \tag{40}$$

So, combined, we have 9 possible outcomes for both systems.
Therefore, an successful unambiguous identification will take place when any of the following outcomes are obtained:

$$\{E_1^A E_1^B, E_1^A E_2^B, E_2^A E_1^B, E_2^A E_2^B\} \tag{41}$$

as getting any outcome $E_Q^{A,B}$ would not be fully unambiguous. The probability we need to maximize is then:

$$P_{unambiguous} = P(E_1^A \cap E_1^B) + P(E_1^A \cap E_2^B) + P(E_2^A \cap E_1^B) + P(E_2^A \cap E_2^B) \tag{42}$$

As both systems are independent and identical:

$$\begin{aligned} P_{unambiguous} &= P(E_1^A)P(E_1^B) + P(E_1^A)P(E_2^B) + P(E_2^A)P(E_1^B) + P(E_2^A)P(E_2^B) = \\ &= P(E_1)^2 + 2P(E_1)P(E_2) + P(E_2)^2 = (P(E_1)+P(E_2))^2 = (P_{unamb}^{sys})^2 \end{aligned} \tag{43}$$

so the total probability of unambiguous identification is actually the product of the probability of unambiguous identification for each system.

Since the maximization for the POVM is done in each individual system, the success probability is then:

$$P_s = \max_{E^A,E^B} P_{unambiguous} = \max_{E^A,E^B} (P^{sys}_{unamb})^2 = (\max_{E} P^{sys}_{unamb})^2 = (P^{sys}_s)^2 \tag{44}$$

We can conclude that the probability of success to unambiguously determine if both systems are in the same state or not is the square of the probability of success for unambiguous identification of each system.

As mentioned before, the probability of success for a single system is developed in Section 1.6 of the lecture notes, where it is found to be $P^{sys}_s = 1 - |c|$ for the overlap between the states of the single system. The complete probability of success is then:

$$\boxed{P_{s,local} = (1 - |c|)^2} \tag{45}$$

We can see that the success probability for local measurements is always bounded by that for global measurements:

$$\boxed{P_{s,local} = (1 - |c|)^2 \leq 1 - |c| = P_{s,global}} \tag{46}$$

for $c \in [-1, 1]$.

As for the minimum error discrimination, we have found that a global measurement outperforms a local one for the unambiguous discrimination task, having the same value only in the cases $c = 0$ and $|c| = 1$. For $c = 0$, the states are orthogonal, so it is always possible to perfectly distinguishing them with certainty. For $c = \pm 1$, both states are actually the same, so trying to distinguish them becomes impossible.

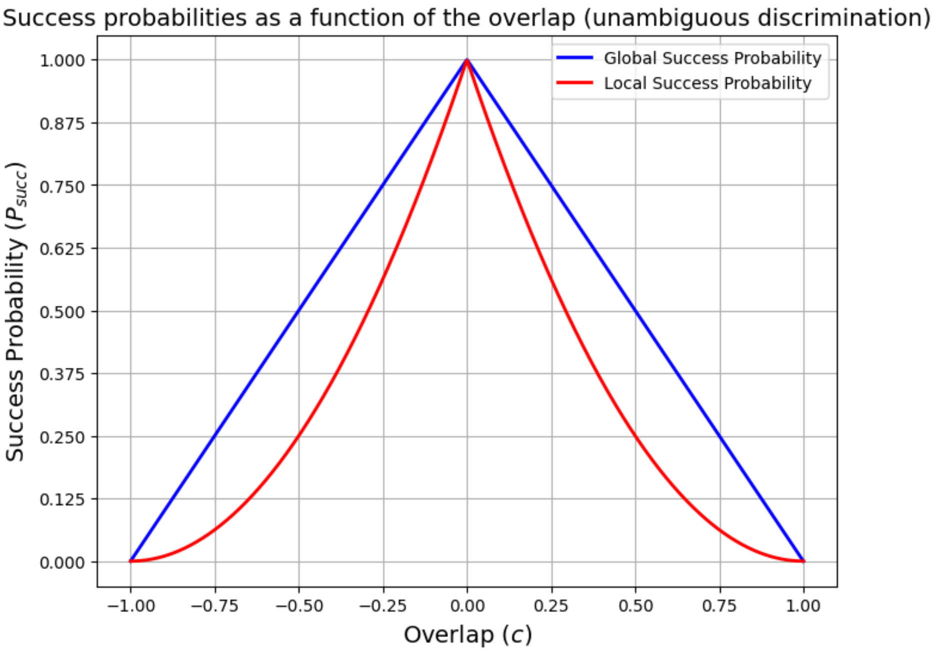

Figure 2: Representation of the success probability to distinguish with complete certainty if both systems are in the same state of not as a function of their overlap, using global measurements over both systems or only local measurements over each single system.

Question 4

Now we are given a set of N states $\{|\psi_k\rangle\}_{k=0}^{N-1}$ for $N \geq 2$ such that:

$$|\psi_k\rangle = \frac{|0\rangle + \omega^k |1\rangle}{\sqrt{2}}, \ \omega = e^{i\frac{2\pi}{N}} \tag{47}$$

We shall assume that each state is equiprobable, so the ensemble for each system is $\{1/N, |\psi_k\rangle_{k=0}^{N-1}\}$. Then the density matrix of each system is:

$$\begin{aligned}
\rho_{sys} &= \sum_{k=0}^{N-1} \eta_k |\psi_k\rangle\langle\psi_k| = \sum_{k=0}^{N-1} \frac{1}{N} \frac{|0\rangle + \omega^k |1\rangle}{\sqrt{2}} \frac{\langle 0| + (\omega^*)^k \langle 1|}{\sqrt{2}} = \\
&= \frac{1}{2N} \sum_{k=0}^{N-1} \big(|0\rangle\langle 0| + (\omega^*)^k |0\rangle\langle 1| + \omega^k |1\rangle\langle 0| + |1\rangle\langle 1| \big) = \\
&= \frac{1}{2N} \left(N |0\rangle\langle 0| + \left(\sum_{k=0}^{N-1} (\omega^*)^k \right) |0\rangle\langle 1| + \left(\sum_{k=0}^{N-1} \omega^k \right) |1\rangle\langle 0| + N |1\rangle\langle 1| \right)
\end{aligned} \tag{48}$$

Using the given hint for a geometrical series, which is valid as $r = e^{i\frac{2\pi}{N}} \neq 1$ because $N > 1$, then:

$$\sum_{k=0}^{N-1} (\omega^*)^k = \frac{1 - (\omega^*)^N}{1 - \omega^*} = \frac{1 - e^{-i2\pi}}{1 - e^{-i\frac{2\pi}{N}}} = 0 \ ; \quad \sum_{k=0}^{N-1} \omega^k = \frac{1 - \omega^N}{1 - \omega} = \frac{1 - e^{i2\pi}}{1 - e^{i\frac{2\pi}{N}}} = 0 \tag{49}$$

Along the exercise, this property will be implicitly used several times to remove irrelevant terms from summations. This, as one would have expected, from the circular symmetry of the system, yields an identity matrix:

$$\rho_{sys} = \frac{1}{2N} \big(N |0\rangle\langle 0| + N |1\rangle\langle 1| \big) = \frac{1}{2} \mathbb{1} \tag{50}$$

which corresponds to the maximally mixed state.

The global density matrix is then:

$$\rho = \rho_{sys_A} \otimes \rho_{sys_B} = \frac{\mathbb{1}_2}{2} \otimes \frac{\mathbb{1}_2}{2} = \frac{1}{4} \mathbb{1}_4 \tag{51}$$

where the subscript under the identity matrix denotes the dimension of the correspondent Hilbert space.

Now, as in Question 1, if the systems are in the same state ($\rho_=$), then they can be described by the pure state $|\psi_k \psi_k\rangle$, $k = 0, ..., N-1$, and, equivalently, if they are in different states ($\rho_{\neq}$), then they can be described by $|\psi_k \psi_j\rangle$, $k \neq j$, $k, j = 0, ..., N-1$. Therefore, we have:

$$\rho_= = \frac{1}{N} \sum_{k=0}^{N-1} |\psi_k \psi_k\rangle\langle\psi_k \psi_k| \ ; \quad \rho_{\neq} = \frac{1}{N(N-1)} \sum_{\substack{k,j=0 \\ j \neq k}}^{N-1} |\psi_k \psi_j\rangle\langle\psi_k \psi_j| \tag{52}$$

where the prefactors have been chosen so that both $\rho_=$ and $\rho_{\neq}$ have trace one, as each term in the summations has trace one:

$$\begin{aligned}
&\mathrm{Tr}\{|\psi_k \psi_j\rangle\langle\psi_k \psi_j|\} = \mathrm{Tr}\{|\psi_k\rangle\langle\psi_k| \otimes |\psi_j\rangle\langle\psi_j|\} = \mathrm{Tr}\{|\psi_k\rangle\langle\psi_k|\} \, \mathrm{Tr}\{|\psi_j\rangle\langle\psi_j|\} = \\
&= \mathrm{Tr}\left\{ \frac{|0\rangle\langle 0| + (\omega^*)^k |0\rangle\langle 1| + \omega^k |1\rangle\langle 0| + |1\rangle\langle 1|}{2} \right\} \mathrm{Tr}\left\{ \frac{|0\rangle\langle 0| + (\omega^*)^j |0\rangle\langle 1| + \omega^j |1\rangle\langle 0| + |1\rangle\langle 1|}{2} \right\} = 1 \cdot 1 = 1
\end{aligned} \tag{53}$$

Notice that, this time, the case when the systems are in different states, contains many more states ($N-1$ times), than the case where the systems are in equal states; and this size difference becomes bigger as N increases.

From the expressions for $\rho_=$ and $\rho_{\neq}$, we can explicitly get the priors for the systems being equal $\eta_=$, or different $\eta_{\neq}$, checking that ρ can be reconstructed through the following expression:

$$
\begin{aligned}
\frac{N-1}{N}\rho_{\neq}+\frac{1}{N}\rho_= &= \frac{1}{N^2}\sum_{\substack{k,j=0\\ j\neq k}}^{N-1}|\psi_k\psi_j\rangle\!\langle\psi_k\psi_j| + \frac{1}{N^2}\sum_{k=0}^{N-1}|\psi_k\psi_k\rangle\!\langle\psi_k\psi_k| = \frac{1}{N^2}\sum_{k,j=0}^{N-1}|\psi_k\psi_j\rangle\!\langle\psi_k\psi_j| = \\
&= \frac{1}{N^2}\sum_{k,j=0}^{N-1}\frac{|00\rangle+\omega^j|01\rangle+\omega^k|10\rangle+\omega^{k+j}|11\rangle}{2}\,\frac{\langle 00|+(\omega^*)^j\langle 01|+(\omega^*)^k\langle 10|+(\omega^*)^{k+j}\langle 11|}{2} = \\
&= \frac{1}{4N}\sum_{j=0}^{N-1}\Big(\big(|00\rangle+\omega^j|01\rangle\big)\big(\langle 00|+(\omega^*)^j\langle 01|\big)+\big(|10\rangle+\omega^j|11\rangle\big)\big(\langle 10|+(\omega^*)^j\langle 11|\big)\Big) = \\
&= \frac{1}{4}\big(|00\rangle\!\langle 00|+|01\rangle\!\langle 01|+|10\rangle\!\langle 10|+|11\rangle\!\langle 11|\big) = \frac{1}{4}\mathbb{1}_4 = \rho
\end{aligned}
\tag{54}
$$

from where we can see that:

$$
\rho = \frac{1}{N}\rho_= + \frac{N-1}{N}\rho_{\neq} = \left\{\frac{1}{N},\rho_=;\frac{N-1}{N},\rho_{\neq}\right\} \;\rightarrow\; \eta_= = \frac{1}{N}\ ,\ \eta_{\neq} = \frac{N-1}{N}
\tag{55}
$$

which is what one would expect, since for a random sampling, once you have the first system fixed, there is just one case were the other system will be equal, and $N-1$ cases were it will be different.

Now, we use the minimum error approach to maximize the probability of a successful identification, i.e.,

$$
P_{\mathrm{s}} = \frac{1}{2}+\frac{1}{2}||\eta_=\rho_= - \eta_{\neq}\rho_{\neq}||_1 = \frac{1}{2}\big(1+||\eta_{\neq}\rho_{\neq}-\eta_=\rho_=||_1\big)
\tag{56}
$$

so we just need to compute this subtraction. To do so, we will first notice that the subtraction is equal to:

$$
\eta_{\neq}\rho_{\neq}-\eta_=\rho_= = \rho - 2\eta_=\rho_=
\tag{57}
$$

So we will only need to express in the computational basis $\rho_=$, which will be easier:

$$
\begin{aligned}
\rho_= &= \frac{1}{N}\sum_{k=0}^{N-1}\frac{|00\rangle+\omega^k\big(|01\rangle+|10\rangle\big)+\omega^{2k}|11\rangle}{2}\,\frac{\langle 00|+(\omega^*)^k\big(\langle 01|+\langle 10|\big)+(\omega^*)^{2k}\langle 11|}{2} = \\
&= \frac{1}{4}\big(|00\rangle\!\langle 00|+\delta_{N2}|00\rangle\!\langle 11|+|01\rangle\!\langle 01|+|01\rangle\!\langle 10|+|10\rangle\!\langle 01|+|10\rangle\!\langle 10|+\delta_{N2}|11\rangle\!\langle 00|+|11\rangle\!\langle 11|\big) = \\
&= \frac{1}{4}\big(\mathbb{1}_4+|01\rangle\!\langle 10|+|10\rangle\!\langle 01|+\delta_{N2}(|00\rangle\!\langle 11|+|11\rangle\!\langle 00|)\big)
\end{aligned}
\tag{58}
$$

where we have used the hint from before, knowing that each term that has a remaining ω will be 0 after the sum, leaving only the pairs that cancel their respective ω's. The coefficient δ_{N2} is a Kronecker delta for $N=2$. These two extra terms are zero in general but not for $N=2$:

$$
\begin{aligned}
&\text{Term } |00\rangle\!\langle 11| : \sum_{k=0}^{N-1}(\omega^*)^{2k} = \sum_{k=0}^{N-1}(e^{-i\frac{4\pi}{N}})^k \;\rightarrow\; \text{if } N=2: \sum_{k=0}^{N-1}(e^{-i2\pi})^k = \sum_{k=0}^{N-1}1 = N \\
&\text{Term } |11\rangle\!\langle 00| : \sum_{k=0}^{N-1}\omega^{2k} = \sum_{k=0}^{N-1}(e^{i\frac{4\pi}{N}})^k \;\rightarrow\; \text{if } N=2: \sum_{k=0}^{N-1}(e^{i2\pi})^k = \sum_{k=0}^{N-1}1 = N
\end{aligned}
\tag{59}
$$

This means that the subtraction ends like:

$$
\begin{aligned}
M = \eta_{\neq}\rho_{\neq}-\eta_=\rho_= &= \rho-2\eta_=\rho_= = \frac{1}{4}\mathbb{1}_4 - \frac{1}{2N}\big(\mathbb{1}_4+|01\rangle\!\langle 10|+|10\rangle\!\langle 01|+\delta_{N2}(|00\rangle\!\langle 11|+|11\rangle\!\langle 00|)\big) = \\
&= \underbrace{\left(\frac{1}{4}-\frac{1}{2N}\right)}_{a}\mathbb{1}_4 \underbrace{-\frac{1}{2N}}_{b}\big(|01\rangle\!\langle 10|+|10\rangle\!\langle 01|+\delta_{N2}(|00\rangle\!\langle 11|+|11\rangle\!\langle 00|)\big) = \begin{pmatrix} a & 0 & 0 & b\delta_{N2} \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b\delta_{N2} & 0 & 0 & a \end{pmatrix}
\end{aligned}
\tag{60}
$$

And the eigenvalues from this are $\{a \pm b, a \pm b\delta_{N/2}\}$, so the norm is:

$$
\begin{aligned}
||\eta_{\neq}\rho_{\neq} - \eta_{=}\rho_{=}||_1 &= \sum_{k=1}^{4} |M_{kk}| = |a+b| + |a-b| + |a+b\delta_{N2}| + |a-b\delta_{N2}| = \\
&= \left|\frac{1}{4} - \frac{1}{N}\right| + \left|\frac{1}{4}\right| + \left|\frac{1}{4} - \frac{(1+\delta_{N2})}{2N}\right| + \left|\frac{1}{4} - \frac{(1-\delta_{N2})}{2N}\right|
\end{aligned} \tag{61}
$$

We need to particularize now for each possible N, taking into account the delta and the absolute values:

$$
||\eta_{\neq}\rho_{\neq} - \eta_{=}\rho_{=}||_1 = \begin{cases} N=2: & \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1 \\ N=3: & \frac{1}{12} + \frac{1}{4} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2} \\ N \geq 4: & \frac{1}{4} - \frac{1}{N} + \frac{1}{4} + \frac{1}{4} - \frac{1}{2N} + \frac{1}{4} - \frac{1}{2N} = 1 - \frac{2}{N} \end{cases} \tag{62}
$$

so finally the success probability for each possible N is:

$$
\boxed{P_s = \begin{cases} N=2: & \frac{1}{2}(1+1) = 1 \\ N=3: & \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4} \\ N \geq 4: & \frac{1}{2}\left(1 + 1 - \frac{2}{N}\right) = 1 - \frac{1}{N} \end{cases}} \tag{63}
$$

For the case $N = 2$, where this question corresponds to two orthogonal states, we can compare it with equation 10 from Question 1 if we take $c = 0$, and verify that we obtain a success probability of 1 with both expressions.

Then, analyzing the behaviour of the expression as a function of N, we see that for $N = 3$ and $N = 4$ we obtain a minimum of $3/4$, and then, the probability of success increases again going asymptotically to one for $N \to \infty$. This should not surprise us, since when $N \to \infty$, the probability of getting different states in the two systems goes to 1, while the probability of getting the same states becomes negligible (it goes to 0). Therefore as $N \to \infty$, the probability of correctly stating that the systems are in different states goes to 1, and the success probability as well:

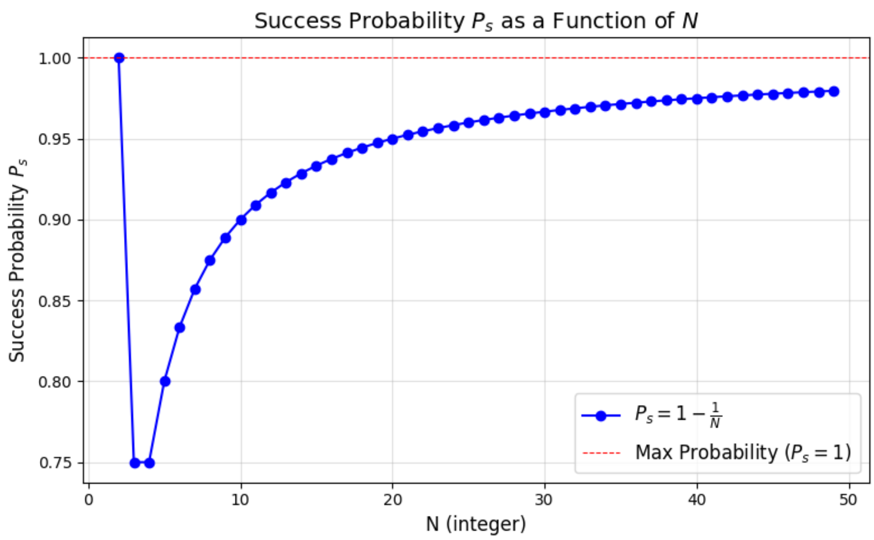

Figure 3: Success probability of telling if both systems are equal or not, as a function of the integer N, where N represents the number of possible quantum states homogeneously distributed around the Bloch sphere's equatorial plane. The success probability P_s measures the likelihood of correctly distinguishing the states in a given quantum measurement setup. As N increases, the states become more densely packed, resulting in a success probability that asymptotically approaches 1. For $N = 2$, $P_s = 1$ (perfect discrimination is possible), while for $N = 3$, $P_s = \frac{3}{4}$, and for larger N, the formula is $P_s = 1 - \frac{1}{N}$ (convergence to $P_s = 1$).