Advanced Quantum Information Theory
(Quantum Master Barcelona)

Homework 1 (19 December 2024, due 12 January 2025)

Solve exercises 1, 2, 3 and 4 and two others of your choice. Justify every step. Each problem is worth 10 points. Your written return is going to be evaluated.

1. Prove that the constant channel $\mathcal{P}_\sigma : A \to B$, acting as $\mathcal{P}_\sigma(\rho) = \sigma(\operatorname{Tr}\rho)$, with a state σ on B, is indeed a cptp linear map.

2. Prove that for the constant channel $\mathcal{P}_\sigma$, its adjoint map is given by $\mathcal{P}_\sigma^*(X) = (\operatorname{Tr}\sigma X)\mathbb{1}$.

3. Prove that the sequential composition, the tensor product and the convex combination of cptp maps is cptp.

4. Prove that for the ideal channel id_A, $C_\epsilon(\mathrm{id}_A) = \log\left\lfloor \frac{|A|}{1-\epsilon} \right\rfloor$.

5. Prove that the hypothesis testing relative entropy is invariant under an isometry applied to both states, i.e. for any isometry $V : A \to B$ it holds that
$$D_h^\epsilon(\mathcal{V}(\rho)\|\mathcal{V}(\sigma)) = D_h^\epsilon(\rho\|\sigma),$$
where $\mathcal{V} : \mathcal{L}(A) \to \mathcal{L}(B)$ with $\mathcal{V}(\rho) = V\rho V^\dagger$ is the isometry channel.

6. Recall that we defined the average error probability of a code $\mathcal{C}$ as $P_e(\mathcal{C}) := \frac{1}{K}\sum_{m=1}^{K} \Pr\left\{\widehat{M} \neq m | M = m\right\} = \Pr\{\widehat{M} \neq M\}$, with uniformly distributed $M \in [K]$. In contrast, define the worst-case error probability as
$$P_{\max}(\mathcal{C}) := \max_{m\in[K]} \Pr\left\{\widehat{M} \neq m | M = m\right\}.$$
Show that for any code $\mathcal{C}$ with average error $P_e(\mathcal{C}) \leq \epsilon$ and rate R, there exists a code $\mathcal{C}'$ with worst-case error $P_{\max}(\mathcal{C}') \leq 2\epsilon$ and rate at least $R - 1$.

   [Hint: Apply Markov inequality to the random variable $p_m := \Pr\left\{\widehat{M} \neq m | M = m\right\}$, which is a function of the random variable $M = m$ corresponding to the message.]

7. Find matrices $A, B \geq 0$ such that $A \leq B$ and $A^2 \leq B^2$, but $A^4 \not\leq B^4$. Show that however for $[A, B] = 0$, it holds $A \leq B \Rightarrow A^2 \leq B^2$.

8. Let $C_\epsilon^{(\text{prod-ass})}(\mathcal{N})$ denote the supremum over the rates of entanglement-assisted codes $(\omega^{T_A T_B}, E, D)$ for the channel $\mathcal{N}$ such that the shared state is $\omega^{T_A T_B} = \omega^{T_A} \otimes \omega^{T_B}$, i.e. it is a product state. Show that $C_\epsilon^{(\text{prod-ass})}(\mathcal{N}) = C_\epsilon(\mathcal{N})$.

   [Hint: For the $\geq$ direction modify the proof of Remark 1.14. The $\leq$ direction requires a new proof.]