Lecture 5:
2.1 How to build circuit Hamiltonians

This lecture will be based on references [1], [2], [3]

0 Introduction

What makes a circuit quantum? A quantum circuit is that which requires quantum mechanics to be described. This implies phenomena such as energy quantization, reversible unitary evolution, quantum superpositions and entanglement. The degrees of freedom of a quantum circuit -current, voltage, charge, flux- are quantum observables. Measurements of these observables lead to probability distributions governed by uncertainty relations. Modelling the circuits requires solving the Schrödinger equation for pure states. This affects greatly how these circuits are designed, controlled and operated.

The quantumness of a physical system may be spoiled by its size, environmental fluctuations and thermal fluctuations.

- *Macroscopicity.* Quantum circuits are built on solid state devices, involving selected (i.e. engineered) degrees of freedom that exhibit observable features -such as a charge differences of $2e$ or flux differences of $\mathrm{m}\Phi_0$- depending on their quantum state, which reside on top of a macroscopic state, the superconducting wave function, involving the entirety of the system. These are typically known as mesoscopic systems, composed of a macroscopic number of particles but exhibiting discrete excitations from quantized degrees of freedom.

  In the macroscopic world, Heisenberg's uncertainty $\Delta x \Delta p \geq \hbar/2$ imposes limitations to observables which are far from our reach. For instance, assuming we can track particles with light, the diffraction limit sets an error of $\Delta x \approx$ 400 nm, leading to $\Delta v \sim 10^{-25}$ m/s, which isn't observable for us.

- *Thermal fluctuations.* A quantum system in a finite temperature will have its quantum states occupied following a Boltzmann distribution $\exp(-\hbar\omega_i n/k_B T)$, with ω_i being the energy to populate state i. This thermal distribution contains no entanglement, in fact it destroys it as the system is brought to the thermal equilibrium, where it ends in a mixed state defined by its density matrix ρ. The probability of being in any excited state is $p_{\mathrm{exc}} = 1 - p_0 = \exp(-\hbar\omega/k_B T)$. A typical circuit of $\omega = 2\pi 5$ GHz will have a reference temperature of $T_r = \hbar\omega/k_B = 0.24$ K. For a temperature of 100 mK, the excitation

probability will be 9%. For 10 mK, 3×10^{-11}. Therefore, it is really necessary to operate at those low temperatures.

- *Environmental fluctuations.* Controlling a qubit system requires accessing its quantum state with on-chip antennas, probes, electromagnetic fields, etc. At the same time, the qubit circuit sits on a substrate full of microscopic phenomena that unavoidably interact with the qubit state. All these phenomena will result in the qubit radiating its energy towards its environment, as well as experience fluctuations in its energy level splitting from the very same interaction with the environment. The processes that we identify as causes of environmental fluctuations can be separated in *dissipation, heating* and *dephasing*. Together they lead to *decoherence* of the quantum state, erasing its quantum coherence and driving it towards a 'classical state', which shows no coherence.

  Part of the environmental fluctuations are mitigated by the usage of superconductors, that dissipate no heat up to the frequencies qubits operate. But there are still lots of microscopic defects present on the substrate that interact with the qubit producing dielectric, charge and flux noise, radiation to free space, etc. The control and readout circuitry also does add noise, which requires heavy engineering to mitigate.

In summary, all efforts in producing good qubits boil down to selecting a set of quantum states, obtaining a high enough internal quality factor defining the qubit (that is, the number of oscillations it can produce within a period of its lifetime) reflected in the linewidth of resonances of the circuit, which need to be as narrow as possible so we do not interfere with undesired resonances.

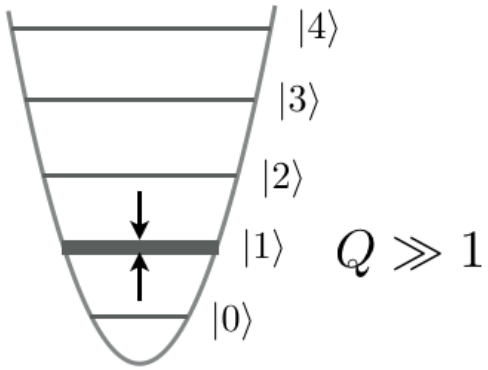

Figure 1: Circuit resonances with linewidth indicated.

1 Circuit elements

A quantum circuit follows very similarly its classical counterpart. A set of dynamic equations, involving either voltage or current, are established which determine the

evolution of all circuit parameters. One can move from this picture to a Hamiltonian that contains essentially the same information, from which the same dynamic equations can be derived. In quantum circuits, though, the circuit variables will be represented by quantum operators obeying commutation relations.

Any electrical circuit can be described as a network of branches consisting of two-terminal components, such as capacitors, inductors, etc. In qubit circuits, the individual circuit components are considered in the lumped-element approximation, which is justified as the size of the component D is much smaller than the wavelength of the frequencies considered, $\lambda \gg D$ since microwave frequencies have wavelengths in the range of centimeters. Each component is described by two branch variables, the electric current intensity I and the voltage drop V across the element. V and I have therefore no spatial dependence across the circuit element in the lumped element approximation.

The current direction is arbitrarily assigned. In the particular drawing in Fig. 2 a positively charged particles moves from A to B. In ordinary circuits, positive charges flow from areas of high potential energy to a low potential energy. This sets a convention that a voltage difference $V = V_B - V_A$ will grow in the opposite direction as the current flow. Negative charges, such as Cooper pairs, flow towards larger potential. Therefore, a moving negative charge flow from A to B leaves behind a positive amount of charge as if there was a positive current from B to A, so the sign of the positive current will still be opposite to the voltage increase.

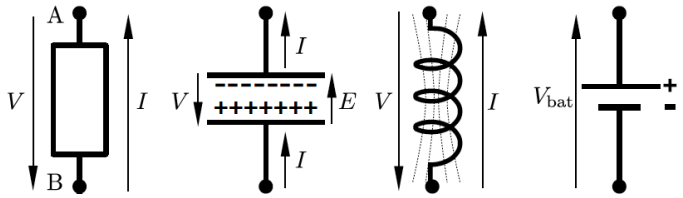

Figure 2: Circuit elements and the usual current and voltage convention. From left to right: generic element, capacitor, inductor, and battery.

In circuit Hamiltonians, it will be natural to introduce branch fluxes and branch charges defined as

$$\Phi(t) = \int_{-\infty}^{t} V(t')dt', \tag{1}$$

$$Q(t) = \int_{-\infty}^{t} I(t')dt'. \tag{2}$$

The circuit is supposed to have been at rest at time $t = -\infty$ with no voltages and currents, and adiabatically switched on until $t = 0$.

Each circuit element is characterized by a constitutive relation of the form $V = f(Q)$ if it is capacitive and $i = g(\Phi)$ if it is inductive. Next, we describe several circuit elements typically used in qubit circuits.

1.1 Capacitors

In a capacitor, the constitutive relation is

$$V = V_B - V_A = \frac{Q}{C} = \frac{1}{C}\int_0^t I(t')dt', \tag{3}$$

where the constant C is the circuit capacitance. We can use the equivalent form

$$I = C\frac{dV}{dt} = C\frac{d^2\Phi}{dt^2} = C\ddot{\Phi}. \tag{4}$$

Charging a capacitor requires some energy to move charges from V_A to V_B:

$$E_{\text{cap}} = \int_0^t I(t')V(t')dt' = \int_0^t \frac{dQ(t')}{dt'}\frac{Q(t')}{C}dt' = \frac{Q^2}{2C} = \frac{CV^2}{2} = \frac{QV}{2}, \tag{5}$$

where $Q(0) = 0$ and $Q(t) \equiv Q$.

In the case of a superconductor, the charge is transferred in units of Coopers pairs. So if n Cooper pairs have been transferred to charge the capacitor, the energy is

$$E_{\text{cap}} = 4E_C n^2, \text{ with } E_C = \frac{e^2}{2C}. \tag{6}$$

E_C is known as the charging energy and will become a relevant energy scale in qubit circuits. Typical values of C range in the femtofarad to picofarad, with E_C taking values in the MHz to GHz range. The value of C will be usually defined by the geometry of the circuit.

1.2 Inductors

The constitutive relation of an inductor is

$$V = L\frac{dI}{dt}, \tag{7}$$

or equivalently

$$\Phi = LI. \tag{8}$$

L is the inductance. The energy stored in the inductor when a current I passes through it is

$$E_{\text{ind}} = \int_0^t V(t')I(t')t' = L\int_0^t \frac{dI(t')}{dt'}I(t')dt' = \frac{LI^2}{2} = \frac{\Phi^2}{2L} = \frac{I\Phi}{2}. \tag{9}$$

L may contain both the geometric inductance as well as the kinetic inductance, especially in superconductors. Typical values of L range in the picoHenry in micron-sized loops to nanoHenry when using superinductors [4].

1.3 Josephson junctions

Until now we have only found linear relations in the constitutive equations between circuit variables. Now we move to nonlinear relations. We have already seen that in a Josephson junction with a phase across its ends φ we can define the equivalent flux as $\Phi = (\Phi_0/2\pi)\varphi$. Therefore, the constitutive relation in this case is the first Josephson relation

$$I = I_C \sin\left(2\pi \frac{\Phi}{\Phi_0}\right). \tag{10}$$

We have already seen that the junction behaves as a nonlinear inductor and for small currents it shows an inductance $L_J \sim \Phi_0/2\pi I_C$.

The energy stored in a junction has both capacitive and inductive components, given its parallel-plate geometry usually found in tunnel junctions used in qubits,

$$E_{\mathrm{JJ}} = \frac{q^2}{2C_J} - E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right). \tag{11}$$

Here, C_J is the geometric junction capacitance and $E_J = (\Phi_0/2\pi)I_C$ is the Josephson inductance. Typical values found in qubits range in the $I_C \sim 10 - 500$ nA, leading to $E_J/h \sim 5 - 250$ GHz.

1.4 Voltage and current sources

Voltage and current sources are active components that set the value of either the voltage or the current at a certain point in the circuit. They will be then treated as boundary conditions, leaving the other variables free to adjust to the rest of circuit constraints. A battery, for instance, will set a voltage $V_A - V_B = V_{\mathrm{bat}}$ and ideally will set no limit to the current it can supply to the circuit.

1.5 Resistors

Until now we only considered reactive elements that introduce no dissipation to the circuit. In general, one also needs to consider resistance, even though they will become potential noise sources since they remove energy from the circuit in an incoherent way, and may also generate heat. For this reason resistors will be avoided, even though there are good reasons to use them as quasiparticle traps, for instance [5].

2 Circuit quantization procedure

The procedure to obtain a quantized Hamiltonian from an electrical circuit obeying quantum mechanical laws consists of a systematic process that begins with Kirchoff's laws to yield the equations of motion of the degrees of freedom, obtains the Lagrangian and finally the Hamiltonian, which is finally quantized. This applies to circuits with reactive elements (capacitors, inductors), Josephson junctions and voltage and current sources. The theory is valid at low temperatures for excitations below the superconducting gap.

Assuming a circuit in the lumped-element approximation, the steps to quantize it are the following:

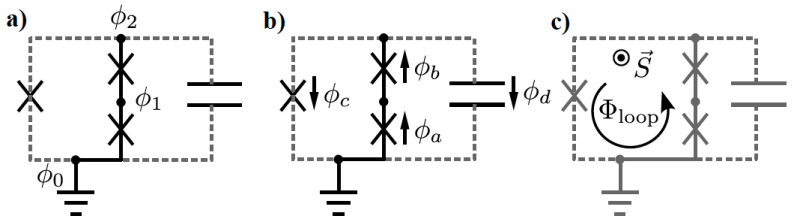

Figure 3: Quantization steps. (a) Open tree and node fluxes; (b) define flux branches; (c) define loop fluxes to obtain independent branches.

1. **Labeling nodes**: Identify all distinct nodes in the circuit -also known as superconducting islands- and associate a flux variable to each of them Φ_j. All points in the same island or circuit line are equivalent and a line can have one distinct node. The flux node is defined as the time integral of the voltage at that node referenced to the ground potential.

2. **Circuit tree**: Define a reference node -or ground- and build open branches (not enclosing loops) that form a tree T connecting all nodes. Each branch is a connection between two nodes. Then create the necessary closing branches that complete circuit loops C. In figure 3, $T = \{a, b\}$, $C = \{c, d\}$. Choosing a circuit tree is equivalent to selecting a gauge in electromagnetism or a particular set of coordinates in newtonian mechanics.

3. **Branch fluxes**: Associate a flux difference $\Phi_{i\to j} = \Phi_j - \Phi_i$, for $i \to j \in T$ to each branch that includes a circuit element. This new variable $\Phi_{i\to j}$ is the difference between the fluxes at the end node j and the beginning node i of the branch. This choice of direction of branches is arbitrary (analogous to current sign convention in Kirchoff's laws) but must be consistent throughout the quantization procedure.

4. **Fluxoid quantization**: For branch elements closing loops, the flux difference $\Phi_{i'\to j'}$ with $i' \to j' \in C$ is determined from fluxoid quantization. For a closed loop threaded by the external flux Φ_{ext} that encloses n fluxoids, fluxoid quantization results in

$$\sum_{\text{x}\in\text{loop}} \Phi_x = n\Phi_0 + \Phi_{\text{ext}}. \tag{12}$$

   When all branches belong to the main tree T except for branch $x' = i' \to j'$, the closure branch flux $\Phi_{x'}$ is expressed as function of the rest of flux branches. The sum of flux differences in each branch around a loop equals the number of fluxoids plus the external flux following the loop orientation defined by the branch flux directions, as in Fig. 3(c)

$$\Phi_{x'} = n\Phi_0 + \Phi_{\text{ext}} - \sum_{x\in T} \Phi_x. \tag{13}$$

Note that capacitors in closing branches cannot enclose a loop and therefore cannot trap flux.

5. **Branch currents**: Compute the currents traversing each circuit element. Consider the current direction on branch $b = i \to j$, which flows from j to i:

   - Capacitor: $I_b = C_b \ddot{\Phi}_b$,
   - Inductor: $I_b = \Phi_b / L_b$,
   - Junction: $I_b = I_{C,b} \sin(2\pi\Phi_b/\Phi_0)$.

   The circuit components $C_b, L_b, I_{C,b}$ do not depend on the orientation of each branch.

6. **Current conservation**: For each node, write a current conservation equation where the sum of all currents must be zero:

$$\sum_j I_{j\to i} = 0. \tag{14}$$

   Each node i will then have its associated equation

$$\sum_{b\in cap} s_i^b C_b \ddot{\Phi}_b + \sum_{b\in ind} s_i^b \frac{1}{L_b}\Phi_b + \sum_{b\in jj} s_i^b I_b \sin(2\pi\Phi_b/\Phi_0) = 0, \tag{15}$$

   where $s^b = \partial\Phi^b/\partial\Phi_i = \pm 1$ is a sign determining whether the branch leaves or enters the node i.

7. **Holonomic constraints**: Real circuits contain an electrical ground, batteries or microwave sources. These elements fix either the voltage of a node $V_k = \dot{\Phi}_k$ or the current along a branch. Using these constraints to eliminate variables and equations.

8. **Lagrangian**: Rewrite Eqs. (15) in the form

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\Phi}_i} = \frac{\partial\mathcal{L}}{\partial\Phi_i}, \tag{16}$$

   in order to identify the circuit Lagrangian $\mathcal{L}$, which will be a function of the node fluxes $\mathbf{\Phi} = (\Phi_0, \Phi_1, \dots)$:

$$\mathcal{L}(\mathbf{\Phi}, \dot{\mathbf{\Phi}}) = E_{cap}(\mathbf{\Phi}, \dot{\mathbf{\Phi}}) - E_{ind}(\mathbf{\Phi}, \dot{\mathbf{\Phi}}), \tag{17}$$

   with

$$E_{cap}(\mathbf{\Phi}, \dot{\mathbf{\Phi}}) = \sum_{b\in cap} \frac{1}{2} C_b \dot{\Phi}_b^2, \tag{18}$$

$$E_{ind}(\mathbf{\Phi}, \dot{\mathbf{\Phi}}) = -\sum_{b\in ind} \frac{\Phi_b^2}{2L_b} + \sum_{b\in jj} E_{J,b} \cos(2\pi\Phi_b/\Phi_0). \tag{19}$$

   Here we are using the branch fluxes but in reality the Lagrangian needs to be expressed as function of the node fluxes Φ_i.

9. **Canonical variables**: Define the node charges as the canonically conjugate momenta of the node fluxes

$$q_i = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i}. \tag{20}$$

Using these variables and a Legendre transformation to obtain a Hamiltonian

$$H(\mathbf{q}, \mathbf{\Phi}) = \sum_i q_i \dot{\Phi}_i - \mathcal{L}(\mathbf{\Phi}, \dot{\mathbf{\Phi}}), \tag{21}$$

and expressing the flux time derivatives as function of charges and fluxes, $\dot{\mathbf{\Phi}}(\mathbf{q}, \mathbf{\Phi})$. As a confirmation, the calculated Hamiltonian leads to the evolution equations

$$\frac{d\mathbf{\Phi}}{dt} = \frac{\partial H}{\partial \mathbf{q}}, \frac{d\mathbf{q}}{dt} = -\frac{\partial H}{\partial \mathbf{\Phi}}. \tag{22}$$

10. **Quantization**: Replace the canonical variables (Φ_i, q_i) with operators $(\hat{\Phi}_i, \hat{q}_i)$ that satisfy commutation relations

$$[\hat{\Phi}_i, \hat{q}_j] = i\hbar\delta_{ij}, \tag{23}$$

and define the Hamiltonian operator $\hat{H}$ with flux and charge operators as variables. This Hamiltonian allows obtaining the wavefunction of the circuit through the Schrödinger equation.

We have derived the Hamiltonian using node fluxes Φ_i but this could have been performed by charge nodes Q_i [1]. When working with Josephson circuits, it is way more convenient to work with fluxes rather than charges.

We have now all tools and machinery to begin quantizing our first quantum circuits. The following sections deal with several examples.

3 The quantized LC oscillator

3.1 Battery powered LC oscillator

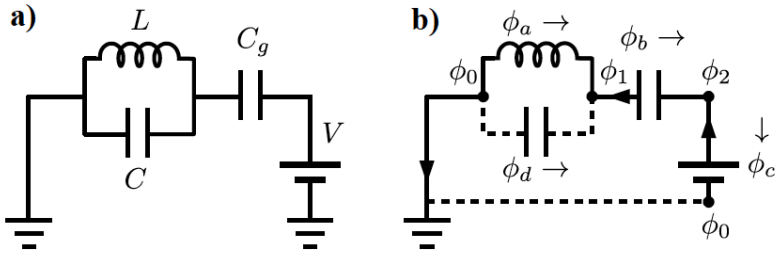

Figure 4: (a) LC resonator circuit connected to an external voltage source. (b) Nodes, node fluxes and branch fluxes for the circuit description and quantization.

[1] See the original circuit quantization work by Yurke and Denker [6]

Let us begin quantizing the LC circuit shown in figure 4 consisting of an LC resonator capacitively coupled to an external voltage source. In Fig. 4(b) we constructed a possible tree with nodes $\{\Phi_0, \Phi_1, \Phi_2\}$ and branches $\{\Phi_a, \Phi_b, \Phi_c\}$, with closing branch Φ_d. The directions chosen lead to the following sign of voltage (or flux) differences:

$$\Phi_a = \Phi_1 - \Phi_0 \tag{24}$$
$$\Phi_b = \Phi_2 - \Phi_1 \tag{25}$$
$$\Phi_c = \Phi_0 - \Phi_2. \tag{26}$$

The ground potential is a reference point. Therefore, we can set it to zero, such that $\Phi_0 = 0$. The battery fixes the branch flux $V = \dot{\Phi}_2 - \dot{\Phi}_0 = \dot{\Phi}_2$. Then, $\dot{\Phi}_c = -V$, and $\dot{\Phi}_b = V - \dot{\Phi}_1$. As there is no trapped flux in the LC loop, $\Phi_d = \Phi_a = \Phi_1$.

Now, at node Φ_1 the currents move to the left through the LC circuit. These currents must equate the current coming from node Φ_2, $C_g\ddot{\Phi}_b$:

$$C\ddot{\Phi}_1 + \frac{\Phi_1}{L} = C_g(\dot{V} - \ddot{\Phi}_1) \tag{27}$$

Introducing $C_\Sigma \equiv C + C_g$,

$$\frac{d}{dt}\left(C_\Sigma\dot{\Phi}_1 - C_gV\right) = -\frac{\Phi_1}{L}. \tag{28}$$

This equation is conservative and derives from the Lagrangian

$$\mathcal{L} = \frac{1}{2}C_\Sigma\dot{\Phi}_1^2 - C_gV\dot{\Phi}_1 - \frac{\Phi_1^2}{2L} \tag{29}$$

The conjugate momentum of the node flux Φ_1 is the charge $q_1 = \partial\mathcal{L}/\partial\dot{\Phi}_1 = C_\Sigma\dot{\Phi}_1 - C_gV$. Using the Legendre transform $H = q_1\dot{\Phi}_1 - \mathcal{L}$, we obtain

$$H = \frac{1}{2C_\Sigma}(q_1 + C_gV)^2 + \frac{1}{2L}\Phi_1^2 = \frac{1}{2C_\Sigma}(q_1 - q_g)^2 + \frac{1}{2L}\Phi_1^2, \tag{30}$$

where the externally induced charge was defined $q_g \equiv -C_gV$. Now all that is left is to add the hat $\hat{}$ to the charge and flux variables to yield the quantized Hamiltonian of the voltage-biased LC oscillator.

3.2 Multi-LC resonator loop

Consider now a set of three LC resonators in series, as shown in Fig. 5. Notice that now the circuit encloses a total flux Φ.

We define the following branch fluxes:

$$\Delta\Phi_a = \Phi_a - \Phi_g, \tag{31}$$
$$\Delta\Phi_b = \Phi_b - \Phi_g, \tag{32}$$
$$\Delta\Phi_c = \Delta\Phi_a - \Delta\Phi_b + \Phi = \Phi_a - \Phi_b + \Phi, \tag{33}$$

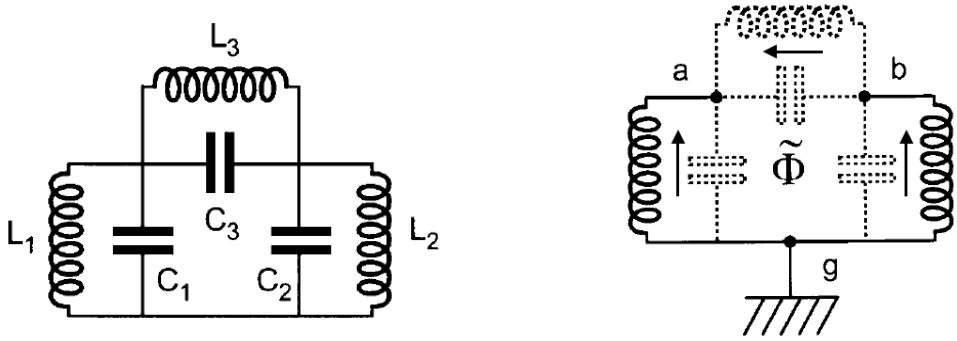

Figure 5: (Left) Multi-LC resonator circuit. (Right) Spanning tree with direction of branch fluxes.

where the last equation comes out of fluxoid quantization

$$-\Delta\Phi_a + \Delta\Phi_b + \Delta\Phi_c = \Phi, \tag{34}$$

following the branch flux direction as in Fig. 5(b) going counter-clockwise around the loop. The current conservation conditions for nodes a and b look

$$C_1\ddot{\Phi}_a + C_3(\ddot{\Phi}_a - \ddot{\Phi}_b) = -\frac{\Phi_a}{L_1} - \frac{\Phi_a - \Phi_b + \Phi}{L_3}, \tag{35}$$

$$C_2\ddot{\Phi}_b - C_3(\ddot{\Phi}_a - \ddot{\Phi}_b) = -\frac{\Phi_b}{L_2} + \frac{\Phi_a - \Phi_b + \Phi}{L_3}. \tag{36}$$

These equations lead from the Lagrangian

$$\mathcal{L}(\Phi_a, \dot{\Phi}_a, \Phi_b, \dot{\Phi}_b) = \frac{C_1\dot{\Phi}_a^2}{2} + \frac{C_2\dot{\Phi}_b^2}{2} + \frac{C_3(\dot{\Phi}_a - \dot{\Phi}_b)^2}{2} - \left[\frac{\Phi_a^2}{2L_1} + \frac{\Phi_b^2}{2L_2} + \frac{(\Phi_a - \Phi_b + \Phi)^2}{2L_3}\right]. \tag{37}$$

The charges in the circuit are defined as usual $q_a = \partial\mathcal{L}/\partial\dot{\Phi}_a$. The Hamiltonian of such a circuit turns out to be

$$\mathcal{H} = \frac{1}{C_1C_2 + C_1C_3 + C_2C_3}\left[\frac{(C_2 + C_3)q_a^2}{2} + \frac{(C_1 + C_3)q_b^2}{2} + \frac{C_3(q_a - q_b)^2}{2}\right] + \left[\frac{\Phi_a^2}{2L_1} + \frac{\Phi_b^2}{2L_2} + \frac{(\Phi_a - \Phi_b + \Phi)^2}{2L_3}\right]. \tag{38}$$

4 Quantization involving Josephson junctions

Most examples of circuits involving junctions will be given in the next chapter on the different qubit types. Nevertheless, there are some typical circuits involving junctions that will be used as complementary elements. For example, the rf-SQUID will reappear later as a flux-tunable coupler.

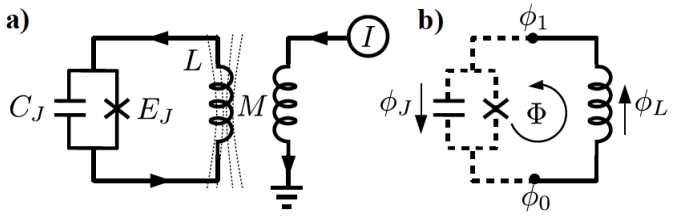

Figure 6: (a) rf-SQUID equivalent circuit including external bias source current. (b) Node fluxes for just the SQUID loop.

4.1 rf-SQUID

We focus now on a circuit containing a single Josephson junction shunted by a linear inductor, a circuit known as the rf-SQUID, shown in Fig. 6. The circuit can be operated as a flux magnetometer, and it is often used in quantum annealers as a tunable flux coupler.

This example shows a realistic scenario where the rf-SQUID loop is floating with respect to a ground potential. The external current source couples flux into the loop through the mutual inductance M. Following the node definitions in Fig. 6(b), let's define

$$\Phi_L = \Phi_1 - \Phi_0, \tag{39}$$

$$\Phi_J = -\Phi_L + \Phi, \tag{40}$$

$$\Phi_{\text{ext}} = MI. \tag{41}$$

Following the sign conventions, the downwards moving current I generated by the source induces a current in the loop $-(MI)/L$ that lowers the magnetic interaction energy between the two circuits. The total flux Φ contains both the external contribution generated by the source I, as well as the current generated by the loop itself Φ_L/L, so $\Phi = \Phi_L - MI$.

The current conservation in node Φ_1 equates the current leaving towards the junction with the current coming from the inductor, including the current induced by the external source:

$$\frac{\Phi_L - IM}{L} = C\ddot{\Phi}_J + I_C \sin(2\pi\Phi_J/\Phi_0). \tag{42}$$

Using the definitions from above to replace the branch fluxes into node fluxes, considering $\dot{\Phi} = 0$,

$$\frac{d}{dt}C(\dot{\Phi}_0 - \dot{\Phi}_1) = \frac{\Phi_1 - \Phi_0 - MI}{L} - I_C \sin\left[\frac{2\pi}{\Phi_0}(\Phi_0 - \Phi_1 + \Phi)\right]. \tag{43}$$

This leads to the Lagrangian

$$\mathcal{L} = \frac{1}{2}C_J(\dot{\Phi}_1 - \dot{\Phi}_0)^2 - \frac{1}{2L}(\Phi_1 - \Phi_0)^2 - \frac{M}{L}(\Phi_1 - \Phi_0)I - E_J \cos\left[2\pi\frac{\Phi_0 - \Phi_1 + \Phi}{\Phi_0}\right], \tag{44}$$

with the Josephson energy defined as $E_J = I_C\Phi_0/2\pi$. Using Φ_L as the independent variable, with conjugate momentum $Q_L = C_J\dot{\Phi}_L$, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2C_J}Q_L^2 + \frac{1}{2L}\Phi_L^2 - \frac{M}{L}I\Phi_L + E_J\cos\left[\frac{2\pi}{\Phi_0}(\Phi_L + \Phi)\right]. \tag{45}$$

The potential energy of this circuit leads to a configuration near maximum frustration $\Phi/\Phi_0 \sim 1/2$ where two local minima become degenerate and may be used as a qubit. This is the basis of flux-like qubits.

5 Number-phase representation

Up until now, we have only been using quantum operators in Hamiltonians obeying commutation relations $[\hat{\Phi}_i, \hat{q}_j] = i\hbar\delta_{ij}$. In this section, we will present two mathematical representation of these operators, with its own set of states in their respective Hilbert spaces. Note that it is necessary to define representations of these operators in order to compute the system wavefunction and eigenenergies.

Charge and flux are not exactly equivalent to momentum and position operators in mechanics. Charge is a discrete operator proportional to the number of Cooper pairs, $\hat{q} = -2e\hat{n}$. The flux operator is connected to the superconducting phase $\hat{\varphi} = 2\pi\hat{\Phi}/\Phi_0$ and therefore has a periodic representation. This means plane waves are not the appropriate representation of these operators and a new phase-number representation is needed.

- **Phase representation**: the eigenstates of the flux $\hat{\Phi}$ or phase $\hat{\varphi}$ operators are used. In this representation, the circuit wavefunction

$$|\Psi\rangle = \int_0^{2\pi} \mathrm{d}\varphi_1 \cdots \int_0^{2\pi} \mathrm{d}\varphi_N \Psi(\varphi_1, \ldots, \varphi_N)|\varphi_1, \ldots, \varphi_N\rangle$$

  expands over states with a well defined value of the node phases $|\varphi_i\rangle$. This is a continuous representation that transforms the Schrödinger equation into a partial-differential equation for $\Psi(\varphi_1, \ldots, \varphi_N)$, leading sometimes to analytical solutions but most of the time to numerical integration.

- **Number representation**: Here we use the eigenstates of the charge operator $\hat{q}$, which are states with a given number of excess of Cooper pairs $|n\rangle$. These states form a discrete basis, leading to a matrix representation that may be of infinite size. This requires introducing a truncation $|n| \leq n_{\max}$, leading sometimes to an analytical formulation or to a numerical diagonalization method.

In the number representation, the charge operator is a sum of projectors onto well-define numbers of Cooper pairs,

$$\hat{q} = \sum_n (-2e)|n\rangle\langle n|, \tag{46}$$

for $n \in \mathrm{Z}$. Any state in the Hilbert space can be expressed as a linear superposition of number states $|\Psi\rangle = \sum_n \Psi_n|n\rangle$. But we need to know how to write flux/phase

operators in this basis. Using the commutation relation, one can show (in the exercise list)

$$e^{i2\pi\hat{\Phi}/\Phi_0}\hat{q} = (\hat{q} - 2e)e^{i2\pi\hat{\Phi}/\Phi_0} \tag{47}$$

This implies that the phase operator $\hat{\varphi} = 2\pi\hat{\Phi}/\Phi_0$ is the generator of displacement in the space of charges. Using the dimensionless representation,

$$\hat{n}e^{i\hat{\varphi}} = e^{i\hat{\varphi}}(\hat{n} - 1), \tag{48}$$

applied on state $|n\rangle$

$$\hat{n}e^{i\hat{\varphi}}|n\rangle = e^{i\hat{\varphi}}(\hat{n} - 1)|n\rangle = (n-1)e^{i\hat{\varphi}}|n\rangle, \tag{49}$$

and defining the state $e^{i\hat{\varphi}}|n\rangle \equiv |\xi\rangle$,

$$\hat{n}|\xi\rangle = (n-1)|\xi\rangle. \tag{50}$$

Therefore,

$$e^{i\hat{\varphi}}|n\rangle = |n-1\rangle, \tag{51}$$

so we can conclude that the exponential of the phase is a ladder operator

$$e^{i\hat{\varphi}} = \sum_n |n-1\rangle\langle n|. \tag{52}$$

Conversely, take the hermitian conjugate of the previous relation to raise a Cooper pair number by 1,

$$e^{-i\hat{\varphi}}|n-1\rangle = |n\rangle. \tag{53}$$

The Josephson term then takes the form

$$\cos\hat{\varphi} = \frac{1}{2}(e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}) = \frac{1}{2}\sum_n \left[|n+1\rangle\langle n| + |n\rangle\langle n+1|\right]. \tag{54}$$

In the basis of the phase operator, the phase and number operators act on a generic wavefunction $\Phi(\varphi)$,

$$\hat{n}\left[e^{i\hat{\varphi}}\Psi(\varphi)\right] = e^{i\varphi}[(\hat{n}-1)\Psi(\varphi)]. \tag{55}$$

This equation identifies $\hat{n} \equiv i\partial_\varphi$. We can derive now the phase eigenstates from

$$\langle m|e^{i\hat{\varphi}}|\varphi\rangle = \langle m+1|\varphi\rangle = e^{i\varphi}\langle m|\varphi\rangle, \tag{56}$$

which implies

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}}\sum_{m\in Z} e^{i\varphi m}|m\rangle. \tag{57}$$

With these tools, we can write down the Hamiltonian of a Josephson junction in both representations. In the phase variable, the state is given by a wavefunction $\Psi(\varphi, t)$ that evolves in time with a partial differential equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -4E_C\frac{\partial^2\Psi}{\partial\varphi^2} - E_J\cos(\varphi)\Psi. \tag{58}$$

This method is useful for circuits with a single degree of freedom that can be readily integrated or even analytically solved. For other types of circuits it is more convenient to use the charge number representation, with a corresponding Hamiltonian

$$H = \sum_n [4E_C n|n\rangle\langle n| - E_J(|n+1\rangle\langle n| + |n\rangle\langle n+1|]. \tag{59}$$

In qubit applications, typically $E_J/E_C \simeq 100$. In this regime, the probability that a state with many Cooper pairs is occupied is very low. This implies that we can safely truncate the Hamiltonian to a finite number of charge number states $|n| \leq n_{\max} \sim 10$ or less.

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