Exercise list ‘Quantum Technologies with Superconducting Qubits’

1 Introductory concepts

1.1 Introductory elements of superconductivity

1. Consider a homogeneous, linear, isotropic, nondispersive, perfectly conducting slab $(\epsilon, \mu_0, \Lambda)$ with finite thickness $2a$ along the $\hat{y}$ axis, and infinitely long in the $\hat{x}, \hat{z}$ axes. Calculate the amplitude of the magnetic field inside the slab $\mathbf{H}_s$ when an external magnetic field $\mathbf{H}_{\text{app}}$ is applied in the following scenarios: (a) $\mathbf{H}_{\text{app}} = \text{Re}\{H_0 e^{j\omega t}\}\hat{x}$. (b) $\mathbf{H}_{\text{app}} = \text{Re}\{H_0 e^{j\omega t}\}\hat{y}$. (c) Consider the general case of a time-dependent, spatially homogeneous external magnetic field $\mathbf{H}_{\text{app}} = H_0(t)\hat{y}$ and calculate the full solution inside the slab $\mathbf{H}_s(y,t)$.

2. The two-fluid model. The penetration depth of many superconductors is found experimentally to follow the following relation with the temperature: $\lambda(T) = \lambda_0/\sqrt{1-(T/T_c)^4}$. Using the relation between λ and n_s, the superelectron density, find $n_s(T)$. Defining the total density of particles in the superconductor as $n(T) = n_e(T) + n_s(T)$, where $n_e(T)$ is the normal electron density, and knowing that a superelectron contains 2 electrons, find $n(T)$. Find the relation between $\mathbf{J}$ and $\mathbf{E}$ in this two-fluid model by considering that $\mathbf{J} = \mathbf{J}_e + \mathbf{J}_s$, where $\mathbf{J}_e = \sigma(\omega, T)\mathbf{E}$ is the normal electron current density and $\partial(\Lambda(T)\mathbf{J}_s)/\partial t = \mathbf{E}$ is the supercurrent density, assuming an oscillating time dependence.

3. The free energy of a superconductor F describes its thermodynamical properties, and the equilibrium between the fraction x of normal electrons and the fraction $1-x$ of Cooper pairs at temperatures below T_c. At any temperature, the thermodynamical equilibrium is reach by minimizing F respect to x. Consider the following free energy of a superconductor

$$F = -\sqrt{x}\frac{1}{2}\gamma T^2 - \beta(1-x),$$

where γ and β are phenomenological given parameters. Calculate the following:
1) The critical temperature T_c as function of γ and β.
2) The fraction of electrons $x(T)$ as function of temperature at a temperature $T < T_c$.
3) Comment the resulting model.

4. Prove that a charged particle with canonical momentum $\mathbf{p} = m\mathbf{v} + e\mathbf{A}$ obeys the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2\psi + q\phi\psi.$$

Prove also that the probability current becomes:

$$\mathbf{J}_{\mathcal{P}} = \mathrm{Re}\left\{\psi^*\left(\frac{\hbar}{im}\nabla - \frac{q}{m}\mathbf{A}\right)\psi\right\}.$$

5. Starting with the second London equation and Lorentz's law, obtain the non-linear first London equation. Notice that it is not necessary to use quantum mechanics to obtain this result.

6. Assume that the local density of the superfluid $n^*(\mathbf{r}, t)$ is not a constant in either space or time. Show that the imaginary part of the Schrödingerlike equation for the MQM is

$$\frac{\partial n^*}{\partial t} = -\nabla\cdot(n^*\mathbf{v}_s),$$

where $\mathbf{v}_s \equiv \left(\frac{\hbar}{m^*}\nabla\theta(\mathbf{r}, t) - \frac{q^*}{m^*}\mathbf{A}(\mathbf{r}, t)\right)$ is the superfluid velocity. Interpret this result physically. Multiply both sides of this relation by q^* and physically interpret the result. What is the constraint on $\mathbf{v}_s$ when n^* is a constant in both space and time?

7. Consider a Josephson junction where an external voltage is applied $V(t) = V_0 + v\sin\omega t$. Calculate the resulting supercurrent running through the junction $I_s(t)$.

8. Consider a DC-SQUID with an asymmetry in its junctions, so that one is α times bigger than the other junction. Calculate the resulting modulation of the critical current when an external flux is applied $I_C(\Phi)$.
Hint: Remember that the definition of the critical current is the phase configuration that maximizes the current, $\partial I_{\mathrm{SQ}}/\partial\varphi$.

9. Consider a DC-SQUID with finite inductance in its loop, L_1 being the inductance in series to junction 1, L_2 being the inductance in series to junction 2, so that the total series loop inductance is $L = L_1 + L_2$. An external flux Φ is applied in the loop, while a bias current I_b is externally supplied to the SQUID.

   (a) Show that fluxoid quantization leads to

   $$2\pi\Phi/\Phi_0 = \varphi_2 - \varphi_1 + \frac{2\pi(L_2 I_2 - L_1 I_1)}{\Phi_0},$$

   $I_{1,2} = I_{C_{1,2}}\sin\varphi_{1,2}$. Show also that the total current through the SQUID obeys $I_b = I_1 + I_2$.

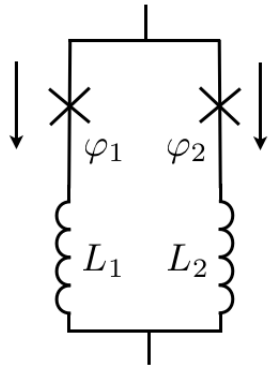


Figure 1: DC-SQUID with geometric inductance. Arrows denote current sign convention.

(b) Consider a symmetric loop $L_1 = L_2 \equiv L/2$, $I_{C_1} = I_{C_2} \equiv I_C$. By defining $\beta \equiv 2\pi L I_C/\Phi_0$, $\varphi \equiv (\varphi_1 + \varphi_2)/2$ and $\Delta\varphi \equiv (\varphi_1 - \varphi_2)/2$, the two relations found above $\Phi(I_1, I_2), I_b(I_1, I_2)$, can be inverted to find the phases φ and $\Delta\varphi$ as function of flux and bias current, $\varphi(\Phi, I_b), \Delta\varphi(\Phi, I_b)$. Perform this inversion numerically for the range $\Phi = -\Phi_0/2 ... \Phi_0/2$, and $I_b = 0 ... I_C$, for $\beta = 0, 0.1, 1$. Note that the inversion of the $I(\varphi)$ relation for a Josephson junction gives rise to 2 solutions (φ, and $\pi - \varphi$), but only one is stable. This is also true for a squid, so that special care has to be taken for a proper inversion.

(c) SQUID equivalent inductance. Calculate the equivalent SQUID inductance, for a fixed applied flux Φ, defined as

$$\frac{1}{L_{\text{eq}}} \equiv \left.\frac{\partial I_b}{\partial \varphi}\right|_{\Phi}.$$

For which values of I_b and Φ the effect of the loop inductance L disappears?

(d) For a given set of values (β, Φ, I_b), the inductance can be plotted as function of Φ and/or I_b by sweeping the values of φ and $\Delta\varphi$ obtained from the previous parts. In one single figure, for $\beta = 0, 0.1, 1, 10$, plot $L_{\text{eq}}(\Phi)$ when $I_b = 0$. Repeat the same in a separate figure for $I_b = 0.1 I_C$. In another two sets of figures, for $\beta = 0, 0.1, 1, 10$, plot $L_{\text{eq}}(I_b)$ when $\Phi = 0$ and when $\Phi = 0.5\Phi_0$.

(e) SQUID critical current. The critical current is defined as the maximum current for a given flux. It can be found by solving: $I_C \equiv \partial I_b/\partial\varphi|_{\Phi} = 0$. Calculate I_C for this circuit and plot it as function of Φ in the range $\{-\Phi_0/2, \Phi_0/2\}$ for $\beta = 0, 0.1, 1, 2$. *Hint: Find* $\cos\varphi$ *as function of* $\Delta\varphi$*, and use the result of part 9a to calculate* $\Delta\varphi$ *as function of* Φ *to later evaluate* I_b*.*

10. Consider a single lumped Josephson junction that is connected by a superconducting loop with inductance L. This circuit is known as the rf-SQUID.

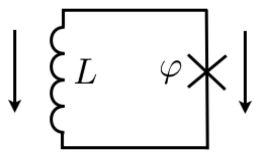


Figure 2: rf-SQUID. Arrows denote current sign convention.

(a) If there are no applied magnetic fields, show that the current in the loop can take on values given by (follow the current sign convention of the drawing)

$$I = -I_C \sin\left(\frac{2\pi LI}{\Phi_0}\right).$$

Find the allowed values of I for $LI_C = 6\Phi_0$.

(b) Show that the energy is given by

$$W = W_{J0} - \frac{\Phi_0 I_C}{2\pi}\cos\left(\frac{2\pi LI}{\Phi_0}\right) + \frac{1}{2}LI^2.$$

Plot $W - W_{J0}$ versus I for $LI_C = 6\Phi_0$. Show that all the allowed values except $I = 0$ are metastable, that is, only $I = 0$ is the true minimum.

(c) Now apply a magnetic field Φ_{ext}. Show that the total flux Φ and the current I are given by

$$\Phi = \Phi_{\text{ext}} - LI_C \sin\left(\frac{2\pi\Phi}{\Phi_0}\right),$$

and

$$I = -I_C \sin\left(\frac{2\pi\Phi}{\Phi_0}\right).$$

(d) For small inductances $L \approx 0$, show that the energy is approximately given by

$$W(I) = W_{J0} - \frac{\Phi_0 I_C}{2\pi}\cos\left(\frac{2\pi\Phi_{\text{ext}}}{\Phi_0}\right),$$

such that

$$W(I) - W(I_C) = -\frac{\Phi_0 I_C}{2\pi}\cos\left(\frac{2\pi\Phi_{\text{ext}}}{\Phi_0}\right),$$

where $W(I_C)$ is the energy of the system when the current is equal to I_C.

(e) When the inductance is large, the total flux is quantized so that

$$\Phi = \Phi_{\text{ext}} + LI = n\Phi_0.$$

Show that the energy is approximately given by

$$W(I) = \frac{1}{2L}(\Phi_{\text{ext}} - n\Phi_0)^2,$$

and that

$$W(I) - W(I_C) = \frac{1}{2L}(\Phi_{\text{ext}} - n\Phi_0)^2 - \frac{\Phi_0^2}{8\pi^2 L}\left(\frac{2\pi LI_C}{\Phi_0} + \frac{\pi}{2}\right)^2.$$

(f) Plot Φ versus Φ_{ext} and also $W(I) - W(I_C)$ versus Φ_{ext} for the two limiting cases considered in parts 10d and 10e. Note that when $W(I) - W(I_C) = 0$ that the system will switch to the normal state, and will then be able to adjust the value of n to be in the lowest energy state as the externally applied flux is changed. The plot of Φ versus Φ_{ext} will be hysteretic for part 10e.

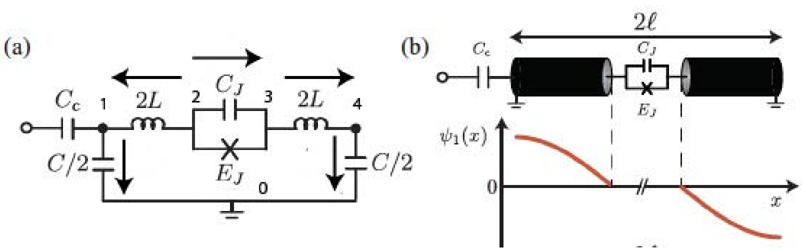
2 Circuit quantization

1. Using the phase-charge commutation relation $[\hat{n}, \hat{\varphi}] = i$, show that

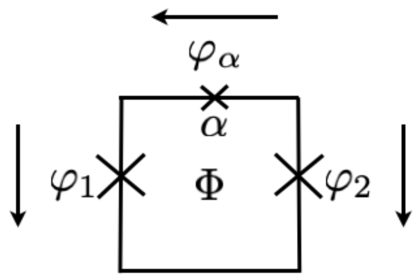
$$\hat{n}e^{i\hat{\varphi}} = e^{i\hat{\varphi}}(\hat{n} - 1).$$

Hint: Use Taylor series of the exponential.

2. Obtain the quantized Hamiltonian of the following circuit, known as the in-line transmon:



3. (a) Consider a 3-junction flux qubit as shown in the figure. Consider the two large junctions to be identical in size with identical E_J and E_C, and the small junction to be smaller by a factor α, so its Josephson energy is αE_J. Proof that the minimum of the potential barrier at $f = 1/2$ takes place when the phase is $\varphi_1 = \varphi_2 = \pm\varphi^*$, where $\cos\varphi^* = 1/2\alpha$.



(b) Compute the flux qubit Hamiltonian and express it in the basis of charge states $|n\rangle$.
Hint: Do use the variables indicated in the figure for the following sections to simplify the calculations.

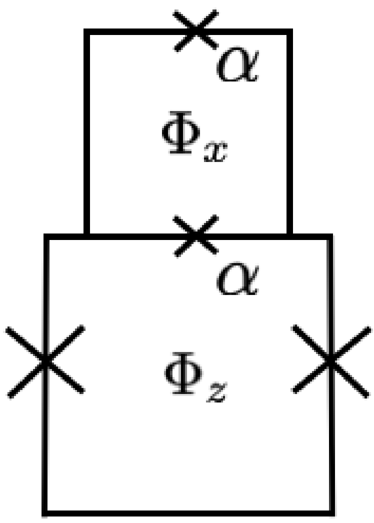
(c) By setting a maximum charge number $n_{\max} = 7$, so that the charge basis runs from $|-7\rangle \ldots |+7\rangle$, represent the Hamiltonian in this basis. Set $E_J/E_C = 100$, with $E_J/h = 100$ GHz, $\alpha = 0.7$, and $f = 1/2$. Note that E_J corresponds to that of the 2 big junctions. For the α junction the Josephson energy is αE_J. The charging energy is defined as $E_C = e^2/2C$, with e being the electron charge.

(d) Now obtain the system energies of the first 2 levels by direct diagonalization of the Hamiltonian. What is the energy splitting in units of GHz?

(e) Compute the spectrum for a range of flux near the symmetry point, $f = \{0.49 \ldots 0.51\}$.

(f) Using the two-level formulation, $H = \hbar\epsilon\sigma_z/2 + \hbar\Delta\sigma_x/2$, with $\hbar\epsilon = 2I_p\Phi_0(f - 1/2)$, fit the spectrum found in the previous section to obtain I_p.
Hint: Rather than fitting the full spectrum, you may just extrapolate the spectrum as if the gap was $\Delta = 0$, and the slope of that line will give you I_p directly.

4. (a) Consider a 4-junction flux qubit as shown in the figure. Consider the two large junctions to be identical in size with identical E_J and E_C, and the two small junctions to be smaller by a factor α, so their Josephson energy is αE_J. Compute the flux qubit Hamiltonian and express it in the basis of charge states $|n\rangle$. *Hint: Here it is crucial how you define the tree branches and closure branches.*



(b) Show that the Josephson potential energy corresponds to that of a 3-junction qubit with the α-junction being tunable by Φ_x.

(c) By setting a maximum charge number $n_{\max} = 7$, so that the charge basis runs from $|-7\rangle \ldots |+7\rangle$, represent the Hamiltonian in this basis. Set $E_J/E_C = 100$, with $E_J/h = 100$ GHz, $\alpha = 0.35$, $\Phi_x/\Phi_0 = 0$ and $\Phi_z/\Phi_0 = 1/2$. Note that E_J corresponds to that of the 2 big junctions.

For the α junction the Josephson energy is αE_J. The charging energy is defined as $E_C = e^2/2C$, with e being the electron charge.

(d) Now obtain the system energies of the first 2 levels by direct diagonalization of the Hamiltonian. What is the energy splitting in units of GHz?

(e) Compute the spectrum for a range of flux near the symmetry point, $f = \{0.49 \ldots 0.51\}$.

(f) Using the two-level formulation, $H = \hbar\epsilon\sigma_z/2 + \hbar\Delta\sigma_x/2$, with $\hbar\epsilon = 2I_p\Phi_0(f - 1/2)$, fit the spectrum found in the previous section to obtain I_p.
Hint: Rather than fitting the full spectrum, you may just extrapolate the spectrum as if the gap was $\Delta = 0$, and the slope of that line will give you I_p directly.