Quantum Statistical Inference. Exercises: Batch 2

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Exercise 1. *Prove the following* **Concentration inequalities**

1. *O1 & O2 Markov's inequality . For any non-negative random variable X and any $t > 0$, show that*

$$\Pr\{X \geq t\} \leq \frac{E(X)}{t}$$

Give a random variable that achieves this inequality with equality. It is often a very useful proof method to use a trivial extension of this inequality:
If $f(x)$ is a strictly increasing non-negative function $\Pr\{X \geq t\} = \Pr\{f(X) \geq f(t)\} \leq \frac{E(f(X))}{f(t)}$.

2. *O1 Chebyshev's inequality. Let Y be a random variable with mean $E(Y) = \mu$ and variance σ^2. By letting $X = (Y - \mu)^2$, show that for any $\epsilon > 0$,*

$$\Pr\{|Y - \mu| > \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}$$

3. *O2* **The weak law of large numbers**. *Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ be the sample mean. Show that*

$$\Pr\left\{\left|\bar{X}_n - \mu\right| > \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2} \tag{1}$$

Thus $\Pr\left\{\left|\bar{X}_n - \mu\right| > \epsilon\right\} \to 0$ as $n \to \infty$. This is known as the weak law of large numbers.

Exercise 2. *O1 (Random volume in large phase-space) Consider an n-dimensional rectangular volume, in a (very) large phase-space, with sides $X_1, X_2, \ldots, X_n$. In addition define an effective linear length-scale ℓ given by length of the edge of n-dimensional cube with the same volume, i.e. $\ell^{(n)} := V_n^{\frac{1}{n}}$ where $V_n = \prod_{i=1}^n X_i$. Now, let $X_1, X_2, \ldots$ be i.i.d. uniform random variables over the unit interval $[0, 1]$. Find the $\lim_{n\to\infty} \ell^{(n)}$ (which is almost surely defined), and compare to another possible length-scale definition $\bar{\ell} := (E[V_n])^{1/n}$. [use strong law of large numbers]*

Exercise 3. *O2 (Gambling) You are invited to participate in a gambling game where you must toss a fair coin a given number of times $n \gg 1$. Every time you get heads your fortune is duplicated, while your fortune is divided by 3 every time you obtain tails. Starting with a fortune of $B_0 = 1$, compute the expected fortunes $\bar{B}_n = E(B_n)$ and also your fortune rate $b := \lim_{n\to\infty} B_n^{\frac{1}{n}}$ (where almost surely convergence is implied). [use strong law of large numbers]*

Based on those calculations would you accept the invitation to play in the game? Express both quantities as either the arithmetic or geometric means of the gain factor (h=2) and loss factor ($t = 1/3$). Give $\bar{B}_n$ and b for a general (biased) coin with $p_h = 1 - p_t := p > 0$, and general gain (g) and loss t factors. Explain the result in the case $p > 1/2$, $h = 2$ and $t = 0$ (you go bankrupt) by computing the probability that you lose your fortune $\Pr(B_n = 0)$.

Exercise 4. *O1 & O2 Suppose you randomly draw $n \gg 1$ balls with replacement from an urn containing 2 red balls and 8 blue balls and 90 white balls. Red balls entail a reward of $R(r) = 100$, while blue get $R(b) = 20$ and white are not rewarded ($R(w) = 0$). What is the probability that the mean reward $\bar{R}_n = \sum_{k=1}^{n} R_k$ is larger or equal than $\bar{R}_n \geq 5$. Give the result as an exponent $\Pr(\bar{R}_n \geq 5) \doteq \mathrm{e}^{-nD}$ and its numerical value $\Pr(\bar{R}_n \geq 5) \approx \mathrm{e}^{-\alpha}$ for, say, $n = 10^5$. [use Sanov's Theorem]*

Exercise 5. *O1 $f(t) = t^p$ for $p > 1$ is not operator monotone. Show it for $f(t) = t^2$ by giving a counter example: use*

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

to show that $A \geq B$ but $A^2 \not\geq B^2$.

O2 Recall that function $f : I \to \mathbb{R}$ is called operator convex, *if for any hermitian operators A, B with* $\operatorname{spec} A, \operatorname{spec} B \subset I \subset \mathbb{R}$ *and $\lambda \in [0,1]$ we have $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$. Use the matrices*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

and $\lambda = 1/2$ to show that $f(t) = t^3$ is not operator convex.

This examples show that functions that are montonone (convex) as real functions need not be operator monotone (operator convex).

Exercise 6. *O1 & O2 Show that if a channel Λ fulfills the covariance property $\Lambda(U\rho U^\dagger) = V\Lambda(\rho)V^\dagger$ then the corresponding Choi operator J_Λ must have the symmetry:*

$$V \otimes U^* J_\Lambda V^\dagger \otimes U^T = J_\Lambda \tag{2}$$

Hint: use the matrix representation $|C\rangle$ presented 2.2.1 in the notes.

Exercise 7. *O1 A bit-flip error channel can be written as*

$$\Lambda_\lambda(\rho) = (1-\lambda)\rho + \lambda \sigma_x \rho \sigma_x \quad \textit{with} \ \ 0 \leq \lambda \leq 1$$

O2 A phase-flip error channel can be written as

$$\Lambda_\lambda(\rho) = (1-\lambda)\rho + \lambda \sigma_z \rho \sigma_z \quad \textit{with} \ \ 0 \leq \lambda \leq 1$$

1. *Show what is its effect on arbitrary qubit state with Bloch vector $\vec{s}$:*

   $$\rho = \tfrac{1}{2}(1 + \vec{s} \cdot \vec{\sigma}) = \frac{1}{2} \begin{pmatrix} 1 + s_z & s_x - i s_y \\ s_x + i s_y & 1 - s_z \end{pmatrix}.$$

   Tip: think of the channel as a convex combination of the identity with a particular rotation)

2. *Is this channel teleportation-covariant? If so, give the 4 "correcting" unitaries, V_i for $i = 0, \ldots, 3$, such that $V_i \Lambda_\lambda(\sigma_i \rho \sigma_i) V_i^\dagger = \Lambda_\lambda(\rho)$. Recall that Paulis anti-commute: e.g. $\sigma_x \sigma_z = -\sigma_x \sigma_z$.*

3. *Give the Choi matrix J_λ of Λ_λ.*

   We wish to discriminate between two (equiprobable) channels Λ_{λ_1} and Λ_{λ_2} ($\lambda_2 > \lambda_1$).

4. *Imagine we do so by sending the input state $\rho = |0\rangle\langle 0|$ through the channel and optimally measuring the output. Compute the resulting probability of error.*

5. *Now consider that we make use of entanglement by sending through the channel qubit A which is maximally entangled with qubit B: $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. What is the probability of error if we perform the optimal measurement on the output state ρ_{AB}?*

6. *In light of (b), would you say that this particular entangled strategy (choice of input state) is optimal, i.e. it gives the smallest possible error probability?*

7. *What can you say about the optimal strategy when n uses of the channel are allowed?*