Quantum Technologies w/ Superconducting circuits, assig. 1

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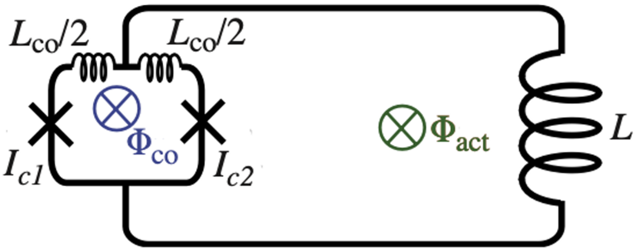
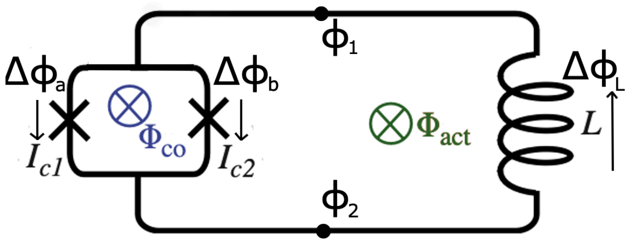
Consider the circuit in the figure:



1 No inductance in the DC-SQUID loop, $L_{co} = 0$. Obtaining the expression for the Hamiltonian of the circuit:

Let's simplify the circuit, and add the node labels and the flux differences:



To simplify the computations further, let's choose the top node as ground, and the inductance path as our main tree path, having two loops at the left:

- $\phi_1 \equiv 0$ and $\phi_2 \equiv \phi$
- $T = \{L\}$ and $C = \{a, b\}$

which means that we also have chosen $\Delta\phi_L = \phi_1 - \phi_2 \equiv -\phi$.

Now, let's determine the other two flux differences, with the two loops flux quantizations:

- $\Delta\phi_a - \Delta\phi_b = \Phi_{co} + n\Phi_0 \;\rightarrow\; \Delta\phi_a = \phi + (\Phi_{co} + \Phi_{act}) + (n + n')\Phi_0$
- $\Delta\phi_b + \Delta\phi_L = \Phi_{act} + n'\Phi_0 \;\rightarrow\; \Delta\phi_b = \phi + \Phi_{act} + n'\Phi_0$

The next step is to define the nodes current conservations, but since both nodes are asymmetric, we only need to compute one:

$$I_a + I_b - I_L = 0 \;\rightarrow\; I_{c1,a}\sin\left(2\pi\frac{\Delta\phi_a}{\Phi_0}\right) + I_{c2,b}\sin\left(2\pi\frac{\Delta\phi_b}{\Phi_0}\right) - \frac{\Delta\phi_L}{L} = 0$$

which substituting the expression for the flux differences, found above, ends up:

$$I_{c1,a}\sin\left(2\pi\frac{\phi+\Phi_{co}+\Phi_{act}}{\Phi_0}\right) + I_{c2,b}\sin\left(2\pi\frac{\phi+\Phi_{act}}{\Phi_0}\right) + \frac{\phi}{L} = 0$$

where we only have dependence on the ϕ parameter, and not on its derivative, which was obvious from the start, due to the system not having capacitors. From this, we could also have deduced, that the system would not evolve in time, since we don't have the charge storage (capacitors), to make the oscillations, against the flux storages (inductances) present on the circuit. But now with the equations of motion, it is even clearer, that there will be only static solutions, given by the external fluxes.

An obvious solution to the equation is given when the external fields (Φ_{act}, Φ_{co}) are a multiple of Φ_0, where one solution will be: $\phi = 0$, which hints to the periodicity of the system with the external fields. Finally, we can also see that the solution for the flux will be bounded by the critical currents, at least by $|\phi/L| \leq |I_{c1,a}| + |I_{c1,a}|$.

So now, let's find the Lagrangian that gives this equation of motion, which since it does not contain any time derivative, will be a pretty straightforward integral of the flux:

$$\begin{aligned} L = \int \frac{\partial L}{\partial\phi} d\phi &= \int \left[I_{c1,a}\sin\left(2\pi\frac{\phi+\Phi_{co}+\Phi_{act}}{\Phi_0}\right) + I_{c2,b}\sin\left(2\pi\frac{\phi+\Phi_{act}}{\Phi_0}\right) + \frac{\phi}{L}\right] d\phi = \\ &= -I_{c1,a}\frac{\Phi_0}{2\pi}\cos\left(2\pi\frac{\phi+\Phi_{co}+\Phi_{act}}{\Phi_0}\right) - I_{c2,b}\frac{\Phi_0}{2\pi}\cos\left(2\pi\frac{\phi+\Phi_{act}}{\Phi_0}\right) + \frac{\phi^2}{2L} \end{aligned}$$

And finally since again, we don't have any derivative term, the Hamiltonian, will just be:

$$\boxed{H = -L = E_{J1}\cos\left(2\pi\frac{\phi+\Phi_{co}+\Phi_{act}}{\Phi_0}\right) + E_{J2}\cos\left(2\pi\frac{\phi+\Phi_{act}}{\Phi_0}\right) - \frac{\phi^2}{2L}}$$

which can be thought of as a correction to the normal inductance of 2nd order in ϕ, plus other linear terms (in ϕ) interacting with the external fields, and higher terms additions that make quartic corrections to the inductance, so you won't have and H.O. anymore.

This can be more easily seen by expanding the cosines:

$$\begin{aligned} H &\approx 2\frac{\pi^2}{\Phi_0^2}\left[-E_{J1}\left(\phi+\Phi_{co}+\Phi_{act}\right)^2 - E_{J2}\left(\phi+\Phi_{act}\right)^2\right] - \frac{\phi^2}{2L} + O^4(\Phi_{co},\Phi_{act},\phi) + ... \approx \\ &\approx 2\frac{\pi^2}{\Phi_0^2}\left[-E_{J1}\left(\phi^2 + 2\phi(\Phi_{co}+\Phi_{act})\right) - E_{J2}\left(\phi^2 + 2\phi\Phi_{act}\right)\right] - \frac{\phi^2}{2L} + ... \approx \\ &\boxed{\approx -\left(\frac{I_{c1}}{2E_{J1}} + \frac{I_{c2}}{2E_{J2}} + \frac{1}{2L}\right)\phi^2 + f(O^1(\Phi_{co},\Phi_{act}))\phi + O^4(\Phi_{co},\Phi_{act},\phi) + ...} \end{aligned}$$

where we see that the correction of second order (without external fields), has a denominator of 2 times the junction inductances, as in the original term. Then we see the interaction terms with the external fields, and finally also the quartic terms, that would bring ϕ^4 and weirder interactions with the external fields $\phi\Phi_{act}^3$, $\phi^2\Phi_{act}^2$, $\phi^3\Phi_{act}$, etc...

2 Finding the potential energy of the system assuming equal Josephson junctions ($I_{c1} = I_{c2} \equiv I_C$):

Again, like in the previous question, since we don't have any derivative or conjugate coordinate in our Lagrangian/Hamiltonian, our life will be much easier.

To start with, our potential energy will be the full Hamiltonian, which assuming both junctions have the same critical current ($E_J \equiv I_C\Phi_0/2\pi$), ends up like:

$$V = H = -\frac{\phi^2}{2L} + E_J\left[\cos\left(2\pi\frac{\phi + \Phi_{co} + \Phi_{act}}{\Phi_0}\right) + \cos\left(2\pi\frac{\phi + \Phi_{act}}{\Phi_0}\right)\right]$$

from where we only need to rearrange the cosine terms in a single product.

To do so, it's as simple as realizing, we actually have a $\cos(a + b) + \cos(a - b)$, with:

- $a = \frac{2\pi}{\Phi_0}(\phi + \frac{\Phi_{co}}{2} + \Phi_{act}) \quad \rightarrow \quad a + b = \frac{2\pi}{\Phi_0}(\phi + \Phi_{co} + \Phi_{act})$
- $b = \frac{2\pi}{\Phi_0}(\frac{\Phi_{co}}{2}) \quad \rightarrow \quad a - b = \frac{2\pi}{\Phi_0}(\phi + \Phi_{act})$

which can then be rewritten as $2\cos(a)\cos(b)$, giving:

$$V = -\frac{\phi^2}{2L} + E_J 2\cos\left(\pi\frac{\Phi_{co}}{\Phi_0}\right)\cos\left(2\pi\frac{\phi + \Phi_{co}/2 + \Phi_{act}}{\Phi_0}\right)$$

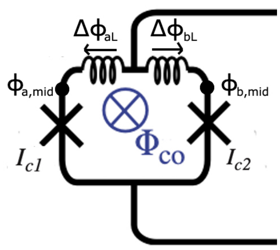
where we see, that this has the structure asked by the statement, with:

- $E_J(\Phi_{co}) = \frac{I_C\Phi_0}{\pi}\cos\left(\pi\frac{\Phi_{co}}{\Phi_0}\right)$
- $f(\phi, \Phi_{co}, \Phi_{act}) = \frac{2\pi}{\Phi_0}(\phi + \frac{\Phi_{co}}{2} + \Phi_{act})$

achieving a regulable Josephson inductance, via the external *co* field Φ_{co}:

$$\boxed{V = -\frac{\phi^2}{2L} + E_J(\Phi_{co})\cos\left(2\pi\frac{\phi + \Phi_{co}/2 + \Phi_{act}}{\Phi_0}\right)}$$

3 Obtaining the Hamiltonian for the general case $L_{co} \neq 0$:



If we add inductances, we will have two new nodes and flux differences, which I will call:

- $\phi_{a,mid}$ and $\phi_{b,mid}$ for the flux in between of the junction-inductance, at path a or b.
- $\Delta\phi_{aL}$ and $\Delta\phi_{bL}$ for the flux difference at each of the new inductances.

and then we need to choose a main tree path, that goes around all the nodes, which opposite to the previous case, we don't have much freedom to chose, it can only be the one that goes around the Φ_{co}, and then the remaining flux differences will form the loops:

$$T = \{I_{c2}, bL_{co}, aL_{co}\} \quad \text{and} \quad C = \{L, I_{c1}\}$$

gives these new relations, when setting the top node as ground again ($\phi_1 = 0$ and $\phi_2 = \phi$):

- $\Delta\phi_{aL} = \phi_{a,mid} - \phi_1 = \phi_{a,mid}$
- $\Delta\phi_{bL} = \phi_{b,mid} - \phi_1 = \phi_{b,mid}$
- $\Delta\phi_b = \phi_2 - \phi_{b,mid} = \phi - \phi_{b,mid}$

and then, the two loops flux quantization are the same as before, with an extra term each:

- $\Delta\phi_a + \Delta\phi_{aL} - \Delta\phi_b - \Delta\phi_{bL} = \Phi_{co} + n\Phi_0 \;\rightarrow\; \Delta\phi_a = \phi - \phi_{a,mid} + \Phi_{co} + n\Phi_0$
- $\Delta\phi_b + \Delta\phi_{bL} + \Delta\phi_L = \Phi_{act} + n'\Phi_0 \;\rightarrow\; \Delta\phi_L = -\phi + \Phi_{act} + n'\Phi_0$

and finally, it's clear we will have two extra, current conservation equations, giving:

- $I_a + I_b - I_L = 0 \quad \rightarrow \quad \frac{2\Delta\phi_{aL}}{L_{co}} + \frac{2\Delta\phi_{bL}}{L_{co}} - \frac{\Delta\phi_L}{L} = 0$
- $-I_{aL} - I_{bL} + I_L = 0 \quad \rightarrow \quad -I_{c1,a}\sin\left(2\pi\frac{\Delta\phi_a}{\Phi_0}\right) - I_{c2,b}\sin\left(2\pi\frac{\Delta\phi_b}{\Phi_0}\right) + \frac{\Delta\phi_L}{L} = 0$
- $I_{aL} - I_a = 0 \;\rightarrow\; I_{aL} = I_a \;\rightarrow\; I_{c1,a}\sin\left(2\pi\frac{\Delta\phi_a}{\Phi_0}\right) = \frac{2\Delta\phi_{aL}}{L_{co}}$
- $I_{bL} - I_b = 0 \;\rightarrow\; I_{bL} = I_b \;\rightarrow\; I_{c2,b}\sin\left(2\pi\frac{\Delta\phi_b}{\Phi_0}\right) = \frac{2\Delta\phi_{bL}}{L_{co}}$

where we again have one redundant equation. Let's then as before, substitute the flux difference expressions from above, to find the new equations of motion (we will also do a change of variable $\phi - \Phi_{act} - n'\Phi_0 \rightarrow \phi$, to get similar equations to before):

- $-I_{c1,a}\sin\left(2\pi\frac{\phi-\phi_{a,mid}+\Phi_{co}+\Phi_{act}}{\Phi_0}\right) - I_{c2,b}\sin\left(2\pi\frac{\phi-\phi_{b,mid}+\Phi_{act}}{\Phi_0}\right) - \frac{\phi}{L} = 0 \;\stackrel{?}{\rightarrow}\; \frac{\partial L}{\partial\phi} = 0$
- $I_{c1,a}\sin\left(2\pi\frac{\phi-\phi_{a,mid}+\Phi_{co}+\Phi_{act}}{\Phi_0}\right) = \frac{2\phi_{a,mid}}{L_{co}} \;\stackrel{?}{\rightarrow}\; \frac{\partial L}{\partial\phi_{a,mid}} = 0$
- $I_{c2,b}\sin\left(2\pi\frac{\phi-\phi_{b,mid}+\Phi_{act}}{\Phi_0}\right) = \frac{2\phi_{b,mid}}{L_{co}} \;\stackrel{?}{\rightarrow}\; \frac{\partial L}{\partial\phi_{b,mid}} = 0$

where we have kept 3 of the 4 (removed the redundancy), which from a visual inspection looked like 3 independent derivatives of the Lagrangian, which would look like:

$$L = E_{J1}\cos\left(2\pi\frac{\phi-\phi_{a,mid}+\Phi_{co}+\Phi_{act}}{\Phi_0}\right) + E_{J2}\cos\left(2\pi\frac{\phi-\phi_{b,mid}+\Phi_{act}}{\Phi_0}\right) - \frac{\phi^2}{2L} - \frac{\phi_{a,mid}^2+\phi_{b,mid}^2}{L_{co}}$$

and which as before, since we don't have derivatives, gives straightforward the Hamiltonian:

$$\boxed{H = \frac{\phi^2}{2L} + \frac{\phi_{a,mid}^2+\phi_{b,mid}^2}{L_{co}} - E_{J1}\cos\left(2\pi\frac{\phi-\phi_{a,mid}+\Phi_{co}+\Phi_{act}}{\Phi_0}\right) - E_{J2}\cos\left(2\pi\frac{\phi-\phi_{b,mid}+\Phi_{act}}{\Phi_0}\right)}$$

where we see how we have the exact same Hamiltonian, plus extra inductance quadratic terms, of the fluxes in the middle at each branch with L_{co}, same terms that also add two new shifts in each of the cosines, respect the previous Hamiltonian.