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1. Consider the two-dimensional Schrödinger equation for a potential $V(r)$ that only depends on the radial variable r. The relations $x = r\cos\theta$ and $y = r\sin\theta$ may be assumed.
   (a) Prove the identity $\partial_x^2 + \partial_y^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$.
   (b) Deduce from this that there is a complete set of eigenfunctions of the form $\psi(r,\theta) = f(r)e^{il\theta}$.
   (c) Prove that the radial part is the solution of the equation
$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{l^2}{r^2} - \frac{2m}{\hbar^2}(V(r) - E)\right] f(r) = 0\,.$$
   (d) Solve this equation for a circular infinite well, $V(r) = 0$ for $r \leqslant r_0$ and $V(r) = \infty$ for $r > r_0$.

   Hint: the solutions of the differential equation that are regular at the origin are the Bessel functions of the first kind, $J_\alpha(x)$ for integer α.

2. Consider states with angular momentum $l = 1$, which generate a 3-dimensional Hilbert space.
   (a) Find the matrix representation of $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$, and $\hat{L}^2$.
   (b) Use matrix multiplications to confirm that $\hat{L}_x$ and $\hat{L}^2$ commute.
   (c) How can you test if these operators are Hermitian?
   (d) Find the eigenvalues of $\hat{L}_x$ and $\hat{L}_y$, and their corresponding eigenvectors.
   (e) A system is prepared in the state vector $|\phi\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} i \\ 1-i \\ 0 \end{pmatrix}$. Express this state as a superposition of eigenstates of $\hat{L}_x$ and show that the expansion coefficients are $c_1 = \frac{1}{\sqrt{6}}\left(i\left(1+\frac{1}{\sqrt{2}}\right) - 1\right)$; $c_2 = \frac{i}{\sqrt{6}}$ and $c_3 = \frac{1}{\sqrt{6}}\left(i\left(1+\frac{1}{\sqrt{2}}\right) + 1\right)$.

3. The operator corresponding to the components of spin in an arbitrary direction $\hat{\mathbf{n}}$ described by the spherical polar angles θ and ϕ is
$$\hat{S}_{\hat{n}} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{+i\phi}\sin\theta & -\cos\theta \end{pmatrix}\,.$$
   (a) Find the eigenvalues of this operator as functions of θ and ϕ.
   (b) Show that the normalized eigenvectors are
$$|+,\hat{\mathbf{n}}\rangle = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{i\phi/2}\sin\theta/2 \end{pmatrix}\,, \quad |-,\hat{\mathbf{n}}\rangle = \begin{pmatrix} -e^{-i\phi/2}\sin\theta/2 \\ e^{i\phi/2}\cos\theta/2 \end{pmatrix}\,.$$
   (c) If an electron is in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, calculate the probability that a measurement of spin at an arbitrary angle will yield the result $\frac{\hbar}{2}$. Plot the probability as a function of θ.

4. Show that $e^{-i\theta\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}} = I\cos\theta - i(\sigma\cdot\hat{\mathbf{n}})\sin\theta$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices and $\hat{\mathbf{n}}$ is a unit vector.

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5. A two-state system is characterized by the Hamiltonian

$$\hat{H} = \epsilon_1|1\rangle\langle 1| + \epsilon_2|2\rangle\langle 2| + \Delta^*|1\rangle\langle 2| + \Delta|2\rangle\langle 1|\,,$$

where $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq \hat{H}$); ϵ_j are real constants and Δ is a possibly complex coupling coefficient.

(a) Find λ_0 and the three-component λ vector such that $\hat{H} = \lambda_0 I + \boldsymbol{\lambda} \cdot \hat{\boldsymbol{\sigma}}$.
(b) Show that the eigenvalues of $\hat{H}$ are $E_\pm = \lambda_0 \pm \lambda$.
(c) Find the corresponding eigenvectors. Make sure your result makes sense for $H_{12} = 0$.

6. Consider a Hamiltonian $\hat{H}_0$ with two closely neighbouring levels, $E_1^{(0)} \approx E_2^{(0)}$, with corresponding eigenkets $|1\rangle$ and $|2\rangle$. We add a perturbation $\hat{H}_1$, so the total hamiltonian is a non-diagonal matrix with elements $\hat{H}_{ij}$.

(a) Show that, on the one hand, only the levels 1 and 2 contribute to the correction of the energy eigenvalues E_1 and E_2.
(b) Check that, on the other hand, perturbation theory is not a good tool to treat this problem.
(c) Plot the exact eigenenergies of the reduced $2-$state Hamiltonian, $E_\pm$, as a function of $\Delta = H_{11} - H_{22}$. Can $\Delta E = E_+ - E_-$ change sign? What is its minimum value?
(d) Find the states $|+\rangle$ and $|-\rangle$ and comment on how they change as a function of Δ.

7. Consider an harmonic oscillator hamiltonian, $\hat{H} = \frac{\hbar\omega}{2}\left(\hat{p}^2 + \hat{x}^2\right)$. The oscillator is perturbed with a linear term, $\hat{H}_1 = \lambda\hat{x}$.

(a) Calculate the energy eigenvalues and the eigenvectors using first-order perturbation theory.
(b) Calculate the energy eigenvalues using second-order perturbation theory.
(c) Find the exact eigenvalues and eigenvectors and discuss the differences.

8. A one dimensional harmonic oscillator has an angular frequency ω_0. Consider a perturbation potential

$$\hat{V}(t) = \begin{cases} 0,\ t \leqslant 0, \\ F_0\omega^2\hat{x}\cos\omega t\,, t > 0, \end{cases}$$

with F_0 constant in time and space.

(a) If the system is originally in the ground state, what states can the system transition to according to lowest-order perturbation theory?
(b) Compute the corresponding transition probability as a function of time for $\omega \approx \omega_0$?

Repeat the steps above for a potential that is quadratic, as opposed to linear, in the spatial variable x. How do the results change?

9. A system of hydrogen atoms in the ground state is contained between the plates of a parallel capacitor. A voltage pulse is applied at $t = 0$, producing a homogeneous electric field,

$$\mathcal{E}(t) = \begin{cases} 0,\ t < 0, \\ \mathcal{E}_0 e^{-t/\tau}\, t \geqslant 0. \end{cases}$$

(a) Show that after a long time the fraction of atoms in the $2p$ ($m = 0$) state is, to first order,

$$\mathcal{P}_{1\text{s}\to 2\text{p}} = \frac{2^{15}}{3^{10}} \frac{a_0^2 e^2 \mathcal{E}_0^2}{\hbar^2 \left(\omega^2 + \tau^{-2}\right)} \,,$$

where a_0 is the Bohr radius and $\hbar\omega$ is the energy difference between the $2p$ state and the $1s$ (ground) state.

(b) What is the fraction of atoms in the $2s$ state?

The wavefunctions of the $1s$ and the $2p$ states are $\psi_{1s} = R_{10}Y_{00} = \frac{2}{\sqrt{4\pi}} \frac{1}{a_0^{3/2}} e^{-\frac{r}{a_0}}$ and $\psi_{2p} = R_{21}Y_{10} = \frac{1}{\sqrt{4\pi}} \frac{1}{(2a_0)^{3/2}} \frac{r}{a_0} e^{-\frac{r}{2a_0}} \cos\theta$.

10. A time-varying Hamiltonian $\hat{H}_1(t')$ brings about transitions of a system from a state $|k\rangle$ at $t' = 0$ to a state $|j\rangle$ at $t' = t$ with probability $\mathcal{P}_{k\to j}(t)$. Use time-dependent first-order perturbation theory to show that $\mathcal{P}_{j\to k}(t) = \mathcal{P}_{k\to j}(t)$ or, in other words, that the probability that the same Hamiltonian brings about the transition $j \to k$ in the same time interval is the same.

11. Find the second-order correction to the transition probability $\mathcal{P}_{i\to f}$ in time-dependent perturbation theory. This is sometimes called a *two-step transition*, why?

12. Consider operators in the different time evolution pictures of quantum mechanics.
    (a) Derive the equation of motion for the operators in the Heisenberg picture.
    (b) In a previous exercise, you were asked to show that, for the harmonic oscillator lowering operator, the relation $e^{+\alpha a^\dagger a} a e^{-\alpha a^\dagger a} = e^{-\alpha} a$ holds for a generic complex α. Find the equation of motion for the lowering operator in the Heisenberg picture, $a_H(t)$, and solve it explicitly.
    (c) Find the equation of motion for $a_H^\dagger(t)$.
    (d) Consider an harmonic oscillator state displaced from the origin by a distance x_0 at $t = 0$. Use the Heisenberg picture to find the evolution of the position operator in time, $\hat{x}(t)$.
    (e) A perturbation of the type $\hat{H}_1 = \lambda \hat{x} f(t)$ acts on the ground state of an harmonic oscillator for a time $0 < t \leqslant T$. Find the equation of motion for the lowering operator in the interaction picture, $a_I(t)$.