Lecture 10: Two-qubit gates

This lecture will be based on references [1], [2].

1 Qubit-qubit interactions

Qubits interact via their degrees of freedom. A charge-like qubit will sense another nearby charge qubit given a finite capacitive coupling (or in general through the presence of an electric field) between them. Similarly for flux qubits but with inductive coupling. In general, superconducting qubits can be coupled in two ways, geometrically or mediated by a coupler. Figure 1 shows schematically all types of couplings considered.

1.1 Geometric direct interactions

The most straightforward way to couple qubits is to position them close enough to each other, so they sense each other's presence. Figure 1 a), c) show capacitive and inductive coupling between charge-like and flux-like qubits respectively. When the coupling element (mutual capacitance or mutual inductance) is not too high (higher compared to the qubit's own capacitance/inductance, then we can use a semi-classical picture to obtain the coupling strength.

The idea is to consider Fig. 1 a) and calculate the energy stored in the coupling capacitor C_g:

$$E_{C_g} = C_g\frac{(V_1 - V_2)^2}{2} = \frac{C_g V_1^2}{2} + \frac{C_g V_2^2}{2} - C_g V_1 V_2. \tag{1}$$

The first two terms in the right hand side correspond to a small renormalization of the capacitance of each qubit coming from the coupling capacitor. This makes sense as the qubit islands now have additional capacitance throught C_g. The last term contains the interaction energy between qubits. Re-writting it in terms of Cooper pair number operators,

$$H_{\text{int}} = 4e^2\frac{C_g}{C_1 C_2}\hat{n}_1\hat{n}_2. \tag{2}$$

This Hamiltonian contains a nonlinearity that couples both qubits through their internal degrees of freedom, $\hat{n}_1$ and $\hat{n}_2$. Remembering that in the charge basis for charge-like qubits with a well-defined number of charges, or equivalently $E_C \sim E_J$, the number operator in the 2-level subspace becomes $\hat{n}_1 \simeq \hat{\sigma}_{z_1}/2 + ct$. For

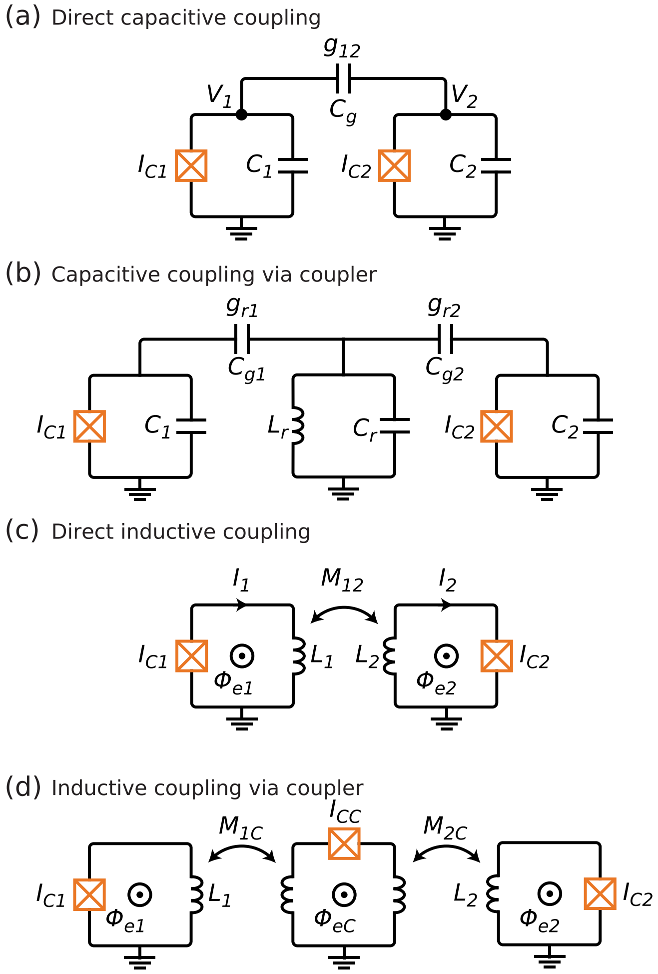

Figure 1: Schematic of capacitive a), b) and inductive c), d) couplings. a), c) correspond to fixed geometric coupling while b), d) correspond to mediated coupling using a coupler circuit. From [1].

transmon qubits, $\hat{n} \simeq -i(E_J/8E_C)^{1/4}(\hat{a}+\hat{a}^\dagger)/\sqrt{2}$, and one can go into the two-state approximation [3]. The complete Hamiltonian in the two-level basis becomes (ignoring constants)

$$H_{2-CPB} = \frac{\hbar\epsilon_1}{2}\hat{\sigma}_{z_1} + \frac{\hbar\Delta_1}{2}\hat{\sigma}_{x_1} + \frac{\hbar\epsilon_2}{2}\hat{\sigma}_{z_2} + \frac{\hbar\Delta_2}{2}\hat{\sigma}_{x_2} - \hbar g\hat{\sigma}_{z_1}\hat{\sigma}_{z_2}, \tag{3}$$

where we defined the qubit-qubit coupling strength as $\hbar g \equiv 4e^2\frac{C_g}{C_1C_2}$. Here, $\epsilon_i = E_{C_i}(n_{g_i} - 1/2)$, with the charging energy given by $E_{C_i} = e^2/2(C_{J_i} + C_g)$. Moving to the diagonal basis for each qubit,

$$H^d_{2-CPB} = \frac{\hbar\omega_{q_1}}{2}\hat{\sigma}_{z_1} + \frac{\hbar\omega_{q_2}}{2}\hat{\sigma}_{z_2} - \hbar g\left(\frac{\Delta_1}{\omega_{q_1}}\hat{\sigma}_{x_1} - \frac{\epsilon_1}{\omega_{q_1}}\hat{\sigma}_{z_1}\right)\left(\frac{\Delta_2}{\omega_{q_2}}\hat{\sigma}_{x_2} - \frac{\epsilon_2}{\omega_{q_2}}\hat{\sigma}_{z_2}\right). \tag{4}$$

Considering flux-like qubits, everything will be analogous to the charge qubits description we have developed. The difference is that now the qubits are inductively coupled via their mutual inductance M, see Fig. 1 b). A qubit with circulating current I_1 generates a flux $\Phi_1 = MI_1$ into qubit 2, which will affect its internal degrees of freedom. The interaction energy can be obtained by calculating the magnetic energy stored in the magnetic field shared between the qubits:

$$H_{\text{int}} = \frac{M(I_1 - I_2)^2}{2} = \frac{MI_1^2}{2} + \frac{MI_2^2}{2} - MI_1I_2. \tag{5}$$

A sign convention was taken since both currents were defined anti-parallel in the figure. This has no relevance to the system energies. The current operator $\hat{I}_i$, in the two-state basis subspace for each qubit, takes the form $\hat{I}_i \simeq I_i\hat{\sigma}_{z_i}$. This leads to a total Hamiltonian in the qubit bases analogous to that of the charge-like qubits, Eq. (3)

$$H_{2-FQ} = \frac{\hbar\epsilon_1}{2}\hat{\sigma}_{z_1} + \frac{\hbar\Delta_1}{2}\hat{\sigma}_{x_1} + \frac{\hbar\epsilon_2}{2}\hat{\sigma}_{z_2} + \frac{\hbar\Delta_2}{2}\hat{\sigma}_{x_2} - \hbar g_{\text{FQ}}\hat{\sigma}_{z_1}\hat{\sigma}_{z_2}, \tag{6}$$

with $\hbar g_{\text{FQ}} \equiv MI_1I_2$. Being the inductive energy.

Flux-like qubits have other means of magnetic interaction, such as being galvanically attached to each other, in this way sharing the kinetic inductance of the shared wire L_K which will add to the mutual geometric inductance M, $g = (M+L_K)I_1I_2/\hbar$. One can add a large Josephson junction playing the role of a linear inductor with inductance $L_J = \Phi_0/(2\pi I_C)$, so that the coupling is $g = (M + L_K + L_J)I_1I_2/\hbar$, and in fact it is quite easy to make the Josephson inductance to dominate $L_K \gg L_K, M$. With galvanic or Josephson interactions g can quite easily reach values proximal to the qubit's own frequencies $g/\omega_{q_i} \sim 1$, in which case the simplified model we have used breaks down and one needs to consider the fully quantized circuit form. The result is still a 2-qubit interaction, but the value of the interaction is not directly proportional to the coupling element, and the qubit resonance frequencies become strongly renormalized towards lower values. This is known as the polariton transformation.

1.2 Mediated interactions: couplers

Two-qubit interactions can be effectively induced when an intermediate object is placed between the qubits. This object, or circuit, may be a linear resonator, a nonlinear resonator (such as a SQUID), or even a qubit. The idea for this object is to behave like a passive circuit element and just its internal properties are used as a kind of susceptibility that mediates the interaction. Figure 1 b), d) shows two examples for charge-like and flux-like qubits, respectively.

We distinguish two types of couplers, ones where the internal level structure is not relevant and only the ground state energy matters. We will refer to these couplers as passive. This is the case of rf-SQUIDs used to mediated interaction between flux qubits, for example. The other type of couplers involve their internal level structure to induce qubit-qubit interactions via virtual photon transitions through the excited states of the coupler. We will call this virtual couplers. This has so far been implemented using linear resonators and qubits.

1.2.1 Passive couplers

We will describe these type of couplers using inductive coupling picture, as it is most illustrative. Imagine a qubit generating a flux $\Phi_1 = M_1 I_1$ on a coupler circuit to which it is coupled with mutual inductance M_1. This flux induces a current in the coupling circuit $I_{cp} = \Phi_1/L_c$. The coupler is coupled to another qubit with inductance M_2, and induces a flux in the second qubit as $\Phi_2 = M_2 I_{cp}$ which changes the qubit energy by an amount $J \equiv I_2\Phi_2 = I_1 I_2 M_1 M_2/L_c$. The effect of the coupler is to behave as a liner effective mutual inductance between qubits of $\tilde{M} = M_1 M_2/L_c$. The only intrinsic variable from the coupler is the inductance L_c. The quantity $1/L_c$ is also defined as the circuit susceptibility χ, which will directly appear in the coupling energy $J = I_1 I_2 M_1 M_2 \chi$. In general, the inductance of a circuit is defined as

$$L^{-1} \equiv \frac{\partial^2 E}{\partial \Phi^2} = \frac{\partial I}{\partial \Phi}, \tag{7}$$

where E is the circuit energy, given the form of the inductive energy $E_{ind} = \Phi^2/2L$ and $I = \Phi/L$. In a quantum circuit, E represents the energy of the circuit in a given quantum state. Usually this will be the ground state. Certain nonlinear circuits exhibit ground state energies that display positive and negative effective inductances, also known as the quantum inductance. This also means that at a certain point the coupling goes to 0.

Let's consider the rf-SQUID fluxoid quantization condition:

$$\Delta\varphi_L + \Delta\varphi_J = \frac{2\pi I L}{\Phi_0} + \sin^{-1}(I/I_C) = 2\pi f. \tag{8}$$

Re-arranging,

$$I = I_C \sin\left(2\pi \frac{f - IL}{\Phi_0}\right). \tag{9}$$

Derivating this relation with respect to flux,

$$\frac{\partial I}{\partial \Phi} = \frac{2\pi I_C}{\Phi_0}\left(1 - L\frac{\partial I}{\partial \Phi}\right)\cos\left(2\pi\frac{\Phi - IL}{\Phi_0}\right). \tag{10}$$

Using the definition of the Josephson inductance $L_J = \Phi_0/2\pi I_C$, the susceptibility of the rf-SQUID becomes

$$\chi = \frac{\partial I}{\partial \Phi} = \frac{1}{L}\frac{\beta\cos(2\pi(f - IL/\Phi_0))}{1 + \beta\cos(2\pi(f - IL/\Phi_0))}, \tag{11}$$

with $\beta \equiv L/L_J$, where I is given by the fluxoid quantization condition, Eq. (9). This susceptibility can be positive, negative and can go to 0. Therefore we have a circuit that mediates qubit interactions and can completely switch them off. Plots from current and susceptibility can be seen in Fig. 2.

The rf-SQUID is the coupler type used in quantum annealers such as those developed by D-Wave [4]. Real data characterizing this type of effective inductance can be seen in Fig. 3. The point $M = 0$ is the decoupling point.

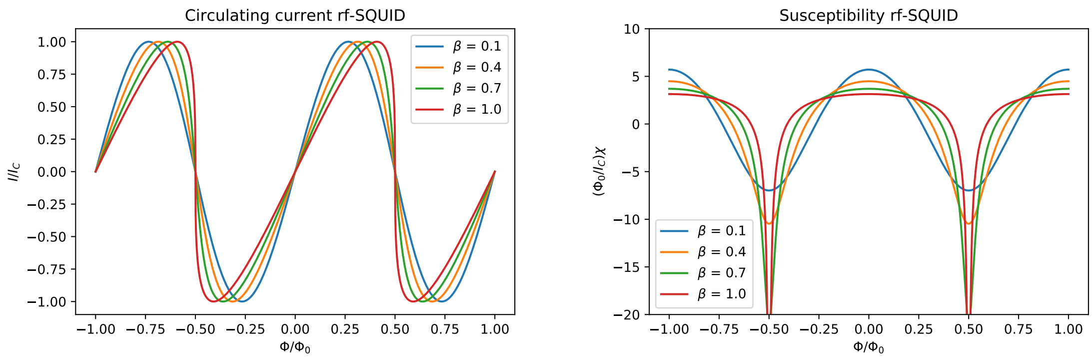

Figure 2: Current (left) and susceptibility (right) of the rf-SQUID coupler.

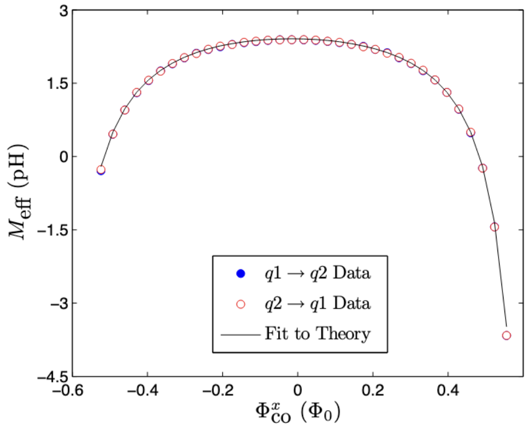

Figure 3: Effective mutual inductance induced by an rf-SQUID coupling two flux qubits. From [4].

1.2.2 Virtual couplers

Virtual couplers are usually employed as a quantum system in their ground state, whose first excited state is not too far from the qubit energies that it is coupled to. A canonic example is displayed in Fig. 4. We see the basic picture where a virtual transition from the first qubit is transferred to the coupler, which is then transferred virtually to the second qubit. A direct qubit-qubit coupling is also included, as it is expected by the qubit geometry. The system Hamiltonian is that of 3 coupled qubits

$$H/\hbar = \sum_{j=1,2} \frac{1}{2}\omega_j \hat{\sigma}_z^j + \frac{1}{2}\omega_c \hat{\sigma}_z^c + \sum_{j=1,2} g_j(\hat{\sigma}_+^j \hat{\sigma}_-^c + \hat{\sigma}_-^j \hat{\sigma}_+^c) + g_{12}(\hat{\sigma}_+^1 \hat{\sigma}_-^2 + \hat{\sigma}_-^2 \hat{\sigma}_+^1). \quad (12)$$

Here, g_j is the qubit j-coupler interaction strength, and g_{12} is the direct qubit-qubit geometric coupling. Consider the particular case where $\Delta_j \equiv \omega_j - \omega_c < 0$, and a dispersive coupling $g_j \ll |\Delta_j|$. Moving into a dispersive frame with the transformation $U = \exp[\sum_{j=1,2} \frac{g_j}{\Delta_j}(\hat{\sigma}_+^j \hat{\sigma}_-^c - \hat{\sigma}_-^j \hat{\sigma}_+^c]$. The transformation up to second

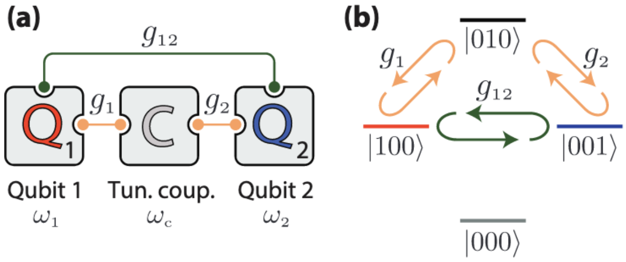

Figure 4: Schematic of a coupler that mediates the interaction between qubits through virtual transitions in the excited state. From [5].

order in g_j/Δ_j results in the Hamiltonian [5]

$$\tilde{H}/\hbar = \sum_{j=1,2} \frac{1}{2}\tilde{\omega}_j \hat{\sigma}_z^j + \left[\frac{g_1 g_2}{\Delta} + g_{12}\right](\hat{\sigma}_+^1 \hat{\sigma}_-^2 + \hat{\sigma}_-^2 \hat{\sigma}_+^1), \tag{13}$$

where $\tilde{\omega} = \omega_j + \frac{g_j^2}{\Delta_j}$, and the detuning $1/\Delta = (1/\Delta_1 + 1/\Delta_2)/2 < 0$. We see here the action of the coupler, which is to induce a SWAP-like interaction between the qubits. The interaction can be tuned with the frequency of the coupler, which changes Δ. The decoupling condition is the point $g_{12} = -g_1 g_2/\Delta$.

Therefore, the virtual couplers are used to tune the interaction strength between qubits using an external parameter, which could be a flux threading the coupler, which could be a SQUID transmon, a tunable resonator, or simply a linear resonator and the coupling is controlled with the qubit frequencies instead.

This tunable interaction allows one to completely turn off the qubit-qubit coupling while it is not used, and activate it to a large value when the coupling is ON. This allows performing very fast 2-qubit gates, as shown in the quantum supremacy paper by Google, where 2-qubit gates where performed in 15ns, while single qubit gates took 25ns. This is in contrast with fixed coupling that always leaves a remanent coupling, and leads to less efficient two-qubit gates. Obviously, the fact that a tunable parameter such as the tunable interaction can induce fluctuations in the qubit frequencies leads to lower coherence properties to the qubits. Still, it appears that better two-qubit gates are still better than worse coherent qubits.

With a tunable coupler, one can parametrically drive it at a certain frequency ω_p, and if this drive matches the qubit difference or the qubit sum, $\omega_p = \omega_1 \pm \omega_2$, this activates two-qubit processes that drives the transitions in swap-like operations for frequency difference, and combined qubit state excitation for frequency sum [6].

1.3 2-qubit gates

Irrespective of charge or flux, the Hamiltonians in Eqs. (3,6) allow several types of 2-qubit gates to be implemented. We will mention a few here:

- Near the symmetry point $\Delta_i \gg \epsilon_i$, the Hamiltonian becomes (with a $\pi/2$ rotation on the qubit operators so $\hat{\sigma}_z \to \hat{\sigma}_x$ and vice-versa)

$$H^d \simeq \frac{\hbar\omega_{q_1}}{2}\hat{\sigma}_{z_1} + \frac{\hbar\omega_{q_2}}{2}\hat{\sigma}_{z_2} - \hbar g \hat{\sigma}_{x_1}\hat{\sigma}_{x_2}. \tag{14}$$

To more clearly understand the effect of the coupling, let's look at it in more detail. The interaction term contains 4 terms: $(\hat{\sigma}_1^+\hat{\sigma}_2^- + \hat{\sigma}_1^-\hat{\sigma}_2^+ + \hat{\sigma}_1^-\hat{\sigma}_2^- + \hat{\sigma}_1^+\hat{\sigma}_2^+)$. If the interaction strength is such that $g \ll \omega_{q_i}$, then the last two terms can be neglected as they average out in the interaction picture where the dynamics is governed by g. This is equivalent to the rotating-wave approximation that was used in qubit-resonator systems. The resulting interaction is a SWAP-like interaction between qubits:

$$H^d \simeq \frac{\hbar\omega_{q_1}}{2}\hat{\sigma}_{z_1} + \frac{\hbar\omega_{q_2}}{2}\hat{\sigma}_{z_2} - \hbar g(\hat{\sigma}_1^+\hat{\sigma}_2^- + \hat{\sigma}_1^-\hat{\sigma}_2^+). \tag{15}$$

The SWAP gate is not an entangling gate. However, the $\sqrt{i\mathrm{SWAP}}$ is entangling. This means that we can generate entangled states between qubits by performing a half a period of the oscillation between qubits. In order to generate such an oscillation, the qubits need to start with different frequencies, such that $\omega_{q_1} - \omega_{q_2} \gg g$, and one of the qubits excited. So the initial state is $|01\rangle$. Then, suddenly and in a time scale much faster than $1/g$ (that is, non-adiabatically), the qubits are brought into resonance, and since the state $|01\rangle$ is not an eigenstate of the coupled system, the state starts to oscillate back and forth between $|01\rangle$ and $|10\rangle$. If the oscillation is stopped at a quarter of the full period, then a $\sqrt{i\mathrm{SWAP}}$ has been performed. Alternatively, if one uses a tunable coupler, the sequence may be repeated but keeping the qubits resonant and initially decoupled, $g = 0$, and just activating the coupling very fast in a time $\tau \gg 1/g$. The same qubit dynamics will follow. An initial experiment using phase qubits (a type of rf-SQUID qubits) observed this type of interaction, except they directly saw the oscillations with the qubits on resonance an initially in the $|00\rangle$ state, possibly since the coupling must have been small [7].

- Staying at the symmetry point, one can apply two separate drives to each qubit and by playing with the relative amplitude of the drives, selectively darken transitions in the 2-qubit spectrum by interference between the two drives on each qubit, in order to implement a CNOT gate. This was implemented with 2 flux qubits [8] and it is quite convenient as the qubits can stay biased at the optimal coherence point.

- Using the same biasing as before, near the symmetry point but far enough from it with the qubits detuned, such that $\omega_{q_1} - \omega_{q_2} \gg g$, the SWAP-like operator becomes dispersive-like (analogous to the dispersive Jaynes-Cummings model), so the resulting Hamiltonian is

$$H = \frac{\hbar\omega_{q_1}}{2}\hat{\sigma}_{z_1} + \frac{\hbar\omega_{q_2}}{2}\hat{\sigma}_{z_2} + \frac{\hbar g^2}{\omega_{q_1} - \omega_{q_2}}\hat{\sigma}_{z_1}\hat{\sigma}_{z_2}. \tag{16}$$

Now by exciting one of the qubits $|01\rangle$ and bringing it closer to resonance (but still far enough from it so there is no population exchange), the coupling operator will be a mixture of $zz + xx$ and keeping the qubit at a finite time at this biasing will induce a relative phase that will depend on the state of the second qubit. Then, bringing back the qubit to the starting point, one will have picked up a relative phase φ in the state depending on the state of qubit 2. So, if for example both qubits are in a superposition initially, $|\psi_i\rangle = \frac{|01\rangle+|00\rangle+|10\rangle+|11\rangle}{2}$ which is not entangled, the resulting state will be $|\psi_f\rangle = \frac{|01\rangle+|00\rangle+e^{i\varphi}(|10\rangle+|11\rangle)}{2}$ which is entangled. In fact, if $e^{i\varphi} = -1$, then we are implementing the C-phase gate. This type of gate is the most standard 2-qubit gate used with transmon qubits. In fact, the gate uses an avoided-level crossing between higher excited states from the manifold of states beyond the qubit subspace. This was first used in Yale to perform the first 2-qubit algorithms [9]. Presently, it is the basis of two-qubit gates used in large-scale transmon circuits used by Google in their Sycamore topology [10].

- If we stay away from the symmetry point (but not too far from it to retain coherence), the Hamiltonian is the full one

$$H = \frac{\hbar\epsilon_1}{2}\hat{\sigma}_{z_1} + \frac{\hbar\Delta_1}{2}\hat{\sigma}_{x_1} + \frac{\hbar\epsilon_2}{2}\hat{\sigma}_{z_2} + \frac{\hbar\Delta_2}{2}\hat{\sigma}_{x_2} - \hbar g\hat{\sigma}_{z_1}\hat{\sigma}_{z_2}. \quad (17)$$

Notice that if we fix on qubit 1, the interaction term can be absorbed in the ϵ_1 term as $\hbar(\epsilon_1 + g\hat{\sigma}_{z_2})/2$. This means we have a qubit-state dependent shift on qubit 1 depending on the state of qubit 2. This is a conditional scenario, in which the resonance frequency of qubit 1 depends on the state of qubit 2. Therefore, if we apply a resonant pulse to qubit 1 when qubit 2 is in the ground state, we rotate qubit 1. Otherwise, our pulse (which should not be too strong to avoid off-resonant driving effects) will be off-resonant and qubit 1 will stay in the same state. This is a CNOT-like operation, a universal entangling gate. The downside is the need to operate outside the symmetry point where qubit coherence is lower, as we rely on the sensitivity of the qubits to changes in charge/flux. The first CNOT gate with flux qubits was implemented with this technique in Delft [11].

Many other 2-qubit gates can be implemented with fixed qubit-qubit couplings, such as the application of multiple rf drives to induce 2-photon transitions [12], which is the protocol used by IBM Q.

References

[1] Krantz, P. *et al.* A quantum engineer's guide to superconducting qubits. *Applied Physics Reviews* **6**, 021318 (2019). URL `https://doi.org/10.1063/1.5089550`. `https://doi.org/10.1063/1.5089550`.

[2] Girvin, S. M. Superconducting qubits and circuits: Artificial atoms coupled to microwave photons (2011). Lectures delivered at Ecole d'Eté Les Houches.

[3] Koch, J. *et al.* Charge-insensitive qubit design derived from the Cooper pair box. *Phys. Rev. A* **76**, 042319 (2007). URL `http://dx.doi.org/10.1103/physreva.76.042319`.

[4] Harris, R. *et al.* Compound josephson-junction coupler for flux qubits with minimal crosstalk. *Phys. Rev. B* **80**, 052506 (2009). URL `https://link.aps.org/doi/10.1103/PhysRevB.80.052506`.

[5] Yan, F. *et al.* Tunable coupling scheme for implementing high-fidelity two-qubit gates. *Phys. Rev. Applied* **10**, 054062 (2018). URL `https://link.aps.org/doi/10.1103/PhysRevApplied.10.054062`.

[6] Bertet, P. *et al.* Dephasing of a Superconducting Qubit Induced by Photon Noise. *Physical Review Letters* **95**, 257002+ (2005). URL `http://dx.doi.org/10.1103/physrevlett.95.257002`.

[7] McDermott, R. *et al.* Simultaneous State Measurement of Coupled Josephson Phase Qubits. *Science* **307**, 1299–1302 (2005). URL `http://dx.doi.org/10.1126/science.1107572`.

[8] de Groot, P. C. *et al.* Selective darkening of degenerate transitions demonstrated with two superconducting quantum bits. *Nature Physics* **6**, 763–766 (2010). URL `http://dx.doi.org/10.1038/nphys1733`.

[9] DiCarlo, L. *et al.* Demonstration of two-qubit algorithms with a superconducting quantum processor. *Nature* **460**, 240–244 (2009). URL `http://dx.doi.org/10.1038/nature08121`.

[10] Arute, F. *et al.* Quantum supremacy using a programmable superconducting processor. *Nature* **574**, 505–510 (2019). URL `https://doi.org/10.1038/s41586-019-1666-5`.

[11] Plantenberg, J. H., de Groot, P. C., Harmans, C. J. P. M. & Mooij, J. E. Demonstration of controlled-NOT quantum gates on a pair of superconducting quantum bits. *Nature* **447**, 836–839 (2007). URL `http://dx.doi.org/10.1038/nature05896`.

[12] Rigetti, C., Blais, A. & Devoret, M. Protocol for universal gates in optimally biased superconducting qubits. *Phys. Rev. Lett.* **94**, 240502 (2005). URL `https://link.aps.org/doi/10.1103/PhysRevLett.94.240502`.