Exam Help Sheet

Schrödinger equation

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = \hat{H}(t)\left|\psi(t)\right\rangle$$

Uncertainty relation

$$\Delta a\Delta b \geqslant \frac{1}{2}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|$$

Eherenfest's theorem

$$\frac{d}{dt}\langle a\rangle = \frac{1}{i\hbar}\langle\psi|[\hat{A},\hat{H}]|\psi\rangle + \langle\psi|\frac{d\hat{A}}{dt}|\psi\rangle\,.$$

Harmonic oscillator

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}e^{-\frac{x^2}{2}}H_n\left(x\right)$$
$$H_0(x) = 1;\quad H_1(x) = 2x;\quad H_2(x) = 4x^2-2;\quad H_3(x) = 8x^3-12x$$

Harmonic oscillator: ladder operators

$$\hat{a} = \frac{\hat{x}+i\hat{p}}{\sqrt{2}};\quad a^\dagger = \frac{\hat{x}-i\hat{p}}{\sqrt{2}};\quad [\hat{a},\hat{a}^\dagger] = 1;\quad |n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle;\quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle;\quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Coherent states expansion

$$\hat{a}|z\rangle = z|z\rangle;\quad |z\rangle = e^{-\frac{|z|^2}{2}}\sum_n\frac{z^n}{\sqrt{n!}}|n\rangle;\quad \mathcal{D}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}};\quad |z(t)\rangle = e^{-i\frac{1}{2}t}|ze^{-it}\rangle\,.$$

Second quantization: fermions

$$\{a_\alpha^\dagger,a_\beta^\dagger\} = \{a_\alpha,a_\beta\} = 0;\quad \{a_\alpha,a_\beta^\dagger\} = \delta_{\alpha,\beta};$$
$$a_\alpha^\dagger|\cdots n_\alpha\cdots\rangle = (-1)^{\sum_{\delta<\alpha}n_\delta}(1-n_\alpha)|\cdots 1_\alpha\cdots\rangle;$$
$$a_\alpha|\cdots n_\alpha\cdots\rangle = (-1)^{\sum_{\delta<\alpha}n_\delta}n_\alpha|\cdots 0_\alpha\cdots\rangle$$

Second quantization: bosons

$$[a_\alpha^\dagger,a_\beta^\dagger] = [a_\alpha,a_\beta] = 0;\quad [a_\alpha,a_\beta^\dagger] = \delta_{\alpha,\beta};$$
$$a_\alpha^\dagger|\cdots n_\alpha\cdots\rangle = \sqrt{n_\alpha+1}|\cdots(n_\alpha+1)\cdots\rangle;$$
$$a_\alpha|\cdots n_\alpha\cdots\rangle = \sqrt{n_\alpha}|\cdots(n_\alpha-1)\cdots\rangle$$

Second quantization: operators

$$\hat{T} = \sum_{\alpha,\beta}t_{\alpha,\beta}a_\alpha^\dagger a_\beta;\quad \hat{V} = \sum_{\alpha,\beta,\gamma,\delta}v_{\alpha\beta,\gamma\delta}a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma.$$

Angular momentum

$$\hat{L}^2|lm\rangle = l(l+1)\hbar^2|lm\rangle;\quad \hat{L}_z|lm\rangle = m\hbar|lm\rangle$$

Spin

$$\mathbf{S} = \frac{\hbar}{2}\sigma; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad [\sigma_x, \sigma_y] = i2\sigma_z \text{ and permutations.}$$

Time-independent perturbation theory

$$E_n^{(1)} = \langle n|\hat{H}_1|n\rangle; \quad |\phi_n^{(1)}\rangle = \sum_{m\neq n} \frac{H_{mn}}{\epsilon_n - \epsilon_m}|m\rangle; \quad E_n^{(2)} = \sum_{m\neq n} \frac{|H_{mn}|^2}{\epsilon_n - \epsilon_m}$$

Time-dependent perturbation theory

$$a_{i\to f}^{(1)}(t) = \frac{\lambda}{i\hbar}\int_{t_0}^{t} d\bar{t}\, H_{fi}(\bar{t})e^{i\omega_{fi}\bar{t}}; \quad \mathcal{W}_{i\to d\epsilon_f} = \frac{2\pi}{\hbar}\,|H_{fi}|^2 \left(\frac{d\nu}{d\varepsilon_f}\right)_{\epsilon_f=\epsilon_i+\hbar\omega}$$

Heisenberg picture

$$|\psi_H\rangle = e^{i\frac{\hat{H}(t-t_0)}{\hbar}}|\psi_S(t)\rangle; \quad \hat{A}_H(t) = e^{i\frac{\hat{H}(t-t_0)}{\hbar}}\hat{A}_S e^{-i\frac{\hat{H}(t-t_0)}{\hbar}}; \quad i\hbar\frac{d\hat{A}_H}{dt} = \left[\hat{A}_H, \hat{H}\right] + i\hbar\left(\frac{d\hat{A}_S}{dt}\right)_H$$

Interaction picture $\hat{H} = \hat{H}_0 + \hat{H}_1$

$$|\psi_I(t)\rangle = e^{i\frac{\hat{H}_0(t-t_0)}{\hbar}}|\psi_S(t)\rangle; \quad \hat{A}_I = e^{i\frac{\hat{H}_0(t-t_0)}{\hbar}}\hat{A}_S e^{-i\frac{\hat{H}_0(t-t_0)}{\hbar}}; \quad i\hbar\frac{d\hat{A}_I}{dt} = \left[\hat{A}_I, \hat{H}_0\right] + i\hbar\left(\frac{d\hat{A}_S}{dt}\right)_I;$$

$$\hat{U}_I(t, t_0) = \mathcal{T}\left[e^{-\frac{i}{\hbar}\int_{t_0}^{t} d\bar{t} V_I(\bar{t})}\right].$$

Born approximation elastic scattering

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |V(\mathbf{q})|^2; \quad V(\mathbf{q}) = \int \mathbf{d}^{(3)}\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r})$$

Formal scattering theory

$$\frac{d^2u_l}{dr^2} - \frac{l(l+1)}{r^2}u_l + \frac{2m}{\hbar^2}\left[E - V(r)\right]u_l = 0$$

$$S_l = e^{2i\delta_l}; \quad \frac{d\sigma}{d\Omega} = |f(\Omega)|^2; \quad \sigma = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l; \quad a_s = -\lim_{k\to 0}\frac{\tan\delta_{l=0}}{k}$$

Hadamard's lemma

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}\left[A, [A, B]\right] + \frac{1}{3!}\left[A, [A, [A, B]]\right] + \cdots$$