Quantum Information

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on the notes of Michalis Skotiniotis

Dirac picture. Abstraction

Definition 5: An inner (or scalar) product over the vector space $\mathscr{V}(F)$ is the map $\langle \cdot | \cdot \rangle : \mathscr{V}(F) \times \mathscr{V}(F) \to F$ such that for arbitrary vectors $|\psi\rangle, |\phi\rangle, |\chi\rangle \in \mathscr{V}(F)$ and scalars $a, b \in F$ the following properties hold

(i) Conjugate symmetry: $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$

(ii) Multiplicative: $\langle\psi|\big(a|\phi\rangle\big) = a\langle\psi|\phi\rangle$

(iii) Linearity in the 2nd argument: $\langle\psi|\big(a|\phi\rangle + b|\chi\rangle\big) = a\langle\psi|\phi\rangle + b\langle\psi|\chi\rangle$

(iv) Positive semi-definite: $\langle\psi|\psi\rangle \geq 0$ with equality if and only if $|\psi\rangle = 0$

Remark: The norm of a vector $|\psi\rangle \in \mathscr{V}(F)$ is defined as

$$|| \, |\psi\rangle \, || := \sqrt{\langle\psi|\psi\rangle}$$

Inner product, bras & kets

The inner product satisfies the following two important properties

1. Schwarz inequality: $|\langle\psi|\phi\rangle| \leq \sqrt{\langle\psi|\psi\rangle\,\langle\phi|\phi\rangle}$

2. Triangle inequality: $||\,|\psi\rangle + |\phi\rangle\,|| \leq ||\,|\psi\rangle|| + ||\,|\phi\rangle||$

with equality if and only if $|\psi\rangle = a|\phi\rangle$.

Definition 6: A set of vectors $\{|\psi_i\rangle \in \mathscr{V}(F)\}$ are said to be orthonormal if they satisfy $\langle\psi_j|\psi_i\rangle = \delta_{ij}$. If the set $\{|\psi_i\rangle \in \mathscr{V}(F)\}$ is also maximally linearly independent then it forms an orthonormal basis of $\mathscr{V}(F)$.

Remark: In quantum mechanics $F = \mathbb{C}$ and $\mathscr{V}(\mathbb{C})$ is called a Hilbert space. Henceforth we shall write $\mathbb{H} \equiv \mathscr{V}(\mathbb{C})$.

Inner product, bras & kets

Let $|\mathbb{H}| = d$. The elements of $\mathbb{H}$ are column vectors of d complex numbers. These column vectors are called "kets" $|\psi\rangle$

The d kets

$$|1\rangle := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad |d\rangle := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

form an orthonormal basis of $\mathbb{H}$ which we shall refer to as the standard (or computational) basis

Any column vector can be written with respect to the standard basis as

$$|\psi\rangle = \sum_{i=1}^{d} \psi_i \, |i\rangle \quad \psi_i \in \mathbb{C}$$

Inner product, bras & kets

Row vectors of d complex numbers belong to the dual space of $\mathbb{H}$ and are called "bras" $\langle\phi|$. The d bras

$$\langle 1| := (1 \quad \ldots \quad 0), \quad \ldots \quad \langle d| := (0 \quad \ldots \quad 1)$$

form an orthonormal basis to the dual space of $\mathbb{H}$. Any bra in the dual space can be written as

$$\langle\phi| = \sum_{i=1}^{d} \phi_i \langle i| \quad \phi_i \in \mathbb{C}$$

The inner product between two vectors $|\psi\rangle, |\phi\rangle \in \mathbb{H}$ is the "braket",

$$\begin{aligned}\langle\phi|\psi\rangle &= \left(\sum_j \phi_j \langle j|\right)\left(\sum_i \psi_i |i\rangle\right)\\ &= \sum_{i,j} \phi_j \psi_i \langle j|i\rangle\\ &= \sum_{i,j} \phi_j \psi_i \delta_{ij}\\ &= \sum_i \phi_i \psi_i \quad \in \mathbb{C}\end{aligned}$$

Exercise 1

Given the ket

$$|\psi\rangle = \sum_{i=1}^{d} \psi_i \, |i\rangle$$

write down the bra $\langle\psi|$, i.e. determine the relation between the coefficients ψ_i of the ket, and those of the bra.

Postulate 1: Associated to any isolated physical system is a complex vector space with an inner product known as a Hilbert space. The system is completely described by its state vector, which is a unit vector in the system's state space.

States of quantum mechanical systems

Postulate 1 says that the mathematical description of a quantum mechanical system is described by a ket $|\psi\rangle \in \mathbb{H}$ such that

$$\langle\psi|\psi\rangle \equiv ||\,|\psi\rangle\,||^2 = 1$$

Remark: Observe that $|\psi\rangle \sim z|\psi\rangle, z \in \mathbb{C}, |z| = 1$. That is the state space of a quantum mechanical system is a projective Hilbert space

Mnemonic: Overall phases do not matter!

Postulates of Quantum Mechanics

Postulate 2: The evolution of the state of a closed quantum system is governed by Schrödinger's equation

$$i\hbar \frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = H|\psi\rangle$$

where H is a Hermitian operator (known as the Hamiltonian) and $\hbar$ is Planck's constant divided by 2π

Remark: Henceforth, $\hbar = 1$

Postulates of Quantum Mechanics

Postulate 2: The evolution of the state of a closed quantum system is described by a unitary operator U, i.e.,

$$|\phi\rangle = U|\psi\rangle$$

where U depends only on the duration of the evolution

Outer product & linear operators

Definition 7: Let $|\psi\rangle \in \mathbb{H}_1$ and $|\phi\rangle \in \mathbb{H}_2$. The outer product $|\psi\rangle\langle\phi|$ is a mapping from $\mathbb{H}_2$ to $\mathbb{H}_1$, i.e., $|\psi\rangle\langle\phi| : \mathbb{H}_2 \to \mathbb{H}_1$. Similarly $|\phi\rangle\langle\psi| : \mathbb{H}_1 \to \mathbb{H}_2$

Let $|\chi\rangle \in \mathbb{H}_2$. Then

$$\big(|\psi\rangle\langle\phi|\big)|\chi\rangle = |\psi\rangle\big(\langle\phi||\chi\rangle\big) = \langle\phi|\chi\rangle\,|\psi\rangle \in \mathbb{H}_1$$

Remark: It will be convenient to call the space associated with the "bra" of the outer product as the input space $\mathbb{H}_{\text{in}}$, and the space associated with the "ket" the output space $\mathbb{H}_{\text{out}}$. Whenever the input and output spaces are the same we shall simply write $\mathbb{H}$.

Outer product & linear operators

Definition 8: An operator $A : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}$ is said to be linear if

$$A\left(a|\psi\rangle + b|\phi\rangle\right) = a\,A|\psi\rangle + b\,A|\phi\rangle$$

for all $a, b \in \mathbb{C}$ and all $|\psi\rangle, |\phi\rangle \in \mathbb{H}_{\text{in}}$.

Remark: In order to specify the action of a linear operator it suffices to specify how it acts on a set of basis vectors

Remark: As linear operators map vectors to vectors, they can be represented by matrices. The converse is also true; every matrix can be associated to a linear operator.

Example: Bra's are linear operators that map vectors to complex numbers

$$\langle\phi| : \mathbb{H} \to \mathbb{C}$$

they are represented by row vectors $\langle\phi| = \begin{pmatrix}\phi_1 & \dots & \phi_d\end{pmatrix}$

WIDITI

Let $\{|i\rangle\}_{i=1}^{d}$ be an orthonormal basis of $\mathbb{H}$. The identity operator is defined as mapping every vector to itself.

$$\mathbb{I}|i\rangle = |i\rangle$$

$$\langle j|\mathbb{I}|i\rangle = \langle j|i\rangle$$

$$\langle j|\mathbb{I}|i\rangle = \delta_{ij}$$

You should convince yourselves that in terms of outer-products (or in-out relations) the identity can be written as

$$\mathbb{I} = \sum_{i=1}^{d} |i\rangle\langle i|$$

and that the identity has the same form in every basis

WIDITY

Let $|\psi\rangle \in \mathbb{H}$. Suppose you want to determine the coefficients of $|\psi\rangle$ in some orthonormal basis $\{|i\rangle\}_{i=1}^{d}$.

$$
\begin{aligned}
|\psi\rangle &= \mathbb{I}|\psi\rangle \\
&= \left(\sum_{i=1}^{d} |i\rangle\langle i| \right) |\psi\rangle \\
&= \sum_{i=1}^{d} \langle i|\psi\rangle |i\rangle \\
&\equiv \sum_{i=1}^{d} \psi_i |i\rangle
\end{aligned}
$$

WIDITI

Let $A : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}$. Suppose you want the matrix representation of A in terms of the orthonormal basis $\{|i\rangle\}_{i=1}^{d_{\text{in}}}$, $\{|j\rangle\}_{j=1}^{d_{\text{out}}}$

$$
\begin{aligned}
A &= \mathbb{I}_{\text{out}}\, A\, \mathbb{I}_{\text{in}} \\
&= \left(\sum_{j=1}^{d_{\text{out}}} |j\rangle\langle j|\right) A \left(\sum_{i=1}^{d_{\text{in}}} |i\rangle\langle i|\right) \\
&= \sum_{j=1}^{d_{\text{out}}} \sum_{i=1}^{d_{\text{in}}} \langle j|A|i\rangle\, |j\rangle\langle i| \\
&\equiv \sum_{j=1}^{d_{\text{out}}} \sum_{i=1}^{d_{\text{in}}} A_{ji}\, |j\rangle\langle i|
\end{aligned}
$$

Exercise 2

Suppose

$$|\phi\rangle = A|\psi\rangle$$

where $|\psi\rangle \in \mathbb{H}$ and $A : \mathbb{H} \to \mathbb{H}$. Determine the coefficients of $|\phi\rangle \in \mathbb{H}$ in terms of the coefficients of $|\psi\rangle \in \mathbb{H}$ and matrix elements of $A : \mathbb{H} \to \mathbb{H}$.

Outer product & linear operators

Definition 9: Let $A : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}$. The adjoint of A, denoted as $A^\dagger$, is the linear operator $A^\dagger : \mathbb{H}_{\text{out}} \to \mathbb{H}_{\text{in}}$.

Recall that bras are linear operators that take vectors to complex numbers. Now let $\langle\psi| : \mathbb{H} \to \mathbb{C}$, $\langle\psi| = \sum_i \psi_i \langle i|$. What is the adjoint?

$$\big(\langle\psi|\big)^\dagger : \mathbb{C} \to \mathbb{H}, \quad \big(\langle\psi|\big)^\dagger = |\psi\rangle = \sum_i \tilde{\psi}_i |i\rangle$$

How is $\tilde{\psi}_i$ related to ψ_i ?

$$\begin{aligned} 1 &= \langle\psi|\psi\rangle \\ &= \sum_{i,j} \psi_i \tilde{\psi}_j \langle i|j\rangle \\ &= \sum_i \tilde{\psi}_i \psi_i \end{aligned} \quad \Rightarrow \quad \tilde{\psi}_i = \psi_i^*$$

Outer product & linear operators

Definition 9: Let $A : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}$. The adjoint of A, denoted as $A^\dagger$, is the linear operator $A^\dagger : \mathbb{H}_{\text{out}} \to \mathbb{H}_{\text{in}}$.

Mnemonic: Given the matrix form of a linear operator A, its adjoint operator is obtained by switching the labels in its outer product and conjugating its coefficients

$$A = \sum_{j=1}^{d_{\text{out}}} \sum_{i=1}^{d_{\text{in}}} A_{ji} \, |j\rangle\langle i|, \quad A^\dagger = \sum_{j=1}^{d_{\text{out}}} \sum_{i=1}^{d_{\text{in}}} A_{ji}^* \, |i\rangle\langle j|$$

Properties:

(i) $(cA)^\dagger = c^* A^\dagger$

(ii) $(A + B)^\dagger = A^\dagger + B^\dagger$

(iii) $(AB)^\dagger = B^\dagger A^\dagger$

Outer product & linear operators

Definition 10: An operator, $A : \mathbb{H} \to \mathbb{H}$, is called Hermitian, or self-adjoint, if $A = A^\dagger$

Definition 11: An operator, $A : \mathbb{H} \to \mathbb{H}$, is said to be positive (negative) semi-definite if $\langle\psi|A|\psi\rangle \geq 0$ ($\langle\psi|A|\psi\rangle \leq 0$) for all $|\psi\rangle \in \mathbb{H}$. When the inequality is strict then the operator is said to be positive (negative) definite.

Remark: We shall often employ the notation $A \geq 0$ to denote positive (semi)-definiteness

Definition 12: An operator, $A : \mathbb{H} \to \mathbb{H}$, is said to be unitary if $A^\dagger A = AA^\dagger = \mathbb{I}$

Remark: We will denote unitary operators by $U : \mathbb{H} \to \mathbb{H}$. Unitary operators preserve inner products. $\langle\phi|\psi\rangle = \langle\phi|\mathbb{I}|\psi\rangle = \langle\phi|U^\dagger U|\psi\rangle$

Outer product & linear operators

Theorem 1 [Spectral Decomposition]: Let $A : \mathbb{H} \to \mathbb{H}$ be a Hermitian linear operator. Then there exists an orthonormal basis $\{|i\rangle\}_{i=1}^{d}$ with respect to which A has the form

$$A = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|$$

where $\lambda_i \in \mathbb{R}$.

Remark: $\{\lambda_i\}_{i=1}^{d}$ are the eigenvalues of A. If all the eigenvalues are unique and non-zero then A is said to be full rank. If $\exists\, \lambda_i = 0$ then A is said to be rank-deficient. If there are $r < d$ distinct eigenvalues then

$$A = \sum_{i=1}^{r} \lambda_i P_i \equiv \sum_{i=1}^{r} \lambda_i \sum_{\nu=1}^{n_i} |i,\nu\rangle\langle i,\nu|$$

where P_i is the projector onto the subspace with eigenvalue λ_i

Outer product & linear operators

Theorem 2 [Polar Decomposition]: Let $A : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}$ be a linear operator. Then there exists positive operators $P = (AA^\dagger)^{\frac{1}{2}} : \mathbb{H}_{\text{out}} \to \mathbb{H}_{\text{out}}$, $Q = (A^\dagger A)^{\frac{1}{2}} : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{in}}$, and isometry operator $U : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}$ such that

$$A = PU = UQ\,.$$

If A is invertible then U is unique.

Remark: The expression $A = UQ$ is called the left polar decomposition whereas $A = PU$ is called the right polar decomposition. In the case $\mathbb{H}_{\text{in}} = \mathbb{H}_{\text{out}} = \mathbb{H}$, U is a unitary operator.

Remark: Observe that $\det A = \det P \det U = re^{i\theta}$.

Outer product & linear operators

Theorem 3 [Singular Value Decomposition]: Let $A : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}$ be a linear operator. Then there exists $V : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{in}}$, $U : \mathbb{H}_{\text{out}} \to \mathbb{H}_{\text{out}}$, and positive semi-definite $D : \mathbb{H}_{\text{in}} \to \mathbb{H}_{\text{out}}$ such that

$$A = UDV$$

Remark: The matrix D is called the matrix of singular-values of A. The SVD generalizes the spectral decomposition to non-square matrices.

Functions of linear operators

Let $f : \mathbb{C} \to \mathbb{C}$ be a function and $A : \mathbb{H} \to \mathbb{H}$ a Hermitian operator. Then

$$f(A) = \sum_{i=1}^{d} f(\lambda_i)\,|i\rangle\langle i|$$

Hence in order to compute a function of any Hermitian matrix you must

1. Diagonalize the matrix
2. Apply the function to its eigenvalues
3. Revert back to the original basis.

$$A = UDV \quad \Rightarrow \quad f(A) = Uf(D)V$$

Example

Suppose

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

$$\lambda_1 = \frac{5}{6}, \quad |\lambda_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\lambda_2 = \frac{1}{6}, \quad |\lambda_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Compute $\log A$

$$\log A = \log\frac{5}{6}|\lambda_1\rangle\langle\lambda_1| + \log\frac{1}{6}|\lambda_2\rangle\langle\lambda_2|$$

In the original basis

$$\log A = U\ \log D\, V$$

$$U = \begin{pmatrix} \langle\lambda_1|0\rangle & \langle\lambda_1|1\rangle \\ \langle\lambda_2|0\rangle & \langle\lambda_2|1\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$V = U^\dagger = U$$

Lecture 2

Previously...

- We talked about linear vector spaces, Hilbert spaces, and quantum states
- We introduced Dirac's notation (bra's and kets) and learned how to algebraically manipulate objects like $|\psi\rangle\langle\phi|$ and $\langle\psi|\phi\rangle$
- We learned the most important rule: WIDITI
- We talked about how to evolve the state of quantum systems via unitary operators
- We learned about positive, Hermitian, and Unitary operators and how to break them up into simpler objects (SD, PD, SVD)

Lecture 2

Today we will...

- Continue learning a few more facts about linear operators (how to combine them to get other linear operators)
- Introduce the measurement postulate and learn all about observables, projective measurements and POVMs
- Introduce the simplest quantum mechanical system and the fundamental unit of quantum information—the qubit
- Do something fun !

Functions of linear operators

Postulate 2: The evolution of the state of a closed quantum system is governed by Schrödinger's equation

$$i\hbar \frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = H|\psi\rangle$$

where H is a Hermitian operator (known as the Hamiltonian) and $\hbar$ is Planck's constant divided by 2π

The solution to this equation is easily seen to be

$$|\psi(t)\rangle = e^{\frac{-i}{\hbar}tH}|\psi\rangle \equiv U(t)|\psi\rangle$$

where $U(t)$ is a unitary operator obtained by exponentiating the Hermitian operator H—the Hamiltonian of the system

Functions of linear operators

Definition 13: The trace of a matrix A, is the sum of its diagonal elements

$$\mathrm{tr}(A) = \sum_{i=1}^{d} A_{ii}$$

Properties:

1. $\mathrm{tr}(aA) = a\mathrm{tr}(A),\ a \in \mathbb{C}$
2. $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
3. $\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$
4. The trace is basis independent (prove it)

Functions of linear operators

$$\mathrm{tr}(A) = \mathrm{tr}\left(\sum_{i,j} A_{ji}\,|j\rangle\langle i|\right)$$

$$\sum_{ij} A_{ij}\,\mathrm{tr}\left(|j\rangle\langle i|\right)$$

$$\sum_{ij} A_{ij}\,\langle i|j\rangle$$

$$= \sum_{i} A_{ii}$$

Using the cyclic property, turn the ketbra into a braket

Functions of linear operators

Definition 14: Let $A, B : \mathbb{H} \to \mathbb{H}$ be two Hermitian operators. The **commutator** of A and B, $[A, B]$, is defined as

$$[A, B] = AB - BA$$

Definition 15: Let $A, B : \mathbb{H} \to \mathbb{H}$ be two Hermitian operators. The **anti-commutator** of A and B, $\{A, B\}$, is defined as

$$\{A, B\} = AB + BA$$

Theorem 4: Let $A, B : \mathbb{H} \to \mathbb{H}$ be two Hermitian operators. Then $[A, B] = 0$ if and only if there exists an orthonormal basis with respect to which A and B are both diagonal.

Postulates of Quantum Mechanics

Postulate 3: Measurements of quantum mechanical systems are described by a collection of positive operators $\{M_m : \mathbb{H} \to \mathbb{H}\}$ where m denotes the measurement outcome. If the system is in state $|\psi\rangle \in \mathbb{H}$ then the probability of observing outcome m is given by

$$p_m = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

and the post-measurement state is

$$|\phi_m\rangle = \frac{M_m|\psi\rangle}{\sqrt{p_m}}$$

Observables and Projective Measurements

Theorem 1 [Spectral Decomposition]: Let $A : \mathbb{H} \to \mathbb{H}$ be a Hermitian linear operator. Then there exists an orthonormal basis $\{|i\rangle\}_{i=1}^{d}$ with respect to which A has the form

$$A = \sum_{i=1}^{d} \lambda_i \, |i\rangle\langle i|$$

where $\lambda_i \in \mathbb{R}$.

Definition 16: Every observable quantity (energy, momentum, angular momentum etc) $O : \mathbb{H} \to \mathbb{H}$, corresponds to a Hermitian operator.

Remark: By Theorem 1 every observable can be written as

$$O = \sum_{i=1}^{r} \lambda_i P_i$$

where P_i are the eigen-projectors corresponding to eigenvalue λ_i

Observables and Projective Measurements

Recall that by the spectral theorem, the eigenbasis of any Hermitian observable forms an orthonormal basis. From this it follows that

Lemma 1: The eigen-projectors $\{P_i\}_{i=1}^r$ of an observable $O : \mathbb{H} \to \mathbb{H}$ corresponding to distinct eigenvalues are orthogonal, i.e.

$$P_i P_j = \delta_{ij} P_j$$

Definition 17: Let $O = \sum_{i=1}^{r} \lambda_i P_i$ be an observable. A measurement with operators $\{P_i\}_{i=1}^r$, is called a projective measurement

Example

$$O = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} = \frac{5}{6}|\lambda_1\rangle\langle\lambda_1| + \frac{1}{6}|\lambda_2\rangle\langle\lambda_2|$$

where $|\lambda_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$ Now $P_1 = |\lambda_1\rangle\langle\lambda_1|$

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$P_2 = |\lambda_2\rangle\langle\lambda_2| \quad \Rightarrow P_1P_2 = 0$$

$$P_1 + P_2 = \mathbb{I}$$

A projective measurement in the standard basis has measurement operators

$$\left\{P_i = |i\rangle\langle i|\right\}_{i=1}^{d}$$

Example

Now suppose that a quantum mechanical system is prepared in the state

$$|\psi\rangle = \sqrt{\frac{2}{3}}\, e^{i\frac{\pi}{6}}|0\rangle + \sqrt{\frac{1}{3}}\, e^{i\frac{4\pi}{6}}|1\rangle$$

Compute

1. $p(\lambda_1)$ and the post measurement state $|\phi_{\lambda_1}\rangle$

2. $p(\lambda_2)$ and the post measurement state $|\phi_{\lambda_2}\rangle$

3. The mean value of the observable O and its corresponding variance

$$O = \frac{5}{6}|\lambda_1\rangle\langle\lambda_1| + \frac{1}{6}|\lambda_2\rangle\langle\lambda_2|$$

Exercise

Let $O = \sum_{i=1}^{r} \lambda_i P_i,$

Show that the mean (or expected) value and variance of observable O with respect to the state $|\psi\rangle \in \mathbb{H}$ are given by

$$\langle O \rangle = \langle \psi | O | \psi \rangle$$

$$\sigma_O^2 = \langle \psi | O^2 | \psi \rangle - \langle \psi | O | \psi \rangle^2$$

The uncertainty principle

Let $O, Q : \mathbb{H} \to \mathbb{H}$ be two observables and let $|\psi\rangle \in \mathbb{H}$ be the state of a quantum mechanical system. Then

$$\sigma_O^2 \sigma_Q^2 \geq \frac{1}{2}\langle\psi|[O,Q]|\psi\rangle$$

Remarks: There is a great deal of misconception about the uncertainty principle

1. It does not say that measuring one quantity causes the value of another quantity to change (or be "disturbed") by some amount

2. It does not say that two non-commuting observables cannot be jointly measured.

3. It does say that there are no "dispersion-free" states, i.e. $\nexists\, |\psi\rangle \in \mathbb{H}$ such that σ_O^2, σ_Q^2 are both zero.

Interlude

The uncertainty principle, and "no measurement-without-disturbance"...

using a deck of cards!!!!!

Postulates of Quantum Mechanics

Postulate 3: Measurements of quantum mechanical systems are described by a collection of positive operators $\{M_m : \mathbb{H} \to \mathbb{H}\}$ where m denotes the measurement outcome. If the system is in state $|\psi\rangle \in \mathbb{H}$ then the probability of observing outcome m is given by

$$p_m = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

As probabilities must sum to unity we have

$$1 = \sum_m p_m = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle$$

$$= \langle\psi|\left(\sum_m M_m^\dagger M_m\right)|\psi\rangle$$

$$\Rightarrow \sum_m M_m^\dagger M_m = \mathbb{I}$$

Define $E_m \equiv M_m^\dagger M_m$

then $p_m = \langle\psi|E_m|\psi\rangle \geq 0 \Rightarrow E_m \geq 0$

and $\sum_m E_m = \mathbb{I}$

Generalised Measurements

Definition 18: The set of positive operators $\left\{ E_m : \mathbb{H} \to \mathbb{H}, E_m \geq 0, \forall m | \sum_m E_m = \mathbb{I} \right\}$ form the elements of a Positive Operator Value Measure (or POVM for short)

Remark: Observe that the elements of the POVM provide only the probabilities of measurement outcomes

$$p_m = \langle \psi | E_m | \psi \rangle, \quad |\psi\rangle \in \mathbb{H}$$

and not the post measurement state. Hence POVMs are not measurement operators as defined in postulate 3.

Generalised Measurements

Definition 18: The set of positive operators $\left\{ E_m : \mathbb{H} \rightarrow \mathbb{H}, E_m \geq 0, \forall m | \sum_m E_m = \mathbb{I} \right\}$ form the elements of a Positive Operator Value Measure (or POVM for short)

We can obtain the requisite measurement operator M_m of the corresponding POVM element E_m by noting that

$$M_m = E_m^{\frac{1}{2}}$$

Remark: Note that the measurement operator corresponding to a given POVM element is not unique (quick proof?)

Examples of POVMs

1. Projective measurements are a special case of POVMs due to the following lemma

Lemma 2: A POVM describes a projective measurement if and only if $E_m^{\frac{1}{2}} = E_m$ for all m.

2. $$E_0 = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} \frac{1}{3} & \sqrt{\frac{1}{6}} \\ \sqrt{\frac{1}{6}} & \frac{1}{2} \end{pmatrix}, \quad E_2 = \begin{pmatrix} \frac{1}{3} & -\sqrt{\frac{1}{6}} \\ -\sqrt{\frac{1}{6}} & \frac{1}{2} \end{pmatrix},$$

$$M_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad M_1 = \begin{pmatrix} \sqrt{\frac{2}{15}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \sqrt{\frac{3}{10}} \end{pmatrix} \quad M_2 = \begin{pmatrix} \sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \sqrt{\frac{3}{10}} \end{pmatrix}$$

Examples of POVMs

3. Continuous POVMs

$$E_\theta = \frac{1}{2\pi}\begin{pmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & 1 \end{pmatrix}$$

$$\int_0^{2\pi} E_\theta \, \mathrm{d}\theta = \mathbb{I}$$

$$M_\theta = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & 1 \end{pmatrix}$$

The qubit

The simplest quantum mechanical system—called a qubit—has a 2-dimensional state space, i.e. $d = |\mathbb{H}| = 2$.

An arbitrary pure state of a two-dimensional system can be written as

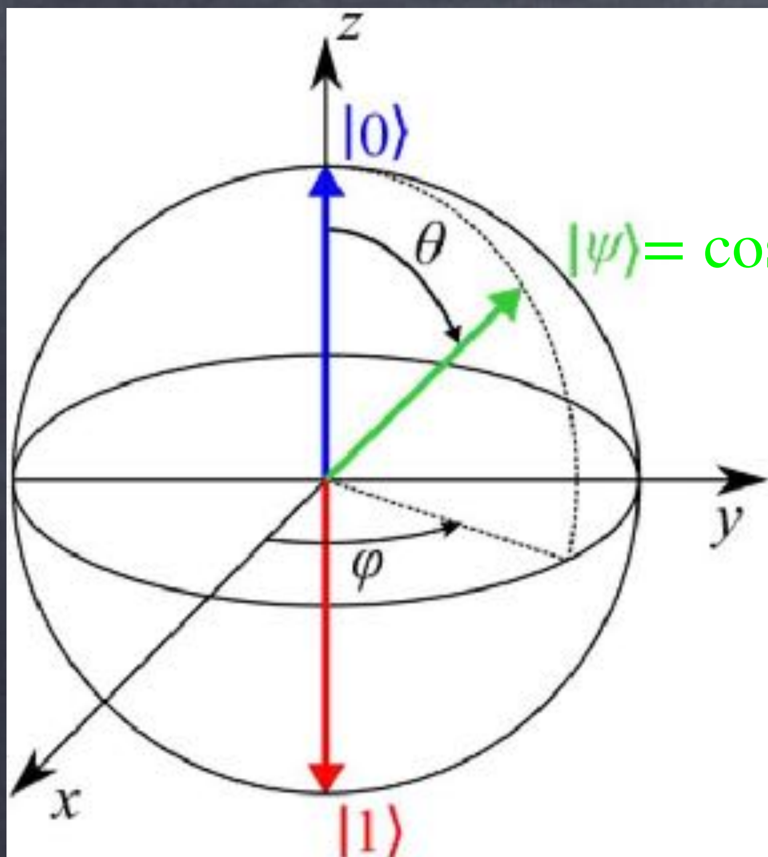
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C}$$

As overall phases do not matter and since $|\alpha|^2 + |\beta|^2 = 1$ an arbitrary state of a two-dimensional quantum system can be parametrised as

$$|\psi\rangle = \cos\theta|0\rangle + \sin\theta\, e^{i\phi}|1\rangle$$

where $\theta \in (0,\pi), \phi \in (0,2\pi]$

The Bloch Sphere representation



$$|\psi\rangle = \cos\theta|0\rangle + \sin\theta\, e^{i\phi}|1\rangle$$

Every point on the surface of the sphere represents a pure quantum state

Unitary operations are rotations about an axis of the Bloch sphere

$$U(\chi,\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i(\chi+\phi)} & -\sin\frac{\theta}{2}e^{-i(\chi-\phi)} \\ \sin\frac{\theta}{2}e^{i(\chi-\phi)} & \cos\frac{\theta}{2}e^{i(\chi+\phi)} \end{pmatrix}$$

Remark: Notice that orthogonal states lie on opposite ends of the Bloch sphere, i.e. they are an angle π apart. The θ appearing in the definition of the unitary is a "real angle". The θ appearing on the Bloch sphere is a "Hilbert space" angle.

Mnemonic: Hilbert space angles are twice the real angles.

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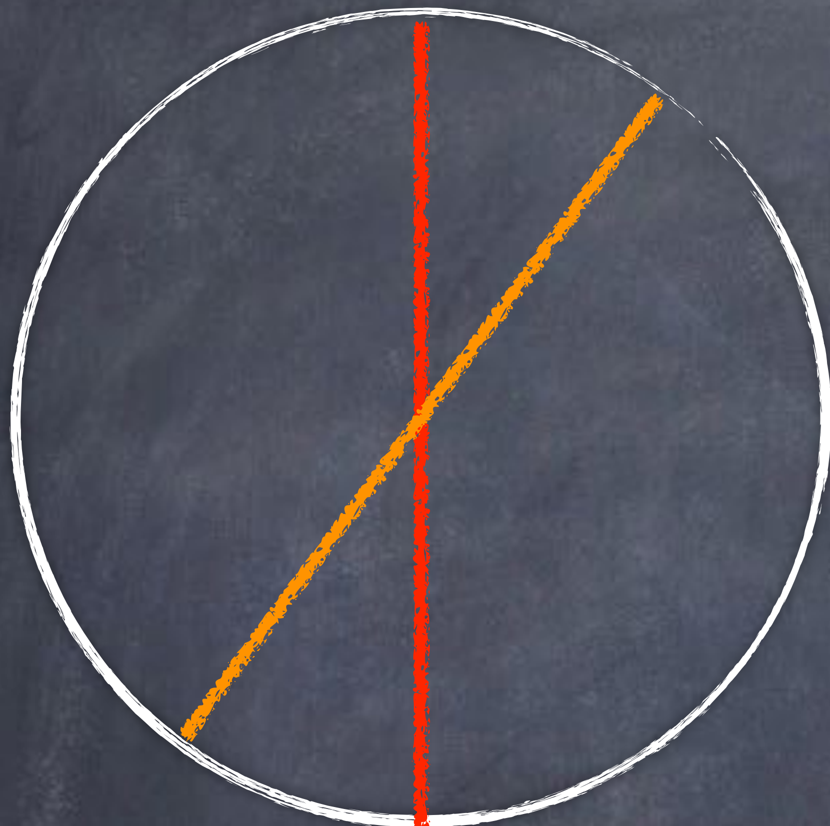

Projective Measurements correspond to idepodents on opposite sides of the Bloch sphere

POVM elements are any arrangement of arrows such that their sum is equal to the center of the sphere