Lectures 3 and 4

Now...

- We will learn about the fourth (and final) postulate of quantum theory concerning composite quantum systems
- We will see how composite quantum systems can exist in states that have no classical analogue (entanglement)
- We introduce the most general description of quantum mechanical systems—the density operator—and reformulate all postulates of the theory in their most general form

Composite systems

Postulate 4: The Hilbert space of a composite physical system is given by the tensor product of the Hilbert spaces of each of its constituent parts

$$\mathscr{H}_{\text{Total}} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \mathscr{H}_3 \otimes \cdots \otimes \mathscr{H}_N \equiv \bigotimes_{k=1}^{N} \mathscr{H}_k$$

The tensor product space inherits all the properties of its constituent parts (linearity, multiplicative & additive identity etc etc)

Remark: Whenever the composite system is comprised of only two parts we shall use the term bipartite system

Remark: Tensor products can be used to describe the total state space of a single physical system. E.g. consider describing both the position as well as the angular momentum of a particle

Composite systems

Definition 19: Let $\mathscr{H}_1, \mathscr{H}_2$ be two vector spaces of dimension d_1, d_2 respectively. Suppose that $\{|i_1\rangle\}_{i_1=1}^{d_1}$ is an orthonormal basis of $\mathscr{H}_1$ and $\{|i_2\rangle\}_{i_2=1}^{d_2}$ an orthonormal basis of $\mathscr{H}_2$. Then an **orthonormal basis** of $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ is

$$\{|i_1\rangle \otimes |i_2\rangle\}, i_1 \in (1,\cdots,i_{d_1}), i_2 \in (1,\cdots,i_{d_2})$$

Remark: notation will be $|i_1\rangle \otimes |i_2\rangle = |i_1, i_2\rangle| = |i_1 i_2\rangle$

Properties of the tensor product

1. Let $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$. Then $|\mathbb{H}| = |\mathbb{H}_1| \times |\mathbb{H}_2|$

2. Suppose $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$, $|\phi\rangle \in \mathscr{H}_1; |\chi\rangle \in \mathscr{H}_2$; and let $\{|i_1\rangle\}_{i_1=1}^{d_1}$, $\{|i_2\rangle\}_{i_2=1}^{d_2}$ be orthonormal basis of $\mathbb{H}_1, \mathbb{H}_2$. Then

$$\begin{pmatrix} \psi_{11} \\ \psi_{12} \\ \vdots \\ \psi_{1d_2} \\ \psi_{21} \\ \vdots \\ \psi_{2d_2} \\ \vdots \\ \psi_{d_1 1} \\ \vdots \\ \psi_{d_1 d_2} \end{pmatrix} = \begin{pmatrix} \phi_1 \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{d_2} \end{pmatrix} \\ \phi_2 \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{d_2} \end{pmatrix} \\ \vdots \\ \phi_{d_1} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{d_2} \end{pmatrix} \end{pmatrix}$$

Properties of the tensor product

1. Let $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$. Then $|\mathbb{H}| = |\mathbb{H}_1| \times |\mathbb{H}_2|$

2. Suppose $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$, and let $\{|i_1\rangle\}_{i_1=1}^{d_1}$, $\{|i_2\rangle\}_{i_2=1}^{d_2}$ be orthonormal basis of $\mathbb{H}_1$, $\mathbb{H}_2$. Then $|\psi\rangle = \displaystyle\sum_{i_1=1}^{d_1} \sum_{i_2=1}^{d_2} \phi_{i_1} \chi_{i_2} |i_1\rangle \otimes |i_2\rangle$

3. Suppose $B : \mathbb{H}_1 \to \mathbb{H}_1, \quad C = \mathbb{H}_2 \to \mathbb{H}_2$. Then $A : \mathbb{H} \to \mathbb{H}, A = B \otimes C$ is given by

$$A = \left(\sum_{i_1, j_1} B_{j_1, i_1} |j_1\rangle\langle i_1| \right) \otimes \left(\sum_{i_2, j_2} C_{j_2, i_2} |j_2\rangle\langle i_2| \right)$$

$$= \sum_{i_1, j_1} \sum_{i_2, j_2} B_{j_1, i_1} C_{j_2, i_2} |j_1\rangle\langle i_1| \otimes |j_2\rangle\langle i_2|$$

Properties of the tensor product

$$A : \mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$$

$$A = B \otimes C$$

$$\begin{pmatrix} A_{11,11} & A_{11,12} & \cdots & A_{11,d_2d_2} \\ A_{12,11} & \cdots & & \\ \vdots & \ddots & & \\ A_{d_1d_1,11} & \cdots & & A_{d_1d_1,d_2d_2} \end{pmatrix} = \begin{pmatrix} B_{11}\begin{pmatrix} C_{11} & \cdots & C_{1d_2} \\ \vdots & \ddots & \\ C_{d_21} & \cdots & C_{d_2d_2} \end{pmatrix} & \cdots & B_{1d_1}\begin{pmatrix} C_{11} & \cdots & C_{1d_2} \\ \vdots & \ddots & \\ C_{d_21} & \cdots & C_{d_2d_2} \end{pmatrix} \\ \vdots & \ddots & \\ B_{d_11}\begin{pmatrix} C_{11} & \cdots & C_{1d_2} \\ \vdots & \ddots & \\ C_{d_21} & \cdots & C_{d_2d_2} \end{pmatrix} & \cdots & B_{d_1d_1}\begin{pmatrix} C_{11} & \cdots & C_{1d_2} \\ \vdots & \ddots & \\ C_{d_21} & \cdots & C_{d_2d_2} \end{pmatrix} \end{pmatrix}$$

$$\sum_{i_1,j_1} \sum_{i_2,j_2} B_{j_1,i_1} C_{j_2,i_2} |j_1\rangle\langle i_1| \otimes |j_2\rangle\langle i_2|$$

Properties of the tensor product

1. Let $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$. Then $|\mathbb{H}| = |\mathbb{H}_1| \times |\mathbb{H}_2|$

2. Suppose $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$, and let $\{|i_1\rangle\}_{i_1=1}^{d_1}$, $\{|i_2\rangle\}_{i_2=1}^{d_2}$ be orthonormal basis of $\mathbb{H}_1$, $\mathbb{H}_2$. Then $|\psi\rangle = \sum_{i_1=1}^{d_1} \sum_{i_2=1}^{d_2} \phi_{i_1} \chi_{i_2} |i_1\rangle \otimes |i_2\rangle$

3. Suppose $B : \mathbb{H}_1 \to \mathbb{H}_1, \quad C = \mathbb{H}_2 \to \mathbb{H}_2$. Then $A : \mathbb{H} \to \mathbb{H}, A = B \otimes C$ is given by $A = \sum_{i_1, j_1} \sum_{i_2, j_2} B_{j_1, i_1} C_{j_2, i_2} |j_1\rangle\langle i_1| \otimes |j_2\rangle\langle i_2|$

4. The tensor product space $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$ inherits all the properties of its constituent parts (linearity, multiplicative & additive identity etc etc)

Composite systems

The more prudent way of understanding the tensor product is that it combines together Hilbert spaces associated to distinct properties; (different particles, position, energy angular momentum of the same particle etc etc etc.)

Definition 20: A composite quantum system is said to be in a product state if

$$|\Psi\rangle = \bigotimes_{i=1}^{N} |\psi_i\rangle$$

where $|\Psi\rangle \in \mathbb{H}_{\text{Total}}$ and $|\psi_i\rangle \in \mathbb{H}_i$.

Exercise

Let $\mathscr{H}_1 = \mathscr{H}_2$ with dimension $d = 2$. Write the state $|\Psi\rangle \in \mathbb{H}_1 \otimes \mathbb{H}_2$ in tensor product form

1. $|\Psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

2. $|\Psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|01\rangle$

3. $|\Psi\rangle = \sqrt{\frac{1}{6}}|00\rangle + \sqrt{\frac{1}{3}}\, e^{i\frac{\pi}{3}}|01\rangle + \sqrt{\frac{1}{6}}\, e^{i\frac{\pi}{4}}|10\rangle + \sqrt{\frac{1}{3}} e^{i\frac{7\pi}{4}}|11\rangle$

4. $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + e^{i\phi}|11\rangle\right)$

Product and Entangled States

Definition 21 A states is called **entangled.** iff cannot be written in tensor product form $|\psi\rangle \neq \otimes |\psi_i\rangle$

example: $|\Psi\rangle = \dfrac{1}{\sqrt{2}}\left(|00\rangle + e^{i\phi}|11\rangle\right)$ is entangled

$|\Psi\rangle = \dfrac{1}{\sqrt{2}}\left(|00\rangle + e^{i\phi}|01\rangle\right)$ is NOT entangled !

Product and Entangled States

Definition 22: Let $\mathscr{H} = \bigotimes_{i=1}^{N} \mathscr{H}_i$. A unitary $U : \mathscr{H} \to \mathscr{H}$ is said to be a **local operation** if $U = \otimes_{i=1}^{N} U_i, \quad U_i : \mathscr{H}_i \to \mathscr{H}_i$. Otherwise the operation is said to be **non-local.**

Definition 23: Let $\mathscr{H} = \bigotimes_{i=1}^{N} \mathscr{H}_i$. A measurement with operators $\{M_k : \mathscr{H} \to \mathscr{H}\}_{k=1}^{M}$ is said to be **local** if **every measurement operator** is of the form

$$M_k = \bigotimes_{i=1}^{N} M_k^{(i)}$$

otherwise the measurement is said to be **non-local**

Bipartite Entanglement

Theorem 4 [Schmidt Decomposition]: Let $|\psi\rangle \in \mathbb{H}_1 \otimes \mathbb{H}_2$. Then there exist orthonormal states $\{|i_1\rangle\}_{i=1}^{d_1} \in \mathbb{H}_1, \quad \{|i_2\rangle\}_{i=1}^{d_2} \in \mathbb{H}_2$ such that

$$|\psi\rangle = \sum_{i=1}^{\min(d_1,d_2)} \lambda_i \, |i_1, i_2\rangle$$

where $\lambda_i \geq 0, \quad \sum_{i=1}^{\min(d_1,d_2)} \lambda_i^2 = 1$ are called the Schmidt coefficients of the state.

Remark: The number of non-zero Schmidt coefficients of the state is called the Schmidt rank of the state, whereas the basis $\{|i_1\rangle\}_{i=1}^{d_1} \in \mathbb{H}_1, \quad \{|i_2\rangle\}_{i=1}^{d_2} \in \mathbb{H}_2$ is known as the Schmidt basis of the state.

Remark: A state is of product form if and only if it has Schmidt number one.

Reduced states of composite systems

Consider two parties—Alice and Bob—each of which hold part of a composite quantum system in some state $|\Psi\rangle_{AB} \in \mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$.

How should Alice (Bob) describe the state of their respective system?

Clearly if $|\Psi\rangle_{AB} = |\phi\rangle_A \otimes |\phi\rangle_B$ where $|\phi\rangle_A \in \mathscr{H}_A, \quad |\phi\rangle_B \in \mathscr{H}_B$ then everything is OK!

What about entangled states?

$$|\psi\rangle = \sum_{i=1}^{\min(d_1,d_2)} \lambda_i \, |i_1, i_2\rangle$$

Reduced states of composite systems

Consider the bipartite state

$$|\Psi\rangle_{AB} = \sqrt{\frac{1}{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

Suppose B measures in the standard basis. What is the probability that B obtains outcome 0 or 1?

$${}_{AB}\langle\Psi|\left(\mathbb{1}_A \otimes |0\rangle\langle 0|\right)|\Psi\rangle_{AB} = \frac{1}{2} \quad \Rightarrow \quad |\Phi\rangle_{AB} = |0\rangle_A|0\rangle_B$$

$${}_{AB}\langle\Psi|\left(\mathbb{1}_A \otimes |1\rangle\langle 1|\right)|\Psi\rangle_{AB} = \frac{1}{2} \quad \Rightarrow \quad |\Phi\rangle_{AB} = |1\rangle_A|1\rangle_B$$

Now suppose that B doesn't tell A the outcome of the measurement. All A can say is that her system is equally likely to be in either state!

Ensembles of quantum states

Definition 24: An **ensemble** (or assemblage) **of pure states** describes a situation where a quantum system can be in any one of a different pure states $|\psi_i\rangle \in \mathbb{H}$ with probability p_i.

Remarks:

1. The subindex i can be discreet or continuous
2. It is customary to represent a particular ensemble of quantum states as
$$\left\{p_i, |\psi_i\rangle\right\}$$
3. Unitarily evolving an ensemble of states by an operator U gives rise to another ensemble
$$\left\{p_i, U|\psi_i\rangle\right\}$$
4. Performing a measurement with operators $\{M_k\}$ on an ensemble gives rise to another ensemble
$$\left\{p_i, \left\{q(k\,|\,i) = \langle\psi_i|M_k^\dagger M_k|\psi_i\rangle, |\phi_i^{(k)}\rangle = \sqrt{\frac{1}{q(k\,|\,i)}} M_k\,|\psi_i\rangle\right\}\right\}$$

Postulates-Recap

1. Associated to any isolated physical system is a Hilbert space. The system is completely described by its state vector, which is a unit vector in the system's state space.

2. The evolution of the state of a closed quantum system is given by

$$|\psi(t)\rangle = U(t)|\psi\rangle$$

 where $U(t)$ is a unitary operator.

3. Measurements are described by any collection of positive operators $\left\{ M_m : \mathbb{H} \to \mathbb{H} \mid \sum_m M_m^\dagger M_m = \mathbb{I} \right\}$ where m denotes the measurement outcome. If the system is in state $|\psi\rangle \in \mathbb{H}$ then the probability of observing outcome m is given by $p_m = \langle\psi|M_m^\dagger M_m|\psi\rangle$ and the post-measurement state is $|\phi_m\rangle = \dfrac{M_m|\psi\rangle}{\sqrt{p_m}}$

Last time...

- We talked about the 4^{th} postulate of quantum theory and how to describe the states of composite systems
- We learned about the tensor product and how to compute it in both algebraic as well as matrix form
- We learned about product states, local operations and measurements and that quantum theory allows for the existence of entangled states
- We derived the Schmidt form of bipartite states and introduced ensembles of quantum states

Today...

- We will learn the most elegant way of describing ensembles of quantum states via the density operator formalism
- We will see how density operators allow us to describe the states of subsystems that are entangled amongst themselves
- We will reformulate the postulates of QT in terms of the density operator
- We will revisit the qubit and see how density operators are described in terms of the Bloch representation.
- Prove the no-cloning theorem.

Ensembles of quantum states

Definition 25: To every ensemble of quantum states $\{p_i, |\psi_i\rangle\}$ one can associate a **density operator** $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in \mathscr{B}(\mathbb{H})$ satisfying the following properties

1. **Bounded trace condition**: $\mathrm{tr}(\rho) < \infty$
2. **Positivity condition**: $\rho > 0$

Remarks:

1. The set $\mathscr{B}(\mathbb{H})$ is known as the set of **bounded operators on a Hilbert space** and is itself a bona fide vector space (check this!)
2. The converse statement above is not true (every density operator ρ may correspond to infinitely many assemblages)
3. Density operators are the most general mathematical description of quantum states

Ensembles of quantum states

Let $\{p_i, |\psi_i\rangle\}$ be an ensemble of quantum states. It's density operator description is

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Remark: Because ρ is a matrix it is often referred to as a density matrix

What is the density operator for the ensemble consisting of a single quantum state?

$$\rho = |\psi\rangle\langle\psi|$$

Theorem 5 [rank-one density operators]: Let $\rho \in \mathscr{B}(\mathbb{H})$. Then $\mathrm{tr}(\rho^2) \leq 1$ with equality if and only if $\rho = |\psi\rangle\langle\psi|$

Nomenclature: If $\rho = |\psi\rangle\langle\psi|$ then we say that the state of the quantum system is pure. Else the state is mixed

The postulates of QT with density operators

Postulate 1: Associated to any physical system is a density operator $\rho \in \mathscr{B}(\mathbb{H})$, $\rho > 0$, $\mathrm{tr}(\rho) = 1$. If the system is known to be in state ρ_i with probability p_i then $\rho = \sum_i p_i \rho_i$.

Postulate 2: The evolution of a closed quantum system is described by a time-dependent unitary operator $U(t) : \mathbb{H} \to \mathbb{H}$ such that

$$\rho(t) = U(t)\rho U^\dagger(t)$$

Postulate 3: Measurements are described by a set of measurement operators $\{M_k : \mathbb{H} \to \mathbb{H}\}$ such that the probability of obtaining measurement outcome k is given by

$$p(k) = \mathrm{tr}\left(M_k^\dagger M_k \rho\right)$$

and the post measurement state is

$$\omega_k = \frac{M_k \rho M_k^\dagger}{p(k)}$$

The postulates of QT with density operators

Postulate 4: The state of a composite quantum system is described by a density operator $\rho \in \mathscr{B}\left(\bigotimes_{i=1}^{N} \mathbb{H}_i\right)$. If the state of each constituent system is given by ρ_i then the state of the composite system is

$$\rho = \bigotimes_{i=1}^{N} \rho_i$$

Let's make sure it all adds up with what we know so far

The space of bounded linear operators

Definition 26: Let $A : \mathbb{H} \to \mathbb{H}$, and $B : \mathbb{H} \to \mathbb{H}$ be two linear operators. Their inner product, called the Hilbert-Schmidt or Frobenius norm, is given by $\langle A, B \rangle = \mathrm{tr}(A^\dagger B) \in \mathbb{C}$.

The space of bounded linear operators is a linear vector space under the Hilbert-Schmidt inner product. Indeed notice that for $\rho = |\psi\rangle\langle\psi|$

$$\mathrm{tr}(\rho^\dagger \rho) = \mathrm{tr}(|\psi\rangle\langle\psi|\psi\rangle\langle\psi|) = |\langle\psi|\psi\rangle|^2 = ||\,|\psi\rangle\,||$$

Moreover as the elements of $\mathscr{B}(\mathbb{H})$ are $|\mathbb{H}| \times |\mathbb{H}|$ one choice of orthonormal basis (the standard basis) is $\{E_{ij} = |i\rangle\langle j|\}$ such that for any $A \in \mathscr{B}(\mathbb{H})$

$$A = \sum_{i,j} \mathrm{tr}(|j\rangle\langle i|A)\,|i\rangle\langle j|$$

$$= \sum_{i,j} \langle i|A|j\rangle\,|i\rangle\langle j|$$

The postulates of QT with density operators

Given an ensemble $\{p_i, |\psi_i\rangle\}$ we know that unitary evolution corresponds to another ensemble given by $\{p_i, U|\psi_i\rangle\}$. The density operator of this ensemble is

$$\begin{aligned}\omega &= \sum_i p_i\, U|\psi_i\rangle\langle\psi_i|U^\dagger \\ &= U\left(\sum_i p_i\, |\psi_i\rangle\langle\psi_i|\right)U^\dagger \\ &= U\rho U^\dagger\end{aligned}$$

The postulates of QT with density operators

What about the measurement postulate! Recall that given an ensemble $\{p_i, |\psi_i\rangle\}$ the ensemble after measuring with operators $\{M_k\}$ is

$$\left\{ p(i), \left\{ q(k\,|\,i) = \langle\psi_i|M_k^\dagger M_k|\psi_i\rangle,\ |\phi_i^{(k)}\rangle = \sqrt{\frac{1}{q(k\,|\,i)}} M_k\,|\psi_i\rangle \right\} \right\}$$

Lets write down what the post measurement state is!

$$\begin{aligned}
\tilde{\omega}_k &= \sum_i p(i) q(k\,|\,i)\, |\phi_i^{(k)}\rangle\langle\phi_i^{(k)}| \\
&= \sum_i p(i) q(k\,|\,i) \frac{M_k|\psi_i\rangle\langle\psi_i|M_k^\dagger}{q(k\,|\,i)} \\
&= \sum_i p_i\, M_k|\psi_i\rangle\langle\psi_i|M_k^\dagger \\
&= M_k \left(\sum_i p_i|\psi_i\rangle\langle\psi_i| \right) M_k^\dagger \\
&= M_k \rho M_K^\dagger
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\mathrm{tr}(\tilde{\omega}_k) &= \mathrm{tr}\left(M_k \rho M_k^\dagger \right) \\
&= \mathrm{tr}\left(M_k^\dagger M_k\, \rho \right) \\
&= p(k)
\end{aligned}$$

Hence

$$\omega_k = \frac{M_k \rho M_k^\dagger}{p(k)}$$

Exercise

Write down the density matrix of the following ensembles

$$\left\{ \frac{1}{2}, |0\rangle;\ \frac{1}{2}, |1\rangle \right\} \qquad \left\{ \frac{1}{2}, \frac{|0\rangle + |1\rangle}{\sqrt{2}};\ \frac{1}{2}, \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right\}$$

Exercise

Write down the density matrix of the following ensembles

$$\left\{\frac{1}{2}, |0\rangle; \ \frac{1}{2}, |1\rangle\right\} \qquad \left\{\frac{1}{2}, \frac{|0\rangle + |1\rangle}{\sqrt{2}}; \ \frac{1}{2}, \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right\}$$

Theorem 6: [Unitary freedom of ensemble decompositions]: Two different ensembles $\{p_i, |\psi_i\rangle\}_{i=1}^{n}$ and $\{q_i, |\phi_i\rangle\}_{i=1}^{m}$, $n \neq m$ give rise to the same density operator if and only if they are unitarily related to each other, i.e.,

$$|\tilde{\phi}_j\rangle = \sum_i U_{ij} |\tilde{\psi}_i\rangle$$

where $|\tilde{\phi}_j\rangle = \sqrt{q_j}|\phi_j\rangle$ and $|\tilde{\psi}_j\rangle = \sqrt{p_j}|\psi_j\rangle$.

Reduced states of composite systems

Consider the bipartite state

$$|\Psi\rangle_{AB} = \sqrt{\frac{1}{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

Suppose B measures in the standard basis. What is the probability that B obtains outcome 0 or 1?

$${}_{AB}\langle\Psi|\left(\mathbb{I}_A \otimes |0\rangle\langle 0|\right)|\Psi\rangle_{AB} = \frac{1}{2} \quad \Rightarrow \quad |\Phi\rangle_{AB} = |0\rangle_A|0\rangle_B$$

$${}_{AB}\langle\Psi|\left(\mathbb{I}_A \otimes |1\rangle\langle 1|\right)|\Psi\rangle_{AB} = \frac{1}{2} \quad \Rightarrow \quad |\Phi\rangle_{AB} = |1\rangle_A|1\rangle_B$$

Now suppose that B doesn't tell A the outcome of the measurement. All A can say is that her system is equally likely to be in either state!

Reduced states of composite systems

Definition 26: Let $A : \bigotimes_{i=1}^{N} \mathbb{H}_i \to \bigotimes_{i=1}^{N} \mathbb{H}_i$. The operator $\tilde{A}$ acting on all state spaces except $\mathbb{H}_k$ is given by

$$\tilde{A} = \mathrm{tr}_k(A)$$

$$\mathrm{tr}_k\left(\sum_{i_1,\ldots i_N}\sum_{j_1,\ldots j_N} A_{i_1,\ldots,i_N;j_1,\ldots,j_N} |i_1\rangle\langle j_1| \otimes \ldots \otimes |i_k\rangle\langle j_k| \otimes \ldots |i_N\rangle\langle j_N|\right)$$

$$\mathbb{1} \otimes \ldots \mathbb{1} \otimes \langle l_k| \otimes \mathbb{1} \ldots \mathbb{1} \left(\sum_{i_1,\ldots i_N}\sum_{j_1,\ldots j_N} A_{i_1,\ldots,i_N;j_1,\ldots,j_N} |i_1\rangle\langle j_1| \otimes \ldots \otimes |i_k\rangle\langle j_k| \otimes \ldots |i_N\rangle\langle j_N|\right) \mathbb{1} \otimes \ldots \mathbb{1} \otimes |l_k\rangle \otimes \mathbb{1} \ldots \mathbb{1}$$

$$= \sum_{i_1,\ldots i_N}\sum_{j_1,\ldots j_N} A_{i_1,\ldots,i_N;j_1,\ldots,j_N} |i_1\rangle\langle j_1| \otimes \ldots \otimes \langle l_k|i_k\rangle\langle j_k|l_k\rangle \otimes \ldots \otimes |i_N\rangle\langle j_N|$$

$$= \sum_{\iota\neg\mathbf{k}}\sum_{\mathbf{j}\neg\mathbf{k}}\sum_{l_k} A_{i_1,\ldots,l_k,\ldots,i_N;j_1,\ldots,l_k,\ldots,j_N} |i_1\rangle\langle j_1| \otimes \ldots \otimes |i_{k-1}\rangle\langle j_{k-1}| \otimes |i_{k+1}\rangle\langle j_{k+1}| \otimes \ldots \otimes |i_1\rangle\langle j_1|$$

Mnemonic: In order to take the partial trace of system k turn the kth ketbra into a braket.

Reduced states of composite systems

Consider the density operator $\rho = |\Psi\rangle_{AB}\langle\Psi| \in \mathscr{B}(\mathbb{H}_{\mathbb{A}} \otimes \mathbb{H}_{\mathbb{B}})$ where

$$|\Psi\rangle_{AB} = \sqrt{\frac{1}{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

Tracing out $\mathbb{H}_B$ gives

$$\begin{aligned}
\mathrm{tr}_B(\rho_{AB}) &= \frac{1}{2}\mathrm{tr}_B\left(|0\rangle_A\langle 0| \otimes |0\rangle_B\langle 0| + |0\rangle_A\langle 1| \otimes |0\rangle_B\langle 1| + |1\rangle_A\langle 0| \otimes |1\rangle_B\langle 0| + |1\rangle_A\langle 1| \otimes |1\rangle_B\langle 1|\right) \\
&= \frac{1}{2}\left(|0\rangle_A\langle 0|\,{}_B\langle 0|0\rangle_B + |0\rangle_A\langle 1|\,{}_B\langle 1|0\rangle_B + |1\rangle_A\langle 0|\,{}_B\langle 0|1\rangle_B + |1\rangle_A\langle 1|\,{}_B\langle 1|1\rangle_B\right) \\
&= \frac{1}{2}\left(|0\rangle_A\langle 0| + |1\rangle_A\langle 1|\right)
\end{aligned}$$

But this corresponds to the ensemble $\left\{\frac{1}{2}, |0\rangle; \frac{1}{2}, |1\rangle\right\}$ which is exactly how we reasoned Alice describes her subsystem.

Reduced states of composite systems

Consider the density operator $\rho = |\Psi\rangle_{AB}\langle\Psi| \in \mathscr{B}(\mathbb{H}_{\mathbb{A}} \otimes \mathbb{H}_{\mathbb{B}})$ where

$$|\Psi\rangle_{AB} = \sqrt{\frac{1}{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

What density operator do we obtain if we trace out $\mathbb{H}_A$ instead?

Reduced states of composite systems

Consider the density operator $\rho = |\Psi\rangle_{AB}\langle\Psi| \in \mathscr{B}(\mathbb{H}_{\mathbb{A}} \otimes \mathbb{H}_{\mathbb{B}})$ where

$$|\Psi\rangle_{AB} = \sqrt{\frac{1}{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

What density operator do we obtain if we trace out $\mathbb{H}_A$ instead?

Remarks

1. Notice that whilst ρ_{AB} is a pure state, the reduced states $\rho_A = \rho_B = \frac{1}{2}\mathbb{I}$ are mixed!

The whole is more than the sum of its parts

2. Mixed quantum states can be understood as bipartite states that are entangled with something which we do not have access to.

Purification of mixed states

Lemma 3: Let $\rho \in \mathscr{B}(\mathbb{H}_1)$ be a density matrix. There exists a reference system, with dimension $d = |H|$ and a pure quantum state $|\Psi\rangle_{12} \in \mathbb{H}_1 \otimes \mathbb{H}_2$ such that $\rho = \mathrm{tr}_2\left(|\Psi\rangle_{12}\langle\Psi|\right)$

Proof: As $\rho \geq 0$, and Hermitian, the spectral theorem says

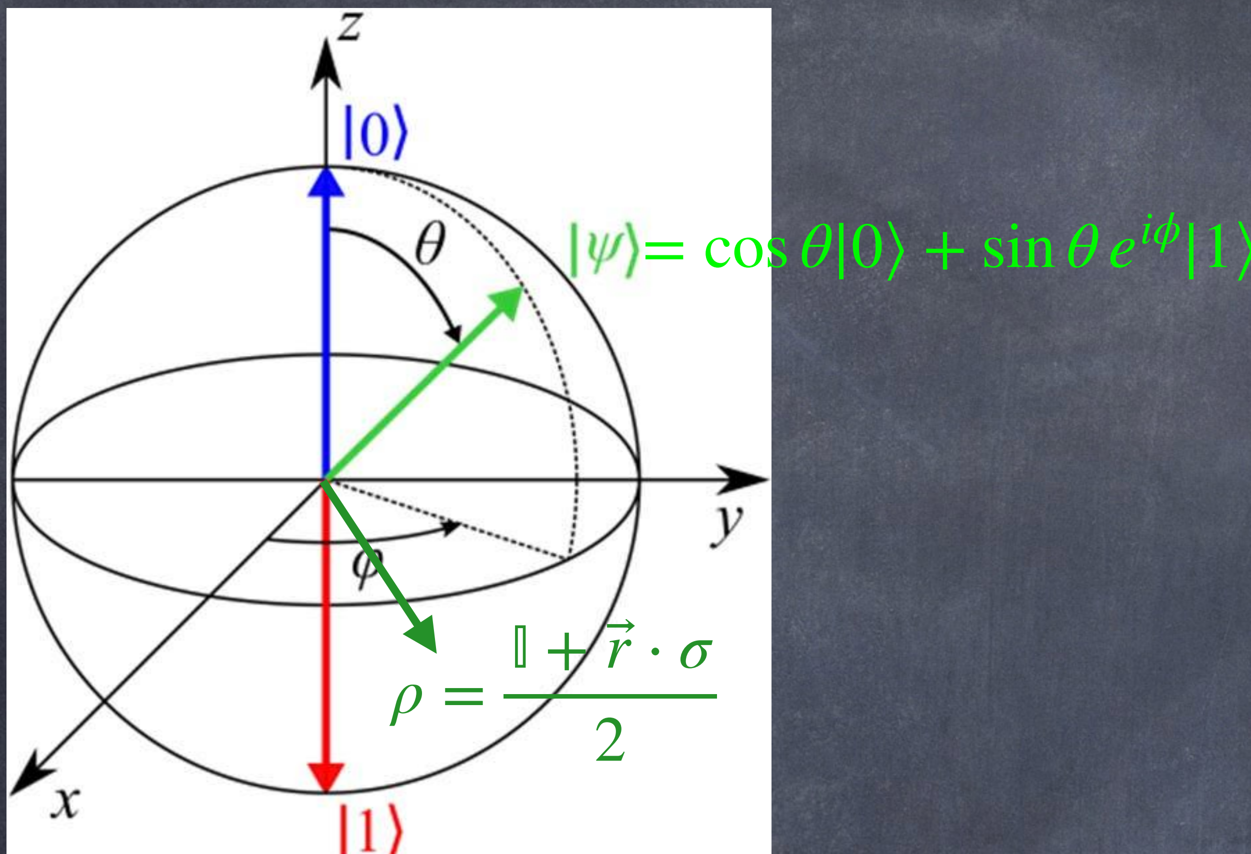
$$\rho = \sum_i p_i |i\rangle\langle i|$$

Now define the same orthonormal basis $\{|i\rangle_2\}_{i=1}^{|\mathbb{H}|} \in \mathbb{H}_2$ and define

$$|\Psi\rangle_{12} = \sum_i \sqrt{p_i}|i\rangle_1|i\rangle_2$$

Then it is easy to show that the result follows.

The Bloch Ball representation



Every point on the surface of the sphere represents a pure quantum state

Every point inside the sphere represents a mixed quantum state

The center of the sphere represents the completely mixed state $\dfrac{\mathbb{I}}{2}$

There is a mapping between quantum states on the Bloch sphere and real vectors in $\vec{r} \in \mathbb{R}_3$

The no-cloning theorem

Theorem 7: There exists no machine that can produce a perfect copies of arbitrary quantum states

Proof: Whatever this cloning machine is its action can be described by a unitary $U : \mathbb{H} \otimes \mathbb{H} \rightarrow \mathbb{H} \otimes \mathbb{H}$ such that

$$U\left(|\psi\rangle \otimes |0\rangle\right) = |\psi\rangle \otimes |\psi\rangle$$

where $|0\rangle$ is the state of an auxiliary qubit. Now since this machine ought to copy any quantum state it follows that

$$U\left(|\phi\rangle \otimes |0\rangle\right) = |\phi\rangle \otimes |\phi\rangle$$

But now notice that

$$\left(\langle 0| \otimes \langle \phi|\right) U^{\dagger} U \left(|0\rangle \otimes |\psi\rangle\right) = \langle \phi|\psi\rangle^2$$

$$\langle \phi|\psi\rangle = \langle \phi|\psi\rangle^2$$

which is true if and only if $\langle \phi|\psi\rangle = 0, \mathrm{or}\ 1$, i.e. only orthogonal (classical) states can be cloned