UNIVERSITAT DE BARCELONA
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1. Consider the properly symmetrized wave-function of two spinless bosons, $\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}}\left[\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2) + \psi_1(\mathbf{r}_2)\psi_2(\mathbf{r}_1)\right]$. Suppose that state 1 has positive parity and state 2 has negative parity.
   (a) Determine the probability of finding a particle at position $\mathbf{r}$, $P(\mathbf{r})$, if the position of the other one is arbitrary.
   (b) Find the probability of finding one particle in the upper half-space, $z \geqslant 0$.
   (c) Find the probability of finding two particles in the upper half-space, $z \geqslant 0$.
   (d) Answer these questions for two fermions in the same states.

2. Use the anticommutation relations of the creation, $a^\dagger_\alpha$, and annihilation, a_α, operators of a system of identical fermions to show that
   (a) $a^\dagger_\alpha \mid 0\rangle$ is an eigenstate of the particle number operator, $\hat{N} = \sum_\alpha a^\dagger_\alpha a_\alpha$.
   (b) $a^\dagger_\beta a^\dagger_\gamma \mid 0\rangle$ is also eigenstate of $\hat{N}$.
   (c) Determine the corresponding eigenvalues in both cases.

3. Consider a Hamiltonian of the one-body type, $\hat{H} = \sum_{\alpha\beta} t_{\alpha\beta} \hat{a}^\dagger_\alpha \hat{a}_\beta$.
   (a) Prove that $\hat{N}$ and $\hat{H}$ commute.
   (b) Consider a one-particle fermionic state, $|\eta\rangle$, and a two-particle fermionic state, $|\delta\gamma\rangle$. Find an expression for the expectation value of $\hat{H}$ in these two states.

4. Use the convention $|11111000\cdots\rangle = \hat{a}^\dagger_5\hat{a}^\dagger_4\hat{a}^\dagger_3\hat{a}^\dagger_2\hat{a}^\dagger_1\,|0\rangle$ to describe a second-quantization state.
   (a) Find the result of the operation $\hat{a}^\dagger_3\hat{a}_6\hat{a}_4\hat{a}^\dagger_6\hat{a}_3\,|11111000\cdots\rangle$.
   (b) Write $|1101100100\cdots\rangle$ in terms of excitations around the "filled Fermi sea" $|1111100000\cdots\rangle$. Interpret your answer in terms of particle and hole excitations.
   (c) Consider the state $|\psi\rangle = A\,|100\rangle + B\,|111000\rangle$. Find the expectation value of the number operator, $\langle\psi|\hat{N}|\psi\rangle$.

5. The density operator in first quantization is given by the expression $\rho(\mathbf{r}) = \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r_i})$.
   (a) Find an expression for this operator in second quantization.
   (b) The field operator is given by $\hat{\psi}(\mathbf{r}) = \sum_l \varphi_l(\mathbf{r})\hat{a}_l$, where $\varphi_l(\mathbf{r}) = \langle\mathbf{r}|l\rangle$ is the real-space single-particle wave function of state l. Show that $\hat{\rho}(\mathbf{r}) = \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})$.
   (c) Show that the particle number operator $\hat{N} = \sum_l \hat{a}^\dagger_l \hat{a}_l$ can be expressed as $\hat{N} = \int d^3\mathbf{r}\hat{\rho}(\mathbf{r})$.

6. Consider two fermion operators, $\hat{a}_1$ and $\hat{a}_2$.
   (a) Show that the Bogolyubov transformation

$$\hat{b}_1 = u\hat{a}_1 + v\hat{a}^\dagger_2\,,$$
$$\hat{b}^\dagger_2 = -v\hat{a}_1 + u\hat{a}^\dagger_2\,,$$

   preserves the canonical anti-commutation relations for u and v real and $u^2 + v^2 = 1$.

(b) Show that the hamiltonian

$$\hat{H} = \epsilon \left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2 \hat{a}_2^\dagger \right) + \Delta \left(\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_2 \hat{a}_1 \right)$$

can be written as a matrix product

$$\hat{H} = \begin{pmatrix} \hat{a}_1^\dagger \hat{a}_2 \end{pmatrix} \begin{pmatrix} \epsilon & \Delta \\ \Delta & -\epsilon \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} .$$

(c) Use the Bogolyubov transformations above to write $\hat{H}$ in a diagonal form,

$$\hat{H} = \bar{\epsilon} \left(\hat{b}_1^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2 - 1 \right) .$$

What is the ground-state energy of this Hamiltonian?

(d) Prove that $\bar{\epsilon} = \sqrt{\epsilon^2 + \Delta^2}$.