Quantum Information Theory —Assignment 2

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Question 1

In this question we're presented with sets of matrices $\{M_m\}$ and asked if each set is a legal generalized measurement, if it represents a projective measurement, or if it represents a POVM.

We recall the following definitions:

Measurement	Matrices	Definition
Generalized measurement	$\{M_m\}$	$\sum_m M_m^\dagger M_m = \mathbb{1}$
Projective measurement	$\{P_m\}$	$\sum_m P_m = \mathbb{1}, \quad P_m P_j = \delta_{ij} P_j$
POVM	$\{E_m\}$	$\sum_m E_m = \mathbb{1}, \quad E_m \geq 0$

Part (i)

First, we're given this set of matrices:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By inspection, these matrices sum to the identity, but they are not projection matrices, since they manifestly do not equal their own squares. However, each of the matrices is positive semidefinite, as may be quickly verified using software.[1] Therefore, this set of matrices forms a POVM.

On the other hand, $\sum_m M_m^\dagger M_m \neq \mathbb{1}$, and so they are not a generalized measurement.[2]

Part (ii)

Next, we're presented with:

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Again, the matrices sum to the identity. Again, they are not projection matrices. But this time, the first two matrices are not positive semidefinite, and so this set of matrices is neither a projective measurement, nor a POVM.

Worse, these matrices do not even form a legal generalized measurement, as is easy to verify using software.

[1] That is, they have non-negative eigenvalues.

[2] On the other hand, since every positive semidefinite matrix can be factorized as $E = M^\dagger M$, we may easily construct a set of measurement operators $\{M_m\}$ from these $\{E_m\}$.

Part (iii)

This time we have four matrices:

$$\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix},\ \frac{1}{4}\begin{pmatrix}1&-1&1&-1\\-1&1&-1&1\\1&-1&1&-1\\-1&1&-1&1\end{pmatrix},\ \frac{1}{4}\begin{pmatrix}1&1&-1&-1\\-1&-1&1&1\\1&1&-1&-1\\-1&-1&1&1\end{pmatrix},\ \frac{1}{4}\begin{pmatrix}1&-1&-1&1\\-1&1&1&-1\\1&-1&-1&1\\1&-1&-1&1\end{pmatrix}$$

These matrices do not sum to the identity, a fact which may be established by staring at them. Thus, they are neither a projective measurement, nor a POVM.

But $\sum_m M_m^\dagger M_m = \mathbb{1}$, and so they are a generalized measurement.

Part (iv)

Finally, we're asked about these four suspicious-looking characters:

$$\begin{pmatrix}\frac{1}{2}&0\\0&0\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}\frac{1}{3}&\frac{\sqrt{2}}{3}e^{-\mathrm{i}\frac{2\pi}{3}}\\\frac{\sqrt{2}}{3}e^{\mathrm{i}\frac{2\pi}{3}}&\frac{2}{3}\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}\frac{1}{3}&\frac{\sqrt{2}}{3}e^{-\mathrm{i}\frac{4\pi}{3}}\\\frac{\sqrt{2}}{3}e^{\mathrm{i}\frac{4\pi}{3}}&\frac{2}{3}\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}\frac{1}{3}&\frac{\sqrt{2}}{3}\\\frac{\sqrt{2}}{3}&\frac{2}{3}\end{pmatrix}$$

These matrices do sum to the identity, though one must stare pretty hard to see it. The first matrix is quite manifestly not a projection matrix, so they are not a projective measurement. But they are all positive semidefinite, and therefore constitute a POVM.

On the other hand, $\sum_m M_m^\dagger M_m = \frac{1}{2}\mathbb{1}$, and so they are not a generalized measurement.[3]

Question 2

Here we consider the following Choi operator for a map Λ:

$$J(\Lambda) = \frac{1}{2}\begin{pmatrix}1&0&0&\sqrt{1-\gamma}\\0&0&0&0\\0&0&\gamma&0\\\sqrt{1-\gamma}&0&0&1-\gamma\end{pmatrix}$$

Part (i)

We diagonalize this matrix, and obtain the eigenvalues $\lambda_1 = \gamma/2$, $\lambda_2 = 1-\gamma/2$, $\lambda_{3,4} = 0$. The non-zero eigenvalues have eigenvectors:

$$|\lambda_1\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |\lambda_2\rangle = \frac{1}{\sqrt{2-\lambda}}\begin{pmatrix}1\\0\\0\\\sqrt{1-\lambda}\end{pmatrix}$$

The Kraus operators are obtained by rearranging the elements of the vectors $\sqrt{\lambda_1}\,|\lambda_i\rangle$ into square matrices.

They are:

$$K_1 = \begin{pmatrix}0&\sqrt{\gamma}\\0&0\end{pmatrix} = \sqrt{\gamma}\,|0\rangle\langle 1| \qquad K_2 = \begin{pmatrix}1&0\\0&\sqrt{1-\gamma}\end{pmatrix} = |0\rangle\langle 0| + \sqrt{1-\gamma}\,|1\rangle\langle 1|$$

These matrices are both manifestly positive semi-definite, and it's easy to verify by direct evaluation that $K_1^\dagger K_1 + K_2^\dagger K_2 = \mathbb{1}$. Finally, we recognize that these are the Kraus operators of the *amplitude damping channel*.[4]

[3] We may obtain a generalized measurement just be scaling these matrices, that is, by taking the set $\{\sqrt{2}M_m\}$.

[4] See Wilde, 2017, section 4.7.

Part (ii)

The matrix representation may be found according to the following formula:

$$M_{ij}^{k\ell} = \mathrm{Tr}\{|\ell\rangle\langle k|\,\Lambda(|i\rangle\langle j|)\} = \langle k|\,\Lambda(|i\rangle\langle j|)\,|\ell\rangle$$

where the matrices $\Lambda(|i\rangle\langle j|)$ are the four 2×2 blocks of $J(\Lambda)$.
So the matrix representation of Λ may be written:

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ \gamma & 0 & 0 & 1-\gamma \end{pmatrix}$$

Question 3

Part (i)

We consider the action of a unitary operator of form $U\otimes U$ on the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and we begin by writing a general unitary matrix U as:

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^\star & \alpha^\star \end{pmatrix} \qquad\qquad \det U = |\alpha|^2 + |\beta|^2 = e^{i\theta}$$

where $e^{i\theta}$ is just a phase, with unit absolute value. The operator $U\otimes U$ acts on the singlet state according to:

$$\begin{aligned}\sqrt{2}\,(U\otimes U)\,|\Psi^-\rangle &= (\alpha\,|0\rangle - \beta^\star\,|1\rangle)\otimes(\beta\,|0\rangle + \alpha^\star\,|1\rangle) - (\beta\,|0\rangle + \alpha^\star\,|1\rangle)\otimes(\alpha\,|0\rangle - \beta^\star\,|1\rangle) \\ &= \alpha\beta\,|00\rangle + \alpha\alpha^\star\,|01\rangle - \beta^\star\beta\,|10\rangle - \beta^\star\alpha^\star\,|11\rangle - \alpha\beta\,|00\rangle + \beta\beta^\star\,|01\rangle - \alpha\alpha^\star\,|10\rangle + \beta^\star\alpha^\star\,|11\rangle \\ &= \big(|\alpha|^2 + |\beta|^2\big)(|01\rangle - |10\rangle) \\ &= \det U(|01\rangle - |10\rangle) \\ &= e^{i\theta}\sqrt{2}\,|\Psi^-\rangle\end{aligned}$$

And so $|\Psi^-\rangle$ is invariant under transformations of this form, at least up to an irrelevant global phase.

Part (ii)

We now consider the action of $U\otimes U\otimes U$ on the tripartite GHZ and W states.

$$\begin{aligned}\sqrt{2}\,(U\otimes U\otimes U)\,|\mathrm{GHZ}\rangle &= \sqrt{2}\,(U\otimes U\otimes U)\tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\ &= (\alpha\,|0\rangle - \beta^\star\,|1\rangle)\otimes(\alpha\,|0\rangle - \beta^\star\,|1\rangle)\otimes(\alpha\,|0\rangle - \beta^\star\,|1\rangle) \\ &\quad + (\beta\,|0\rangle + \alpha^\star\,|1\rangle)\otimes(\beta\,|0\rangle + \alpha^\star\,|1\rangle)\otimes(\beta\,|0\rangle + \alpha^\star\,|1\rangle) \\ &= \big(\alpha^3 + \beta^3\big)\,|000\rangle + \big(-(\beta^\star)^3 + (\alpha^\star)^3\big)\,|111\rangle \\ &\quad + \big(-\alpha^2\beta^\star + \beta^2\alpha^\star\big)(|001\rangle + |010\rangle + |100\rangle) \\ &\quad + \big(\alpha(\beta^\star)^2 + \beta(\alpha^\star)^2\big)(|011\rangle + |101\rangle + |110\rangle)\end{aligned}$$

and:

$$\begin{aligned}\sqrt{3}\,(U\otimes U\otimes U)\,|\mathrm{W}\rangle &= \sqrt{3}\,(U\otimes U\otimes U)\tfrac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \\ &= (\alpha\,|0\rangle - \beta^\star\,|1\rangle)\otimes(\alpha\,|0\rangle - \beta^\star\,|1\rangle)\otimes(\beta\,|0\rangle + \alpha^\star\,|1\rangle) \\ &\quad + (\alpha\,|0\rangle - \beta^\star\,|1\rangle)\otimes(\beta\,|0\rangle + \alpha^\star\,|1\rangle)\otimes(\alpha\,|0\rangle - \beta^\star\,|1\rangle) \\ &\quad + (\beta\,|0\rangle + \alpha^\star\,|1\rangle)\otimes(\alpha\,|0\rangle - \beta^\star\,|1\rangle)\otimes(\alpha\,|0\rangle - \beta^\star\,|1\rangle) \\ &= 3\alpha^2\beta\,|000\rangle - 3(\beta^\star)^2\alpha^\star\,|111\rangle \\ &\quad + \big(\alpha^2\alpha^\star - 2\alpha\beta^\star\beta\big)(|001\rangle + |010\rangle + |100\rangle) \\ &\quad + \big((\beta^\star)^2\beta - 2\alpha\beta^\star\alpha^\star\big)(|011\rangle + |101\rangle + |110\rangle)\end{aligned}$$

Neither of these states are invariant under this transformation.

Part (iii)

We now ask whether a transformation of form $U_A \otimes U_B \otimes U_C$, with U_A, U_B, U_C all unitary, could transform the GHZ state into a W state. We begin by noting that such a transformation is invertible and so the operation $U_A^\dagger \otimes U_B^\dagger \otimes U_C^\dagger$—also composed of local unitaries—would transform a W state into a GHZ state.

To see if this is possible, we must consider these states as bipartite systems on $\mathcal{H}_A \otimes \mathcal{H}_{CB}$:

$$\rho_{ABC}^{\text{GHZ}} = \tfrac{1}{2}\Big(|000\rangle + |111\rangle\Big)\Big(\langle 000| + \langle 111|\Big)$$
$$\rho_{ABC}^{\text{W}} = \tfrac{1}{3}\Big(|100\rangle + |010\rangle + |001\rangle\Big)\Big(\langle 100| + \langle 010| + \langle 001|\Big)$$

The reduced density matrices may be computed by taking the partial traces:

$$\rho_A^{\text{GHZ}} = \mathrm{Tr}_{BC}\, \rho_{ABC}^{\text{GHZ}} = \tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \qquad\qquad \rho_A^{\text{W}} = \mathrm{Tr}_{BC}\, \rho_{ABC}^{\text{W}} = \tfrac{1}{2}(|1\rangle\langle 1| + 2\,|0\rangle\langle 0|)$$

We may now compute the entanglement entropy of each state:

$$H(A)_{\text{GHZ}} = -2\,\tfrac{1}{2}\log\tfrac{1}{2} = \log 2$$
$$H(A)_{\text{W}} = -\tfrac{1}{2}\log\tfrac{1}{3} - \tfrac{2}{3}\log\tfrac{2}{3} = \log 3 - \tfrac{2}{3}\log 2$$

That is, $H(A)_{\text{W}} < H(A)_{\text{GHZ}}$.

If a transformation of form $U_A \otimes U_B \otimes U_C$ taking GHZ to W existed, then we could reverse it, and increase entanglement using local unitary operations, which is impossible. We conclude that no such operation exists.

Question 4

Here we consider the family of isotropic states:

$$\rho_{\text{iso}}(\alpha) \equiv \alpha\,|\Phi_{00}\rangle\langle\Phi_{00}| + (1-\alpha)\frac{\mathbb{1}}{d^2} \qquad\qquad |\Phi_{00}\rangle \equiv \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$$

which for any real α is a linear combination of a maximally entangled state and a maximally mixed state.

Part (i) and part (ii)

For a state $\rho_{\text{iso}}(\alpha)$ to be a well-defined quantum state, it must have unit trace, and be positive semidefinite.

Since both $|\Phi_{00}\rangle\langle\Phi_{00}|$ and $\frac{\mathbb{1}}{d^2}$ have unit trace, the first condition is satisfied for every value of α.

It's possible to find the bounds on α just by evaluating the sandwich of ρ_{iso} with an arbitrary vector, but since we've also been asked to find the eigenvalues, it's quicker to just do that first.

To find the eigenvalues of ρ_{iso} we must solve the eigenvalue equation:

$$\rho_{\text{iso}}(\alpha)\,|\lambda\rangle = \left[\alpha\,|\Phi_{00}\rangle\langle\Phi_{00}| + (1-\alpha)\frac{\mathbb{1}}{d^2}\right]|\lambda\rangle = \lambda\,|\lambda\rangle$$

which may be rearranged to:

$$|\Phi_{00}\rangle\langle\Phi_{00}|\,|\lambda\rangle = \frac{1}{\alpha}\left(\lambda - \frac{1-\alpha}{d^2}\right)|\lambda\rangle$$

and this is just the eigenvalue equation of the projection matrix $|\Phi_{00}\rangle\langle\Phi_{00}|$.

Therefore, we have eigenvalues λ_0, λ_1 satisfying:

$$\frac{1}{\alpha}\left(\lambda_0 - \frac{1-\alpha}{d^2}\right) = 0 \qquad\qquad \frac{1}{\alpha}\left(\lambda_1 - \frac{1-\alpha}{d^2}\right) = 1$$

Solving, we find:

$$\lambda_0 = \frac{1-\alpha}{d^2} \qquad\qquad \lambda_1 = \alpha + \frac{1-\alpha}{d^2}$$

A matrix is positive definite when its eigenvalues are non-negative. Thus, from the expressions for the eigenvalues, we obtain the following bounds on α:

$$\frac{-1}{d^2-1} \leq \alpha \leq 1$$

Part (iii)

Here we wish to find the general form of density matrices for bipartite systems which are invariant under transformations of type $U \otimes U^\star$. We consider a basis where U is diagonal, that is:

$$U = \sum_k \lambda_k |k\rangle\langle k| \qquad\qquad |\lambda_k| = 1$$

We begin by writing such an arbitrary density matrix ρ in this basis:

$$\rho = \sum_{ijk\ell} \rho_{ijk\ell} |ij\rangle\langle k\ell| \tag{1}$$

We now operate on ρ with the transformation:

$$(U \otimes U^\star)\rho(U \otimes U^\star)^\dagger = \sum_{ijk\ell} \lambda_i \lambda_j^\star \lambda_k^\star \lambda_\ell \, \rho_{ijk\ell} |ij\rangle\langle k\ell|$$

Now, if $\rho = (U \otimes U^\star)\rho(U \otimes U^\star)^\dagger$, the matrix elements in this basis must match:

$$\rho_{ijk\ell} = \lambda_i \lambda_j^\star \lambda_k^\star \lambda_\ell \, \rho_{ijk\ell}$$

These condition must be satisfied for arbitrary U—that is, for any set of eigenvalues λ_i—which can only happen if either:

- $i = j$ and $k = \ell$,
- $i = k$ and $j = \ell$, or
- $\rho_{ijk\ell} = 0$.

Thus, we can split the sum (1) into three terms, one with $i = j, k = \ell$, one with $i = k, j = \ell$, and a third with terms for which $i = j = k = \ell$:

$$\rho = \sum_{i \neq k} \rho_{iikk} |ii\rangle\langle kk| + \sum_{i \neq j} \rho_{ijij} |ij\rangle\langle ij| + \sum_i \rho_{iiii} |ii\rangle\langle ii|$$

Now, the matrix elements of ρ must be invariant under any unitary transformation, and in particular, under any permutation of the basis vectors. Thus, the elements ρ_{iikk} cannot depend on i, k, the elements ρ_{ijij} cannot depend on i, j, and the elements ρ_{iiii} cannot depend on i.

If you're inclined to doubt this argument, take $U \otimes U = X \otimes X$ and act on matrices of the form we're considering:

$$(X \otimes X)\rho(X \otimes X)^\dagger = (X \otimes X)\begin{pmatrix} C_1 & 0 & 0 & A1 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ A_4 & 0 & 0 & C4 \end{pmatrix}(X \otimes X)^\dagger = \begin{pmatrix} C_4 & 0 & 0 & A_4 \\ 0 & B_3 & 0 & 0 \\ 0 & 0 & B_2 & 0 \\ A_1 & 0 & 0 & C_1 \end{pmatrix}$$

We see that As, Bs, and Cs cannot depend on their indices. Dear reader, your skepticism was unfounded.

Therefore, ρ must take the form:

$$\rho = A \sum_{i \neq k} |ii\rangle\langle kk| + B \sum_{i \neq j} |ij\rangle\langle ij| + C \sum_i |ii\rangle\langle ii| = \begin{pmatrix} C & 0 & 0 & A \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ A & 0 & 0 & C \end{pmatrix}$$

We now consider the action of the Hadamard $H \otimes H$ on this state:

$$(H \otimes H)\rho(H \otimes H)^\dagger = \frac{1}{2}\begin{pmatrix} A+B+C & 0 & 0 & A-B+C \\ 0 & -A+B+C & -A-B+C & 0 \\ 0 & -A-B+C & -A+B+C & 0 \\ A-B+C & 0 & 0 & A+B+C \end{pmatrix}$$

and we see that we must have $C = A + B$. Finally:

$$\rho = A \sum_{ik} |ii\rangle\langle kk| + B \sum_{ij} |ij\rangle\langle ij| = A\, d\, |\Phi_{00}\rangle\langle\Phi_{00}| + B\, \mathbb{1}$$

and, since ρ must have unit trace, we see that it must be of form $\rho = \rho_{\text{iso}}(\alpha)$ for some α.

Finally, the identity matrix is invariant under any unitary transformation, and it's trivial to verify that $|\Phi_{00}\rangle$ is invariant under transformations of form $U \otimes U^\star$. So every isotropic state is invariant under these transformations.

Thus, a density matrix ρ is invariant under such transformations if and only if it is an isotropic state.

Part (iv)

According to the entanglement reduction criterion, a bipartite state ρ_{AB} is entangled if $\rho_A \otimes \mathbb{1}_B \not\geq \rho_{AB}$.

We begin by calculating the reduced density matrix of our state ρ_{iso}.

$$\text{Tr}_B\, \rho_{\text{iso}} = \text{Tr}_B \left[\alpha \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle\langle jj| + (1-\alpha)\frac{\mathbb{1}}{d^2} \right] = \alpha \frac{1}{d} \mathbb{1}_A + \frac{1-\alpha}{d^2}\, d\mathbb{1}_A = \frac{\mathbb{1}_A}{d}$$

The eigenvalue equation for the operator $\rho_A \otimes \mathbb{1}_B - \rho_{AB}$ becomes:

$$\left[\frac{1}{d}\mathbb{1}_A \otimes \mathbb{1}_B - \alpha\, |\Phi_{00}\rangle\langle\Phi_{00}| - \frac{1-\alpha}{d^2}\mathbb{1}_A \otimes \mathbb{1}_B \right] |\lambda\rangle = \lambda\, |\lambda\rangle$$

which, following the same procedure as before, we may rearrange to obtain:

$$|\Phi_{00}\rangle\langle\Phi_{00}|\, |\lambda\rangle = \frac{-1}{\alpha}\left(\lambda - \frac{1}{d} + \frac{1-\alpha}{d^2} \right) |\lambda\rangle$$

so our eigenvalues are found by solving:

$$\frac{-1}{\alpha}\left(\lambda_0 - \frac{1}{d} + \frac{1-\alpha}{d^2} \right) = 0 \qquad\qquad \frac{-1}{\alpha}\left(\lambda_1 - \frac{1}{d} + \frac{1-\alpha}{d^2} \right) = 1$$

which produces:

$$\lambda_0 = \frac{1}{d} - \frac{1-\alpha}{d^2} \qquad\qquad \lambda_1 = \frac{1}{d} - \frac{1-\alpha}{d^2} - \alpha$$

Now, ρ_{iso} is entangled either of these eigenvalues is negative. But $\lambda_0 < 0$ when $\alpha < 1 - d$, which never occurs for $d \geq 2$, and $\lambda_1 < 0$ when $\alpha > 1/(1+d)$. So we have entanglement when:

$$\alpha > \frac{1}{1+d}$$

Part (v)

According to the PPT criterion, bipartite state ρ_{AB} is entangled if $\rho_{AB}^{T_A} \not\geq 0$.

We must calculate the partial transpose of our density matrix:

$$\rho_{\text{iso}}^{T_A} = \left[\alpha \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle\langle jj| + (1-\alpha)\frac{\mathbb{1}}{d^2} \right]^{T_A} = \alpha \frac{1}{d} \sum_{i,j=0}^{d-1} |ji\rangle\langle ij| + (1-\alpha)\frac{\mathbb{1}}{d^2}$$

we again rearrange the eigenvalue equation to obtain:

$$\sum_{i,j=0}^{d-1} |ji\rangle\langle ij| \, |\lambda\rangle = \frac{d}{\alpha}\left(\lambda - \frac{1-\alpha}{d^2}\right)|\lambda\rangle$$

We now notice that the matrix on the left hand side squares to the identity, and so its eigenvalues are ± 1, and the eigenvalues we're looking for are:

$$\lambda_{\pm 1} = \frac{1-\alpha}{d^2} \pm \frac{\alpha}{d}$$

Our state ρ_{iso} is entangled either of these eigenvalues is negative. But $\lambda_{+1} < 0$ when $\alpha < \frac{1}{1-d}$, which never occurs for $d \geq 2$, and $\lambda_{-1} < 0$ when $\alpha > 1/(1+d)$, exactly as we found in the previous part.

Part (vi)

Entanglement witness for isotropic states

We now narrow our attention to $d = 2$, where ρ_{iso} is entangled for $\frac{1}{3} < \alpha \leq 1$, and we are asked whether the operator $W \equiv |\Phi_{00}\rangle\langle\Phi_{00}|^{T_A}$ is an entanglement witness for ρ_{iso}. We must evaluate:

$$\begin{aligned}
\mathrm{Tr}[W\rho_{\text{iso}}] &= \mathrm{Tr}\left[|\Phi_{00}\rangle\langle\Phi_{00}|^{T_A}\left(\alpha\,|\Phi_{00}\rangle\langle\Phi_{00}| + (1-\alpha)\frac{\mathbb{1}}{d^2}\right)\right] \\
&= \alpha\,\mathrm{Tr}\left[|\Phi_{00}\rangle\langle\Phi_{00}|^{T_A}\,|\Phi_{00}\rangle\langle\Phi_{00}|\right] + \frac{1-\alpha}{d^2}\,\mathrm{Tr}\left[|\Phi_{00}\rangle\langle\Phi_{00}|^{T_A}\right] \\
&= \alpha\cdot\frac{1}{2} + \frac{1-\alpha}{4}\cdot 1 = \frac{\alpha+1}{4}
\end{aligned}$$

This quantity is negative only when $\alpha < -1$, which is disallowed, so this is not a good entanglement witness.

On the other hand, the operator $W \equiv \frac{\mathbb{1}}{2} - |\Phi_{00}\rangle\langle\Phi_{00}|$ is a good entanglement witness for this family of states, since:

$$\begin{aligned}
\mathrm{Tr}[W\rho_{\text{iso}}] &= \mathrm{Tr}\left[\left(\frac{\mathbb{1}}{2} - |\Phi_{00}\rangle\langle\Phi_{00}|\right)\left(\alpha\,|\Phi_{00}\rangle\langle\Phi_{00}| + (1-\alpha)\frac{\mathbb{1}}{d^2}\right)\right] \\
&= \mathrm{Tr}\left[\frac{1}{2}\alpha\,|\Phi_{00}\rangle\langle\Phi_{00}| - |\Phi_{00}\rangle\langle\Phi_{00}|\,\alpha\,|\Phi_{00}\rangle\langle\Phi_{00}| + \frac{1}{2}\frac{1-\alpha}{4}\mathbb{1} - |\Phi_{00}\rangle\langle\Phi_{00}|\,\frac{1-\alpha}{4}\right] \\
&= \frac{\alpha}{2} - \alpha + \frac{1}{2}\frac{1-\alpha}{4}\cdot 4 - \frac{1-\alpha}{4} = \frac{1-3\alpha}{4}
\end{aligned}$$

This quantity is negative when $\alpha > \frac{1}{3}$, as required.

Furthermore, for any separable state $|\psi\rangle$:

$$\begin{aligned}
\mathrm{Tr}[W\,|\psi\rangle\langle\psi|] &= \mathrm{Tr}\left[\left(\frac{\mathbb{1}}{2} - |\Phi_{00}\rangle\langle\Phi_{00}|\right)|\psi\rangle\langle\psi|\right] \\
&= \tfrac{1}{2}\,\mathrm{Tr}[|\psi\rangle\langle\psi|] - \mathrm{Tr}[|\Phi_{00}\rangle\langle\Phi_{00}|\,|\psi\rangle\langle\psi|] \\
&= \tfrac{1}{2} - |\langle\Phi_{00}\psi|\Phi_{00}\psi\rangle|^2 \geq 0
\end{aligned} \tag{2}$$

since the maximum overlap between a separable state and the maximally entangled state is $\frac{1}{\sqrt{2}}$.

Entanglement witness for Werner states

Next, we're asked for an entanglement witness for the family of states:

$$\rho_w(\alpha) \equiv \alpha\,|\psi\rangle\langle\psi| + (1-\alpha)\frac{\mathbb{1}}{d^2} \qquad\qquad |\psi\rangle = a\,|01\rangle + b\,|10\rangle$$

Since $\frac{\mathbb{1}}{2} - |\Phi_{00}\rangle\langle\Phi_{00}|$ was a good entanglement witness for the states $\rho_{\text{iso}}(\alpha)$, we're motivated to try the operator:

$$W \equiv \frac{\mathbb{1}}{2} - |\Psi^+\rangle\langle\Psi^+|$$

for the states $\rho_w(\alpha)$. Since this operator has almost the same structure as the previous operator, the proof that $\mathrm{Tr}[W\,|\psi\rangle\langle\psi|] > 0$ for all separable states $|\psi\rangle$ is identical to the one we gave above in (2).

It remains to show that $\mathrm{Tr}[W\rho_w(\alpha)] < 0$ for entangled states from the family $\rho_w(\alpha)$. We start by writing $\rho_w(\alpha)$ as a matrix:

$$\rho_w(\alpha) = \frac{1}{4}\begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 1-\alpha+4|a|^2\alpha & 4ab^\star\alpha & 0 \\ 0 & 4a^\star b\alpha & 1-\alpha+4|b|^2\alpha & 0 \\ 0 & 0 & 0 & 1-\alpha \end{pmatrix} \qquad \text{with} \qquad |a|^2+|b|^2=1$$

The partial transpose $\rho_w^{T_B}(\alpha)$ is:

$$\rho_w^{T_B}(\alpha) = \frac{1}{4}\begin{pmatrix} 1-\alpha & 0 & 0 & 4ab^\star\alpha \\ 0 & 1-\alpha+4|a|^2\alpha & 0 & 0 \\ 0 & 0 & 1-\alpha+4|b|^2\alpha & 0 \\ 4a^\star b\alpha & 0 & 0 & 1-\alpha \end{pmatrix}$$

Which has eigenvalues λ_i given by:

$$\lambda_1 = \tfrac{1}{4}\Big(1-\alpha+4|a|^2\alpha\Big) \qquad \lambda_2 = \tfrac{1}{4}\Big(1-\alpha+4|b|^2\alpha\Big)$$
$$\lambda_3 = \tfrac{1}{4}\Big(1-\alpha+4|ab|\alpha\Big) \qquad \lambda_4 = \tfrac{1}{4}\Big(1-\alpha-4|ab|\alpha\Big)$$

The state $\rho_w(\alpha)$ is separable when all the eigenvalues are non-negative, that is, when:

$$\alpha\Big(1-4|a|^2\Big) \le 1 \qquad \alpha\Big(1-4|b|^2\Big) \le 1$$
$$\alpha\Big(1-4|ab|\Big) \le 1 \qquad \alpha\Big(1+4|ab|\Big) \le 1$$

The first three conditions are always satisfied for $0 \le \alpha \le 1$. We conclude that the state is entangled when:

$$\frac{1}{1+4|ab|} < \alpha \le 1 \tag{3}$$

But we may also evaluate:

$$\mathrm{Tr}[W\rho_w(\alpha)] = \frac{1}{8}\begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & -4a^\star b\alpha & -1+\alpha-4|b|^2\alpha & 0 \\ 0 & -1+\alpha-4|a|^2\alpha & -4ab^\star\alpha & 0 \\ 0 & 0 & 0 & 1-\alpha \end{pmatrix}$$
$$= \tfrac{1}{4}\Big[1-\alpha(1+2a^\star b+2ab^\star)\Big]$$

We can see that this expression is negative when the following inequality holds:

$$\frac{1}{1+4\,\mathrm{Re}\,ab^\star} < \alpha \le 1$$

And this result matches inequality (3), our criterion for entangled states in the case where a, b have the same complex phase.

Disappointed, we now try a different approach. We begin by finding the eigenvector of the partial transpose $\rho_w^{T_A}(\alpha)$ with the negative eigenvalue. With Mathematica we obtain the eigenvalue $\lambda_- = -|a||b|$, which is manifestly negative, and the corresponding eigenvector:

$$|\lambda_-\rangle = \frac{1}{\sqrt{2}}\left(-\frac{|a||b|}{ab^\star}\,|00\rangle + |11\rangle\right)$$

And we construct the witness according to the prescription:

$$W = |\lambda_-\rangle\langle\lambda_-|^{T_A} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11| - \frac{|a||b|}{ab^\star}\,|01\rangle\langle 10| - \frac{|a||b|}{a^\star b}\,|01\rangle\langle 10|\right)$$

It's now straightforward to verify that:

$$\text{Tr}[W\rho_w(\alpha)] = \frac{1}{4}(1 - \alpha - 4|a||b|\alpha)$$

which is negative when:

$$\frac{1}{1+4|ab|} < \alpha \leq 1$$

Question 5

The completely dephasing channel is:

$$\Lambda[\rho] \equiv \text{Tr}_2 \left[U_{\text{CNOT}}(\rho \otimes |0\rangle\langle 0|) U_{\text{CNOT}}^\dagger \right] \qquad U_{\text{CNOT}} \equiv |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$$

It's straightforward to expand and simplify this definition:

$$\begin{aligned}
\Lambda[\rho] &= \text{Tr}_2 \left[(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X)(\rho \otimes |0\rangle\langle 0|)(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X) \right] \\
&= \text{Tr}_2 \left[(|0\rangle\langle 0| \rho |0\rangle\langle 0|) \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \rho |1\rangle\langle 1|) \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \rho |0\rangle\langle 0|) \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \rho |1\rangle\langle 1|) \otimes |1\rangle\langle 1| \right] \\
&= |0\rangle\langle 0| \rho |0\rangle\langle 0| + |1\rangle\langle 1| \rho |1\rangle\langle 1|
\end{aligned}$$

Luckily, $|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$, and these projectors are positive semidefinite, and so the Kraus operators are just:

$$K_0 = \Pi_0 = |0\rangle\langle 0|, \qquad K_2 = \Pi_1 = |1\rangle\langle 1|$$

Question 6

We're asked to show that partial transpose of any density matrix representing a bipartite system of two cubits has at most one negative eigenvalue. Our proof follows the outline given in Sanpera 1998.[5]

Condition for a state to be a product state

First, we obtain a condition for arbitrary pure state $W|00\rangle + Z|11\rangle + X|01\rangle + Y|10\rangle$ of two qubits to be a product state. An arbitrary product state may be written:

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = ac|00\rangle + bd|11\rangle + ad|01\rangle + bc|10\rangle$$

and by considering the two expressions, we see that the condition we seek must be:

$$WZ = XY \qquad \Leftrightarrow \qquad W|00\rangle + Z|11\rangle + X|01\rangle + Y|10\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

That is, the pure state is a product state, and may be factorized by choosing a, b, c, d such that $ac = W$, $bd = Z$, $ad = X$, $bc = Y$, if and only if $WZ = XY$.

This condition agrees with the PPT criterion, which we obtain by computing the eigenvalues of the partial transpose of the density matrix of the pure state. They are:

$$\pm|WZ - XY|^2$$

$$\frac{1}{2}\left(|W|^2 + |X|^2 + |Y|^2 + |Z|^2\right) \pm \sqrt{\left(|W|^2 + |X|^2 + |Y|^2 + |Z|^2\right)^2 - 4|WZ - XY|^2}$$

These eigenvalues are all non-negative if and only if $WZ = XY$.

[5] Sanpera, Tarrach, and Vidal, Physical Review A, 58, 1998.

Every two-dimensional subspace contains a product state

We now ask if it's possible for the two-dimensional subspace spanned by two eigenvectors, $|\lambda_1\rangle$ and $|\lambda_2\rangle$, of a Hermitian matrix to contain only entangled states. To investigate this question, we choose a basis $\{|i\rangle \otimes |j\rangle\}$ of $\mathcal{H}_A \otimes \mathcal{H}_B$, with $\{|i\rangle\}$ a basis of $\mathcal{H}_A$ and $\{|j\rangle\}$ a basis of $\mathcal{H}_B$, so that $|\lambda_1\rangle$ takes the simple form:

$$|\lambda_1\rangle = \alpha\,|00\rangle + \beta\,|11\rangle$$

and then, since the eigenvectors are orthogonal, we must be able to write $|\lambda_2\rangle$ in the form:

$$|\lambda_2\rangle = \epsilon(\beta\,|00\rangle + \gamma\,|01\rangle + \delta\,|10\rangle - \alpha\,|11\rangle)$$

We will not insist that $|\lambda_2\rangle$ be normalized, and so, without loss of generality, we may take $\epsilon = 1$, leaving:

$$|\lambda_2\rangle = \beta^\star\,|00\rangle + \gamma\,|01\rangle + \delta\,|10\rangle - \alpha^\star\,|11\rangle$$

Consider an arbitrary element of the two-dimensional subspace, which must be a linear combination of $|\lambda_1\rangle$ and $|\lambda_2\rangle$:

$$\begin{aligned} A\,|\lambda_1\rangle + B\,|\lambda_2\rangle &= A(\alpha\,|00\rangle + \beta\,|11\rangle) + B(\beta^\star\,|00\rangle + \gamma\,|01\rangle + \delta\,|10\rangle - \alpha^\star\,|11\rangle) \\ &= (A\alpha + B\beta^\star)\,|00\rangle + (A\beta - B\alpha^\star)\,|11\rangle + B\gamma\,|01\rangle + B\delta\,|10\rangle \end{aligned}$$

According to the condition established above, this is a product state if:

$$(A\alpha + B\beta^\star)\cdot(A\beta - B\alpha^\star) = B\gamma \cdot B\delta$$

Now, our subspace contains only entangled states if there is no choice of A, B which satisfies this equation. But this is just a single equation, quadratic in the variables A and B, and may always be solved over the complex numbers.

So the subspace contains at least one product state.

The partial transpose cannot have two negative eigenvalues

The remaining part of the proof is by contradiction.

Suppose we have a density matrix $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ representing the bipartite state of two cubits, and suppose that ρ^{T_A} has two negative eigenvalues $\lambda_1, \lambda_2 < 0$ for corresponding eigenvectors $|\lambda_1\rangle, |\lambda_2\rangle$, which are necessarily orthogonal, since ρ^{T_A} is Hermitian.

For some coefficients A, B we have a linear combination which is a product state:

$$A\,|\lambda_1\rangle + B\,|\lambda_2\rangle = |a\rangle \otimes |b\rangle$$

It's clear that the following sandwich is strictly negative:

$$\begin{aligned} \langle a|\,\langle b|\,\rho^{T_A}\,|a\rangle\,|b\rangle &= (A^\star\,\langle\lambda_1| + B^\star\,\langle\lambda_2|)\,\rho^{T_A}\,(A\,|\lambda_1\rangle + B\,|\lambda_2\rangle) \\ &= (A^\star\,\langle\lambda_1| + B^\star\,\langle\lambda_2|)(A\lambda_1\,|\lambda_1\rangle + B\lambda_2\,|\lambda_2\rangle) \\ &= |A|^2\lambda_1 + |B|^2\lambda_2 < 0 \end{aligned}$$

But, writing $\rho = \sum_{ijl\ell} \rho_{ij,k\ell}\,|ij\rangle\langle k\ell|$, we may expand the meat of this sandwich as follows:

$$\begin{aligned} 0 > \langle a|\,\langle b|\,\rho^{T_A}\,|a\rangle\,|b\rangle &= \langle a|\,\langle b|\left[\sum_{ijl\ell} \rho_{kj,k\ell}\,|ij\rangle\langle i\ell|\right]|a\rangle\,|b\rangle \\ &= \sum_{ijl\ell} \rho_{kj,k\ell}\,\langle a|k\rangle\,\langle b|j\rangle\,\langle i|a\rangle\,\langle \ell|b\rangle \\ &= \sum_{ijl\ell} \rho_{kj,k\ell}\,\langle a|i\rangle^\star\,\langle b|j\rangle\,\langle k|a\rangle^\star\,\langle \ell|b\rangle \\ &= \langle a^\star|\,\langle b|\,\rho\,|a^\star\rangle\,|b\rangle \end{aligned}$$

And so ρ is not positive semidefinite, contradicting the assumption that it is a density matrix.

We conclude that ρ^{T_A} has at most one negative eigenvalue.

Question 7

We're presented with the family of maps parameterized by $0 \leq \alpha \leq 1$ and defined by:

$$\Lambda_\alpha(\rho) = \tfrac{1}{2}\mathbb{1} + \alpha(X\rho Z + Z\rho X)$$

This is manifestly not a linear function of 2×2 matrices, nor trace-preserving, nor is it completely positive, so we need to turn it into something that is all of these things.[6]

Fortunately, we see that the maximally-mixed state $\frac{1}{2}\mathbb{1}$ can be written as a map acting on a unit trace density matrix ρ:

$$\tfrac{1}{2}\mathbb{1} = \pi(\rho) \qquad \text{where} \qquad \pi(\rho) \equiv \tfrac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z) = \tfrac{1}{2}\,\mathrm{Tr}\{\rho\}\,\mathbb{1}$$

So we can replace the definition of Λ_α with the following redefinition:

$$\Lambda_\alpha(\rho) = \pi(\rho) + \alpha(X\rho Z + Z\rho X) \tag{4}$$

whenever we need a proper linear map.

Part (i)

We may express the eigenvalue equation for the matrix $\Lambda_\alpha(\rho)$ as:

$$(X\rho Z + Z\rho X)\,|\lambda\rangle = \frac{1}{\alpha}(\lambda - \tfrac{1}{2})\,|\lambda\rangle$$

To solve this equation, we will express the density matrix ρ in the Pauli basis $\rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma})$ where $\|\boldsymbol{r}\| \leq 1$. Then:

$$\begin{aligned} X\rho Z &= X\tfrac{1}{2}(r_0\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma})Z = \tfrac{1}{2}(ir_2\mathbb{1} + r_3X - iY + r_1Z) \\ Z\rho X &= (X\rho Z)^\dagger = \tfrac{1}{2}(-ir_2\mathbb{1} + r_3X + iY + r_1Z) \end{aligned}$$

and so:

$$X\rho Z + Z\rho X = r_3X + r_1Z$$

The eigenvalues of this matrix are easily found to be $\pm\sqrt{r_1^2 + r_3^2}$, and therefore:

$$\lambda = \tfrac{1}{2} \pm \alpha\sqrt{r_1^2 + r_3^2}$$

The maximum possible value of $\sqrt{r_1^2 + r_3^2}$ is 1, and so both eigenvalues are non-negative for every ρ when $\alpha \leq \frac{1}{2}$.
The resulting density matrix is:

$$\Lambda_\alpha(\rho) = \tfrac{1}{2}(\mathbb{1} + 2\alpha r_3X + 2\alpha r_1Z)$$

in the Pauli basis, and so we see that the effect of the map was to:

- project away the Y component of ρ,
- swap and—depending on α—rescale the X and Z components.

When $\alpha > \frac{1}{2}$, the rescaling is an unphysical expansion of the Bloch sphere; if $\alpha < \frac{1}{2}$, it is a contraction of the Bloch sphere, and the resulting state is more mixed than the original state.

The final state lies in the $X - Z$ plane of the Bloch sphere.

[6] A quantum map must be linear; it does not make sense to talk about a "linear" map that operates only on unit trace matrices, because the set of unit trace matrices is not closed under vector addition, nor under scalar multiplication.

Part (ii)

To find the Choi matrix for these maps, we work from the definition:

$$J(\Lambda_\alpha) \equiv (\mathbb{1} \otimes \Lambda_\alpha)(|\Omega\rangle\langle\Omega|) = \sum_{i,j=0}^{d-1} |i\rangle\langle j| \otimes \Lambda_\alpha(|i\rangle\langle j|)$$

so all we need to do is apply Λ_α as redefined in (4) to each $|i\rangle\langle j|$, resulting in:

$$J(\Lambda_\alpha) = \frac{1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} + \alpha \begin{pmatrix} X & Z \\ Z & -X \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

This is manifestly the Choi state for a trace-preserving map, and the eigenvalues of this matrix are:

$$\frac{1}{2}, \frac{1}{2}(1 \pm 4\alpha)$$

which are all non-negative when $\alpha \leq \frac{1}{4}$. Thus, Λ_α is a completely positive map when $\alpha \leq \frac{1}{4}$.

Part (iii)

For $\alpha = \frac{1}{4}$, the eigenvalues and eigenvectors of the Choi matrix are:

$$\frac{1}{2}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \frac{1}{2}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad 0, \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad 1, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

and we may rearrange the elements of these vectors to obtain the Kraus operators:

$$K_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\mathbb{1}}{2} \qquad K_2 = \frac{i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{Y}{2} \qquad K_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{H}{\sqrt{2}}$$

(The eigenvector with eigenvalue 0 does not give rise to a Kraus operator.)

It's trivial to verify that $\sum K_i^\dagger K_i = \mathbb{1}$ as befits a list of Kraus operators and that each $K_i^\dagger K_i$ is positive definite. Therefore:

$$\Lambda_{\frac{1}{4}}(\rho) = \tfrac{1}{4}\rho + \tfrac{1}{4}Y\rho Y + \tfrac{1}{2}H\rho H$$