Homework Lecture 6

November 7, 2023

Consider the transmission of a classical random variable X through a classical-quantum channel with pure outputs such that the joint density matrix at the output of the channel is given by:

$$\rho_{XB} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes |\theta_x\rangle\langle\theta_x|_B.$$

A measurement POVM is applied to the B share to yield Y:

$$\begin{aligned}\rho_{XY} &= \sum_{x,y} p_X(x)|x\rangle\langle x|_X \otimes \mathrm{tr}\{\Lambda_y|\theta_x\rangle\langle\theta_x|_B\}|y\rangle\langle y|_Y \\ &= \sum_{x,y} p_X(x)p_{Y|X}(y|x)|x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y.\end{aligned}$$

- For the binary, uniformly distributed case, $X \sim \mathsf{Bern}(\frac{1}{2})$, $|\mathcal{Y}| = |\mathcal{X}| = 2$, $\dim(\mathcal{H}_B) = 2$ and

  $$|\theta_0\rangle = \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix};\, |\theta_1\rangle = \begin{bmatrix}\cos\frac{\theta}{2}\\ -\sin\frac{\theta}{2}\end{bmatrix},$$

  the optimal POVM (in terms of minimizing the error probability) is obtained in the Lecture 5 slides and is given by $\{|+\rangle\langle+|, |-\rangle\langle-|\}$. In this case, the probability of error is $\mathrm{P}_e = \frac{1}{2}(1-\sin\theta)$.

  - Obtain and plot the accessible information $I(X;Y)$ and the quantum mutual information $I(X;B)_\rho$ for $\theta \in (0,\pi]$.

- We now consider $X \sim \mathsf{Unif}([0,1,2,3])$, $|\mathcal{Y}| = |\mathcal{X}| = 4$. For every random variable realization x we use three parallel quantum channels like the one employed before such that:

  $$\rho_{XB^3} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes |\psi_x\rangle\langle\psi_x|_{B^3},$$

  where

  $$\begin{aligned}|\psi_0\rangle_{B^3} &= |\theta_0\rangle_B \otimes |\theta_0\rangle_B \otimes |\theta_0\rangle_B\\ |\psi_1\rangle_{B^3} &= |\theta_0\rangle_B \otimes |\theta_1\rangle_B \otimes |\theta_1\rangle_B\\ |\psi_2\rangle_{B^3} &= |\theta_1\rangle_B \otimes |\theta_0\rangle_B \otimes |\theta_1\rangle_B\\ |\psi_3\rangle_{B^3} &= |\theta_1\rangle_B \otimes |\theta_1\rangle_B \otimes |\theta_0\rangle_B.\end{aligned}$$

Again, a measurement POVM is applied to the B^3 share to yield Y:

$$\begin{aligned}\rho_{XY} &= \sum_{x,y} p_X(x)|x\rangle\langle x|_X \otimes \mathrm{tr}\{\Lambda_y|\psi_x\rangle\langle\psi_x|_{B^3}\}|y\rangle\langle y|_Y \\ &= \sum_{x,y} p_X(x)p_{Y|X}(y|x)|x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y.\end{aligned}$$

In this case, the POVM known as the *square-root measurement* becomes optimal meaning:

$$\Lambda_y = \frac{1}{4}(\rho_{B^3})^{-\frac{1}{2}}|\psi_y\rangle\langle\psi_y|(\rho_{B^3})^{-\frac{1}{2}}, \text{ for } y \in [0,1,2,3],$$

and where $\rho_{B^3} = \mathrm{tr}_X\{\rho_{XB^3}\}$. Note that since ρ_{B^3} is not full rank the inverse operation should be replaced by the pseudo inverse, available in any numerical software.

  - Show that $\{\Lambda_y\}$ is a proper POVM.
  - Obtain (numerically) and plot the accessible information $I_3(X;Y)$ and the quantum mutual information $I_3(X;B^3)_\rho$ for $\theta \in (0,\pi]$.

- Finally plot $I_3(X;Y) - 3I(X;Y)$ and $I_3(X;B^3)_\rho - 3I(X;B)_\rho$ for $\theta \in (0,\pi]$.
- Analyze and discuss the results.