Lecture 6: Classical Communication over quantum channels

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Overview

Introduction

Introduction

- We begin by introducing a **Discrete Memoryless Channel** (DMC) model for the transmission of **classical information** making use of one **preparation** channel, an arbitrary **quantum channel** and a **measurement** channel.
- The **capacity** of the resulting DMC, the **accessible information**, is then formulated as a (highly complex) optimization problem.
- The **quantum data processing inequality** is applied to derive an upper bound, the **Holevo information** of the quantum channel which is the solution to a **simpler** optimization.
- We then analyze the **additivity** of the **Holevo information** of a channel to find out that the **additivity** that characterizes the mutual information and private information of **classical** channels is not preserved (in general) in **quantum**.
- We finish presenting the **capacity** for the transmission of **classical** information over quantum channels or HSW Theorem, and some **examples**.

Accesible information

DMC formulation for classical communication

- The transmission of Classical information over quantum channels can be modeled as a DMC by a **preparation** stage in which a quantum state is generated according to a given ensemble $\mathcal{E} \equiv \{p_X(x), \rho_A^x\}$:

$$\rho_{XA} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho_A^x, \;\; \rho_A = \sum_x p_X(x)\rho_A^x.$$

- The modulated quantum state is **transmitted** through the channel $\mathcal{N}_{A\to B}$:

$$\rho_{XB} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}_{A\to B}(\rho_A^x), \;\; \rho_B = \sum_x p_X(x)\mathcal{N}_{A\to B}(\rho_A^x).$$

- A **measurement POVM** is applied to the state at B to yield Y:

$$\begin{aligned}\rho_{XY} &= \sum_{x,y} p_X(x)|x\rangle\langle x|_X \otimes \mathrm{tr}\{\Lambda_y \mathcal{N}_{A\to B}(\rho_A^x)\}|y\rangle\langle y|_Y \\ &= \sum_{x,y} p_X(x)p_{Y|X}(y|x)|x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y.\end{aligned}$$

Accesible information

- Note that this procedure makes **one use** of channel $\mathcal{N}_{A\to B}$ for each DMC use thus does **not exploit any potential entanglement** between the transmitted states at A or any **joint measurement** of the received states at B.
- Incorporating the effect of $\mathcal{N}_{A\to B}$ in the conditional pmf $p_{Y|X}(y|x)$, the communication limits are **well known** (Shannon Capacity), and the DMC channel capacity will be given by the **maximization** of $I(X;Y)$ with respect to the **ensemble** $\mathcal{E} = \{p_X(x), \rho_A^x\}$ and the **measurement** POVM $\{\Lambda_y\}$.

Accesible information $I_{\mathrm{acc}}(\mathcal{E})$ and I^*_{acc}

The accesible information is:

$$I_{\mathrm{acc}}(\mathcal{E}) = \max_{\{\Lambda_y\}} I(X;Y),$$

$$I^*_{\mathrm{acc}} = \max_{\mathcal{E}} I_{\mathrm{acc}}(\mathcal{E}) = \max_{\mathcal{E},\{\Lambda_y\}} I(X;Y).$$

Accesible information upper bound

- **Computation** of the **accessible information** $I_{\text{acc}}(\mathcal{E})$ or I^*_{acc} is **difficult** and would still be subject to **additional optimization** regarding the general strategy (possibly more that one quantum channel use per DMC use - discussion on this issue follows later).
- The quantum data processing inequality $I(A;B)_\rho \geq I(\mathcal{N}(A);\mathcal{M}(B))_\sigma$ provides an upper bound to the accessible information noting that Y is obtained applying a measurement channel to the state at B.

Upper bound to the accesible information

The accessible information of $\mathcal{E} \equiv \{p_X(x), \rho_A^x\}$ when transmitted through $\mathcal{N}_{A\to B}$ is upper bounded by the quantum mutual information $I(X;B)_\rho$:

$$I_{\text{acc}}(\mathcal{E}) = \max_{\{\Lambda_y\}} I(X;Y) \leq I(X;B)_\rho = H(\rho_B) - \sum_x p_X(x) H(\rho_B^x),$$

where $\rho_B = \sum_x p_X(x)\rho_B^x = \sum_x p_X(x)\mathcal{N}_{A\to B}(\rho_A^x)$.

The information of quantum channels

Holevo information of a quantum channel

Holevo information of a quantum channel

The Holevo information $\chi(\mathcal{N})$ of channel $\mathcal{N}_{A\to B}$ is:

$$\chi(\mathcal{N}) \equiv \max_{\rho_{XA}} I(X;B)_\rho \geq \max_{\mathcal{E}} I_{\text{acc}}(\mathcal{E}) = I^*_{\text{acc}}$$

where $\mathcal{E} = \{p_X(x), \rho_A^x\}$ and

$$\rho_{XA} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho_A^x,$$
$$\rho_{XB} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}_{A\to B}(\rho_A^x).$$

The **Holevo Information of a quantum channel** provides an upper bound to the **accessible information**.

Computation of the Holevo information (I)

Pure states are sufficient

The Holevo information of channel $\mathcal{N}_{A\to B}$ can be obtained restricting the maximization to pure states

$$\chi(\mathcal{N}) = \max_{\rho_{XA}} I(X;B)_\rho = \max_{\tau_{XA}} I(X;B)_\tau,$$

where ρ_{XA} and ρ_{XB} are defined as in previous slide and

$$\tau_{XA} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes |\phi_x\rangle\langle\phi_x|_A,$$

and

$$\tau_{XB} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}_{A\to B}(|\phi_x\rangle\langle\phi_x|_A).$$

Computation of the Holevo information (and II)

The proof follows considering the spectral decomposition,

$$\rho_A^x = \sum_z p_{Z|X}(z|x)|\psi_{x,z}\rangle\langle\psi_{x,z}|_A,$$

defining

$$\sigma_{XZA} = \sum_{x,z} p_X(x)p_{Z|X}(z|x)|x\rangle\langle x|_X \otimes |z\rangle\langle z|_Z \otimes |\psi_{x,z}\rangle\langle\psi_{x,z}|_A,$$

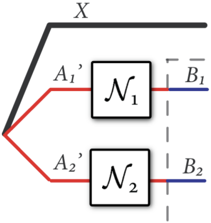
where $\rho_{XA} = \mathrm{tr}_Z\{\sigma_{XZA}\}$. Note that σ_{XZA} is a state of the form τ_{XA} with joint random variable $\mathcal{X} \times \mathcal{Z}$ as the classical part. Let σ_{XZB} be the state that results from sending A through $\mathcal{N}_{A\to B}$:

$$I(X;B)_\rho = I(X;B)_\sigma \leq I(XZ;B)_\sigma.$$

The equality follows from $\rho_{XB} = \mathrm{tr}_Z\{\sigma_{XZB}\}$ and the inequality from the quantum data processing inequality. ■

Superadditivity of the Holevo information (I)

- A more general strategy for the transmission of X through quantum channel $\mathcal{N}_{A\to B}$ is by using more than one quantum channel uses for each x, for instance for **two** quantum channel uses.



$$\rho_{XA_1A_2} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho^x_{A_1A_2},$$

the joint state between X and B is

$$\rho_{XB_1B_2} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes (\mathcal{N}_{A_1\to B_1} \otimes \mathcal{N}_{A_2\to B_2})(\rho^x_{A_1A_2}).$$

Superadditivity of the Holevo information (II)

- Is the Holevo information of the product channel $\mathcal{N}_{A_1\to B_1} \otimes \mathcal{N}_{A_2\to B_2}$ equal to twice the Holevo information of the channel $\mathcal{N}_{A\to B}$?

$$\chi(\mathcal{N}_{A_1\to B_1} \otimes \mathcal{N}_{A_2\to B_2}) \stackrel{?}{=} \chi(\mathcal{N}_{A_1\to B_1}) + \chi(\mathcal{N}_{A_2\to B_2}) = 2\chi(\mathcal{N}_{A\to B}),$$

or

$$\max_{\rho_{XA_1A_2}} I(X;B_1B_2)_\rho \stackrel{?}{=} 2\max_{\rho_{XA}} I(X;B)_\rho,$$

or, in general, for arbitrary channels $\mathcal{N}$ and $\mathcal{M}$,

$$\chi(\mathcal{N}_{A_1\to B_1} \otimes \mathcal{M}_{A_2\to B_2}) \stackrel{?}{=} \chi(\mathcal{N}_{A_1\to B_1}) + \chi(\mathcal{M}_{A_2\to B_2}).$$

Superadditivity of the Holevo information (III)

- Since independent coding is a particular case of joint coding the Holevo information $\chi(\mathcal{N} \otimes \mathcal{M})$ is at least superadditive:

$$\chi(\mathcal{N}_{A_1 \to B_1} \otimes \mathcal{M}_{A_2 \to B_2}) \geq \chi(\mathcal{N}_{A_1 \to B_1}) + \chi(\mathcal{M}_{A_2 \to B_2}).$$

- It was believed to be additive for many years and it is indeed additive for many channels. However, a counterexample demonstrated **strict** superadditivity [Hastings, 2009].

$$\chi(\mathcal{N} \otimes \mathcal{M}) > \chi(\mathcal{N}) + \chi(\mathcal{M}),$$

which yields the **regularized** definition (possibly with strict inequality),

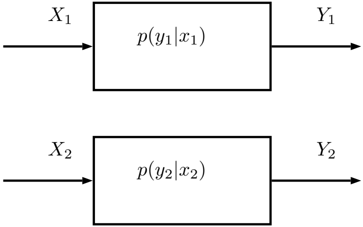
$$\chi_{\mathrm{reg}}(\mathcal{N}) \equiv \lim_{n \to \infty} \frac{1}{n} \chi(\mathcal{N}^{\otimes n}) \geq \chi(\mathcal{N}).$$

- This property has no counterpart in classical information theory where the capacity of the product DMC is additive.

Additivity in classical channels

For two **parallel** DMCs, $p(y_1, y_2|x_1, x_2) = p(y_1|x_1)p(y_2|x_2)$ each of which is characterized by a given **capacity** C_1 and C_2. The **total capacity** C is **equal** to $C_1 + C_2$:

$$C = \max_{p(x_1,x_2)} I(X_1, X_2; Y_1, Y_2) = \max_{p(x_1)} I(X_1; Y_1) + \max_{p(x_2)} I(X_2; Y_2) = C_1 + C_2$$



The HSW theorem

The Holevo-Schumacher-Westmoreland theorem (I)

The Holevo-Schumacher-Westmoreland theorem

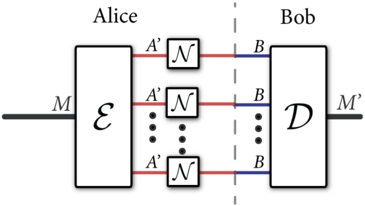
The classical capacity of a quantum channel is equal to the regularization of the Holevo information of the channel:

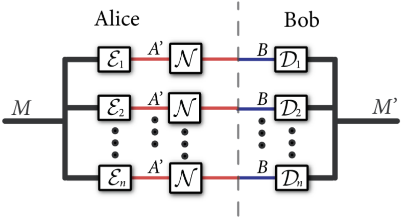
$$\mathrm{C}(\mathcal{N}) = \lim_{k\to\infty} \frac{1}{k}\chi(\mathcal{N}^{\otimes k}) \equiv \chi_{\mathrm{reg}}(\mathcal{N}),$$

where

$$\mathcal{N}^{\otimes k}_{A_1\cdots A_k \to B_1\cdots B_k} = \mathcal{N}_{A_1\to B_1} \otimes \cdots \otimes \mathcal{N}_{A_k\to B_k}.$$

- Regularization is equivalent to **multiletter** characterization in classical information theory $\frac{1}{k}I(X^k;Y^k)$ which, in contrast to **single** letter characterizations, $I(X;Y)$, does not provide much insight into practical coding techniques and is difficult to compute.
- Note that if the Holevo information is **additive** the regularization limit is not necessary and thus, $\mathrm{C}(\mathcal{N}) = \chi(\mathcal{N}) = \max_{\rho_{XA}} I(X;B)_\rho$.





- The **most general** strategy for classical information communication is illustrated **left**. One can make use of entanglement at Alice side and joint decoding at Bob's. On the contrary, the **suboptimal** strategy illustrated **right** uses the block DMC defined by a single use of a quantum channel only. Figures from [Wilde, 2017].
- The HSW theorem requires regularization, and thus investigating the expression $\lim_{k\to\infty} \dfrac{1}{k}\chi(\mathcal{N}^{\otimes k})$, since both strategies are in general not equivalent.

Channel code for classical communication

Channel code for communication over quantum channels

A $(2^{nR}, n, \epsilon)$ code $\mathcal{C}$ for quantum channel $\mathcal{N}_{A\to B}$ is:

- a message set $\mathcal{M} \equiv [1:2^{nR}]$.
- an encoding function $[1:2^{nR}] \to \rho_{A^n}$: that assigns a codeword $\rho^m_{A^n}$ to each message $m \in [1:2^{nR}]$. This state is transmitted over n independent uses on the quantum channel $\mathcal{N}_{A\to B}$ so that the state at Bob end is $\mathcal{N}^{\otimes n}(\rho^m_{A^n})$.
- a decoding POVM $\{\Lambda_m\}$ that assigns an estimate $\hat{m} \in [1:2^{nR}]$ to each received state $\mathcal{N}^{\otimes n}(\rho^m_{A^n})$. The conditional probability of error assuming message m was transmitted is:

$$P_e(m) \equiv \Pr\{\hat{M} \neq m | M = m\} = \mathrm{tr}\{(I - \Lambda_m)\mathcal{N}^{\otimes n}(\rho^m_{A^n})\}$$

The code has ϵ error if $\max_m\{P_e(m)\} \equiv P_e^* \leq \epsilon$.

Random codebook generation

Let $\mathcal{E} = \{p_X(x), \phi_A^x\}$ be the ensemble of pure states $\phi_A^x \equiv |\phi_x\rangle\langle\phi_x|_A$ that attains the Holevo information of $\mathcal{N}$. Generate $x^n(m)$ for $m \in [1 : 2^{nR}]$ according to $p_{X^n}(x^n) = \prod_{i=1}^n p_X(x_i)$. The generated sequences yield the following product density matrix at A:

$$\begin{aligned}
\phi_{A^n} &\equiv \phi_{A_1} \otimes \cdots \otimes \phi_{A_n} \\
&= \sum_{x_1 \in \mathcal{X}} p_X(x_1)\phi_{A_1}^{x_1} \otimes \cdots \otimes \sum_{x_n \in \mathcal{X}} p_X(x_n)\phi_{A_n}^{x_n} \\
&= \sum_{x^n \in \mathcal{X}^n} p_{X^n}(x^n)\phi_{A^n}^{x^n} \equiv \phi_A^{\otimes n}.
\end{aligned}$$

and B, for $\sigma_{B_i}^{x_i} \equiv \mathcal{N}(\phi_{A_i}^{x_i})$:

$$\sigma_{B^n} \equiv \sigma_{B_1} \otimes \cdots \otimes \sigma_{B_n} = \sum_{x^n \in \mathcal{X}^n} p_{X^n}(x^n)\sigma_{B^n}^{x^n} \equiv \sigma_B^{\otimes n}$$

Encoding and decoding

Encoding

To send message $m \in [1:2^{nR}]$ take $\rho_{A^n}^m = \phi_{A^n}^{x^n(m)}$ where:

$$\phi_{A^n}^{x^n(m)} \equiv \phi_{A_1}^{x_1(m)} \otimes \cdots \otimes \phi_{A_n}^{x_n(m)}.$$

Decoding strategy

Let

$$\sigma_{B^n}^{x^n(m)} \equiv \mathcal{N}^{\otimes n}(\phi_{A^n}^{x^n(m)}) \equiv \sigma_{B_1}^{x_1(m)} \otimes \cdots \otimes \sigma_{B_n}^{x_n(m)}$$

be the received quantum state, where $\sigma_{B_i}^{x_i(m)} \equiv \mathcal{N}(\phi_{A_i}^{x_i(m)})$, for $i \in [1:n]$. The receiver declares that $\hat{m} \in [1:2^{nR}]$ was transmitted using a detection POVM $\{\Lambda_m\}$. The quantum typicality properties are used to define the $\{\Lambda_m\}$, where $\Lambda_m \succeq 0$ and $\sum_{m=1}^{2^{nR}} \Lambda_m = I$.

Decoding projectors

- The Shannon typicality decoder would check typicality between the received sequence and the codewords corresponding to each of the m possible messages.
- We will follow a somehow similar procedure in the quantum setting by considering the observed density matrix $\sigma_B^{\otimes n}$, the ones corresponding to the m possibly transmitted messages $\sigma_{B^n}^{x^n(m)}$, and then defining their respective projectors.

Total subspace $\Pi_{B^n}^{\delta}$ and message subspace $\Pi_{B^n|x^n(m)}^{\delta}$ projectors

- The total subspace projector $\Pi_{B^n}^{\delta}$ is defined as the (weakly) typical projector for $\sigma_B^{\otimes n}$.
- The message subspace $\Pi_{B^n|x^n(m)}^{\delta}$ projector is defined as the (weakly) conditional typical projector for $\sigma_{B^n}^{x^n(m)}$.

Quantum Packing Lemma

Quantum Packing Lemma

Assuming strong typicality, we can now apply the properties of quantum typical and conditionally typical sequences:

$$
\begin{aligned}
\mathrm{tr}\{\Pi^{\delta}_{B^n}\rho^{x^n}_{B^n}\} &\geq 1-\epsilon, \\
\mathrm{tr}\{\Pi^{\delta}_{B^n|x^n}\rho^{x^n}_{B^n}\} &\geq 1-\epsilon, \\
\mathrm{tr}\{\Pi^{\delta}_{B^n|x^n}\} &\leq 2^{n(H(B|X)+c\delta)}, \\
\Pi^{\delta}_{B^n}\sigma_B^{\otimes n}\Pi^{\delta}_{B^n} &\preceq 2^{-n(H(B)-c'\delta)}\Pi^{\delta}_{B^n},
\end{aligned}
$$

and then the (derandomized) quantum packing lemma, (see Chapter 16 of [Wilde, 2017] for details) which states that the maximum P_e is:

$$
\begin{aligned}
P_e^* &= \max_m \mathrm{tr}\{(I-\Lambda_m)\sigma^{x^n(m)}_{B^n}\} \\
&\leq 4(\epsilon+2\sqrt{\epsilon}) + 16(1-\epsilon)^{-1}2^{-n(H(B)-H(B|X)-(c+c')\delta)}|\mathcal{M}|.
\end{aligned}
$$

Probability of error and design of the POVM $\{\Lambda_m\}$

Probability of error

The quantum packing lemma shows that P_e^* can be made as small as desired as long as the cardinality of the message set $|\mathcal{M}|$ is small enough,

$$P_e^* \leq 4(\epsilon + 2\sqrt{\epsilon}) + 16(1-\epsilon)^{-1} 2^{-n(I(X;B)-(c+c')\delta)} 2^{nR}$$

and $\lim_{n\to\infty} P_e^* = 0$ if $R < I(X;B) - (c+c')\delta$.

Design of the POVM $\{\Lambda_m\}$

Projectors $\Pi^{\delta}_{B^n|x^n}$ can not be used directly as POVM since there is no guarantee that $\sum_{m=1}^{2^{nR}} \Pi^{\delta}_{B^n|x^n(m)} \equiv \bar{\Pi}^{\delta}_{B^n|x^n} = I$. In order to satisfy this condition one possible solution is:

$$\Lambda_m = (\bar{\Pi}^{\delta}_{B^n|x^n})^{-\frac{1}{2}} \Pi^{\delta}_{B^n|x^n(m)} (\bar{\Pi}^{\delta}_{B^n|x^n})^{-\frac{1}{2}}$$

Remarks on the achievability result

- Note the proof shows achievability for a rate equal to the Holevo information of the quantum channel $\chi(\mathcal{N})$ and not to the possibly higher rate $\chi_{\mathrm{reg}}(\mathcal{N}) = \lim_{k\to\infty} \dfrac{1}{k}\chi(\mathcal{N}^{\otimes k})$.
- But nothing prevents applying the same encoding and decoding procedure to the product channel $\mathcal{N}^{\otimes k}$ showing that for arbitrary k the rate $\dfrac{1}{k}\chi(\mathcal{N}^{\otimes k})$ is achievable.

Randomness distribution over quantum channels

For uniformly distributed messages the ideal **randomness distribution** joint state between Alice and Bob is:

$$\bar{\phi}_{M\hat{M}} \equiv \frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}} |m\rangle\langle m|_M \otimes |m\rangle\langle m|_{\hat{M}},$$

whereas the actual state after encoding and decoding is:

$$\omega_{M\hat{M}} = \frac{1}{|\mathcal{M}|}\sum_{m,\hat{m}\in\mathcal{M}} \mathrm{tr}\{\Lambda_{\hat{m}}\mathcal{N}^{\otimes n}(\rho^m_{A^n})\}|m\rangle\langle m|_M \otimes |\hat{m}\rangle\langle \hat{m}|_{\hat{M}}.$$

Code for randomness distribution over channel $\mathcal{N}_{A\to B}$

A $(2^{nR}, n, \epsilon)$ code for randomness distribution is achievable if there is an encoding-decoding procedure as indicated before for the channel code and

$$\frac{1}{2}\|\bar{\phi}_{M\hat{M}} - \omega_{M\hat{M}}\|_1 \leq \epsilon.$$

- The capacity for randomness distribution cannot be smaller than the one for communication since an error-free classical channel can be used to obtain shared randomness.
- This implies that randomness distribution can be used in converse proofs of classical communication.
- Invoking the AFW inequality:

$$\frac{1}{2}\|\bar{\phi}_{M\hat{M}} - \omega_{M\hat{M}}\|_1 \leq \epsilon,$$

implies

$$\begin{aligned} |H(M|\hat{M})_{\bar{\phi}} - H(M|\hat{M})_{\omega}| &\leq \epsilon \log|\mathcal{M}| + (1+\epsilon)H(\epsilon/(1+\epsilon)) \\ &\equiv f(|\mathcal{M}|, \epsilon). \end{aligned}$$

Converse proof (and II)

- Now applying a procedure similar in spirit to Fano's inequality,

$$\begin{aligned} nR = \log|\mathcal{M}| &= H(M) \\ &= H(M) - H(M|\hat{M})_{\bar{\phi}} \\ &\leq H(M) - H(M|\hat{M})_{\omega} + f(|\mathcal{M}|, \epsilon) \\ &= I(M;\hat{M})_{\omega} + f(|\mathcal{M}|, \epsilon) \\ &\leq I(M;B^n)_{\omega} + f(|\mathcal{M}|, \epsilon) \\ &\leq \chi(\mathcal{N}^{\otimes n}) + f(|\mathcal{M}|, \epsilon). \end{aligned}$$

- Recovering the expression for $f(|\mathcal{M}|, \epsilon)$ and rearranging terms,

$$R(1-\epsilon) \leq \frac{1}{n}\chi(\mathcal{N}^{\otimes n}) + \frac{1}{n}(1+\epsilon)H(\epsilon/(1+\epsilon))$$

meaning that for $n \to \infty$, $R \leq \chi_{\mathrm{reg}}(\mathcal{N})$. ■

Why is regularization needed?

What is different in this converse proof with respect to the classical channel coding converse?

- In classical the mutual information is additive for the block DMC, and in the classical converse proof:

$$nR \leq I(X^n; Y^n) + n\epsilon_n \leq nC + n\epsilon_n$$

- But the Holevo information is not additive!

$$I(M; B^n)_\omega \nleq nI(M; B)_\omega \leq n\chi(\mathcal{N}).$$

Capacity examples

Entanglement-breaking channels

Entanglement-breaking channels

An entanglement breaking channel $\mathcal{N}^{\mathsf{EB}}_{A\to B}$ takes any arbitrary composite state $\rho_{RA} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_A)$ to a separable state, i.e.

$$(\mathrm{id}_R \otimes \mathcal{N}^{\mathsf{EB}}_{A\to B})(\rho_{RA}) = \sum_z p_Z(z)\rho^z_R \otimes \tau^z_B$$

One can proof that it suffices to check this condition for the maximally entangled state Φ_{RA} and that a channel is entanglement-breaking iff its Kraus operators are unit rank.

In fact, this proof leads to the corollary that any entanglement-breaking channel is a serial concatenation of a *qc* channel followed by a *cq* channel, i.e., a measurement channel followed by a preparation channel:

$$\mathcal{N}^{\mathsf{EB}}_{A\to B}(\rho_A) = (\mathcal{P}_{Z\to B} \circ \mathcal{M}_{A\to Z})(\rho_A)$$

Capacity of entanglement-breaking channels (I)

Capacity of entanglement-breaking channels

Let $\mathcal{N}^{EB}$ be an entanglement-breaking channel. The capacity of $\mathcal{N}^{EB}$ equals the Holevo information of the channel, i.e.,

$$\mathrm{C}(\mathcal{N}^{EB}) = \chi(\mathcal{N}^{EB}).$$

The proof follows directly from the additivity of the Holevo information of the entanglement-breaking channels.

Additivity of the Holevo information for EB channels

Let $\mathcal{N}^{EB}$ be an entanglement-breaking channel and $\mathcal{M}$ an arbitrary channel. Then,

$$\chi(\mathcal{N}^{EB} \otimes \mathcal{M}) = \chi(\mathcal{N}^{EB}) + \chi(\mathcal{M}).$$

Capacity of entanglement-breaking channels (II)

- The proof requires to show that $\chi(\mathcal{N}^{EB} \otimes \mathcal{M}) \leq \chi(\mathcal{N}^{EB}) + \chi(\mathcal{M})$ since $\chi(\mathcal{N}^{EB} \otimes \mathcal{M}) \geq \chi(\mathcal{N}^{EB}) + \chi(\mathcal{M})$ by definition.
- Let $\rho_{XB_1B_2}$ the state in $\chi(\mathcal{N}^{EB} \otimes \mathcal{M}) = \max_{\rho_{XA_1A_2}} I(X;B_1B_2)_\rho$,

$$\rho_{XB_1B_2} = (\mathrm{id}_X \otimes \mathcal{N}^{EB}_{A_1\to B_1} \otimes \mathcal{M}_{A_2\to B_2})(\rho_{XA_1A_2}),$$
$$\rho_{XA_1A_2} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho^x_{A_1A_2}.$$

After applying $\mathcal{N}^{EB}_{A_1\to B_1}$ to $\rho_{XA_1A_2}$,

$$\begin{aligned}\rho_{XB_1A_2} &= \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}^{EB}_{A_1\to B_1}(\rho^x_{A_1A_2}) \\ &= \sum_x p_X(x)|x\rangle\langle x|_X \otimes \sum_z p_{Z|X}(z|x)\sigma^{x,z}_{B_1} \otimes \theta^{x,z}_{A_2} \\ &= \sum_{x,z} p_X(x)p_{Z|X}(z|x)|x\rangle\langle x|_X \otimes \sigma^{x,z}_{B_1} \otimes \theta^{x,z}_{A_2}.\end{aligned}$$

Capacity of entanglement-breaking channels (III)

- Then, after applying $\mathcal{M}_{A_2 \to B_2}$ to $\rho_{XB_1A_2}$,

$$\rho_{XB_1B_2} = \sum_{x,z} p_{Z|X}(z|x) p_X(x) |x\rangle\langle x|_X \otimes \sigma_{B_1}^{x,z} \otimes \mathcal{M}(\theta_{A_2}^{x,z}).$$

  State for which we can define the extension,

$$\omega_{XZB_1B_2} = \sum_{x,z} p_{Z|X}(z|x) p_X(x) |x\rangle\langle x|_X \otimes |z\rangle\langle z|_Z \otimes \sigma_{B_1}^{x,z} \otimes \mathcal{M}(\theta_{A_2}^{x,z}),$$

  satisfying $\rho_{XB_1B_2} = \mathrm{tr}_Z\{\omega_{XZB_1B_2}\}$.

- Now, by the definition of $\rho_{XB_1B_2}$, the chain rule for mutual information and the definition of $\chi(\mathcal{N}^{EB})$,

$$\begin{aligned} \chi(\mathcal{N}^{EB} \otimes \mathcal{M}) = I(X;B_1B_2)_\rho &= I(X;B_1)_\rho + I(X;B_2|B_1)_\rho \\ &\leq \chi(\mathcal{N}^{EB}) + I(X;B_2|B_1)_\rho. \end{aligned}$$

Capacity of entanglement-breaking channels (IV)

- We concentrate now on the term $I(X;B_2|B_1)_\rho$.

$$\begin{aligned}
I(X;B_2|B_1)_\rho &= I(X;B_2|B_1)_\omega \\
&= I(XB_1;B_2)_\omega - I(B_1;B_2)_\omega \\
&\leq I(XB_1;B_2)_\omega \\
&\leq I(XZB_1;B_2)_\omega \\
&= I(XZ;B_2)_\omega + I(B_1;B_2|XZ)_\omega \\
&= I(XZ;B_2)_\omega \\
&\leq \chi(\mathcal{M}).
\end{aligned}$$

where we have used the chain rule and the positivity for mutual information, the data processing inequality, the chain rule again, the fact that, conditioned on XZ, the $\omega_{XZB_1B_2}$ state is product, and the definition of $\chi(\mathcal{M})$. ■

Capacity of entanglement-breaking channels (and V)

- We showed any $\mathcal{N}^{EB}$ channel can be expressed as a serial concatenation of a measurement channel followed by a preparation channel. Then for $\mathcal{N}$ an arbitrary channel, Alice can simulate an $\mathcal{N}^{EB}$ channel by performing a **measurement** in the $\{|x\rangle\langle x|\}$ basis and **preparing** a state ρ^x conditioned on the outcome so that the resulting density matrix is:

$$\rho_{XB} = \sum_x \langle x|\sigma_A|x\rangle |x\rangle\langle x|_X \otimes \mathcal{N}(\rho^x) = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}(\rho^x).$$

  the capacity of which is,

$$\max_{p_X(x),\rho^x} I(X;B)_\rho.$$

- This is known as the product-state capacity of $\mathcal{N}$, a lower bound to its true capacity (no entanglement at the encoder).

Capacity of quantum erasure channel (I)

The capacity of the (classical) erasure channel (EC) for $\mathcal{Y} = \mathcal{X} \cup \{e\}$,

$$p_{Y|X}(y|x) = (1-\epsilon)\mathbb{1}\{y = x\} + \epsilon\mathbb{1}\{y = e\}$$

where $\epsilon \in [0, 1]$, is well known, $\mathrm{C}_{\mathrm{EC}} = (1-\epsilon)\log|\mathcal{X}|$. A similar result is obtained in the quantum case.

Capacity of quantum erasure channel

The capacity of the quantum erasure channel defined as

$$\mathcal{N}^{\epsilon}_{A\to B}(\rho_A) = (1-\epsilon)\mathcal{I}_{A\to B}(\rho_A) + \epsilon|e\rangle\langle e|_B$$

where $\epsilon \in [0, 1]$, $d_A = d$, $d_B = d + 1$, and $\{|0\rangle, \cdots, |d-1\rangle\}$ and $\{|0\rangle, \cdots, |d-1\rangle, |e\rangle\}$ form an orthonormal basis in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, is

$$\mathrm{C}(\mathcal{N}^{\epsilon}) = (1-\epsilon)\log d.$$

Capacity of quantum erasure channel (II)

Achievability proof (I)

Consider the ensemble $\mathcal{E} \equiv \{1/d; |0\rangle_A, \cdots, |d-1\rangle_A\}$, this yields:

$$\rho_{XB} = \frac{1}{d}\sum_{x=0}^{d-1} |x\rangle\langle x|_X \otimes ((1-\epsilon)|x\rangle\langle x|_B + \epsilon|e\rangle\langle e|_B),$$

$$\begin{aligned}\rho_B = \mathrm{tr}_X\{\rho_{XB}\} &= \frac{1}{d}\sum_{x=0}^{d-1}((1-\epsilon)|x\rangle\langle x|_B + \epsilon|e\rangle\langle e|_B) \\ &= \frac{1-\epsilon}{d}(I_B - |e\rangle\langle e|_B) + \epsilon|e\rangle\langle e|_B.\end{aligned}$$

The eigenvalues of ρ_B are $\frac{1-\epsilon}{d}$ with multiplicity d and ϵ with multiplicity 1,

$$H(B)_\rho = -(1-\epsilon)\log(\frac{1-\epsilon}{d}) - \epsilon\log\epsilon = H(\epsilon) + (1-\epsilon)\log d.$$

Capacity of quantum erasure channel (III)

Achievability proof (II)

$$H(B|X)_\rho = \frac{1}{d}\sum_{x=0}^{d-1} H((1-\epsilon)|x\rangle\langle x|_B + \epsilon|e\rangle\langle e|_B)$$
$$= -(1-\epsilon)\log(1-\epsilon) - \epsilon\log\epsilon = H(\epsilon),$$

since the eigenvalues are $(1-\epsilon)$ and ϵ both with multiplicity 1. Therefore,

$$I(X;B)_\rho = H(B)_\rho - H(B|X)_\rho = (1-\epsilon)\log d.$$

meaning this rate is achievable. ∎

See [Wilde, 2017] for the converse where it is proved that the regularized Holevo information can not exceed $(1-\epsilon)\log d$ an thus it is indeed the capacity.

Private classical and quantum information

Private information of a quantum channel

- We defined the **Holevo information** of a quantum channel $\mathcal{N}_{A\to B}$ as an upper bound to the **accessible information** that can be **transmitted** through the channel:

$$\chi(\mathcal{N}) \equiv \max_{\rho_{XA}} I(X;B)_\rho \geq \max_{\mathcal{E}} I_{\mathrm{acc}}(\mathcal{E}) = I^*_{\mathrm{acc}},$$

  where the **maximization** took place with respect to a classical-quantum state:

$$\rho_{XA} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho_A^x,$$

  i.e. the ensemble $\mathcal{E} = \{p_X(x), \rho_A^x\}$.
- Now we want to consider **how much** information can be **transmitted** to Bob **keeping it secret to the rest of the world**, **extending** the concept of the classical **wiretap** channel to the quantum context.

Private information of a quantum channel: definition

Private information of a quantum channel

The private information $P(\mathcal{N})$ of channel $\mathcal{N}_{A\to B}$ is defined as:

$$P(\mathcal{N}_{A\to B}) \equiv \max_{\rho_{XA}} (I(X;B)_\rho - I(X;E)_\rho),$$

for

$$\rho_{XA} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho_A^x.$$

The mutual information terms are computed with respect to:

$$\rho_{XBE} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{U}^{\mathcal{N}}_{A\to BE}(\rho_A^x),$$

where $\mathcal{U}^{\mathcal{N}}_{A\to BE}$ is an **isometric** extension of channel $\mathcal{N}_{A\to B}$.

Private information of a quantum channel: remarks

- Note that the private information of a quantum channel is **non-negative**, $P(\mathcal{N}) \geq 0$ since for $\rho_{XA} = |0\rangle\langle 0|_X \otimes |\phi\rangle\langle\phi|_A$ we have $P(\mathcal{N}) = 0$.
- Like the Holevo information, the private information is not **additive** in general, but can be additive in **some** cases.
- The **regularized** private information is then defined as

$$P_{\mathrm{reg}}(\mathcal{N}) \equiv \lim_{n\to\infty} \frac{1}{n} P(\mathcal{N}^{\otimes n}).$$

Private information of a classical channel

- The expression for the private information is reminiscent of the capacity of the (classical) **degraded wiretap** channel, i.e. where $X \leftrightarrow Y \leftrightarrow Z$ form a Markov chain,

$$C_S \equiv \max_{p_X(x)} (I(X;Y) - I(X;Z)).$$

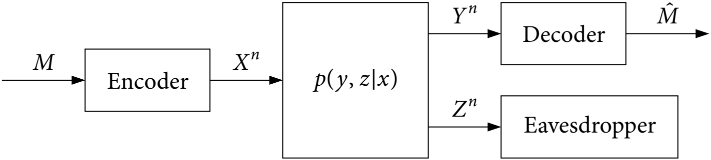


Figure: Wiretap Channel Model

- However, the **private information of a classical channel** is indeed always **additive**.

Coherent information of a quantum channel

Coherent information of a quantum channel

The **coherent information** $Q(\mathcal{N})$ of channel $\mathcal{N}_{A\to B}$ is defined as:

$$Q(\mathcal{N}_{A\to B}) \equiv \max_{\phi_{AA'}} I(A\rangle B)_\rho = -\min_{\phi_{AA'}} H(A|B)_\rho$$

where the maximization takes place with respect to all input **pure** states $\phi_{AA'}$ and

$$\rho_{AB} = \mathcal{N}_{A'\to B}(\phi_{AA'}).$$

- Note that $Q(\mathcal{N})$ is also non-negative, $Q(\mathcal{N}) \geq 0$ since for $\phi_{AA'} = \psi_A \otimes \varphi_{A'}$, we have $\rho_{AB} = \psi_A \otimes \mathcal{N}(\varphi_{A'})$ and,

$$H(A|B) = H(A) = 0.$$

Coherent information and private information

Coherent information and private information

The coherent information $Q(\mathcal{N})$ of any channel $\mathcal{N}$ is never greater than its private information $P(\mathcal{N})$,

$$Q(\mathcal{N}) \leq P(\mathcal{N}).$$

Exercise

Prove that the coherent information of any channel is never greater than its private information, $Q(\mathcal{N}) \leq P(\mathcal{N})$.

- Consider $\phi_{AA'}$ as the **pure state** that maximizes $Q(\mathcal{N})$
- Consider ϕ_{ABE} the state after sending the A' system through $\mathcal{U}^{\mathcal{N}}_{A' \to BE}$, an **isometric** extension of channel $\mathcal{N}_{A' \to B}$.
- Using the spectral decomposition

$$\phi_{A'} = \sum_x |\phi_x\rangle\langle\phi_x|_{A'}$$

create an augmented classical-quantum state of the form:

$$\sigma_{XA'} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes |\phi_x\rangle\langle\phi_x|_{A'}$$

and its extension σ_{XBE} after $\mathcal{U}^{\mathcal{N}}_{A' \to BE}$.

The Private classical capacity

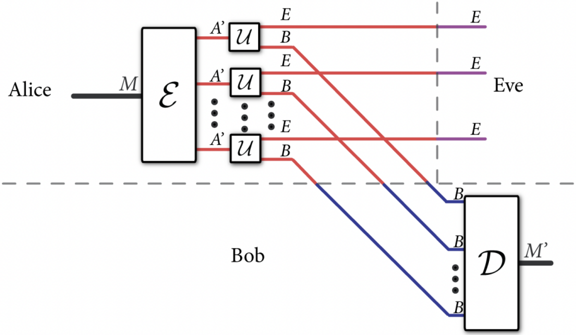


Figure: Private Capacity context

The objective remains as the **maximization** of the classical information transmission **rate** to Bob but without any information **leakage** to Eve.

Reliability and secrecy requirements

- The channel code for private communication over quantum channels must satisfy the same requirements in terms of error probability at Bob:

$$P_e(m) \equiv \Pr\{\hat{M} \neq m | M = m\} = \mathrm{tr}\{(I - \Lambda_m)\mathcal{N}^{\otimes n}(\rho^m_{A^n})\}$$

- In addition there is the secrecy requirement towards Eve. Let $\omega^m_{E^n}$ be the state observed by Eve:

$$\omega^m_{E^n} = \mathrm{tr}_B\{\mathcal{U}^{\mathcal{N}}_{A\to BE}(\rho^m_{A^n})\}$$

- Secrecy is measured by the normalized trace distance between $\omega^m_{E^n}$ and an constant (with respect to m) state σ_{E^n}:

$$\forall m \in [1 : 2^{nR}] : \frac{1}{2}\|\omega^m_{E^n} - \sigma_{E^n}\|_1 \leq \epsilon$$

The Devetak-Cai-Winter-Yeung theorem

The Devetak-Cai-Winter-Yeung theorem

The private classical capacity of a quantum channel is equal to the regularization of the private information of the channel:

$$\mathrm{C}_P(\mathcal{N}) = \lim_{k\to\infty} \frac{1}{k} P(\mathcal{N}^{\otimes k}) \equiv P_{\mathrm{reg}}(\mathcal{N}),$$

with

$$P(\mathcal{N}) \equiv \max_{\rho}(I(X;B)\sigma - I(X;E)_{\sigma}),$$

$$\rho_{XA} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho_A^x,$$

$$\sigma_{XBE} = \mathcal{U}^{\mathcal{N}}_{A\to BE}(\rho_{XA}),$$

where $\mathcal{U}^{\mathcal{N}}_{A\to BE}$ is an isometric extension of channel $\mathcal{N}_{A\to B}$.

Sketch of the achievability proof

- Alice constructs a codebook using two-index codewords (m, k), $m \in \mathcal{M}$ and $k \in \mathcal{K}$ that she communicates through n independent uses of the channel $\mathcal{N}_{A\to B}$.
- As long as the condition established by the quantum packing lemma for channel $\mathcal{N}_{A\to B}$ is satisfied, i.e., $|\mathcal{M}||\mathcal{K}| \approx 2^{nI(X;B)}$ Bob can detect both m and k with vanishing maximum error probability. This is known as the **reliability** requirement.
- Also, as long as the condition determined for the quantum covering lemma for the channel $\mathcal{N}^c_{A\to E}$, i.e., $|\mathcal{K}| \approx 2^{nI(X;E)}$ is fulfilled, then the information from the message m leaked to Eve can be made as small as desired. This is the **secrecy** requirement.

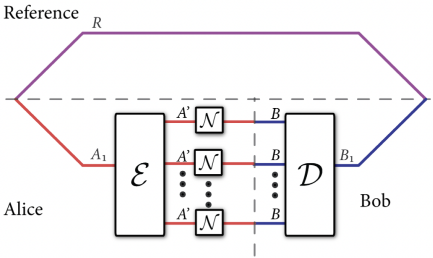


Figure: Information-processing task for entanglement transmission

The objective is that Bob performs a **decoding** of the systems he receives, and the state at the end of the protocol is **close** to the original state **shared** between Alice and the reference.

Quantum Communication requirements

- The state at Bob is:

$$\omega_{RB_1} \equiv \mathcal{D}_{B^n \to B_1}(\mathcal{N}_{A^n \to B^n}(\mathcal{E}_{A_1 \to A^n}(\varphi_{RA_1})))$$

- The encoding and decoding procedure must guarantee:

$$\frac{1}{2}\|\varphi_{RA_1} - \omega_{RB_1}\|_1 \leq \epsilon$$

- The quantum rate Q of this scheme is measured as the number of qbits transmitted per quantum channel use:

$$Q \equiv \frac{1}{n} \log \dim(\mathcal{H}_{A_1})$$

- The quantum capacity $C_Q(\mathcal{N})$ is defined as the supremum of all achievable rates for $\mathcal{N}$.

Quantum capacity theorem

Quantum capacity theorem

The quantum capacity $C_Q(\mathcal{N})$ of a quantum channel is equal to the regularized coherent information of the channel:

$$\mathrm{C}_Q(\mathcal{N}) = \lim_{k\to\infty} \frac{1}{k} Q(\mathcal{N}^{\otimes k}) \equiv Q_{\mathrm{reg}}(\mathcal{N})$$

with

$$Q(\mathcal{N}) \equiv \max_{\phi} I(A\rangle B)_{\sigma}$$

where the maximization is with respect to all pure states $\phi_{AA'}$ and $\sigma_{AB} = \mathcal{N}_{A'\to B}(\phi_{AA'})$

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