The goal of this exercise is to apply the theory of quantum state identification to determine whether two quantum states are identical.

Q1. You are given two quantum systems, each independently prepared in one of two possible pure states, $|\phi_1\rangle$ and $|\phi_2\rangle$, with the same prior probability $\eta = 1/2$. These states are known and can be assumed to be qubit states with a real overlap $\langle\phi_1|\phi_2\rangle = c$. Your objective is to determine if both systems are in the same state, regardless of which specific state is. To achieve this, consider maximizing the probability of a successful identification (minimum error approach) assuming that you can perform global measurements on the combined system. Show that this problem can be cast as a discrimination task between two mixed states, $\rho_=$ and $\rho_{\neq}$, which correspond to the cases where the systems are in the same or different states, respectively. Derive the optimal success probability in terms of the overlap c.

Q2. Now, assume that only local measurements (measuring each system independently) are allowed. Here, your strategy is to identify the state in which each individual system has been prepared, using a minimum error approach. Determine the maximum success probability in this setup and compare it to the probability found in **Q1**. Discuss your results.

Q3. Next, explore the possibility of unambiguously identifying whether the two systems are in the same quantum state. Note that this requires distinguishing between the two mixed states, $\rho_=$ and $\rho_{\neq}$, in an unambiguous manner, which, as discussed in theory, may not always be feasible.

a) Begin by expressing the states $\rho_=$ and $\rho_{\neq}$ in terms of the following unnormalized states

$$\begin{aligned} |\varphi_1\rangle &= |\phi_1\phi_1\rangle + |\phi_2\phi_2\rangle \ , \\ |\varphi_2\rangle &= |\phi_1\phi_2\rangle + |\phi_2\phi_1\rangle \ , \\ |\varphi_3\rangle &= |\phi_1\phi_1\rangle - |\phi_2\phi_2\rangle \ , \\ |\varphi_4\rangle &= |\phi_1\phi_2\rangle - |\phi_2\phi_1\rangle \ . \end{aligned} \tag{1}$$

b) By examining the orthogonality relations among the states $\{|\varphi_i\rangle\}$, argue that the problem can be reduced to the task of unambiguously discriminating between $|\varphi_1\rangle$ and $|\varphi_2\rangle$, and find the optimal success probability in terms of the overlap.

c) Compare this approach with the strategy of unambiguously identifying the state of each system independently.

Q4. Returning to the minimum error approach and allowing global measurements, consider that each system is now prepared in one of $N \geq 2$ possible states $\{|\psi_k\rangle\}_{k=0}^{N-1}$, where

$|\psi_k\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + \omega^k |1\rangle\right)$ and $\omega = e^{i\frac{2\pi}{N}}$. Your task remains to certify whether the two systems are in the same state. Following similar steps as in **Q1**, derive the optimal success probability as a function of N, and discuss your results.

(*Hint: It may be useful to recall that for* $r \neq 1$, $\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r}$)