Quantum Information Theory —Assignment 1

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Question 1

Here we consider the properties of the density matrix ρ defined by:

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad \text{with} \quad \sum_i p_i = 1$$

where $\{p_i, |\psi_i\rangle\}$ is an ensemble of quantum states and $0 \le p_i \le 1$ are probabilities.
We begin by recalling that ρ is Hermitian, since:

$$\rho^\dagger = \sum_i (p_i |\psi_i\rangle\langle\psi_i|)^\dagger = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho$$

and that ρ has unit trace since:

$$\operatorname{Tr}\rho = \sum_i p_i \operatorname{Tr}|\psi_i\rangle\langle\psi_i| = \sum_i p_i \operatorname{Tr}\langle\psi_i|\psi_i\rangle = \sum_i p_i = 1$$

and finally that ρ is positive semi-definite, and with eigenvalues $\lambda_i \ge 0$ since, for arbitrary $|\chi\rangle$:

$$\langle\chi|\,\rho\,|\chi\rangle = \sum_i p_i \langle\chi|\,|\psi_i\rangle\langle\psi_i|\,|\chi\rangle = \sum_i p_i |\langle\chi|\psi_i\rangle|^2 \ge 0$$

Part (i)

If $\rho = |\psi_1\rangle\langle\psi_1|$ represents a pure state then:

$$\operatorname{Tr}\rho^2 = \operatorname{Tr}(|\psi_1\rangle\langle\psi_1|\,|\psi_1\rangle\langle\psi_1|) = \operatorname{Tr}(\langle\psi_1|\psi_1\rangle\,\langle\psi_1|\psi_1\rangle) = 1$$

Otherwise, ρ, being positive semi-definite, may be diagonalized by a change of basis and written in the form:

$$\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| \quad \text{where} \quad \langle\lambda_i|\lambda_j\rangle = \delta_{ij} \quad \text{and} \quad \lambda_i \ge 0$$

This operation preserves the trace of the matrix, so:

$$\sum_i \lambda_i = \operatorname{Tr}\rho = 1 \qquad \text{and so each eigenvalue satisfies} \qquad 0 \le \lambda_i \le 1 \quad \Rightarrow \quad \lambda_i^2 \le \lambda_i$$

In this basis where ρ is diagonal it's easy to compute ρ^2[1] and:

$$\operatorname{Tr}\rho^2 = \sum_i \lambda_i^2 \le \sum_i \lambda_i = 1 \tag{1}$$

The inequality is saturated only when there is exactly one nonzero eigenvalue $\lambda_1 = 1$, that is, when $\rho = |\lambda_1\rangle\langle\lambda_1|$ represents a pure state.

[1] $\operatorname{Tr}\rho^2 = \operatorname{Tr}\left(\sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|\right) = \sum_{ij} \lambda_i\lambda_j \operatorname{Tr}(|\lambda_i\rangle\langle\lambda_i|\,|\lambda_j\rangle\langle\lambda_j|) = \sum_{ij} \lambda_i\lambda_j \operatorname{Tr}(\langle\lambda_j|\lambda_i\rangle\,\langle\lambda_i|\lambda_j\rangle) = \sum_i \lambda_i^2$

Parts (ii) and (iii)

We're asked to show that if ρ is a 2×2 matrix, then it can be written in the form:

$$\rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma}) \tag{2}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector containing the Pauli matrices and $\boldsymbol{r} = (r_x, r_y, r_z)$ is a vector of coefficients.

The Pauli matrices—together with the identity—are, by construction, a basis for 2×2 Hermitian matrices, and we may therefore expand ρ in this basis, choosing the coefficients of $\mathbb{1}$ and σ_w so that:

$$\rho = \frac{1}{2}(r_0 \mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma})$$

It remains to show that $r_0 = 1$ and that $\|\boldsymbol{r}\| \leq 1$.

The Pauli matrices are traceless and so, as required:

$$1 = \operatorname{Tr} \rho = \frac{1}{2}\left(r_0 \operatorname{Tr} \mathbb{1} + \sum_w r_w \operatorname{Tr} \sigma_w \right) = \frac{1}{2}(2r_0 + 0) = r_0$$

To find the norm $\|\boldsymbol{r}\|$ of $\boldsymbol{r}$, we now calculate:

$$\begin{aligned}
4 \operatorname{Tr} \rho^2 &= \operatorname{Tr}\left[(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma})(\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma})\right] \\
&= \operatorname{Tr} \mathbb{1} + 2 \operatorname{Tr}(\boldsymbol{r} \cdot \boldsymbol{\sigma}) + \operatorname{Tr}(\boldsymbol{r} \cdot \boldsymbol{\sigma}\, \boldsymbol{r} \cdot \boldsymbol{\sigma}) \\
&= \operatorname{Tr} \mathbb{1} + 2 \sum_w r_w \operatorname{Tr} \sigma_w + \sum_{w,v} r_w r_v \operatorname{Tr}(\sigma_w \sigma_v) \\
&= \operatorname{Tr} \mathbb{1} + 2 \sum_w r_w \operatorname{Tr} \sigma_w + \sum_{w,v} r_w r_v (\delta_{wv} + \epsilon_{wvu} \operatorname{Tr} \sigma_u) \\
&= 2 + 2 \sum_w r_w^2
\end{aligned}$$

and so, making use of the bound (1) on $\operatorname{Tr} \rho^2$ we derived above:

$$\|\boldsymbol{r}\|^2 = \sum_w r_w^2 = 2 \operatorname{Tr} \rho^2 - 1 \begin{cases} = 1 & \text{if } \rho \text{ represents a pure state} \\ < 1 & \text{otherwise} \end{cases}$$

Alternative approach to (ii) and (iii)

Another way to see this is to consider an ensemble of generalized states of form:

$$|\psi_i\rangle = \begin{bmatrix} \cos(\theta_i/2) \\ e^{i\phi_i} \sin(\theta_i/2) \end{bmatrix}$$

The corresponding density matrix is given by:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i \frac{p_i}{2} \begin{bmatrix} \cos^2(\theta_i/2) & e^{-i\phi_i} \sin(\theta_i/2)\cos(\theta_i/2) \\ e^{i\phi_i} \sin(\theta_i/2)\cos(\theta_i/2) & \sin^2(\theta_i/2) \end{bmatrix}$$

$$= \sum_i \frac{p_i}{2} \begin{bmatrix} 1 + \cos\theta_i & \sin\theta_i \cos\phi_i - i \sin\theta_i \sin\phi_i \\ \sin\theta_i \cos\phi_i + i \sin\theta_i \sin\phi_i & 1 - \cos\theta_i \end{bmatrix}$$

$$= \sum_i p_i \frac{\mathbb{1} + \boldsymbol{n_i}\boldsymbol{\sigma}}{2} = \frac{\mathbb{1} + \left(\sum_i p_i \boldsymbol{n_i}\right)\boldsymbol{\sigma}}{2} = \frac{\mathbb{1} + \boldsymbol{n}\boldsymbol{\sigma}}{2}$$

where $\boldsymbol{n_i} = (\sin\theta_i \cos\phi_i,\ \sin\theta_i \sin\phi_i,\ \cos\theta_i)$ is a unit vector.

The weighted sum of unit vectors $\boldsymbol{n} = \sum_i p_i \boldsymbol{n_i}$ with $\sum_i p_i = 1$ is a vector with $\|\boldsymbol{n}\| \leq 1$, where the equality occurs only when that vectors are parallel, that is, for a pure state.

Part (iv)

Here we show that every orthogonal rotation of the Bloch sphere corresponds to a rotation of the density matrix, and that any such transformation may be implemented as a unitary operator on qubits. We may label this operator $U_{\boldsymbol{r}}(\varphi)$ where $\boldsymbol{r}$ is a 3-vector and φ is an angle. This mapping amounts to a homomorphism from $SU(2)$ to $SO(3)$, since unitary operations on qubits belong to the defining representation of $SU(2)$, but rotations of the Bloch sphere must belong to a representation of $SO(3)$.

Representation of qubit states on the Bloch sphere, and relation to density matrix

We first consider a rotation parameterized by angles θ, ϕ, representing rotations about the y and z axes respectively. Such a rotation takes the vector $|0\rangle$ at the north pole of the Bloch sphere to the vector:

$$|\theta, \phi\rangle = \cos\theta\,|0\rangle + e^{i\phi}\sin\theta\,|1\rangle \tag{3}$$

Here we pause to note an important ambiguity—the vector associated with a location on the Bloch sphere is unique only up to the overall phase of the vector. In (3) we have made a conventional choice of phase. If we were to make a different choice of global phase $e^{i\alpha}$, we would obtain the same density matrix, since: $e^{i\alpha}\,|\theta,\phi\rangle\,\left(e^{i\alpha}\right)^{\star}\,\langle\theta,\phi| = |\theta,\phi\rangle\langle\theta,\phi|$. We'll now forget this global phase, which for a single qubit has no physical significance anyway.

The density matrix $\rho_{\theta,\phi}$ corresponding to the pure state $|\theta,\phi\rangle$ is:

$$\begin{aligned}
\rho_{\theta,\phi} = |\theta,\phi\rangle\langle\theta,\phi| &= \Big(\cos\theta\,|0\rangle + e^{i\phi}\sin\theta\,|1\rangle\Big)\Big(\cos\theta\,\langle 0| + e^{-i\phi}\sin\theta\,\langle 1|\Big) \\
&= \cos^2\theta\,|0\rangle\langle 0| + \sin^2\theta\,|1\rangle\langle 1| + e^{i\theta}\cos\theta\sin\theta\,|1\rangle\langle 0| + e^{-i\theta}\cos\theta\sin\theta\,|0\rangle\langle 1| \\
&= \tfrac{1}{2}\cos^2\theta(\mathbb{1}+\sigma_z) + \tfrac{1}{2}\sin^2\theta(\mathbb{1}-\sigma_z) + \cos\theta\sin\theta\cos\phi\,\sigma_x + \cos\theta\sin\theta\sin\phi\,\sigma_y \\
&= \tfrac{1}{2}(\mathbb{1} + \cos 2\theta\,\sigma_z + \sin 2\theta\cos\phi\,\sigma_x + \sin 2\theta\sin\phi\,\sigma_y)
\end{aligned}$$

By comparison with (2), we've now found an explicit formula for $\boldsymbol{r}$, for *every* state on the Bloch sphere:

$$\boldsymbol{r}(\theta,\phi) = (\sin 2\theta\cos\phi, \sin 2\theta\sin\phi, \cos 2\theta)$$

It's easy to verify that $\boldsymbol{r}$ is a unit 3-vector with $\|\boldsymbol{r}\| = 1$.

This formula associates qubit states $|\theta,\phi\rangle$, which transform in $SU(2)$, with positions $\boldsymbol{r}$ on the unit sphere in $\mathbb{R}^3$, which transform in $SO(3)$. It is a *double cover*, because the function $\boldsymbol{r}(\theta,\phi)$ has periodicity π in θ, but equation (3) has periodicity 2π. That is, each $\boldsymbol{r}$ is associated with two distinct values of θ.

We observe that the density matrix for a pure state is completely determined by $\boldsymbol{r}$. It therefore inherits the geometry and periodicity of $\boldsymbol{r}$. The density matrix for a mixed state is simply a linear combination of such objects, and has the same geometry. That is, ρ must transform in a representation of $SO(3)$.

Homomorphism between $SU(2)$ and $SO(3)$

We now wish to make the mapping between orthogonal rotations and unitary operations on qubits explicit, by building a representation of each orthogonal rotation $U_{\boldsymbol{r}}(\varphi)$ of angle φ about an arbitrary axis $\boldsymbol{r}$ from elements of $SU(2)$.

We first determine a matrix which implements the mapping $|0\rangle \mapsto |\theta,\phi\rangle$, and we demand that this matrix be unitary, with unit determinant, as befits a member of the group $SU(2)$. It's easy to verify that the following matrix fits the requirements, and less easy to verify that it's unique:

$$U_{\theta,\phi} = \begin{bmatrix} \cos\theta & -e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & \cos\theta \end{bmatrix} = \cos\theta\,\mathbb{1} + i\sin\phi\sin\theta\,\sigma_x - i\cos\phi\sin\theta\,\sigma_y$$

We will also need a matrix $U_{\hat{\boldsymbol{z}}}(\varphi)$ which implements a rotation about the z axis. The eigenvectors of this matrix are $|0\rangle$ and $|1\rangle$, and so the matrix is diagonal in the computational basis. Again demanding that the matrix be unitary with unit determinant, we obtain:

$$U_{\hat{\boldsymbol{z}}}(\varphi) \equiv e^{-i\varphi/2}\,|0\rangle\langle 0| + e^{i\varphi/2}\,|1\rangle\langle 1| = \cos\tfrac{\varphi}{2}\,\mathbb{1} - i\sin\tfrac{\varphi}{2}\,\sigma_z$$

It's now easy to construct $U_{\boldsymbol{r}}(\varphi)$ explicitly.

$$U_{\boldsymbol{r}(\theta,\phi)}(\varphi) = U_{\theta,\phi}\, U_{\hat{\boldsymbol{z}}}(\varphi)\, U^\dagger_{\theta,\phi}$$

This operator, being a product of unitaries, is itself clearly unitary.

It acts on the density matrix according to:

$$\begin{aligned}
\rho = \sum_i p_i \,|\psi_i\rangle\langle\psi_i| \mapsto \rho' &= \sum_i p_i\, |\psi_i'\rangle\langle\psi_i'| \\
&= \sum_i p_i \Big(U_{\boldsymbol{r}}(\varphi)\, |\psi_i\rangle\langle\psi_i|\, U^\dagger_{\boldsymbol{r}}(\varphi) \Big) \\
&= U_{\boldsymbol{r}}(\varphi) \Big(\sum_i p_i\, |\psi_i\rangle\langle\psi_i| \Big) U^\dagger_{\boldsymbol{r}}(\varphi) \\
&= U_{\boldsymbol{r}}(\varphi)\, \rho\, U^\dagger_{\boldsymbol{r}}(\varphi)
\end{aligned}$$

It's important to clarify that our $U_{\boldsymbol{r}}(\varphi)$ is manifestly an element of the fundamental representation of $SU(2)$, but that what we've shown is that it's *also* an element of a representation of $SO(3)$. In fact, every element of $SO(3)$ can be expressed via some choice of $\boldsymbol{r}$ and φ.

Thus, we have a homomorphism from $SU(2)$ to $SO(3)$. If we identify qubit states by the equivalence relation $|\theta,\phi\rangle \sim |\theta+\pi,\phi\rangle$, we obtain the quotient group $SU(3)/(\mathbb{1},-\mathbb{1})$ which is isomorphic to $SO(3)$.

Part (v)

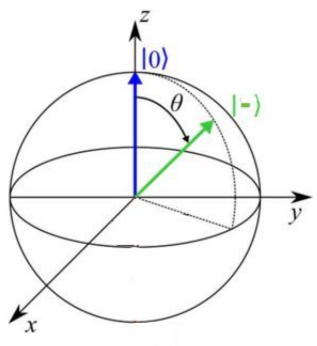
The state $|-\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ is located on the negative x axis of the Bloch sphere, since:

$$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \cos\tfrac{\pi}{4}\, |0\rangle + \sin\tfrac{\pi}{4}\, e^{i\pi}\, |1\rangle \quad \Rightarrow \quad \theta = \tfrac{\pi}{4}, \phi = \pi$$

according to (3). Therefore, the rotation matrix:

$$U_{\frac{\pi}{4},\pi} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

maps $|0\rangle \mapsto |-\rangle$.



Alternatively, to sanity-test the construction above, we may use our $U_{\boldsymbol{r}}$. To transform $|0\rangle \mapsto |-\rangle$ we must perform a rotation of $\varphi = -\frac{\pi}{2}$ about the y axis, which we identify with the axis:

$$\boldsymbol{r} = (0,1,0) \qquad \text{or} \qquad \theta = \tfrac{\pi}{4}, \phi = \tfrac{\pi}{2}$$

in the coordinate system we've used above. With these angles locked in, we obtain:

$$U_{\frac{\pi}{4},\frac{\pi}{2}} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \qquad U_{\hat{\boldsymbol{z}}}(-\tfrac{\pi}{2}) = \begin{bmatrix} e^{i\pi/4} & \\ & e^{-i\pi/4} \end{bmatrix} \qquad \Rightarrow \quad U_{(0,1,0)}(\tfrac{\pi}{2}) = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

which is the same matrix obtained previously. That is, we've implemented a rotation around y as a rotation about x, followed by a rotation about z, followed by the reverse rotation around x.

Question 2

Via Schmidt decomposition, any state $|\psi\rangle_{AB}$ of a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$ may be written in the form:

$$|\psi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |e_i\rangle_A\, |f_i\rangle_B \quad \text{where} \quad d \equiv \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B), \quad \sum_i \lambda_i = 1, \quad \lambda_i \geq 0 \tag{4}$$

Here $\{|e_i\rangle_A\}$ is an orthonormal basis for $\mathcal{H}_A$ and $\{|f_i\rangle_B\}$ is an orthonormal basis for $\mathcal{H}_B$.

Part (i)

It's clear that for this state to be *maximally* entangled, the λ_i should be equal, since that's the configuration which maximises entropy, and therefore:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |e_i\rangle_A\, |f_i\rangle_B$$

We now recall that every d-dimensional Hilbert space is isomorphic to $\mathbb{C}^d$, so without loss of generality we may take $\mathcal{H}_A = \mathbb{C}^{d_A}$ and $\mathcal{H}_B = \mathbb{C}^{d_B}$.

Now, either $\mathcal{H}_A$ is a subspace of $\mathcal{H}_B$, or vice versa, and so there exists a unitary change of basis U which aligns the first d basis elements $|e_i\rangle_A$ with the first d basis elements $|f_i\rangle_B$, so that $|f_i\rangle = U\,|e_i\rangle$. Then:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |e_i\rangle \otimes U\,|e_i\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} (\mathbb{1} \otimes U)(|e_i\rangle \otimes |e_i\rangle) \equiv (\mathbb{1} \otimes U) \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle\,|i\rangle$$

where at the last equality we've relabelled $|e_i\rangle$ to $|i\rangle$. We note that $\{|i\rangle\}_{i=1}^{d}$ is an orthonormal basis of $\mathbb{C}^d$.

Part (ii)

Here we again make use of the Schmidt decomposition (4) of the pure state $|\psi\rangle_{AB}$.

The density matrix corresponding to $|\psi\rangle_{AB}$ is:

$$\rho_{AB} = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |e_i\rangle\, |f_i\rangle \sum_{j=1}^{d} \sqrt{\lambda_j}\, \langle e_j|\, \langle f_j| = \sum_{i,j=1}^{d} \sqrt{\lambda_i \lambda_j}\, |e_i\rangle\, |f_i\rangle\, \langle e_j|\, \langle f_j|$$

We now take the partial trace over A:

$$\begin{aligned}
\rho_B = \mathrm{Tr}_A\, \rho_{AB} = \sum_k \langle e_k|\, \rho_{AB}\, |e_k\rangle &= \sum_k \sum_{i,j=1}^{d} \sqrt{\lambda_i \lambda_j} ((\langle e_k| \otimes \mathbb{1})(|e_i\rangle\, |f_i\rangle\, \langle e_j|\, \langle f_j|)(|e_k\rangle \otimes \mathbb{1}) \\
&= \sum_k \sum_{i,j=1}^{d} \sqrt{\lambda_i \lambda_j}\, \langle e_k|e_i\rangle\, |f_i\rangle\, \langle f_j|\, \langle e_j|e_k\rangle \\
&= \sum_k \sum_{i,j=1}^{d} \sqrt{\lambda_i \lambda_j}\, \delta ik\, |f_i\rangle\, \langle f_j|\, \delta_{jk} \\
&= \sum_{k=1}^{d} \lambda_k\, |f_k\rangle\langle f_k|
\end{aligned}$$

Similarly, $\rho_A = \mathrm{Tr}_B\, \rho_{AB} = \sum_{k=1}^{d} \lambda_k\, |e_k\rangle\langle e_k|$.

We see that in the basis of the Schmidt decomposition, the elements of the d-dimensional upper left blocks of the reduced density matrices are identical—and diagonal. The larger reduced density matrix is padded with zeros.

That is to say, we may always choose a basis where the reduced density matrices ρ_A and ρ_B have the same elements, up to padding rows/columns of zeros, and the reduced density matrices are diagonal in this basis.

Question 3

We're given the single-qubit state $|\psi\rangle_1$ and the bipartite entangled state $|\Phi^+\rangle_{23}$ where:

$$|\psi\rangle_1 = \alpha\,|0\rangle_1 + \beta\,|1\rangle_1 \qquad\qquad |\Phi^+\rangle_{23} = \tfrac{1}{\sqrt{2}}(|00\rangle_{23} + |11\rangle_{23})$$

We assume that Bob has never had access to the qubit $|\psi\rangle$, which is in the possession of Alice, and that Alice would like to transmit it to Bob. They will use the entangled state $|\Phi^+\rangle$ as a resource.

Part (i)

The composite state is:

$$\begin{aligned}|\Psi\rangle = |\psi\rangle_1 \otimes |\Phi^+\rangle_{23} &= (\alpha\,|0\rangle_1 + \beta\,|1\rangle_1) \otimes \tfrac{1}{\sqrt{2}}(|00\rangle_{23} + |11\rangle_{23}) \\ &= \tfrac{1}{\sqrt{2}}\big(\alpha\,|000\rangle_{123} + \alpha\,|011\rangle_{123} + \beta\,|100\rangle_{123} + \beta\,|111\rangle_{123}\big)\end{aligned}$$

We now assign the first two qubits to Alice, and the third to Bob. Bob will reconstruct the state of ψ in the third qubit. We may now rewrite Alice's qubits in the Bell basis, leading it:

$$\begin{aligned}|\Psi\rangle = &\tfrac{1}{2}\,|\Phi^+\rangle_{12}\,(\alpha\,|0\rangle_3 + \beta\,|1\rangle_3) \\ &+ \tfrac{1}{2}\,|\Phi^-\rangle_{12}\,(\alpha\,|0\rangle_3 - \beta\,|1\rangle_3) \\ &+ \tfrac{1}{2}\,|\Psi^+\rangle_{12}\,(\beta\,|0\rangle_3 + \alpha\,|1\rangle_3) \\ &+ \tfrac{1}{2}\,|\Psi^-\rangle_{12}\,(-\beta\,|0\rangle_3 + \alpha\,|1\rangle_3)\end{aligned} \tag{5}$$

Part (ii)

The Bell basis for the four-dimensional Hilbert space of the two qubits belonging to Alice is:

$$|\Phi^\pm\rangle_{12} = \tfrac{1}{\sqrt{2}}(|00\rangle_{12} \pm |11\rangle_{12}) \qquad\qquad |\Psi^\pm\rangle_{12} = \tfrac{1}{\sqrt{2}}(|01\rangle_{12} \pm |10\rangle_{12})$$

If Alice performs a measurement in this basis, there are four possible outcomes, which we may find using the projection operators $\Pi_{\Phi^\pm}, \Pi_{\Psi^\pm}$ onto the corresponding basis states:

$$p_{\Phi^\pm} = \langle\Psi|\,\Pi_{\Phi^\pm}\,|\Psi\rangle = \big|\big(\langle\Phi^\pm|_{12} \otimes \mathbb{1}_3\big)\,|\Psi\rangle\big|^2 = \tfrac{1}{4}|\alpha\,|0\rangle_3 \pm \beta\,|1\rangle_3|^2 = \tfrac{1}{4}\Big(|\alpha|^2 + |\beta|^2\Big) = \tfrac{1}{4}$$

Similarly:

$$p_{\Psi^\pm} = \langle\Psi|\,\Pi_{\Psi^\pm}\,|\Psi\rangle = \big|\big(\langle\Psi^\pm|_{12} \otimes \mathbb{1}_3\big)\,|\Psi\rangle\big|^2 = \tfrac{1}{4}|\alpha\,|1\rangle_3 \pm \beta\,|0\rangle_3|^2 = \tfrac{1}{4}\Big(|\alpha|^2 + |\beta|^2\Big) = \tfrac{1}{4}$$

Thus, the outcomes have equal probability $p_{\Psi^-} = p_{\Psi^+} = p_{\Phi^-} = p_{\Phi^+} = \frac{1}{4}$. The post-measurement states are:

$$(p_{\Phi^\pm})^{-\frac{1}{2}}\,\Pi_{\Phi^\pm}\,|\Psi\rangle = (p_{\Phi^\pm})^{-\frac{1}{2}}\,\big(|\Phi^\pm_{12}\rangle \otimes \mathbb{1}_3\big)\big(\langle\Phi^\pm_{12}| \otimes \mathbb{1}_3\big)\,|\Psi\rangle = |\Phi^\pm\rangle_{12}\,(\alpha\,|0\rangle_3 \pm \beta\,|1\rangle_3)$$

and:

$$(p_{\Psi^\pm})^{-\frac{1}{2}}\,\Pi_{\Psi^\pm}\,|\Psi\rangle = (p_{\Psi^\pm})^{-\frac{1}{2}}\,\big(|\Psi^\pm_{12}\rangle \otimes \mathbb{1}_3\big)\big(\langle\Psi^\pm_{12}| \otimes \mathbb{1}_3\big)\,|\Psi\rangle = |\Psi^\pm\rangle_{12}\,(\alpha\,|1\rangle_3 \pm \beta\,|0\rangle_3)$$

Of course, all this was already obvious from (5).

Part (iii)

If Alice performs a measurement in the Bell basis, but does not communicate the result to Bob, then Bob's description of the system must be a mixed state. From Bob's point of view, then, the mixed state is an ensemble of four equally-likely states:

$$\big\{|\Phi^\pm\rangle_{12}\,(\alpha\,|0\rangle_3 \pm \beta\,|1\rangle_3),\, |\Psi^\pm\rangle_{12}\,(\alpha\,|1\rangle_3 \pm \beta\,|0\rangle_3)\big\}$$

That is, the density matrix is:

$$\begin{aligned}\rho_{123} = \tfrac{1}{4} \cdot \left|\Phi^+\middle\rangle\!\middle\langle\Phi^+\right|_{12} \left(\alpha\,|0\rangle_3 + \beta\,|1\rangle_3\right)\left(\alpha^\star\,\langle 0|_3 + \beta^\star\,\langle 1|_3\right)\\ + \tfrac{1}{4} \cdot \left|\Phi^-\middle\rangle\!\middle\langle\Phi^-\right|_{12} \left(\alpha\,|0\rangle_3 - \beta\,|1\rangle_3\right)\left(\alpha^\star\,\langle 0|_3 - \beta^\star\,\langle 1|_3\right)\\ + \tfrac{1}{4} \cdot \left|\Psi^+\middle\rangle\!\middle\langle\Psi^+\right|_{12} \left(\beta\,|0\rangle_3 + \alpha\,|1\rangle_3\right)\left(\beta^\star\,\langle 0|_3 + \alpha^\star\,\langle 1|_3\right)\\ + \tfrac{1}{4} \cdot \left|\Psi^-\middle\rangle\!\middle\langle\Psi^-\right|_{12} \left(-\beta\,|0\rangle_3 + \alpha\,|1\rangle_3\right)\left(-\beta^\star\,\langle 0|_3 + \alpha^\star\,\langle 1|_3\right)\end{aligned}$$

The state of Bob's qubit is described by his reduced density matrix ρ_3, that is, by the partial trace of the full density matrix:

$$\rho_3 = \mathrm{Tr}_{12}\,\rho_{123} = \tfrac{1}{2}\left(|\alpha|^2 + |\beta|^2\right)\left(|0\rangle\!\langle 0|_3 + |1\rangle\!\langle 1|_3\right) = \tfrac{1}{2}\mathbb{1}_3$$

where $\mathbb{1}_3$ is the 2×2 identity matrix corresponding to Bob's qubit.

This result—a maximally mixed state—is sensible, because Bob has essentially no information about the state held by Alice. Note that it's exactly the same result we would obtain by calculating $\mathrm{Tr}_{12}\,|\Psi\rangle\!\langle\Psi|$, the reduced density matrix for the pure state *before* Alice's measurement.

Part (iv)

Bob would like to recover the state $|\psi\rangle$ in his qubit. The operation he must apply to his qubit depends on the state obtained by Alice:

Measurement result (Alice)	State (Bob)	Operation (Bob)
Φ^+	$\frac{1}{2}\left(\alpha\,|0\rangle_3 + \beta\,|1\rangle_3\right)$	$\mathbb{1}$
Φ^-	$\frac{1}{2}\left(\alpha\,|0\rangle_3 - \beta\,|1\rangle_3\right)$	Z
Ψ^+	$\frac{1}{2}\left(\beta\,|0\rangle_3 + \alpha\,|1\rangle_3\right)$	X
Ψ^-	$\frac{1}{2}\left(-\beta\,|0\rangle_3 + \alpha\,|1\rangle_3\right)$	$ZX = iY$

Part (v)

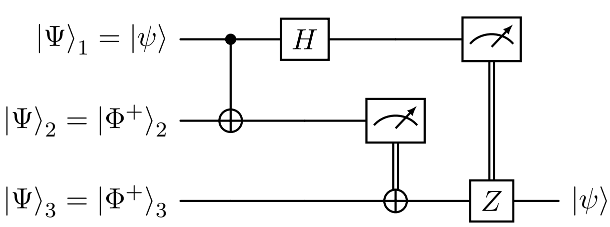
Alice must send Bob the result of her measurement, which requires $\log_2 4 = 2$ classical bits. It's impossible for Alice to completely specify the state $|\psi\rangle$ using classical communication, since it requires a countably-infinite number of binary digits to completely specify a complex coefficient α or β.

On the other hand, the quantum teleportation protocol described here can't be implemented ideally with perfect precision either, and so, taking into account these errors, it *would* be possible to achieve equivalent precision with a finite number of classical bits. Furthermore, the teleportation protocol consumes an ebit which—at least for the foreseeable future—is a *much* more expensive resource than the resources required to transmit some classical bits.

But this analysis completely misses the really interesting thing about quantum teleportation: the reason it's difficult to "teleport" a quantum state classically is not that it's costly to transmit classical bits; it's that it's impossible to obtain a classical representation of the quantum state without performing many destructive measurements of a large number of identically-prepared qubits.

Part (vi)

To implement this protocol as a circuit we need to rotate from the Bell basis to the computational basis before measuring in the computational basis. The gates which perform this basis change are $(H \otimes \mathbb{1}) \cdot \mathrm{CNOT}$.



Question 4

We start with the entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, to which Alice applies the operations $\mathbb{1}$, σ_x, σ_y, and $\sigma_x\sigma_y = i\sigma_z$.

Operation	State	Density matrix
$\mathbb{1}$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle) = \|\Phi^+\rangle$	$\frac{1}{2}(\|00\rangle\langle 00\| + \|11\rangle\langle 11\| + \|00\rangle\langle 11\| + \|11\rangle\langle 00\|)$
σ_x	$\frac{1}{\sqrt{2}}(\|10\rangle + \|01\rangle) = \|\Psi^+\rangle$	$\frac{1}{2}(\|10\rangle\langle 01\| + \|01\rangle\langle 10\| + \|10\rangle\langle 10\| + \|01\rangle\langle 01\|)$
σ_y	$\frac{i}{\sqrt{2}}(\|10\rangle - \|01\rangle) = i\,\|\Psi^-\rangle$	$\frac{1}{2}(\|10\rangle\langle 01\| + \|01\rangle\langle 10\| - \|10\rangle\langle 10\| - \|01\rangle\langle 01\|)$
$\sigma_x\sigma_y = i\sigma_z$	$\frac{i}{\sqrt{2}}(\|00\rangle - \|11\rangle) = i\,\|\Phi^-\rangle$	$\frac{1}{2}(\|00\rangle\langle 00\| + \|11\rangle\langle 11\| - \|00\rangle\langle 11\| - \|11\rangle\langle 00\|)$

We now obtain the reduced density matrices for each of the resulting pure states in the table above. In every case, we obtain the same result whether we take the partial trace over A or B.

Operation	State	Reduced density matrix
$\mathbb{1}$	$\|\Phi^+\rangle$	$\frac{1}{2}(\|0\rangle\langle 0\| + \|1\rangle\langle 1\|) = \frac{1}{2}\mathbb{1}$
σ_x	$\|\Psi^+\rangle$	$\frac{1}{2}(\|0\rangle\langle 0\| + \|1\rangle\langle 1\|) = \frac{1}{2}\mathbb{1}$
σ_y	$i\,\|\Psi^-\rangle$	$\frac{1}{2}(\|0\rangle\langle 0\| + \|1\rangle\langle 1\|) = \frac{1}{2}\mathbb{1}$
$\sigma_x\sigma_y = i\sigma_z$	$i\,\|\Phi^-\rangle$	$\frac{1}{2}(\|0\rangle\langle 0\| + \|1\rangle\langle 1\|) = \frac{1}{2}\mathbb{1}$

Here we observe that every one of the Bell states leads to the same reduced density matrix $\frac{1}{2}\mathbb{1}$. So any one of the *four* Bell states could be chosen as a purification of the mixed state represented by that reduced density matrix.

Question 5

Here we're given the definition that an observable is any linear combination $\sum_i a_i \Pi_i$ with real coefficients a_i of orthogonal projection matrices Π_i which sum to the identity:

$$\sum_i \Pi_i = \mathbb{1} \qquad \Pi_i \Pi_j = \delta_{ij}\Pi_i \qquad \Pi_i^2 = \Pi_i \qquad \Pi_i^\dagger = \Pi_i$$

We note that since projection matrices are Hermitian, observables are also Hermitian, by this definition.

Part (i)

We're given the matrix:

$$\Pi = \frac{1}{5}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By inspection, we notice that:

$$\Pi = |\psi\rangle\langle\psi| \quad \text{where} \quad |\psi\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Part (ii)

We're now presented the matrices:

$$B = \frac{1}{2}\begin{bmatrix} -1 & 0 & -3i \\ 0 & 2 & 0 \\ 3i & 0 & -1 \end{bmatrix} \qquad C = \frac{1}{6}\begin{bmatrix} 1 & 2 & i \\ 2 & 4 & 2i \\ i & -2i & 1 \end{bmatrix}$$

and we're asked which matrix represents an observable. The matrix B is Hermitian, and C is not, so B is the observable. The eigenvectors of B are:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} \qquad |\psi_3\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

with eigenvalues $a_1 = -2$ and $a_{2,3} = 1$.

Part (iii)

The projection matrix Π_1 corresponding to the first eigenvalue is simply:

$$\Pi_1 = |\psi_1\rangle\langle\psi_1| = \frac{1}{2}\begin{bmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 1 \end{bmatrix}$$

and the projection matrix $\Pi_{2,3}$ corresponding to the second eigenvalue is:

$$\Pi_{2,3} = |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| = \frac{1}{2}\begin{bmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 1 \end{bmatrix}$$

We observe with relief that $\Pi_1 + \Pi_{2,3} = \mathbb{1}$.

Part (iv)

We now relabel $\{\Theta_1, \Theta_2\} \equiv \{\Pi_1, \Pi_{2,3}\}$ from the previous section and $\{\Pi_1, \Pi_2\} \equiv \{\Pi, \mathbb{1} - \Pi\}$ from earlier. We're given the state $|\psi\rangle = \frac{1}{\sqrt{6}}(|0\rangle + 2\,|1\rangle + i\,|2\rangle)$.

Subpart (a)

We're asked for the probability of obtaining the measurement outcomes Θ_1 and then Π_2. But $|\psi\rangle$ is in the null space of the matrix Θ_1, that is, $\Theta_1\,|\psi\rangle = 0$, so this probability vanishes.

Subpart (b)

We're now asked for the probability of measuring outcomes Π_2 then Θ_1. This time, $|\psi\rangle$ is in the row space of Π_2, that is, $\Pi_2\,|\psi\rangle = |\psi\rangle$, so the first outcome has probability 1 and leaves the state $|\psi\rangle$ unchanged. But then the second measurement outcome again occurs with zero probability, and so the overall probability of the sequence of measurements is again zero.

Question 6

We're given a permutation matrix P of unknown dimension d, along with a circulant matrix C of the same dimension:

$$P \equiv \sum_i |i\rangle\langle i{+}1(\text{mod}\,d)| \qquad C \equiv \sum_{i,j} c_{i-j+1\,(\text{mod}\,d)}\,|i\rangle\langle j|$$

where the indexes are interpreted modulo d, that is, with $|i + d(\text{mod}\,d)\rangle = |i\rangle$, and $c_{i+d\,(\text{mod}\,d)} = c_i$.

We could have—and, indeed, *have*—solved this problem in bra-ket notation, but it's more convenient to write things in index form. So we start again with:

$$P_{ij} = \delta^{\text{mod}\,d}_{i+1,j} \qquad C_{ij} = \sum_{k=1}^{d} c_k\,\delta^{\text{mod}\,d}_{k-1,j-i}$$

where $\delta^{\text{mod}\,d}$ is the Kronecker delta in modular arithmetic.

Part (i)

We're asked to show that $C = \sum_{k=1}^{d} c_k \, P^{k-1}$ where $P^0 = \mathbb{1}$.
We first observe that:

$$\sum_k \left(P^{n+1}\right)_{ij} = \sum_k \left(P^n\right)_{ik} P_{kj} = \sum_k \left(P^n\right)_{ik} \delta^{\operatorname{mod} d}_{k+1,j} = \left(P^n\right)_{i,j-1(\operatorname{mod} d)} = \left(P^n\right)_{i+1(\operatorname{mod} d),j}$$

and so, by induction on n, we obtain that $\left(P^n\right)_{ij} = \delta^{\operatorname{mod} d}_{i+n,j}$. Thus:

$$\left(\sum_{k=1}^{d} c_k \, P^{k-1}\right)_{ij} = \sum_{k=1}^{c} c_k \left(P^{k-1}\right)_{ij} = \sum_{k=1}^{d} c_k \, \delta^{\operatorname{mod} d}_{i+k-1,j} = \sum_{k=1}^{d} c_k \, \delta^{\operatorname{mod} d}_{k-1,j-i} = C_{ij}$$

as required.

Part (ii)

It's easy to show that P is unitary, that is, that $P^\dagger P = PP^\dagger = \mathbb{1}$:

$$\begin{aligned}
\left(P^\dagger P\right)_{ij} &= \sum_k \left(P^\dagger\right)_{ik} P_{kj} = \sum_k P^\star_{ki} P_{kj} = \sum_k \delta^{\operatorname{mod} d}_{k+1,i} \, \delta^{\operatorname{mod} d}_{k+1,j} = \delta_{ij} \\
\left(PP^\dagger\right)_{ij} &= \sum_k P_{ik} \left(P^\dagger\right)_{kj} = \sum_k P_{ik} P^\star_{jk} = \sum_k \delta^{\operatorname{mod} d}_{i+1,k} \, \delta^{\operatorname{mod} d}_{j+1,k} = \delta_{ij}
\end{aligned}$$

We have already shown above that C is obtained by applying a real function $f(x) = \sum_{k=1}^{d} c_k \, x^{k-1}$ to P, and it must therefore share eigenvectors with P. The eigenvectors of P are orthogonal, since P is unitary, and so the eigenvectors of C are also orthogonal. Thus, C is a normal matrix with $C^\dagger C = CC^\dagger$.
If you don't believe us, we can even show this explicitly:

$$\begin{aligned}
\left(C^\dagger C\right)_{ij} &= \sum_k \left(C^\dagger\right)_{ik} C_{kj} = \sum_k C^\star_{ki} C_{kj} = \sum_k \left(\sum_{p=1}^{d} c^\star_p \, \delta^{\operatorname{mod} d}_{p-1,i-k}\right)\left(\sum_{q=1}^{d} c_q \, \delta^{\operatorname{mod} d}_{q-1,j-k}\right) \\
&= \sum_k \sum_{p=1}^{d} \sum_{q=1}^{d} c^\star_p \, c_q \, \delta^{\operatorname{mod} d}_{p-1,i-k} \, \delta^{\operatorname{mod} d}_{q-1,j-k} = \sum_{p=1}^{d} \sum_{q=1}^{d} c^\star_p \, c_q \, \delta^{\operatorname{mod} d}_{p-q,i-j} \\
\left(CC^\dagger\right)_{ij} &= \sum_k C_{ik} \left(C^\dagger\right)_{kj} = \sum_k C^\star_{jk} C_{ik} = \sum_k \left(\sum_{p=1}^{d} c^\star_p \, \delta^{\operatorname{mod} d}_{p-1,k-j}\right)\left(\sum_{q=1}^{d} c_q \, \delta^{\operatorname{mod} d}_{q-1,k-i}\right) \\
&= \sum_k \sum_{p=1}^{d} \sum_{q=1}^{d} c^\star_p \, c_q \, \delta^{\operatorname{mod} d}_{p-1,k-j} \, \delta^{\operatorname{mod} d}_{q-1,k-i} = \sum_{p=1}^{d} \sum_{q=1}^{d} c^\star_p \, c_q \, \delta^{\operatorname{mod} d}_{p-q,i-j}
\end{aligned}$$

But, phew, that was much harder.

Part (iii)

We write out $P - \lambda\mathbb{1}$ as:

$$P - \lambda\mathbb{1} = \begin{bmatrix} -\lambda & 1 & 0 & 0 & \cdots \\ 0 & -\lambda & 1 & 0 & \cdots \\ 0 & 0 & -\lambda & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and then use the cofactor expansion to compute the determinant, arriving at the characteristic equation for the eigenvalues of P:

$$(-\lambda)^d - (-1)^d = 0 \qquad \Rightarrow \qquad \lambda^d - 1 = 0$$

The solutions of this equation are the d^{th} roots of unity, which we'll label $\omega_k = \exp(2\pi i k/d)$ for $k = 1 \dots d$. We then have equations for the eigenvectors $\boldsymbol{v}$ of P, of form:

$$v_{j+1} = \lambda v_j$$

It's easy to see that the eigenvector $\boldsymbol{v}^{(\boldsymbol{k})}$ associated with ω_k has elements:

$$v_j^{(k)} = \omega_k^j = \exp(2\pi i k j/d)$$

We've already shown above that C shares eigenvectors with P. Furthermore, the eigenvalues of C are obtained by applying f to the eigenvalues of P, that is, they are just

$$f(\omega_k) = \sum_{j=1}^{d} c_j\, \omega_k^{j-1} = \sum_{k=1}^{d} c_j\, \exp\left(\frac{2\pi i k(j-1)}{d}\right)$$

which is essentially just the discrete Fourier transform of the coefficients c_j.

Question 7

We begin by applying the unitary operation $U \equiv U(\theta_A) \otimes U(\theta_B)$ to the state:

$$\left|\Phi^-\right\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

where:

$$U(\theta) \equiv \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \theta_A \equiv -\tfrac{\pi}{16} + \tfrac{\pi}{4}\, x_0 \qquad \theta_B \equiv -\tfrac{\pi}{16} + \tfrac{\pi}{4}\, y_0$$

We obtain:

$$\begin{aligned} U\left|\Phi^-\right\rangle &= \tfrac{1}{\sqrt{2}}\left[(\cos\theta_A\,|0\rangle_A + \sin\theta_A\,|1\rangle_A) \otimes (\cos\theta_B\,|0\rangle_B + \sin\theta_B\,|1\rangle_B)\right. \\ &\qquad \left. -(-\sin\theta_A\,|0\rangle_A + \cos\theta_A\,|1\rangle_A) \otimes (-\sin\theta_B\,|0\rangle_B + \cos\theta_B\,|1\rangle_B)\right] \\ &= \tfrac{1}{\sqrt{2}}\left[\cos\theta_{AB}(|00\rangle - |11\rangle) + \sin\theta_{AB}(|01\rangle + |10\rangle)\right] \\ &= \cos\theta_{AB}\left|\Phi^-\right\rangle + \sin\theta_{AB}\left|\Psi^+\right\rangle \end{aligned}$$

where $\theta_{AB} \equiv \theta_B + \theta_A = -\frac{\pi}{8} + \frac{\pi}{4}(x_0 + y_0)$.

Suppose Alice measures her qubit and obtains a, while Bob measures his and obtains b. Since $|\Phi^-\rangle$ results in $a \oplus b = 0$, and $|\Psi^+\rangle$ results in $a \oplus b = 1$, may read off the following probabilities:

$$\mathbb{P}(a \oplus b = 0) = \cos^2\theta_{AB} \qquad \mathbb{P}(a \oplus b = 1) = \sin^2\theta_{AB}$$

where a, b are the results of measuring qubits A, B in the computational basis.

x_0	y_0	θ_{AB}	$\mathbb{P}(a \oplus b = 0)$	$\mathbb{P}(a \oplus b = 1)$	$x_0 \cdot y_0$	$\mathbb{P}(a \oplus b = x_0 \cdot y_0)$
0	0	$-\frac{\pi}{8}$	0.85	0.15	0	0.85
1	0	$\frac{\pi}{8}$	0.85	0.15	0	0.85
0	1	$\frac{\pi}{8}$	0.85	0.15	0	0.85
1	1	$\frac{3\pi}{8}$	0.15	0.85	1	0.85

Therefore, $\mathbb{P}(a \oplus b = x_0 \cdot y_0) = \cos^2\frac{\pi}{8} \simeq 0.85$ in all cases.

Question 8

We're asked to find the Schmidt decomposition of the vector:

$$\boldsymbol{v} = \frac{1}{2}\begin{bmatrix}1\\0\\1\\1\\0\\1\end{bmatrix}$$

first in the space $\mathbb{C}^2 \otimes \mathbb{C}^3$, and then in the space $\mathbb{C}^3 \otimes \mathbb{C}^2$.
The first decomposition can be done just by visual inspection. Obviously:

$$\boldsymbol{v} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

For the second decomposition it helps to perform SVD on this matrix:

$$A = \begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix}$$

We start with A^TA:

$$A^TA = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix} \quad \text{which has eigenvalues} \quad \lambda_1 = 3, \lambda_2 = 1$$

So Σ is:

$$\Sigma = \begin{bmatrix}\sqrt{3} & 0\\0 & 1\\0 & 0\end{bmatrix}$$

And the corresponding eigenvectors are:

$$\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}3a\\3b\end{bmatrix}$$
$$2a + b = 3a, a + 2b = 3b \;\Rightarrow\; \vec{v}_1 = \begin{bmatrix}1 & 1\end{bmatrix}$$
$$\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$
$$2a + b = a, a + 2b = b \;\Rightarrow\; \vec{v}_2 = \begin{bmatrix}-1 & 1\end{bmatrix}$$

Normalizing them, we construct V:

$$V = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

Before calculating U, we find the remaining eigenvector:

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
$$a + b = 0, b + c = 0 \;\Rightarrow\; \vec{x} = \begin{bmatrix}1 & -1 & 1\end{bmatrix}$$

And we may construct U:

$$U = \begin{bmatrix}\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}}\\ \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}}\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

Finally, we have:

$$\frac{1}{2}\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}1 & -\sqrt{3} & \sqrt{2}\\ 2 & 0 & -\sqrt{2}\\ 1 & \sqrt{3} & \sqrt{2}\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}\sqrt{3} & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}^T$$

and we see that:

$$\boldsymbol{v} = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{6}}\begin{bmatrix}1\\ 2\\ 1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}$$

We notice that $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$, as expected, and that the sets:

$$\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}\right\} \qquad \left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\right\}$$

each contain two orthonormal vectors, which is the same as the minimum of the dimensions.