ICCUB
Institut de Ciències del Cosmos
UNIVERSITAT DE BARCELONA
EXCELENCIA MARÍA DE MAEZTU

1. Consider an Hermitian operator, $\hat{A}^\dagger = \hat{A}$.
   (a) Prove that any expectation value of $\hat{A}$ is a real number.
   (b) Prove that the eigenvalues of $\hat{A}$ are real numbers.
   (c) Prove that two eigenvectors, ψ_α and ψ_β, associated to two different eigenvalues, a_α and a_β are orthogonal, $\langle \psi_\alpha | \psi_\beta \rangle = 0$.

2. Consider a one-dimensional system where x represents the spatial coordinate. The momentum operator is $\hat{p}_x = -i\hbar\partial_x$. Prove that this operator is Hermitian.

3. An operator $\hat{A}$, corresponding to observable A, has two normalised eigenstates $|a_1\rangle$ and $|a_2\rangle$, with eigenvalues a_1 and a_2. An operator $\hat{B}$, corresponding to observable B, has two normalised eigenstates $|b_1\rangle$ and $|b_2\rangle$ with eigenvalues b_1 and b_2. The eigenstates are related by

$$\begin{aligned} |a_1\rangle &= c_1|b_1\rangle + c_2|b_2\rangle \,, \\ |a_2\rangle &= c_2|b_1\rangle - c_1|b_2\rangle \,, \end{aligned}$$

   where $c_1 = \frac{2}{\sqrt{13}}$ and $c_2 = \frac{3}{\sqrt{13}}$. A is measured and the value a_1 is obtained. If B is measured immediately after, and A is measured immediately after that, what is the probability of obtaining a_1 again?

4. A Hamiltonian $\hat{H}$ has two eigenstates, $|1\rangle$ and $|2\rangle$, associated to the two eigenvalues E_1 and E_2. A physical quantity A associated to observable $\hat{A}$ has two eigenvalues, a_1 and a_2. Its eigenstates can be expanded in the Hamiltonian basis according to:

$$\begin{aligned} |a_1\rangle &= \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle \,, \\ |a_2\rangle &= \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|2\rangle \,. \end{aligned}$$

   If the system is in state $|\psi\rangle = |a_1\rangle$ at $t = 0$,
   (a) Find the wave function as a function of time, $|\psi(t)\rangle$.
   (b) Show that the expectation value of A as a function of time is:

$$\langle a \rangle = \frac{a_1 + a_2}{2} + \frac{a_1 - a_2}{2} \cos\left(\frac{(E_1 - E_2)\, t}{\hbar} \right) \,.$$

   (c) Sketch the expectation value $\langle a \rangle$ as a function of time and provide a physical interpretation for the time evolution.

5. At time $t = 0$, the state function of a free particle in a 1D system is a gaussian wavepacket,

$$\psi(x, t = 0) = c \exp\left(-\frac{x^2}{4\Delta_0^2} \right) \,.$$

   Show that the uncertainty in position is $\Delta^2(t) = \Delta_0^2 + (\Delta v)^2 t^2$, where Δv is the uncertainty in velocity at $t = 0$. How does the uncertainty in velocity evolve over time?

6. The spectral decomposition of the position $\hat{x}$ and momentum $\hat{p}$ operators is given by the expressions

$$\hat{x} = \int \mathrm{d}x\, x|x\rangle\langle x|, \quad \hat{p} = \int \frac{\mathrm{d}p}{2\pi\hbar}\, p|p\rangle\langle p|.$$

Computing the matrix elements of the commutator $[\hat{x}, \hat{p}]$, prove that these definitions lead to the standard relation $[\hat{x}, \hat{p}] = i\hbar$.
Hint: you may use the wavefunction $\langle x|p\rangle = \exp\left(\frac{i}{\hbar}px\right)$*; the Fourier transform* $|p\rangle = \int dy e^{i\frac{py}{\hbar}}|y\rangle$ *and the formal relation* $\int \frac{dp}{2\pi\hbar} p e^{-i\frac{p(y-x)}{\hbar}} = i\hbar\partial_y\delta(x-y)$.