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1. A quantum mechanical system is associated to an unperturbed hamiltonian $\hat{H}_0$, with spectrum $\hat{H}_0|n\rangle = \epsilon_n|n\rangle$. The system is perturbed at $t = 0$ by a constant hamiltonian, $\hat{H}_1(t) = \hat{H}'$, up until $t = T$, when the perturbation is switched off. Consider a transition between an initial state, $|i\rangle$ at $t < 0$, and a final state, $|f\rangle$ at $t > T$.
   (a) Show that the transition probability from first-order perturbation theory is given by the expression
   $$\mathcal{P}_{i\to f}(T) = \frac{\left|H'_{fi}\right|^2}{\hbar^2} \frac{\sin^2\left(\frac{\omega_{fi}}{2}T\right)}{\left(\frac{\omega_{fi}}{2}\right)^2} ,$$
   with the Bohr frequencies $\omega_{fi} = \frac{\epsilon_f - \epsilon_i}{\hbar}$; and H'_{fi}, the matrix element $H'_{fi} = \langle f|\hat{H}'|i\rangle$.
   (b) For early times, $T \ll \frac{2}{\omega_{fi}}$, how does the transition probability depend on the final state? Provide a physical explanation of this result.
   (c) Find an expression for $\mathcal{P}_{i\to f}(T)$ when $T \gg 1$ and discuss it in the context of the Fermi golden rule.
   (d) How does the expression change for a Gaussian perturbation, $\hat{H}_1(t) = H'e^{-\frac{t^2}{T^2}}$, active between $t = -\infty$ and $t = \infty$?

2. An unperturbed two-state system Hamiltonian gives rise to two degenerate states with energy $E = \epsilon_1 = \epsilon_2$. A constant perturbation, $\hat{H}_1 = v(|1\rangle\langle 2| + |2\rangle\langle 1|)$ acts on the system for a time T. Show, by adding up perturbation theory terms, that the transition amplitudes are given by:
   $$a_{1\to 1} = 1 - \frac{v^2T^2}{2\hbar^2} + \frac{v^4T^4}{24\hbar^4} + \cdots = \cos\frac{vT}{\hbar} ,$$
   $$a_{1\to 2} = -i\frac{vT}{\hbar} + i\frac{v^3T^3}{6\hbar^3} + \cdots = -i\sin\frac{vT}{\hbar} .$$

3. Use the Born approximation to derive the differential elastic scattering cross section, $\frac{d\sigma}{d\Omega}$, for a particle interacting with a target through:
   (a) A square well potential, $V(r) = V_0$ for $r < R$ and $V(r) = 0$ elsewhere.
   (b) A gaussian potential, $V(r) = \frac{V_0}{2\pi R^{3/2}} e^{-\frac{r^2}{2R^2}}$.
   (c) Compare the two results above with the Yukawa potential expression. Can you extract some generic conclusions of the cross section dependence as a function of V_0 and R?

4. **Resonances**
   In some energy regime, the asymptotic wave function of $l = 0$ partial waves is $u(k, r) = e^{-ikr} - S_0 e^{ikr}$, with $S_0 = e^{2i\delta_0}$.
   (a) Compute the logarithmic derivative, $r\frac{u'}{u}$, of this wave function.
   (b) By matching this logarithmic derivative to a value $\mathcal{L}(E) = a(E) - ib(E)$ at $r = R$, find S_0.
   (c) The scattering amplitude is $f_0 = \frac{1}{2ik}(S_0 - 1)$. Show that it can be separated in two terms, $f_{\text{pot}} = \frac{1}{2ik}\left(e^{-2ikR} - 1\right)$ and $f_{\text{res}} = \frac{e^{-2ikR}}{k}\left(\frac{kR}{a - i(kR + b)}\right)$.

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(d) What is the phase-shift associated to f_{pot}?

(e) A resonance occurs at an energy E_r such that $a(E_r) = 0$. Taylor expanding around this energy, one finds $a(E_r) \approx a'(E_r)(E - E_r)$. Prove that the cross section associated to the scattering amplitude f_{res}, $\sigma = 4\pi |f_{\text{res}}|^2$, has a Breit-Wigner shape,

$$\sigma = \frac{\pi}{k^2} \frac{\Gamma_e^2}{(E - E_r)^2 + \Gamma^2/4}$$

close to $E \approx E_r$. The scattering width is $\Gamma_e = -\frac{2kR}{a'(E_r)}$ and the total width $\Gamma = \Gamma_e - \frac{2b}{a'(E_r)}$.

5. Consider the $l = 0$ (or $s-$wave) scattering off a potential that reads

$$V(r) = \begin{cases} \infty, \text{ for } r < R_0, \\ -U_0 \text{ for } R_0 < r < R_1, \\ 0, \text{ for } r > R_1. \end{cases}$$

This model represents a rough van der Waals or nuclear interaction.

(a) Find the solutions of the Schrödinger equation in each of the three regions of space.

(b) Match the solutions at $r = R_0$ to find that $u_0(r) = A \sin[\bar{k}(r - R_0)]$ for $R_0 < r < R_1$. What is $\bar{k}$?

(c) Find the scattering phase shift by matching the solution at $r = R_1$.

(d) Find the scattering length of this potential. How does it compare to the range of the potential, R_1?

(e) What phenomenon may occur as the momentum increases?