UNIVERSITAT DE BARCELONA
Facultat de Física

1. The raising operator for harmonic oscillator states is defined as $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$.
   (a) Express it as a differential operator in the real space representation, using harmonic oscillator (dimensionless) units.
   (b) Taking the ground-state wavefunction $\psi_0(x)$, find $a^\dagger\psi_0(x)$. How does this compare to $\psi_1(x)$?
   (c) What happens when you apply $\hat{a}^\dagger$ twice to $\psi_0(x)$? What can you conclude?

   Hints: the eigenstates of the harmonic oscillator are $\psi_n(x) = \mathcal{N}_n e^{-\frac{x^2}{2}} H_n(x)$ in harmonic oscillator units, with $\mathcal{N}_n = (2^n n! \sqrt{\pi})^{-1/2}$. The first few Hermite polynomials are $H_0(x) = 1$, $H_1(x) = 2x$ and $H_2(x) = 4x^2 - 2$.

2. Write the one-dimensional harmonic oscillator eigenvalue equation in momentum representation. Calculate the corresponding eigenfunctions, $\langle p|\psi_n\rangle$.

3. Consider an harmonic oscillator $\hat{H} = \frac{\hbar\omega}{2}(\hat{p}^2 + \hat{x}^2)$ and its first two normalized states, $\phi_0(x)$ and $\phi_1(x)$. Consider a system with initial wave function $\psi(x, t = 0) = \cos\theta\phi_0(x) + \sin\theta\phi_1(x)$ with $0 < \theta \leqslant \pi$.
   (a) Find the wave function at time t, $\psi(x,t)$.
   (b) Calculate the expectation values $\langle E\rangle$, $\langle E^2\rangle$ and the dispersion ΔE. Explain their time dependence.
   (c) Calculate the time evolution of $\langle x\rangle$, $\langle x^2\rangle$ and the dispersion Δx.

4. **Virial theorem**
   Consider a one-dimensional system with the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$ and $V(x) = \lambda x^\alpha$.
   (a) Calculate the commutator $[\hat{H}, \hat{x}\hat{p}]$.
   (b) By taking the expectation value of this commutator, show that for any eigenstate of $\hat{H}$, one has $2\langle T\rangle = \alpha\langle V\rangle$, with $\langle T\rangle$ and $\langle V\rangle$ the kinetic and potential energies of the system, respectively.
   (c) Check this relation for the harmonic oscillator.

5. Consider a coherent state $|z\rangle$. Starting from the definition of the Hamiltonian and the projection into Fock states,
$$|z\rangle = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle,$$
   find the expectation values $\langle E\rangle$, $\langle E^2\rangle$ and the dispersion ΔE. Discuss the precision of the energy as a function of $|z|$.

6. **Minimum uncertainty states**
   (a) Consider a coherent state $|z\rangle$. Find the expectation value of the position operator on this state, $\langle z|\hat{x}|z\rangle$.
   (b) Find the same expectation value for the momentum operator $\hat{p}$.
   (c) Considering $\langle z|\hat{x}^2|z\rangle$ and $\langle z|\hat{p}^2|z\rangle$, find the uncertainties of both operators over a coherent state, Δx_z^2 and Δp_z^2.
   (d) What do you find for the product $\Delta x_z^2 \Delta p_z^2$? Comment on this result.