(iv) We first consider a rotation about the y-axis by an angle θ, which is given by the rotation matrix

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Writing $\vec{r}$ in spherical coordinates as $(r_0 \sin\theta_0 \cos\varphi_0, r_0 \sin\theta_0 \sin\varphi_0, r_0 \cos\theta_0)$ where $r_0 \leq 1$ by part (ii), we get that $R_y(\theta)$ maps $\vec{r}$ to

$$\vec{r}\,' = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} r_0 \sin\theta_0 \cos\varphi_0 \\ r_0 \sin\theta_0 \sin\varphi_0 \\ r_0 \cos\theta_0 \end{pmatrix} = r_0 \begin{pmatrix} \cos\theta \sin\theta_0 \cos\varphi_0 + \sin\theta \cos\theta_0 \\ \sin\theta_0 \sin\varphi_0 \\ -\sin\theta \sin\theta_0 \cos\varphi_0 + \cos\theta \cos\theta_0 \end{pmatrix}$$

On the other hand, by part (ii), $\vec{r}$ and $\vec{r}\,'$ correspond to the density matrices given by

$$\begin{aligned} \rho = \frac{\mathbb{1} + \vec{r}\cdot\vec{\sigma}}{2} &= \frac{1}{2} \begin{pmatrix} 1 + r_0 \cos\theta_0 & r_0 \sin\theta_0 \cos\varphi_0 - i r_0 \sin\theta_0 \sin\varphi_0 \\ r_0 \sin\theta_0 \cos\varphi_0 + i r_0 \sin\theta_0 \sin\varphi_0 & 1 - r_0 \cos\theta_0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 + r_0 \cos\theta_0 & r_0 e^{-i\varphi_0} \sin\theta_0 \\ r_0 e^{i\varphi_0} \sin\theta_0 & 1 - r_0 \cos\theta_0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \rho' &= \frac{\mathbb{1} + \vec{r}\,' \cdot \vec{\sigma}}{2} \\ &= \frac{1}{2} \begin{pmatrix} 1 + r_0(-\sin\theta \sin\theta_0 \cos\varphi_0 + \cos\theta \cos\theta_0) & r_0(\cos\theta \sin\theta_0 \cos\varphi_0 + \sin\theta \cos\theta_0 - i \sin\theta_0 \sin\varphi_0) \\ r_0(\cos\theta \sin\theta_0 \cos\varphi_0 + \sin\theta \cos\theta_0 + i \sin\theta_0 \sin\varphi_0) & 1 + r_0(\sin\theta \sin\theta_0 \cos\varphi_0 - \cos\theta \cos\theta_0) \end{pmatrix} \end{aligned}$$

By setting

$$U_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

we get

$$\begin{aligned} U_y(\theta)\rho U_y^\dagger(\theta) &= \frac{1}{2} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 + r_0 \cos\theta_0 & r_0 e^{-i\varphi_0} \sin\theta_0 \\ r_0 e^{i\varphi_0} \sin\theta_0 & 1 - r_0 \cos\theta_0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2}(1 + r_0 \cos\theta_0) - r_0 e^{-i\varphi_0} \sin\frac{\theta}{2} \sin\theta_0 & \sin\frac{\theta}{2}(1 + r_0 \cos\theta_0) + r_0 e^{-i\varphi_0} \cos\frac{\theta}{2} \sin\theta_0 \\ r_0 e^{i\varphi_0} \cos\frac{\theta}{2} \sin\theta_0 - \sin\frac{\theta}{2}(1 - r_0 \cos\theta_0) & r_0 e^{i\varphi_0} \sin\frac{\theta}{2} \sin\theta_0 + \cos\frac{\theta}{2}(1 - r_0 \cos\theta_0) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 + r_0 \cos\theta_0 \cos\theta - r_0 \sin\theta \sin\theta_0 \cos\varphi_0 & r_0 \sin\theta \cos\theta_0 + r_0 \sin\theta_0 (e^{-i\varphi_0} \cos^2\frac{\theta}{2} - e^{i\varphi_0} \sin^2\frac{\theta}{2}) \\ r_0 \sin\theta \cos\theta_0 + r_0 \sin\theta_0 (e^{i\varphi_0} \cos^2\frac{\theta}{2} - e^{-i\varphi_0} \sin^2\frac{\theta}{2}) & 1 - r_0 \cos\theta_0 \cos\theta + r_0 \sin\theta \sin\theta_0 \cos\varphi_0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 + r_0 \cos\theta_0 \cos\theta - r_0 \sin\theta \sin\theta_0 \cos\varphi_0 & r_0 \sin\theta \cos\theta_0 + r_0 \sin\theta_0 (\cos\theta \cos\varphi_0 - i \sin\varphi_0) \\ r_0 \sin\theta \cos\theta_0 + r_0 \sin\theta_0 (\cos\theta \cos\varphi_0 + i \sin\varphi_0) & 1 - r_0 \cos\theta_0 \cos\theta + r_0 \sin\theta \sin\theta_0 \cos\varphi_0 \end{pmatrix} \\ &= \rho' \end{aligned}$$

Thus, the rotation $R_y(\theta)$ in the Bloch sphere corresponds to the unitary transformation $U_y(\theta)$ of the density matrix.

Now consider a rotation about the z-axis by an angle φ, which is given by the rotation matrix

$$R_z(\theta) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This maps $\vec{r}$ to

$$
\begin{aligned}
\vec{r}' &= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_0 \sin\theta_0 \cos\varphi_0 \\ r_0 \sin\theta_0 \sin\varphi_0 \\ r_0 \cos\theta_0 \end{pmatrix} \\
&= r_0 \begin{pmatrix} \cos\varphi \sin\theta_0 \cos\varphi_0 - \sin\varphi \sin\theta_0 \sin\varphi_0 \\ \sin\varphi \sin\theta_0 \cos\varphi_0 + \cos\varphi \sin\theta_0 \sin\varphi_0 \\ \cos\theta_0 \end{pmatrix} \\
&= r_0 \begin{pmatrix} \sin\theta_0 \cos(\varphi_0 + \varphi) \\ \sin\theta_0 \sin(\varphi_0 + \varphi) \\ \cos\theta_0 \end{pmatrix}
\end{aligned}
$$

The density matrix corresponding to $\vec{r}'$ is given by

$$
\begin{aligned}
\rho' &= \frac{\mathbb{1} + \vec{r}' \cdot \vec{\sigma}}{2} \\
&= \frac{1}{2} \begin{pmatrix} 1 + r_0 \cos\theta_0 & r_0 \sin\theta_0 (\cos(\varphi_0 + \varphi) - i \sin(\varphi_0 + \varphi)) \\ r_0 \sin\theta_0 (\cos(\varphi_0 + \varphi) + i \sin(\varphi_0 + \varphi)) & 1 - r_0 \cos\theta_0 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 + r_0 \cos\theta_0 & r_0 e^{-i(\varphi_0 + \varphi)} \sin\theta_0 \\ r_0 e^{i(\varphi_0 + \varphi)} \sin\theta_0 & 1 - r_0 \cos\theta_0 \end{pmatrix}
\end{aligned}
$$

By setting

$$
U_z(\varphi) = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}
$$

we get

$$
\begin{aligned}
U_z(\varphi) \rho U_z^\dagger(\varphi) &= \frac{1}{2} \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} \begin{pmatrix} 1 + r_0 \cos\theta_0 & r_0 e^{-i\varphi_0} \sin\theta_0 \\ r_0 e^{i\varphi_0} \sin\theta_0 & 1 - r_0 \cos\theta_0 \end{pmatrix} \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} \begin{pmatrix} e^{i\varphi/2}(1 + r_0 \cos\theta_0) & r_0 e^{-i(\varphi_0 + \varphi/2)} \sin\theta_0 \\ r_0 e^{i(\varphi_0 + \varphi/2)} \sin\theta_0 & e^{-i\varphi/2}(1 - r_0 \cos\theta_0) \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 + r_0 \cos\theta_0 & r_0 e^{-i(\varphi_0 + \varphi)} \sin\theta_0 \\ r_0 e^{i(\varphi_0 + \varphi)} \sin\theta_0 & 1 - r_0 \cos\theta_0 \end{pmatrix} \\
&= \rho'
\end{aligned}
$$

Thus, the rotation $R_z(\varphi)$ in the Bloch sphere corresponds to the unitary transformation $U_z(\varphi)$ of the density matrix.

Any orthogonal rotation in $SO(3)$ can be decomposed into products of R_y and R_z rotations, which have corresponding unitary representations in $SU(2)$, and thus itself has a corresponding unitary representation. It follows that $SO(3) \simeq SU(2)/(-\mathbb{1}, \mathbb{1})$, where the quotient is due to the fact that $U_y(2\pi)$ and $U_z(2\pi)$ introduce a -1 phase, whereas $R_y(2\pi)$ and $R_z(2\pi)$ are both equal to the identity.