Master in Quantum Science and Technology Barcelona

UNIVERSITAT DE BARCELONA
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Quantum Statistical Inference: Homework 2

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Sunday 2nd February, 2025

Exercise 1

Prove the following Concentration inequalities:

1. **Markov's inequality.** For any non-negative random variable X and any $t > 0$, show that:

$$\Pr\{X \geq t\} \leq \frac{\mathbb{E}(X)}{t}.$$

Give a random variable that achieves this inequality with equality. It is often a very useful proof method to use a trivial extension of this inequality:

$$\Pr\{X \geq t\} = \Pr\{f(X) \geq f(t)\} \leq \frac{\mathbb{E}(f(X))}{f(t)},$$

where $f(x)$ is a strictly increasing non-negative function.

Solution: Let X be a non-negative random variable and $t > 0$. Consider the indicator function $\mathbf{1}_{\{X \geq t\}}$, which equals 1 when $X \geq t$ and 0 otherwise. Then:

$$\Pr\{X \geq t\} = \mathbb{E}[\mathbf{1}_{\{X \geq t\}}].$$

Since $\mathbf{1}_{\{X \geq t\}} \leq \frac{X}{t}$ whenever $X \geq t$ and $\mathbf{1}_{\{X \geq t\}} = 0$ otherwise, we can write:

$$\mathbf{1}_{\{X \geq t\}} \cdot t \leq X.$$

Taking the expectation on both sides:

$$\mathbb{E}[\mathbf{1}_{\{X \geq t\}} \cdot t] \leq \mathbb{E}[X].$$

Since t is a constant, it can be factored out of the expectation:

$$t \cdot \mathbb{E}[\mathbf{1}_{\{X \geq t\}}] \leq \mathbb{E}[X].$$

Rearranging, we obtain Markov's inequality:

$$\Pr\{X \geq t\} = \mathbb{E}[\mathbf{1}_{\{X \geq t\}}] \leq \frac{\mathbb{E}[X]}{t}.$$

Equality in Markov's inequality holds, for example, if:

$$X = \begin{cases} t, & \text{with probability } p; \\ 0, & \text{with probability } 1-p. \end{cases}$$

Then:

$$\mathbb{E}[X] = t \cdot p, \quad \Pr\{X \geq t\} = p.$$

Substituting into the inequality:

$$\Pr\{X \geq t\} = p = \frac{\mathbb{E}[X]}{t},$$

showing that equality is achieved.

2. **Chebyshev's inequality.** Let Y be a random variable with mean $\mathbb{E}(Y) = \mu$ and variance σ^2. By letting $X = (Y-\mu)^2$, show that for any $\epsilon > 0$:

$$\Pr(|Y-\mu| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}.$$

Solution: Let Y be a random variable with mean $\mathbb{E}(Y) = \mu$ and variance σ^2. Define $X = (Y-\mu)^2$. By definition of variance:

$$\sigma^2 = \mathbb{E}[(Y-\mu)^2] = \mathbb{E}[X].$$

Now, consider the probability $\Pr(|Y-\mu| > \epsilon)$. Using the equivalence $|Y-\mu| > \epsilon \iff (Y-\mu)^2 > \epsilon^2$, we can rewrite this probability as:

$$\Pr(|Y-\mu| > \epsilon) = \Pr\big((Y-\mu)^2 > \epsilon^2\big) = \Pr\big(X > \epsilon^2\big).$$

By applying Markov's inequality to the non-negative random variable X, we have:

$$\Pr\big(X > \epsilon^2\big) \leq \frac{\mathbb{E}[X]}{\epsilon^2}.$$

Substituting $\mathbb{E}[X] = \sigma^2$, we get:

$$\Pr(|Y-\mu| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}.$$

This completes the proof.

Exercise 2

Consider an n-dimensional rectangular volume in a (very) large phase-space with sides $X_1, X_2, \ldots, X_n$. In addition, define an effective linear length-scale ℓ given by the length of the edge of an n-dimensional cube with the same volume, i.e.,

$$\ell(n) := V_n^{1/n}, \quad \text{where } V_n = \prod_{i=1}^{n} X_i.$$

Now, let $X_1, X_2, \ldots$ be i.i.d. uniform random variables over the unit interval $[0,1]$. Find:

$$\lim_{n\to\infty} \ell(n),$$

which is almost surely defined, and compare it to another possible length-scale definition:

$$\bar{\ell} := \left(\mathbb{E}[V_n]\right)^{1/n}.$$

(Use the strong law of large numbers.)

Solution: First, we calculate the desired limit. Let,

$$y = \lim_{n\to\infty} \ell(n) = \lim_{n\to\infty} V_n^{1/n} = \lim_{n\to\infty} \left(\prod_{i=1}^{n} X_i\right)^{1/n}.$$

This limit has the indeterminate form 0^0. Therefore, we take the natural logarithm of both sides to find:

$$\ln y = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln X_i = \mathbb{E}[\ln X] \tag{1}$$

where $X \sim$ Uniform[0,1] similarly to X_i. This follows from the strong law of large numbers, since $\mathbb{E}[\ln X] < \infty$. We can calculate the aforementioned expectation value as follows:

$$\begin{aligned}
\mathbb{E}[\ln X] &= \int_{-\infty}^{\infty} \ln x f_X(x)dx \qquad \text{where } f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \int_0^1 \ln x dx \\
&= x \ln x - x \Big|_0^1 \\
&= -1
\end{aligned}$$

Substituting into eq. 1 and refactoring yields

$$\lim_{n\to\infty} \ell(n) = \frac{1}{e}.$$

Next, we calculate the limit for $\bar{\ell}$ using the observation that $\mathbb{E}[X_i] = \frac{1}{2}$:

$$\lim_{n\to\infty} \bar{\ell}(n) = \lim_{n\to\infty} (\mathbb{E}[V_n])^{\frac{1}{n}} = \lim_{n\to\infty} \left(\mathbb{E}\Big[\prod_{i=1}^{n} X_i\Big]\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\prod_{i=1}^{n} \mathbb{E}[X_i]\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{1}{2}\right)^n\right)^{\frac{1}{n}} = \frac{1}{2}$$

Comparing the two results, we find that

$$\lim_{n\to\infty} \bar{\ell}(n) = \frac{1}{2} > \frac{1}{e} = \lim_{n\to\infty} \ell(n).$$

This result is a manifestation of Jensen's inequality and the concavity of $\ln(\cdot)$, meaning that typically, the geometric mean is smaller than the arithmetic mean.

Exercise 4

Suppose you randomly draw $n \gg 1$ balls with replacement from an urn containing 2 red balls (reward $R(r) = 100$), 8 blue balls (reward $R(b) = 20$), 90 white balls (reward $R(w) = 0$).
Let the mean reward be defined as:

$$\bar{R}_n = \frac{1}{n} \sum_{k=1}^{n} R_k.$$

Find the probability that the mean reward satisfies $\bar{R}_n \ge 5$. Express the result as:

$$\Pr\big(\bar{R}_n \ge 5\big) \approx e^{-nD},$$

where D is the rate function. Compute the numerical value of:

$$\Pr\big(\bar{R}_n \ge 5\big) \approx e^{-\alpha},$$

for $n = 10^5$. (Use Sanov's theorem.)

Solution: First, let's define the problem and its constraints. Let $\mathcal{X} = \{r, b, w\}$, representing the possible colours for ball $X_k,\ k = 1, ..., n$. The empirical probability distribution $Q = (Q(r), Q(b), Q(w))$ is given by:

$$Q(r) = \frac{2}{100}, \qquad Q(b) = \frac{8}{100}, \qquad \text{and} \qquad Q(w) = \frac{90}{100}.$$

Let $P = (P(r), P(b), P(w))$ represent some other empirical probability distribution satisfying the constraint,

$$P(r) + P(b) + P(w) = 1 \qquad (\{P(x)\ \forall x \in \mathcal{X}\} \text{ exhausts our options for sampling the urn}) \tag{2}$$

In order to find $\Pr\big(\bar{R}_n \geq 5\big)$, we also require

$$\bar{R}_n = 100P(r) + 20P(b) = 5 \tag{3}$$

According to Sanov's Theorem,

$$\Pr\big(\bar{R}_n \geq 5\big) \approx e^{-nD(P^*||Q)} \tag{4}$$

where $P^* = (P^*(r), P^*(b), P^*(w))$ is the probability distribution that minimizes the relative entropy $D(P||Q)$ with respect to the constraints (2) and (3). We can express this optimization problem by constructing the Lagrangian:

$$L(P) = \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)} + \lambda \left(\sum_{x \in \mathcal{X}} P(x)R(x) - 5 \right) + \nu \left(\sum_{x \in \mathcal{X}} P(x) - 1 \right).$$

Differentiating gives the solution

$$P^*(x) = \frac{Q(x)e^{\lambda R(x)}}{\sum_a Q(a)e^{\lambda R(a)}}.$$

Thus, our optimal probability distribution P^* is given by:

$$\begin{aligned} P^*(r) &= \frac{Q(r)e^{\lambda R(r)}}{Z} = \frac{0.02e^{\lambda 100}}{Z} \\ P^*(b) &= \frac{Q(b)e^{\lambda R(b)}}{Z} = \frac{0.08e^{\lambda 20}}{Z} \\ P^*(w) &= \frac{Q(w)e^{\lambda R(w)}}{Z} = \frac{0.9e^{\lambda 0}}{Z} = \frac{0.9}{Z} \end{aligned}$$

where

$$\begin{aligned} Z &= \sum_a Q(a)e^{\lambda R(a)} \\ &= Q(r)e^{\lambda R(r)} + Q(b)e^{\lambda R(b)} + Q(w)e^{\lambda R(w)} \\ &= 0.02e^{\lambda 100} + 0.08e^{\lambda 20} + 0.9. \end{aligned}$$

Solving for λ subject to constraint (3) yields

$$\lambda \approx 0.00512064.$$

The solution for λ can be seen on the following Desmos plot, which is in agreement with the numerical solution provided by WolframAlpha. Substituting λ into the formula for relative entropy yields

$$D(P^*||Q) = \sum_{x \in \mathcal{X}} P^*(x) \log \frac{P^*(x)}{Q(x)} \approx 0.00383985.$$

Finally, our probability reads

$$\Pr\big(\bar{R}_n \geq 5\big) \approx e^{-0.00383985n}$$

For $n = 10^5$, the probability becomes

$$\Pr\big(\bar{R}_n \geq 5\big) \approx e^{-0.00383985 \cdot 10^5} \approx e^{-384}$$

which is in accordance with our expectation that the probability of observing an atypical sequence of events decays exponentially as the length of the sequence increases.

Exercise 5

The function $f(t) = t^p$ for $p > 1$ is not operator monotone. Show this for $f(t) = t^2$ by providing a counterexample. Use the matrices:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Show that $A \geq B$, but $A^2 \ngeq B^2$.

Solution: The function $f(t) = t^2$ is not operator monotone. A function $f(t)$ is operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for all Hermitian matrices A and B. To show that $f(t) = t^2$ is not operator monotone, consider the matrices:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

First, verify $A \geq B$. Compute $A - B$:

$$A - B = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1-i \\ 1+i & 1 \end{pmatrix}.$$

To check if $A - B$ is positive semidefinite, calculate its eigenvalues. The characteristic polynomial of $A - B$ is:

$$\det\left(\begin{pmatrix} 3 & 1-i \\ 1+i & 1 \end{pmatrix} - \lambda I\right) = \det\begin{pmatrix} 3-\lambda & 1-i \\ 1+i & 1-\lambda \end{pmatrix}.$$

$$\det = (3-\lambda)(1-\lambda) - (1-i)(1+i) = (3-\lambda)(1-\lambda) - 2.$$

$$\det = \lambda^2 - 4\lambda + 1.$$

The eigenvalues are the roots of $\lambda^2 - 4\lambda + 1 = 0$, given by:

$$\lambda = 2 \pm \sqrt{3}.$$

Since both eigenvalues are positive, $A - B$ is positive semidefinite, and $A \geq B$.
Next, compute A^2 and B^2. For A^2:

$$A^2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix}.$$

For B^2:

$$B^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now compare A^2 and B^2. Compute $A^2 - B^2$:

$$A^2 - B^2 = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 4 \\ 4 & 1 \end{pmatrix}.$$

To check if $A^2 - B^2$ is positive semidefinite, calculate its eigenvalues. The characteristic polynomial of $A^2 - B^2$ is:

$$\det\left(\begin{pmatrix} 9 & 4 \\ 4 & 1 \end{pmatrix} - \lambda I\right) = \det\begin{pmatrix} 9-\lambda & 4 \\ 4 & 1-\lambda \end{pmatrix}.$$

$$\det = (9-\lambda)(1-\lambda) - 16 = \lambda^2 - 10\lambda - 7.$$

The eigenvalues are the roots of $\lambda^2 - 10\lambda - 7 = 0$, given by:

$$\lambda = 5 + 4\sqrt{2}, \quad \lambda = 5 - 4\sqrt{2}.$$

Since the second eigenvalue is negative, $A^2 \ngeq B^2$, violating the operator monotonicity condition.
Thus, $f(t) = t^2$ is not operator monotone because $A \geq B$ does not imply $A^2 \geq B^2$.

Exercise 6

Show that if a channel Λ fulfills the covariance property:

$$\Lambda(U\rho U^\dagger) = V\Lambda(\rho)V^\dagger,$$

then the corresponding Choi operator J_Λ must have the symmetry:

$$V \otimes U^* J_\Lambda V^\dagger \otimes U^T = J_\Lambda.$$

Hint: Use the matrix representation $|C\rangle$ presented in section 2.2.1 of the notes.

Solution: First of all, let us recall the following two relations:

$$\begin{aligned} (A \otimes B)\,|C\rangle &= |ACB^T\rangle \\ \langle C|\,(A^\dagger \otimes B^\dagger) &= \langle ACB^T|\,. \end{aligned} \tag{5}$$

Furthermore, we recall that the Choi operator, J_Λ, associated to channel Λ can be expressed as:

$$J_\Lambda = (\Lambda \otimes I)(|I\rangle\langle I|) = \sum_i |K_i\rangle\langle K_i|\,. \tag{6}$$

Now, let us compute the following quantity:

$$(\Lambda \otimes I)\left[(U \otimes I)\,|I\rangle\langle I|\,(U^\dagger \otimes I)\right] = (V \otimes I)\,\underbrace{(\Lambda \otimes I)\,[|I\rangle\langle I|]}_{J_\Lambda}(V^\dagger \otimes I), \tag{7}$$

where in the equality we used the fact that the channel is covariant.

Next, we multiply both sides of Eq.(7) by $(I \otimes U^*) * (I \otimes U^T)$. Hence:

$$(V \otimes U^*)J_\Lambda(V^\dagger \otimes U^T) = (I \otimes U^*)\,\underbrace{(\Lambda \otimes I)\left[(U \otimes I)\,|I\rangle\langle I|\,(U^\dagger \otimes I)\right]}_{\tau}(I \otimes U^T). \tag{8}$$

Massaging τ one gets:

$$\begin{aligned} \tau &= (\Lambda \otimes I)[|U\rangle\langle U|] = \sum_i |K_i U\rangle\langle K_i U| = \sum_i (I \otimes U^T)\,|K_i\rangle\langle K_i|\,(I \otimes U^*) \\ &= (I \otimes U^T)J_\Lambda(I \otimes U^*), \end{aligned} \tag{9}$$

where the third equality readily follows as we are rotating the input state, hence the rotation in the eigenvectors of J_Λ.

Lastly, plugging in Eq.(9) in Eq.(8), follows that:

$$(V \otimes U^*)J_\Lambda(V^\dagger \otimes U^T) = J_\Lambda. \tag{10}$$

Exercise 7

A bit-flip error channel can be written as:

$$\Lambda_\lambda(\rho) = (1-\lambda)\rho + \lambda\sigma_x\rho\sigma_x, \quad 0 \le \lambda \le 1.$$

1. Show the effect of these channels on an arbitrary qubit state with Bloch vector $\vec{s}$:

$$\rho = \frac{1}{2}(1 + \vec{s}\cdot\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+s_z & s_x - is_y \\ s_x + is_y & 1-s_z \end{pmatrix}.$$

Tip: Think of the channel as a convex combination of the identity with a particular rotation.

Solution: Recalling the relations $X^3 = X$, $XYX = -Y$ and $XZX = -Z$ follows that:

$$\begin{aligned}\Lambda_\lambda\left(\frac{1}{2}(I + s_x X + s_y Y + s_z Z)\right) &= (1-\lambda)\rho + \frac{\lambda}{2}(I + s_x X - s_y Y - s_z Z) \\ &= \frac{1}{2}(I + s_x X + (1-2\lambda)s_y Y + (1-2\lambda)s_z Z).\end{aligned} \tag{11}$$

That is,

$$\mathbf{s} = (s_x, s_y, s_z) \mapsto \mathbf{s}' = (s_x, (1-2\lambda)s_y, (1-2\lambda)s_z). \tag{12}$$

2. Is this channel teleportation-covariant? If so, give the 4 "correcting" unitaries V_i for $i = 0, \ldots, 3$, such that:

$$V_i \Lambda_\lambda(\sigma_i \rho \sigma_i) V_i^\dagger = \Lambda_\lambda(\rho).$$

Solution: Let us first recall the definition of a teleportation-covariant channel:

$$\Lambda(U\rho U^\dagger) = V\Lambda(\rho)V^\dagger. \tag{13}$$

Hence, by finding the four unitaries $\{V_i\}$ s.t. $V_i^\dagger \Lambda_\lambda(\sigma_i\rho\sigma_i)V_i = \Lambda_\lambda(\rho)$, we prove that Λ_λ is teleportation-covariant.

- $i = 0$: $V_0^\dagger \Lambda_\lambda(\rho) V_0 = \Lambda_\lambda(\rho) \iff V_0 = I$.
- $i = 1$: $V_1^\dagger \Lambda_\lambda(X\rho X)V_1 = V_1^\dagger \Lambda_\lambda(1/2(I + s_x X - s_y Y - s_z Z))V_1 = V_1^\dagger 1/2(I + s_x X - (1-2\lambda)s_y Y - (1-2\lambda)s_z Z)V_1 = \Lambda_\lambda(\rho) \iff V_1 = X$.
- $i = 2$: $V_2^\dagger \Lambda_\lambda(Y\rho Y)V_2 = V_2^\dagger 1/2(I - s_x X + (1-2\lambda)s_y Y - (1-2\lambda)s_z Z)V_2 = \Lambda_\lambda(\rho) \iff V_2 = Y$.
- $i = 3$: $V_3^\dagger \Lambda_\lambda(Z\rho Z)V_3 = V_3^\dagger 1/2(I - s_x X - (1-2\lambda)s_y Y + (1-2\lambda)s_z Z)V_3 = \Lambda_\lambda(\rho) \iff V_3 = Z$.

In short,

$$\{V_i\}_{i=0}^3 = \{I, X, Y, Z\}. \tag{14}$$

3. Give the Choi matrix J_λ of Λ_λ.

Solution: From the definition of the channel one can easily see that:

$$\{K_i\} = \{\sqrt{1-\lambda}I, \sqrt{\lambda}X\}, \tag{15}$$

the Kraus operators set s.t. $\Lambda_\lambda(\rho) = \sum_i K_i \rho K_i^\dagger$ and $\sum_i K_i^\dagger K_I = I$.

Then, J_{Λ_λ} follows from its eigendecomposition:

$$J_{\Lambda_\lambda} = |K_0\rangle\langle K_0| + |K_1\rangle\langle K_1| = \begin{pmatrix} 1-\lambda & 0 & 0 & 1-\lambda \\ 0 & \lambda & \lambda & 0 \\ 0 & \lambda & \lambda & 0 \\ 1-\lambda & 0 & 0 & 1-\lambda \end{pmatrix}. \tag{16}$$

Where we note that the vectorisation of each Kraus operator corresponds to:

$$\begin{aligned}|K_0\rangle &= \sqrt{1-\lambda}(|00\rangle + |11\rangle) \doteq \sqrt{1-\lambda}(1,0,0,1)^T, \\ |K_1\rangle &= \sqrt{\lambda}(|01\rangle + |10\rangle) \doteq \sqrt{\lambda}(0,1,1,0)^T.\end{aligned} \tag{17}$$

We wish to discriminate between two equiprobable channels Λ_{λ_1} and Λ_{λ_2}, where $\lambda_2 > \lambda_1$.

4. Imagine we do so by sending the input state $\rho = |0\rangle\langle 0|$ through the channel and optimally measuring the output. Compute the resulting probability of error.

Solution: As the task corresponds to discriminating between the two output states, we compute the action of each channel on the input state $\rho_{\text{in}} = |0\rangle\langle 0|$, that is:

$$\Lambda_{\lambda_i}(\rho_{\text{in}}) = (1-\lambda_i)\,|0\rangle\langle 0| + \lambda_i\,|1\rangle\langle 1| = \rho_{\text{out,i}}. \tag{18}$$

Then, given a Helstrom measurement:

$$P_e = \frac{1}{2}\left(1 - \frac{1}{2}||\rho_{\text{out},1} - \rho_{\text{out},2}||_1\right). \tag{19}$$

Recall that the trace norm of an Hermitian matrix is simply the sum of the eigenvalues in absolute value. besides, as each $\rho_{\text{out,i}}$ is diagonal, Eq.(18), readily follows that:

$$||\rho_{\text{out},1} - \rho_{\text{out},2}||_1 = |1-\lambda_1-1+\lambda_2| + |\lambda_1-\lambda_2| = 2(\lambda_2-\lambda_1), \tag{20}$$

for $\lambda_2 > \lambda_1$.

Therefore,

$$P_e = \frac{1}{2}(1-\lambda_2+\lambda_1). \tag{21}$$

5. Now consider using entanglement by sending qubit A, which is maximally entangled with qubit B, through the channel:

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$$

Perform the optimal measurement on the output state ρ_{AB} and compute the probability of error.

Solution: By definition of the Choi operator, Eq.(6), the output state of each channel is:

$$\rho_{\text{out,i}} = (\Lambda_{\lambda_i} \otimes I)[\phi^+_{AB}] = \frac{1}{2}J_{\Lambda_{\lambda_i}}. \tag{22}$$

Hence,

$$||\rho_{\text{out},1} - \rho_{\text{out},2}||_1 = 2(\lambda_2-\lambda_1), \tag{23}$$

as the non-zero eigenvalues of $\rho_{\text{out},1} - \rho_{\text{out},2}$ are $\{-\lambda_1+\lambda_2, \lambda_1-\lambda_2\}$.

So that the error probability is as before, i.e.,

$$P_e = \frac{1}{2}(1-\lambda_2+\lambda_1). \tag{24}$$

6. Based on part (5), would you say this particular entangled strategy (choice of input state) is optimal, i.e., it gives the smallest possible error probability?

Solution: As shown in the lectures, in the setting of quantum teleportation-covariant channels discrimination task it is equivalent to discriminate between the two output states and to discriminate the two Choi operators. This fact is reflected in the equality of the optimal error probability of both approaches.

7. What can you say about the optimal strategy when n uses of the channel are allowed?

Solution: When n-channel uses are allowed the error probability reads as:

$$P_e^{(n)} = \frac{1}{2}\left(1 - \frac{1}{2}||\rho_{\text{out},1}^{\otimes n} - \rho_{\text{out},2}^{\otimes n}||_1\right), \tag{25}$$

which, as seen, gives the same expression when discriminating between the Choi operators of each channel.

From the expression of $\rho_{\text{out,i}}$, Eq.(18), follows that:

$$\rho_{\text{out,i}}^{\otimes n} = \text{diag}((1-\lambda_i)^{n-k}\lambda_i^k), \qquad k = 0, \dots, n, \tag{26}$$

where each element has multiplicity $\binom{n}{k}$.

Hence,

$$||\rho_{\text{out},1}^{\otimes n} - \rho_{\text{out},2}^{\otimes n}||_1 = \sum_{k=0}^{n}\binom{n}{k}|(1-\lambda_1)^{n-k}\lambda_1^k - (1-\lambda_2)^{n-k}\lambda_2^k|, \tag{27}$$

which yields:

$$P_e^{(n)} = \frac{1}{2}\left(1 - \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}|(1-\lambda_1)^{n-k}\lambda_1^k - (1-\lambda_2)^{n-k}\lambda_2^k|\right). \tag{28}$$

Fig.1 shows the graphic representation of the previous quantity for a range of values of n. Clearly, $P_e \to 0$ as $n \to \infty$, as we expected.

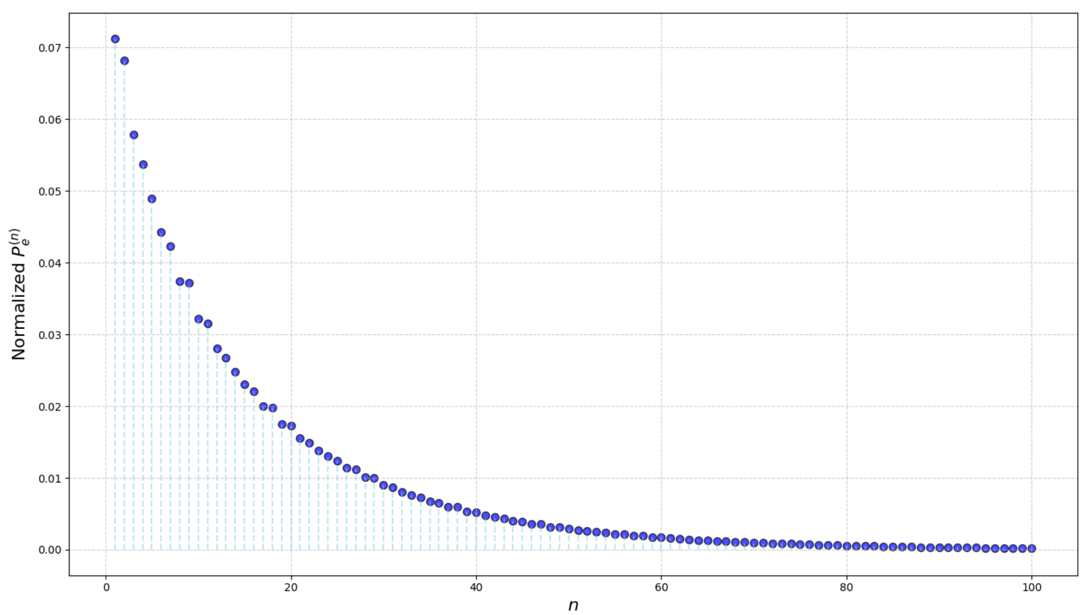

Figure 1: Normalised $P_e^{(n)}$ vs n for $\lambda_1 = 0.3$ and $\lambda_2 = 0.6$.

Moreover, we compute the Quantum Chernoff Bound (QCB) to fully characterise the discrimination task:

$$QCB(\Lambda_{\lambda_1}, \Lambda_{\lambda_2}) = -\lim_{n\to\infty}\frac{1}{n}\log P_e^{(n)} = -\min_{0\leq s\leq 1}\log tr(\rho_{\text{out},1}^s\rho_{\text{out},2}^{1-s}). \tag{29}$$

As $\rho_{\text{out,i}}$ is diagonal in the computational basis one can easily see that:

$$\log tr(\rho_{\text{out},1}^s\rho_{\text{out},2}^{1-s}) = \log f(s), \qquad f(s) = (1-\lambda_1)^s(1-\lambda_2)^{1-s} + \lambda_1^s\lambda_2^{1-s}. \tag{30}$$

Due to the fact that $\log(\cdot)$ is a monotonically increasing function, we minimise $f(s)$:

$$f'(s) = (1-\lambda_1)^s(1-\lambda_2)^{1-s}\log\frac{1-\lambda_1}{1-\lambda_2} + \lambda_1^s\lambda_2^{1-s}\log\frac{\lambda_1}{\lambda_2} = 0, \tag{31}$$

achieved for:

$$s^* = \frac{\log \frac{\lambda_2}{1-\lambda_2}}{\log \frac{1-\lambda_1}{\lambda_1} - \log \frac{\lambda_2}{1-\lambda_2}}, \tag{32}$$

which can easily be seen that corresponds to a minimum as $f''(s^*) > 0$.

From this optimum s^*, in Fig.2 we see how the QCB provides the correct error exponent for $P_e^{(n)}$ in the asymptotic limit.

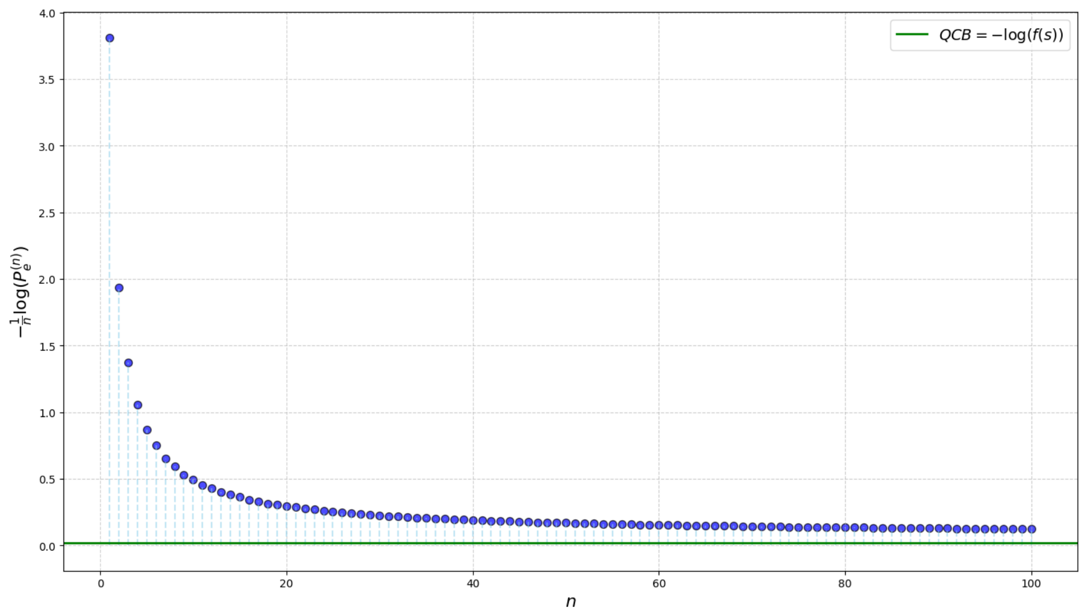

Figure 2: $-\frac{1}{n}\log P_e^{(n)}$ vs n and the QCB for $\lambda_1 = 0.3$ and $\lambda_2 = 0.6$.