Universitat de Barcelona
Facultat de Física

MOCK EXAM Instructions

Write your name in all the pages, preferably at the top end.

Answer **ALL** questions in Section A.

Answer **TWO** of THREE questions in Section B. If you attempt more than two questions, the best two solutions will be considered.

The mark carried by an individual part of a question is indicated in square brackets [ ].

Section A - answer all questions

A1. (a) State in your own words the sixth postulate of quantum mechanics.

[3 marks]

(b) A system evolves in time as dictated by a time-independent hamiltonian $\hat{H}$. At $t = 0$, the system is in an eigenstate $|\alpha\rangle$. Find an expression for $|\alpha\rangle$ as a function of time.

[3 marks]

(c) How does $\langle\alpha|\alpha\rangle$ evolve in time?

[2 marks]

(d) Consider a state $|\psi\rangle$, which is not necessarily an eigenstate of $\hat{H}$. What is the transition probability $P_{\alpha\to\psi}$ as a function of time?

[2 marks]

A2. The harmonic oscillator raising and lowering operators fulfil the commutation relation $\left[a, \left(a^\dagger\right)^n\right] = n\left(a^\dagger\right)^{n-1}$.

(a) Show that $ae^{\beta a^\dagger} = e^{\beta a^\dagger}(\beta + a)$ for any complex β.

[3 marks]

(b) Using this result, or any other method, show that $|\psi\rangle = e^{\beta a^\dagger}|0\rangle$ is an eigenstate of the annihilation operator $\hat{a}$. How do we call such states?

[5 marks]

(c) Find the normalization constant of $|\psi\rangle$ as a function of β.

[2 marks]

A3. Consider a two-level system with energy levels $E_+ = \epsilon + \delta/2$ and $E_- = \epsilon - \delta/2$. We add a perturbation to the hamiltonian reading $H_1 = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}$.

(a) What is the first-order perturbation theory correction to the energies?

[3 marks]

(b) What is the second-order perturbation theory correction to the energies?

[3 marks]

(c) Find the exact result. Show that the separation between levels increases as a function of v.

[4 marks]

A4. A spin $\frac{1}{2}$ particle travels across a region of space with constant magnetic field, $\mathbf{B} = B_0\hat{\mathbf{k}}$. The hamiltonian is $\hat{H} = -\mu_0 B_0 \sigma_z$.

(a) The time evolution of any given spin state can be expressed as

$$|S(t)\rangle = a_+(t)|+\rangle + a_-(t)|-\rangle\,,$$

where $|+\rangle$ and $|-\rangle$ are the eigenstates of $\hat{S}_z$. Find the equations of motion for $a_\pm(t)$ and solve them.

[3 marks]

(b) Originally, the particle is in an eigenstate of $\hat{S}_x$, $|S_x, \pm\rangle$. Express these two possible initial states in terms of $|+\rangle$ and $|-\rangle$.

[3 marks]

(c) Find the expectation value of $\hat{S}_x$ as a function of time assuming that at $t = 0$ the system is in the state $|S_x, +\rangle$. When does the expectation value become zero?

[4 marks]

A5. In second quantization, one defines the particle number operator as $\hat{N} = \sum_\alpha a^\dagger_\alpha a_\alpha$. Consider a Hamiltonian of the one-body type, $\hat{H} = \sum_{\alpha\beta} t_{\alpha\beta} a^\dagger_\alpha a_\beta$.

(a) What is the expectation of $\hat{N}$ over a a one-particle fermionic state $|\delta\rangle$.

[2 marks]

(b) Consider now a two-particle fermionic state $|\delta\gamma\rangle$. Find the expectation of $\hat{N}$ over this state.

[4 marks]

(c) Find an expression for the expectation value of $\hat{H}$ in these two states.

[4 marks]

Section B - answer TWO of THREE questions

B1. The eigenfunctions of the one-dimensional harmonic oscillator of frequency ω, $\psi_n(x) = \langle x|n\rangle$, transform under parity as $\psi_n(-x) = (-1)^n \psi_n(x)$. Consider the modified potential

$$V(x) = \begin{cases} +\infty, x \leqslant 0, \\ \frac{1}{2}x^2, x > 0\,, \end{cases}$$

in natural oscillator units. The eigenstates of this potential are $\phi_n(x)$.

(a) Can the wave functions $\phi_n(x)$ extend over all space? What boundary condition does this impose?

[4 marks]

(b) Find a relationship between $\psi_n(x)$ and $\phi_m(x)$. Sketch the wave function of the first 2 energy eigenstates, $\phi_0(x)$ and $\phi_1(x)$.

[6 marks]

(c) Find the energy eigenvalues of particle in this potential.

[6 marks]

(d) Consider a perturbation λx on top of $V(x)$. Write the first-order correction to the energy for the lowest-energy state.

[9 marks]

You may use the integral result $\int_0^\infty dx x^{2n+1} e^{-x^2} = \frac{n!}{2}$.

B2. A one-dimensional harmonic oscillator of frequency ω_0 is placed in a time-dependent spatially homogeneous electric field $E = E_0 \frac{\tau^2}{t^2+\tau^2}$. This gives rise to a time-dependent perturbation $\hat{H}_1(t) = E_0 \hat{x} \frac{\tau^2}{t^2+\tau^2}$.

(a) Show that the transition matrix elements have the form

$$H_{mn}(t) = \frac{E_0}{\sqrt{2}} \frac{\tau^2}{t^2+\tau^2} \left[\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right] .$$

[7 marks]

(b) At $t = -\infty$, the atom is in the ground state. Use time-dependent perturbation theory to compute the transition amplitude to the first excited state at $t = \infty$.

[10 marks]

(c) Discuss physically what happens in the limit of $\tau \to \infty$.

[4 marks]

(d) Can the atom perform a transition with an energy difference $\Delta\epsilon = 2\omega_0$?

[4 marks]

You may use the integral result $\int_{-\infty}^{\infty} dt \frac{1}{1+t^2} e^{i\alpha t} = \pi e^{-|\alpha|}$.

B3. Consider the $s-$wave elastic scattering properties of a particle of mass m and energy E.

(a) If the interaction potential is very short-ranged, we can approximate it as $V(\mathbf{r}) = g\delta(\mathbf{r})$, with $\delta(\mathbf{r})$ a δ-function. Compute the differential cross section in the Born approximation.

[8 marks]

(b) Find the total cross section at low energies and deduce from this a relation between the scattering length, a_s, and the coupling, g.

[5 marks]

(c) Consider now an interaction potential that is an attractive square well of depth V_0 and radius R. The scattering phase-shift is $\delta_0 = \arctan\left[\frac{k}{\sqrt{k_0^2+k^2}} \tan\left(\sqrt{k_0^2+k^2}R\right)\right] - kR$, with $k^2 = \frac{2mE}{\hbar^2}$ and $k_0^2 = \frac{2mV_0}{\hbar^2}$. Find an expression for the scattering length of this potential as a function of R and k_0. Is it positive or negative?

[5 marks]

(d) Matching the expressions for the scattering lengths of the two potentials, find and expression for g as a function of V_0 and R.

[5 marks]

(e) Explain why this procedure can only work for relatively shallow square wells.

[2 marks]