Homework Lecture 4

October 20, 2023

1. **Let** $(X^n, Y^n) \sim p(x^n, y^n)$. Prove:

$$\sum_{i=1}^{n} I(X_{i+1}^n; Y_i | Y_1^{i-1}) = \sum_{i=1}^{n} I(X_i; Y_1^{i-1} | X_{i+1}^n),$$

   where $X_i^j = (X_i, \ldots, X_j)$, $Y_i^j = (Y_i, \ldots, Y_j)$ and $X_{n+1}^n = Y_1^0 = \emptyset$.
   Hint: Note that by application of the chain rule of mutual information:

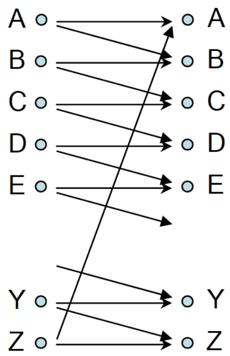
$$I(X_{i+1}^n; Y_1^i) = I(X_{i+1}^n; Y_1^{i-1}) + I(X_{i+1}^n; Y_i | Y_1^{i-1}),$$

   and also:

$$I(X_i^n; Y_1^{i-1}) = I(X_{i+1}^n; Y_1^{i-1}) + I(X_i; Y_1^{i-1} | X_{i+1}^n)$$

2. **Inequalities**. Label each of the following statements with $=, \leq$, or $\geq$. Justify each step, unjustified responses will be considered wrong.
   (a) $H(X|Z)$ versus $H(X|Y) + H(Y|Z)$. **Hint**: Consider $H(X, Y|Z)$.
   (b) $H(X + Y)$ versus $H(X)$, if X and Y are independent. **Hint**: Consider $H(X + Y|Y)$.
   (c) $I(X_1, X_2; Y_1, Y_2)$ versus $I(X_1; Y_1) + I(X_2; Y_2)$, if $p(y_1, y_2|x_1, x_2) = p(y_1|x_1)p(y_2|x_2)$.
   Hint: Apply the chain rule sequentially and note that $Y_1 \leftrightarrow X_1 \leftrightarrow X_2 \leftrightarrow Y_2$ form a Markov chain.
   (d) $I(X_1, X_2; Y_1, Y_2)$ versus $I(X_1; Y_1) + I(X_2; Y_2)$, if $p(x_1, x_2) = p(x_1)p(x_2)$.
   Hint: Apply the chain rule sequentially again.

3. Z **channel**. The Z channel has binary input and output alphabets, and conditional pmf $p(0|0) = 1, p(1|1) = p(0|1) = 1/2$. Find the capacity C.



4. **The Noisy typewriter channel**. Compute the capacity of the noisy typewriter channel defined above. Consider that all lines represent a $1/2$ probability and that there are 26 characters in the alphabet. Provide two solutions for the maximizing pmf $p(x)$, that yield capacity.

- You are welcome to cooperate among yourselves, but cooperation must be declared. You can even make a joint delivery for up to four students.
- Use preferably LaTeX for the solutions and deliver them as a .pdf file in the corresponding task in Campus Virtual. In the case of joint delivery, only one of the students should deliver it.