Quantum Information Theory
- Homework 6 -

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Part I

The first part of the exercise asks us to determine $I(X;Y)$ aswell as $I(X;B)_\rho$ for the given classical-quantum state before and after a POVM measurement.

We start with the slightly easier part of obtaining the mutual information before the POVM measurement, so $I(X;B)_\rho$.

$$I(X;B)_\rho = H(B)_\rho - H(B \mid X)_\rho = H(B)_{\rho_B}$$

where we have taken advantage of the conditioned on the classical variable the state is pure. To determine $H(B)_\rho$, we need to acquire the ρ_B first by taking the partial trace.

$$\begin{aligned}\rho_B = \mathrm{Tr}_X\left\{\rho_{XB}\right\} &= \frac{1}{2}\left(\left|\theta_0\right\rangle\left\langle\theta_0\right| + \left|\theta_1\right\rangle\left\langle\theta_1\right|\right) = \\ &= \begin{bmatrix}\cos^2\left(\frac{\theta}{2}\right) & 0 \\ 0 & \sin^2\left(\frac{\theta}{2}\right)\end{bmatrix}\end{aligned}$$

Which finally leads to:

$$I(X;Y) = \frac{1}{2}(1+\sin\theta)\log(1+\sin\theta) + \frac{1}{2}(1-\sin\theta)\log(1-\sin\theta).$$

To compute $I(X;Y)$, we once again have to compute $H(Y)$ and $H(Y \mid X)$, but without the luxury of the conditional entropy being zero. On the other hand we notice two important aspects. Firstly, that $p_Y(0) = p_Y(1)$.

$$p_Y(0) = \frac{1}{2}\left(p_{Y|X}(0 \mid 0) + p_{Y|X}(0 \mid 1)\right) = \frac{1}{2}\left(1 - P_e + P_e\right) = \frac{1}{2}\left(p_{Y|X}(1 \mid 0) + p_{Y|X}(1 \mid 1)\right) = \frac{1}{2} = p_Y(1)$$

which leads to

$$\begin{aligned}H(Y) &= -\sum_y p_Y(y)\log p_Y(y) \\ &= -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} \\ &= 1.\end{aligned}$$

And secondly, do to the symmetry of the channel, we can express the conditional entropy simply by using the given error probability, $P_e = \frac{1}{2}(1-\sin\theta)$.

$$\begin{aligned}H(Y \mid X) &= H(Y \mid x=0)p_X(x=0) - H(Y \mid x=1)p_X(x=1) \\ &= -\left(1-P_e\right)\log\left(1-P_e\right) - P_e\log P_e\end{aligned}$$

This eventually gives us the following expression for the mutual information

$$I(X;Y) = 1 + \left(1 - \frac{1}{2}(1-\sin\theta)\right)\log\left\{1 - \frac{1}{2}(1-\sin\theta)\right\} + \frac{1}{2}(1-\sin\theta)\log\left\{\frac{1}{2}(1-\sin\theta)\right\}$$