# Quantum Technologies w/ Superconducting circuits, assig. 2

Guillermo Abad

12 February, 2024

## 1 Superconductivity question

### The kinetic inductance is a source of magnetic flux in a superconducting circuit:

No, although the Josephson junction kinetic behaviour is similar to that of an inductor, it has no associated magnetic field. The behaviour, instead of the energy in a magnetic field, is derived from the kinetic energy of the charge carriers.

When the current and phase change in time, the voltage across the junction will also change. This behaviour can be modeled by the kinetic inductance (Josephson Inductance).

For superconducting flux qubits, the parameter $\beta$ is normally the ratio of the kinetic inductance of the Josephson junctions, to the geometrical inductance of the flux qubit. A design with a low beta behaves more like a simple inductive loop, while a design with a high beta is dominated by the Josephson junctions and has more hysteretic behaviour.

## 2 Circuit quantization question

### A Josephson junction is necessary in a circuit to exhibit quantization of its levels:

False, the Josephson junction is necessary to break the harmonicity, so the levels are not all equally spaced, and we can have a controllable 2-level system!

Basically, the Josephson junction adds a sinusoidal term to the Hamiltonian, that brings terms of higher order than the normal quadratic ones in an harmonic oscillator.

## 3 Qubit control question

### A discrete LC resonator needs to be as large as the wavelength of its microwave resonance:

Strictly it does not need to be "as large as" the wavelength, but it needs to be of the same order of magnitude if that was the question, since depending on if it is open or shorted, it needs to be $\lambda/2$ or $\lambda/4$ respectively.

## Problem

**Consider a two-level system driven by an external field with a drive where the phase $\phi$ is controllable $H/\hbar = \frac{\Delta}{2}\sigma_x + \frac{\epsilon}{2}\sigma_z + H_d/\hbar$ ($H_d = \hbar\Omega_R\cos(\omega_d t + \phi)\sigma_z$):**

### a) Transform the Hamiltonian $H$ into the diagonal basis of the qubit.

First of all, let's define the qubit frequency as $\omega_q = \sqrt{\epsilon^2 + \Delta^2}$, which is the module of the Pauli's vector. And now we can write the qubit Hamiltonian as:

$$H/\hbar = \frac{\omega_q}{2}M + H_d \quad \text{with} \quad M = \frac{1}{\omega_q}\begin{pmatrix} \epsilon & \Delta \\ \Delta & -\epsilon \end{pmatrix}$$

where we clearly see that the vectors of the matrix $M$ are an orthonormal basis, so $|\det(M)| = 1$ of course, since the area between the vectors is the unit square.

Concretely we see that $\det(M) = -1$, which together with the fact that both eigenvalues need to be equal in module (orthonormality), tell us directly that the eigenvalues of the $M$ matrix are $\lambda = 1, -1$ (also $\lambda = |\sigma_0| \pm |\vec{\sigma}| = \pm(\epsilon^2 + \Delta^2)/\omega_q^2 = \pm 1$).

So in its diagonal basis, $M$ is actually just $M' = \sigma_z'$, telling us that our matrix is its own inverse ($\sigma_z^2 = \mathbb{I} \ \rightarrow \ \sigma_z = \sigma_z^{-1} \ \rightarrow \ M = M^{-1}$).

With this we can skip any complicated vector transformation if we realize that to go to the diagonal basis, we have done an $M^{-1}$, plus a negation of an axis ($\sigma_z$):

$$\frac{\Delta\sigma_x + \epsilon\sigma_z}{\omega_q} \xrightarrow{\text{Basis change } = \ \sigma_z M^{-1} \ = \ \sigma_z M} \sigma_z$$

So now it's obvious where the drive component will go:

$$\sigma_z \xrightarrow{\text{Basis change } = \ \sigma_z M} \sigma_z M \sigma_z = \sigma_z \frac{\Delta\sigma_x + \epsilon\sigma_z}{\omega_q}\sigma_z = \frac{\Delta\sigma_z\sigma_x\sigma_z + \epsilon\sigma_z}{\omega_q} = \frac{-\Delta\sigma_x + \epsilon\sigma_z}{\omega_q}$$

Giving a final Hamiltonian in the diagonal basis of the qubit:

$$\boxed{H/\hbar \xrightarrow{\text{Basis change}} H'/\hbar = \frac{\omega_q}{2}\sigma_z' + \Omega_R\cos(\omega_d t + \phi)\frac{\epsilon\sigma_z' - \Delta\sigma_x'}{\omega_q}}$$

### b) Perform the rotating-wave approximation (RWA) to the external drive.

I will actually do this step rigorously in the next section, at c), but a quick intuition can be gained if we express the cosines with its imaginary exponential form:

$$H/\hbar = \frac{\omega_q}{2}\sigma_z + \Omega_R(e^{i\omega_d t}e^{i\phi} + e^{-i\omega_d t}e^{-i\phi})\frac{\epsilon\sigma_z - \Delta\sigma_x}{2\omega_q}$$

Where to apply the RWA, we can just extract a single phase outside the parenthesis:

$$H/\hbar = \frac{\omega_q}{2}\sigma_z + \Omega_R e^{i\omega_d t}(e^{i\phi} + \cancel{e^{-2i\omega_d t}e^{-i\phi}})\frac{\epsilon\sigma_z - \Delta\sigma_x}{2\omega_q}$$

where if we get rid of this extra $e^{i\omega_d t}$ phase, as we will do later going to the rotating frame, we see that we will have a static term, plus an oscillating one, and in enough large time integrations the last one won't contribute compared to the static term, giving:

$$\boxed{H/\hbar \approx \frac{\omega_q}{2}\sigma_z + \Omega_R e^{i\omega_d t}e^{i\phi}\frac{\epsilon\sigma_z - \Delta\sigma_x}{2\omega_q} \quad \left(\text{same H with } \cos(\omega_d t + \phi) \approx e^{i\omega_d t}e^{i\phi}\right)}$$

**c) Consider now the qubit to be at its symmetry point $\epsilon = 0$. With the RWA applied to the drive, transform the total Hamiltonian to the frame rotating at the drive frequency, $U = e^{i(\omega_d/2)t\sigma_z}$, and show that the transformed Hamiltonian $H$ becomes: $H'/\hbar = \frac{1}{2}(\omega_q - \omega_d)\sigma_z - \frac{\Omega_R}{2}[\cos(\phi)\sigma_x - \sin(\phi)\sigma_y]$**

Let's now start again from the final Hamiltonian in a), in the symmetry point ($\epsilon = 0$ and $\Delta = \omega_q$):

$$H/\hbar = \frac{\omega_q}{2}\sigma_z - \Omega_R \cos(\omega_d t + \phi)\sigma_x$$

And now let's move to the drive frequency rotating frame, $H' = U^\dagger H U - iU\frac{\partial U^\dagger}{\partial t}$:

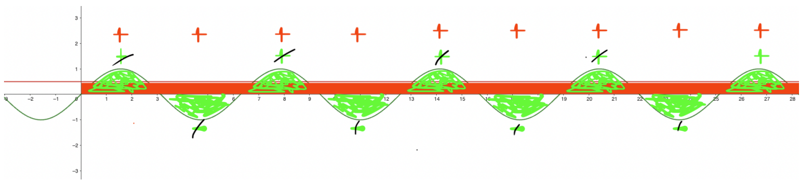
$$\begin{aligned}
H'/\hbar &= \frac{\omega_q}{2}\left(U^\dagger \sigma_z U\right) - \Omega_R \cos(\omega_d t + \phi)\left(U^\dagger \sigma_x U\right) - iU\frac{-i}{2}\omega_d \sigma_z U^\dagger = \\
&= \frac{\omega_q}{2}\sigma_z - \Omega_R \cos(\omega_d t + \phi)\left(e^{-\frac{i}{2}\omega_d t\sigma_z}\sigma_x e^{+\frac{i}{2}\omega_d t\sigma_z}\right) - \frac{\omega_d}{2}\sigma_z = \\
&= \frac{\omega_q - \omega_d}{2}\sigma_z - \Omega_R c_{\omega_d t+\phi}\left(c_{\omega_d t/2}\mathbb{I} - is_{\omega_d t/2}\sigma_z\right)\sigma_x\left(c_{\omega_d t/2}\mathbb{I} + is_{\omega_d t/2}\sigma_z\right) = \\
&= \frac{\omega_q - \omega_d}{2}\sigma_z - \Omega_R c_{\omega_d t+\phi}\left(c^2_{\omega_d t/2}\sigma_x + s^2_{\omega_d t/2}(\sigma_z\sigma_x\sigma_z) - is_{\omega_d t/2}c_{\omega_d t/2}[\sigma_z, \sigma_x]\right) = \\
&= \frac{\omega_q - \omega_d}{2}\sigma_z - \Omega_R c_{\omega_d t+\phi}\left(\left(c^2_{\omega_d t/2} - s^2_{\omega_d t/2}\right)\sigma_x - is_{\omega_d t/2}c_{\omega_d t/2}(2i\sigma_y)\right) = \\
&= \frac{\omega_q - \omega_d}{2}\sigma_z - \Omega_R \cos(\omega_d t + \phi)\left(\cos(\omega_d t)\sigma_x + \sin(\omega_d t)\sigma_y\right)
\end{aligned}$$

where in the end I've used $c_x \equiv \cos(x)$, and the identities $c_x^2 - s_x^2 = c_{2x}$ and $s_x c_x = \frac{1}{2}s_{2x}$.

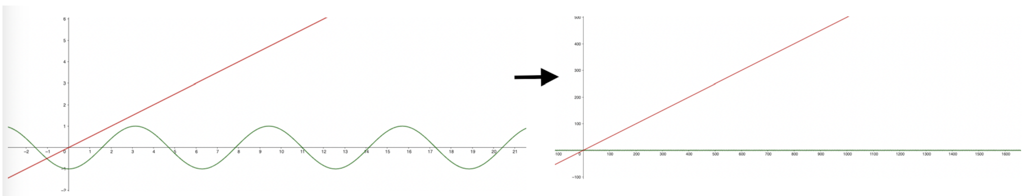
Now to simplify it further and get the proposed expression, we will need to apply the RWA. I'm going to do so, in the following expressions ($a = \omega_d t$ and $b = \phi$):

$$\cos(a)\cos(a+b) = \frac{\cos(2a+b) + \cos(b)}{2} \xrightarrow{\int^{\inf} da \quad \text{(RWA)}} \frac{\cos(b)}{2}$$
$$\sin(a)\cos(a+b) = \frac{\sin(2a+b) - \sin(b)}{2} \xrightarrow{\int^{\inf} da \quad \text{(RWA)}} -\frac{\sin(b)}{2}$$

where the oscillations areas cancel themselves in each period, while the constant term area just keeps accumulating (red function is $f(x) = 0.5$ and green function is g(x)=sin(x)):



giving an integration, where for long enough intervals the oscillation part is redundant:



So, finally, the Hamiltonian in the RWA, ends up like the desired expression:

$$\boxed{H'/\hbar \approx \frac{\omega_q - \omega_d}{2}\sigma_z - \frac{\Omega_R}{2}\left(\cos(\phi)\sigma_x - \sin(\phi)\sigma_y\right)}$$