## Problem 1: Homogeneous Fermi gas

Consider an ideal Fermi gas with a single spin component and dispersion relation $\epsilon_k = \hbar^2 k^2/2m$.

1. The ground state is a Fermi sea filled up to the Fermi wavevector $k_F$ and Fermi energy $\epsilon_k = \epsilon(k_F)$. Compute the density $n$ and total energy $E$ in $d = 1, 2, 3$ dimensions. Give the polytropic index $\gamma$ in the equation of state $\mu(T = 0) = \epsilon_k \propto n^\gamma$ in dimension $d$.

2. Compute the density of states

$$\rho(\epsilon) = \frac{1}{V} \sum \delta(\epsilon - \epsilon_k) \tag{1}$$

   in d = 1, 2, 3 dimensions. Give $\rho(\epsilon_F)$ both in terms of $k_F$, and in terms of $n$ and $\epsilon_F$.

## Problem 2: Phonons

A monoatomic linear chain consisting of atoms of mass $m = 6.81\text{x}10^{-26} kg$, with an equilibrium separation of 4.85 Å, has a propagation speed for sound waves of $1.08\text{x}10^4 m/s$. Assuming a classical model with nearest neighbor interaction, determine the value of the elastic constant and the maximum frequency of the modes.

## Problem 1

1. The total number of electrons $N$ is

$$N = \sum_i 1 \quad , \text{ a sum over all the different states with } k_i = \frac{2\pi n_i}{L}$$

We can approximate the sum to an integral:

$$N = \int \left(\frac{L}{2\pi}\right)^d dk^d$$

$$= V \frac{1}{(2\pi)^d} \int dk^d$$

$\rightarrow d=1$: $$n = \frac{1}{2\pi}\int dk = \frac{1}{2\pi}\int 2\, dk = \frac{1}{\pi}\int \sqrt{\frac{m}{2\varepsilon}}\frac{d\varepsilon}{\hbar}$$

$\rightarrow d=2$: $$n = \frac{1}{4\pi^2}\int dk^2 = \frac{1}{4\pi^2}\int 2\pi k\, dk = \frac{1}{2\pi}\int \frac{\sqrt{2m\varepsilon}}{\hbar}\sqrt{\frac{m}{2\varepsilon}}\frac{d\varepsilon}{\hbar} = \frac{1}{2\pi}\int \frac{m}{\hbar^2}\, d\varepsilon$$

$\rightarrow d=3$: $$n = \frac{1}{8\pi^3}\int dk^3 = \frac{1}{8\pi^3}\int 4\pi k^2\, dk = \frac{1}{2\pi^2}\int \frac{2m\varepsilon}{\hbar^2}\sqrt{\frac{m}{2\varepsilon}}\frac{d\varepsilon}{\hbar} = \frac{1}{2\pi^2}\int \frac{m^{3/2}}{\hbar^3}\sqrt{2\varepsilon}\, d\varepsilon$$

For the ground state,

$d=1$: $$n = \frac{1}{2}\frac{\sqrt{2m}}{\pi\hbar}\int_0^{\varepsilon_F}\frac{d\varepsilon}{\sqrt{\varepsilon}} = \frac{\sqrt{2m}}{\pi\hbar}\sqrt{\varepsilon_F} \qquad \propto \varepsilon_F^{1/2}$$

$d=2$: $$n = \frac{m}{2\pi\hbar^2}\int_0^{\varepsilon_F} d\varepsilon = \frac{m}{2\pi\hbar^2}\varepsilon_F \qquad \propto \varepsilon_F$$

$d=3$: $$n = \frac{m^{3/2}}{2\pi^2\hbar^3}\int_0^{\varepsilon_F}\sqrt{2\varepsilon}\, d\varepsilon = \frac{m^{3/2}}{\pi^2\hbar^3}\sqrt{2}\,\frac{1}{3}\varepsilon_F^{3/2} \qquad \propto \varepsilon_F^{3/2}$$

Therefore, the polytropic index is
$$\begin{cases} \gamma^{1D} = 2 \\ \gamma^{2D} = 1 \\ \gamma^{3D} = 2/3 \end{cases}$$

The total energy $E$ is calculated like:

$$E = V\int_0^{\varepsilon_F}\rho(\varepsilon)\,\varepsilon\, d\varepsilon$$

$d=1$: $$\frac{E}{V} = \frac{\sqrt{2m}}{2\pi\hbar}\int_0^{\varepsilon_F}\sqrt{\varepsilon}\, d\varepsilon = \frac{\sqrt{2m}}{\pi\hbar}\,\frac{1}{3}\varepsilon_F^{3/2}$$

$d=2$: $$\frac{E}{V} = \frac{m}{2\pi\hbar^2}\int_0^{\varepsilon_F}\varepsilon\, d\varepsilon = \frac{m}{\pi\hbar^2}\,\frac{1}{4}\varepsilon_F^2$$

$d=3$: $$\frac{E}{V} = \frac{m^{3/2}}{2\pi^2\hbar^3}\int_0^{\varepsilon_F}\sqrt{2}\,\varepsilon^{3/2}\, d\varepsilon = \frac{m^{3/2}}{\pi^2\hbar^3}\sqrt{2}\,\frac{1}{5}\varepsilon_F^{5/2}$$

2. The density of states, from the expresions above, are

$d=1$: $$\rho(\varepsilon) = \frac{\sqrt{2m}}{2\pi\hbar}\frac{1}{\sqrt{\varepsilon}} \Rightarrow \rho(\varepsilon_F) = \frac{\sqrt{2m}}{2\pi\hbar}\frac{1}{\sqrt{\varepsilon_F}} = \frac{m}{\pi\hbar^2}\frac{1}{k_F} = \frac{n}{2\varepsilon_F}$$

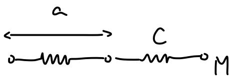
$d=2$: $$\rho(\varepsilon) = \frac{m}{2\pi\hbar^2} \Rightarrow \rho(\varepsilon_F) = \frac{m}{2\pi\hbar^2} = \frac{n}{\varepsilon_F}$$

$d=3$: $$\rho(\varepsilon) = \frac{m^{3/2}}{2\pi^2\hbar^3}\sqrt{2\varepsilon} \Rightarrow \rho(\varepsilon_F) = \frac{m^{3/2}}{2\pi^2\hbar^3}\sqrt{2\varepsilon_F} \underset{\uparrow}{=} \frac{m^2}{2\pi^2\hbar^2}k_F = \frac{3n}{2\varepsilon_F}$$

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}$$

$$\left.\right\} \rho(\varepsilon_F) = \frac{n}{\gamma\varepsilon_F}$$

## Problem 2




$M = 6'81 \cdot 10^{-26}$ kg

$a = 4'85$ Å

$V_s = 1'08 \cdot 10^4$ m/s

A nearest neighbor approximation gives:

$$V_s = \sqrt{\frac{C}{M}}\, a \Rightarrow C = \left(\frac{V_s}{a}\right)^2 M = 33'8 \ \frac{N}{m}$$

The dispersion relation is:

$$\omega^2(k) = \frac{2C}{M}\left(1 - \cos(ka)\right) \Rightarrow \omega_{max} = \omega\left(\frac{\pi}{a}\right) = 2\sqrt{\frac{C}{M}} = 44'5 \text{ THz}$$

*Nearest neighbor model:*

From the harmonic potential $U_{ar}$, we can reach the equation of motion of the ion:

$$U_{ar} = \frac{1}{2}\left[C\,(u_{n+1} - u_n)^2 + C\,(u_n - u_{n-1})^2\right]$$

$$M\ddot{u}_n = -\frac{\partial U_{ar}}{\partial u_n} = C\,(u_{n+1} + u_{n-1} - 2u_n)$$

By applying the following ansatz we obtain the phonon dispersion relation:

$$u_n = A e^{i(kx - \omega t)} \Rightarrow -M\omega^2 = C\,(e^{ika} + e^{-ika} - 2)$$

$$\Rightarrow -\omega^2 = \frac{2C}{M}\left(\cos(ka) - 1\right)$$

The speed of sound is obtained by inspecting the form of the dispersion relation when $k \to 0$, which should be like $\omega(k) = |V_s|\, k$.

$$k \to 0 \Rightarrow \cos(ka) \approx 1 - \frac{k^2a^2}{2}, \quad \cos(2ka) \approx 1 - 2k^2a^2$$

$$\Rightarrow \omega^2 = \frac{2C}{M}\frac{k^2a^2}{2} = \frac{C}{M}k^2a^2$$

$$\Rightarrow \omega = |V_s|\, k = \sqrt{\frac{C}{M}} \cdot a k \Rightarrow V_s = \sqrt{\frac{C}{M}} \cdot a$$