1. (a) **Hadamard's lemma**: for $\hat{A}$ and $\hat{B}$ two arbitrary operators, prove that:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A},\hat{B}\right] + \frac{1}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \frac{1}{3!}\left[\hat{A},\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\right] + \cdots$$

   (b) **Glauber formula** If two operators $\hat{A}$ and $\hat{B}$ do not commute, there is no simple relation between $e^{\hat{A}}e^{\hat{B}}$ and $e^{\hat{A}+\hat{B}}$. Suppose that both $\hat{A}$ and $\hat{B}$ commute with their commutator, so that $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$. Prove the formula $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{[\hat{A},\hat{B}]/2}$.

   Hint: introduce the operators $\hat{G}(t) = e^{t\hat{A}}\hat{B}e^{-t\hat{A}}$ and $\hat{F}(t) = e^{t\hat{A}}e^{t\hat{B}}$ and take derivatives with respect to t.

2. Consider the lowering, a, and raising, $a^\dagger$, operators.
   (a) Prove that $[\hat{a},(\hat{a}^\dagger)^2] = 2\hat{a}^\dagger$ and that $[\hat{a}^2,\hat{a}^\dagger] = 2\hat{a}$.
   (b) Using the result above, show that $[\hat{a},(\hat{a}^\dagger)^n] = n(\hat{a}^\dagger)^{n-1}$ and $[\hat{a}^n,\hat{a}^\dagger] = n\hat{a}^{n-1}$.
   (c) Prove that for an arbitrary function $f(x)$, the following relations hold

$$\left[\hat{a},f(\hat{a}^\dagger)\right] = \frac{\partial f(\hat{a}^\dagger)}{\partial \hat{a}^\dagger}\,, \quad \left[\hat{a}^\dagger,f(\hat{a})\right] = -\frac{\partial f(\hat{a})}{\partial \hat{a}}\,.$$

   (d) Prove that $\left[\hat{a},e^{\alpha\hat{a}^\dagger}\right] = \alpha e^{\alpha\hat{a}^\dagger}$ for α complex.
   (e) Use the previous result to express $e^{-\alpha\hat{a}^\dagger}\hat{a}e^{\alpha\hat{a}^\dagger}$ as a function of $\hat{a}$ and α.
   (f) Prove that $e^{+\alpha\hat{a}^\dagger a}\hat{a}e^{-\alpha\hat{a}^\dagger a} = e^{-\alpha}\hat{a}$.

3. **Squeezed states**
   Consider the squeezing operator $\hat{S}(\epsilon) = e^{\frac{\epsilon^*}{2}\hat{a}^2 - \frac{\epsilon}{2}(\hat{a}^\dagger)^2}$ where $\epsilon = re^{i2\phi}$ is a complex number. A squeezed state is defined as $|z,\epsilon\rangle = \hat{D}(z)\hat{S}(\epsilon)|0\rangle$, with $\hat{D}(z)$ the standard displacement operator $\hat{D}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}}$.
   (a) Show that the effect of the displacement operator on $\hat{a}$ and $\hat{a}^\dagger$ is

$$\hat{D}(z)^\dagger\hat{a}\hat{D}(z) = \hat{a} + z\,, \quad \hat{D}(z)^\dagger\hat{a}^\dagger\hat{D}(z) = \hat{a}^\dagger + z^*\,.$$

   (b) Using Hadamard's Lemma as written above, and the commutators in Problem 3.b, show that

$$\hat{S}(\epsilon)^\dagger\hat{a}\hat{S}(\epsilon) = \cosh r\hat{a} - e^{2i\phi}\sinh r\hat{a}^\dagger\,.$$

   (c) Prove also that

$$\hat{S}(\epsilon)^\dagger\hat{a}^\dagger\hat{S}(\epsilon) = \cosh r\hat{a}^\dagger - e^{-2i\phi}\sinh r\hat{a}\,.$$

   (d) Calculate the expectation values $\langle z,\epsilon|\hat{a}|z,\epsilon\rangle$ and $\langle z,\epsilon|\hat{a}^\dagger|z,\epsilon\rangle$. What are these values when $z = 0$ but $\epsilon \neq 0$?
   (e) Similarly, compute the expectation values of the squared lowering and raising operators, $\langle z,\epsilon|\hat{a}^2|z,\epsilon\rangle$ and $\langle z,\epsilon|(\hat{a}^\dagger)^2|z,\epsilon\rangle$. What are these values when $z = 0$ but $\epsilon \neq 0$?

UNIVERSITAT DE BARCELONA
Facultat de Física

(f) Finally, use the results above to find the dispersion of the position and momentum operators on a squeezed state. Find an expression for Heisenberg's uncertainty principle as a function of z, r and ϕ.

4. Consider the permutation operator of identical particles j and k, $\hat{P}_{jk}$.
   (a) Prove that $P_{jk}^{-1} = P_{jk}$.
   (b) Consider the expectation value of an operator $\hat{B}$ over a many-body wave function, $\langle\Psi|\hat{B}|\Psi\rangle$. Convince yourself that the identity $\hat{B} = P_{jk}^{\dagger}\hat{B}P_{jk}$ holds.
   (c) Use the previous result to prove that P_{jk} is Hermitian.
   (d) Use the result in (b) to find $\left[\hat{B}, \hat{P}_{jk}\right]$. Provide a physical interpretation.

5. The symmetrization and anti-symmetrization operators for $N = 2$ identical particles read:

$$\hat{S} = \frac{1}{2}\left(1 + \hat{P}_{12}\right), \qquad \hat{A} = \frac{1}{2}\left(1 - \hat{P}_{12}\right).$$

   Here, $\hat{P}_{12}$ is the exchange operators of particles 1 and 2. Show explicitly that:
   (a) $\hat{S}^2 = \hat{S}$ and $\hat{A}^2 = \hat{A}$;
   (b) $\hat{S}\hat{A} = \hat{A}\hat{S} = 0$;
   (c) $\hat{P}_{12}\hat{S} = \hat{S}$ and $\hat{P}_{12}\hat{A} = -\hat{A}$.
   (d) Can you guess what $\hat{S}$ and $\hat{A}$ look like for systems with $N > 2$ particles?