CHAPTER 3: COHERENT CONTROL OF SUPERCONDUCTING QUBIS: 3.1 Introductory elements of quantum optics

This lecture will be based on references [1], [2], [3].

0 Introduction

In this chapter 3 we will enter the coherent control of superconducting qubits. The previous chapters have provided us with a set of components and materials from which to start building superconducting quantum circuits. Now we need to understand the methods to engineer superconducting quantum circuits for quantum information.

In this section, we will first introduce the microwave fields being the relevant energy scale in superconducting circuits. Then, we will review the basics of field quantization so that everyone has sufficient tools to follow the coming lectures where this techniques naturally appearing in quantum optics will be widely used. We will particularly quantize electromagnetic fields in microwave structures, and later we will describe the most relevant states of the electromagnetic field found in quantum information using superconducting circuits, such as the coherent states and the thermal states. In the second part of this lecture we will give an overview of atom-photon interactions in a general sense, particularly looking at the fully quantized form represented by the Jaynes-Cummings model. The interaction between atoms and classical fields is postponed to a future lecture we will study coherent control of superconducting qubits.

Maxwell's equations in free space are scale-invariant, that is, they have the same form for the longest wavelengths (radio) to the shortest ones (gamma rays). However, what determines the behaviour of the different energy scales is given by the response of the materials. In particular, microwave frequencies are between the infrared-optical waves and the radio frequencies used in conventional low frequency electronics. Microwaves comprise the frequency band from 300 MHz to 300 GHz with a corresponding wavelength $\lambda = c/f$ between 1 m and 1 mm, respectively. Given that the wavelength is comparable or smaller than the typical size of electrical components, it is not correct to treat them as discrete objects, and one needs to invoke wave interference to model their behaviour appropriately. The phase of a voltage or current significantly changes over the physical extent L of the device. When the contrary occurs where the wavelength is much longer $\lambda \gg L$ and the phase barely changes across the dimensions of the component, the objects are called 'lumped'. In the case where $\lambda \ll L$, it is the domain of optics and geometrical optics. In reality, microwave circuits can be treated with a simpler circuit theory than the full field theory.

One important difference with the standard treatments found in quantum optics textbooks is that microwave fields have much lower mode energies. Even in the standard operating temperatures in experiments in dilution refrigerators, where temperatures are typically 10 mK, thermal effects must be considered to analyze the circuits properly. In chapter 4, the modes in a transmission line will be quantized following a treatment motivated with specific experimental implementations from qubits in mind, even though the procedures will be completely general and can be applicable to any 1D quantum field problem.

1 Microwave fields

In superconducting quantum circuits we will be dealing mainly with two types of microwave structures: resonators/cavities and transmission lines/waveguides. In order to understand these structures as media to propagate and store quantum information, it is worth looking at them classically.

1.1 Transmission lines / Waveguides

Prior to defining resonators, it is more instructive to understand signal propagation and guiding in the microwave regime following transmission line theory. Transmission line theory is typically introduced in advanced electromagnetism courses as part of the propagation of electromagnetic waves, in this case in conductors which is the right material to guide the propagation of microwave signals, besides free space.

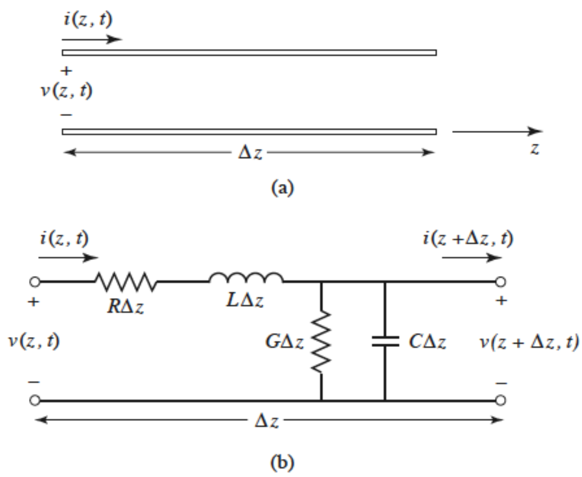

Figure 1: (a) Transmission line and (b) discrete model. Source: Pozar.

Let's consider first the typical textbook lossless (no resistive nor capacitive losses) transmission line (Ch. 2, 3 Ref. [1]), discretized in differential elements of length Δz. The line has inductance per unit length l_0 and capacitance per unit length c_0.

Figure 1 shows a small section of such a transmission line. The voltage and current drop between z and $z + dz$ are given by

$$V(z + dz, t) = V(z, t) - l_0 \frac{dI(z, t)}{dt} dz, \tag{1}$$

$$I(z + dz, t) = I(z, t) - c_0 \frac{dV(z, t)}{dt} dz. \tag{2}$$

Taking the limit $dz \to 0$,

$$\frac{\partial V(z, t)}{\partial z} = -l_0 \frac{\partial I(z, t)}{\partial t}, \tag{3}$$

$$\frac{\partial I(z, t)}{\partial z} = -c_0 \frac{\partial V(z, t)}{\partial t}. \tag{4}$$

These equations are the time-domain form of the transmission line, also known as the telegraph equations. Combining them leads to wave equations for V and I,

$$v_0^2 \frac{\partial^2 V(z, t)}{\partial z^2} = \frac{\partial^2 V(z, t)}{\partial t^2}, \tag{5}$$

$$v_0^2 \frac{\partial^2 I(z, t)}{\partial z^2} = \frac{\partial^2 I(z, t)}{\partial t^2}, \tag{6}$$

where $v_0 \equiv 1/\sqrt{c_0 l_0}$ is the phase velocity. The equations accept traveling wave solutions:

$$V(z, t) = e^{-i\omega t}(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}), \tag{7}$$

$$I(z, t) = e^{-i\omega t}(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}), \tag{8}$$

with $\gamma \equiv i\omega\sqrt{l_0 c_0}$ is the propagation constant. The relation between voltage and current can be found from Eq. (3)

$$I(z, t) = \frac{1}{Z_0}[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}], \tag{9}$$

after definition of the characteristic impedance

$$Z_0 \equiv \sqrt{\frac{l_0}{c_0}}. \tag{10}$$

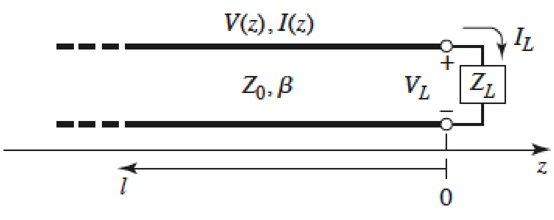

Figure 2: Transmission line terminated by a load Z_L. Source: Pozar.

When the line is terminated by a certain load impedance Z_L (see Fig. 2), reflections may occur. We thus define the reflection coefficient Γ as

$$\Gamma \equiv \frac{Z_L - Z_0}{Z_L + Z_0}, \tag{11}$$

where Z_0 is the characteristic impedance of the line. For a matched load $Z_0 = Z_L$ no reflection occurs and all power is absorbed by the load. A mismatch load will lead to reflections, and those will interfere with the incoming field, leading to standing wave. We define the standing wave ratio as

$$\mathrm{SWR} \equiv \frac{1 + |\Gamma|}{1 - |\Gamma|}. \tag{12}$$

This fraction is also known as voltage standing wave ratio, or VSWR, and is a characteristic of every microwave component. In a mismatched load, an oscillating voltage will thus lead to an oscillating impedance in the line. Taking the load at the origin $z = 0$ and looking towards $z < 0$, at a distance $z = -l$, the input impedance seen looking towards the load looks (after some algebra)

$$Z_{\mathrm{in}} = Z_0 \frac{Z_L + jZ_0 \tan \gamma l}{Z_0 + jZ_L \tan \gamma l}. \tag{13}$$

This relation is called the impedance equation of the line and will be very useful to define resonating structures.

So far, we only described transmission lines from purely electrical circuit considerations. In the real world, there are many possible ways in which to manufacture a transmission line, in structures known as waveguides. Examples include coaxial cables, rectangular waveguides, microstrip waveguides, etc. While coaxial cables used in experimental qubit setups do support different polarizations, the on-chip transmission lines are typically designed using a coplanar waveguide configuration (see figure 3) and therefore the symmetry is such that different polarizations correspond to very different mode profiles and mode frequencies. The modes of interest are usually the quasi-TEM shown in figure 4. Coplanar waveguides are particularly

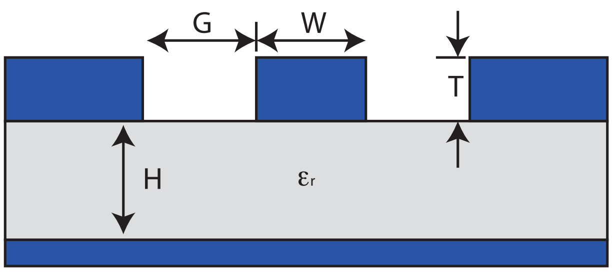

Figure 3: Transmission line cross section. W is the center line width, G is the gap between center line and ground plane, T is the film thickness, ϵ_r the relative dielectric constant of the substrate and H is the substrate thickness.

useful since the field is tightly confined between the center trace and the ground planes. This will be very important later on to interface these type of structures

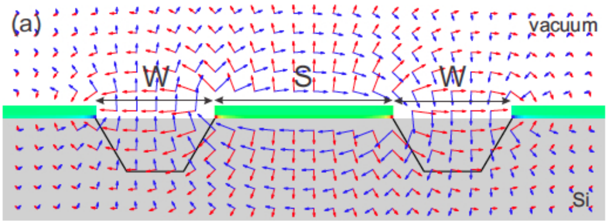

Figure 4: Transmission line fundamental mode. Red arrows represent the electric field while blue arrows represent the magnetic field. From Ref. [4].

with qubits. Care must be taken to design these type of waveguides with the right impedance as function of its geometry and material properties. As the electric field is roughly distributed over the substrate with relative permittivity constant ϵ_r and air (or vacuum) $\epsilon_{r_{\text{air,vacuum}}} \simeq 1$, a rule of thumb to estimate the effective dielectric constant is taking the average $\epsilon_{\text{eff}} = (\epsilon_r + 1)/2$. A typical substrate is silicon with $\epsilon_{r_{\text{Si}}} = 12$, leading to $\epsilon_{\text{eff}} = 6.5$.

1.2 Resonators

Having obtained a basic description of transmission lines, let us now explore the different types of resonating structures one can find in microwave circuits.

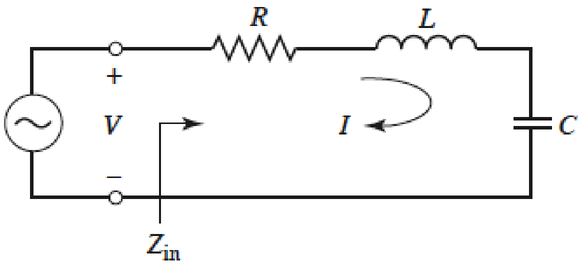

Figure 5: RLC series circuit. Source: Pozar.

First, we start by reviewing the basics of resonators. The simplest case is the series RLC network driven by a voltage source, as in Fig. (5). The impedance of such a circuit is given by

$$Z_s = R + i\omega L - i/\omega C. \tag{14}$$

The resonant condition is given when the energy stored in the inductor equals that of the capacitor or, equivalently, when the impedances are matched, which only happens when

$$\omega_{LC} \equiv \frac{1}{\sqrt{LC}}. \tag{15}$$

At this resonant frequency the impedance of the circuit is $Z_s(\omega_{LC}) = R$ and is purely real. Another important parameter to be defined from this circuit is the quality factor Q as

$$Q \equiv \omega \frac{\text{Average energy stored}}{\text{energy dissipated/second}}. \tag{16}$$

For the RLC series circuit this is $Q_s = 1/(RC\omega_{LC})$. Another practical relation is when the system is near resonance, $\omega = \omega_{LC} + \Delta\omega$,

$$Z_s \simeq R + j2L\Delta\omega = R + j\frac{2RQ_s\Delta\omega}{\omega_{LC}}. \tag{17}$$

A last parameter to be defined is the resonator fractional bandwidth BW response of the resonator corresponding to the frequency when half the power is delivered to the circuit with respect to resonance, $BW/2 \equiv \Delta\omega/\omega_{LC}$, which corresponds to $BW = 1/Q_s$.

By contrast, we can also consider the RLC parallel circuit in Fig. 6, driven by a current source. In this case, the impedance is

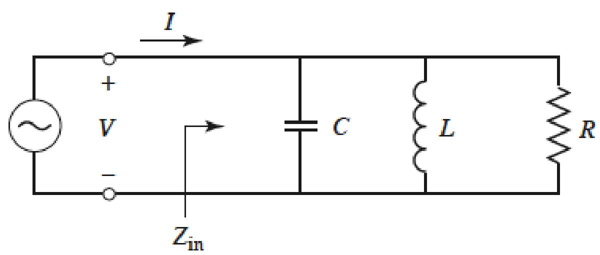

Figure 6: RLC parallel circuit. Source: Pozar.

$$Z_p = \left(\frac{1}{R} + \frac{1}{j\omega L} + i\omega C\right)^{-1}. \tag{18}$$

In this case the resonant frequency is the same as the series case $\omega_{LC} = 1/\sqrt{LC}$, when the impedance of the inductor and capacitor cancel each other and the total impedance is purely real $Z(\omega_{LC}) = R$. In this case, the quality factor is $Q_p = \omega_{LC}RC$. Near resonance, this circuit impedance is

$$Z_p \simeq \frac{R}{1 + 2jQ_p\Delta\omega/\omega_{LC}}. \tag{19}$$

Finally, the bandwidth of this circuit is also $BS = 1/Q_p$.

Now we have enough tools to understand transmission line resonators. Let us recall the input impedance of a transmission line

$$Z_{\text{in}} = Z_0\frac{Z_L + jZ_0\tan\gamma l}{Z_0 + jZ_L\tan\gamma l}. \tag{20}$$

1.2.1 Short-circuited $\lambda/2$ CPW resonator

Let us first take the case of a short-ended line, $Z_L = 0$,

$$Z_{\lambda/2} = jZ_0\tan(\gamma l). \tag{21}$$

From this expression, we say that a $\lambda/2$-type resonance will occur when the boundary condition at the input l coincides with that at the end (in this particular case, a short circuit). Then we need $Z_{\lambda/2} = Z_L = 0$, which corresponds to

$$\tan(\gamma l) = \tan[(2\pi/\lambda)l] = 0, \tag{22}$$

which will occur for $l = n\lambda_n/2$, as shown in Fig. (7), which is fitting n half-wavelengths $\lambda_n/2$ in the resonator length l. The resonance frequency for mode n is then

$$\omega_{\lambda/2,n} = n\frac{\pi v_0}{l}. \tag{23}$$

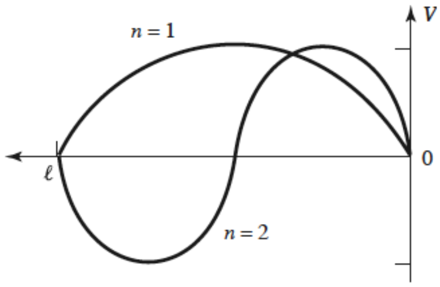

Figure 7: $\lambda/2$ short-circuited transmission line and the first 2 modes shown in the voltage amplitude. Source: Pozar.

Let's examine the impedance. Take a frequency close to the first resonance $\omega = \omega_{\lambda/2,1} + \Delta\omega$. In this case, $\gamma l = (\omega/v_0)\lambda/2 = \pi + \pi\Delta\omega/\omega_{\lambda/2,1}$. Then, $\tan\gamma l \simeq \pi\Delta\omega/\omega_{\lambda/2,1}$. The impedance in this case is

$$Z_{\lambda_1/2} \simeq jZ_0\pi\Delta\omega/\omega_{\lambda/2,1}. \tag{24}$$

Comparing this expression with the series RLC, Eq. (17), we identify the inductance of this equivalent series RLC circuit for the short-circuited $\lambda/2$ transmission line resonator near the resonance of its first mode:

$$L_s = \frac{Z_0\pi}{2\omega_{\lambda/2,1}}. \tag{25}$$

The effective capacitance will then be

$$C_s = \frac{1}{\omega^2_{\lambda/2,1}L}. \tag{26}$$

If we had considered an open-circuit transmission line on both ends, the equivalent inductance and capacitance would have been

$$C_o = \frac{\pi}{2\omega_{\lambda/2,1}Z_0}, \tag{27}$$

$$L_o = \frac{1}{\omega^2_{\lambda/2,1}C}. \tag{28}$$

1.2.2 Short-circuited $\lambda/4$ CPW resonator

Let's repeat the process of the previous section, but now consider a line that is short-circuited in one end $Z_L = 0$ but we look for $\lambda/4$-type resonance, which is the case where the other end of the line is an open-circuit, $Z_{\lambda/4} = \infty$. The impedance equation now is

$$Z_{\lambda/4} = jZ_0 \tan(\gamma l) = \infty, \tag{29}$$

which corresponds to $\gamma l = (2\pi/\lambda_n)l = n(\pi/2)$. Therefore, the modes must satisfy $l = n\lambda_n/4$. For $n = 1$ the lowest mode corresponds to a quarter wavelength in the length l. The resonance frequency of mode n will be

$$\omega_{\lambda/4,n} = n\frac{\pi v_0}{2l}. \tag{30}$$

Note that for the same length l, the $\lambda/4$ resonance will have half the frequency of the $\lambda/2$ resonance.

In order to study the impedance, let's take again a value close to resonance $\omega = \omega_{\lambda/4,1} + \Delta\omega$. In this case, $\gamma l = \pi/2 + (\pi/2)\Delta\omega/\omega_{\lambda/4,1}$. The impedance then looks

$$Z_{\lambda/4} = jZ_0 \tan\left(\frac{\pi}{2} + \frac{\pi}{2}\frac{\Delta\omega}{\omega_{\lambda/4,1}}\right) \simeq -jZ_0 \cot\left(\frac{\pi}{2}\frac{\Delta\omega}{\omega_{\lambda/4,1}}\right) \simeq -jZ_0\frac{2\omega_{\lambda/4,1}}{\pi\Delta\omega}. \tag{31}$$

The equivalent RLC circuit in this case corresponds to a parallel network. If we take Eq. (19) for a lossless system ($R_p \to \infty$),

$$Z_p \simeq -\frac{j}{2C\Delta\omega}, \tag{32}$$

we identify the corresponding $L_{\lambda/4}$ and $C_{\lambda/4}$ to be

$$C_{\lambda/4} = \frac{\pi}{4\omega_{\lambda/4,1}Z_0}, L_{\lambda/4} = \frac{1}{\omega_{\lambda_4,1}^2 C}. \tag{33}$$

1.3 Cavities

Having studied the transmission line resonators, which are planar 2D structures, we now move to 3D resonant structures, known as resonant cavities. The idea is rather simple: one needs to confine the electromagnetic field in a closed volume, and given its dimensions and topology different modes will be excited. The most commonly used geometry in superconducting qubits is the rectangular waveguide cavity. We will explore it here in this section. Other geometries (such as the cylindrical) will be explored in exercises or assignments.

A rectangular waveguide is a hollow tube with rectangular cross section where electromagnetic fields can propagate as waves, just like those on coplanar transmission lines mentioned above. Here the fields propagate mostly through the dielectric medium inside the waveguide (vacuum in our case) and they may dissipate power on the walls. By producing a closed waveguide at both ends, one obtains a resonant cavity, where modes can reside and be excited through small apertures. The electromagnetic fields can be obtained from Maxwell's equations without sources

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{34}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{35}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{36}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{37}$$

Since the walls are metallic, this imposes boundary conditions on the electric field, where the transverse component must vanish $\mathbf{E} \cdot \mathbf{n} = 0$. Applying this condition to all dimensions leads to a resonant wavenumber for the cavity defined as

$$k_{mnl} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2}. \tag{38}$$

These correspond to the transverse electric field TE_{mnl} and transverse magnetic field TM_{mnl} modes. The resonant frequencies of those modes are given by

$$f_{mnl} = \frac{c}{2\pi\sqrt{\epsilon_r \mu_r}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2}. \tag{39}$$

If $b < a < d$, the fundamental mode is the TE_{101} mode. The lowest TM mode is the TM_{110} mode. We will return to cavities and resonators once we start to quantize

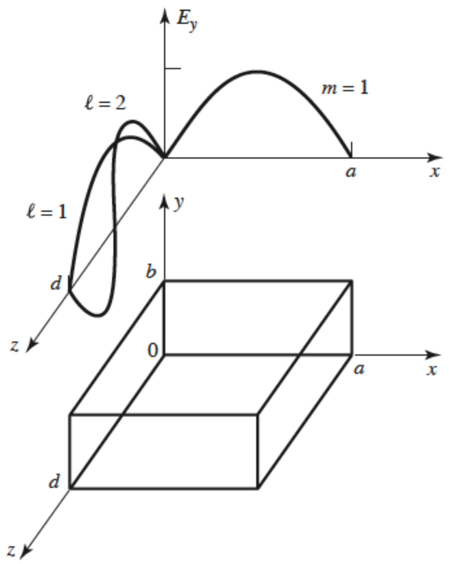

Figure 8: Resonant modes of a rectangular waveguide cavity resonator. Source: Pozar.

electromagnetic circuits.

2 Field quantization

Having the knowledge of the type of resonant structures we are going to deal with, in this section we are going to learn how to quantize the electromagnetic field in generic resonant structures. Later in the chapter of circuit quantization we will specifically quantize transmission line electromagnetic fields, which will be the ones coupling to qubits. We will also use this chapter to give an overview of certain types of states of the electromagnetic field, such as the number states and the coherent states.

2.1 Quantization of an LC oscillator

Let us begin with the simplest circuit to be quantized, an LC circuit in the lumped element regime. The Hamiltonian of the circuit will be that of the sum of energies inside the capacitor and the inductor,

$$\mathcal{H} = \frac{q^2}{2C} + \frac{\Phi^2}{2L}. \tag{40}$$

The resonance frequency of the system takes place at $\omega_0 = 1/\sqrt{LC}$, and the circuit has an impedance $Z_0 = \sqrt{L/C}$. Flux Φ and charge q are canonical dynamic variables. We now associate each variable a quantum operator by adding a caret $\hat{}$:

$$\hat{\mathcal{H}} = \frac{\hat{q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}. \tag{41}$$

This Hamiltonian is exactly analogous to that of the quantum mechanical oscillator if we associate the mass to C, and identify $\hat{q}$ as the momentum and $\hat{\Phi}$ as the position operators. Following the analogy, we will now have commutation relations for charge and flux, $[\hat{\Phi}, \hat{q}] = i\hbar$. We may now introduce the creation and annihilation operators, following the analogy to the harmonic oscillator

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{42}$$

We then have

$$\hat{\Phi} = \sqrt{\frac{\hbar Z_0}{2}}(\hat{a} + \hat{a}^\dagger), \tag{43}$$

$$\hat{q} = \frac{1}{i}\sqrt{\frac{\hbar}{2Z_0}}(\hat{a} - \hat{a}^\dagger). \tag{44}$$

With these operators the Hamiltonian becomes the well-known form

$$\hat{\mathcal{H}} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + 1/2). \tag{45}$$

It can be shown, by either correlation function calculations [5] or by using the Hermite polynomials and the position representation of the harmonic oscillator states, that the flux and charge operators exhibit fluctuations even at zero temperature, known as zero point fluctuations or vacuum fluctuations:

$$\langle 0|\hat{\Phi}^2|0\rangle = \frac{\hbar Z_0}{2}, \tag{46}$$

$$\langle 0|\hat{q}^2|0\rangle = \frac{\hbar}{2Z_0}. \tag{47}$$

These relations clearly show an asymmetry between flux and charge fluctuations. For a medium with high impedance, the flux fluctuations will be significantly larger than charge fluctuations. This has important consequences when considering interaction with qubits as well as wave propagation.

We can also calculate the form of current $\hat{I}$ and voltage $\hat{V}$ operators in the LC circuit from $\hat{I} = d\hat{q}/dt$, $\hat{V} = d\hat{\Phi}/dt$:

$$\hat{I} = \omega_0 \sqrt{\frac{\hbar}{2Z_0}} (\hat{a} - \hat{a}^\dagger) \tag{48}$$

$$\hat{V} = i\omega_0 \sqrt{\frac{\hbar Z_0}{2}} (\hat{a} + \hat{a}^\dagger). \tag{49}$$

We can also compute their corresponding vacuum fluctuations:

$$V_{\text{rms}} = (\langle 0|\hat{V}^2|0\rangle)^{1/2} = \omega_0 \sqrt{\frac{\hbar Z_0}{2}}, \tag{50}$$

$$I_{\text{rms}} = (\langle 0|\hat{I}^2|0\rangle)^{1/2} = \omega_0 \sqrt{\frac{\hbar}{2Z_0}}. \tag{51}$$

These relations will become very relevant when considering qubit-resonator interactions in the circuit QED architecture.

2.2 Quantization of a transmission line resonator field

As detailed in earlier sections, the circuit model of a transmission line is that of an infinite series of capacitor and inductor elements of differential length Δx, with capacitance per unit length c_0 and inductance per unit length l_0, which define the speed of light in the medium $v_0 = (c_0 l_0)^{-1/2}$ and the medium impedance $Z_0 = (l_0/c_0)^{1/2}$. A transmission line resonator has finite length L, as it is intercepted at both ends by input and output capacitors C_{in} and C_{out}, respectively. This circuit model is shown in Fig.fig:TLQ.

The geometry employed in superconducting qubit experiments is that of a coplanar waveguide, where a center trace conducts the signal which is confined between two ground planes on each side of the center trace. This system is quasi-one dimensional, as the field extension is mostly confined to the gaps between center trace and ground planes. The geometry of the problem results quite asymmetric in the vertical axis, as there is a substrate with high dielectric constant $\epsilon_r \gg \epsilon_0$ and air above the metal of the circuit leads. Due to this lack of symmetry, we do not have to take into account a polarization degree of freedom.

With these considerations and following the same procedure used to describe the transmission line classically, we can build the equations of motion of cell n in the circuit with inductors and capacitors. The flux across the inductor $\Phi(x_n + \Delta x, t) - \Phi(x_n, t)$ defines a current that equals the current going to ground,

$$\frac{\Phi(x_n + \Delta x, t) - \Phi(x_n, t)}{l_0 \Delta x} = c_0 \Delta x \ddot{\Phi}(x_n + \Delta x, t). \tag{52}$$

Taking the limit $\Delta x \to 0$, this equation becomes

$$c_0 \frac{d}{dt}\left[\frac{\partial \Phi(x,t)}{\partial t}\right] = \frac{1}{l_0}\frac{d}{dx}\left[\frac{\partial \Phi(x,t)}{\partial x}\right]. \tag{53}$$

This equation of motion defines a total circuit Lagrangian,

$$\mathcal{L} = \int_{-L/2}^{L/2} dx \left[\frac{c_0}{2}\left(\frac{\partial \Phi(x,t)}{\partial t}\right)^2 - \frac{1}{2l_0}\left(\frac{\partial \Phi(x,t)}{\partial x}\right)^2\right] \equiv \int_{-L/2}^{L/2} \mathcal{L}_d(x)dx, \tag{54}$$

where $\mathcal{L}_d(x)$ is a lagrangian density per unit length.

In order to find a more convenient form of the Lagrangian, we expand the flux density $\Phi(x,t)$ in eigenmodes in which the time and space dependence can be factored:

$$\Phi(x,t) = \sum_{n=1}^{\infty} \Phi_n(t)u_n(x). \tag{55}$$

Here, $u_n(x)$ are unitless normal modes, and $\Phi_n(t)$ is the flux amplitude of eigenmode n of frequency ω_n. Plugging this into the equation of motion Eq. (53), leads to

$$v_0^2 \frac{u_n''(x)}{u_n(x)} = \frac{\ddot{\Phi}_n(t)}{\Phi_n(t)}, \tag{56}$$

which is a Sturm-Liouville form of the equation of motion and can be decomposed into eigenvalues and eigenvectors. As there are no boundary conditions for the time dependence, we can assume the general solution $\Phi_n(t) = e^{-i\omega_n t}\Phi_n(0)$. Using this in Eq. (56) leads to the eigenvalue equation for $u(x)$,

$$v_0^2 \frac{u_n''(x)}{u_n(x)} = -\omega_n^2. \tag{57}$$

The most general form of the solution of this equation for $u_n(x)$ can be written as

$$u_n(x) = A\cos(k_n x) + B\sin(k_n x), \tag{58}$$

which then leads to the dispersion relation $\omega_n = k_n v_0$. These modes will satisfy the orthogonality condition

$$\int_{-L/2}^{L/2} u_n(x)u_m(x)dx = \delta_{nm}. \tag{59}$$

Now we need to insert the boundary condition to determine A and B. As this is a $\lambda/2$-waveguide there is an open circuit at both ends of the line, implying that the current will be 0[1]. In other words $\partial\Phi_n(x,t)/\partial x|_{x=\pm L/2} = 0$. This leads to the following equations:

$$\frac{\partial u_n(-L/2,t)}{\partial x} = k_n A\sin(Lk_n/2,t) + k_n B\cos(Lk_n/2,t) = 0, \tag{60}$$

$$\frac{\partial u_n(L/2,t)}{\partial x} = -k_n A\sin(Lk_n/2,t) + k_n B\cos(Lk_n/2,t) = 0. \tag{61}$$

[1]For simplicity we are not including the coupling capacitors which lead to more complicated boundary conditions but result in the same qualitative picture.

Combining these equations leads to a first condition $\sin(k_n L/2) = 0$, which corresponds to $k_n = 2n\pi/L$, the even modes. The second condition is $\cos(k_n L/2) = 0$, leading to $k_n = n\pi/L$, the odd modes. Without loss of generality, A and B are taken identical and using the normalization condition Eq. (59), $A = B = \sqrt{2/L}$. We can now write the explicit form of the eigenmodes

$$\Phi(x,t) = \sqrt{\frac{2}{L}} \sum_{k_o=1}^{k_{o,c}} \Phi_{k_o}(t) \cos\frac{k_o \pi x}{L} + \sqrt{\frac{2}{L}} \sum_{k_e=2}^{k_{e,c}} \Phi_{k_e}(t) \sin\frac{k_e \pi x}{L}. \tag{62}$$

k_o, k_e are odd/even integers and $k_{o,c}, k_{e,c}$ are cutoffs defined by the physics of the circuit. Usually at high enough mode number there will be physical effects which limit the maximum mode. In this case it may be the fact that modes will not be quasi-1D at high enough energy, but for instance the superconducting gap may define another limit in the energy of the modes, or the dielectric constant becoming imaginary at some high energy and the corresponding mode stops propagating.

Each mode will have resonant frequency $\omega_k = k\pi v_0/L$, with k being a positive integer. Using the normal mode expansion Eq. (62) in the Lagrangian Eq. (54) and integrating over x,

$$\mathcal{L} = \sum_k \left[\frac{c_0}{2} \dot{\Phi}_k^2(t) - \frac{1}{2l_0} \left(\frac{k\pi}{L} \right)^2 \Phi_k^2(t) \right]. \tag{63}$$

The canonical conjugate charge is $\theta_k \equiv \partial\mathcal{L}/\partial\dot{Phi}_k = c_0\dot{\Phi}_k$. This way we can write the circuit Hamiltonian

$$\mathcal{H} = \sum_k \left[\frac{1}{2c_0} \theta_k^2(t) + \frac{1}{2l_0} \left(\frac{k\pi}{L} \right)^2 \Phi_k^2(t) \right]. \tag{64}$$

We have found that the transmission line resonator Hamiltonian is a collection of harmonic oscillators, each corresponding to a given mode k. Now we can impose the quantization condition of circuit variables θ_k and Φ_k into non-commuting operators, $[\hat{\Phi}_k, \hat{\theta}_l] = i\hbar\delta_{kl}$. These can be expressed in terms of creation and annihilation operators, just like it was done with the quantized LC resonator,

$$\hat{\Phi}_k = \sqrt{\frac{\hbar\omega_k l_0}{2}} \left(\frac{L}{k\pi} \right) (\hat{a}_k + \hat{a}_k^\dagger), \tag{65}$$

$$\hat{\theta}_k = i\sqrt{\frac{\hbar\omega_k c_0}{2}} (\hat{a}_k - \hat{a}_k^\dagger), \tag{66}$$

which lead to the quantized Hamiltonian

$$\hat{\mathcal{H}} = \sum_k (\hat{a}_k^\dagger \hat{a}_k + 1/2). \tag{67}$$

2.3 Quantization of a single-mode field

Let us start with a one dimensional waveguide with metallic walls at its ends, as in Fig. 9. This will be a simplified version of the case studied in the previous section,

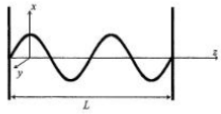

Figure 9: 1D cavity localized along the z direction. Source: Gerry&Knight.

but it will be sufficiently illustrative. The boundary conditions impose that the tangential component of the electric field vanishes at the ends $z = 0, L$. Consider a field polarized along the x-direction, $\mathbf{E} = \mathbf{e}_x E_x(z,t)$. Maxwell's equations without sources are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{68}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{69}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{70}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{71}$$

A single-mode field solution is given by

$$E_x(z,t) = q(t)[A\cos(kz) + B\sin(kz)]. \tag{72}$$

The boundary condition $E_x(0,t) = 0$ imposes $A = 0$, while $E_x(L,t) = 0$ leads to define the allowed mode wavevectors $k_n = n\pi/L$ and frequencies $\omega_n = nc\pi/L$, with $n = 1, 2, \dots$. B will be defined later on. $q(t)$ has dimensions of a length.

From here on we just take the lowest mode $k_1 \equiv k$. The magnetic field will be given by $\mathbf{B}(\mathbf{r},t) = \mathbf{e}_y B_y(z,t)$, with

$$B_y(z,t) = B\frac{\mu_0\epsilon_0}{k}\dot{q}(t)\cos(kz). \tag{73}$$

The classical field energy, the Hamiltonian H, is given by

$$H = \frac{1}{2}\int dV \left[\epsilon_0 \mathbf{E}^2(\mathbf{r},t) + \frac{1}{\mu_0}\mathbf{B}^2(\mathbf{r},t)\right] = \frac{1}{2}\int dV \left[\epsilon_0 E_x^2(z,t) + \frac{1}{\mu_0}B_y^2(z,t)\right]. \tag{74}$$

Plugging in the solution obtained,

$$\begin{aligned} H &= \frac{B^2}{2}S\int_0^L dz \left[\epsilon_0 q(t)^2 \sin^2(kz) + \frac{\epsilon_0^2\mu_0}{k^2}\dot{q}(t)\cos^2(kz)\right] \\ &= \frac{\epsilon_0 B^2 S}{2}\left(q^2(t)\left[\frac{z}{2} - \frac{\sin(2kz)}{4k}\right]\Bigg|_0^L \frac{\mu_0\epsilon_0}{k^2}\left[\frac{z}{2} + \frac{\sin(2kz)}{4k}\right]\Bigg|_0^L\right). \end{aligned} \tag{75}$$

We defined the transverse area of the field as $\int\int dxdy \equiv S$. Using $\omega = kc$, and defining the volume of the field as $V \equiv SL$

$$H = \frac{B^2}{2}\frac{\epsilon_0 V}{2\omega^2}[\dot{q}(t)^2 + \omega^2 q(t)^2]. \tag{76}$$

By choosing $B = \sqrt{2\omega^2/\epsilon_0 V}$,

$$H = \frac{1}{2}\left[p(t)^2 + \omega^2 q(t)^2\right]. \tag{77}$$

Here we identified $\dot{q}(t) \equiv p(t)$ with a canonical momentum of the mode. This form of the Hamiltonian is equivalent to the one of the mechanical oscillator of unit mass. In order to describe the electromagnetic field in quantum mechanics, we associate Hilbert space operators with the dynamic variables, some of which do not commute. From now on, the classical variables become quantum operators with the caret $\hat{}$. That is, the momentum $p(t)$ and position $q(t)$ classical variables become operators $\hat{p}(t)$ and $\hat{q}(t)$. Following the postulates of quantum mechanics, each pair of canonically conjugate operators has non-zero commutator

$$[\hat{p}(t), \hat{q}(t)] = i\hbar. \tag{78}$$

Now the state of the system, the single cavity mode, is described by a state vector $|\psi\rangle$ in Hilbert space, and all the quantum mechanics of operators applies to the electromagnetic field, that is, observables, eigenvalues, eigenstates of operators, etc. In particular, the electric and magnetic fields become quantum operators

$$\hat{E}_x(z,t) = \sqrt{\frac{2\omega^2}{\epsilon_0 V}}\hat{q}(t)\sin(kz), \tag{79}$$

$$\hat{B}_y(z,t) = \sqrt{\frac{2\omega^2}{\epsilon_0 V}}\left(\frac{\mu_0\epsilon_0}{k}\right)\hat{p}(t)\cos(kz), \tag{80}$$

the expectation value of which depends on the quantum state, $\langle\psi|\hat{E}|\psi\rangle$.

The Hamiltonian of the quantized field becomes

$$\hat{H} = \frac{1}{2}[\hat{p}(t)^2 + \omega^2\hat{q}(t)^2]. \tag{81}$$

Usually, the Hermitian operators $\hat{p}(t)$, $\hat{q}(t)$ are replaced by the non-Hermitian operators

$$\hat{a}(t) = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q}(t) + i\hat{p}), \tag{82}$$

$$\hat{a}^\dagger(t) = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q}(t) - i\hat{p}), \tag{83}$$

which satisfy the commutation relation from Eq. (78)

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1. \tag{84}$$

Note that the value of this commutator is subject to the normalization condition chosen to define H. This has no real consequences but it affects the units of measure of operators. It is an issue normally dealt with in high energy physics. The inverse relation

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\omega}}[\hat{a}(t) + \hat{a}^\dagger(t)], \tag{85}$$

$$\hat{p}(t) = i\sqrt{\frac{\hbar\omega}{2}}[-\hat{a}(t) + \hat{a}^\dagger(t)], \tag{86}$$

Plugging this into the Hamiltonian, the form becomes more familiar

$$\hat{H} = \frac{1}{2}\left[-\frac{\hbar\omega}{2}[\hat{a}(t)\hat{a}(t) - \hat{a}(t)\hat{a}^\dagger(t) - \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}^\dagger(t)\hat{a}^\dagger(t)]\right.$$
$$\left. + \omega^2\frac{\hbar}{2\omega}[\hat{a}(t)\hat{a}(t) + \hat{a}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}^\dagger(t)\hat{a}^\dagger(t)]\right]$$
$$= \hbar\omega\left[\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right], \quad (87)$$

where we already used the commutating relations Eq. (84). The 1/2 term corresponds to the zero point contribution of the energy of the vacuum, no quantum mechanical system can be at rest, not even in the ground state!

The time dependence of $p(t), q(t)$ could be obtained from Maxwell's equations (prove it!), but we are adopting a more classical approach by using Heisenberg's equations of motion of the operators $\hat{a}(t), \hat{a}^\dagger(t)$

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}] = \frac{i}{\hbar}(\hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}) = i\omega[\hat{a}, \hat{a}^\dagger]\hat{a} = -i\omega\hat{a}. \quad (88)$$

Then,

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t}. \quad (89)$$

Similarly for $\hat{a}^\dagger(t)$,

$$\hat{a}^\dagger(t) = \hat{a}(0)e^{i\omega t}. \quad (90)$$

With these operators, the electric and magnetic fields become

$$\hat{E}_x(z,t) = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}[\hat{a}(0)e^{-i\omega t} + \hat{a}^\dagger(0)e^{i\omega t}]\sin(kz), \quad (91)$$

$$\hat{B}_y(z,t) = \frac{i}{c}\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}[\hat{a}(0)e^{-i\omega t} - \hat{a}^\dagger(0)e^{i\omega t}]\cos(kz). \quad (92)$$

The general case of a multimode field in free space will be analyzed in chapter 4 when we consider microwave photonics for quantum communications purposes.

2.4 Number states of the electromagnetic field

We will omit the explicit time dependence of operators in what follows. The operator that appears in the Hamiltonian is known as the number operator $\hat{n} \equiv \hat{a}^\dagger\hat{a}$, and has an associated set of eigenstates $|n\rangle$, also known as Fock states, and eigenvalues n as

$$\hat{n}|n\rangle = n|n\rangle. \quad (93)$$

Since the Hamiltonian is the energy of the system in state $|\psi\rangle$ we want to know the energy of the system in state $|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)|n\rangle = \hbar\omega(n + 1/2)|n\rangle = E_n|n\rangle, \quad (94)$$

where $E_n \equiv \hbar\omega(n + 1/2)$ is the energy of state $|n\rangle$.

What is the energy now of state $\hat{a}^\dagger|n\rangle$? We need to calculate

$$\hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)\hat{a}^\dagger|n\rangle. \tag{95}$$

We need to express the operator in the so-called normal order in which all operators $\hat{a}$ must move to the right. Using the commutation relation, $\hat{a}^\dagger\hat{a}\hat{a}^\dagger = \hat{a}^\dagger(1 + \hat{a}^\dagger\hat{a})$

$$\hat{a}^\dagger(1 + \hat{a}^\dagger\hat{a})|n\rangle = \hat{a}^\dagger|n\rangle + \hat{a}^\dagger\hat{n}|n\rangle = \hat{a}^\dagger(1 + n)|n\rangle. \tag{96}$$

Then, the energy of this state is

$$\hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)\hat{a}^\dagger|n\rangle = (E_n + \hbar\omega)\hat{a}^\dagger|n\rangle \tag{97}$$

Therefore the eigenvalue of state $\hat{a}^\dagger|n\rangle$ is $E_n+\hbar\omega$. $\hat{a}^\dagger$ creates a quantum of energy $\hbar\omega$. That is why it is called the creation operator. **We call this quantum of energy a photon**[2]. This operator can be applied multiple times, so it is not bounded from above. Conversely, let's consider the energy of the state $\hat{a}|n\rangle$

$$\hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)\hat{a}|n\rangle = \hbar\omega(\hat{a}\hat{a}^\dagger - 1 + 1/2)\hat{a}|n\rangle = (E_n - \hbar\omega)\hat{a}|n\rangle. \tag{98}$$

$\hat{a}$ destroys or annihilates a quantum of energy $\hbar\omega$ or a photon. Repeating the application of $\hat{a}$ will eventually result in a negative energy, but the harmonic oscillator can only have positive energies[3]. Thus the minimum eigenvalue of $\hat{n}$ has to be 0, $\hat{a}|0\rangle = 0$. The energy of state $|0\rangle$ is

$$\hat{H}|0\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)|0\rangle = \hbar\omega/2|0\rangle. \tag{99}$$

So the energy of state $|0\rangle$ is that of the vacuum $\hbar\omega/2$, also called zero-point energy.

The set of states $|n\rangle$ are normalized, $\langle n|n\rangle = 1$. If $\hat{a}|n\rangle$ takes away an energy quantum, it implies that the resulting state must be proportional to $|n-1\rangle$,

$$\hat{a}|n\rangle = b|n-1\rangle. \tag{100}$$

This factor b can be obtained by calculating the norm of such a state, $|\langle n|\hat{a}^\dagger\hat{a}|n\rangle| = |b|^2|\langle n-1|n-1\rangle| = |b|^2$, then $|b| = \sqrt{n}$, leading to

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle. \tag{101}$$

Similarly, $\hat{a}^\dagger|n\rangle = c|n+1\rangle$. Calculating the modulus, $\langle n|\hat{a}\hat{a}^\dagger|n\rangle = \langle n|(1 + \hat{a}^\dagger\hat{a})|n\rangle = |c|^2|\langle n+1|n+1\rangle| = |c|^2$. Therefore, $|c| = \sqrt{n+1}$, so

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{102}$$

This last relation implies that we can build any number state $|n\rangle$ by application of the operator $\hat{a}^\dagger$ on the vacuum state $|0\rangle$ n times with the correct normalization factor $\sqrt{n}\sqrt{n-1}\ldots 1 = \sqrt{n!}$

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{103}$$

[2]These excitations of the single-mode field are not localized in space, rather they span over the entire mode volume. This is in contrast to the old view of quantum theory seeing photons as corpuscles of light. [2]

[3]One can also show that the eigenvector $|m\rangle$ of operator $\hat{n}$, has a norm $\langle m|\hat{a}|m\rangle \geq 0$. [3]

From Eqs. (101, 102), one obtains the only nonzero matrix elements of the annihilation and creation operators,

$$\langle n-1|\hat{a}|n\rangle = \sqrt{n}, \tag{104}$$
$$\langle n+1|\hat{a}^\dagger|n\rangle = \sqrt{n+1}. \tag{105}$$

Finally, the number states is a complete, orthonormal set,

$$\sum_n |n\rangle\langle n| = 1, \tag{106}$$

implying that it can be used as a basis to represent any other state of the electromagnetic field.

2.5 Quantum fluctuations of a single-mode field

The number state $|n\rangle$ has no well-defined value of the electric or magnetic fields of a single-mode cavity, Eqs. (91),

$$\langle n|\hat{E}_x(z,t)|n\rangle = \langle n|\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\sin(kz)[\hat{a}+\hat{a}^\dagger]|n\rangle = 0. \tag{107}$$

Similarly, $\langle n|\hat{B}_y(z,t)|n\rangle = 0$. The mean value of the fields is 0. The uncertainty of the fields, however, does have a finite value, $[\Delta\hat{E}_x(z,t)]^2 = \langle n|\hat{E}_x^2(z,t)|n\rangle - [\langle n|\hat{E}_x(z,t))|n\rangle]^2$, as

$$\begin{aligned}\langle n|\hat{E}_x^2(z,t)|n\rangle &= \frac{\hbar\omega}{\epsilon_0 V}\sin^2(kz)\langle n|(\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)|n\rangle \\ &= \frac{\hbar\omega}{\epsilon_0 V}\sin^2(kz)\langle n|(1+2\hat{a}^\dagger\hat{a})|n\rangle = \frac{\hbar\omega}{\epsilon_0 V}\sin^2(kz)(1+2n).\end{aligned} \tag{108}$$

Hence, the field uncertainty is

$$\Delta\hat{E}_x(z,t) = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\sin(kz)\sqrt{1+2n}, \tag{109}$$

which even when $n=0$ has a finite value. These are known as the vacuum fluctuations and will become extremely relevant for the coupling of superconducting qubits to cavities and resonators. In the next chapter we will begin quantizing electromagnetic circuits which will lead to analogous expressions for the vacuum fluctuations of voltages and currents from electrical circuits.

2.6 Coherent states of the electromagnetic field

Another class of states of the electromagnetic field exist which find a correspondence to the dynamics of classical operators. These are known as coherent states, and correspond to eigenstates of the annihilation operator. As this operator is non-Hermitian, its eigenvalues will in general be complex. Define the coherent state $|\alpha\rangle$ as

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{110}$$

The Hermitian conjugate is

$$\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^*. \tag{111}$$

In the basis of number state $|n\rangle$,

$$|\alpha\rangle = \sum_n C_n|n\rangle. \tag{112}$$

Applying $\hat{a}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle = \alpha\sum_n^\infty C_n|n\rangle = \sum_{n=1}^\infty C_n\sqrt{n}|n-1\rangle. \tag{113}$$

Equating the n terms of the sums leads to $C_n = C_{n-1}\alpha/\sqrt{n}$. This relation can be iterated further $C_n = \alpha^2 C_{n-2}/\sqrt{n(n-1)} = \cdots = \alpha^n C_0/\sqrt{n!}$. The coherent state now looks

$$|\alpha\rangle = C_0\sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{114}$$

The constant C_0 may be found by the normalization condition of the coherent states $\langle\alpha|\alpha\rangle = 1$,

$$\langle\alpha|\alpha\rangle = \sum_{n,m}\frac{\alpha^n(\alpha^*)^m}{\sqrt{n!m!}}|C_0|^2\langle m|n\rangle = |C_0|^2\sum_n\frac{|\alpha|^{2n}}{n!} = |C_0|^2 e^{|\alpha|^2} = 1. \tag{115}$$

Therefore, $|C_0| = e^{-|\alpha|^2/2}$. Finally,

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^\infty\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{116}$$

Now we understand why coherent states are eigenstates of the annihilation operator: since they are a superposition of number states, the effect of $\hat{a}$ removing one photon leaves the state unchanged! Here we see the controversy in the photon concept, since coherent states are electromagnetic fields that propagate, and they are composed of superpositions of number states, but we usually say that electromagnetic fields are built out of photons.

The expectation value of the electric field in a 1D cavity in a coherent state is then

$$\begin{aligned}\langle\alpha|\hat{E}_x(z,t)|\alpha\rangle &= \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}[\langle\alpha|\hat{a}(0)|\alpha\rangle e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \langle\alpha|\hat{a}^\dagger(0)|\alpha\rangle e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] \\ &= \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}(\alpha e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \alpha^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}) = 2|\alpha|\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\cos(\omega t - \mathbf{k}\cdot\mathbf{r} - \theta),\end{aligned} \tag{117}$$

where we wrote α in polar form $\alpha = |\alpha|e^{i\theta}$. We also used the more general 3D form of the propagation phase $\mathbf{k}\cdot\mathbf{r}$ to obtain a more general result. This result looks very much like the classical evolution of the electric field. We also see the role played by α, being the amplitude of the expectation value of the field. Despite providing expectation values that match to those of classical variables, $|\alpha\rangle$ is very much a quantum state. We can regain the quantumness by looking at the fluctuations of the electric field $(\Delta\hat{E}_x)^2 = \langle\hat{E}_x^2\rangle - \langle\hat{E}_x\rangle^2$. The result (left as an exercise) is

$$\Delta\hat{E}_x = \left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2}, \tag{118}$$

which coincides with the fluctuations of the vacuum state. Therefore it is a minimum uncertainty state[4].

We can calculate what is the expectation value of the photon number operator in a coherent state, $\langle\alpha|\hat{n}|\alpha\rangle$, otherwise called the average photon number. It is easy to see that

$$\bar{n} = \langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2. \tag{119}$$

Thus $|\alpha|^2$ is the average photon number of the field in a coherent state. Next, the fluctuations, for which we need the second moment $\langle\alpha|\hat{n}^2|\alpha\rangle$

$$\langle\alpha|\hat{n}^2|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2\langle\alpha|(1+\hat{a}^\dagger\hat{a})|\alpha\rangle = |\alpha|^2 + |\alpha|^4. \tag{120}$$

The uncertainty (also known as the variance) is thus

$$\Delta\hat{n} = (\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2)^{1/2} = |\alpha| = \sqrt{\bar{n}}. \tag{121}$$

A statistical distribution exhibiting this kind of behavior, where the variance scales with the average is a Poissonian distribution. To see this further, let's calculate the probability to find n photons in state $|\alpha\rangle$,

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\sum_{n'}^{\infty}\frac{\alpha^{2n'}}{n'!}|\langle n|n'\rangle|^2 = e^{-|\alpha|^2}\frac{\alpha^{2n}}{n!} = e^{-\bar{n}}\frac{\bar{n}^n}{n!}, \tag{122}$$

which is indeed a Poissonian distribution with average $\bar{n}$. The fractional uncertainty is

$$\frac{\Delta n}{\bar{n}} = (\bar{n})^{-1/2}, \tag{123}$$

therefore it becomes increasingly smaller as the field amplitude grows, becoming more and more classical and exhibiting less fluctuations, see Fig. 10 for some example of distributions. The time average of the electric field will thus oscillate in time at the frequency of the cavity or mode excited, with the vacuum noise superimposed. This can be sketch as in Fig. 11. It turns out that many macroscopically observable states of light correspond to coherent states. These states can be generated by a harmonic oscillator-type excitation. Examples include the state of a laser or that of a microwave source, as will be used to control superconducting qubits.

Another possibility to produce (and hence define) coherent states is by displacing the vacuum state in phase space using the displacement operator defined as

$$\hat{D}(\alpha) \equiv e^{\alpha\hat{a}-\alpha^*\hat{a}^\dagger}. \tag{124}$$

[4] In fact, its quadratures can be shown to yield a minimum Heisenberg uncertainty product state [2]. This can be in fact used as an alternative definition of the coherent states.

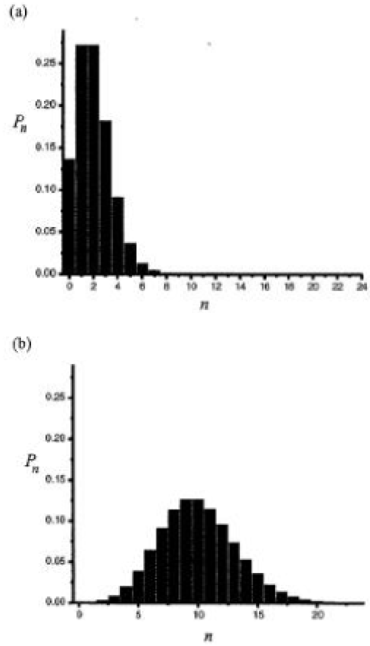

Figure 10: Poissonian distribution of different photon average number $\hat{n}$ for a coherent state. Source: Gerry&Knight.

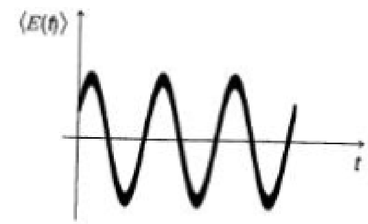

Figure 11: Time evolution of the average electric field for a coherent state, including the vacuum noise. Source: Gerry&Knight.

It will be show in an exercise that one can obtain the coherent state $|\alpha\rangle$ as $\hat{D}(\alpha)|0\rangle$. This operator will be useful when we consider excitation of qubits located inside cavities and resonators.

There is a convenient representation of coherent states using the quadrature plane defined by the quadrature operators $\hat{X}_1 = (\hat{a} + \hat{a}^\dagger)/2$ and $\hat{X}_2 = (\hat{a} - \hat{a}^\dagger)/2i$. A coherent state α with $\alpha = |\alpha|e^{i\theta}$ may be represented by a circle with diameter $1/2$ since $\Delta\hat{X}_{1,2} = 1/2$ in a coherent state, displaced by an amount $|\alpha|$ (from the origin at an angle θ. This is seen in Fig. 12. The vacuum state is then a coherent state centered at the origin since $|\alpha| = 0$ in that state. By contrast, a number state $|n\rangle$ has $\langle n|\hat{X}_{1,2}|n\rangle = 0$ but its photon average is very well defined $\langle n|\hat{n}|n\rangle = n$, without uncertainty $\Delta n = 0$. The phase of the state θ is completely randomized, making it a highly nonclassical state, as it can be represented by a circle of null width and radius n. This is a qualitative representation. Other types of true phase space representation exist, such as Wigner-, Husimi-, and Q-functions [2].

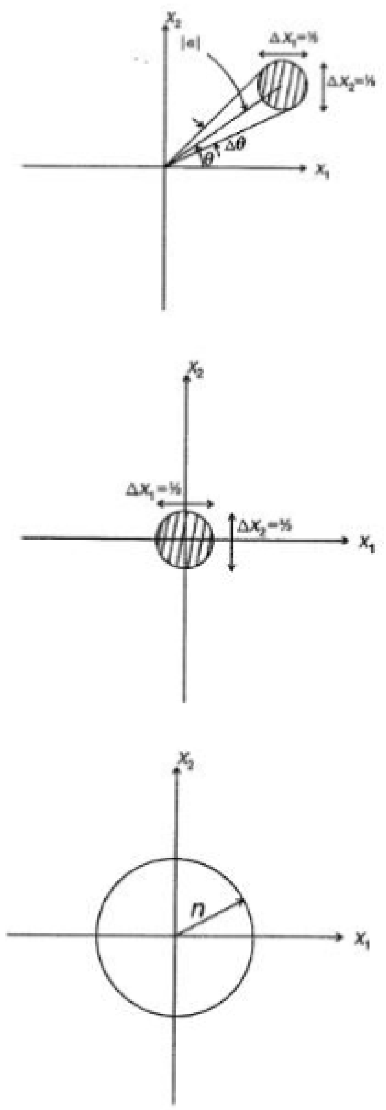

Figure 12: Examples of phase space representations of (a) a coherent state;(b) a vacuum state; and (c) a number state. Source: Gerry&Knight.

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