Quantum Information Theory. Assignment 2

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Q1. For each of the following set of matrices determine whether they are admissible quantum measurements. Determine also whether they are projective measurements or POVMs.

(i)

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(1 point)

(ii)

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

(1 point)

(iii)

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

(1 point)

(iv)

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3}e^{-\mathrm{i}\frac{2\pi}{3}} \\ \frac{\sqrt{2}}{3}e^{\mathrm{i}\frac{2\pi}{3}} & \frac{2}{3} \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3}e^{-\mathrm{i}\frac{4\pi}{3}} \\ \frac{\sqrt{2}}{3}e^{\mathrm{i}\frac{4\pi}{3}} & \frac{2}{3} \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix}$$

(1 point)

Q2 Consider the following Choi-Jamiołkowski state of a map Λ

$$J(\Lambda) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \sqrt{1-\gamma} & 0 & 0 & 1-\gamma \end{pmatrix}$$

(i) Determine the Kraus operators of the map Λ

(2 points)

(ii) Give the matrix representation of the map Λ.

(2 points)

Q3. *Maximally entangled states under unitary transformations*

(i) Show that the singlet state $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)$ is invariant under unitaries of the form $U \otimes U$. (1 point)

(ii) Show the effect a transformation of the type $U \otimes U \otimes U$ in the GHZ-state of 3 qubits $|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and in the W-state $|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. (1point)

(iii) Provide, if possible, a unitary transformation of the type $U_A \otimes U_B \otimes U_C$ such that $(U_A \otimes U_B \otimes U_C) |GHZ\rangle_{ABC} = |W\rangle_{ABC}$ (1point)

Q4. Let us define the following family of bipartite states

$$\rho_{iso}(\alpha) = \alpha |\Phi_{00}\rangle \langle\Phi_{00}| + \frac{1-\alpha}{d^2}(\mathbb{1}_A \otimes \mathbb{1}_B)$$

$\rho_{iso} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ with $d_A = d_B = d$, and $|\Phi_{00}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle |i\rangle$ is a maximally entangled state.

(i) For which values of α, is $\rho_{iso}(\alpha)$ a proper quantum state? (1 point)

(ii) What are the eigenvalues of $\rho_{iso}(\alpha)$ for arbitrary dimensions d? (1 point)

(iii) Demonstrate that ρ is invariant under the transformation $(U \otimes U^*)\rho(U \otimes U^*)^\dagger$ for any unitary matrix U if and only if $\rho = \rho_{iso}$ 2 points)

(iv) The entanglement reduction criterion states that if:

$$\rho_A \otimes \mathbb{1}_B - \rho_{AB} \ngeq 0,$$

or

$$\mathbb{1}_A \otimes \rho_B - \rho_{AB} \ngeq 0,$$

then ρ_{AB} is entangled. Using such criterion, determine, for a given dimension d, for which values of α, the isotropic state ρ_{iso} is entangled (2 point)

(v) Using the PPT criterion, determine, for a given dimension d, for which values of α, the isotropic state ρ_{iso} is entangled (1 point)

(vi) For dimensions $d = 2$ the istotropic states are called Werner states. For which values of α is the state entangled? Demonstrate if the operator $W = (|\Phi\rangle_{00} \langle\Phi|_{00})^{T_A}$ is an entanglement witness for Werner states. If the Werner state is now $\rho_w = \alpha\rho + \frac{(1-\alpha)}{4}\mathbb{1} \otimes \mathbb{1}$, with $\rho = |\psi\rangle \langle\psi|$ and $|\psi\rangle = a|01\rangle + b|10\rangle$ with $a \neq b$, provide an entanglement witness for such state. (2 points).

Q5. Find out the Kraus operators for the complete dephasing channel of a single qubit, which can be expressed as: 1point

$$\Lambda[\rho] = Tr_2[U_{CNOT}(\rho \otimes |0\rangle \langle 0|)U_{CNOT}^\dagger]$$

Q6. Show that the partial transpose of any density matrix acting on $\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^2 \otimes \mathbb{C}^2$ can have at most one negative eigenvalue. (2 points)

Q7. Kraus, Choi and CPPT maps. Consider the following family of maps:

$$\Lambda_\alpha(\rho) = \frac{1}{2}\mathbb{1} + \alpha(X\rho Z + Z\rho X); \;\; 0 \leq \alpha \leq 1,$$

where as usual X,Y,Z denotes the Pauli matrices.

(i) Determine the values of α for which these maps are positive. Use the Bloch representation to see the effect of the map and indicate how the Bloch sphere changes. (1 point)

(ii) Which is the corresponding Choi state of the $\Lambda_\alpha(\rho)$? For which values of α is the map a CPTP map? (1 point)

(iii) Find the Kraus representation of the map for $\alpha = 1/4$. (2 points)