Solución Lab 8
FISI 6510

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Exercise 6.13: Wien's displacement constant

Planck's radiation law tells us that the intensity of radiation per unit area and per unit wavelength λ from a black body at temperature T is

$$I(\lambda) = \frac{2\pi hc^2\lambda^{-5}}{e^{hc/\lambda k_B T} - 1},$$

where h is Planck's constant, c is the speed of light, and k_B is Boltzmann's constant.

a) Show by differentiating that the wavelength λ at which the emitted radiation is strongest is the solution of the equation

$$5e^{-hc/\lambda k_B T} + \frac{hc}{\lambda k_B T} - 5 = 0.$$

Make the substitution $x = hc/\lambda k_B T$ and hence show that the wavelength of maximum radiation obeys the *Wien displacement law*:

$$\lambda = \frac{b}{T},$$

where the so-called *Wien displacement constant* is $b = hc/k_B x$, and x is the solution to the nonlinear equation

$$5e^{-x} + x - 5 = 0.$$

b) Write a program to solve this equation to an accuracy of $\epsilon = 10^{-6}$ using the binary search method, and hence find a value for the displacement constant.

c) The displacement law is the basis for the method of *optical pyrometry*, a method for measuring the temperatures of objects by observing the color of the thermal radiation they emit. The method is commonly used to estimate the surface temperatures of astronomical bodies, such as the Sun. The wavelength peak in the Sun's emitted radiation falls at $\lambda = 502$ nm. From the equations above and your value of the displacement constant, estimate the surface temperature of the Sun.

a)

Tenemos que diferenciar la ecuación

$$I(\lambda) = \frac{2\pi hc^2\lambda^{-5}}{e^{hc/\lambda k_B T} - 1}$$

$$
\begin{aligned}
\frac{\mathrm{d}I(\lambda)}{\mathrm{d}\lambda} =& \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{2\pi hc^2\lambda^{-5}}{e^{hc/\lambda k_B T}-1}\right) = 0\\
=& 2\pi hc^2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\lambda^{-5}}{e^{hc/\lambda k_B T}-1}\right) = 0\\
=& 2\pi hc^2\left(-5\lambda^{-6}(e^{hc/\lambda k_B T}-1)^{-1}+\lambda^{-5}(e^{hc/\lambda k_B T}-1)^{-2}e^{hc/\lambda k_B T}\frac{hc}{\lambda^2 k_B T}\right) = 0\\
=& \left(-5+\frac{e^{hc/\lambda k_B T}}{e^{hc/\lambda k_B T}-1}\frac{hc}{\lambda k_B T}\right) = 0\\
=& \frac{-5e^{-hc/\lambda k_B T}(e^{hc/\lambda k_B T}-1)+hc/\lambda k_B T}{e^{hc/\lambda k_B T}-1} = 0\\
=& 5e^{-hc/\lambda k_B T}+\frac{hc}{\lambda k_B T}-5 = 0
\end{aligned}
$$

Ahora hacemos la sustitución $x = \frac{hc}{\lambda k_B T}$ y despejamos para λ

$$x = \frac{hc}{\lambda k_B T} \to \lambda = \frac{hc}{x k_B T}$$

Si $b = \frac{hc}{x k_B}$ entonces

$$\lambda = \frac{b}{T}$$

b)

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

Para la función $f(x) = 5e^{-x} + x - 5$

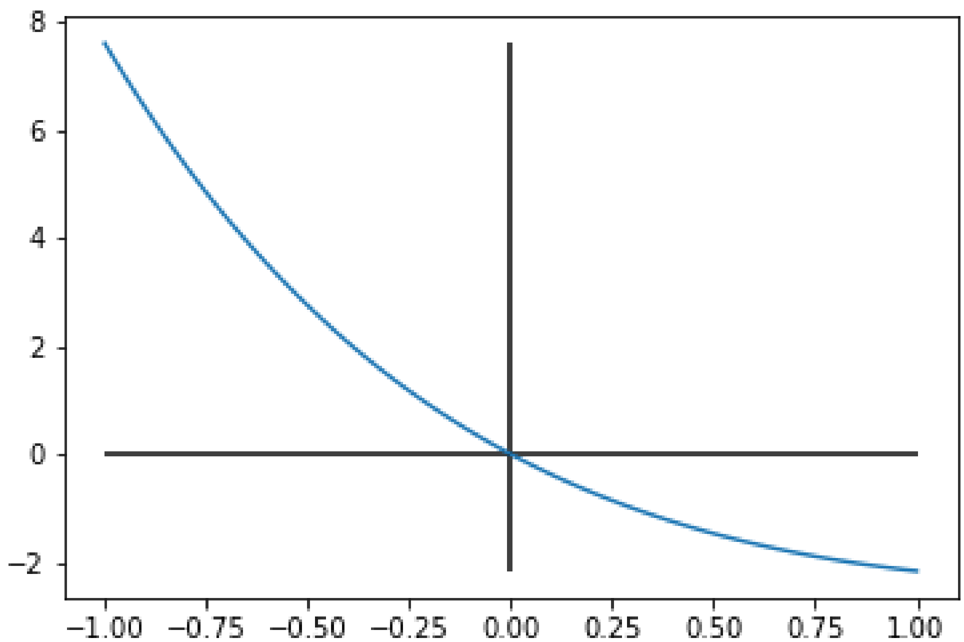
[2]:
```
def f(x):
    return 5*np.exp(-x)+x - 5



def binary_search(f,a,b,acc=1e-6):
    err=1
    if f(a)*f(b) <0:
        print("there is a root between {} and {}".format(a,b))
        while err> acc:
            mid=(a+b)/2
            if f(mid)*f(a)>0:
                a=mid
            else:
                b=mid
            err=abs(a-b)
```

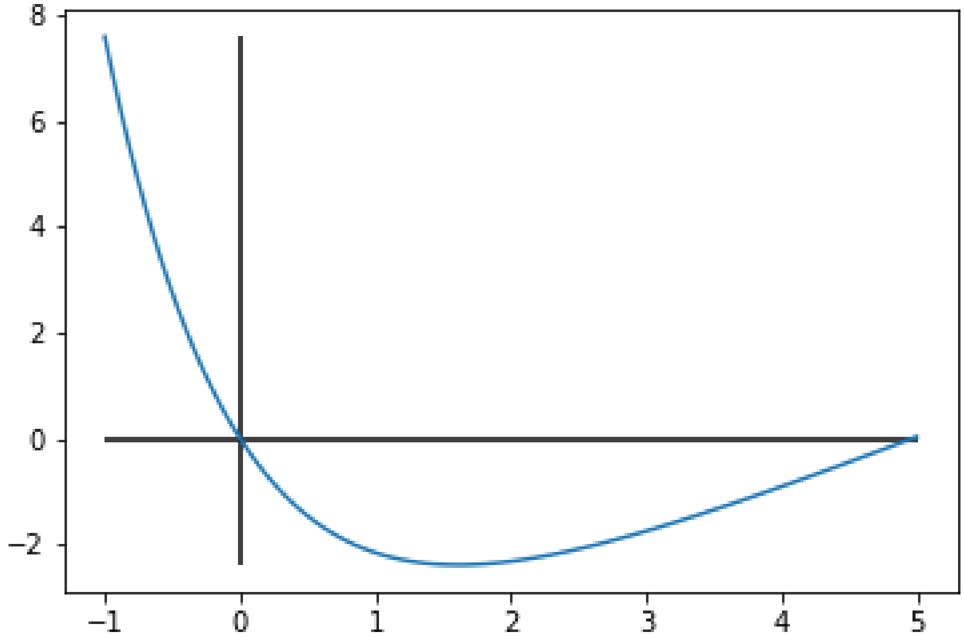
```
        return mid

    else:
        print("ERROR\nchoose other interval\n")
        return None
```

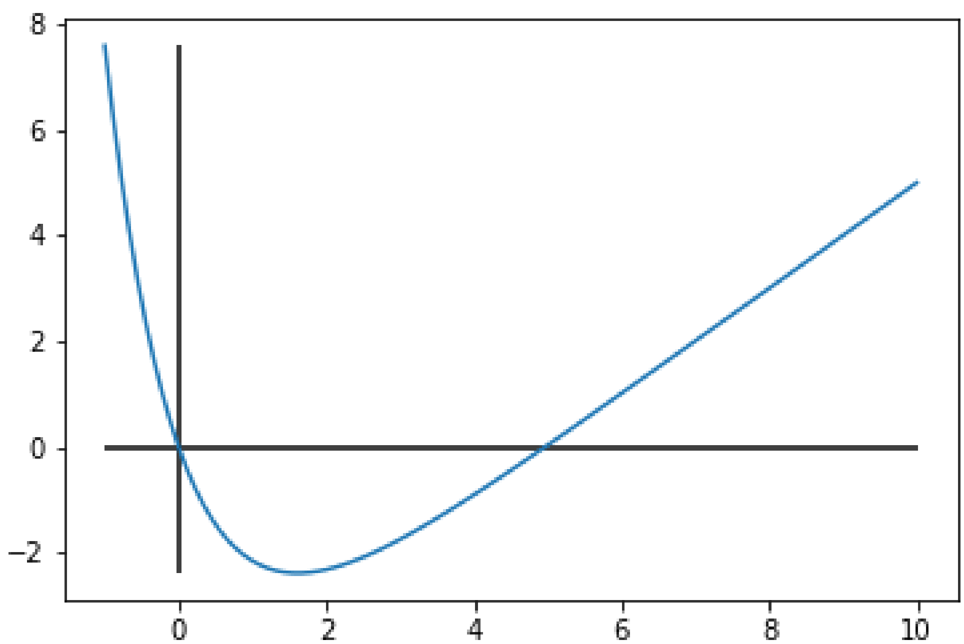
Primero inspeccionamos el comportamiento de esta función para tener una idea de un intervalo para el estimado de la raíz.

Por inspección sabemos que una raíz debe ser $x = 0$, dado que esta función cruza por el origen y sabemos que en el intevalo $x \in [0, \infty)$ el término $x - 5$ domina la función así que esperaría que la segunda raíz debe estar cerca de $x = 5$

[3]:
```
for a,b in zip([-1,-1,-1],[1,5,10]):
    x=np.linspace(a,b,100)

    plt.plot(x,f(x))
    plt.vlines(0,min(f(x)),max(f(x)),colors="k")
    plt.hlines(0,min(x),max(x),"k")
    plt.show()
```







Vemos que hay 2 raíces donde anticipamos

[4]:
```
x1=binary_search(f,-1,1)
print("la primera raíz es x =",x1,"\nDebe ser 0\n")
```

```
x2=binary_search(f,4,6)
print("La segunda raíz es x =",x2)
```

```
there is a root between -1 and 1
la primera raíz es x = -9.5367431640625e-07
Debe ser 0

there is a root between 4 and 6
La segunda raíz es x = 4.965113639831543
```

Determinamos ahora la constante de desplazameinto de Wein $b = \frac{hc}{xk_B}$. Notamos que no tiene sentido utilizar la raíz $x = 0$ por lo tanto tenemos que

[5]:
```
h=6.626e-34
c=3e8
kB= 1.38e-23
A=h*c/kB

b= A/x2
print("la constante b es",b)
```

```
la constante b es 0.0029011114087160485
```

c)

Estimamos la temperatura de la superficie del sol con la expresión

$$\lambda = \frac{b}{T} \rightarrow T = \frac{b}{\lambda}$$

[6]:
```
lamda= 502e-9 #m
T = b/lamda
print("La temperatura superficial del Sol es",int(T),"K")
```

```
La temperatura superficial del Sol es 5779 K
```