PH - 541
Quantum Mechanics
Homework 1

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1. **Problem 1.3** For the spin $\frac{1}{2}$ state $|S_x;+\rangle$ evaluate both sides of the inequality (1.146), that is

$$\left\langle(\Delta A)^2\right\rangle\left\langle(\Delta B)^2\right\rangle \geq \frac{1}{4}|\left\langle[A,B]\right\rangle|^2$$

for the operators $A = S_x$ and $B = S_y$, and show that the inequality is satisfied. Repeat for the operators $A = S_z$ and $B = S_y$.

Solution

We start by stating the state

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

and the operators S_x, S_y, S_z and S_i^2

$$S_x = \frac{\hbar}{2}[|+\rangle\langle-| + |-\rangle\langle+|]$$
$$S_y = \frac{\hbar}{2}[-i\,|+\rangle\langle-| + i\,|-\rangle\langle+|]$$
$$S_z = \frac{\hbar}{2}[|+\rangle\langle+| - |-\rangle\langle-|]$$
$$S_i^2 = \frac{1}{4}\hbar^2\mathbb{I}$$

We now substitue and evaluate the left hand side of the equation (L.H.S.)

$$\left\langle(\Delta S_x)^2\right\rangle\left\langle(\Delta S_y)^2\right\rangle \geq \frac{1}{4}|\left\langle[S_x,S_y]\right\rangle|^2$$
$$\left\langle(\Delta S_x)^2\right\rangle = \left\langle S_x^2\right\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - \langle S_x\rangle^2$$
$$\langle S_x\rangle = \langle S_x;+|S_x|S_x;+\rangle = \frac{1}{\sqrt{2}}(\langle+| + \langle-|)\left[\frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|)\right]\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$
$$= \frac{\hbar}{4}[0+0] = 0$$
$$\left\langle(\Delta S_x)^2\right\rangle = \frac{\hbar^2}{4}$$

Now for $(\Delta S_y)^2$, we compute S_y^2 and $\left\langle S_y\right\rangle$ first

$$S_y^2 = \frac{\hbar^2}{4}$$
$$\left\langle S_y\right\rangle = \langle S_x;+|S_y|S_x;+\rangle = \frac{1}{\sqrt{2}}(\langle+| + \langle-|)\left[\frac{\hbar}{2}(-i\,|+\rangle\langle-| + i\,|-\rangle\langle+|)\right]\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$
$$\left\langle S_y\right\rangle = \frac{\hbar}{4}[-i+i] = 0$$
$$\left\langle(\Delta S_y)^2\right\rangle = \frac{\hbar^2}{4}$$
$$\therefore \left\langle(\Delta S_x)^2\right\rangle\left\langle(\Delta S_y)^2\right\rangle = \frac{\hbar^2}{4}\frac{\hbar^2}{4} = \frac{\hbar^4}{16}$$

Now we evaluate the R.H.S of the equation

$$[S_x, S_y] = i\varepsilon_{x,y,z}\hbar \langle S_z \rangle = i\hbar \langle S_z \rangle = i\hbar\frac{\hbar}{2}[|+\rangle\langle+| - |-\rangle\langle-|]$$
$$i\frac{\hbar^2}{2}\frac{1}{\sqrt{2}}(\langle+| + \langle-|)[|+\rangle\langle+| - |-\rangle\langle-|]\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$