QM 1

January 8, 2025

Fundamental concepts

Stern-Gerlach Experiment

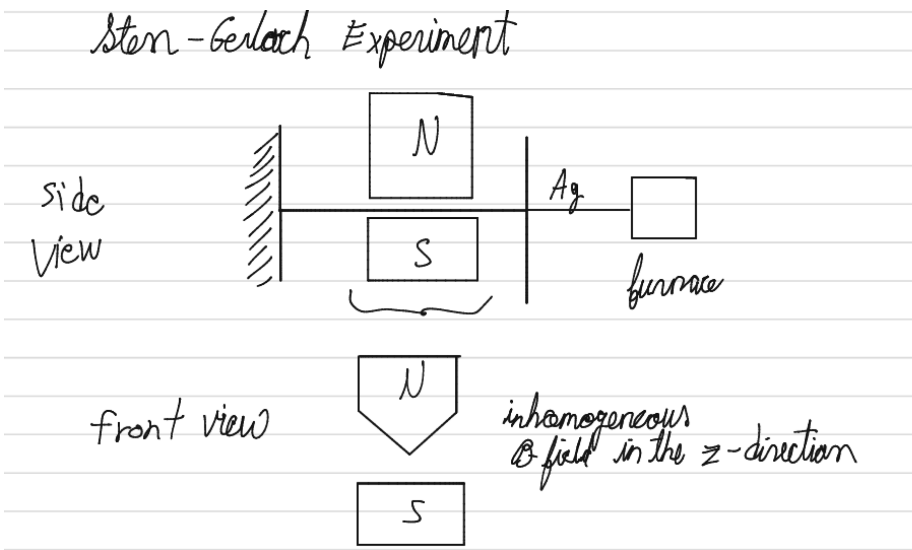


Figure 1: image.png

- Ag atoms coming out of screen becoming a beam after passing through collimator.
- Beam subjected to inhomogeneous B-field.
- Beam hits the screen & pattern is observed.

Ag atom carries a magnetic moment, equal to spin magnetic moment of the unpaired 5s e^-:

$$\vec{\mu} \propto \vec{S} \qquad (= \frac{e}{m_e c}\vec{S})$$

$$\begin{array}{cc} \text{Interaction} & E = -\vec{\mu} \cdot \vec{B} \\ \text{Force} & F = -\nabla E \end{array}$$

By symmetry, total force is along the z direction

$$F_z = \frac{\partial}{\partial z} \vec{\mu} \cdot \vec{B}$$

$$F_z \simeq \mu \frac{\partial B_z}{\partial z} \quad \text{since, } \frac{\partial B_x}{\partial z}, \frac{\partial B_y}{\partial z} << \frac{\partial B_z}{\partial z}$$

$\frac{\partial B_z}{\partial z} > 0$ since field is stronger near sharper edge of the "N" pole piece.

- Atoms with $\mu_z > 0 \quad (S_z < 0)$ experience upward force
- Atoms with $\mu_z < 0 \quad (S_z > 0)$ experience downward force

  $\therefore$ Beam split according to value of μ_z of individual Ag atom. The positions where beam hits the screen measure the z component of $\vec{\mu} \propto \vec{S}$.

  Note that the orientation of $\vec{\mu}$ of an atom coming out of the oven is random since atoms inside oven are randomly oriented.

Classically: expect all values of μ_z between $|\mu_z|$ and $-|\mu_z|$ since all orientations of $\vec{\mu}$ are realizable.

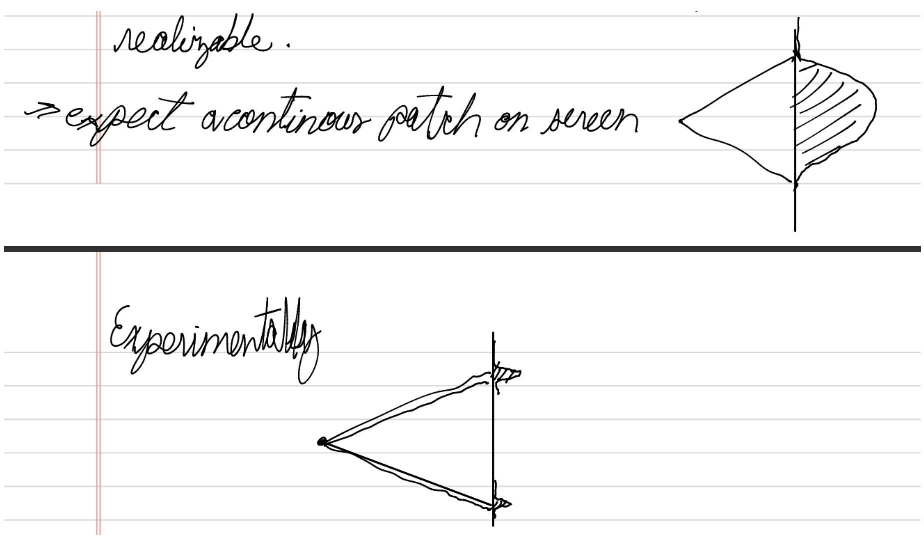


Figure 2: image.png

Observe two separate spots on screen. The SG apparatus splits the original beam into 2 distinct components, labelled as "up" or "down" components.

Therefore only two distinct values of S_z are possible, labelled as S_z up (or S_{z+}) and S_z down (or S_{z-})

$$S_{z+} = \frac{\hbar}{2} \quad , \quad S_{z-} = -\frac{\hbar}{2} \qquad \hbar = 1.0546 \times 10^{-34} Js.$$

Sequential SG experiments.

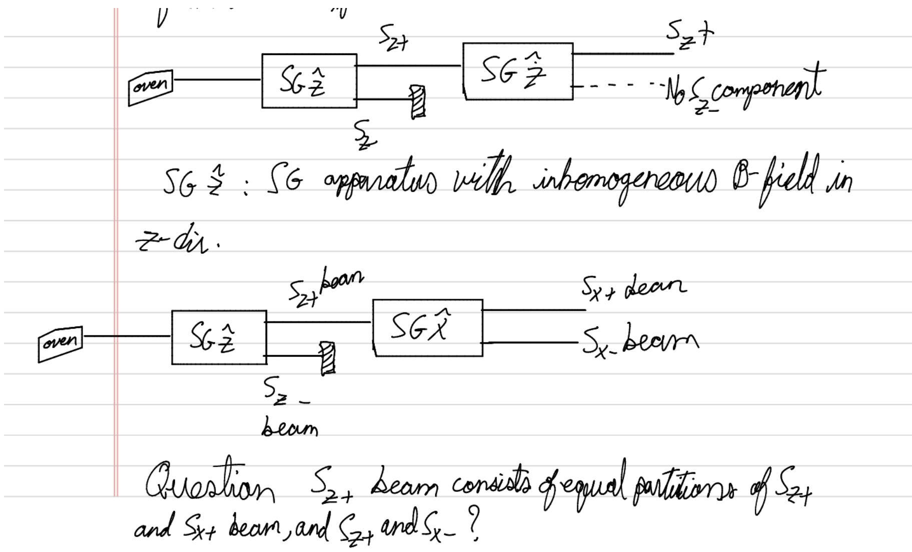


Figure 3: image.png

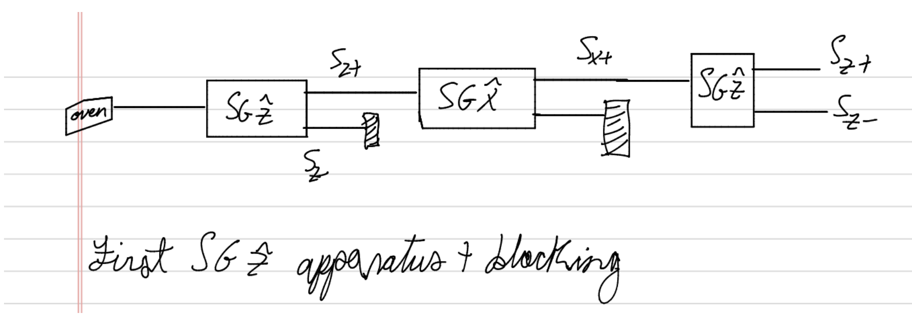


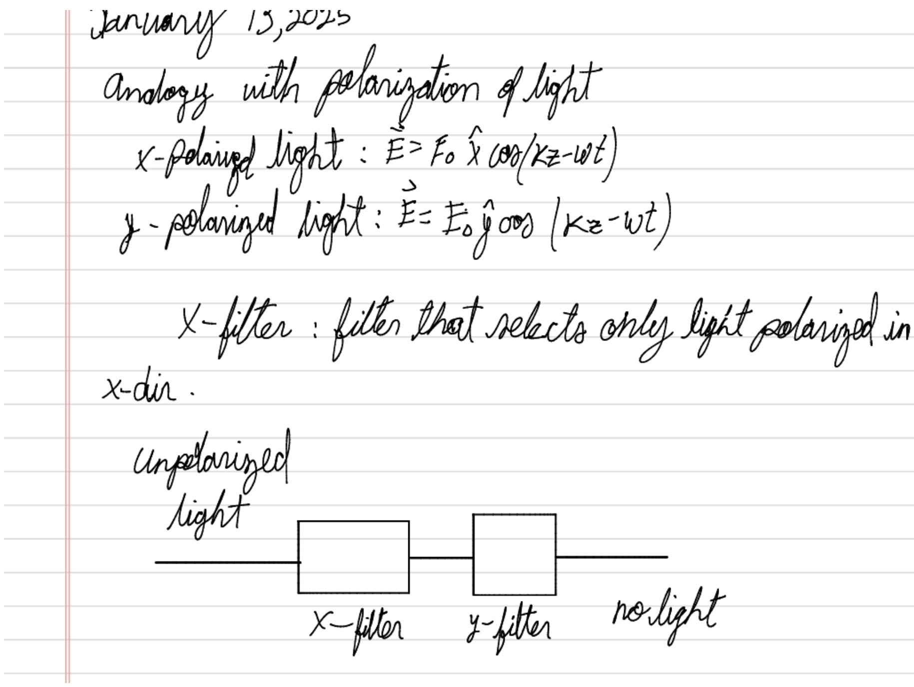
Figure 4: image.png

January 13, 2025

$S_z\pm$ atoms $x-, y-$ polarized light

$S_x\pm$ atoms $x'-, y'-$ polarized light

Idea: can represent spin state of Ag atom by a vector in anew kind of vector space. This vector is called a ket and the vector space is called a ket space. A ket is denoted as $|\cdots>$.



January 13, 2025

Analogy with polarization of light

x-polarized light : $\vec{E} = E_0 \hat{x} \cos(kz - \omega t)$

y-polarized light : $\vec{E} = E_0 \hat{y} \cos(kz - \omega t)$

X-filter : filter that selects only light polarized in x-dir.

Figure 5: image.png

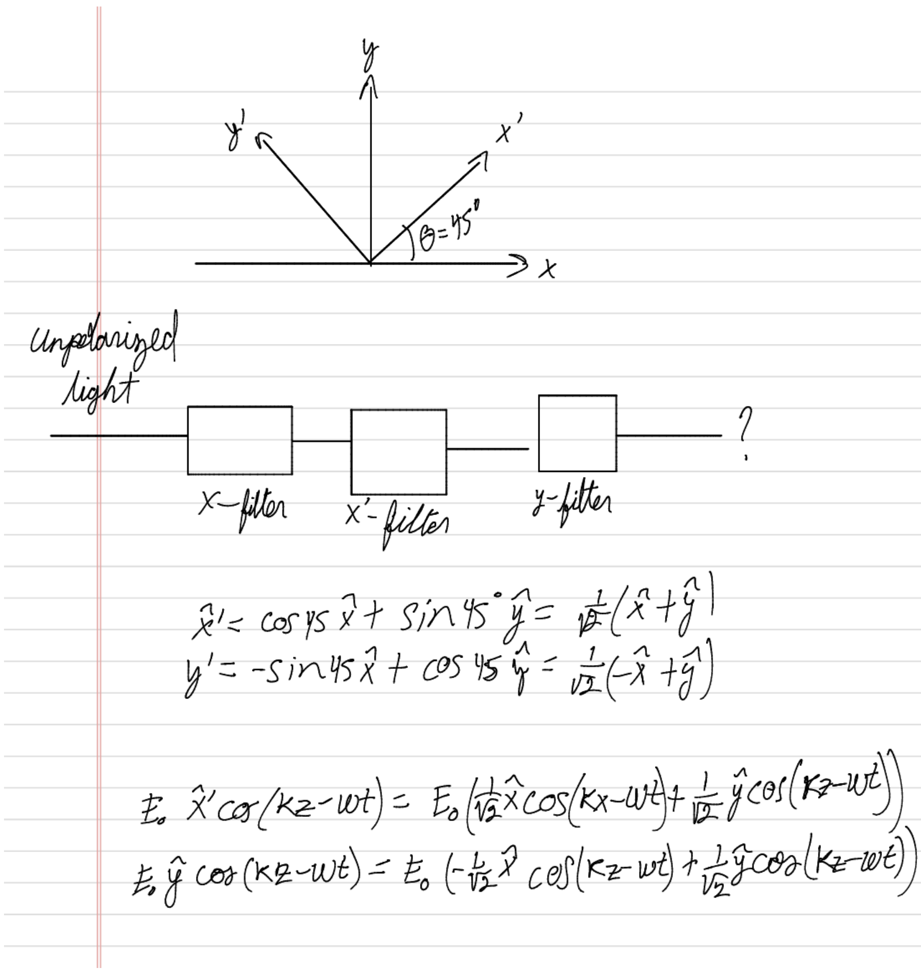


Figure 6: image.png

$|S_z; \pm>$ $-$spin "up" and "down" components of S along z - dir.

$$|S_x; +>= \frac{1}{\sqrt{2}}|S_z; +> + \frac{1}{\sqrt{2}}|S_z; -> |S_x; ->= -\frac{1}{\sqrt{2}}|S_z; +> + \frac{1}{\sqrt{2}}|S_z; ->$$

- $S_y\pm$ states? Introduce complex coefficients

$$\bar{\varepsilon} = E_0 \left[\frac{1}{\sqrt{2}}\hat{x}e^{i(kz-\omega t)} \pm \frac{i}{\sqrt{2}}\hat{y}e^{i(kz-\omega t)}\right] - \begin{pmatrix} R & CPL \\ L & CPL \end{pmatrix}$$

Analogy:

$$S_y\pm \rightarrow \begin{pmatrix} R \\ L \end{pmatrix} CPL |S_y; \pm>= \frac{1}{\sqrt{2}}|S_z; +> \pm \frac{i}{\sqrt{2}}|S_z; \pm>$$

$\therefore$ We need a complex vector space to describe the spin states of Ag atoms.

Kets, Bras and Operators

Ket Space

A ket space is a complex vector space with dimensionality specific to the nature of the considered system.

Eg. SG experiments : ket space has a dimension of 2 correspondign to the two values of S_z

— An example of a ket space with finite, and discrete degrees of freedom.

Eg. x and p of a particle

— an example of a ket space with infinite and continuous degrees of freedom. (Hilbert Space).

- A physical state is represented by a state vector in a complex vector space

$$|\alpha> \text{— ket}$$
$$|\alpha> + |\beta>= |\delta> \text{— addition of kets (in the same space)}$$
$$C|\alpha = |\alpha> C \text{— multiplication by a complex number}$$

  Note: $|\alpha>$ and $C|\alpha>$, with $c \neq 0$, represent the same state.

- An observable is represented by an operator in the vector space under consideration.

$$A \cdot (|\alpha>) = \underbrace{A|\alpha>}_{\text{generally a different state than } |\alpha>.} \qquad A|\alpha>= a|\alpha>$$

There exists eigenstates of A

$$\begin{gathered} |a'>, |a''>, |a'''>, \dots \\ \text{i.e.} \quad A|a'> = a'|a'> \\ A|a''> = a''|a''> \\ \vdots \\ \{a', a'', \dots\} = \{a'\} \text{ are eigenvlues of A} \\ |a'>, |a''>, \dots \text{ are each an eigenstate of A.} \end{gathered}$$

Eg.

$$S_z|S_z; +>= \frac{\hbar}{2}|S_z; +> \; S_z|S_z; ->= -\frac{\hbar}{2}|S_z; ->$$

January 15, 2025

Bra space and inner product

- Bra space is a vector space that is dual to the ket space.

Dual correspondance (DC) $|\alpha> \quad \longleftrightarrow \quad <\alpha|$

$|\alpha'>, |\alpha''>, \dots \iff <\alpha'|, <\alpha''|, \dots$ DC

$$C_\alpha|\alpha> + C_\beta|\beta> \longleftrightarrow C^*_\alpha <\alpha| + C^*_\beta <\beta|$$

where C_α and C_β are complex numbers.

- Inner product of a Bra and a Ket

$$<\beta|\alpha>= (<\beta|) \cdot (|\alpha>)$$

- Complex number postulated to have:

$$<\beta|\alpha>=<\alpha|\beta>^*$$

if $\alpha = \beta$:

$$<\alpha|\alpha>=<\alpha|\alpha>^* \to <\alpha|\alpha> \text{ is real}$$

- $< \alpha|\alpha > \geq 0$
- Two kets $|\alpha >$ and $|\beta >$ are orthogonal if $< \alpha|\beta >= 0$ or $< \beta|\alpha >= 0$
- $\sqrt{< \alpha|\alpha >}$ is called the norm of $|\alpha >$

Normalized ket:

$$|\hat{\alpha} >= \left(\frac{1}{\sqrt{< \alpha|\alpha >}} \right) |\alpha >$$

so that $< \hat{\alpha}|\hat{\alpha} >= 1$

Operators

$$X \cdot (|\alpha >) = X|\alpha >$$

Operators X and Y are equal if

$$X|\alpha >= Y|\alpha > \qquad \text{for any arbitrary ket}$$

Operator X is a null ooperator if

$$X|\alpha >= 0 \quad \text{for any arbitrary ket}$$

- Addition of operators:

$$X+Y = Y+X \quad \text{— commutativity } X+(Y+Z) = (X+Y)+Z \quad \text{— associativity}$$

- Linear operators:

$$X(C_\alpha|\alpha > +C_\beta|\beta >) = C_\alpha X|\alpha > +C_\beta X|\beta >$$

- An operator acts on a bra from the right side

$$(< \alpha|) \cdot X =< \alpha|X$$

Note $< \alpha|X$ and $X|\alpha >$ are not dual to each other. But

$$X|\alpha > \longleftrightarrow < \alpha|X^\dagger$$

where $X^\dagger$ is called the Hermitian adjoint of X. An operator is Hermitian if

$$X = X^\dagger$$

- Multiplication

Multiplication is generally non-commutative:

$$XY \neq YX$$

but it is associative

$$X(YX) = (XY)Z = XYZ\text{eg.} \quad X(Y|\alpha >) = (XY)|\alpha > (< \beta|X)Y =< \beta|(XY) =< \beta|XY$$

$$XY|\alpha >= X(Y|\alpha >) \longleftrightarrow (< \alpha|Y^\dagger)X^\dagger =< \alpha|Y^\dagger X^\dagger \text{because} \quad X|\alpha > \longleftrightarrow < \alpha|X^\dagger \Longrightarrow (XY)^\dagger = Y^\dagger X^\dagger$$

- Outer product

$$(|\beta > \cdot < \alpha|) = \underbrace{|\beta >< \alpha|}_{\text{operator}}$$

$$\text{eg.} \quad (|\beta >< \alpha|) \cdot |\gamma >= |\beta >< \alpha|\gamma >= (< \alpha|\gamma >) \cdot |\beta >$$

$$\begin{aligned} \underbrace{|\alpha >< \beta}_{X}|\gamma > \Longleftrightarrow \ &< \alpha|(< \beta|\gamma >)^* \\ &=< \alpha|(< \gamma|\beta >) \\ &=< \gamma\underbrace{|\beta >< \alpha|}_{X^\dagger} \end{aligned}$$

$$\therefore X = |\alpha >< \beta| \rightarrow X^\dagger = |\beta >< \alpha|$$

$$(< \beta|) \cdot (X|\alpha >) = (< \beta|X) \cdot (|\alpha >) =< \beta|X|\alpha >$$

$$\begin{aligned} \text{and} \quad < \beta|X|\alpha >=< \beta| \cdot \underbrace{(X\alpha)}_{\text{a ket}} &= \left\{[< \beta| \cdot (X|\alpha >)]^*\right\}^* \\ &=\{(< \alpha|X^\dagger) \cdot |\beta >\}^* \\ &= < \alpha|X^\dagger|\beta >^* \end{aligned}$$

if $X = X^\dagger$ then $< \beta|X|\alpha >=< \alpha|X|\beta >^*$