PH - 541
Quantum Mechanics
Homework 1

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February 3, 2025

1. **Problem 1.3** For the spin $\frac{1}{2}$ state $|S_x;+\rangle$ evaluate both sides of the inequality (1.146), that is

$$\left\langle(\Delta A)^2\right\rangle\left\langle(\Delta B)^2\right\rangle \geq \frac{1}{4}|\left\langle[A,B]\right\rangle|^2$$

for the operators $A = S_x$ and $B = S_y$, and show that the inequality is satisfied. Repeat for the operators $A = S_z$ and $B = S_y$.

Solution

We start by stating the state

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

and the operators S_x, S_y, S_z and S_i^2

$$S_x = \frac{\hbar}{2}[|+\rangle\langle-| + |-\rangle\langle+|]$$
$$S_y = \frac{\hbar}{2}[-i\,|+\rangle\langle-| + i\,|-\rangle\langle+|]$$
$$S_z = \frac{\hbar}{2}[|+\rangle\langle+| - |-\rangle\langle-|]$$
$$S_i^2 = \frac{1}{4}\hbar^2\mathbb{I}$$

We now substitute and evaluate the left hand side of the equation (L.H.S.)

$$\left\langle(\Delta S_x)^2\right\rangle\left\langle(\Delta S_y)^2\right\rangle \geq \frac{1}{4}|\left\langle[S_x,S_y]\right\rangle|^2$$
$$\left\langle(\Delta S_x)^2\right\rangle = \left\langle S_x^2\right\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - \langle S_x\rangle^2$$
$$\langle S_x\rangle = \langle S_x;+|S_x|S_x;+\rangle = \frac{1}{\sqrt{2}}(\langle+| + \langle-|)\left[\frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|)\right]\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$
$$= \frac{\hbar}{4}[0+0] = 0$$
$$\left\langle(\Delta S_x)^2\right\rangle = \frac{\hbar^2}{4}$$

Now for $(\Delta S_y)^2$, we compute S_y^2 and $\left\langle S_y\right\rangle$ first

$$S_y^2 = \frac{\hbar^2}{4}$$
$$\left\langle S_y\right\rangle = \langle S_x;+|S_y|S_x;+\rangle = \frac{1}{\sqrt{2}}(\langle+| + \langle-|)\left[\frac{\hbar}{2}(-i\,|+\rangle\langle-| + i\,|-\rangle\langle+|)\right]\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$
$$\left\langle S_y\right\rangle = \frac{\hbar}{4}[-i+i] = 0$$
$$\left\langle(\Delta S_y)^2\right\rangle = \frac{\hbar^2}{4}$$
$$\therefore \left\langle(\Delta S_x)^2\right\rangle\left\langle(\Delta S_y)^2\right\rangle = \frac{\hbar^2}{4}\frac{\hbar^2}{4} = \frac{\hbar^4}{16}$$

Now we evaluate the R.H.S of the equation

$$\begin{aligned}
\left\langle [S_x,S_y] \right\rangle &= i\varepsilon_{x,y,z}\hbar \left\langle S_z \right\rangle = i\hbar \left\langle S_z \right\rangle = i\hbar\frac{\hbar}{2} \left\langle S_x;+\right| [|+\rangle\langle+| - |-\rangle\langle-|] \left|S_x;+\right\rangle \\
&= i\frac{\hbar^2}{2}\frac{1}{\sqrt{2}}(\langle+| + \langle-|)[|+\rangle\langle+| - |-\rangle\langle-|]\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \\
&= i\frac{\hbar^2}{4}(1-1) = 0 \\
|\left\langle [S_x,S_y] \right\rangle|^2 &= 0
\end{aligned}$$

So now we have

$$\left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle \geq \frac{1}{4}|\left\langle [S_x,S_y] \right\rangle|^2 \longrightarrow \frac{\hbar^4}{16} \geq 0$$

So the relation holds. Now for $A = S_z$ and $B = S_y$. Starting with the L.H.S.

$$\begin{aligned}
\left\langle (\Delta S_z)^2 \right\rangle &= \left\langle S_z^2 \right\rangle - \left\langle S_z \right\rangle^2 \\
\left\langle (\Delta S_y)^2 \right\rangle &= \left\langle S_y^2 \right\rangle - \left\langle S_y \right\rangle^2
\end{aligned}$$

We already know that $\langle S_z \rangle = 0$ and we have already stated above $S_i^2 = \frac{1}{4}\hbar^2\mathbb{I}$.

$$\begin{aligned}
\left\langle S_z^2 \right\rangle &= \frac{\hbar^2}{4} \\
\therefore \left\langle (\Delta S_z)^2 \right\rangle &= \frac{\hbar^2}{4} \\
\left\langle (\Delta S_y)^2 \right\rangle &= \frac{\hbar^2}{4} \quad \text{As we have seen before.}
\end{aligned}$$

Now we work on the R.H.S.

$$\begin{aligned}
\left\langle [S_z,S_y] \right\rangle &= i\varepsilon_{z,y,x}\hbar \left\langle S_x \right\rangle = -i\hbar \left\langle S_x \right\rangle = -i\hbar \left\langle S_x;+|S_x|S_x;+ \right\rangle \\
&= -i\hbar\frac{\hbar}{2}\frac{1}{2} \left\langle S_x;+\right| (|+\rangle\langle-| + |-\rangle\langle+|) \left|S_x;+\right\rangle \\
&= -i\frac{\hbar^2}{2}\frac{1}{2}(\langle+| + \langle-|)[|+\rangle\langle-| + |-\rangle\langle+|](|+\rangle + |-\rangle) \\
&= -i\frac{\hbar^2}{2}\frac{1}{2}(\langle+|+\rangle\langle-|-\rangle + \langle-|-\rangle\langle+|+\rangle) = -i\frac{\hbar^2}{2}\frac{1}{2}(2) = -i\frac{\hbar^2}{2} \\
|\left\langle [S_z,S_y] \right\rangle|^2 &= \left(\frac{-i\hbar^2}{2}\right)\left(\frac{i\hbar^2}{2}\right) = \frac{\hbar^4}{4}
\end{aligned}$$

So finally

$$\left\langle (\Delta S_z)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle \geq \frac{1}{4}|\left\langle [S_z,S_y] \right\rangle|^2 \longrightarrow \frac{\hbar^4}{16} \geq \frac{1}{4}\frac{\hbar^4}{4} \longrightarrow \frac{\hbar^4}{16} \geq \frac{\hbar^4}{16}$$

The relation also holds.

2. **Problem 1.6** Using the rules of bra-ket algebra, prove or evaluate the following:

   (a) $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$, where X and Y are operators.
   Solution:

$$\begin{aligned}\mathrm{Tr}(XY) &= \sum_{a'} \langle a'|XY|a'\rangle = \sum_{b'}\sum_{a'} \langle a'|X|b'\rangle \langle b'|Y|a'\rangle \\ &= \sum_{b'a'} \langle b'|Y|a'\rangle \langle a'|X|b'\rangle = \sum_{b'} \langle b'|YX|b'\rangle = \mathrm{Tr}(YX)\end{aligned}$$

   (b) $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators.
   Solution:
   Using two arbitrary kets $|\alpha\rangle, |\beta\rangle$

$$\begin{aligned}\langle\alpha|(XY)^\dagger|\beta\rangle = \langle\beta|XY|\alpha\rangle^* &= \sum_{a'} \left(\langle\beta|X|a'\rangle \langle a'|Y|\alpha\rangle\right)^* = \sum_{a'} \langle a'|Y|\alpha\rangle^* \langle\beta|X|a'\rangle^* \\ &= \sum_{a'} \langle\alpha|Y^\dagger|a'\rangle \langle a'|X^\dagger|\beta\rangle = \langle\alpha|Y^\dagger X^\dagger|\beta\rangle\end{aligned}$$

   So we have

$$\langle\alpha|(XY)^\dagger|\beta\rangle = \langle\alpha|Y^\dagger X^\dagger|\beta\rangle$$

   for arbitrary $|\alpha\rangle, |\beta\rangle$, therefore we must have

$$(XY)^\dagger = Y^\dagger X^\dagger$$

   (c) $\exp[if(A)] =?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known.
   Solution:

$$\begin{aligned}\exp[if(A)] &= \exp\left[if\left(\sum_{a'a''} |a'\rangle\langle a'|A|a''\rangle\langle a''|\right)\right] \\ = \exp\left(if\begin{bmatrix} a' & 0 & 0 \\ 0 & a'' & 0 \\ 0 & 0 & \ddots \end{bmatrix}\right) &= \begin{bmatrix} \exp(if(a')) & 0 & 0 \\ 0 & \exp(if(a'')) & 0 \\ 0 & 0 & \ddots \end{bmatrix} \\ &= \sum_{a'} \exp\left[if(a')\right] |a'\rangle\langle a'|\end{aligned}$$

3. **Problem 1.7**

(a) Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle, \langle a''|\alpha\rangle, \ldots$ and $\langle a'|\beta\rangle, \langle a''|\beta\rangle, \ldots$ are all known, where $|a'\rangle, |a''\rangle, \ldots$ form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$.

$$|\alpha\rangle = \sum_{a'} |a'\rangle \langle a'|\alpha\rangle$$
$$|\beta\rangle = \sum_{a''} |a''\rangle \langle a''|\beta\rangle$$

Now we evaluate $|\alpha\rangle\langle\beta|$ and set $|\alpha\rangle\langle\beta| = Z$

$$\sum_{a''a'} |a'\rangle \langle a'|\alpha\rangle \langle\beta|a''\rangle \langle a''| = \sum_{a'a''} |a'\rangle\langle a'|Z|a''\rangle\langle a''|$$
$$\begin{pmatrix} \langle a^1|Z|a^1\rangle & \langle a^1|Z|a^2\rangle & \cdots & \langle a^1|Z|a^m\rangle \\ \langle a^2|Z|a^1\rangle & \langle a^2|Z|a^2\rangle & \cdots & \langle a^2|Z|a^m\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a^m|Z|a^1\rangle & \cdots & \cdots & \langle a^m|Z|a^m\rangle \end{pmatrix}$$

(b) We know consider a spin $\frac{1}{2}$ system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|S_z;+\rangle$ and $|S_x;+\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (s_z diagonal) basis.

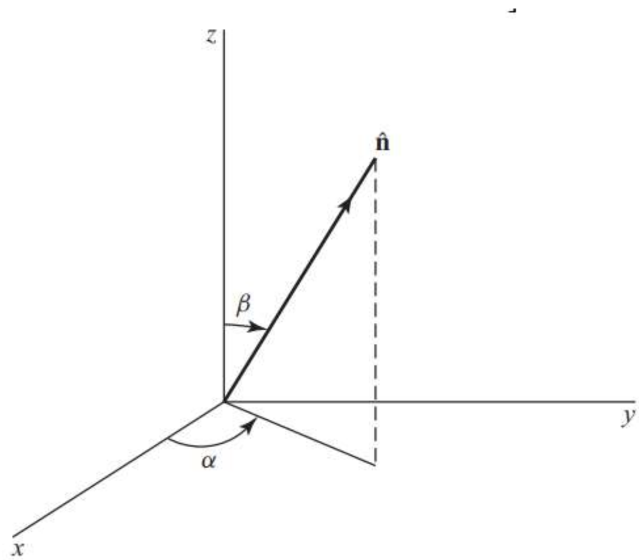
$$|\alpha\rangle = |S_z;+\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$|\beta\rangle = |S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$|\alpha\rangle\langle\beta| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

4. **Problem 1.11** Construct $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$ such that

$$\mathbf{S}\cdot\hat{\mathbf{n}}\,|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \left(\frac{\hbar}{2}\right)|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$$

where $\hat{\mathbf{n}}$ is characterized by the angles shown in the figure. Express your answer as a linear combination of $|+\rangle$ and $|-\rangle$. [Note: The answer is

$$\cos\left(\frac{\beta}{2}\right)|+\rangle \sin\left(\frac{\beta}{2}\right)e^{i\alpha}\,|-\rangle\,.$$



Solution

We begin by writing the matrix representation of this eigenvalue problem and then solving the eigenvalue problem stated above with the given eigenvalue $\frac{\hbar}{2}$ to find the eigenvector.

$$\mathbf{S}\cdot\hat{\mathbf{n}} = S_x\sin\beta\cos\alpha + S_y\sin\beta\sin\alpha + S_z\cos\beta$$
$$= \frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta\cos\alpha - i\sin\beta\sin\alpha \\ \sin\beta\cos\alpha + i\sin\beta\sin\alpha & -\cos\beta\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta e^{-i\alpha} \\ \sin\beta e^{i\alpha} & -\cos\beta\end{pmatrix}$$

We use this matrix and solve the eigenvalue equation

$$(\mathbf{S}\cdot\hat{\mathbf{n}} - \frac{\hbar}{2}\mathbb{I})X = 0$$
$$\frac{\hbar}{2}\begin{pmatrix}\cos\beta - 1 & \sin\beta e^{-i\alpha} \\ \sin\beta e^{i\alpha} & -\cos\beta - 1\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

This gives us 2 equations

$$(\cos\beta - 1)x + (\sin\beta e^{-i\alpha})y = 0 \tag{1}$$
$$(\sin\beta e^{i\alpha})x - (\cos\beta + 1)y = 0 \tag{2}$$

From equation (2) we get the expression

$$y = \frac{\sin\beta e^{i\alpha}}{\cos\beta + 1}x$$

And so our eigenvector can be written as

$$\begin{bmatrix} 1 \\ \frac{\sin\beta e^{i\alpha}}{\cos\beta+1} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\sin\beta e^{i\alpha}}{2\cos^2(\beta/2)} \end{bmatrix} = \begin{bmatrix} \cos(\beta/2) \\ \frac{\sin\beta e^{i\alpha}}{2\cos(\beta/2)} \end{bmatrix} = \begin{bmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2}e^{i\alpha} \end{bmatrix}$$

Where I have used the relations

$$\cos\beta + 1 = 2\cos^2\left(\frac{\beta}{2}\right) \qquad \sin\beta = 2\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)$$

and so using $x \to |+\rangle$ and $y \to |-\rangle$ we have

$$|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}$$

1.13 A two-state system is characterized by the Hamiltonian

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}[|1\rangle\langle 2| + |2\rangle\langle 1|]$$

where H_{11}, H_{22}, and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and corresponding energy eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$.

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \qquad \det(H - \lambda I) = 0$$

$$\rightarrow \det\begin{pmatrix} H_{11}-\lambda & H_{12} \\ H_{12} & H_{22}-\lambda \end{pmatrix} = (H_{11}-\lambda)(H_{22}-\lambda) - H_{12}^2 = 0$$

$$H_{11}H_{22} + \lambda^2 - \lambda(H_{11}+H_{22}) - H_{12}^2 = 0$$

$$\lambda^2 - \lambda(H_{11}+H_{22}) + H_{11}H_{22} - H_{12}^2 = 0$$

$$\lambda = \frac{(H_{11}+H_{22}) \pm \sqrt{(H_{11}+H_{22})^2 - 4(1)(H_{11}H_{22} - H_{12}^2)}}{2}$$

$$= \frac{(H_{11}+H_{22}) \pm \sqrt{H_{11}^2 + 2H_{11}H_{22} + H_{22}^2 - 4H_{11}H_{22} + 4H_{12}^2}}{2}$$

$$\lambda_{1,2} = \frac{(H_{11}+H_{22}) \pm \sqrt{H_{11}^2 - 2H_{11}H_{22} + H_{22}^2 + 4H_{12}^2}}{2} = \frac{(H_{11}+H_{22}) \pm \sqrt{(H_{11}-H_{22})^2 + 4H_{12}^2}}{2}$$

for λ_1 we find the eigenvector

$$(H - \lambda_{1,2}\mathbb{1})\vec{X} = \vec{0} \rightarrow \begin{pmatrix} H_{11} - \frac{(H_{11}+H_{22})}{2} \pm \frac{\sqrt{(H_{11}-H_{22})^2 + 4H_{12}^2}}{2} & H_{12} \\ H_{12} & H_{22} - \frac{(H_{11}+H_{22}) \pm \sqrt{(H_{11}-H_{22})^2 + 4H_{12}^2}}{2} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eq (1)

$$\hookrightarrow X_1\left(H_{11}-\frac{(H_{11}+H_{22})}{2}\pm\frac{\sqrt{(H_{11}-H_{22})^2+4H_{12}^2}}{2}\right)+H_{12}X_2=0$$

$$X_2=\frac{X_1}{H_{12}}\left(H_{11}-\frac{(H_{11}+H_{22})}{2}\pm\frac{\sqrt{(H_{11}-H_{22})^2+4H_{12}^2}}{2}\right)$$

into Eq (2)

$$\hookrightarrow H_{12}X_1+\left(H_{22}-\frac{(H_{11}+H_{22})}{2}\pm\frac{\overbrace{\sqrt{(H_{11}-H_{22})^2+4H_{12}^2}}^{A}}{2}\right)\left(\frac{X_1}{H_{12}}\right)\Bigg($$

$$H_{11}-\frac{(H_{11}+H_{22})}{2}\pm\frac{\sqrt{(H_{11}-H_{22})^2+4H_{12}^2}}{2}\Bigg)=0$$

$$\rightarrow H_{12}^2+\left(H_{22}-\frac{(H_{11}+H_{22})}{2}\pm\frac{A}{2}\right)\left(H_{11}-\frac{(H_{11}+H_{22})}{2}\pm\frac{A}{2}\right)=0$$

$$=H_{12}^2+\left(H_{22}H_{11}+\left(\frac{H_{11}+H_{22}\pm A}{2}\right)^2-\frac{H_{22}[(H_{11}+H_{22})\pm A]}{2}\right.$$

$$\left.-\frac{H_{11}}{2}\left[H_{11}+H_{22}\pm A\right]\right)=0$$

$$=2H_{12}^2+2H_{22}H_{11}+2\left(\frac{H_{11}+H_{22}\pm A}{2^2}\right)^2-H_{22}(H_{11}+H_{22}\pm A)-H_{11}(H_{11}+H_{22}\pm A)$$

$$=0$$

$$=2H_{12}^2+2H_{22}H_{11}+2\left(\frac{(H_{11}+H_{22})^2\pm A^2\pm 2A(H_{11}+H_{22})}{2^2}\right)$$

$$a=\frac{-H_{12}}{H_{11}-\lambda}\,b\rightarrow\begin{bmatrix}\frac{-H_{12}}{H_{11}-\lambda}\\1\end{bmatrix}=\begin{bmatrix}-H_{12}\\H_{11}-\lambda\end{bmatrix}$$

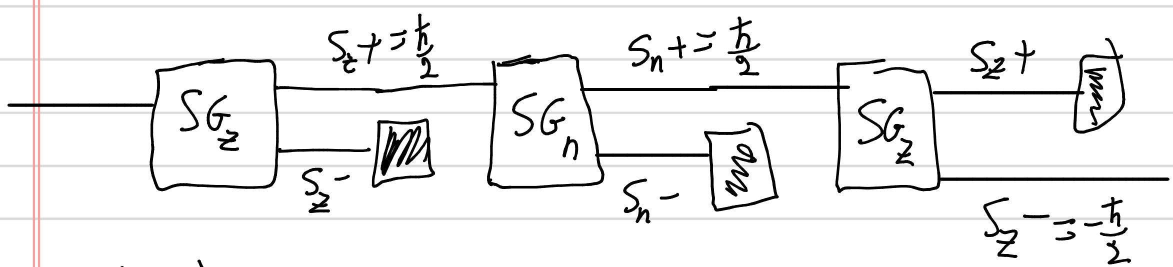
1.15 A beam of spin $\frac{1}{2}$ atoms goes through a series of Stern–Gerlach type measurements as follows.

a. The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.

b. The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where s_n is the eigenvalue of the operator $\mathbf{S}\cdot\hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ making an angle β in the xz-plane with respect to the z-axis.

c. The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam?



$|S_z \pm\rangle = |\pm\rangle$

$S_z|\pm\rangle = \frac{\hbar}{2}|\pm\rangle$

Fist measurement is $\langle +|+\rangle = 1$

Then $\langle S\cdot\hat{n};+|+\rangle = ?$

$\hat{n} = \sin\beta\cos\alpha\,\hat{x} + \sin\beta\sin\alpha\,\hat{y} + \cos\beta\,\hat{z}$ but $\hat{n}$ is on the xz-plane

so $\alpha = 0 \rightarrow \hat{n} = \sin\beta\,\hat{x} + \cos\beta\,\hat{z}$

$|S\cdot\hat{n};+\rangle = \cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}e^{i\alpha}|-\rangle = \cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}|-\rangle$

$\langle S\cdot\hat{n};+|+\rangle = \cos\frac{\beta}{2} \rightarrow |\langle S\cdot\hat{n}|+\rangle|^2 = \cos^2\left(\frac{\beta}{2}\right)$

Find measurement is $\langle -|S\cdot\hat{n};+\rangle = \sin\frac{\beta}{2}$

So $|\langle -|S_n\cdot\hat{n};+\rangle|^2 = \sin^2\left(\frac{\beta}{2}\right)$

The final intensity is $|\langle -|S\cdot\hat{n};+\rangle|^2\,|\langle S\cdot\hat{n};+|+\rangle|^2 = \sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2}$

if $\sin\frac{\beta}{2}\cos\frac{\beta}{2} = \frac{1}{2}\sin(\beta) \rightarrow \frac{1}{4}\sin^2\beta$

Inorder to maximize the final beam $\beta = \frac{\pi}{2}$ so then the beam is $\frac{1}{4}$.