

Useful formulas

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ — number of ways to choose k objects out of n

$\binom{n+k-1}{k-1}$ — number of ways to choose k objects out of n with repetitions

$[n]_k$ — Stirling numbers of the first kind; number of permutations of n elements with k cycles

$$[n+1]_k = n[n]_k + [n]_{k-1}$$

$$(x)_n = x(x-1)\dots x-n+1 = \sum_{k=0}^n (-1)^{n-k} [n]_k x^k$$

$\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ — Stirling numbers of the second kind; number of partitions of set $1, \dots, n$ into k disjoint subsets.

$$\left\{ \begin{smallmatrix} n+1 \\ m \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\}$$

$$\sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} (x)_k = x^n$$

$$C_n = \frac{1}{n+1} \binom{2n}{n} \text{ — Catalan numbers}$$

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

Binomial transform

If $a_n = \sum_{k=0}^n \binom{n}{k} b_k$, then $b_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k$

- $a = (1, x, x^2, \dots), b = (1, (x+1), (x+1)^2, \dots)$

- $a_i = i^k, b_i = \left\{ \begin{smallmatrix} n \\ i \end{smallmatrix} \right\} i!$

Burnside's lemma

Let G be a group of *action* on set X (Ex.: cyclic shifts of array, rotations and symmetries of $n \times n$ matrix, ...)

Call two objects x and y *equivalent* if there is an action f that transforms x to y : $f(x) = y$.

The number of equivalence classes then can be calculated as follows: $C = \frac{1}{|G|} \sum_{f \in G} |X^f|$, where X^f

is the set of *fixed points* of f : $X^f = \{x | f(x) = x\}$

Generating functions

Ordinary generating function (o.g.f.) for sequence

$$a_0, a_1, \dots, a_n, \dots \text{ is } A(x) = \sum_{n=0}^{\infty} a_n x^n$$

Exponential generating function (e.g.f.) for

$$\text{sequence } a_0, a_1, \dots, a_n, \dots \text{ is } A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

$$B(x) = A'(x), b_{n-1} = n \cdot a_n$$

$$c_n = \sum_{k=0}^n a_k b_{n-k} \text{ (o.g.f. convolution)}$$

$$c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} \text{ (e.g.f. convolution, compute with FFT using } \widetilde{a_n} = \frac{a_n}{n!})$$

General linear recurrences

If $a_n = \sum_{k=1}^n b_k a_{n-k}$, then $A(x) = \frac{a_0}{1-B(x)}$. We also can compute all a_n with Divide-and-Conquer algorithm in $O(n \log^2 n)$.

Inverse polynomial modulo x^l

Given $A(x)$, find $B(x)$ such that $A(x)B(x) = 1 + x^l \cdot Q(x)$ for some $Q(x)$

1. Start with $B_0(x) = \frac{1}{a_0}$

2. Double the length of $B(x)$:

$$B_{k+1}(x) = (-B_k(x)^2 A(x) + 2B_k(x)) \bmod x^{2^{k+1}}$$

Fast subset convolution

Given array a_i of size 2^k , calculate $b_i = \sum_{j \& i = i} b_j$

```
for b = 0..k-1
  for i = 0..2^k-1
    if (i & (1 << b)) != 0:
      a[i + (1 << b)] += a[i]
```

Hadamard transform

Treat array a of size 2^k as k -dimensional array of size $2 \times 2 \times \dots \times 2$, calculate FFT of that array:

```
for b = 0..k-1
  for i = 0..2^k-1
    if (i & (1 << b)) != 0:
      u = a[i], v = a[i + (1 << b)]
      a[i] = u + v
      a[i + (1 << b)] = u - v
```