Defining the Fourier Transform Using Complex Numbers

Valerio Velardo

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Previously...

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$$\varphi_f = argmax_{\varphi \in [0,1)} \left(\int s(t) \cdot sin(2\pi \cdot (ft - \varphi)) \cdot dt \right)$$

$$d_f = \max_{\varphi \in [0,1)} \left(\int s(t) \cdot sin(2\pi \cdot (ft - \varphi)) \cdot dt \right)$$

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- Use magnitude and phase as polar coordinates

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- Encode both coefficients in a single complex number

Complex Fourier transform coefficients

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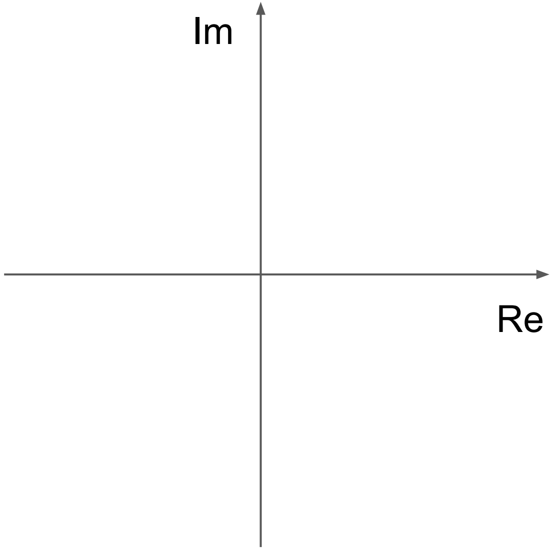

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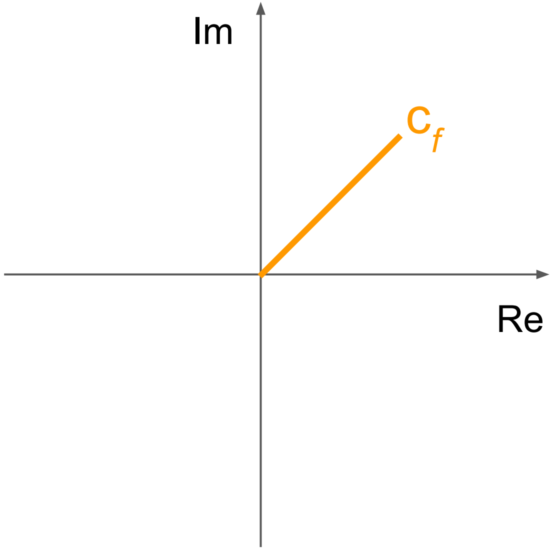

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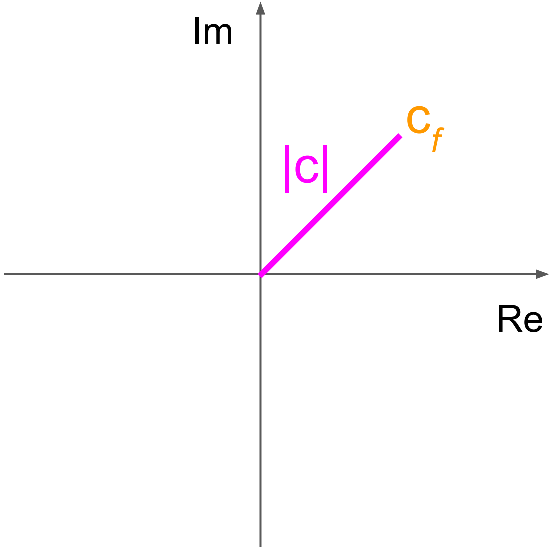

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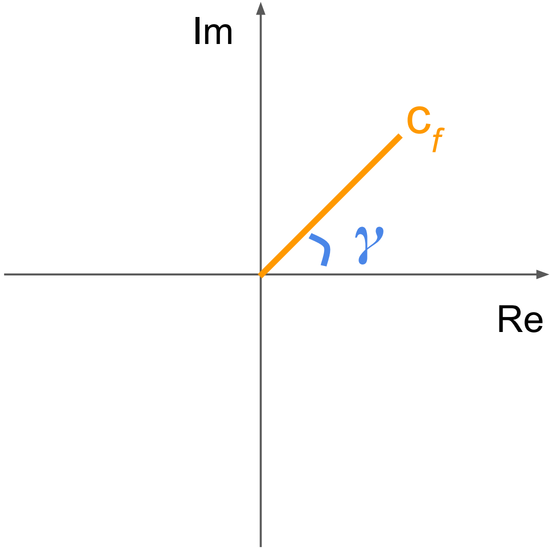

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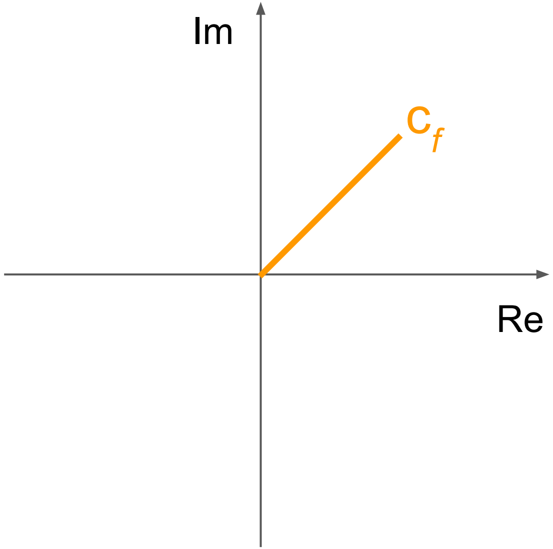

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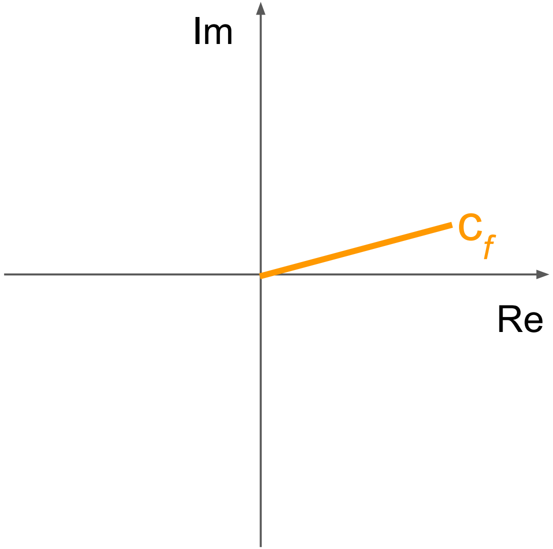

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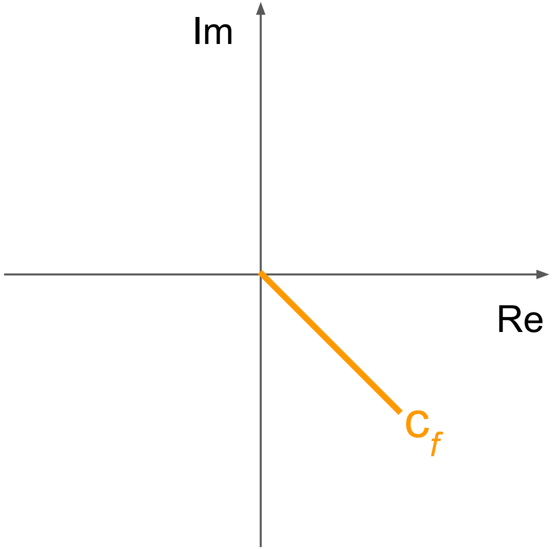

Continuous audio signal

$$g(t)$$

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$$g(t) \qquad\qquad g : \mathbb{R} \to \mathbb{R}$$

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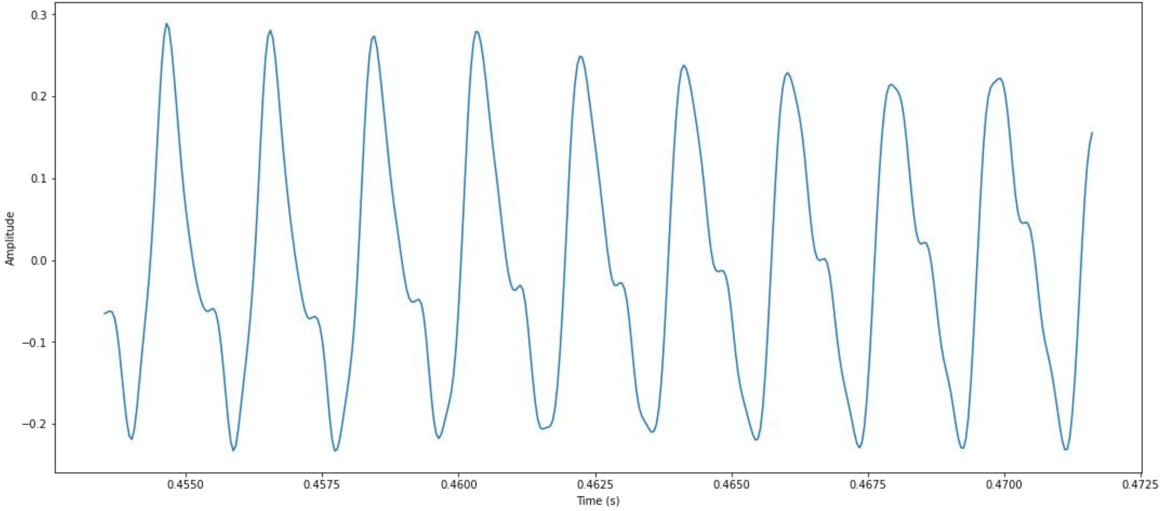

Complex Fourier transform

$$\hat{g}(f) = c_f$$

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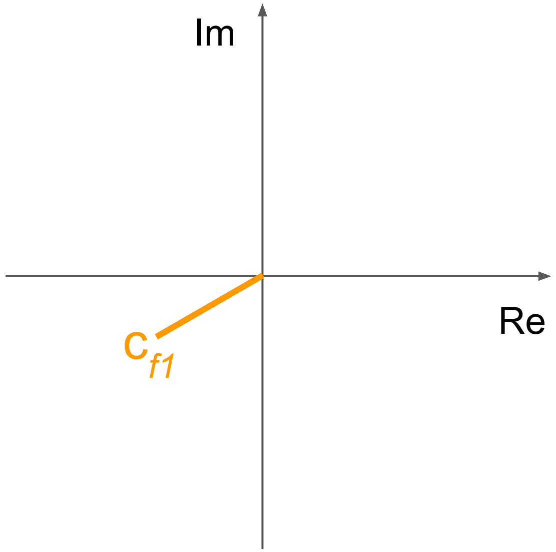

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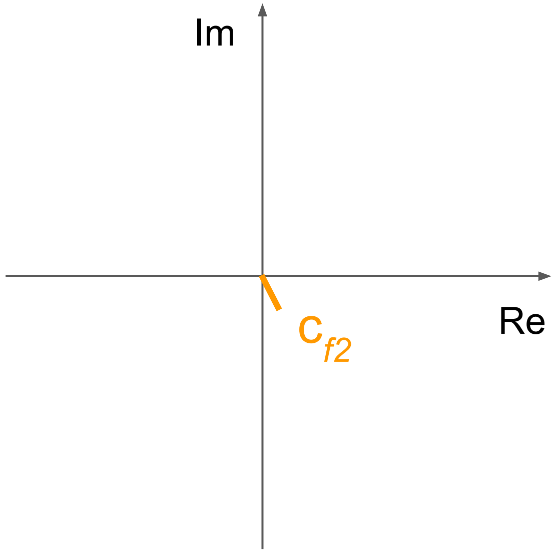

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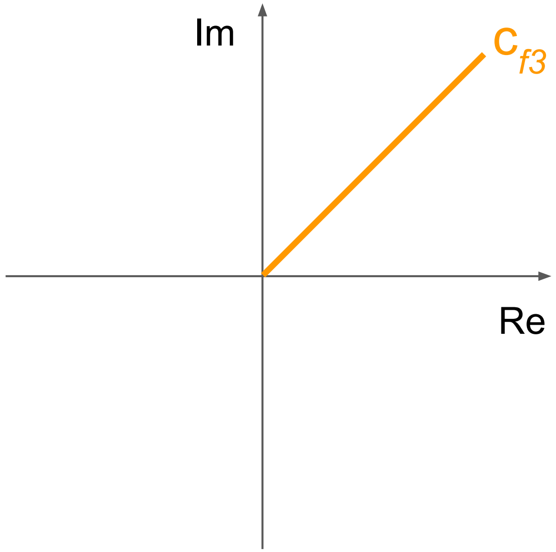

Complex Fourier transform

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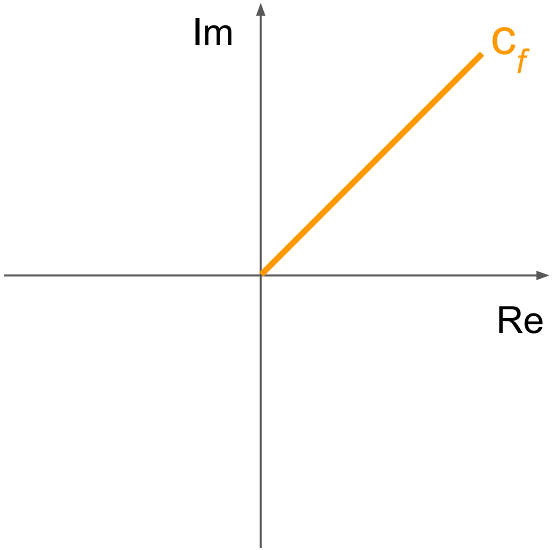
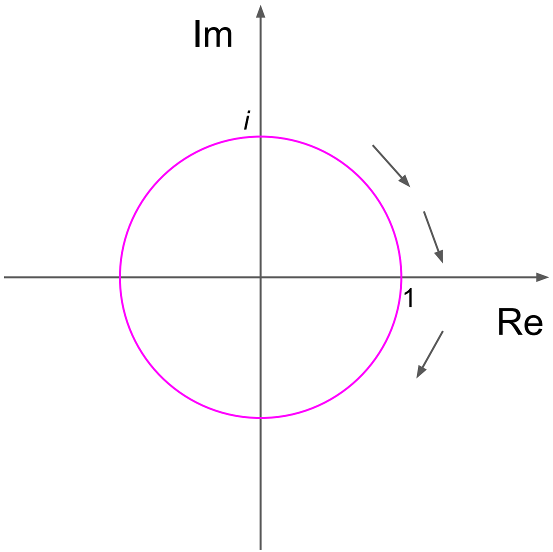


Complex Fourier transform

$$\hat{g}(f) = \int g(t) \cdot \boxed{e^{-i2\pi f t}} dt$$



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Complex Fourier transform

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$$\hat{g}(f) = \int g(t) \cdot e^{-i2\pi ft} dt = \boxed{\int g(t) \cdot \cos(-2\pi ft) dt + i \int g(t) \cdot \sin(-2\pi ft) dt}$$

Complex Fourier transform

Real part

Imaginary part

$$\hat{g}(f) = \int g(t) \cdot e^{-i2\pi f t} dt = \boxed{\int g(t) \cdot \cos(-2\pi f t) dt} + \boxed{i \int g(t) \cdot \sin(-2\pi f t) dt}$$

Magnitude Fourier transform

$$|\hat{g}(f)|$$

Magnitude and phase

$$c_f = \frac{d_f}{\sqrt{2}} \cdot e^{-i2\pi\varphi_f}$$

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$$\boxed{d_f = \sqrt{2} \cdot |\hat{g}(f)|}$$

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$$\boxed{\varphi_f = -\frac{\gamma_f}{2\pi}}$$

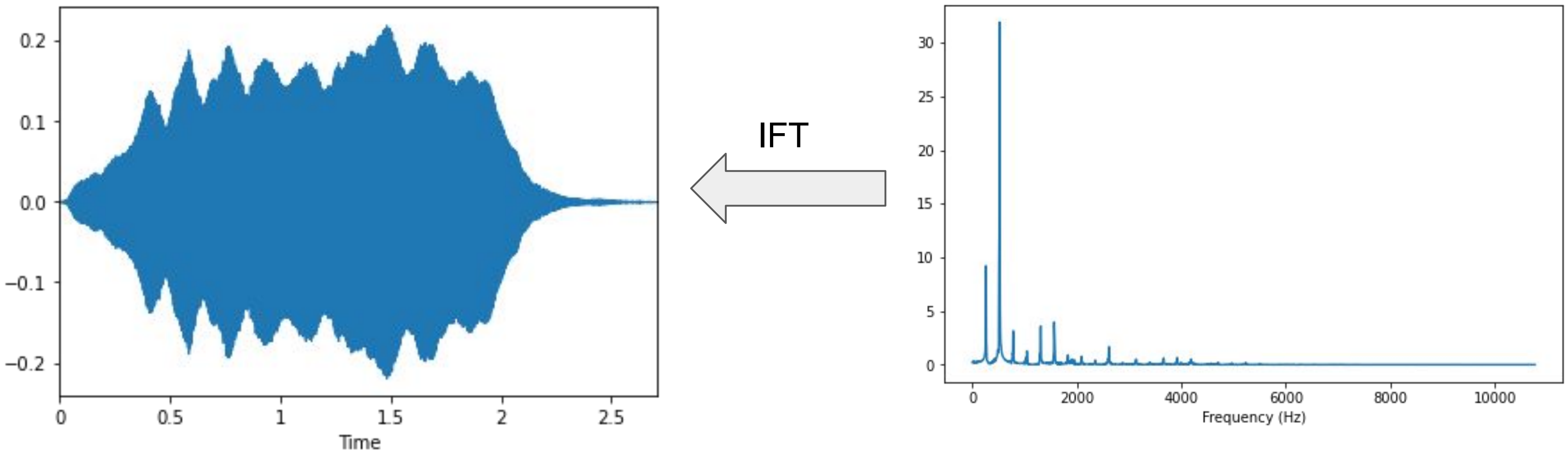
Magnitude and phase

$$c_f = \frac{d_f}{\sqrt{2}} \cdot e^{-i\boxed{2\pi\varphi_f}}$$

$$d_f = \sqrt{2} \cdot |\hat{g}(f)|$$

$$\varphi_f = -\frac{\boxed{\gamma_f}}{2\pi}$$

Inverse Fourier transform



Fourier representation

$$g(t) = \int c_f \cdot e^{i2\pi f t} df$$

Fourier representation

Pure tone of frequency *f*

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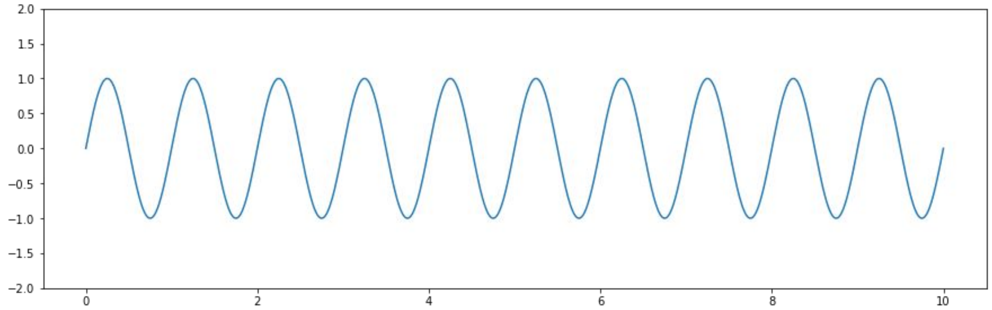
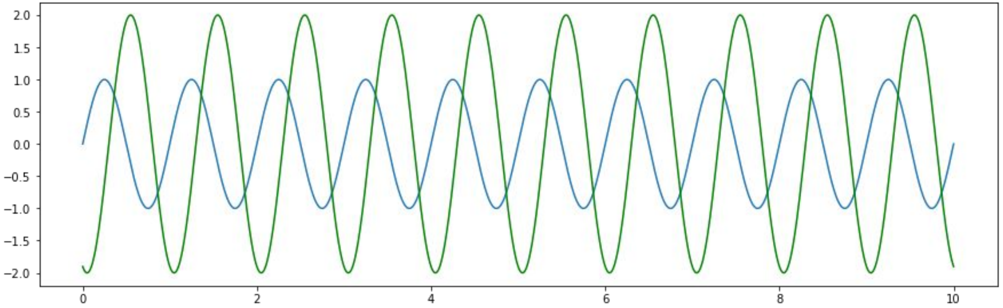


Fourier representation

Weight pure tone with magnitude and add phase

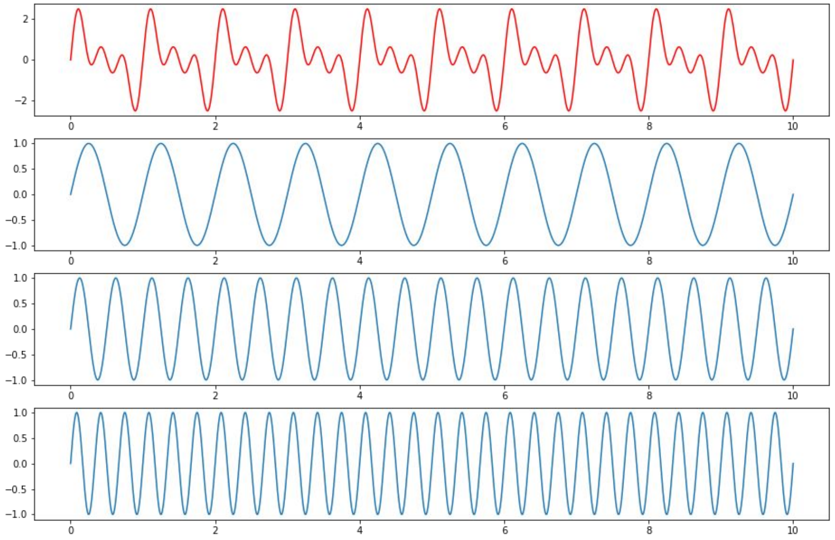
$$g(t) = \int \boxed{c_f \cdot e^{i2\pi ft}} df$$



Fourier representation

Add up all (weighted) sinusoids

$$g(t) = \boxed{\int c_f \cdot e^{i2\pi ft} df}$$



A Fourier roundtrip

$$\hat{g}(f) = \int g(t) \cdot e^{-i2\pi ft} dt$$

$$g(t) = \int c_f \cdot e^{i2\pi ft} df$$