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## A Numerical treatment of Fisher Equation

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### Abstract

Fisher equation is commonly arises in chemistry, heat and mass transfer, biology and ecology. In mathematics Fisher equation is also known as Kolmogorov Petrovsky-Piscounov equation, KPP equation or Fisher KPP equation. Fisher equation describes the process of interaction between diffusion and reaction. In this paper a semi implicit method is used to solve the Fisher equation. A semi implicit finite difference scheme has been designed for numerical solution of one dimensional nonlinear Fisher equation. The designed scheme accuracy is first order in time and second order in space. Numerical results are calculated for different values of Diffusion coefficient and time steps are matching with exact solution.

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**Keywords:** Fisher equation; Diffusion Coefficient; Reaction and diffusion equation; semi implicit scheme; method of lagging

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### 1. Introduction

In 1937 Fisher [1] and Kolmogorov et al. [2] investigated independently the Fisher- Kolmogorov-Petrovsky-Piscounov (Fisher-KPP) equation, after that it is widely known as Fisher equation. This equation has many applications in science and engineering fields [3-6]. The researchers studied some meaningful generalization of this equation [7-9]. Here we considered one generalization of this equation which is called as one component reaction-diffusion equation. Many reaction-diffusion equations have travelling wave fronts which play an important role in the understanding of physical, chemical, and biological phenomena [7], [10], [11]. Reaction-diffusion systems are mathematical models which explains how the concentration of one or more substances distributed in space changes under the influence of two processes, first one is local chemical reactions in which the substances are transformed into each other and second is the diffusion which causes the substances to spread out over a surface in space.

Reaction-diffusion systems are naturally applied in chemistry. However, the system can also describe the dynamical processes of non-chemical nature. Mathematically, reaction diffusion system takes the form of semi-linear parabolic partial differential equation. They can be represented in general form,

$$\partial_t u = D \nabla^2 u + R(u) \quad (1)$$

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Where each component of vector  $u(x, t)$  represents the concentration of one substance,  $D$  is diffusion coefficients and  $R$  accounts for all local reactions. The solution of reaction diffusion equations display a wide range of behaviours, including the formation of travelling waves and wave like phenomena as well as other self -organized patterns like stripes, hexagons or more intricate structure like dissipative solutions. Reaction diffusion equation will be classified as one component, two component or three or more component diffusion equations depending upon the component taking part in reaction.

The simplest reaction-diffusion equation concerning the concentration  $u$  of a single substance in one spatial dimension,

$$u_t = Du_{xx} + R(u) \quad (2)$$

If the reaction term vanishes, then the equation represents pure diffusion process and if thermal diffusivity term  $\alpha$  will come instead of diffusion term  $D$  then equation will convert into parabolic partial differential equation in one dimensional in space. The choice  $R(u) = u(1 - u)$  yields Fisher equation that is used to describe the spreading of biological populations. Fisher KPP equation with advection is used to describe the population dynamics in advective environments [12].

Fisher proposed nonlinear partial differential equation,

$$u_t = Du_{xx} + su(1 - u)$$

As a model, for propagation of a mutant gene with an advantageous selection intensity  $s$ . Same equation also arises in flame propagation, in the branching Brownian motion process and in nuclear reactor theory. Because of the use of Fisher equation in wide variety of applications it is more concerned equation in the field of engineering.

The paper is structured as follows. Section 2, incorporate details of proposed differences scheme for Fisher equation. To obtain the finite difference solution of Fisher equation, the differential equation at each node is replaced by a difference equation. After considering boundary conditions in the difference equations, the resulting algebraic system of equations is solved. In section 3, accuracy of proposed scheme is verified by performing several numerical experiments.

## 2. A Semi Implicit Difference Scheme

The solution domain is discretized with uniform meshes. The space interval  $[0, 1]$  is divided into  $N$  equal subintervals. The time interval  $[0, \tau]$  is divided into  $M$  equal subintervals. Assuming  $\Delta x = 1/N$  as the mesh width in space and  $x_i$  set as  $x_i = i \Delta x$  for  $i = 0, 1, 2, ., N$ . Assuming  $\Delta t = \tau/M$  as the mesh width in time and  $t^n$  is set as  $t^n = n \Delta t$  for  $n = 0, 1, 2, ., M$ .

In this semi implicit scheme derivatives are calculated at  $n+1$  time level whereas in explicit method derivatives are calculated in  $n$  time level. We use central difference formula in space and forward difference formula for time. In equation (3) if  $s = 0$ , then equation will be similar as parabolic heat equation which is used in heat and mass transfer. But in equation (3) nonlinear term is also there in right hand side. For converting this nonlinear term into linear term we used method of lagging. In method of lagging one is calculated at  $n$  time level and other is calculated at  $n + 1$  time level.

Fisher equation is given by,

$$u_t = Du_{xx} + su(1 - u) \quad (3)$$

Fisher equation will becomes,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = D \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} + su_i^{n+1}(1 - u_i^n) \quad (4)$$

$$u_i^{n+1} = \frac{D \Delta t}{\Delta x^2} [u_{i+1}^{n+1} - 2u_i^n + 1 + u_{i-1}^{n+1}] + s \Delta t u_i^{n+1} (1 - u_i^n) + u_i^n \quad (5)$$

Assuming,

$$\frac{D \Delta t}{\Delta x^2} = p \quad \& \quad s \Delta t = q$$

Equation will be converted into,

$$u_i^{n+1} = p(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + qu_i^{n+1}(1 - u_i^n) + u_i^n \quad (6)$$

Above equation can be written in tridiagonal form as,

$$A_i u_{i-1}^{n+1} + B_i u_i^{n+1} + C_i u_{i+1}^{n+1} = D_i \quad (7)$$

Where,  $A_i$  and  $C_i$  are constants. so  $A_i = C_i = (-p)$  and  $B_i = 1 + 2p - q(1 - u_i^n)$

Actually the constant  $p$  is a dimensionless number which is called Fourier number and it is used in the study of unsteady-state mass transfer, equal to the product of diffusion coefficient and characteristic time divided by the square of a characteristic length. After assembling the entire system of equations and applying boundary conditions, the matrix form  $\vec{A}\vec{X} = \vec{B}$ . Where  $\vec{A}$  is tridiagonal matrix of order  $M \times N$  and  $\vec{X}$  &  $\vec{B}$  are the vectors, where vector  $\vec{B}$  contains all known values and vector  $\vec{X}$  contains the value which we want to find out.

$$\vec{A} = \begin{pmatrix} B_1 & -p & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -p & B_2 & -p & \cdots & \cdots & \vdots & \vdots & \vdots \\ \cdots & -p & B_3 & -p & \cdots & \vdots & \vdots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -p & B_{N-2} & -p & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & -p & B_{N-1} & -p \end{pmatrix}$$

$$\vec{X} = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \\ u_N^{n+1} \end{pmatrix} \quad \vec{B} = \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ u_4^n \\ \vdots \\ u_{N-1}^n \\ u_N^n \end{pmatrix}$$

## 2.1. Numerical Experiments

After presenting the proposed scheme, here the simulation of test examples is undertaken to validate the theoretical results obtained by the new scheme. Due to its transient nature, Fisher equation is difficult for carrying out the analysis. The boundary conditions have sufficient smoothness for maintaining the accuracy of the proposed scheme. MATLAB software is used to programme and generate the numerical solutions of boundary value problems.

Example-1

Fisher equation with initial condition and homogeneous boundary conditions are as follows [13]:

$$u_t = u_{xx} + 6u(1-u) \quad 0 < x < 1, t > 0$$

$$u(x, 0) = \frac{1}{(1 + e^x)^2} \quad 0 \leq x \leq 1$$

$$u(0, t) = \frac{1}{(1 + e^{5t})^2} \quad 0 \leq t \leq \tau$$

$$u(1, t) = \frac{1}{(1 + e^{1-5t})^2} \quad 0 \leq t \leq \tau$$

## 2.2. Results and Discussion

Numerical results are computed using the proposed scheme in order to validate our proposed scheme with exact solution of the Fisher equation [14-17]. The graphs are the results obtained for example 1. Here we observe the changes in the result with variation in time and space steps. The different parameter values which considered are as follows: diffusion coefficient ( $D$ ) = 1 and intesity ( $s$ ) = 6, and Fig(1) are the results when no. of time steps = 50, no. of space steps = 50. Fig (2) are the results when no of time spaces =100 and no of space steps=100.

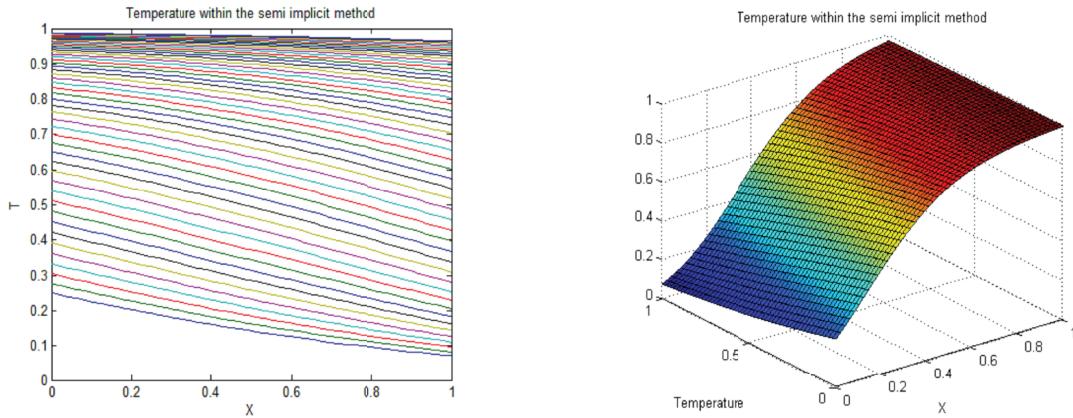


Fig. 1.

Table 1. Numerical results generated by semi-implicit method with  $x=0.25$ ,  $t=0.001$ , at final time  $t=0.01$ .

| Value of x | Numerical Solution | Exact Solution |
|------------|--------------------|----------------|
| 0.00       | 0.2627             | 0.2626         |
| 0.25       | 0.2027             | 0.2026         |
| 0.50       | 0.1516             | 0.1516         |
| 0.75       | 0.1101             | 0.1100         |
| 1.00       | 0.0778             | 0.0777         |

From the above result we can say that our proposed scheme provides almost accurate values related to exact solution.

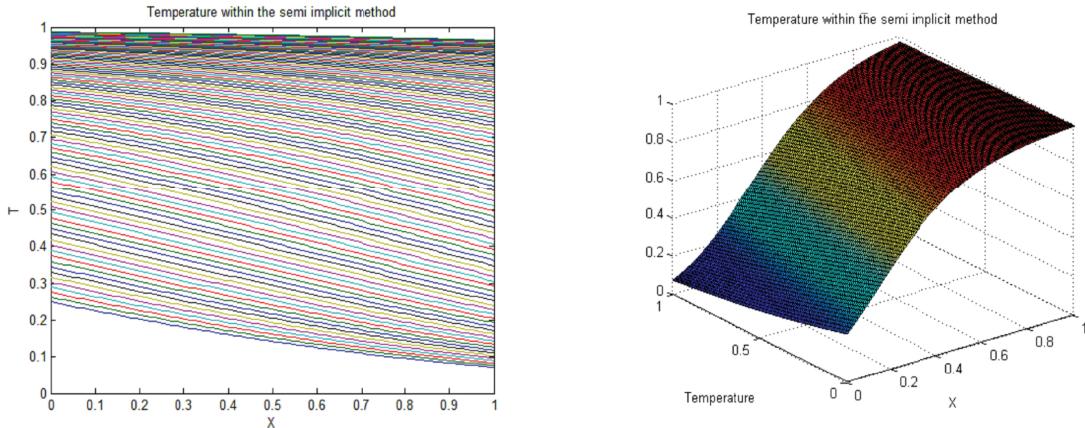


Fig. 2.

### 3. Stability analysis

For finding stability of Fisher equation we are using matrix analysis, Fisher equation is a nonlinear equation and to find its stability we are assuming  $u_i^n = k$  as a constant, therefore writing the equation (7) in terms of constants  $k$ .

So, coefficient matrix for equation (6) can be given by,

$$\begin{pmatrix} F & -p & \cdots & \cdots & \cdots & \cdots & \cdots \\ -p & F & -p & \cdots & \cdots & \cdots & \cdots \\ \cdots & -p & F & -p & \cdots & \cdots & \cdots \\ \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -p & F & -p \end{pmatrix}$$

Here,  $F = 1 + 2p - q(1 - k)$  which is also a constant.

The coefficient matrix is symmetric and positive definite, then its Eigenvalues are also positive and to minimize the errors Eigenvalue of coefficient matrix must be less than or equal to one, so  $\lambda_l \leq 1$ . Eigen value for above matrix can be represented as [18],

$$1 + 2p - q(1 - k) + 2pcos\frac{\pi l}{r + 1} \leq 1 \quad (8)$$

where  $l = 1, 2, 3, \dots, r$

when value of

$$(a) \ cos\frac{\pi l}{r + 1} = 0, \text{ then } \frac{p}{q(1 - k)} \leq \frac{1}{2}$$

$$(b) \ cos\frac{\pi l}{r + 1} = 1, \text{ then } \frac{p}{q(1 - k)} \leq \frac{1}{4}$$

### 4. Consistency and Errors

#### 4.1. LTE(local Truncation Errors )

The local truncation error (LTE) of a numerical method is an estimate of the error introduced in a single iteration of the method, assuming that everything fed into the method was perfectly accurate.

Expanding the coefficients  $u_i^{n+1}, u_{i+1}^{n+1}, u_{i-1}^{n+1}$  by Taylor series Method.

$$u_i^{n+1} = u_i^n + \frac{(\Delta t)}{1!} \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^3}{3!} \frac{\partial^3 u}{\partial t^3} + \dots + O(\Delta t)^4 \quad (9)$$

$$u_{i+1}^{n+1} = u_i^n + \frac{(\Delta x)}{1!} \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3} + \dots + O(\Delta x)^4 \quad (10)$$

$$u_{i-1}^{n+1} = u_i^n - \frac{(\Delta x)}{1!} \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{3!} \frac{\partial^3 u}{\partial x^3} + \dots + O(\Delta x)^4 \quad (11)$$

Now substituting the value of equation (9), (10), (11) in equation (5)

We get,

$$\frac{1}{\Delta t} \left[ \frac{\partial u}{\partial t} (\Delta t) + O(\Delta t)^2 \right] = \frac{D}{(\Delta x)^2} \left[ \frac{\partial^2 u}{\partial x^2} (\Delta x)^2 + O(\Delta x)^4 \right] + s u_i^n (1 - u_i^n) \quad (12)$$

Local Truncation error for above equation can be written as,

$$LTE = \lim_{\Delta x, \Delta t \rightarrow 0} \frac{(\Delta t)^2}{2!} \frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^3}{3!} \frac{\partial^3 u}{\partial t^3} + \dots + \frac{(\Delta x)^4}{4!} \frac{\partial^4 u}{\partial x^4} + \frac{(\Delta x)^6}{6!} \frac{\partial^6 u}{\partial x^6} + \dots = 0 \quad (13)$$

#### 4.2. Consistency

A finite difference representation of PDE is said to be consistent if we can show that the difference between PDE and its FDE representation vanishes as mesh is refined.

So we can write as,

$$\lim_{mesh \rightarrow 0} (PDE - FDE) = \lim_{mesh \rightarrow 0} (LTE) = 0 \quad (14)$$

Since  $\Delta t$  and  $\Delta x$  approaches to zero, So from equation (13), local truncation error becomes zero, therefore proposed scheme is consistent. And solving the equation (12) and comparing with equation (5) we can say order of our proposed scheme is first order in time and second order in space.

## 5. Conclusions

Numerical scheme proposed to solve Fisher/KPP equation by using semi implicit finite difference method in combination with method of lagging. The numerical results obtained by the proposed scheme are quite satisfactory for a wide range of time steps in comparison of numerical results have been made in order to validate the scheme. The proposed scheme is a semi implicit scheme which is second order accurate in space and first order accurate in time. Stability and consistency of the proposed scheme is also validated.

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