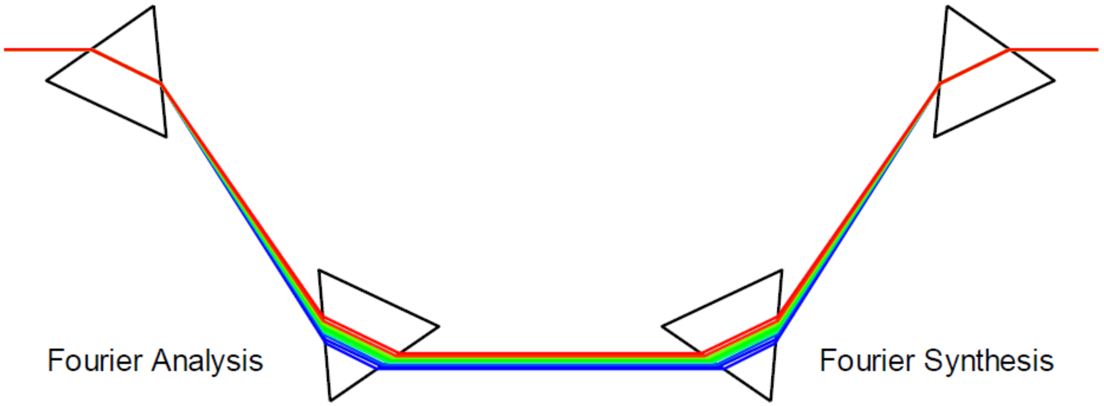
Week 2
Frequency Domain Representation of Signals

What is frequency?

How to characterize frequency?

Optical Fourier Transform

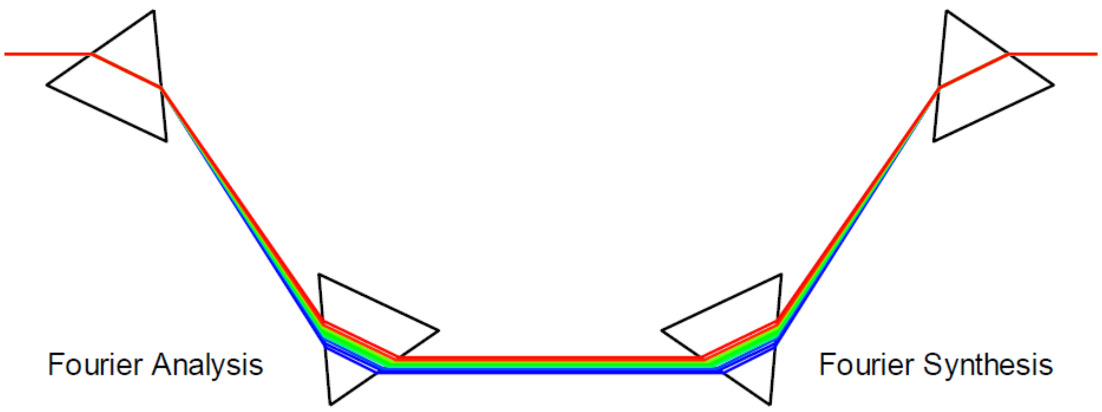
- A pair of prisms (棱镜) can split light up into its component frequencies (colors)
  - This is called Fourier Analysis
- A second pair can re-combine the frequencies.
  - This is called Fourier Synthesis

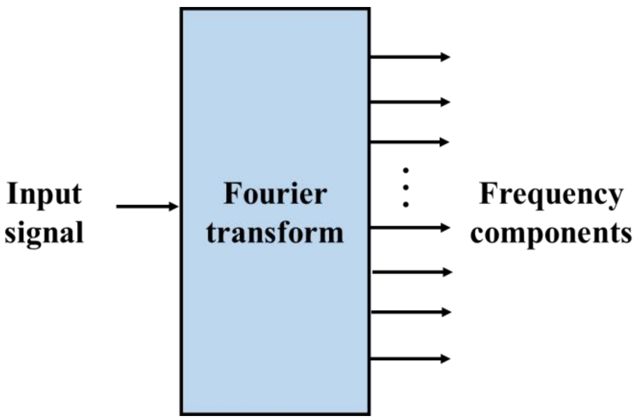


http://www.ee.ic.ac.uk/hp/staff/dmb/courses/E1Fourier/Handouts.pdf

Optical Fourier Transform

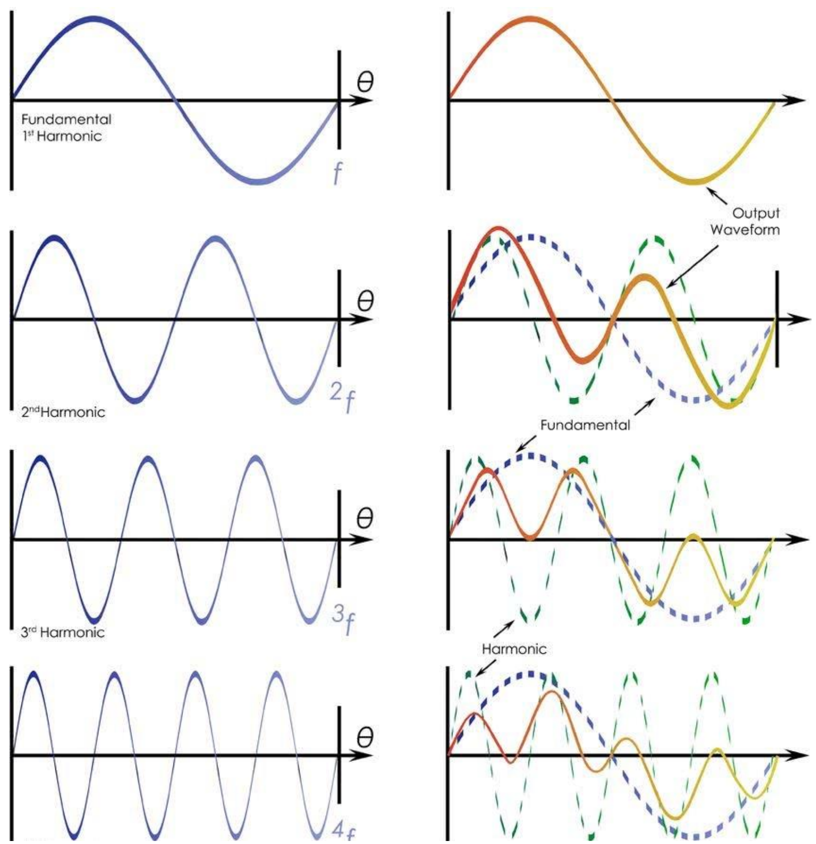
- We want to do the same thing with mathematical signals instead of light



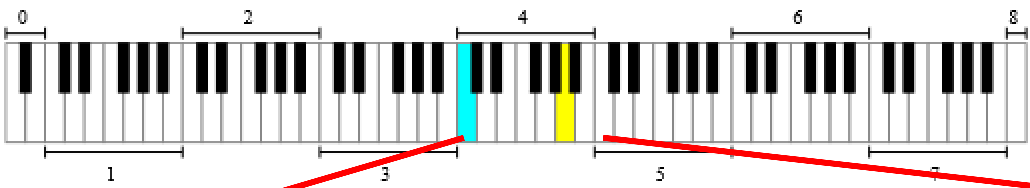


What is Music?

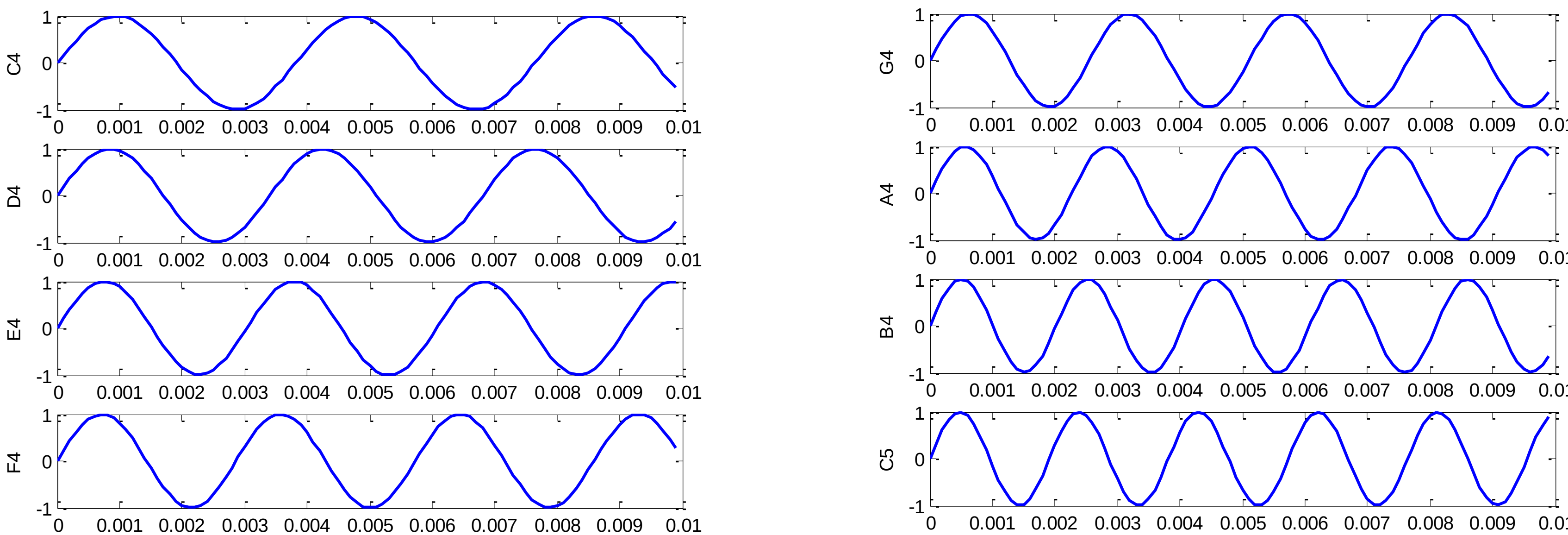
- For musician, music is
- For engineer, music is



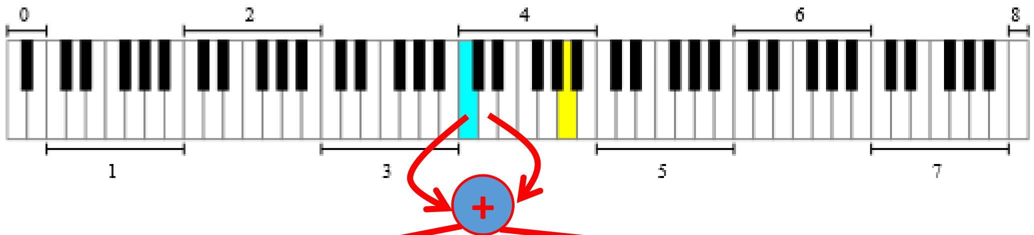
Signals in Time Domain



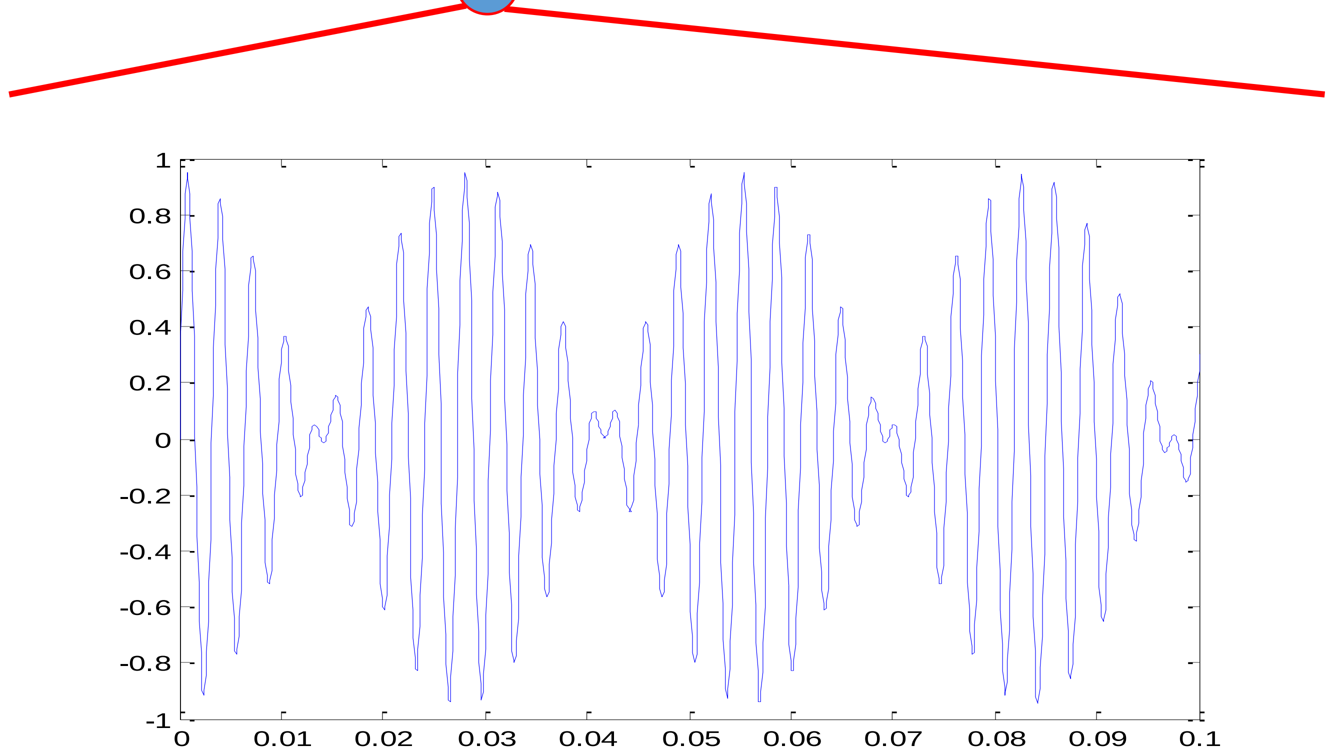
Sound of Musical Notes



Signals in Time Domain

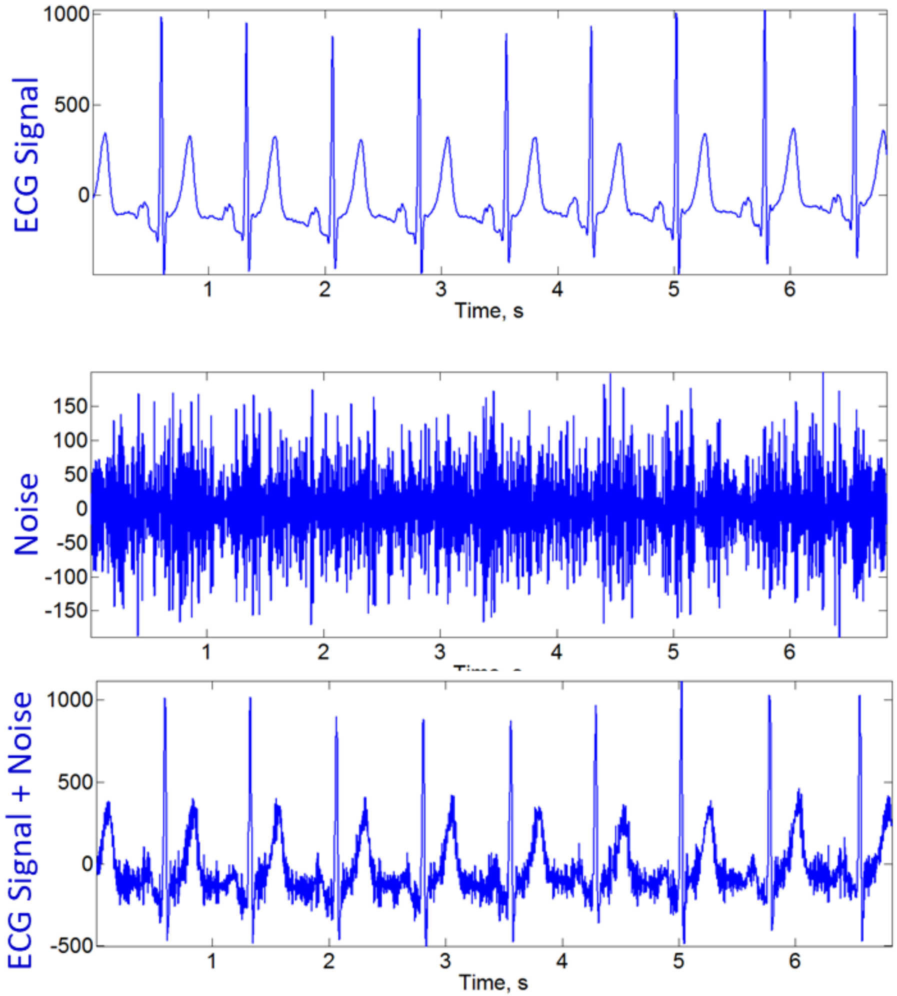


Sound of Beats

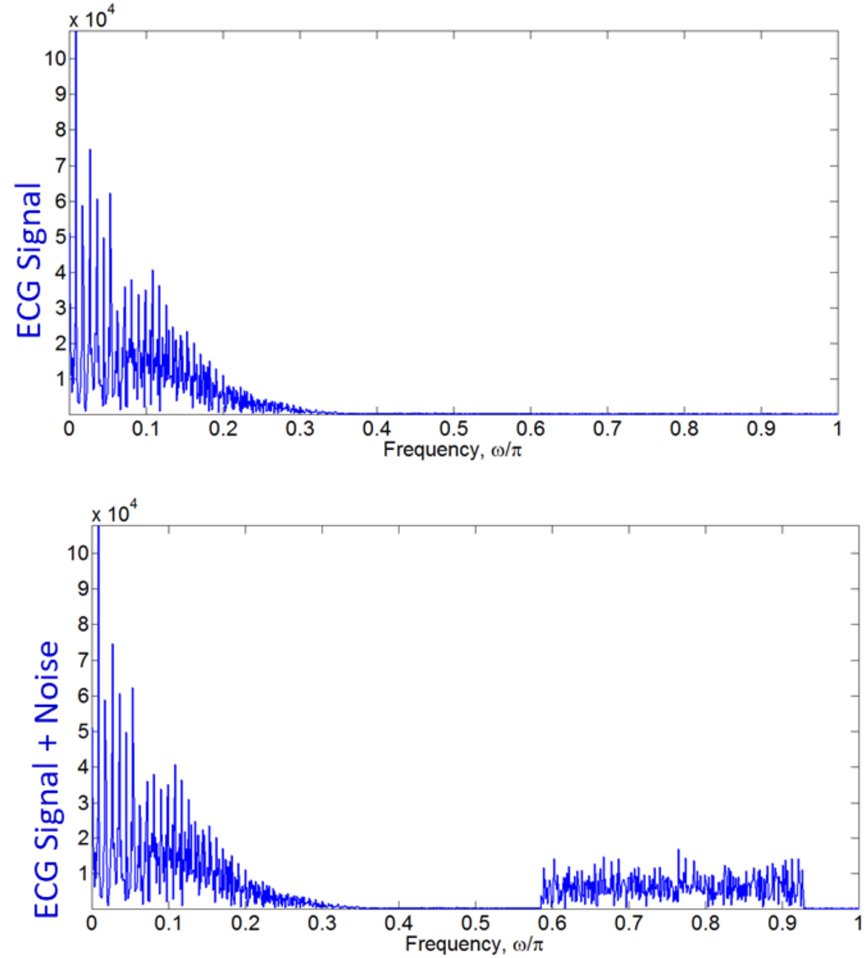


Signals in Frequency Domain

- Time domain



- Frequency domain



Question:

How to analyze frequency in a signal?

Answer:

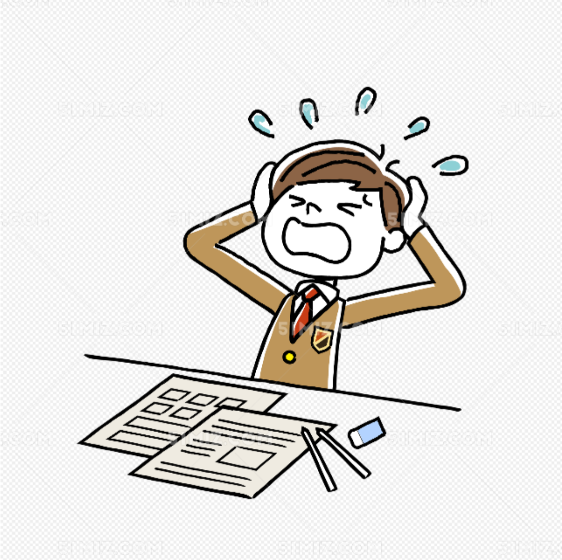
Fourier Transform

/ˈfʊəriˌeɪ, -iər/

1768-1830
French Mathematician, Physicist, Historian

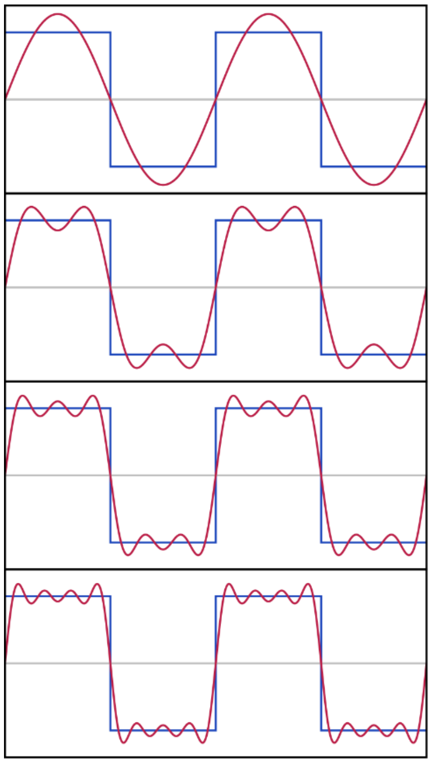
Types of *Fourier*

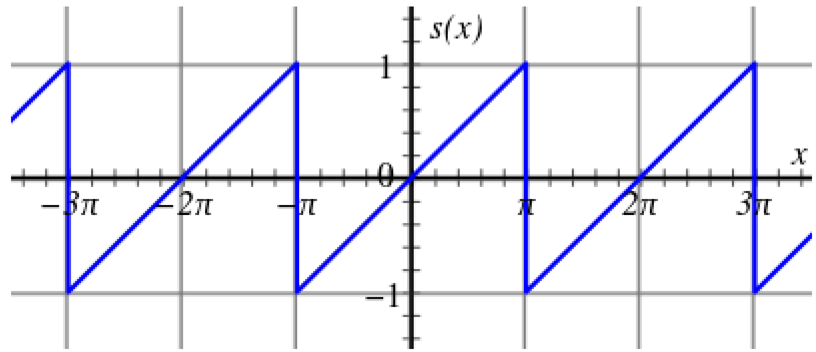
- Fourier series
- Fourier transform
  - Continuous Fourier transform
  - Discrete-time Fourier transform
  - Discrete Fourier transform
  - Fast Fourier transform

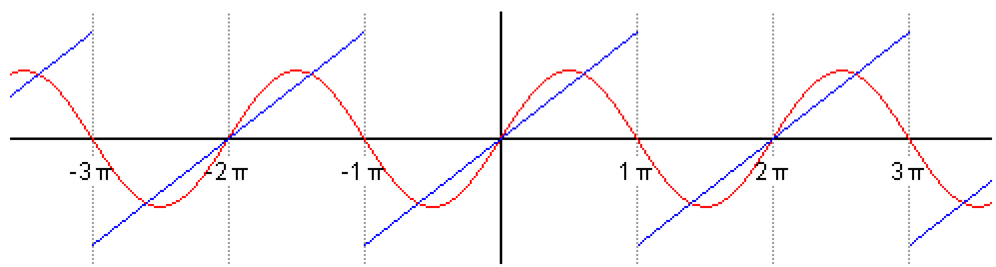


Fourier Series

- To represent a periodic signal as the (possibly infinite) sum of sine and cosine functions







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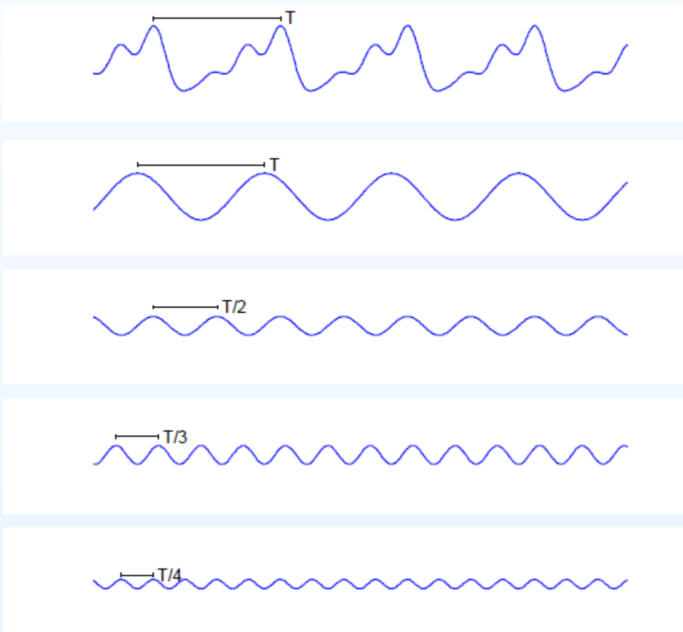


$$u(t) =$$
$$\sin 2\pi f t$$
$$-0.4 \sin 2\pi 2 f t$$
$$+0.4 \sin 2\pi 3 f t$$
$$-0.2 \cos 2\pi 4 f t$$

The Fourier series for $u(t)$ is

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi n f t + b_n \sin 2\pi n f t\right)$$

Fourier Series

- Another representation – continuous case

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{2\pi}{T})t}$$

Fourier Series

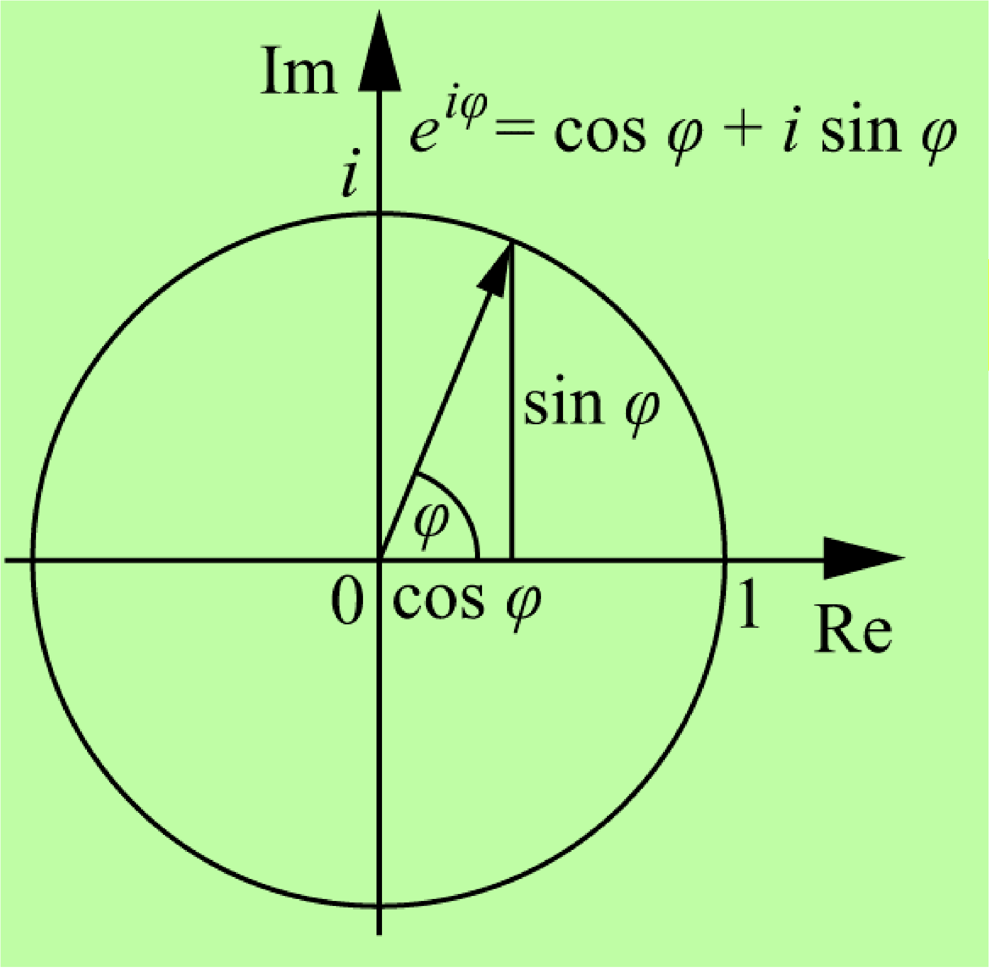
- Discrete case

$$x[n] = \sum_{k \in <N>} a_k e^{jk\omega_0 n} = \sum_{k \in <N>} a_k e^{jk(\frac{2\pi}{N})n}$$

$$a_k = \sum_{n \in <N>} x[n]\, e^{-jk\omega_0 n} = \sum_{k \in <N>} x[n] e^{-jk(\frac{2\pi}{N})n}$$

Where are the sign and cosine functions?

Euler's Formula



$$e^{j\varphi} = \cos\varphi + j\sin(\varphi)$$

$$\cos\varphi = \frac{1}{2}(e^{j\varphi} + e^{-j\varphi})$$

$$\sin\varphi = \frac{1}{2j}(e^{j\varphi} - e^{-j\varphi})$$

Tools to Play With

- A lot of on-line tools and resources
  - https://en.wikipedia.org/wiki/Fourier_series
  - https://bl.ocks.org/jinroh/7524988

- Use Matlab or other programming languages, e.g, Python
  - An example

How About Non-periodic signals?

- Non-periodic signals can be treated as a periodic signal with infinite period

Fourier series → Fourier transform

$$x[n]=\sum_{k\in<N>} a_k e^{jk\omega_0 n}=\sum_{k\in<N>} a_k e^{jk(\frac{2\pi}{N})n}$$

Fourier Transform
Continuous-time

- Fourier transform (continuous-time)

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \qquad \Omega = 2\pi f$$

Signal analysis: to analyze the frequency components

- Inverse Fourier transform (continuous-time)

$$x(t) = \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \qquad \Omega = 2\pi f$$

Signal synthesis: to recover the time-domain signal

Fourier Transform
Discrete-time

- Fourier Transform (discrete-time)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \omega = 2\pi f/f_s$$

ω is a continuous variable in the range of $-\infty < \omega < \infty$

- Inverse Fourier transform (discrete-time)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \omega = 2\pi f/f_s$$

Why one is sum and the other integral?

Why Fourier Works?

- Fourier series as an example

$$x[n] = \sum_{k\in<N>} a_k e^{jk\omega_0 n} = \sum_{k\in<N>} a_k e^{jk(\frac{2\pi}{N})n} \qquad a_k = \sum_{n\in<N>} x[n]\, e^{-jk\omega_0 t} = \sum_{k\in<N>} x[n] e^{-jk(\frac{2\pi}{N})n}$$

$$\sum_{n\in<N>} e^{jk_1\omega_0 n}\ e^{-jk_2\omega_0 n} = \sum_{k\in<N>} e^{j(k_1-k_2)\omega_0 n}$$

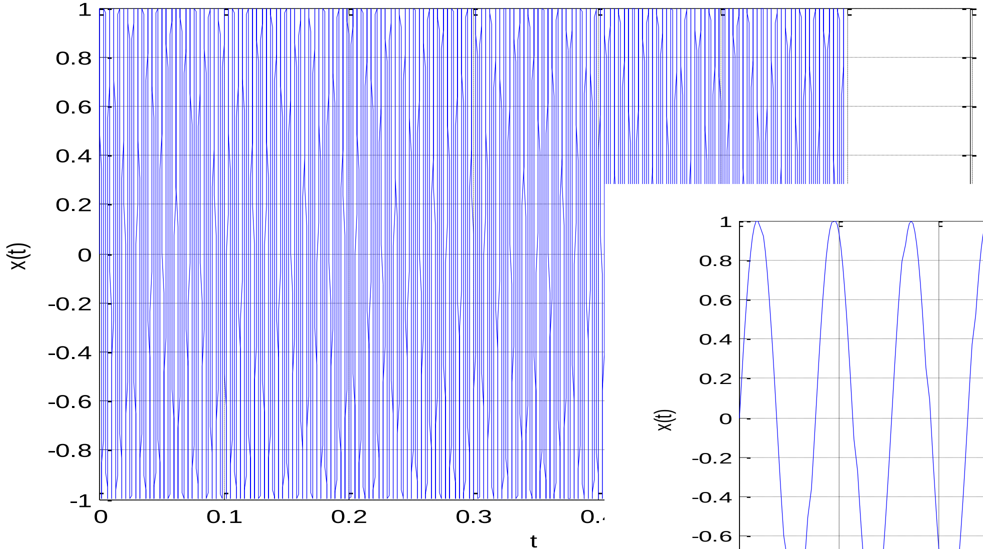
$$= \begin{matrix} N \ \text{for}\ k_1 = k_2 \\ 0 \ \text{for}\ k_1 \neq k_2 \end{matrix}$$

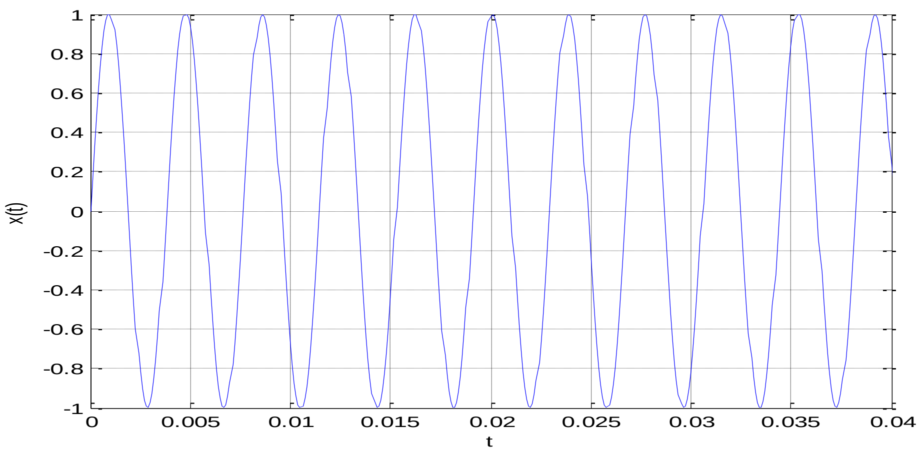
Orthogonality of complex exponentials

Frequency Domain

- Example 1: Fourier Transform of the C4 tone

$$x(t) = \begin{cases} \sin(2\pi \cdot f_0\, t), & t \in [0,1] \\ 0, & o.w. \end{cases} \qquad f_0 = 261.626\text{Hz}$$





Frequency Domain

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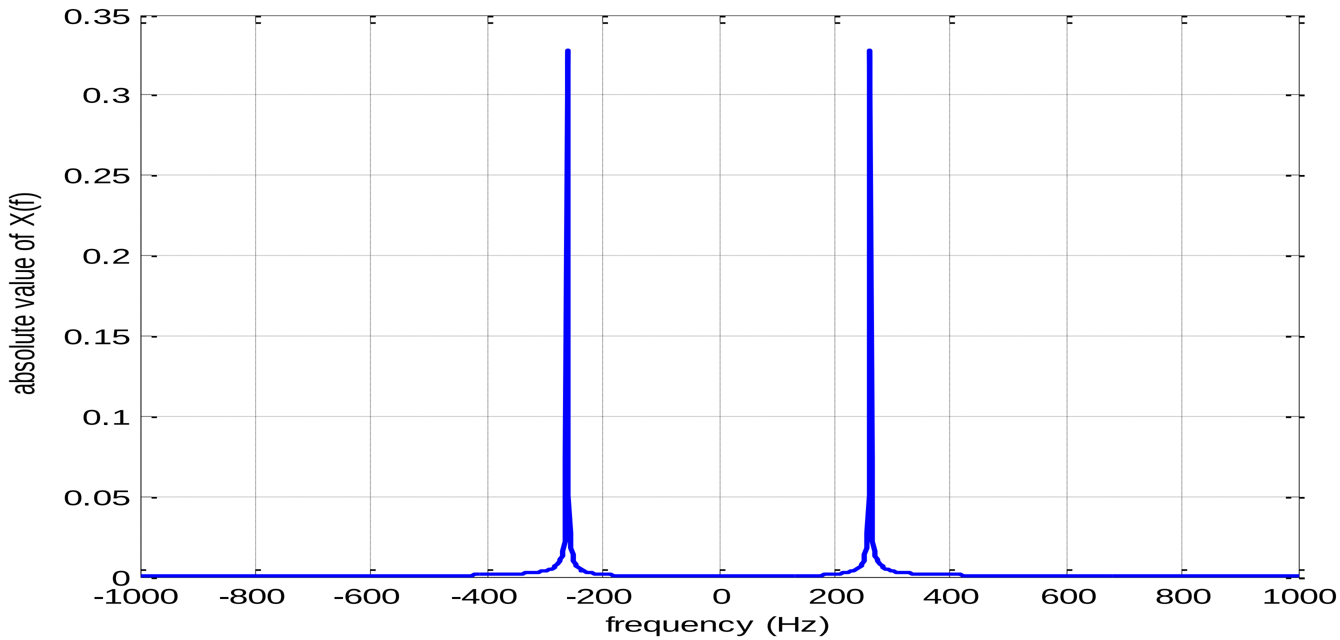
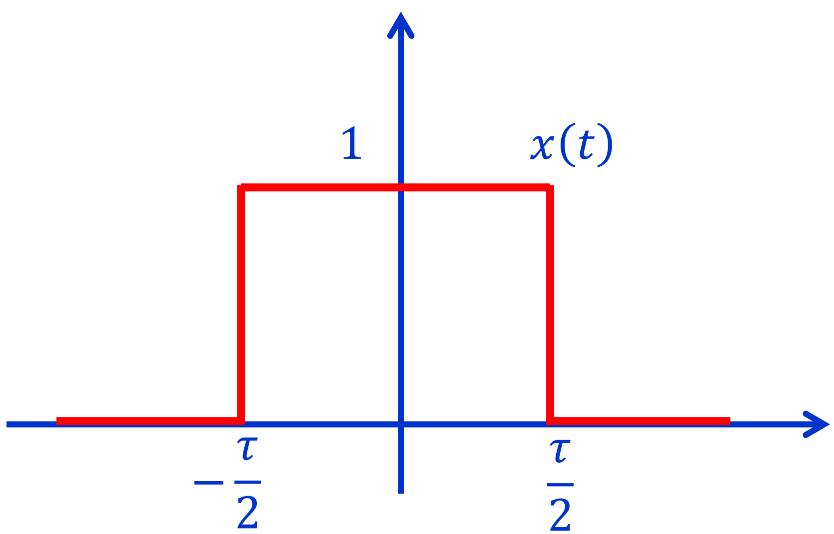


Frequency Domain

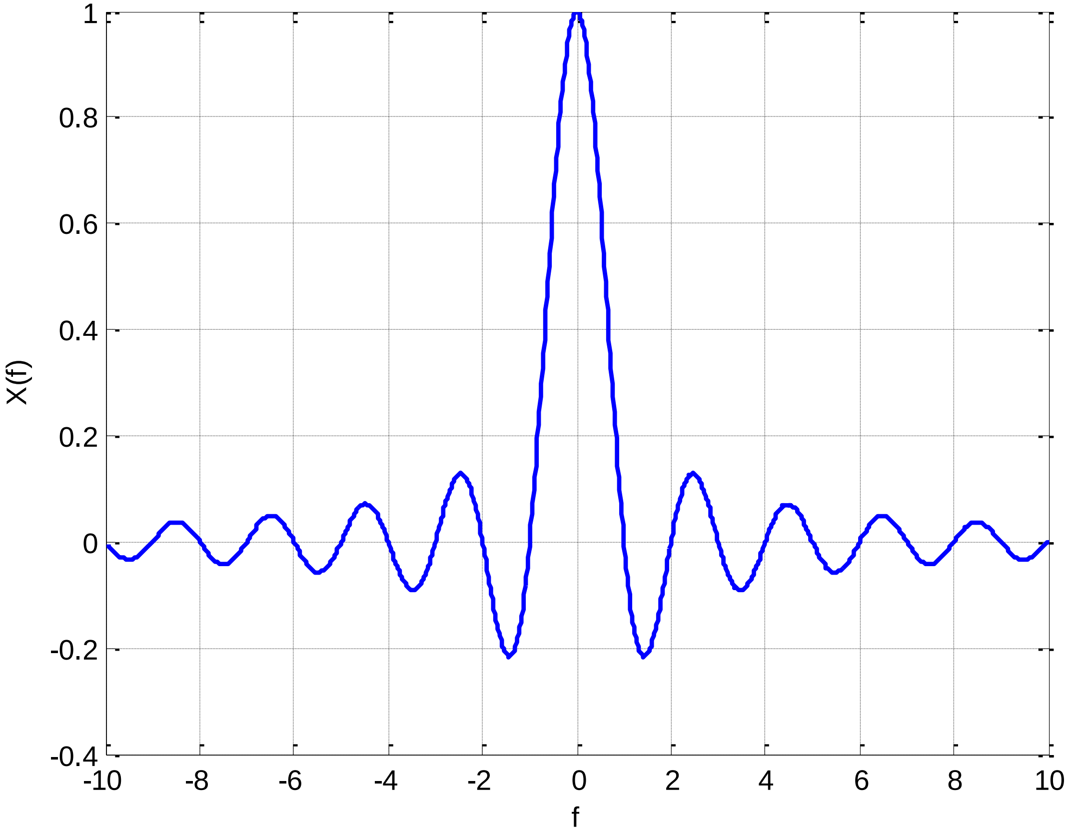
- Example 2: Fourier Transform of the rectangular pulse

$$x(t) = \begin{cases} 1, & t \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right] \\ 0, & o.w. \end{cases}$$



Frequency Domain

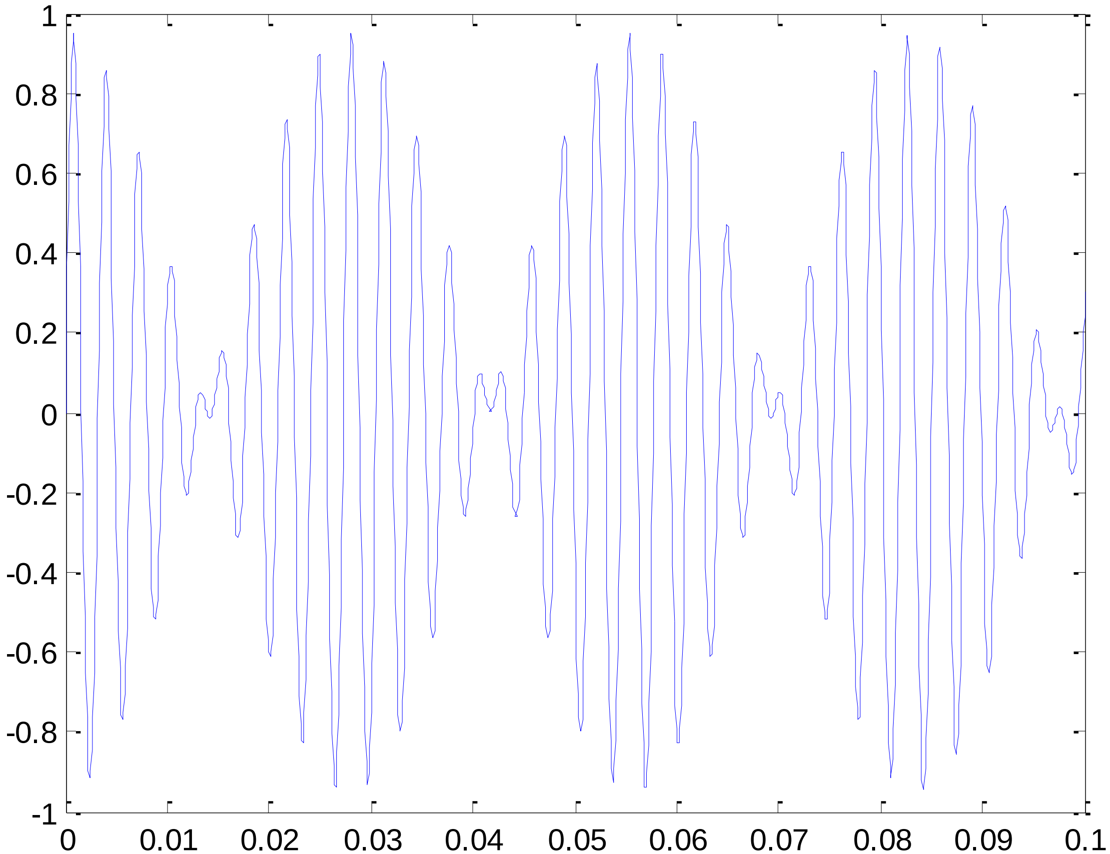
- Example 2: Fourier Transform of the rectangular pulse



$\tau = 1$

Back to Where We Begin

- Time domain



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Back to Where We Begin

- Frequency domain