

# **Week 2**

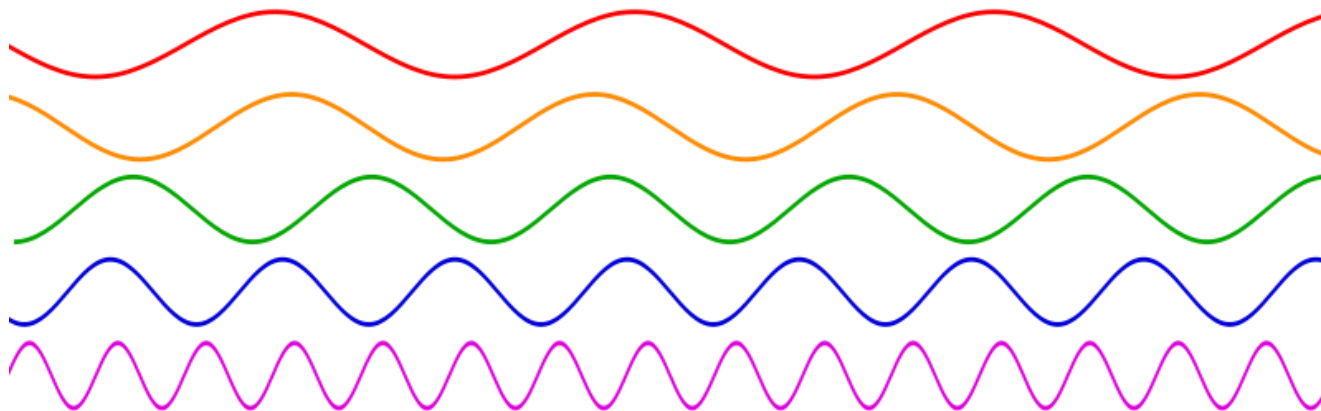
## **Frequency Domain**

### **Representation of Signals**

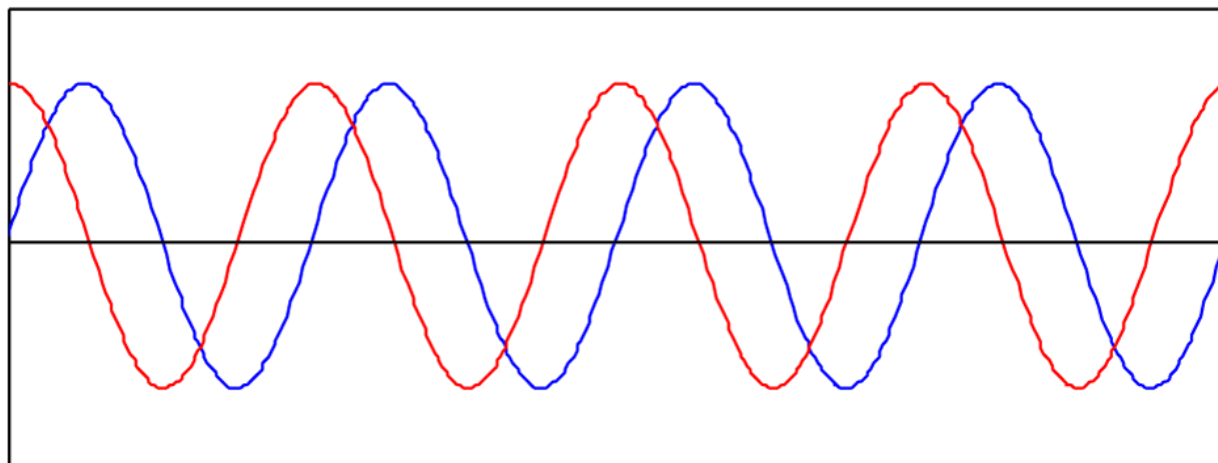
What is frequency?

How to characterize frequency?

Frequency

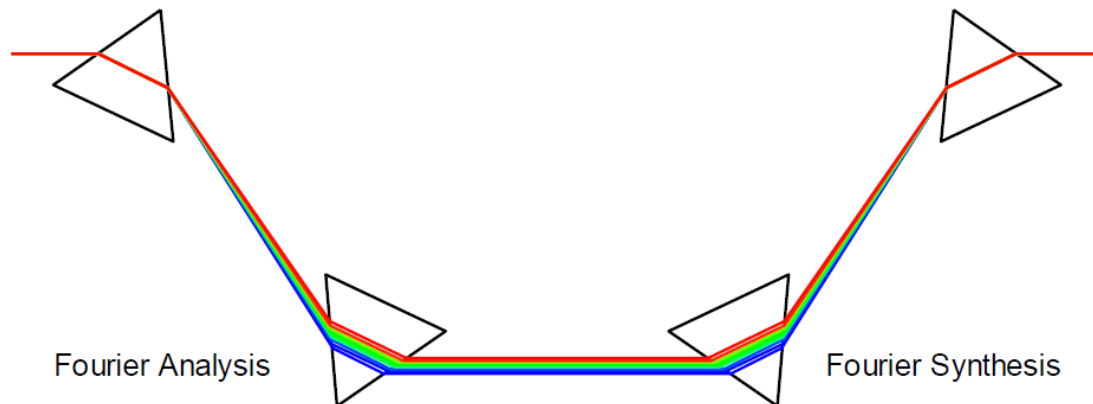


Phase



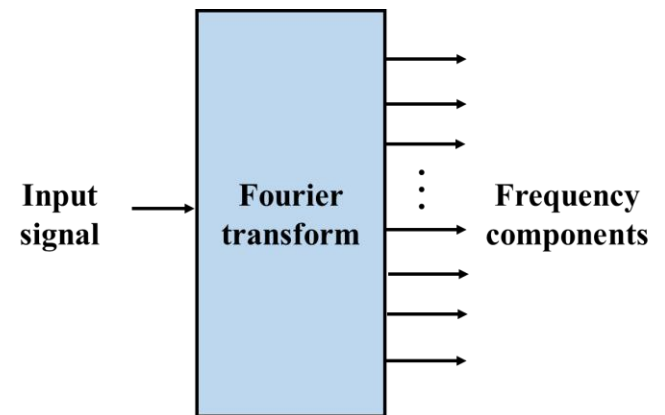
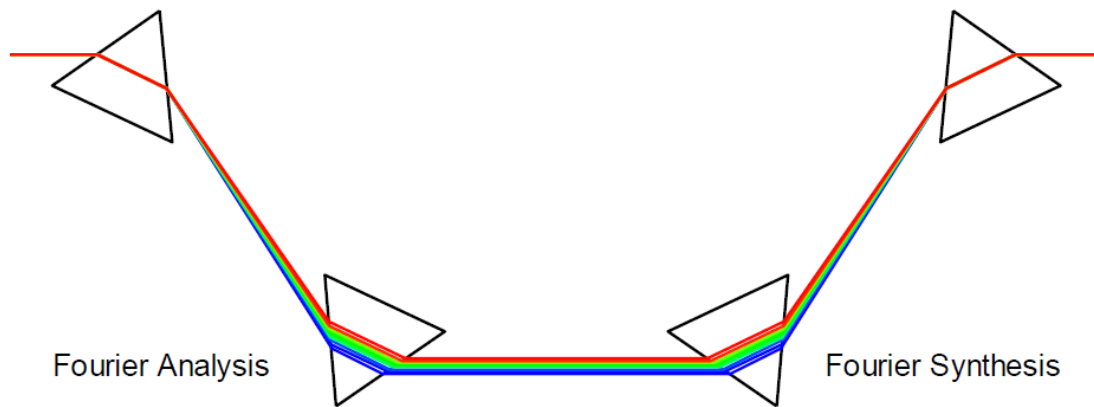
# Optical Fourier Transform

- ❑ A pair of prisms (棱镜) can split light up into its component frequencies (colors)
  - This is called Fourier Analysis
- ❑ A second pair can re-combine the frequencies.
  - This is called Fourier Synthesis



# Optical Fourier Transform

- We want to do the same thing with mathematical signals instead of light

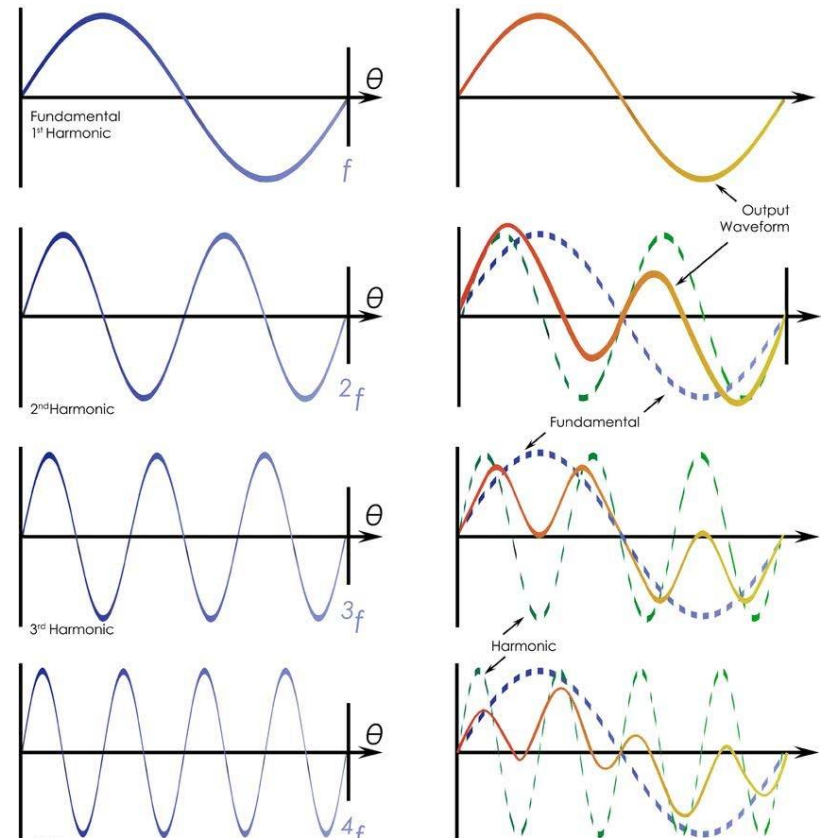


# What is Music?

❑ For musician, music is

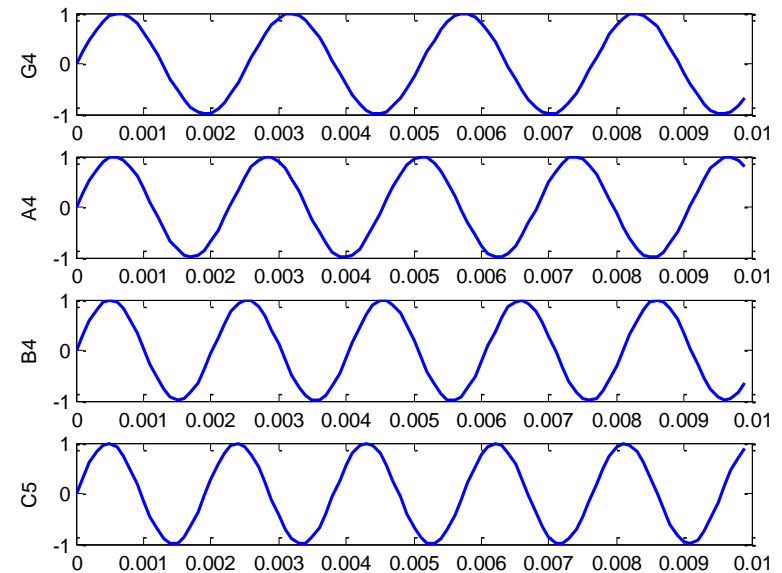
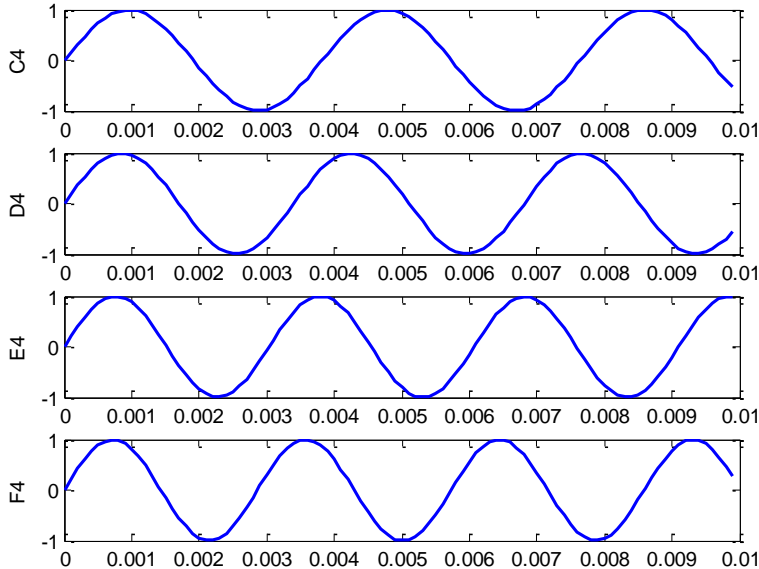


❑ For engineer, music is

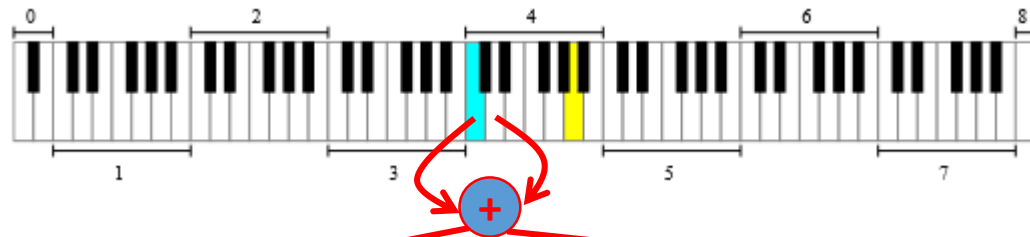


# Signals in Time Domain

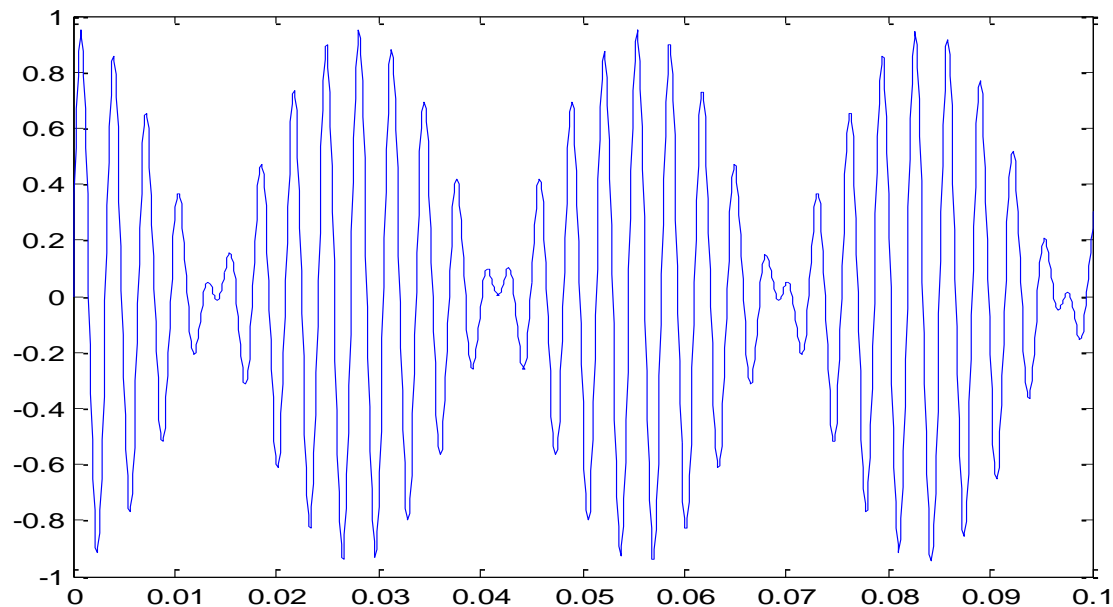
## Sound of Musical Notes



# Signals in Time Domain



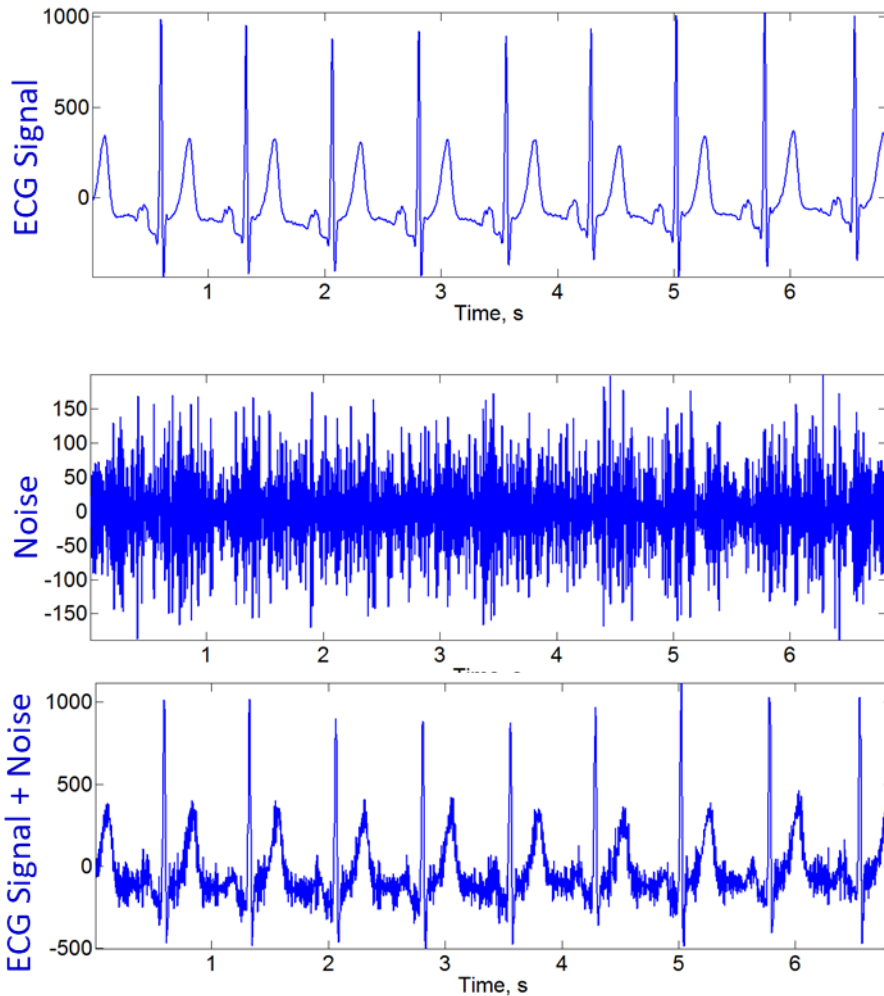
Sound of Beats



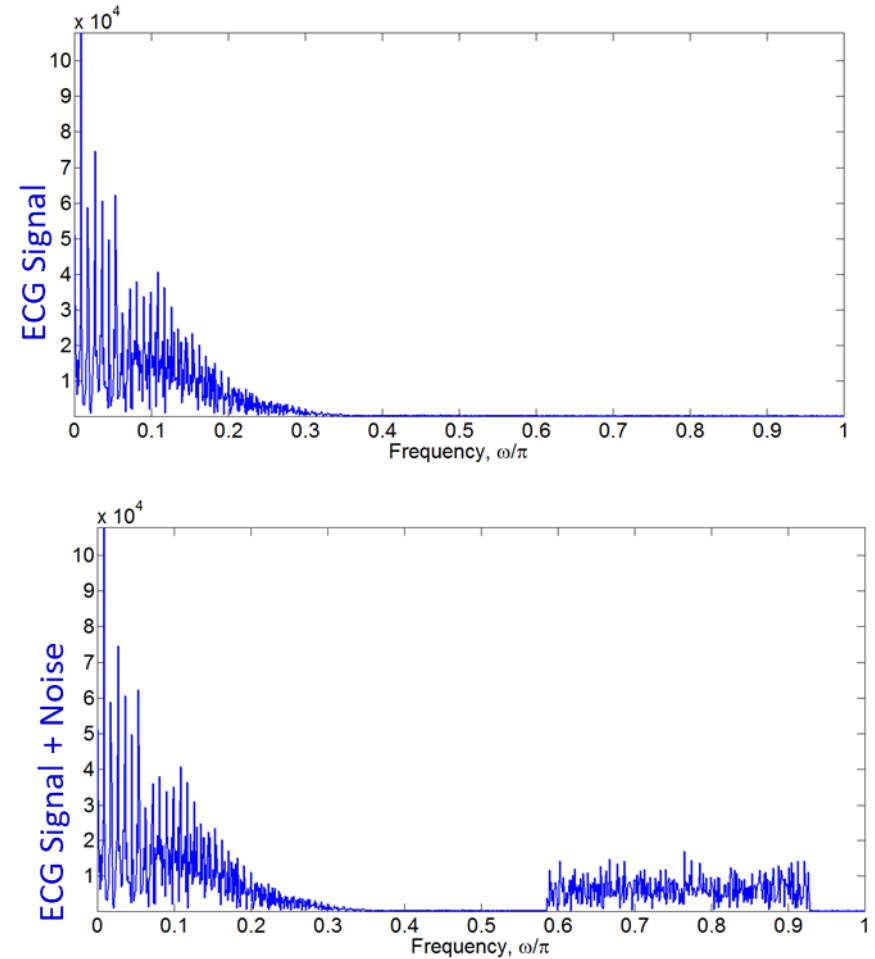


# Signals in Frequency Domain

## Time domain



## Frequency domain



# Question:

How to analyze **frequency** in a signal?

# Answer:

## Fourier Transform



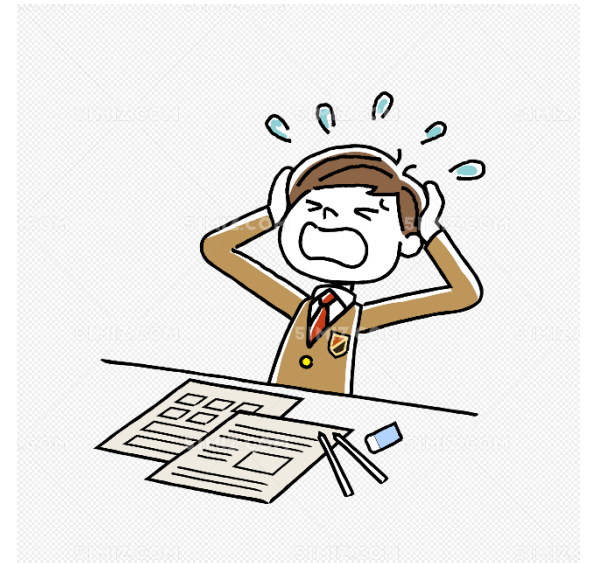
**$/\text{'f}\text{u}\text{ər}\text{i}, \text{e}\text{ɪ}, -\text{i}\text{ər}/$**

**1768-1830**

**French Mathematician,  
Physicist, Historian**

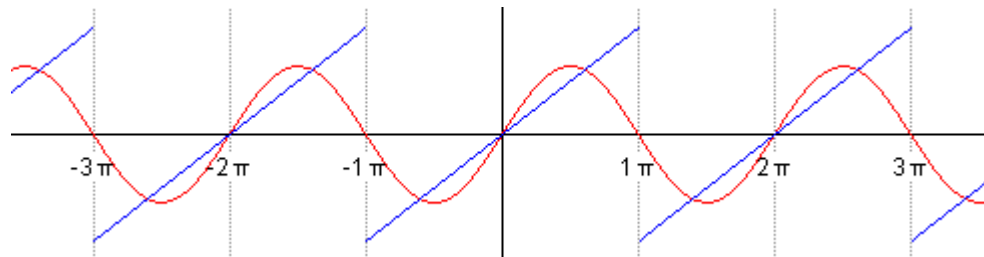
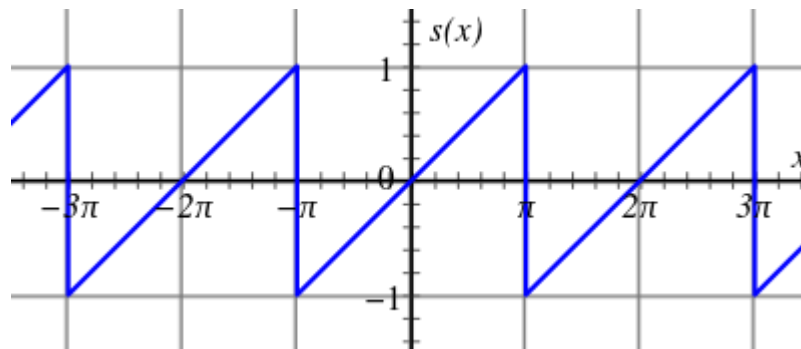
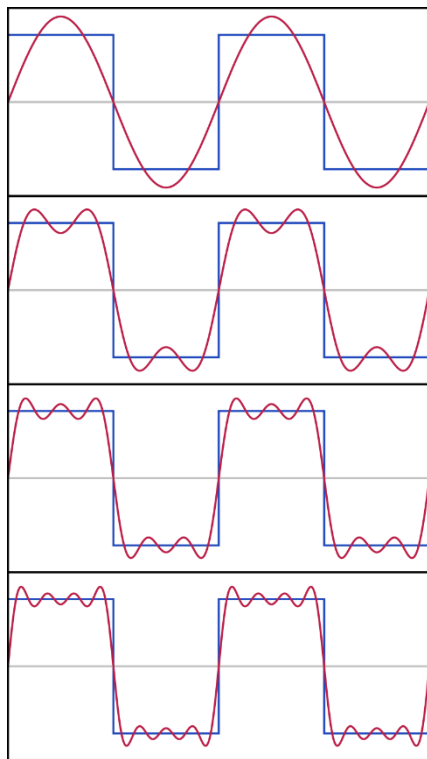
# Types of *Fourier*

- ❑ Fourier series
- ❑ Fourier transform
  - Continuous Fourier transform
  - Discrete-time Fourier transform
  - Discrete Fourier transform
  - Fast Fourier transform



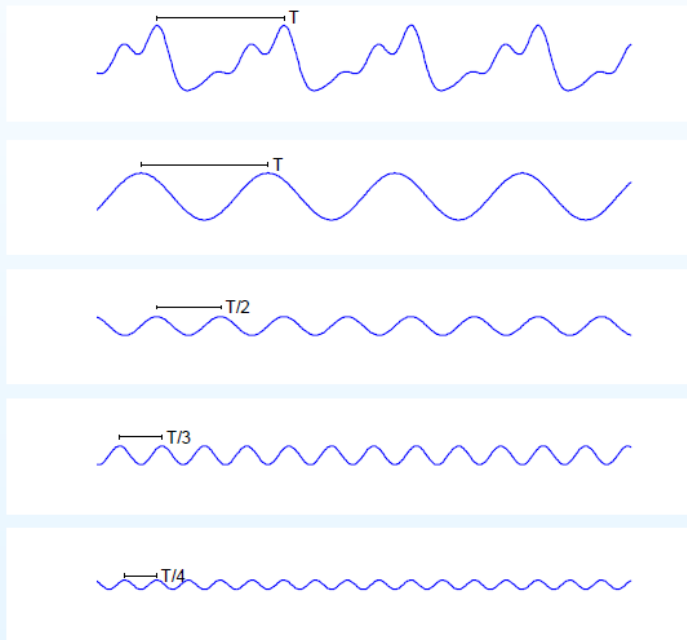
# Fourier Series

- To represent a **periodic signal** as the (possibly infinite) sum of sine and cosine functions



# Fourier Series

- To represent a **periodic signal** as the (possibly infinite) sum of sine and cosine functions



$$u(t) =$$

$$\sin 2\pi f t$$

$$-0.4 \sin 2\pi 2 f t$$

$$+0.4 \sin 2\pi 3 f t$$

$$-0.2 \cos 2\pi 4 f t$$

The Fourier series for  $u(t)$  is

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n f t + b_n \sin 2\pi n f t)$$

# Why Sine and Cosine Waves?

- ❑ A sine wave remains a sine wave of the same frequency when you
  - Multiply by a constant
  - Add onto to another sine wave of the same frequency
  - Differentiate or integrate or shift in time
- ❑ Almost any function can be expressed as a sum of sine waves
  - Periodic→Fourier Series
  - Aperiodic→Fourier Transform
- ❑ Many physical and electronic systems are
  - Composed entirely of constant-multiply/add/differentiate
  - Linear:  $u(t) \rightarrow x(t)$  and  $v(t) \rightarrow y(t)$  means  $u(t)+v(t) \rightarrow x(t)+y(t)$

# Fourier Series

□ Another representation – continuous case

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$



# Fourier Series

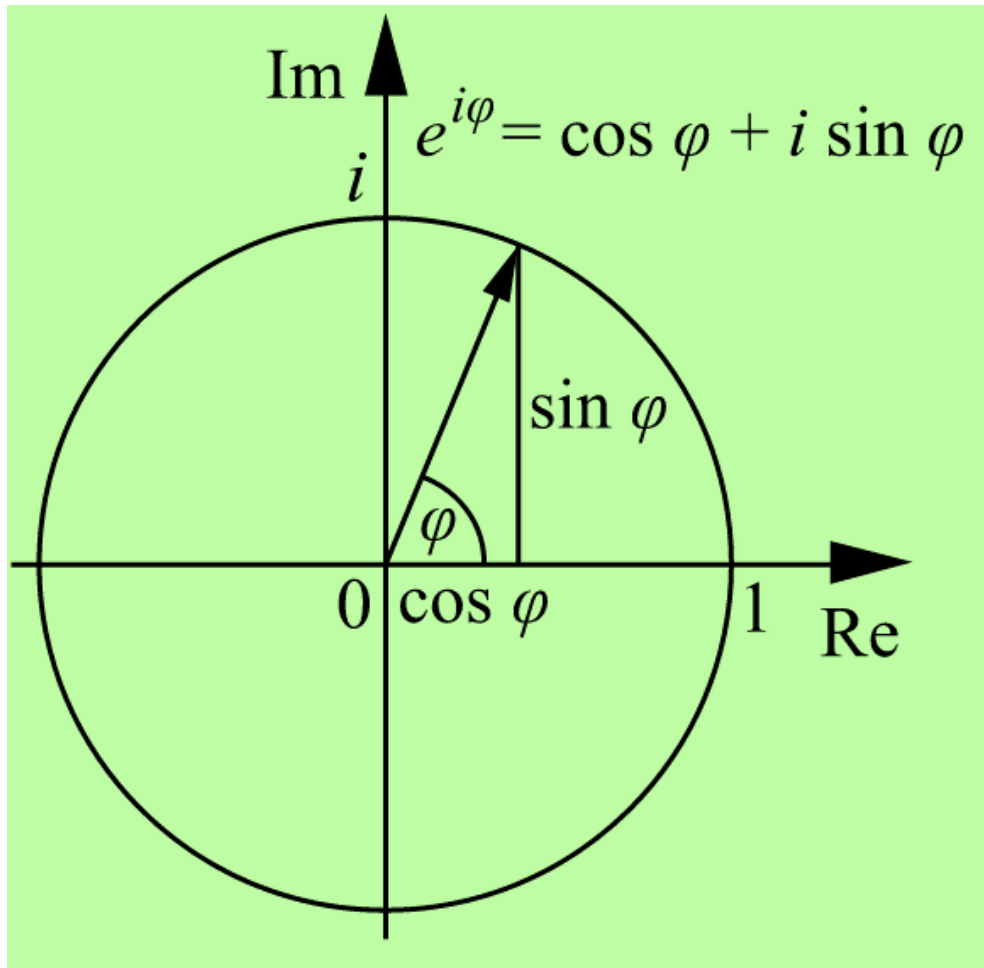
## □ Discrete case

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k \in \langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$$

$$a_k = \sum_{n \in \langle N \rangle} x[n] e^{-jk\omega_0 n} = \sum_{n \in \langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$$

Where are the sign and cosine functions?

# Euler's Formula



$$e^{j\varphi} = \cos \varphi + j \sin(\varphi)$$

$$\cos \varphi = \frac{1}{2} (e^{j\varphi} + e^{-j\varphi})$$

$$\sin \varphi = \frac{1}{2j} (e^{j\varphi} - e^{-j\varphi})$$

# Tools to Play With

- ❑ A lot of on-line tools and resources
  - [https://en.wikipedia.org/wiki/Fourier\\_series](https://en.wikipedia.org/wiki/Fourier_series)
  - <https://bl.ocks.org/jinroh/7524988>
  
- ❑ Use Matlab or other programming languages, e.g, Python
  - [An example](#)

# How About Non-periodic signals?

- Non-periodic signals can be treated as a periodic signal with **infinite period**

Fourier series



Fourier transform

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k \in \langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

# Fourier Transform

## Continuous-time

### □ Fourier transform (continuous-time)

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt \quad \Omega = 2\pi f$$

Signal analysis: to analyze the frequency components

### □ Inverse Fourier transform (continuous-time)

$$x(t) = \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t} d\Omega \quad \Omega = 2\pi f$$

Signal synthesis: to recover the time-domain signal

# Fourier Transform Discrete-time

## □ Fourier Transform (discrete-time)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \omega = 2\pi f/f_s$$

$\omega$  is a continuous variable in the range of  $-\infty < \omega < \infty$

## □ Inverse Fourier transform (discrete-time)

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega \quad \omega = 2\pi f/f_s$$

Why one is sum and the other integral?

# Why Fourier Works?

## □ Fourier series as an example

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k \in \langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n} \quad a_k = \sum_{n \in \langle N \rangle} x[n] e^{-jk\omega_0 n} = \sum_{k \in \langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$$

$$\sum_{n \in \langle N \rangle} e^{jk_1\omega_0 n} e^{-jk_2\omega_0 n} = \sum_{k \in \langle N \rangle} e^{j(k_1 - k_2)\omega_0 n}$$

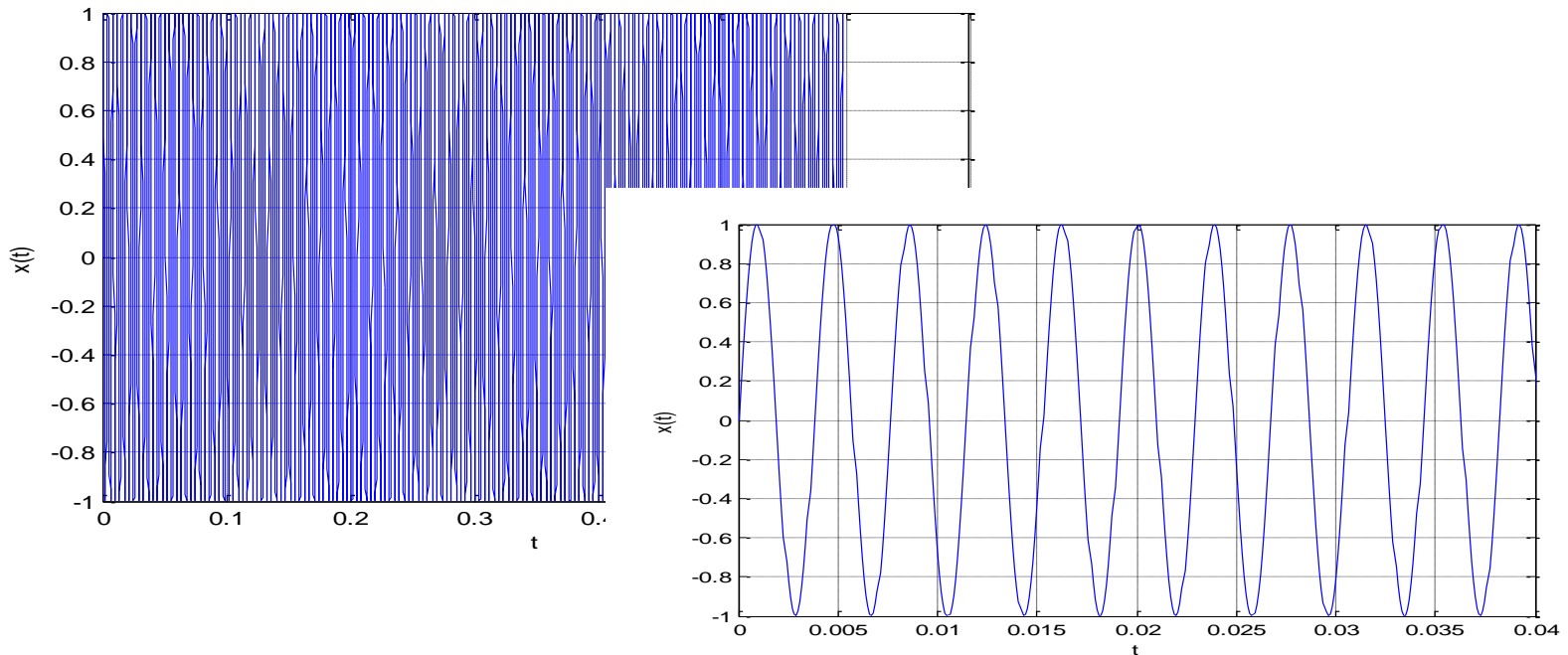
$$= \begin{aligned} & N \text{ for } k_1 = k_2 \\ & 0 \text{ for } k_1 \neq k_2 \end{aligned}$$

Orthogonality of complex exponentials

# Frequency Domain

## □ Example 1: Fourier Transform of the C4 tone

$$x(t) = \begin{cases} \sin(2\pi \cdot f_0 t), & t \in [0,1] \\ 0, & \text{o.w.} \end{cases} \quad f_0 = 261.626\text{Hz}$$

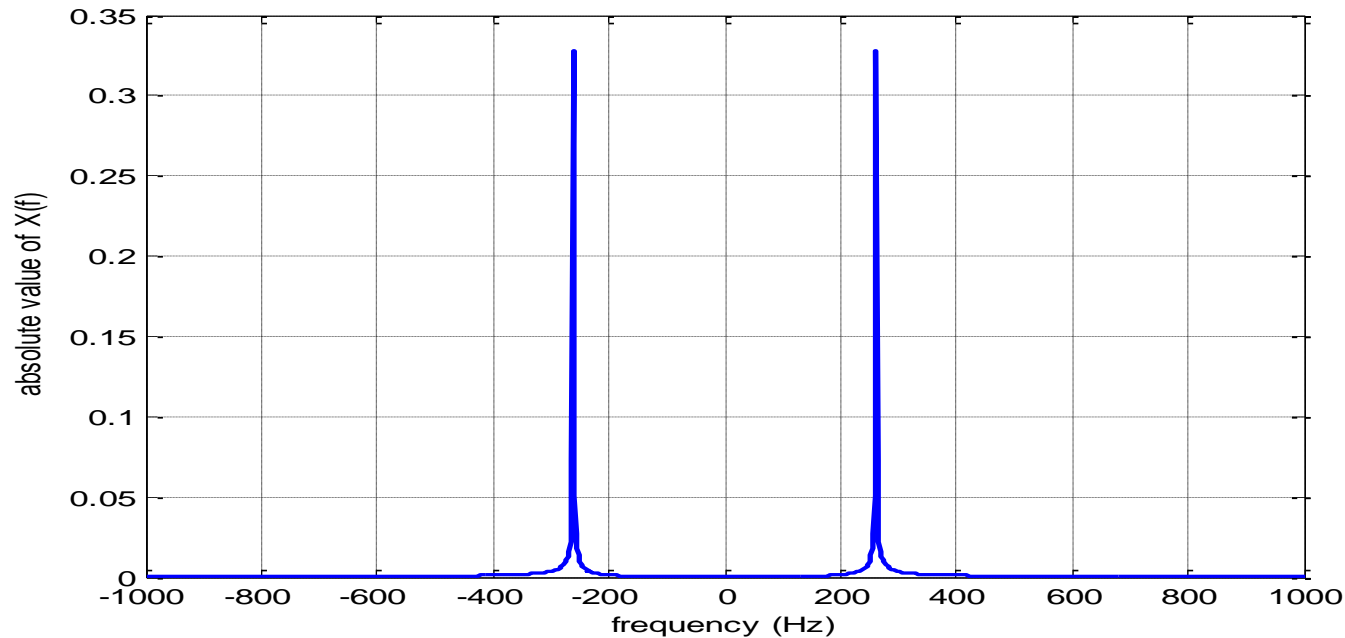




# Frequency Domain

## □ Example 1: Fourier Transform of the C4 tone

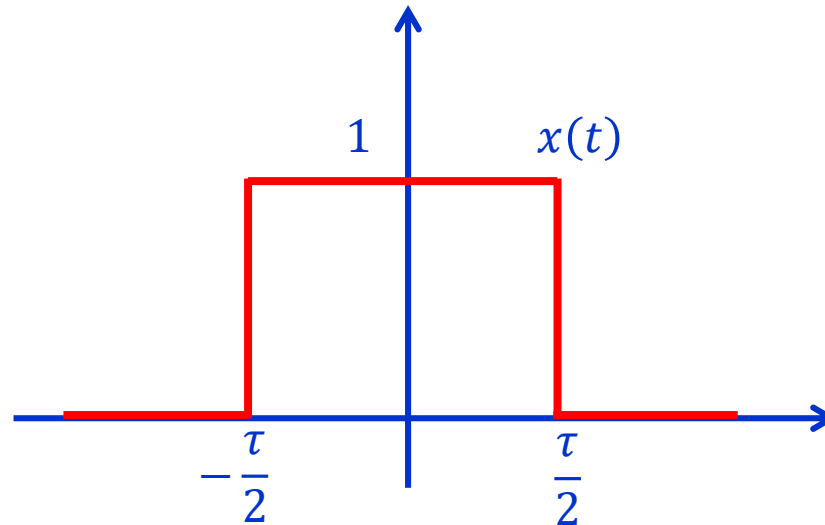
$$x(t) = \begin{cases} \sin(2\pi \cdot f_0 t), & t \in [0,1] \\ 0, & o.w. \end{cases} \quad f_0 = 261.626\text{Hz}$$



# Frequency Domain

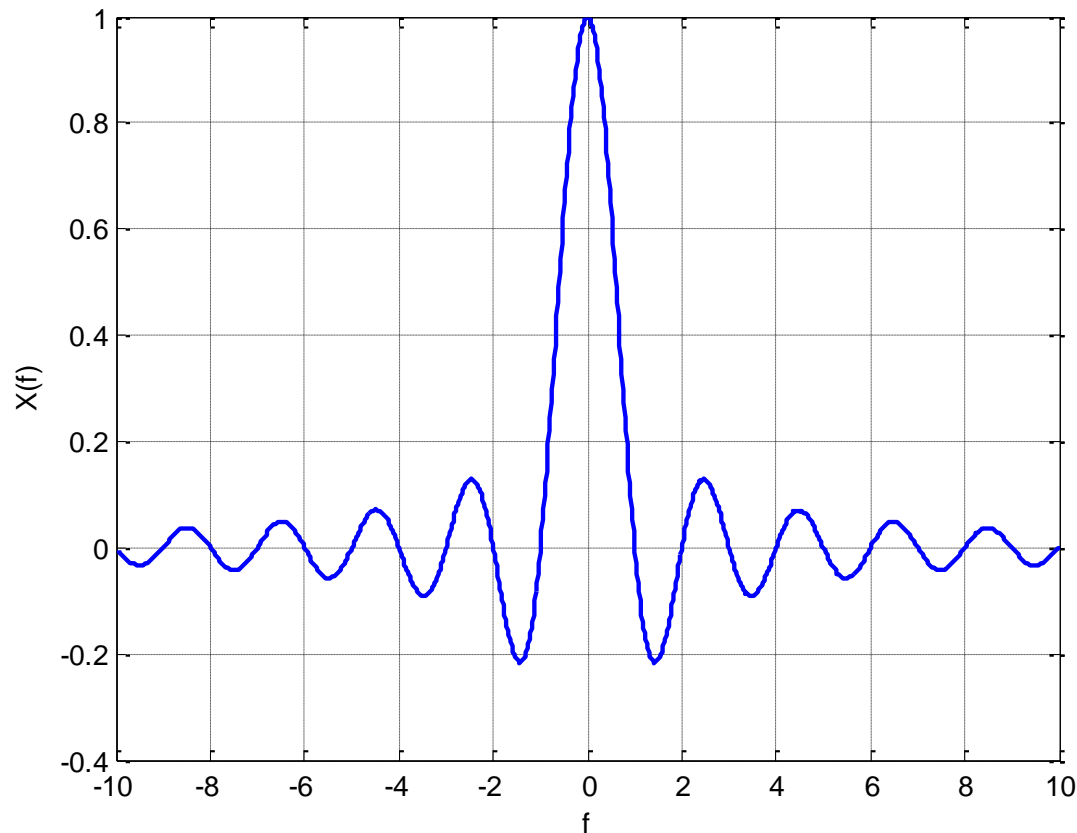
## □ Example 2: Fourier Transform of the rectangular pulse

$$x(t) = \begin{cases} 1, & t \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right] \\ 0, & o.w. \end{cases}$$



# Frequency Domain

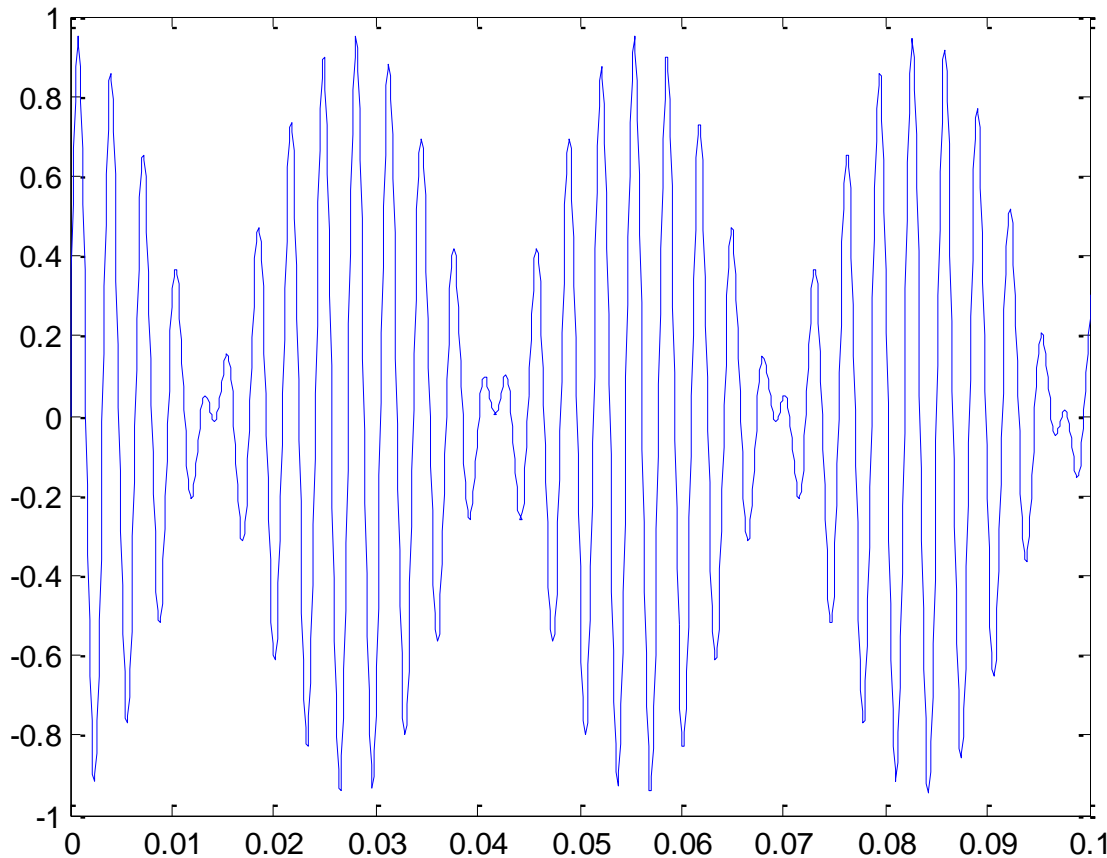
## □ Example 2: Fourier Transform of the rectangular pulse



$$\tau = 1$$

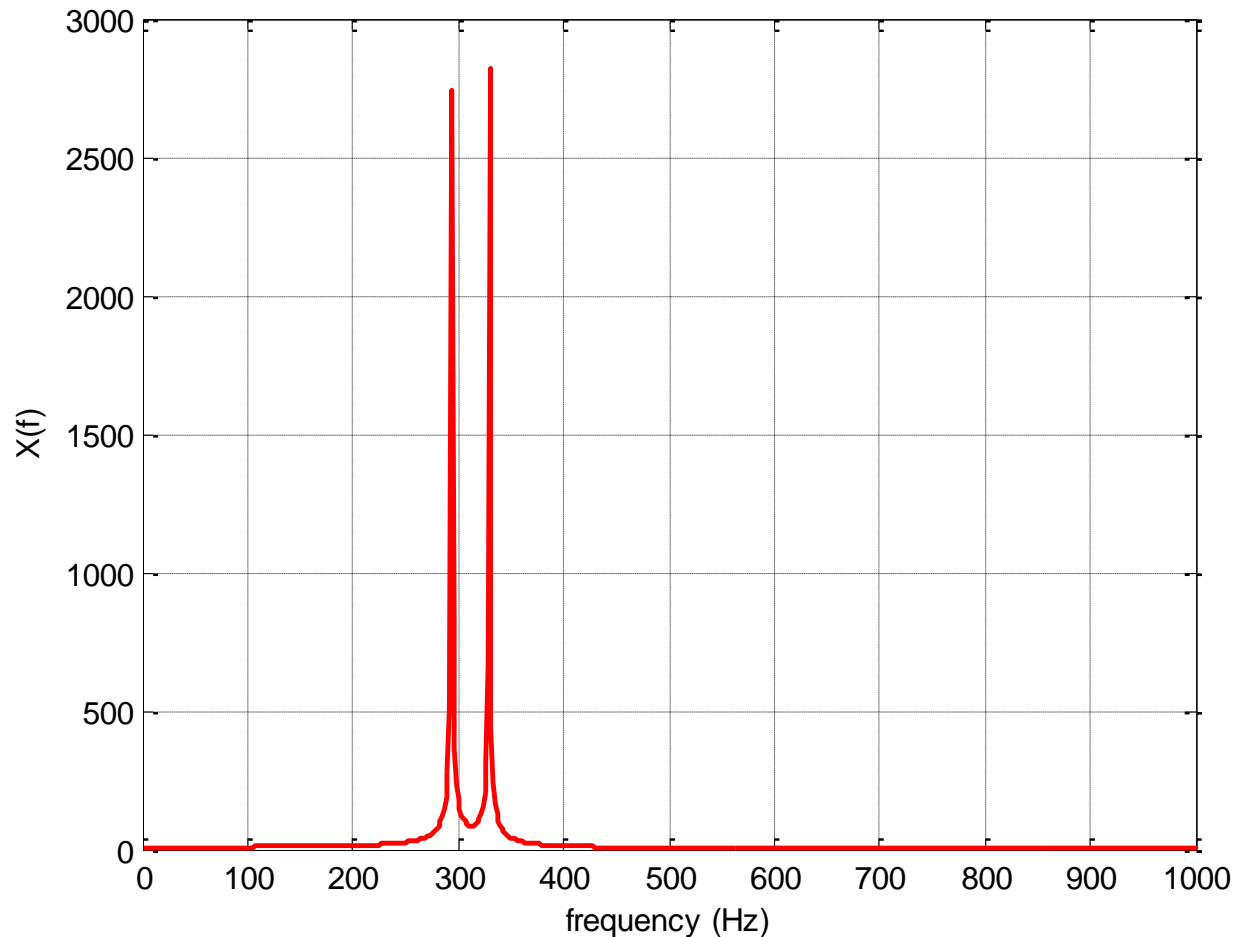
# Back to Where We Begin

## □ Time domain



# Back to Where We Begin

## □ Frequency domain



# Applications of Fourier Transform: An Example

## 大学生据拨号音破译周鸿祎手机号 李开复求贤

2012年09月01日 06:31 南方都市报[微博] 刘靖康 我要评论(0)

字号： T | T

[导读]听手机拨号音能破解电话号码？这个在许多影视剧和动漫中出现的“传说”被一位南京大学学生证实，他竟从采访视频中破解出奇虎360董事长周鸿祎的手机号。有网友为此感慨“技术宅要逆天了！”



神不神奇？

# Applications of Fourier Transform: An Example

## □ Dual-tone multi-frequency (DTMF) signaling

Frequency (Hz)	1209	1336	1477
697	1	2	3
770	2	5	6
852	7	8	9
941	*	0	#

一点都不神奇?

# Applications of Fourier Transform

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- ❑ Methods based on Fourier are used in almost **all areas of engineering and science**
  - Electrical and electronic engineering
  - Computer science
  - Communication engineering

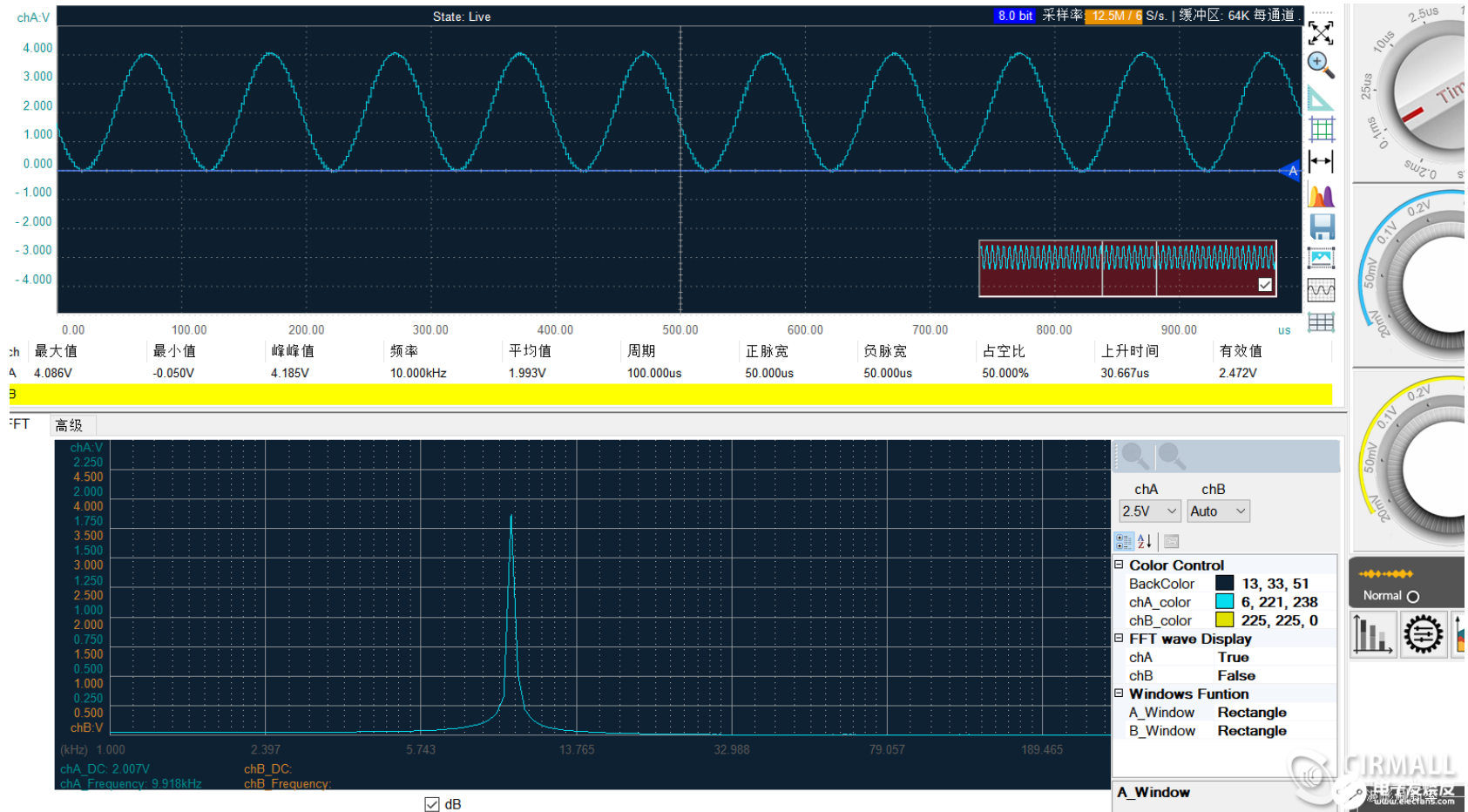


# Oscilloscope



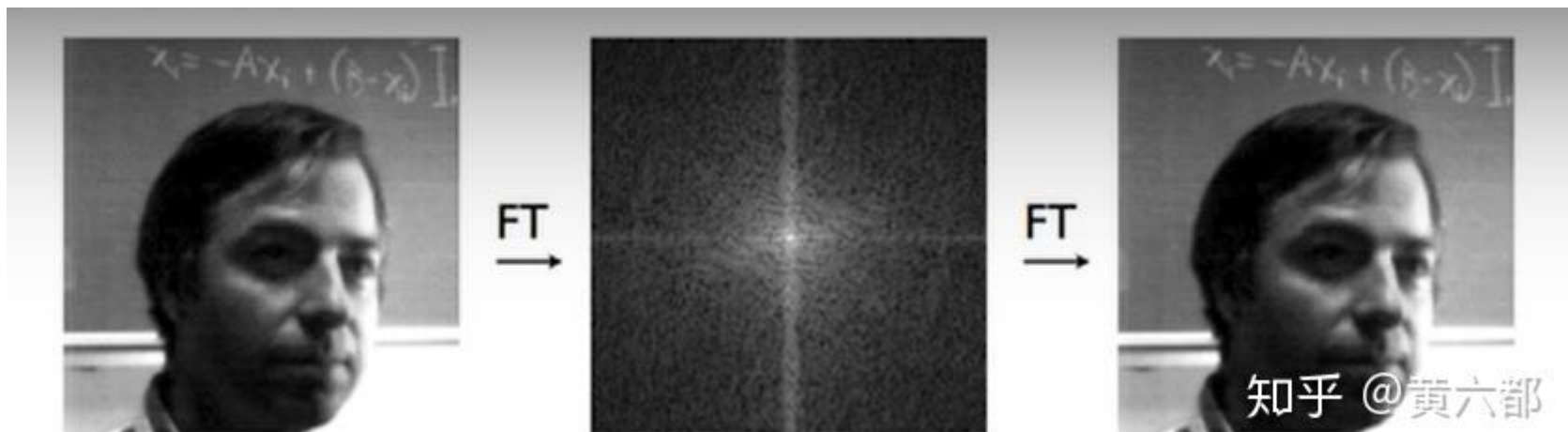
# Oscilloscope

## □ Frequency analysis

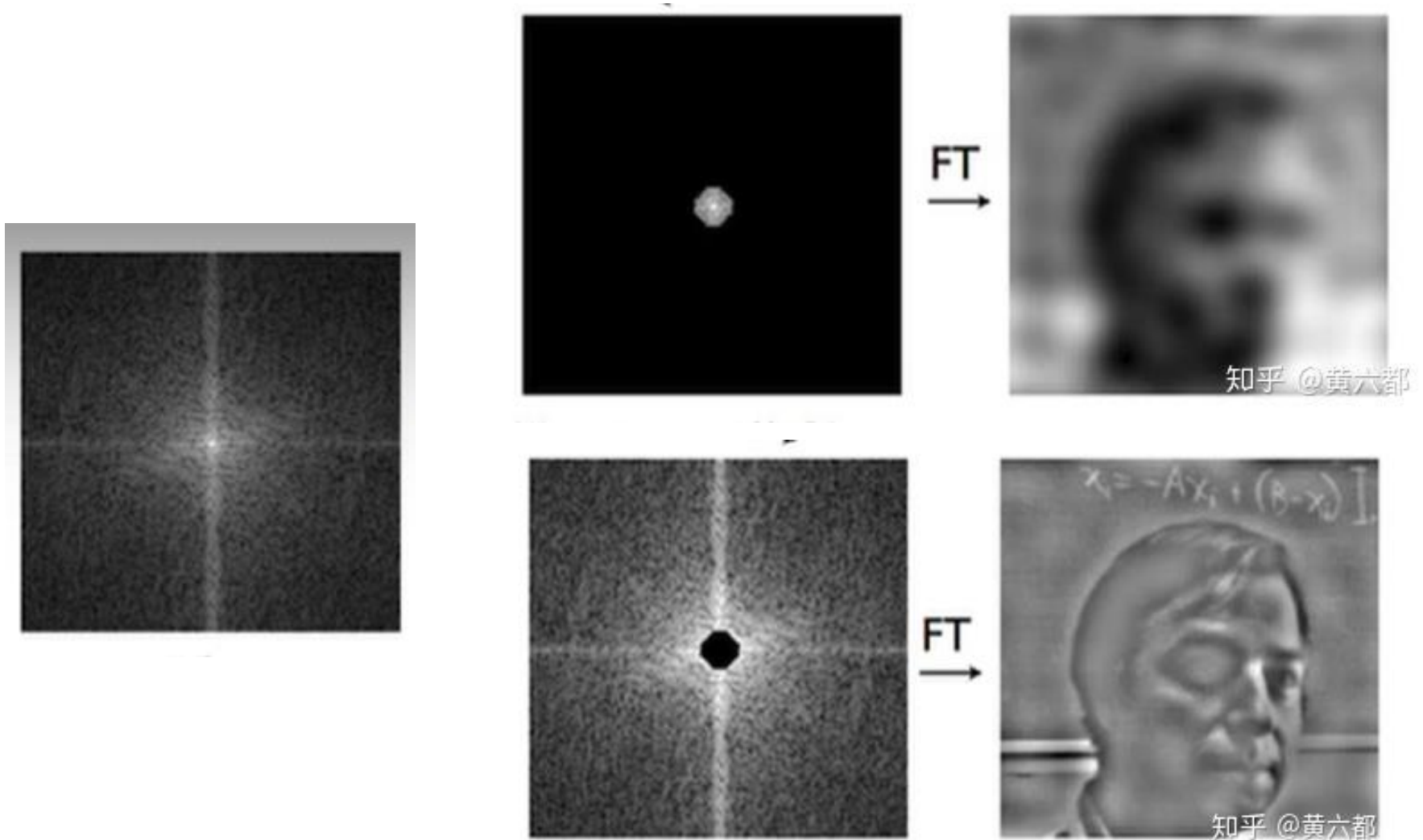


# Applications

## □ Digital image processing



# Digital Image Processing



灰度图



保留DC



保留DC和5个低频AC



保留DC和15个低频AC



灰度图



保留全部AC



保留58个高频AC

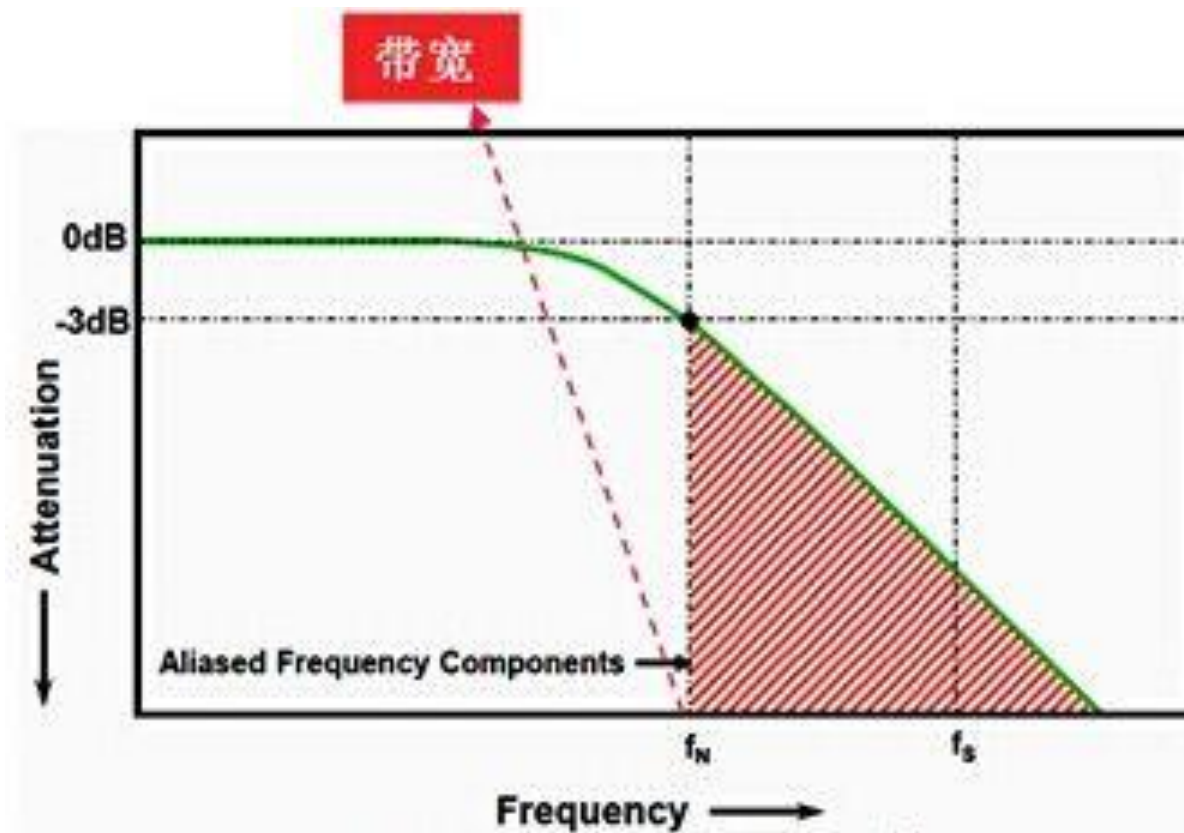


保留49个高频AC



# Applications

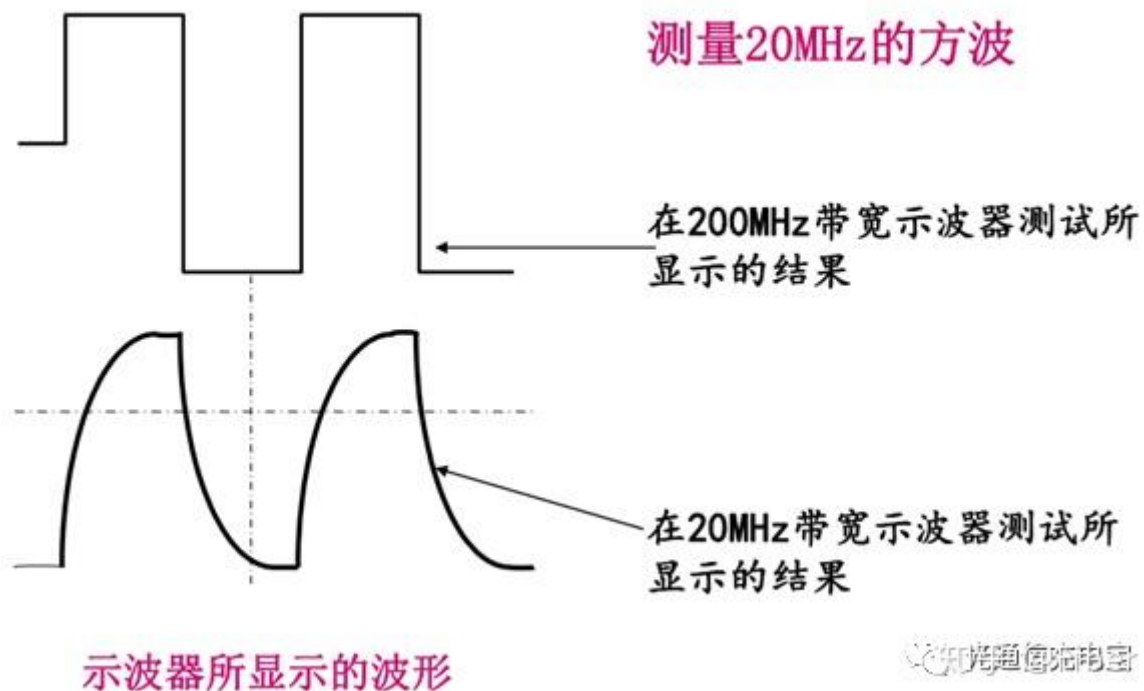
## □ Communication





# Applications

## □ Communication



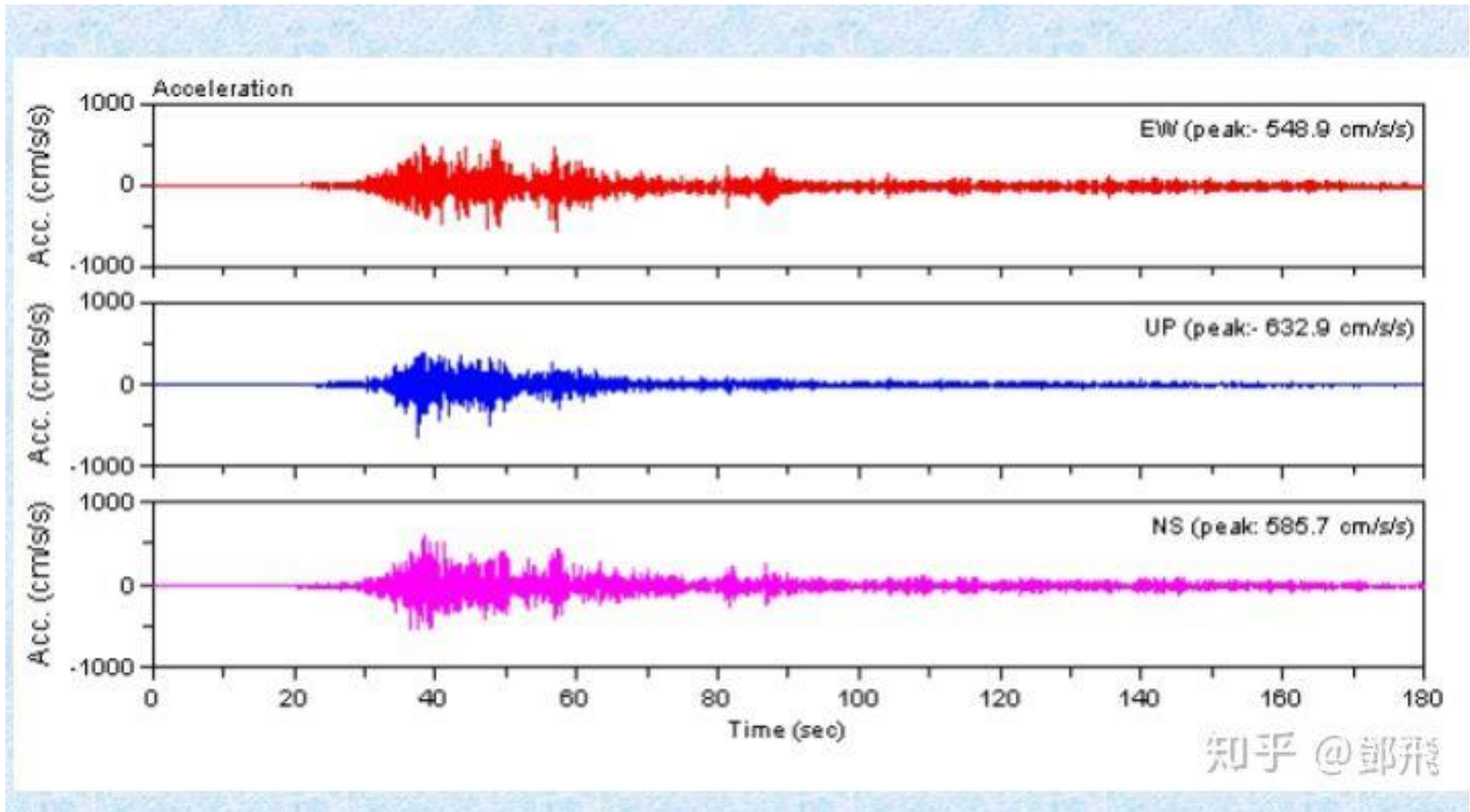
知光通信实验室

# Applications

- ❑ Geology—seismic research (地震研究)
- ❑ Original uses of FFT (fast Fourier transform)
  - Distinguish between natural seismic events and nuclear test explosions

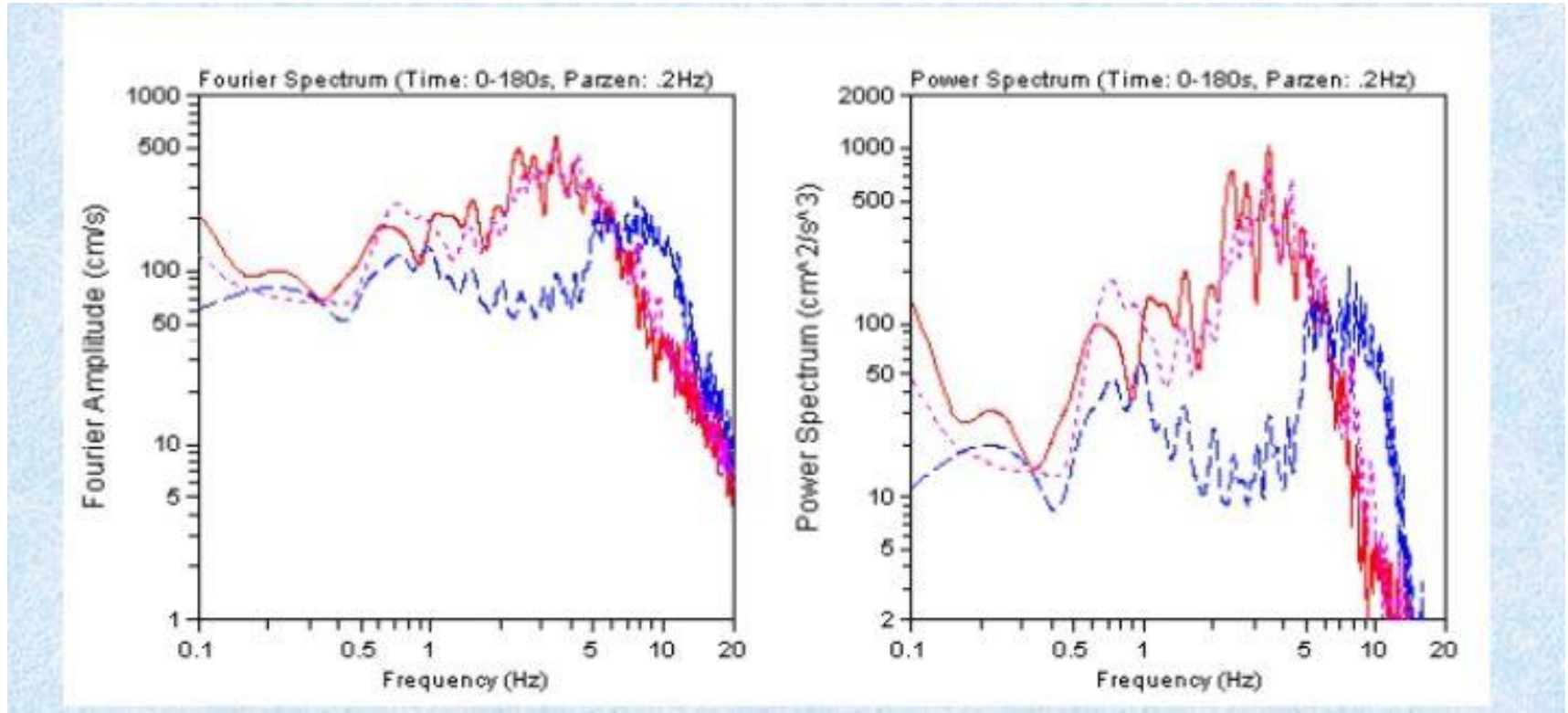


# Seismic Research



作者：绿豆蛙链接：<https://www.zhihu.com/question/288904048/answer/465881216>

# Seismic Research



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# Seismic Research

□ Interested?

