

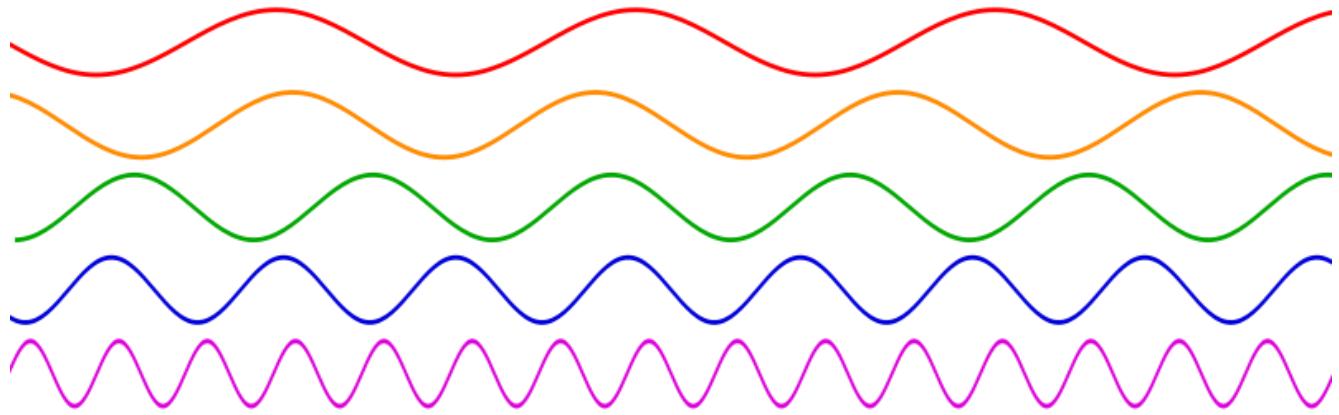
Week 2

Frequency Domain

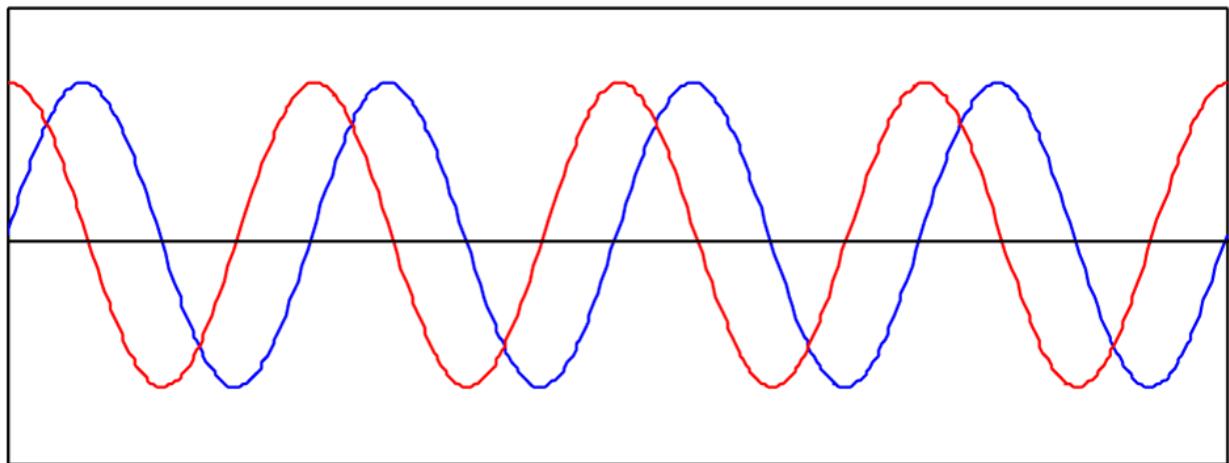
Representation of Signals

What is frequency?
How to characterize frequency?

Frequency

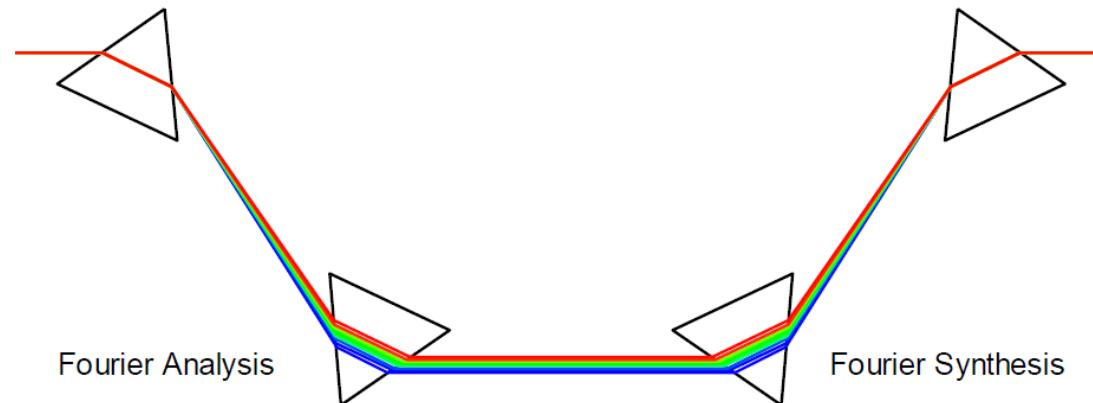


Phase



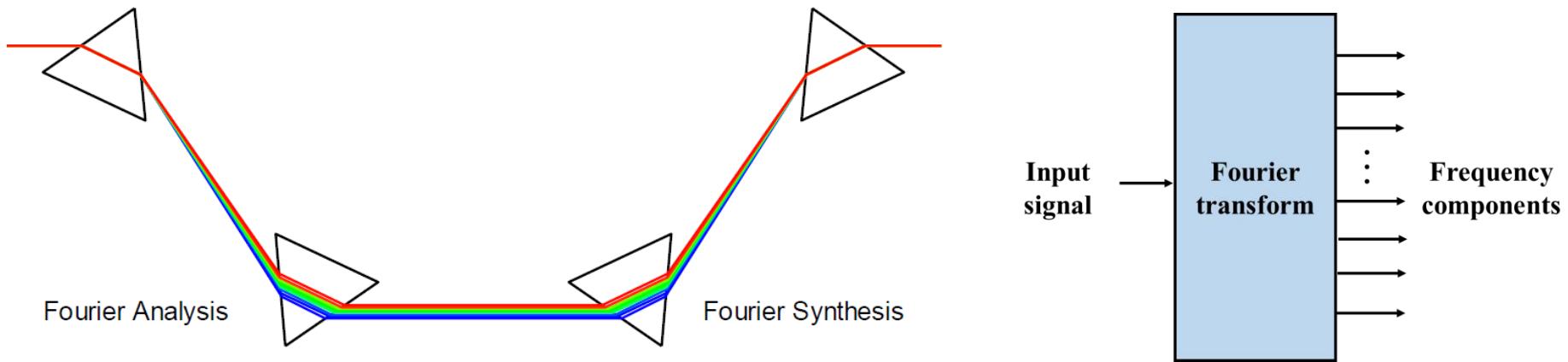
Optical Fourier Transform

- A pair of prisms (棱镜) can split light up into its component frequencies (colors)
 - This is called Fourier Analysis
- A second pair can re-combine the frequencies.
 - This is called Fourier Synthesis



Optical Fourier Transform

- We want to do the same thing with mathematical signals instead of light

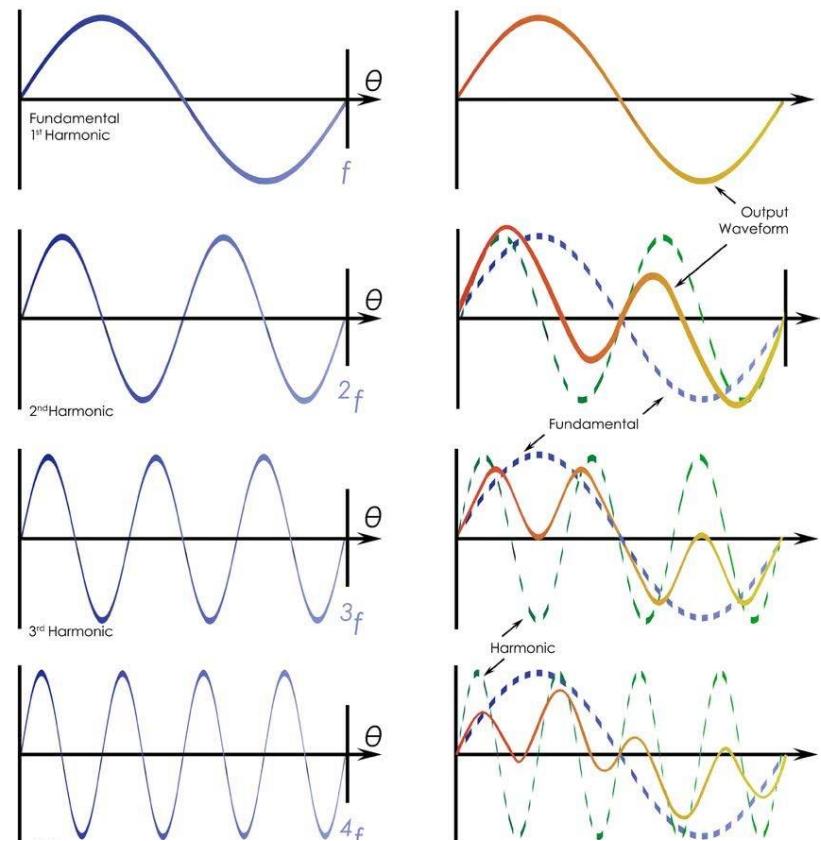


What is Music?

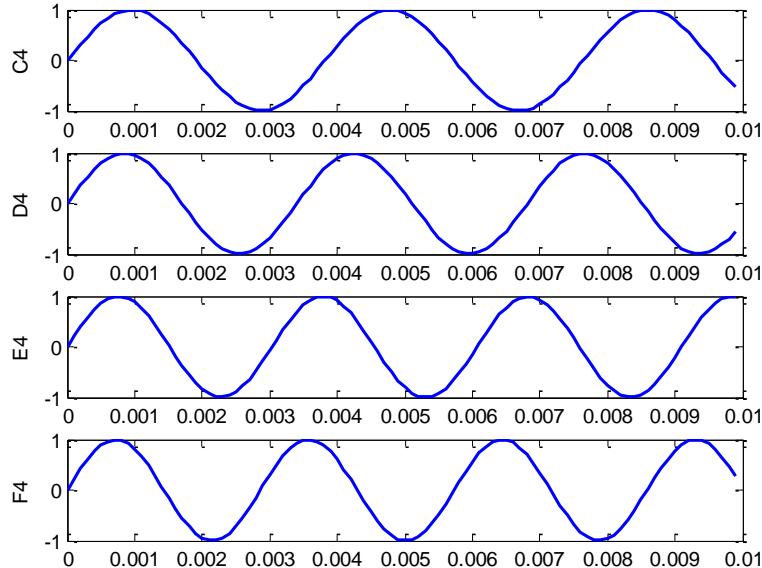
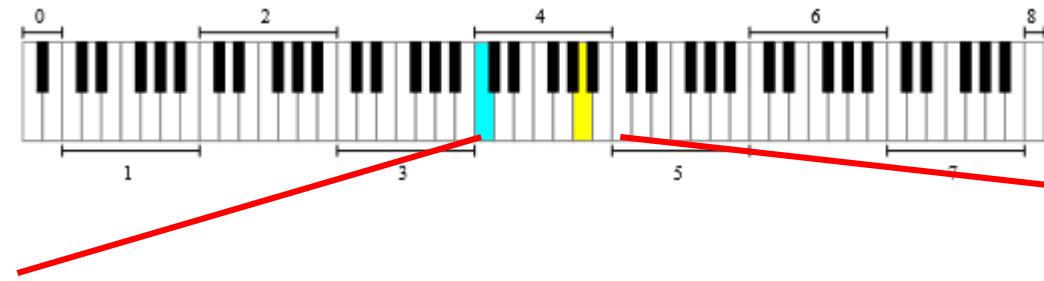
□ For musician, music is



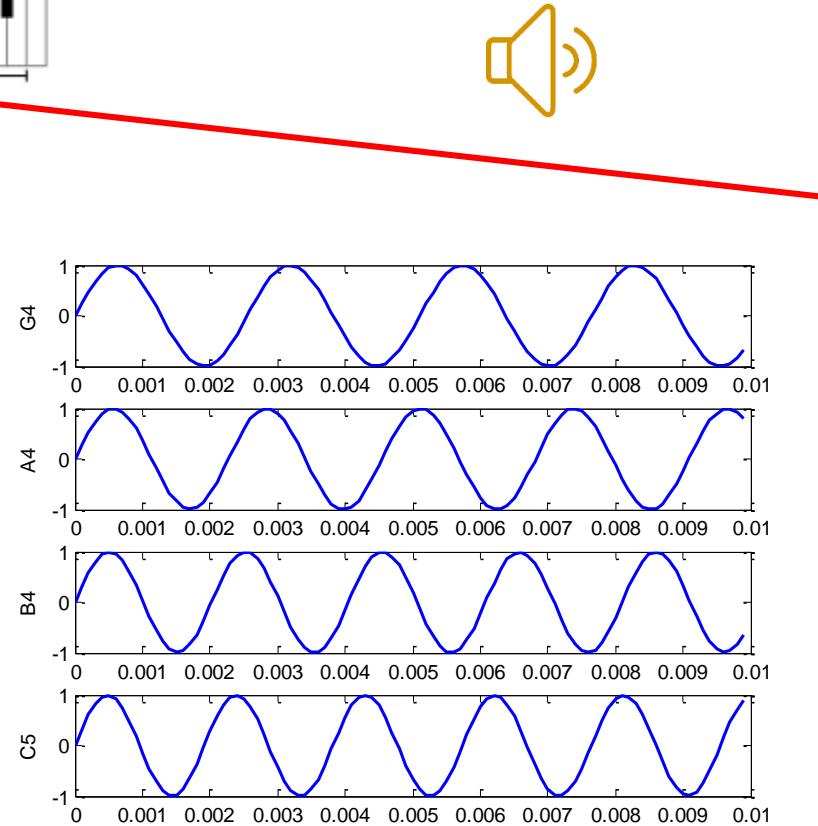
□ For engineer, music is



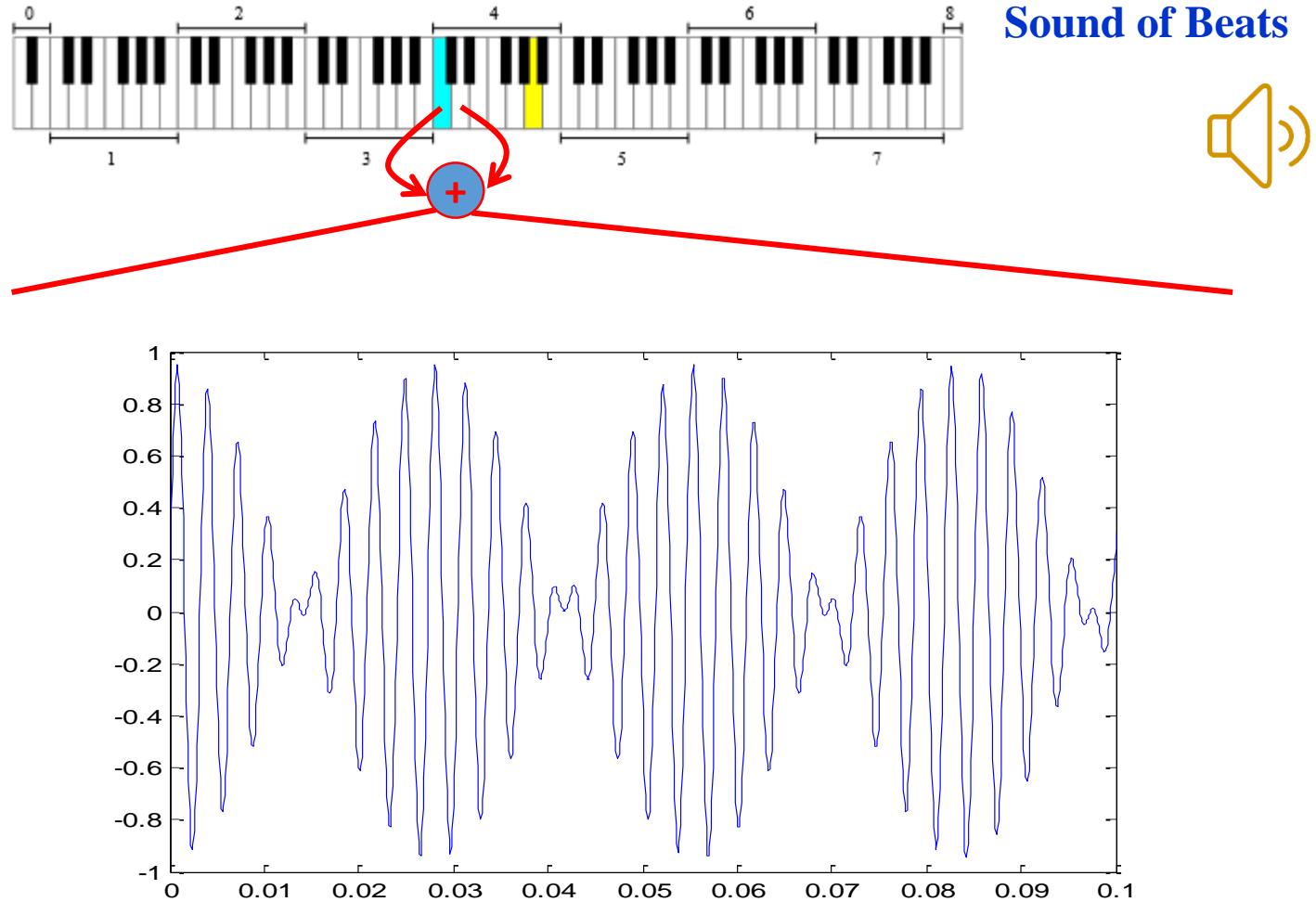
Signals in Time Domain



Sound of Musical Notes

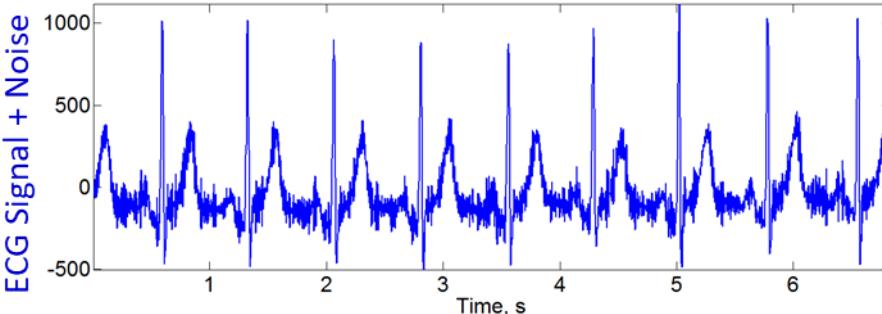
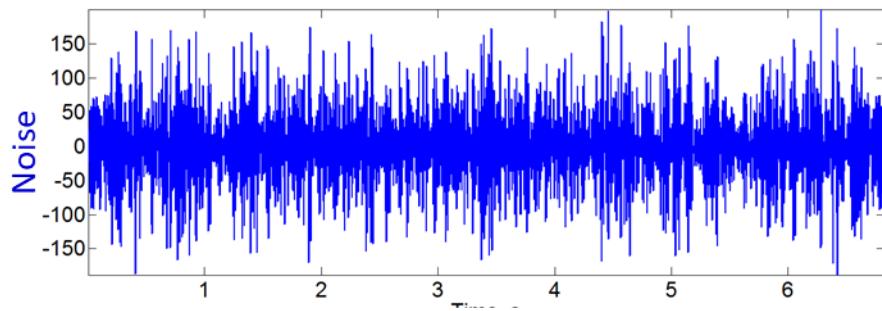
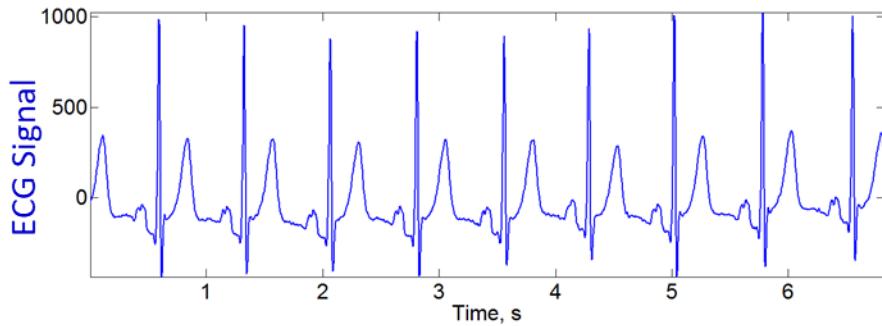


Signals in Time Domain

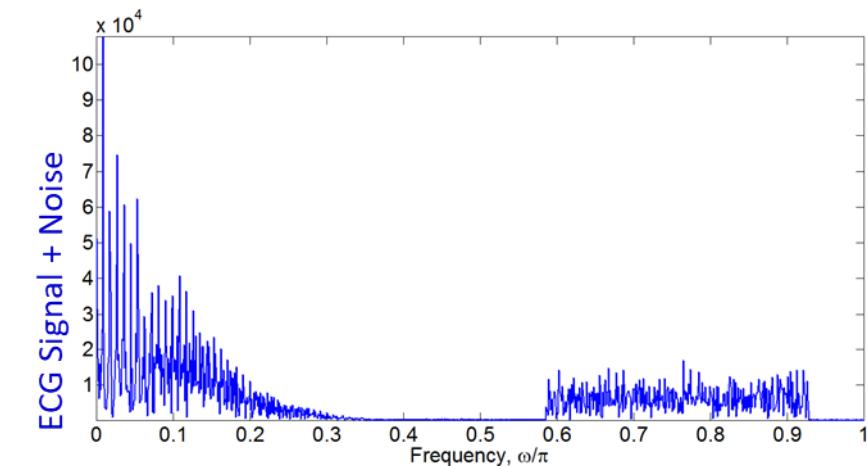
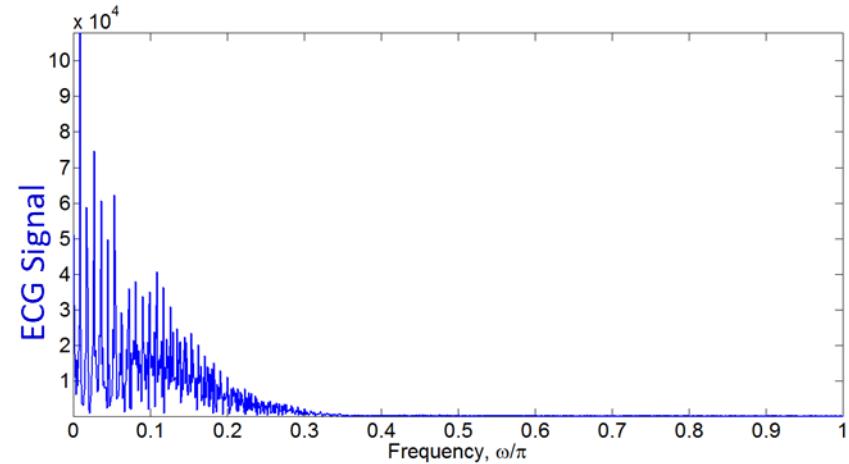


Signals in Frequency Domain

□ Time domain



□ Frequency domain



Question:

How to analyze frequency in a signal?

Answer:

Fourier Transform



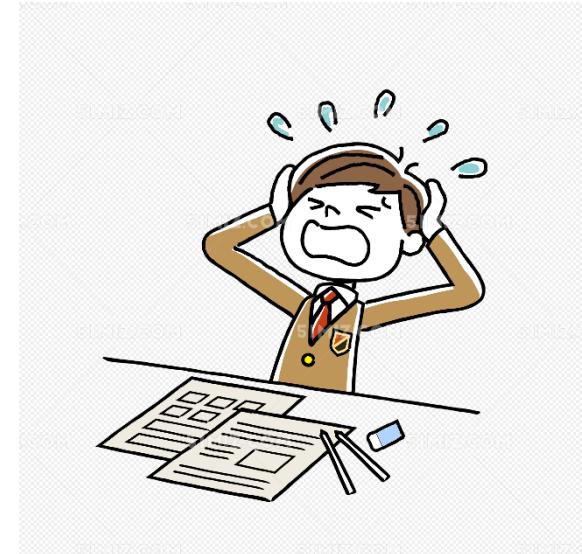
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1768-1830

French Mathematician,
Physicist, Historian

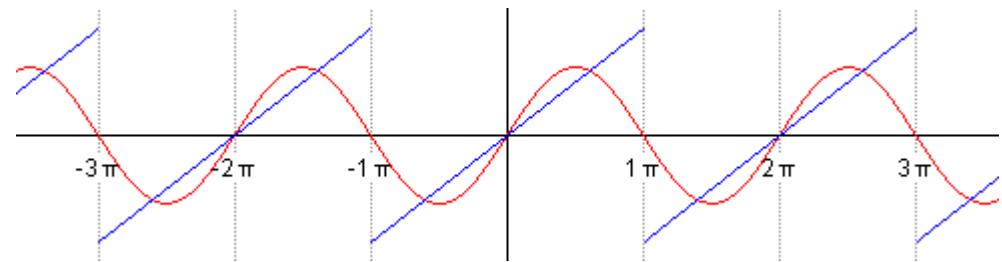
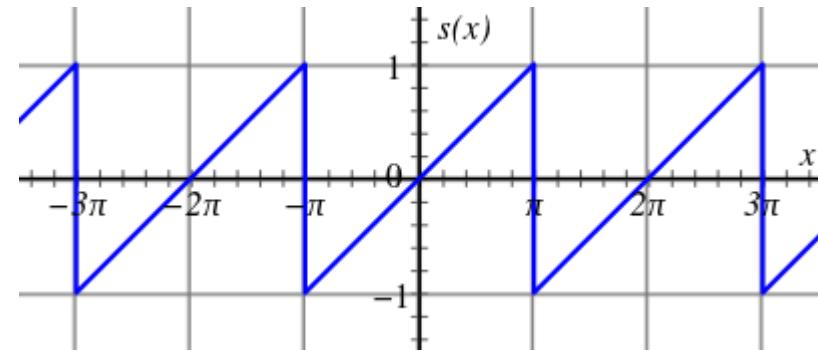
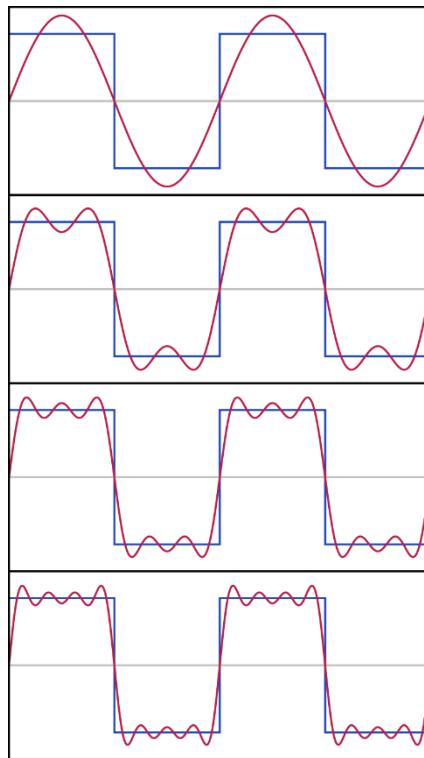
Types of Fourier

- Fourier series
- Fourier transform
 - Continuous Fourier transform
 - Discrete-time Fourier transform
 - Discrete Fourier transform
 - Fast Fourier transform



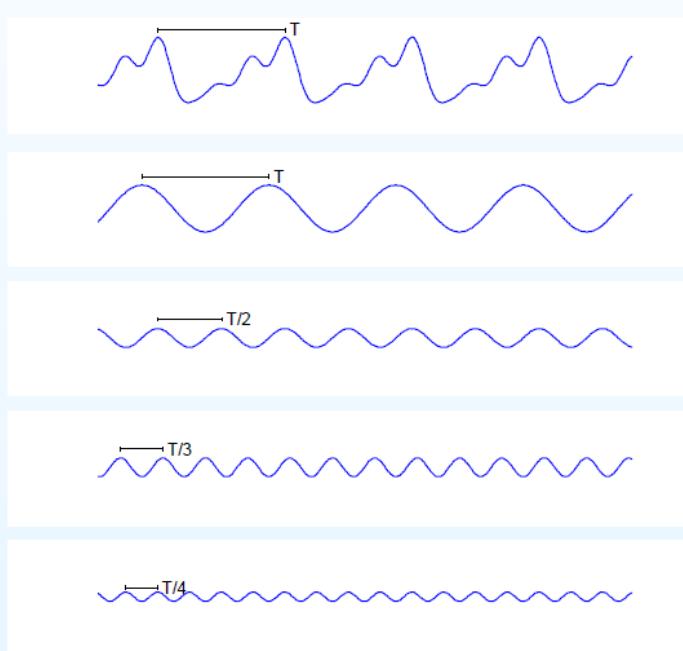
Fourier Series

- To represent a periodic signal as the (possibly infinite) sum of sine and cosine functions



Fourier Series

- To represent a periodic signal as the (possibly infinite) sum of sine and cosine functions



$$u(t) =$$

$$\sin 2\pi ft$$

$$-0.4 \sin 2\pi 2ft$$

$$+0.4 \sin 2\pi 3ft$$

$$-0.2 \cos 2\pi 4ft$$

The Fourier series for $u(t)$ is

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nft + b_n \sin 2\pi nft)$$

Why Sine and Cosine Waves?

- A sine wave remains a sine wave of the same frequency when you
 - Multiply by a constant
 - Add onto to another sine wave of the same frequency
 - Differentiate or integrate or shift in time
- Almost any function can be expressed as a sum of sine waves
 - Periodic → Fourier Series
 - Aperiodic → Fourier Transform
- Many physical and electronic systems are
 - Composed entirely of constant-multiply/add/differentiate
 - Linear: $u(t) \rightarrow x(t)$ and $v(t) \rightarrow y(t)$ means $u(t)+v(t) \rightarrow x(t)+y(t)$

Fourier Series

- Another representation – continuous case

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{2\pi}{T})t}$$

Fourier Series

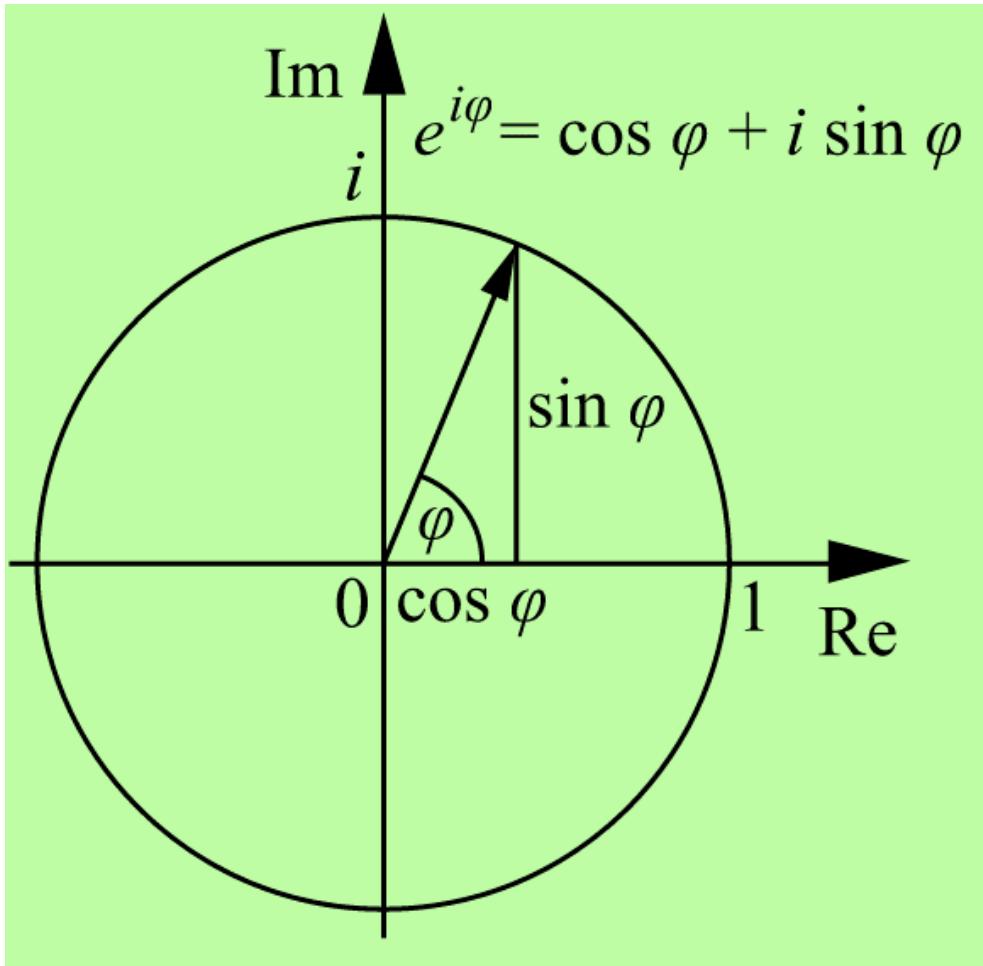
□ Discrete case

$$x[n] = \sum_{k \in \{N\}} a_k e^{jk\omega_0 n} = \sum_{k \in \{N\}} a_k e^{jk(\frac{2\pi}{N})n}$$

$$a_k = \sum_{n \in \{N\}} x[n] e^{-jk\omega_0 n} = \sum_{k \in \{N\}} x[n] e^{-jk(\frac{2\pi}{N})n}$$

Where are the sign and cosine functions?

Euler's Formula



$$e^{j\varphi} = \cos \varphi + j \sin(\varphi)$$

$$\cos \varphi = \frac{1}{2}(e^{j\varphi} + e^{-j\varphi})$$

$$\sin \varphi = \frac{1}{2j}(e^{j\varphi} - e^{-j\varphi})$$

Tools to Play With

- A lot of on-line tools and resources
 - https://en.wikipedia.org/wiki/Fourier_series
 - <https://bl.ocks.org/jinroh/7524988>
- Use Matlab or other programming languages, e.g, Python
 - [An example](#)

How About Non-periodic signals?

- Non-periodic signals can be treated as a periodic signal with infinite period

Fourier series  Fourier transform

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k \in \langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$$

Fourier Transform

Continuous-time

□ Fourier transform (continuous-time)

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt \quad \Omega = 2\pi f$$

Signal analysis: to analyze the frequency components

□ Inverse Fourier transform (continuous-time)

$$x(t) = \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t} d\Omega \quad \Omega = 2\pi f$$

Signal synthesis: to recover the time-domain signal

Fourier Transform

Discrete-time

□ Fourier Transform (discrete-time)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \omega = 2\pi f/f_s$$

ω is a continuous variable in the range of $-\infty < \omega < \infty$

□ Inverse Fourier transform (discrete-time)

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \omega = 2\pi f/f_s$$

Why one is sum and the other integral?

Why Fourier Works?

□ Fourier series as an example

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k \in \langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$$
$$a_k = \sum_{n \in \langle N \rangle} x[n] e^{-jk\omega_0 t} = \sum_{k \in \langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$$

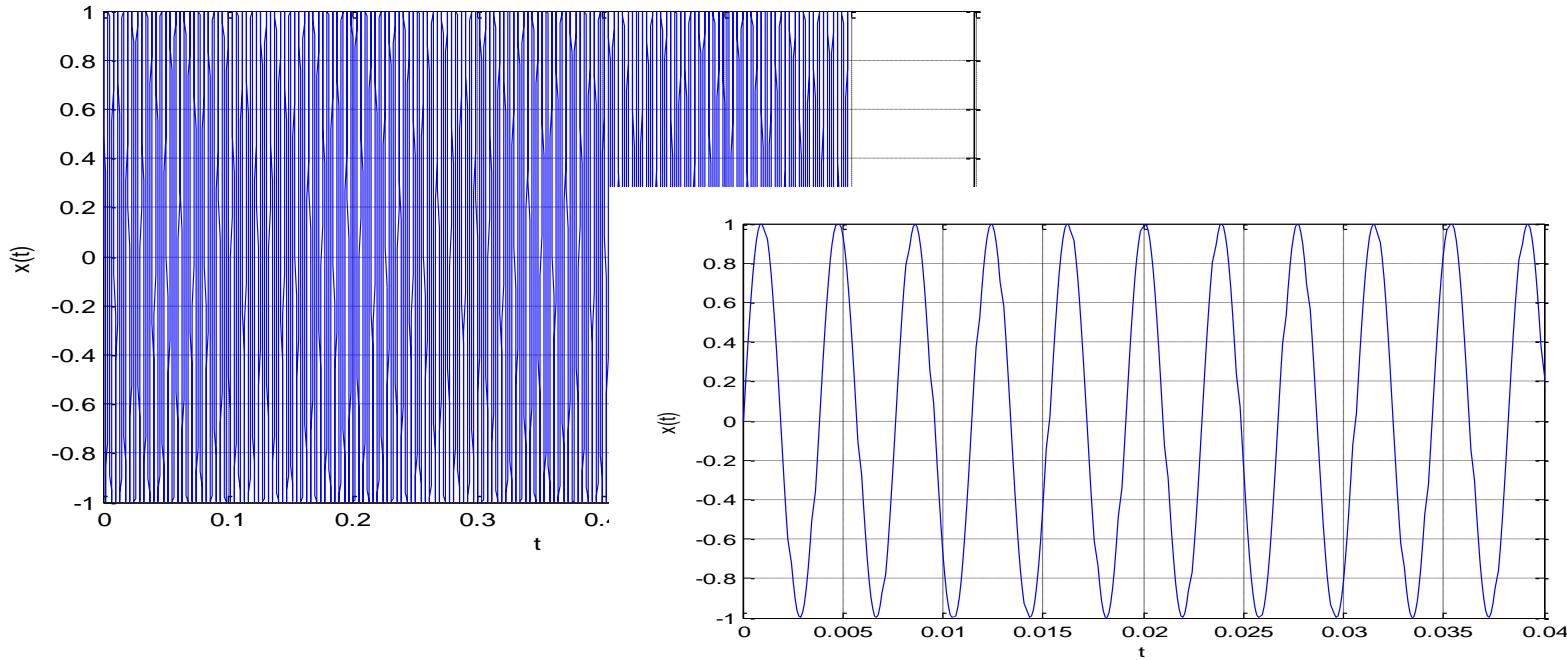
$$\sum_{n \in \langle N \rangle} e^{jk_1 \omega_0 n} e^{-jk_2 \omega_0 n} = \sum_{k \in \langle N \rangle} e^{j(k_1 - k_2)\omega_0 n}$$
$$= \begin{cases} N & \text{for } k_1 = k_2 \\ 0 & \text{for } k_1 \neq k_2 \end{cases}$$

Orthogonality of complex exponentials

Frequency Domain

- Example 1: Fourier Transform of the C4 tone

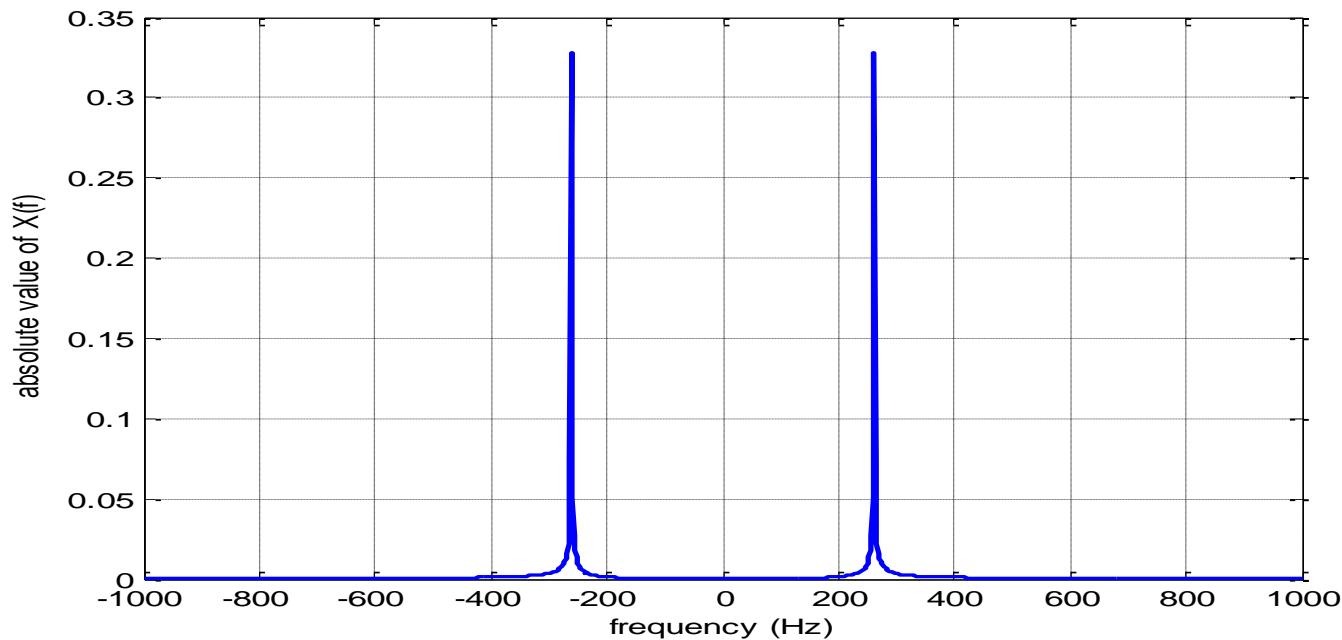
$$x(t) = \begin{cases} \sin(2\pi \cdot f_0 t), & t \in [0,1] \\ 0, & \text{o.w.} \end{cases} \quad f_0 = 261.626\text{Hz}$$



Frequency Domain

- Example 1: Fourier Transform of the C4 tone

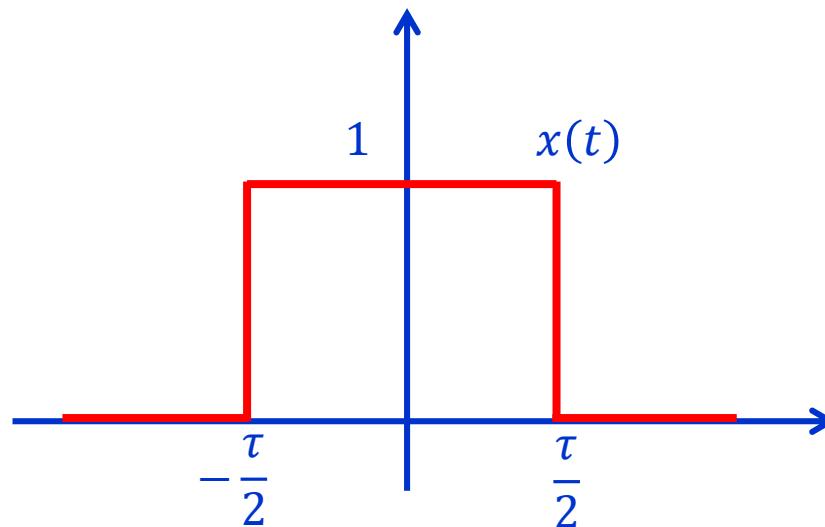
$$x(t) = \begin{cases} \sin(2\pi \cdot f_0 t), & t \in [0,1] \\ 0, & o.w. \end{cases} \quad f_0 = 261.626\text{Hz}$$



Frequency Domain

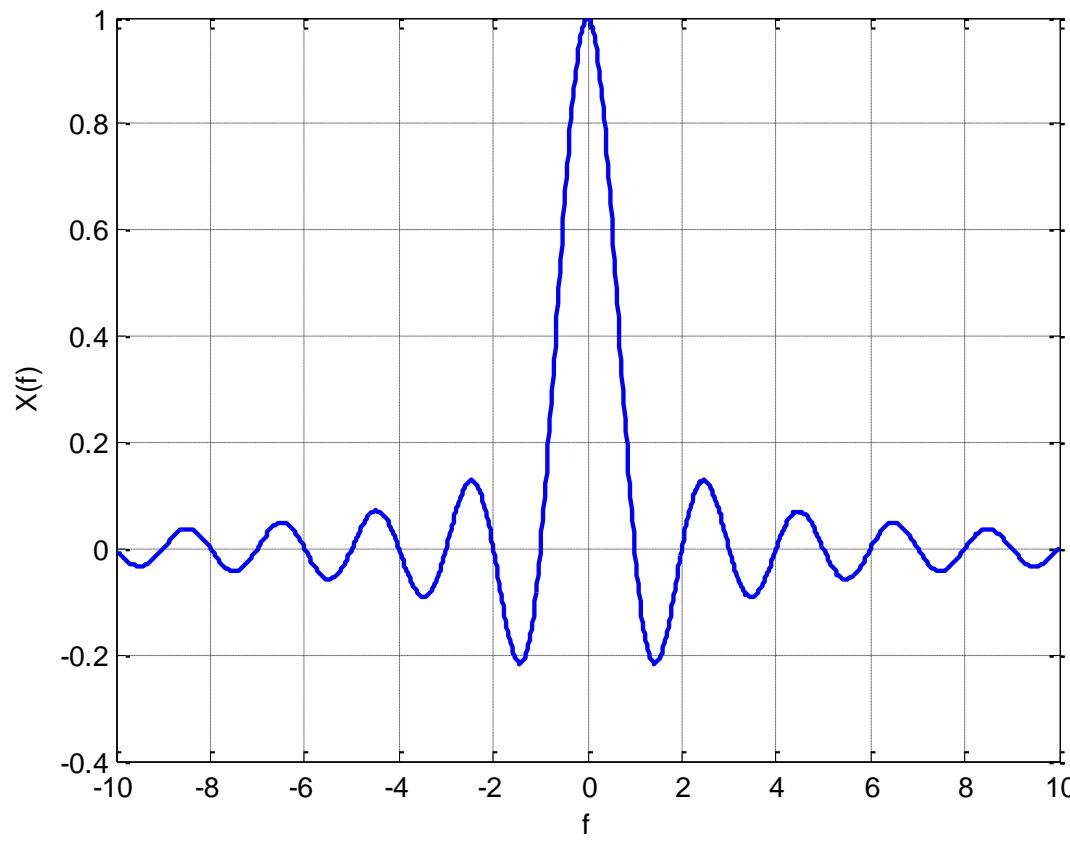
- Example 2: Fourier Transform of the rectangular pulse

$$x(t) = \begin{cases} 1, & t \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right] \\ 0, & o.w. \end{cases}$$



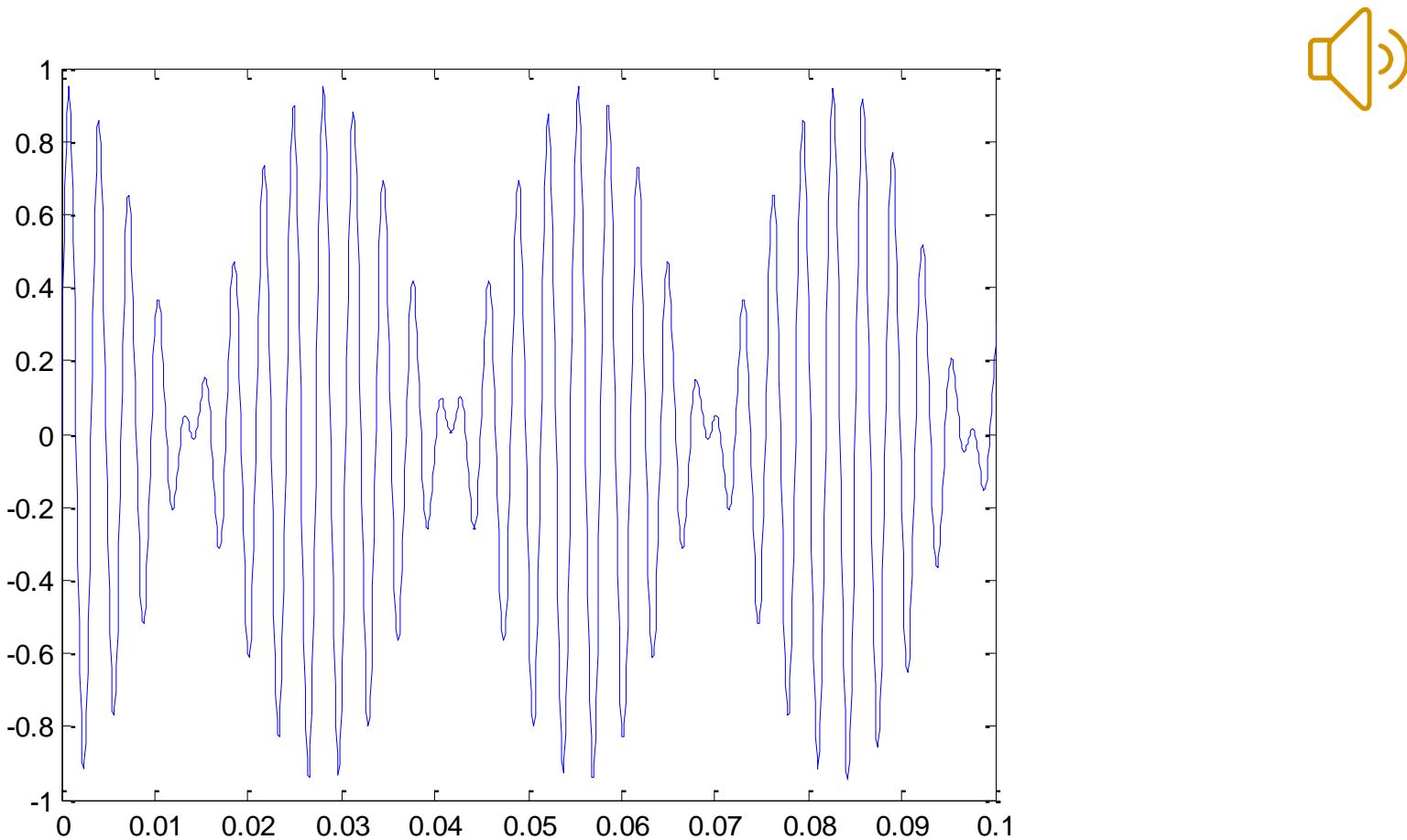
Frequency Domain

- Example 2: Fourier Transform of the rectangular pulse



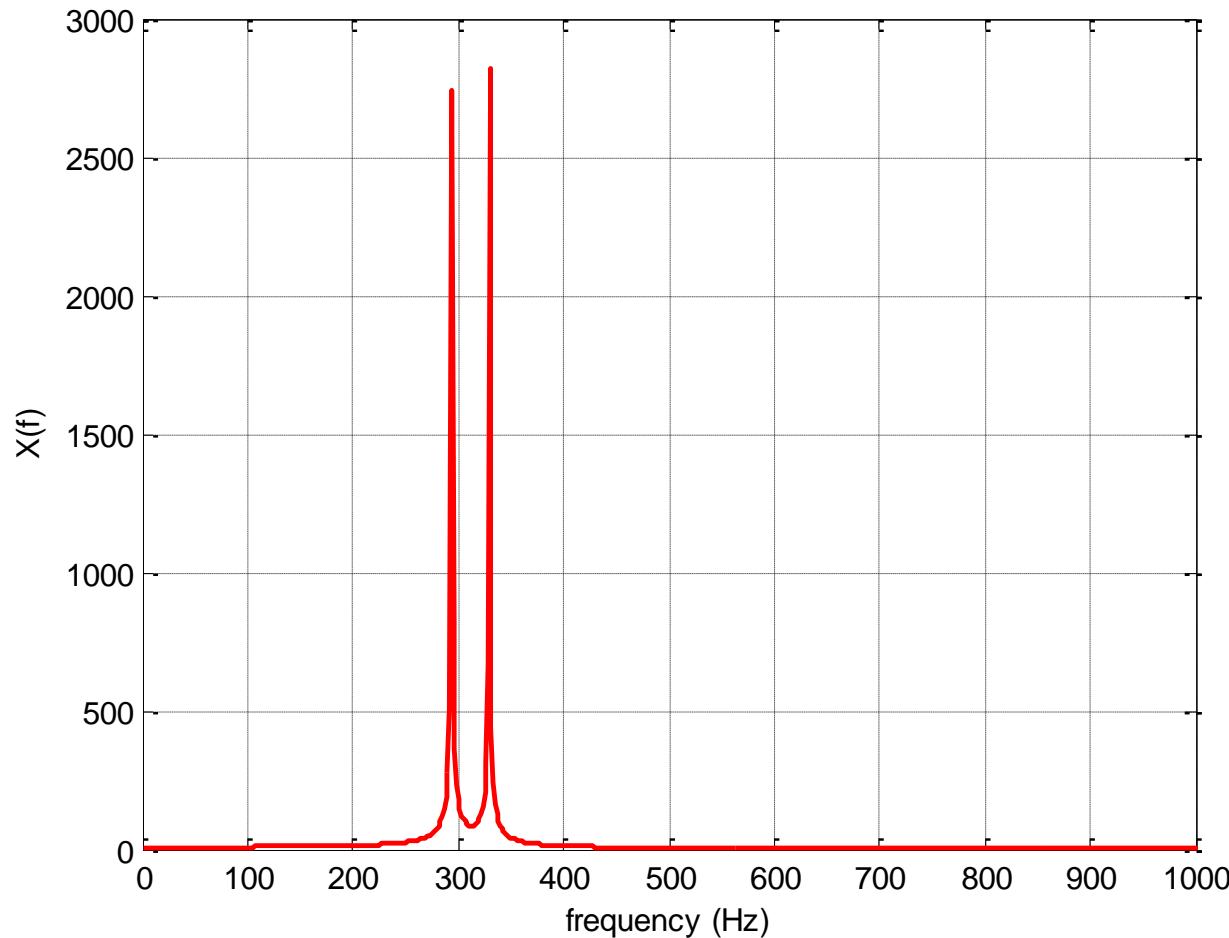
Back to Where We Begin

□ Time domain



Back to Where We Begin

□ Frequency domain



Applications of Fourier Transform: An Example

大学生据拨号音破译周鸿祎手机号 李开复求贤

2012年09月01日 06:31 南方都市报[微博] 刘靖康 我要评论(0)

字号: T | T

[导读]听手机拨号音能破解电话号码？这个在许多影视剧和动漫中出现的“传说”被一位南京大学学生证实，他竟从采访视频中破解出奇虎360董事长周鸿祎的手机号。有网友为此感慨“技术宅要逆天了！”



神不神奇？

Applications of Fourier Transform: An Example

- Dual-tone multi-frequency (DTMF) signaling

Frequency (Hz)	1209	1336	1477
697	1	2	3
770	2	5	6
852	7	8	9
941	*	0	#

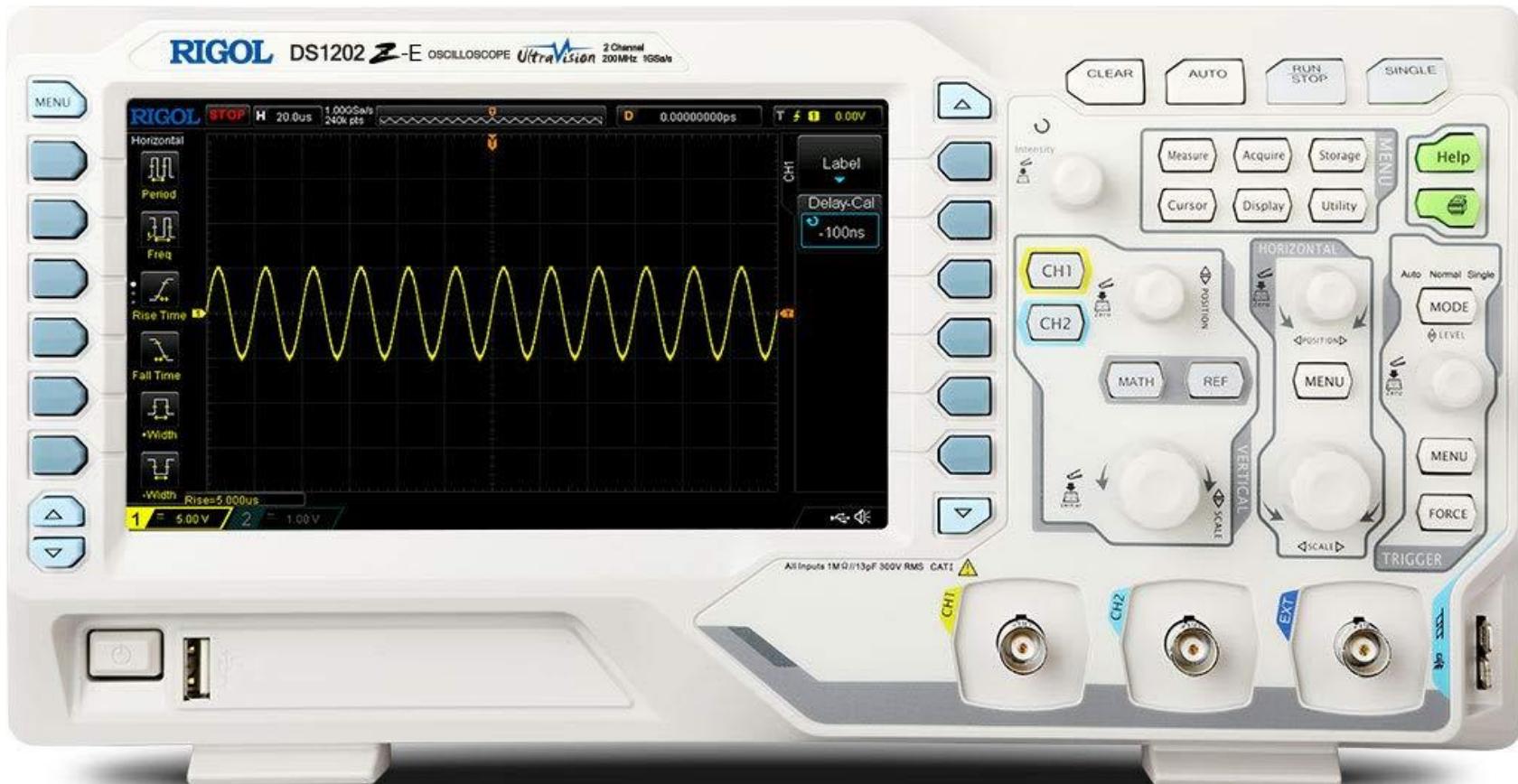
一点都不神奇？

Applications of Fourier Transform

- Methods based on Fourier are used in almost all areas of engineering and science
 - Electrical and electronic engineering
 - Computer science
 - Communication engineering

Applications

□ Oscilloscope



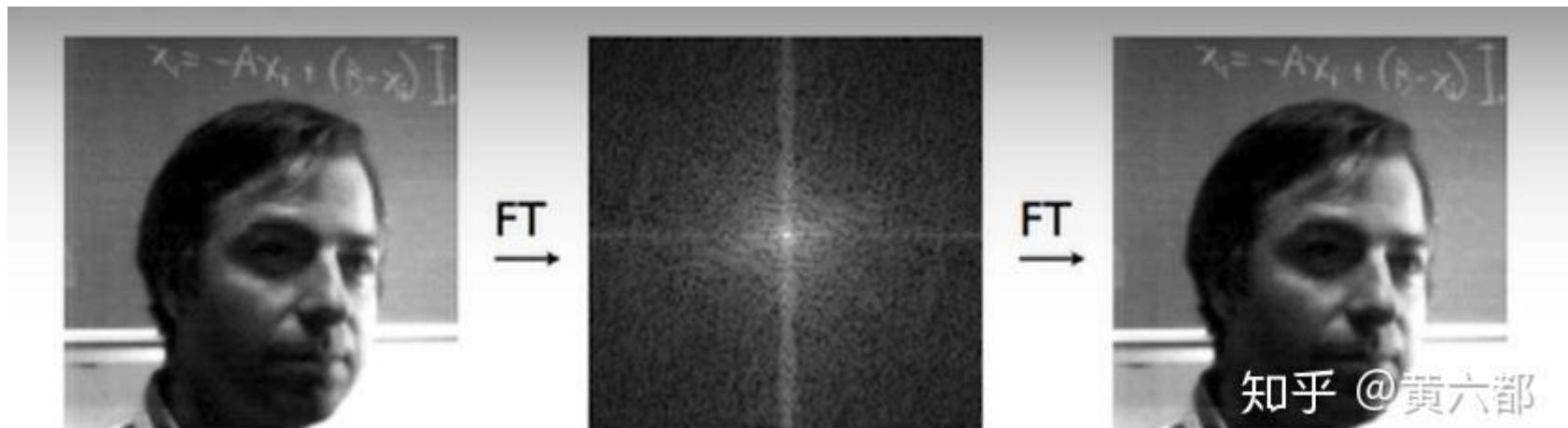
Oscilloscope

□ Frequency analysis

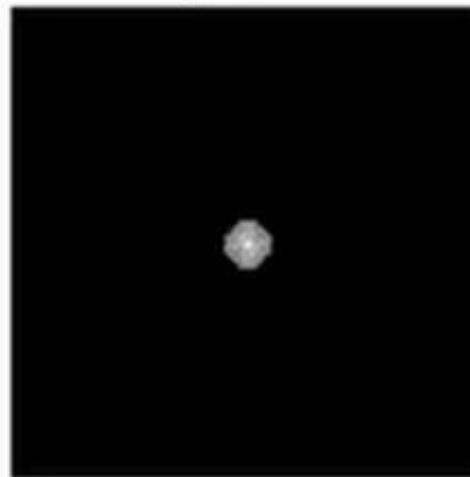
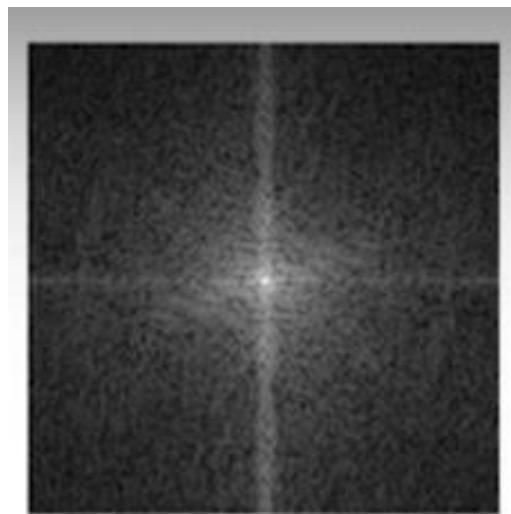


Applications

□ Digital image processing



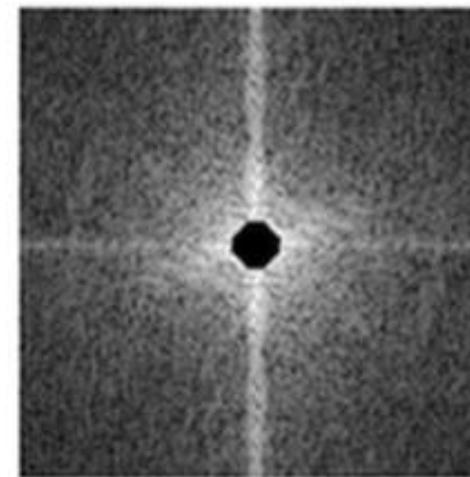
Digital Image Processing



FT
→



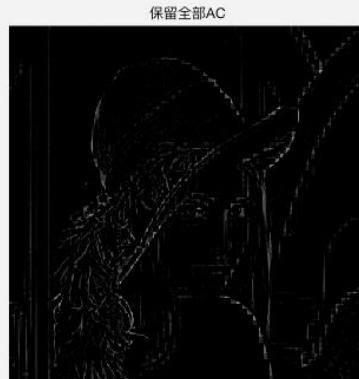
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FT
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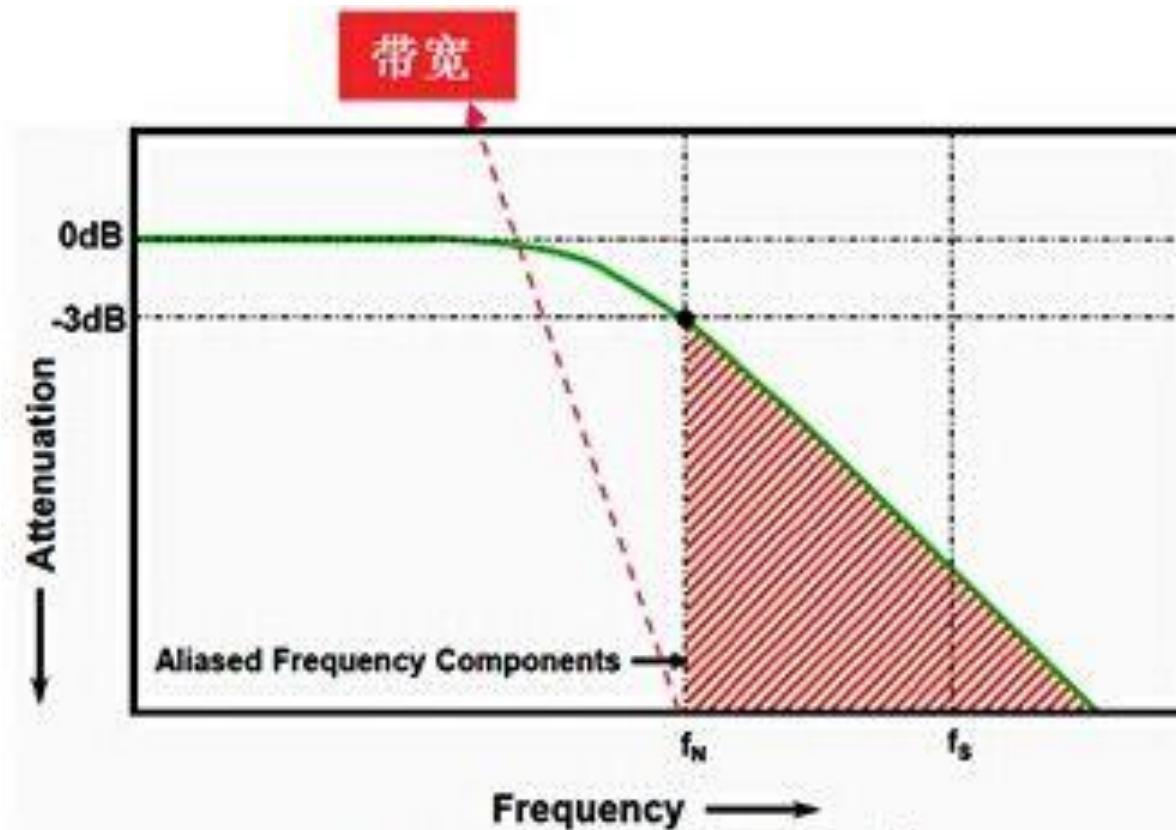


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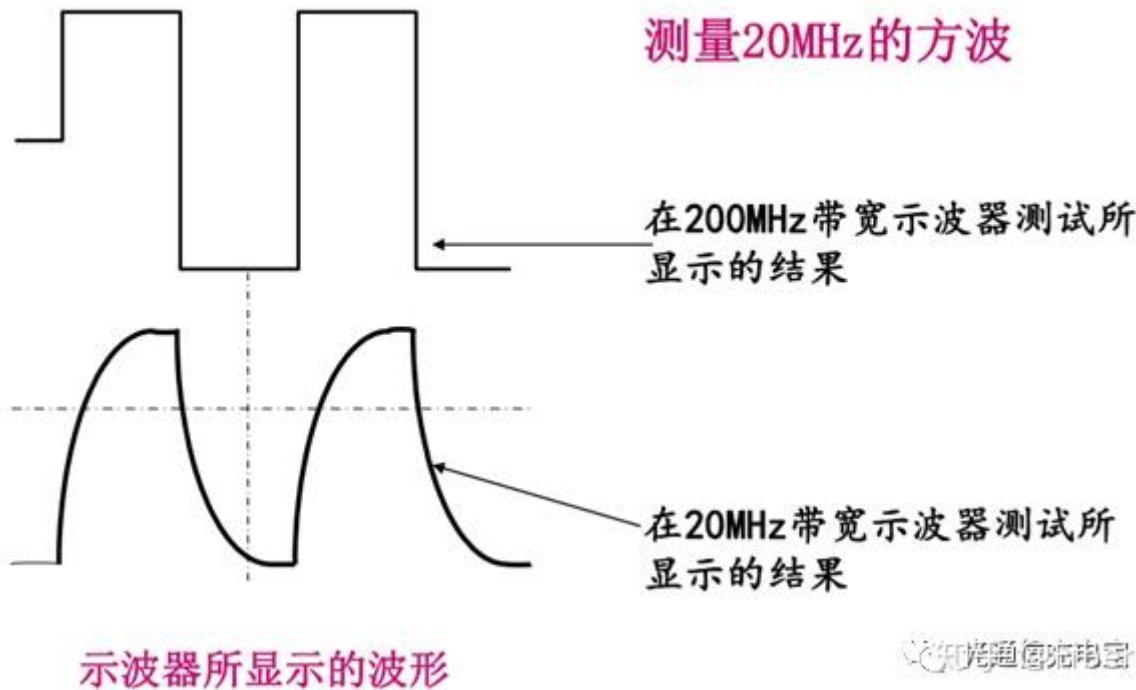
Applications

□ Communication



Applications

□ Communication



光通信与电学

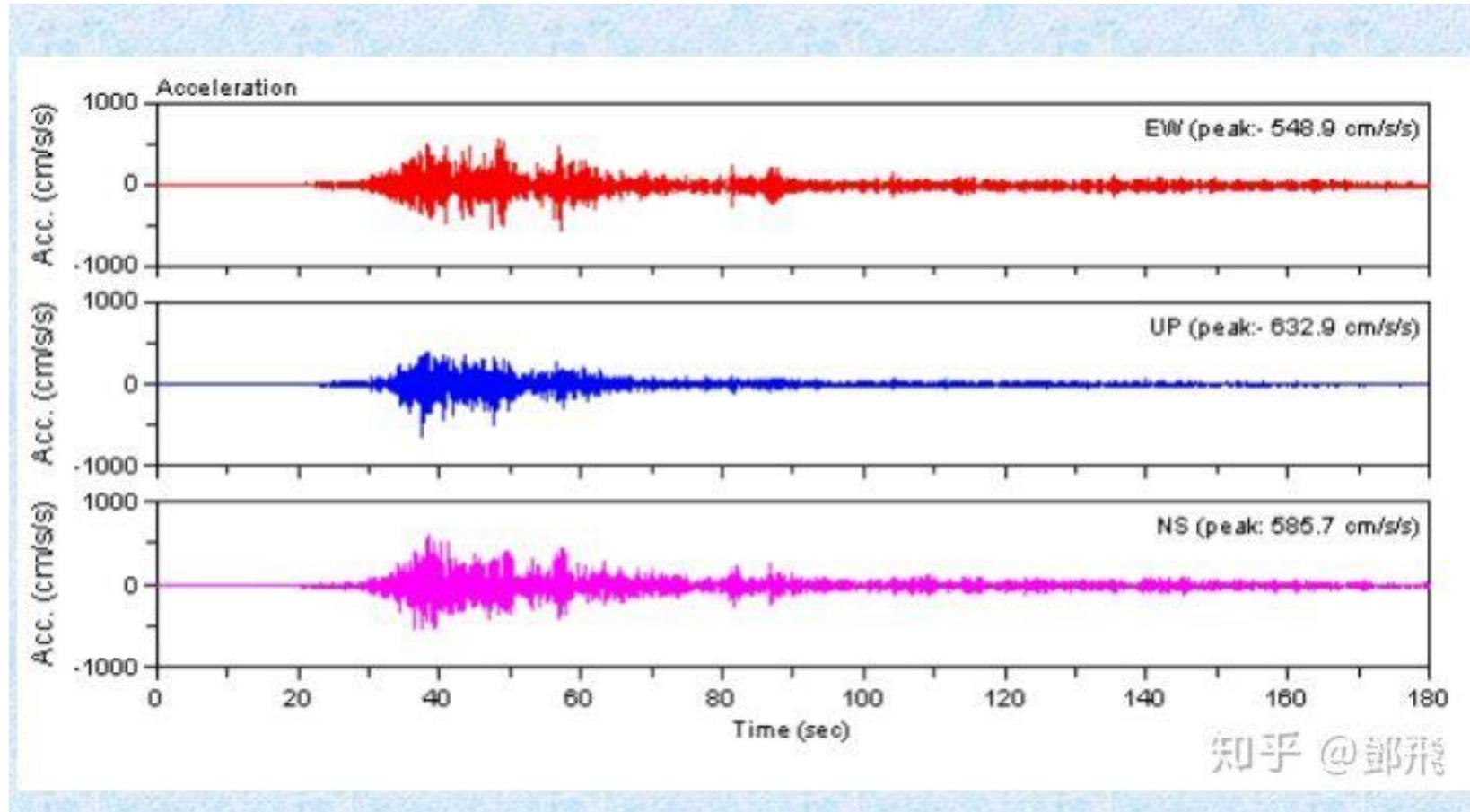


上海科技大学
ShanghaiTech University

Applications

- Geology—seismic research (地震研究)
- Original uses of FFT (fast Fourier transform)
 - Distinguish between natural seismic events and nuclear test explosions

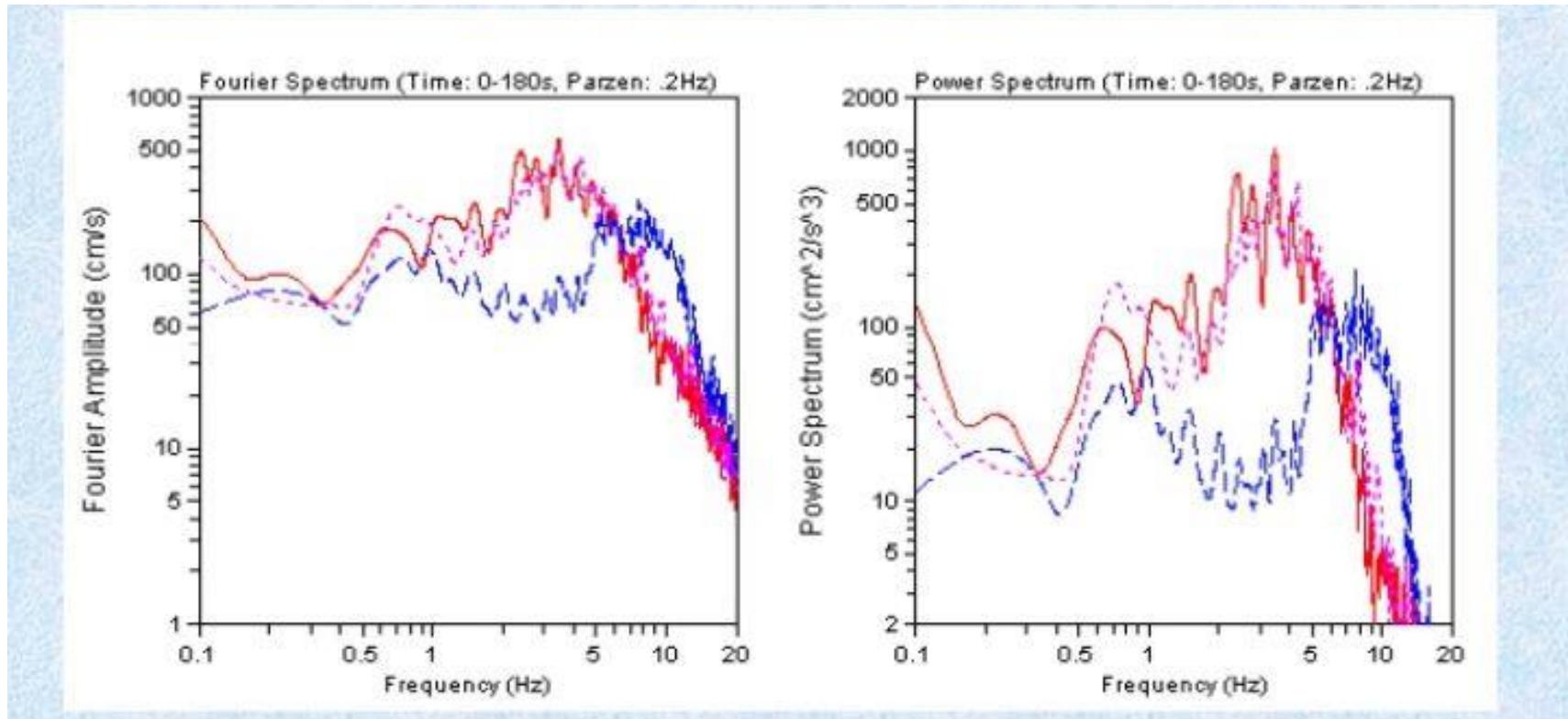
Seismic Research



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Seismic Research



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Seismic Research

□ Interested?

