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hic sunt futura

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Expressiveness issues in Interval Temporal Logics

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To my sister,
to always believe in me

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Introduction

Every area of artificial intelligence (*AI*) has to do with time. Medical diagnosis systems may need to reason about the time at which a virus infected the blood system. In automatic verification, the evolution of variables along time is important. In planning, one wants to achieve one goal before another to meet deadlines, and so on. The representation of temporal information and reasoning about such information requires a language that can capture the concept of change over time and can express the truth or falsity of statements at different times. This has led researchers to develop temporal logics.

In this thesis we will deal with the interval-based temporal logic. In *Interval Temporal Logics*, *time intervals (periods)*, rather than *time instants (points)*, are assumed to be the basic ontological temporal entities. All atomic propositions, and therefore all formulae, are interpreted as true or false on intervals, rather than points. Often time points are not suitable to reason about real-world events, which have a duration. Indeed, many practical aspects of temporality, occuring, for instance, in hardware specifications and real-time process, are better modeled and dealt with if the underlying temporal ontology is based on time intervals.

From a technical point of view, Interval Temporal Logics turn out to be much more expressive than point-based ones [16, 8, 18] and more appropriate for a number of applications. In particular, interval-based temporal logics are applied in many areas of computer science and artificial intelligence, such as, for instance, formal specification and verification of sequential, concurrent, reactive real-time systems, temporal databases and natural language analysis and processing [10]. As an example, consider a typical safety requirement of traffic light systems at road intersections as the following one: *'For every time interval I during which the green light is on for the traffic on either road at the intersection, the green light must be continuously off and the red light must be continuously on for the traffic on the other intersecting road, for a time interval beginning strictly before and ending strictly after I.*

One of the first examples of an interval-based formalism is *Propositional Interval Temporal Logic (*PITL*)* introduced by Moszkowski in [17]. An extension of PITL is called Duration Calculus (DC), [8, 7]. The high expressive power of Interval Temporal Logics comes at the price of high computational complexity, compared to point-based ones. This is due to the fact that the relations between intervals are more complex than those between points. The first systematic study of binary relations between intervals was carried out by Allen [3], who identified the set of 12 possible binary relations between intervals on a linear order (excluding identity). Based on them, *Halpern and Shoham's Modal Logic of*

Time Intervals (HS, for short) was introduced in [12]. HS features one modal operator for each one of the 12 relations identified by Allen.

While decidability has been widely shown to be a feature of most (point-based) temporal logics studied and used in computer science, it turned out that *undecidibility* is not a surprise for interval-based logics. In particular, Halpern and Shoham in [12] proved that the validity of HS formulae over the classes of all linear models, all discrete linear models, and all dense linear models is undecidable. For a long time, results of this nature have discouraged attempts for practical applications and further research on interval-based temporal logics. The light appears with the discovery of expressive decidable fragments of HS; see [4, 5, 6, 15]. Halpern and Shoham's work initiated an active research on the family of fragments of HS, especially concerning *expressiveness*, *(un)decidability*, and *complexity* issues.

The classification of the fragments of HS with respect to expressiveness will be the main focus of the present thesis. As already pointed out, the language of HS features 12 modal operators, one for each Allen relation. By restricting the set of modal operators, 2^{12} syntactically different languages arise. A comparison of the expressive power of different HS fragments is not trivial; some HS modal operators are definable in terms of others and such inter-definabilities may depend on the class of linear order in which the logic is interpreted. Generally, such inter-definabilities cannot be easily transferred from one class of linear orders to another. Indeed, while a definability does transfer from a class to all its sub-classes, there might be definabilities holding in some sub-classes but not in the class itself. Therefore, different linear orders give rise, in general, to different sets of inter-definabilities.

In order to show that a modal operator of HS is definabile in terms of others, it is enough to exhibit an equation witnessing the definability. Instead, a non definability result, i.e., proving that a modal operator is *not* definabile in terms of others, is much more technically involved and amounts to providing a counterexample based on concrete linear orders. To this end, a useful tool is the notion of *bisimulation* (and the related notion of N*-bisimulation*, for some natural number N) between models, which allowed to identify the complete set of inter-definabilities among modal operators of HS (and thus a complete characterization of all expressively different fragments of HS) over the class of *all* linear orders (the *general* case) [9]. As for the class of *all dense* linear orders (the *dense* case), a complete classification of the expressiveness of HS fragments was given in [1]. Using these results, it was shown that there are exactly 1347 expressively different HS fragments in the *general* case, and 966 ones in the *dense* case, out of the $2^{12} = 4096$ subsets of modal operator of HS. The cases of *discrete* and *finite* linear orders turn out to be much more involved. A complete set of definabilities has been identified for the modal operators of HS corrisponding to the Allen's relations *meets*, *later*, *begins*, *finishes*, and *during*, as well as the ones corresponding to their inverse relations [2]. The only missing piece of the expressiveness puzzle is that of the definibilities for the modal operator of HS corresponding to the Allen relation *overlaps* (those for the inverse relation *overlapped by* would immediately follow by symmetry). In this thesis, we conjecture that the set of known definabilities for the modal operator of HS $\langle O\rangle$, corresponding to Allen relation *overlaps* is complete for the class of all discrete linear orders, and we make a significant step towards a formal proof of the conjecture.

The thesis is structured as follows. In Chapter 2, we introduce some basic concepts and notations in the field of interval temporal logics; in particular, we define syntax and semantics of the interval temporal logic HS and its fragments. In Chapter 3, we first provide the basic notion of definability of a modal operator in terms of others and then we introduce the problem of comparing the expressive power of HS fragments; we also give a hint about the expressiveness abilities of HS and its fragments, before presenting the standard technique used to disprove definability of a modal operator in terms of others, which is based on the notion of bisimulation and the classic property of bisimulation invariance of modal formulae. Finally, in Chapter 4 we apply such a technique to obtain our technical and original result, that is, a bisimulation proposal, along with a (partial) formal proof to support our claim about the completeness of the set of known definabilities for the modal operator $\langle O \rangle$ in the class of all discrete linear orders. Proofs are rather difficult and technically involved; therefore, we refer the reader interested in the technical details to Appendix A.

2

Preliminaries

2.1 Intervals and interval structures

Given a strict partial order $\mathbb{D} = \langle D, < \rangle$, where D is a non-empty set, and $<$ is an irreflexive, asymmetric, and transitive binary relation on D. The idea is that D is the set of *time points*, and $<$ is the *earlier-later relation* on D. For time points $t, u \in D$, $t < u$ means intuitively that t is earlier than u, and that u is later that t. This explains the requirements that $<$ be irreflexive and transitive: no time points should be in the past or future of itself and if t is earlier that u, and u earlier that v, then we expect t to be earlier that v. An *interval* in $\mathbb{D}$ is an ordered pair $[d_0, d_1]$ such that $d_0, d_1 \in D$ and $d_0 \leq d_1$. A point d *belongs to an interval* $[d_0, d_1]$ if $d_0 \leq d \leq d_1$. If $d_0 < d_1$, then $[d_0, d_1]$ is called a *strict*, or *proper*, interval; otherwise, it is called a *point interval.* The set of all intervals in $\mathbb{D}$, including both strict and point intervals, is denoted by $\mathbb{I}(\mathbb{D})^+$, while the set of all strict intervals is denoted by $\mathbb{I}(\mathbb{D})^-$. Finally, we call a pair $\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+ \rangle$ (resp., $\langle \mathbb{D}, \mathbb{I}(\mathbb{D})^- \rangle$) a *non-strict interval structure* (resp., *strict interval structure*).

2.2 Linear orders

All interval structures considered here will be assumed to be *linear*, that is, every two points in it are comparable. This restriction can usually be relaxed without essential complications to partial orderings with the *linear interval property*, that is, partial orderings in which every interval is linear. Here is the formal definition in first-order logic:

$$\forall x \forall y (x \leq y \to \forall z_1 \forall z_2 (x < z_1 < y \wedge x < z_2 < y \to z_1 < z_2 \vee z_1 = z_2 \vee z_2 < z_1)),$$

In the figure below an interval structure with the linear interval property is given on the left and an interval structure violating that property is given on the right.

Definition 1. *A linear order, and the associated interval structure, is called:*

- ***finite****, if it has finitely many points;*

- ***unbounded above*** *or* ***to right*** *(resp.,* ***below*** *or* ***to left****), if every point has a successor (resp., predecessor);*
- ***dense****, if for all $t, u \in D$, if $t < u$ then there is $v \in D$ with $t < v < u$*
- ***discrete****, if for each $t \in D$,*
  - *if there is any $u \in D$ with $u > t$, then there is a first such u: one such that there is no $v \in D$ with $t < v < u$, and*
  - *if there is any $u \in D$ with $u < t$, then there is a last such u: one such that there is no $v \in D$ with $u < v < t$.*
- ***Dedekind complete****, if every non-empty subset $S \subseteq D$ that has an upper bound in D also has a least upper bound in D: there is $t \in D$ such that*
  - *$t \geq s$ for all $s \in S$, and*
  - *there is no $t' < t$ with $t' \geq s$ for all $s \in S$*

2.3 Allen's interval relations

Table 2.1 depicts all possible binary relations between two strict intervals on a linear order, known as *Allen's relations*. Besides the identity relation *equal* ($=$), these are (in Allen's original terminology [3]): *before* ($<$), *meets* (m), *overlaps* (o), *finishes* (f), *during* (d), *starts* (s), plus their inverses *after* ($>$), *met-by* (mi), *overlapped-by* (oi), *finished-by* (fi), *contains* (di), *started-by* (si). These 13 relations are *mutually exclusive* and *jointly exhaustive*, meaning that exactly one Allen relation holds between any given pair of strict intervals. Each Allen relation gives rise to a corresponding unary modal operator.

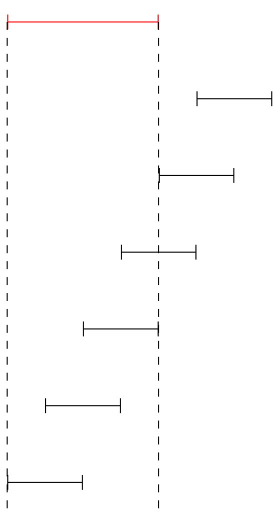
Interval's relations	**Allen's notation**	**HS notation**
	*equals* $\{=\}$	
	before $\{<\}$ / *after* $\{>\}$	$\langle L\rangle$ / $\langle\overline{L}\rangle$ (*Later*)
	meets $\{m\}$ / *met-by* $\{mi\}$	$\langle A\rangle$ / $\langle\overline{A}\rangle$ (*After*)
	overlaps $\{o\}$ / *overlapped-by* $\{oi\}$	$\langle O\rangle$ / $\langle\overline{O}\rangle$ (*Overlaps*)
	finished-by $\{fi\}$ / *finishes* $\{f\}$	$\langle E\rangle$ / $\langle\overline{E}\rangle$ (*Ends*)
	contains $\{di\}$ / *during* $\{d\}$	$\langle D\rangle$ / $\langle\overline{D}\rangle$ (*During*)
	started-by $\{si\}$ / *starts* $\{s\}$	$\langle B\rangle$ / $\langle\overline{B}\rangle$ (*Begins*)

Table 2.1: Relations between pairs of strict intervals.

2.4 Syntax and semantics of Halpern-Shoham's logic HS

The *Halpern and Shoham's Modal Logic of Time Intervals* (HS) includes a set of propositional letters $\mathcal{AP}$, the classical propositional connectives $\neg$ and $\vee$ (all others, including the propositional constants $\top$ and $\bot$, are assumed definable as usual), the modal constant π (whose meaning will be made clear soon), and a family of *interval temporal modal operators (modalities)* of the form $\langle X \rangle$, one for each Allen's relation. Formulae are defined by the following grammar:

$$\varphi ::= p \mid \pi \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X \rangle \varphi$$

where $p \in \mathcal{AP}$ and $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$. We use R_X to denote the Allen relation associated with modal operator $\langle X \rangle$, for every $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$. For instance, R_A denotes Allen relation *meets*, associated with modal operator $\langle A \rangle$.

The remaining connectives are derived from the others; for instance:

- $\top \equiv \neg p \vee p$,
- $\bot \equiv \neg\top$,
- $\varphi_1 \wedge \varphi_2 \equiv \neg(\neg\varphi_1 \vee \neg\varphi_2)$,
- $\varphi_1 \rightarrow \varphi_2 \equiv \neg\varphi_1 \vee \varphi_2$.

Below, we give the definition of *modal depth* of an HS formula, that is, the largest nesting of modal operators in it, that will come handy in later chapters.

Definition 2. *The* modal depth *of an* HS *formula* φ*, denoted by* $d(\varphi)$*, is defined inductively as follows:*

- $d(p) = 0$*, for all* $p \in \mathcal{AP}$*,*
- $d(\pi) = 0$*,*
- $d(\neg\psi) = d(\psi)$*,*
- $d(\psi \vee \phi) = max\{d(\psi), d(\phi)\}$*,*
- $d(\langle X \rangle \psi) = d(\psi) + 1$ *for each* $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$.

As already pointed out, there are two different natural semantics for interval logics, namely, a *strict* one, which excludes point-intervals, and a *non-strict* one, which includes them. A *non-strict interval model* is a pair $M^+ = \langle \mathbb{I}(\mathbb{D})^+, V \rangle$ where $\mathbb{I}(\mathbb{D})^+$ is the set of all intervals (including point intervals) over linear order $\mathbb{D}$ and $V : \mathbb{I}(\mathbb{D})^+ \rightarrow 2^{\mathcal{AP}}$ is an *evaluation function* assigning to each interval a set of atomic propositions considered true at it. Respectively, a *strict interval model* is a structure $M^- = \langle \mathbb{I}(\mathbb{D})^-, V \rangle$ defined likewise. Formally, both the strict and non-strict semantics of HS modal operators can be defined inductively by the following rules, for every interval model M, interval $[d_0, d_1]$ in M, and HS formula φ:

- $M, [d_0, d_1] \Vdash \langle A \rangle \varphi$ iff $M, [d_1, d_2] \Vdash \varphi$ for some d_2;

- $M, [d_0, d_1] \Vdash \langle L\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_1 < d_2$;
- $M, [d_0, d_1] \Vdash \langle B\rangle\varphi$ iff $M, [d_0, d_2] \Vdash \varphi$ for some d_2 such that $d_2 < d_1$;
- $M, [d_0, d_1] \Vdash \langle E\rangle\varphi$ iff $M, [d_2, d_1] \Vdash \varphi$ for some d_2 such that $d_0 < d_2$;
- $M, [d_0, d_1] \Vdash \langle D\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_0 < d_2$ and $d_3 < d_1$;
- $M, [d_0, d_1] \Vdash \langle O\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_0 < d_2 < d_1 < d_3$;
- $M, [d_0, d_1] \Vdash \langle \overline{A}\rangle\varphi$ iff $M, [d_2, d_0] \Vdash \varphi$ for some d_2;
- $M, [d_0, d_1] \Vdash \langle \overline{L}\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_3 < d_0$;
- $M, [d_0, d_1] \Vdash \langle \overline{B}\rangle\varphi$ iff $M, [d_0, d_2] \Vdash \varphi$ for some d_2 such that $d_2 > d_1$;
- $M, [d_0, d_1] \Vdash \langle \overline{E}\rangle\varphi$ iff $M, [d_2, d_1] \Vdash \varphi$ for some d_2 such that $d_2 < d_0$;
- $M, [d_0, d_1] \Vdash \langle \overline{D}\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_2 < d_0$ and $d_1 < d_3$;
- $M, [d_0, d_1] \Vdash \langle \overline{O}\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_2 < d_0 < d_3 < d_1$.

In general, it holds that $M, [a, b] \Vdash \langle X\rangle\psi$ iff there exists an interval $[c, d]$ in M such that $[a, b]\ R_X\ [c, d]$, and $M, [c, d] \Vdash \psi$.

For each of the above-defined existential HS modal operators, the corresponding universal modality is defined as a usual, e.g., $[A]\varphi$ is defined as $\neg\langle A\rangle\neg\varphi$. Moreover, in the non-strict semantics, it makes sense to consider the additional *modal constant for point intervals*, denoted π, interpreted as follows:

- $M, [d_0, d_1] \Vdash \pi$ iff $d_0 = d_1$.

The *truth of a formula over a given interval* $[a, b]$ *in an interval model* M is defined by structural induction on formulae:

- $M, [a, b] \Vdash p$ iff $p \in V([a, b])$, for all $p \in \mathcal{AP}$;
- $M, [a, b] \Vdash \pi$ iff $a = b$;
- $M, [a, b] \Vdash \neg\psi$ iff it is not the case that $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \varphi \vee \psi$ iff $M, [a, b] \Vdash \varphi$ or $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \langle X\rangle\psi$ iff there exists an interval $[c, d]$ in M such that $[a, b]\ R_X\ [c, d]$, and $M, [c, d] \Vdash \psi$, for every $X \in \{A, L, B, E, D, O, \overline{A}, \overline{L}, \overline{B}, \overline{E}, \overline{D}, \overline{O}\}$.

Interval models can be classified depending on the kind of linear orders they are built on. For the sake of brevity, for a given class of linear orders $\mathcal{C}$, we identify the class of interval models over linear orders in $\mathcal{C}$ with $\mathcal{C}$ itself. Thus, we will use, for example, the expression "formulae of HS are interpreted over the class $\mathcal{C}$ of linear orders". In particular, we mention the following important classes of linear orders:

- the class of all linear orders Lin;

- the class of all *dense* linear orders Den (e.g., $\mathbb{Q}$ and $\mathbb{R}$);
- the class of all *discrete* linear orders Dis (e.g., $\mathbb{N}$ and $\mathbb{Z}$) ;
- the class of all finite linear orders Fin.

All the classes of linear orders we consider are (left/right) *symmetric*, that is, if a class $\mathcal{C}$ contains a linear order $\mathbb{D} = \langle D, < \rangle$, then it also contains its dual linear order $\mathbb{D}^d = \langle D, <^d \rangle$, where it holds $x <^d y$ if and only if $y < x$, for all x, y . Consequently, definabilities transfer between symmetric modal operators of HS. This will be made precise in the next chapter.

Validity and *satisfiability* are defined as usual, that is, a formula φ of HS is *satisfiable* over a class $\mathcal{C}$ of linear orders, if there exists an interval model M belonging to $\mathcal{C}$ and an interval $[a, b]$ such that φ is true on $[a, b]$, i.e., $M, [a, b] \Vdash \varphi$; analogously, φ is *valid* over a class $\mathcal{C}$ of linear orders, denoted by $\models_{\mathcal{C}} \varphi$, if it is true on every interval in every interval model belonging to $\mathcal{C}$. Two formulae φ and ψ are *equivalent* relative to the class $\mathcal{C}$ of linear orders, denoted by $\varphi \equiv_{\mathcal{C}} \psi$, if $\models_{\mathcal{C}} \varphi \leftrightarrow \psi$. When the class of linear orders is clear from the context, we omit it, thus simply writing, e.g., $\varphi \equiv \psi$ instead of $\varphi \equiv_{\mathcal{C}} \psi$.

2.5 Fragments of HS

While HS features the whole set of modal operators listed in 2.1, its fragments feature (strict) subsets of it. With every subset $\mathcal{X} = \{\langle X_1 \rangle, \ldots, \langle X_k \rangle\}$ of modal operators of HS, we associate the *fragment* $\mathcal{F}_{\mathcal{X}}$ of HS denoted $\mathsf{X_1 X_2 \ldots X_k}$, with formulae built on the same set of propositional letters $\mathcal{AP}$, but only using modal operators from $\mathcal{X}$. For example, $\mathsf{A\overline{A}}$ denotes the fragment involving the modalities $\langle A \rangle$ and $\langle \overline{A} \rangle$ only. For any given fragment $\mathcal{F} = \mathsf{X_1 X_2 \ldots X_k}$ and a modal operator $\langle X \rangle$, we write $\langle X \rangle \in \mathcal{F}$ if $\langle X \rangle \in \{\langle X_1 \rangle, \ldots, \langle X_k \rangle\}$. For any given pair of fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X \rangle \in \mathcal{F}_1$ implies $\langle X \rangle \in \mathcal{F}_2$, for every modal operator $\langle X \rangle$.

3

Expressiveness of HS fragments

The analysis of the expressiveness of of interval logics has been an active research direction. In particular, the natural and important problems arise to identify the definabilities between the modal operators of the logic HS and to classify the fragments of HS with respect to their expressiveness. We will introduce and discuss these problems here.

3.1 Inter-definabilities among HS modal operators

Some of the HS modal operators are definable in terms of others and thus it is possible, for every HS modal operator, to identify the minimal fragments that are expressive enough to define it, in each of the two considered (strict and non-strict) semantics. For instance:

- In the strict semantics, the six modalities $\langle A\rangle$, $\langle B\rangle$, $\langle E\rangle$, $\langle \overline{A}\rangle$, $\langle \overline{B}\rangle$, $\langle \overline{E}\rangle$ suffice to express all the others, as shown by the following equalities [12]:

$$
\begin{aligned}
\langle L\rangle\varphi &\equiv \langle A\rangle\langle A\rangle\varphi, & \langle \overline{L}\rangle\varphi &\equiv \langle \overline{A}\rangle\langle \overline{A}\rangle\varphi,\\
\langle D\rangle\varphi &\equiv \langle B\rangle\langle E\rangle\varphi, & \langle \overline{D}\rangle\varphi &\equiv \langle \overline{B}\rangle\langle \overline{E}\rangle\varphi,\\
\langle O\rangle\varphi &\equiv \langle E\rangle\langle \overline{B}\rangle\varphi, & \langle \overline{O}\rangle\varphi &\equiv \langle B\rangle\langle \overline{E}\rangle\varphi.
\end{aligned}
$$

- In the non-strict semantics, the four modalities $\langle B\rangle$, $\langle E\rangle$, $\langle \overline{B}\rangle$, $\langle \overline{E}\rangle$ suffice to express all the others, as shown by the following equalities [18]:

$$
\begin{aligned}
\langle A\rangle\varphi &\equiv ([E]\bot \wedge (\varphi \vee \langle \overline{B}\rangle\varphi)) \vee \langle E\rangle([E]\bot \wedge (\varphi \vee \langle \overline{B}\rangle\varphi)),\\
\langle \overline{A}\rangle\varphi &\equiv ([B]\bot \wedge (\varphi \vee \langle \overline{E}\rangle\varphi)) \vee \langle B\rangle([B]\bot \wedge (\varphi \vee \langle \overline{E}\rangle\varphi)),\\
\langle L\rangle\varphi &\equiv \langle A\rangle(\langle E\rangle\top \wedge \langle A\rangle\varphi),\\
\langle \overline{L}\rangle\varphi &\equiv \langle \overline{A}\rangle(\langle B\rangle\top \wedge \langle \overline{A}\rangle\varphi),\\
\langle D\rangle\varphi &\equiv \langle B\rangle\langle E\rangle\varphi,\\
\langle \overline{D}\rangle\varphi &\equiv \langle \overline{B}\rangle\langle \overline{E}\rangle\varphi,\\
\langle O\rangle\varphi &\equiv \langle E\rangle(\langle E\rangle\top \wedge \langle \overline{B}\rangle\varphi),\\
\langle \overline{O}\rangle\varphi &\equiv \langle B\rangle(\langle B\rangle\top \wedge \langle \overline{E}\rangle\varphi).
\end{aligned}
$$

Also, the modal constant π is definable both in terms of $\langle B\rangle$ and $\langle E\rangle$, respectively as $[B]\bot$ and $[E]\bot$. Indeed, $\pi \equiv [B]\bot$ (resp., $\pi \equiv [E]\bot$) holds, because point-intervals are exactly the intervals for which it (vacuously) holds $[B]\bot$ (resp., $[E]\bot$), i.e, every prefix (resp., suffix) satisfies $\bot$.

3.2 Definability and expressiveness

Now, we introduce some formal notions used for comparing the expressiveness of logical languages, adapted to fragments of HS. The following definition formalizes the notion of definability of a modal operator in terms of others.

Definition 3. *A modal operator $\langle X\rangle$ of* HS *is* definable *in an* HS *fragment $\mathcal{F}$ relative to a class $\mathcal{C}$ of linear orders, denoted $\langle X\rangle \lhd_{\mathcal{C}} \mathcal{F}$, if $\langle X\rangle p \equiv_{\mathcal{C}} \psi$ for some formula ψ of $\mathcal{F}$, for any propositional variable $p \in \mathcal{AP}$. In such a case, the equivalence $\langle X\rangle p \equiv_{\mathcal{C}} \psi$ is called an* inter-definability equation *(or, simply,* inter-definability *or* definability*) of $\langle X\rangle$ in $\mathcal{F}$ relative to $\mathcal{C}$. We write $\langle X\rangle \not\lhd_{\mathcal{C}} \mathcal{F}$ if $\langle X\rangle$ is not definable in $\mathcal{F}$ relative to $\mathcal{C}$.*

Notice that smaller classes of linear orders inherit the inter-definabilities holding for larger classes of linear orders. Formally, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are classes of linear orders such that $\mathcal{C}_1 \subset \mathcal{C}_2$, then all definabilities holding for $\mathcal{C}_2$ are also valid for $\mathcal{C}_1$. However, more definabilities can possibly hold for $\mathcal{C}_1$. On the other hand, undefinability results for $\mathcal{C}_1$ hold also for $\mathcal{C}_2$.

Let $\mathcal{C}$ be a class of linear orders, and $\mathcal{F}_1$ and $\mathcal{F}_2$ be a pair of fragments of HS. We say that:

- $\mathcal{F}_2$ is *at least as expressive as* $\mathcal{F}_1$ over $\mathcal{C}$, denoted $\mathcal{F}_1 \preceq_{\mathcal{C}} \mathcal{F}_2$, if $\langle X\rangle \lhd_{\mathcal{C}} \mathcal{F}_2$, for every modal operator $\langle X\rangle$ in $\mathcal{F}_1$.

- $\mathcal{F}_1$ is *strictly less expressive* than $\mathcal{F}_2$ over $\mathcal{C}$, denoted $\mathcal{F}_1 \prec_{\mathcal{C}} \mathcal{F}_2$, if $\mathcal{F}_1 \preceq_{\mathcal{C}} \mathcal{F}_2$ but not $\mathcal{F}_2 \preceq_{\mathcal{C}} \mathcal{F}_1$.

- $\mathcal{F}_1$ and $\mathcal{F}_2$ are *equally expressive* (or, *expressively equivalent*) over $\mathcal{C}$, denoted $\mathcal{F}_1 \equiv_{\mathcal{C}} \mathcal{F}_2$, if $\mathcal{F}_1 \preceq_{\mathcal{C}} \mathcal{F}_2$ and $\mathcal{F}_2 \preceq_{\mathcal{C}} \mathcal{F}_1$.

- $\mathcal{F}_1$ and $\mathcal{F}_2$ are *expressively incomparable* over $\mathcal{C}$, denoted $\mathcal{F}_1 \not\equiv_{\mathcal{C}} \mathcal{F}_2$, if neither $\mathcal{F}_1 \preceq_{\mathcal{C}} \mathcal{F}_2$ nor $\mathcal{F}_2 \preceq_{\mathcal{C}} \mathcal{F}_1$.

In order to show that $\mathcal{F}_1 \preceq_{\mathcal{C}} \mathcal{F}_2$, it suffices to prove that every modal operator of $\mathcal{F}_1$ is definable in $\mathcal{F}_2$, while in order to show that $\mathcal{F}_1 \not\preceq_{\mathcal{C}} \mathcal{F}_2$, we must show that some modal operator in $\mathcal{F}_1$ is not definable in $\mathcal{F}_2$.

Now, we define the notion of optimal definability, relative to a class $\mathcal{C}$ of linear orders, as follows.

Definition 4. *We say that a definability $\langle X\rangle \lhd_{\mathcal{C}} \mathcal{F}$ is* optimal *if $\langle X\rangle \not\lhd_{\mathcal{C}} \mathcal{F}'$ for any fragment $\mathcal{F}'$ such that $\mathcal{F}' \prec_{\mathcal{C}} \mathcal{F}$; a set of definabilities is* optimal *if it consists of optimal definabilities.*

As usual, we omit the subscript for the class of linear orders when it is clear from the context, thus simply writing, e.g., $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\langle X\rangle \lhd \mathcal{F}$ in place of $\mathcal{F}_1 \preceq_{\mathcal{C}} \mathcal{F}_2$ and $\langle X\rangle \lhd_{\mathcal{C}} \mathcal{F}$, respectively.

3.3 Expressiveness classification of the fragments of HS

From now on, we only consider strict intervals and thus we omit the superscript $^-$ (e.g., we simply denote by $\mathbb{I}(\mathbb{D})$, rather than $\mathbb{I}(\mathbb{D})^-$, the set of strict, i.e., non-point intervals in $\mathbb{D}$). Consequently, from now on we only talk about strict semantics, strict interval structures (simply denoted by $\langle \mathbb{D}, \mathbb{I}(\mathbb{D})\rangle$), and strict interval models (simply denoted by $M = \langle \mathbb{I}(\mathbb{D}, V\rangle)$.

As we already pointed out, every subset of set of the 12 modal operators corresponding to Allen's relations gives rise to a logic, namely, a fragment of HS. There are 2^{12} such fragments. Due to possible inter-definabilities among modal operators, not all these fragments are expressively different. Here, we provide the complete set of optimal inter-definabilities between HS modal operators over the class of all linear orders and all dense linear orders. In this way, given two HS fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we are able to decide how they relate to each other with respect to expressiveness (that is, whether $\mathcal{F}_1$ is strictly less expressive than $\mathcal{F}_2$, $\mathcal{F}_1$ is strictly more expressive than $\mathcal{F}_2$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are expressively equivalent, or $\mathcal{F}_1$ and $\mathcal{F}_2$ are incomparable).

$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$	$\langle L\rangle \lhd \mathsf{A}$
$\langle \overline{L}\rangle p \equiv \langle \overline{A}\rangle\langle \overline{A}\rangle p$	$\langle \overline{L}\rangle \lhd \overline{\mathsf{A}}$
$\langle O\rangle p \equiv \langle E\rangle\langle \overline{B}\rangle p$	$\langle O\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{O}\rangle p \equiv \langle B\rangle\langle \overline{E}\rangle p$	$\langle \overline{O}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$
$\langle D\rangle p \equiv \langle E\rangle\langle B\rangle p$	$\langle D\rangle \lhd \mathsf{BE}$
$\langle \overline{D}\rangle p \equiv \langle \overline{E}\rangle\langle \overline{B}\rangle p$	$\langle \overline{D}\rangle \lhd \overline{\mathsf{B}}\overline{\mathsf{E}}$
$\langle L\rangle p \equiv \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{L}\rangle p \equiv \langle \overline{E}\rangle[B]\langle \overline{E}\rangle\langle B\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$

Table 3.1: The complete set of optimal definabilities for the class of all linear orders Lin.

$\langle A\rangle p \quad \equiv \varphi(p) \vee \langle E\rangle \varphi(p)$	$\langle A\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
where $\varphi(p) := [E]\bot \wedge \langle \overline{B}\rangle([E][E]\bot \wedge \langle E\rangle(p \vee \langle \overline{B}\rangle p))$	
$\langle \overline{A}\rangle p \quad \equiv \overline{\varphi}(p) \vee \langle B\rangle \overline{\varphi}(p)$	$\langle \overline{A}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$
where $\overline{\varphi}(p) := [B]\bot \wedge \langle \overline{E}\rangle([B][B]\bot \wedge \langle B\rangle(p \vee \langle \overline{E}\rangle p))$	
$\langle L\rangle p \quad \equiv \langle O\rangle(\langle O\rangle\top \wedge [O]\langle D\rangle\langle O\rangle p)$	$\langle L\rangle \lhd \mathsf{DO}$
$\langle \overline{L}\rangle p \quad \equiv \langle \overline{O}\rangle(\langle \overline{O}\rangle\top \wedge [\overline{O}]\langle D\rangle\langle \overline{O}\rangle p)$	$\langle \overline{L}\rangle \lhd \mathsf{D}\overline{\mathsf{O}}$
$\langle L\rangle p \quad \equiv \langle \overline{B}\rangle[D]\langle \overline{B}\rangle\langle D\rangle\langle \overline{B}\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{D}$
$\langle \overline{L}\rangle p \quad \equiv \langle \overline{E}\rangle[D]\langle \overline{E}\rangle\langle D\rangle\langle \overline{E}\rangle p$	$\langle \overline{L}\rangle \lhd \overline{\mathsf{E}}\mathsf{D}$
$\langle L\rangle p \quad \equiv \langle O\rangle[E]\langle O\rangle\langle O\rangle p$	$\langle L\rangle \lhd \mathsf{EO}$
$\langle \overline{L}\rangle p \quad \equiv \langle \overline{O}\rangle[B]\langle \overline{O}\rangle\langle \overline{O}\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{O}}$
$\langle L\rangle p \quad \equiv \langle O\rangle(\langle O\rangle\top \wedge [O]\langle B\rangle\langle O\rangle\langle O\rangle p)$	$\langle L\rangle \lhd \mathsf{BO}$
$\langle \overline{L}\rangle p \quad \equiv \langle \overline{O}\rangle(\langle \overline{O}\rangle\top \wedge [\overline{O}]\langle E\rangle\langle \overline{O}\rangle\langle \overline{O}\rangle p)$	$\langle \overline{L}\rangle \lhd \mathsf{E}\overline{\mathsf{O}}$
$\langle L\rangle p \quad \equiv \langle O\rangle[O][\overline{L}]\langle O\rangle\langle O\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{L}}\mathsf{O}$
$\langle \overline{L}\rangle p \quad \equiv \langle \overline{O}\rangle[\overline{O}][\overline{L}]\langle \overline{O}\rangle\langle \overline{O}\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{L}\overline{\mathsf{O}}$

Table 3.2: More definabilities for $\langle A\rangle$ and $\langle \overline{A}\rangle$ over the class of all discrete linear orders Dis (top) and for $\langle L\rangle$ and $\langle \overline{L}\rangle$ over the class of all dense linear order Den (bottom).

Table 3.1 collects the definabilities that hold in every linear order. Most of them are known from the seminal work of Halpern and Shoham [12], while the definability $\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ and its symmetric one, $\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$ are first obtained in [9]. Moreover, in [9], it was proved that the set of definabilities given in Table 3.1 is sound, complete, and optimal for the class of all linear order Lin. It is important to highlight that the completeness of the set of definabilities in Table 3.1 does not necessarily hold any longer if the semantics is restricted to a specific class of linear orders. Indeed, while validity of definabilities of Table 3.1 transfers from the class of all linear orders Lin to both the class of all discrete linear orders Dis and the one of all dense linear orders Den, the definabilities in Table 3.2 only hold for the class Dis (top) or Den (bottom), but not for Lin. In [1], it was proved that definabilities in Table 3.1 and in the bottom of Table 3.2 constitutes the complete set of optimal definabilities over Den. The problem of identifying the complete set of optimal definabilities over Dis is still open. In [2], a complete set of definabilities has been identified for the modal operators of HS corrisponding to the Allen's relations *meets* $\langle A\rangle$, *later* $\langle L\rangle$, *starts* $\langle B\rangle$, *finishes* $\langle E\rangle$, and *during* $\langle D\rangle$, as well as the ones corresponding to their inverse relations $\langle \overline{A}\rangle$, $\langle \overline{L}\rangle$, $\langle \overline{B}\rangle$, $\langle \overline{E}\rangle$, and $\langle \overline{D}\rangle$. The only missing piece of the expressiveness puzzle is that of the definibilities for the modal operator of HS corresponding to the Allen relation *overlaps* $\langle O\rangle$ (those for the inverse relation *overlapped by* $\langle \overline{O}\rangle$ would immediately follow by symmetry). This thesis is devoted to closing this gap. We claim that definabilities in Table 3.1 and in the top of Table 3.2 constitutes the complete set of optimal definabilities over Dis, and, in the next chapter, we give strong evidence and a partial formal proof to support such a claim.

Whereas proving the *soundeness* of a set of optimal definabilities, i.e., showing that each definability holds, relative to a class of linear orders, is quite straightforward, proving the *completeness* is the hard task; *optimality* is established together with it. Intuitively, to prove the completeness, we identify a set D of optimal definabilities and then we show that no more optimal definabilities exist. To this end, for each HS modal operator $\langle X\rangle$, we show that $\langle X\rangle$ is not definable in any fragment of HS that does not contain $\langle X\rangle$ and does not contain as definable (according to definabilities in D) all operators of some of the fragments in which $\langle X\rangle$ is definable (again, according to D). This process will be made more precise later, after the introduction of the crucial notion of bisimulation.

3.4 Expressing properties in HS and its fragments

In this section, we illustrate a meaningful temporal property, *counting*, which can be expressed in HS and some of its fragments, when they are interpreted over discrete linear orders (Table 3.3). This gives a hint about the expressiveness capabilities of HS modal operators. For instance, in some fragments, assuming discreteness suffices to constrain the length of intervals (this is the case with the fragments E and B); other fragments rely on additional assumptions (this is the case with the fragment DO, which requires the linear orders to be right-unbounded). This gives further evidence of how expressiveness results can be affected by the features of the linear orders.

3.4.1 Counting property

When the linear order is assumed to be discrete, some HS fragments are powerful enough to constrain the *length* of an interval, that is, the number of its points minus one. Let $\sim\in \{<,\leq,=,\geq,>\}$. For every $k \in \mathbb{N}$, we define $\ell_{\sim k}$ as an atomic proposition which is true over all and only those intervals whose length is $\sim$-related to k. Moreover, for a modal operator $\langle X\rangle$, we denote by $\langle X\rangle^k\varphi$ the formula $\langle X\rangle \ldots \langle X\rangle\varphi$, with k occurences of $\langle X\rangle$ before φ. It is well known that the fragments E and B can express $\ell_{\sim k}$, for every natural number k and every $\sim\in \{<,\leq,=,\geq,>\}$ (see, e.g., [12]). As an example, the formulae $\langle B\rangle^k\top$ and $[B]^k\bot$ are equivalent to $\ell_{>k}$ and $\ell_{\leq k}$, respectively. The fragment D features limited counting properties, as, for every k, $\langle D\rangle^k\top \wedge [D]^{k+1}\bot$ is true over intervals whose length is either $2\cdot k+1$ or $2\cdot(k+1)$. Notice that $[D]\bot$ is true over intervals whose length is either 1 or 2. The counting capabilities of the fragment O are limited as well: it allows one to discriminate between *unit intervals* (intervals whose length is 1) and *non-unit intervals* (which are longer than 1), with the constraint that the underlying linear order is right-unbounded, like $\mathbb{Z}$ or $\mathbb{N}$ (fragment $\overline{\mathsf{O}}$ owns the same capability, with the constraint that the underlying linear order has to be left-unbounded, like $\mathbb{Z}$ or the linear order of the negative integers). In this thesis, we let $\mathbb{Z}^{\sim c} = \{x \in \mathbb{Z} \mid x \sim c\}$, for each $\sim\in \{\leq,<,=,>,\geq\}$ and $c \in \mathbb{Z}$. Moreover, $\mathbb{Z}^+$ and $\mathbb{Z}^-$ denote the sets $\mathbb{Z}^{>0}$ and $\mathbb{Z}^{<0}$, respectively.

Counting properties		
$\ell_{>k}$	$\equiv \langle B\rangle^k\top$	
$\ell_{=k}$	$\equiv \langle B\rangle^{k-1}\top \wedge [B]^k\bot$	
$\ell_{>2\cdot k}$	$\equiv \langle D\rangle^k\top$	
$\ell_{\leq 2\cdot k}$	$\equiv [D]^k\bot$	
$\ell_{>1}$	$\equiv \langle O\rangle\top$	*only on right-unbounded domains*
$\ell_{>2\cdot k+1}$	$\equiv \langle D\rangle^k\langle O\rangle\top$	*only on right-unbounded domains*
$\ell_{=2\cdot(k+1)}$	$\equiv \langle D\rangle^k\langle O\rangle\top \wedge [D]^{k+1}\bot$	*only on right-unbounded domains*

Table 3.3: Expressiveness of HS modal operators over discrete linear orders.

3.5 Proof techniques to disprove definability

In order to show non-definability of a given modal operator of HS in a certain fragment, the notion of bisimulation (or its cousin N-bisimulation) [11, 13, 14] can be used.

First of all, we give formal definitions, then we explain how they will be used.

Definition 5 (N-bisimulations for an HS fragment $\mathcal{F}$ [11]). *Let $\mathcal{F}$ be an HS fragment. An N-bisimulations for $\mathcal{F}$ (or, simply, an $\mathcal{F}_N$-bisimulation) between two models $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V'\rangle$ over a set of proposition letters $\mathcal{AP}$ is a sequence of N relations $Z_N, \ldots, Z_1 \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ satisfying the following properties:*

- (local condition) *for every $([x, y], [x', y']) \in Z_h$, with $N \geq h \geq 1$, $M, [x, y] \Vdash p$ if and only if $M', [x', y'] \Vdash p$ for all $p \in \mathcal{AP}$;*
- (forward condition) *for every $([x, y], [x', y']) \in Z_h$, with $N \geq h > 1$, if $[x, y]R_X[v, w]$ for some $[v, w] \in \mathbb{I}(\mathbb{D})$ and some $\langle X\rangle \in \mathcal{F}$, then there exists $([v, w], [v', w']) \in Z_{h-1}$ such that $[x', y']R_X[v', w']$;*
- (backward condition) *for every $([x, y], [x', y']) \in Z_h$, with $N \geq h > 1$, if $[x', y']R_X[v', w']$ for some $[v', w'] \in \mathbb{I}(\mathbb{D}')$ and some $\langle X\rangle \in \mathcal{F}$, then there exists $([v, w], [v', w']) \in Z_{h-1}$ such that $[x, y]R_X[v, w]$.*

The property of N-bisimulations that is crucial to our purposes is the invariance of modal formulae (up to modal depth $N - 1$) over (N-)bisimilar models.

Theorem 1 (N-bisimulation invariance [11]). *Given an $\mathcal{F}_N$-bisimulation $Z_N, \ldots, Z_1$ between M and M', the truth of $\mathcal{F}$-formulae of modal depth at most $h - 1$ is invariant for pairs of intervals belonging to Z_h, for every $N \geq h \geq 1$. More precisely, for every $h \in \{1, \ldots, N\}$, $\mathcal{F}$-formula φ of modal depth at most $h - 1$, and $([x, y], [x', y']) \in Z_h$, it holds that $M, [x, y] \Vdash \varphi$ if and only if $M', [x', y'] \Vdash \varphi$.*

Thanks to this property, it is possible to prove that an HS modal operator $\langle X\rangle$ is not definable in an HS fragment $\mathcal{F}$. To this end, it suffices to provide, for every natural number N, a pair of models M and M' and an $\mathcal{F}_N$-bisimulation between them for which there exists a pair $([x, y], [x', y']) \in Z_N$ such that $M, [x, y] \Vdash \langle X\rangle p$ while $M', [x', y'] \nVdash \langle X\rangle p$, for some $p \in \mathcal{AP}$ (in this case, we say that the $\mathcal{F}_N$-bisimulation *violates* $\langle X\rangle$). Then, the desired undefinability result follows by contradiction. Let us assume that ϕ is an $\mathcal{F}$-formula of modal depth n such that $\langle X\rangle p \equiv \phi$. Since, for each N, there is an $\mathcal{F}_N$-bisimulation that violates $\langle X\rangle$, there exists, in particular, one such bisimulation for $N = n + 1$. Let $([x, y], [x', y']) \in Z_N$ be the pair of intervals that violates $\langle X\rangle$, that is, $M, [x, y] \Vdash \langle X\rangle p$ and $M', [x', y'] \nVdash \langle X\rangle p$. For the definition of $\mathcal{F}_N$-bisimulation, the truth value of ϕ over $[x, y]$ (in M) and $[x', y']$ (in M') is the same, and this is in contradiction with $\langle X\rangle p \equiv \phi$ and the fact that $M, [a, b] \Vdash \langle X\rangle p$ while $M', [a', b'] \nVdash \langle X\rangle p$.

It is worth to notice that the more standard notion of bisimulation [11] (for an HS fragment $\mathcal{F}$), that identifies a relation between intervals of two models (rather than a sequence of relations, as it is the case with N-bisimulations), guarantees a stronger properties, that is, intervals related by a bisimulation for an HS fragment $\mathcal{F}$ satisfy exactly the same $\mathcal{F}$-formulae, with no bound on their modal depth.

Bisimulations are easier to deal with than N-bisimulations, and thus it would be preferable to use former, rather than the latter, to show undefinability results. However, undefinability results obtained using bisimulations also apply to infinitary extensions of HS, like, for example, the extensions of HS that allows for infinite disjunctions and conjunctions. Since in such extensions it is possible to write formulae with unbounded modal depth, the use of bisimulation may give misleading results. Indeed, there are example of modal operators not definable in some HS fragments, but definable in its infinitary extension. In these cases, one is forced to resort to a proof via N-bisimulation. As an example, while $\langle E \rangle$ is not definable in fragment BA, it is definable in its infinitary extension, as witnessed by the following formula whose modal depth is unbounded:

$$\langle E \rangle p \equiv \bigvee_{k \in \mathbb{N}} (\ell_{=k} \wedge \bigvee_{i<k} (\langle B \rangle (\ell_{=i} \wedge \langle A \rangle (\ell_{=k-i} \wedge p))),$$

where length constraints of the forms $\ell_{=k}$, $\ell_{=i}$ and $\ell_{=k-i}$ can be expressed using $\langle B \rangle$ (see Section 3.4, Table 3.3).

In the next chapter, we claim that $\langle O \rangle$ is not definable in $\mathsf{ABD}\overline{\mathsf{ABE}}$, and we present an idea for a proof. It is not difficult to convince oneself that $\langle O \rangle$ is indeed definable in the infinitary extension of $\mathsf{ABD}\overline{\mathsf{ABE}}$, and thus a proof based on bisimulations do not exist; N-bisimulations are necessary.

4

Towards a closure of the cases $\langle O \rangle$ and $\langle \overline{O} \rangle$

In this section, we claim that the set of definabilities in Table 3.1 and the top of Table 3.2 is sound, complete, and optimal for Dis. As already pointed, complete sets of definabilities have been already identified in [2] for all modal operators but $\langle O \rangle$ and $\langle \overline{O} \rangle$, which are dealt with in this chapter.

It is known from 3.1 that $\langle O \rangle$ is definable in terms of the fragment $\overline{\mathsf{B}}\mathsf{E}$, i.e., $\langle O \rangle \lhd \overline{\mathsf{B}}\mathsf{E}$. We conjecture that this is the only optimal definability for $\langle O \rangle$ in Dis, that is, for every fragment $\mathcal{F}$, with $\langle O \rangle \notin \mathcal{F}$, if $\overline{\mathsf{B}}\mathsf{E} \not\subseteq \mathcal{F}$ then $\langle O \rangle \not\lhd F$.

A formal proof of this conjecture consists of the following steps.

1. Identify the set of fragments $\mathcal{F}_i$ such that $\langle O \rangle \lhd \mathcal{F}_i$, according to the known definabilities for the class of all discrete linear orders Dis (see Table 3.1 and the top of Table 3.2). Such set contains exactly those fragments featuring both $\langle \overline{B} \rangle$ and $\langle E \rangle$.

2. Identify the set of $\subseteq$-maximal fragments of HS that contain neither the operator $\langle O \rangle$ nor any of the fragments identified in the previous step. It is easy to see that there are only two fragments that do not contain $\langle O \rangle$ and do not contain both modal operators $\langle \overline{B} \rangle$ and $\langle E \rangle$ together, namely, $\mathsf{ALBD}\overline{\mathsf{ALBEDO}}$ and $\mathsf{ALBED}\overline{\mathsf{ALEDO}}$.

3. Provide, for each fragment $\mathcal{F} \in \{\mathsf{ALBD}\overline{\mathsf{ALBEDO}}, \mathsf{ALBED}\overline{\mathsf{ALEDO}}\}$ and each $N \in \mathbb{N}$, an $\mathcal{F}_N$-bisimulation that violates $\langle O \rangle$.

Step 3 above is the only missing piece towards a proof of the above conjecture. According to Table 3.1, it is possible to replace $\mathsf{ALBD}\overline{\mathsf{ALBEDO}}$ and $\mathsf{ALBED}\overline{\mathsf{ALEDO}}$, with the simpler, equivalent fragments $\mathsf{ABD}\overline{\mathsf{ABE}}$ and $\mathsf{ABE}\overline{\mathsf{AED}}$, respectively. In this chapter, we present a proposal for an $\mathsf{ABD}\overline{\mathsf{ABE}}_N$-bisimulation that violates $\langle O \rangle$, and we provide strong evidence to support its correctness. A complete correctness proof is very technically involved and requires dealing with a large number of cases; while we formally address most cases, few of them are left open. The correctness of our proposal would imply that $\langle O \rangle \not\lhd \mathsf{ABD}\overline{\mathsf{ABE}}$, thus proving half of the above conjecture (the missing part would be to show that $\langle O \rangle \not\lhd \mathsf{ABE}\overline{\mathsf{AED}}$). Notice that, by symmetry, from the correctness of our proposal it would also follow

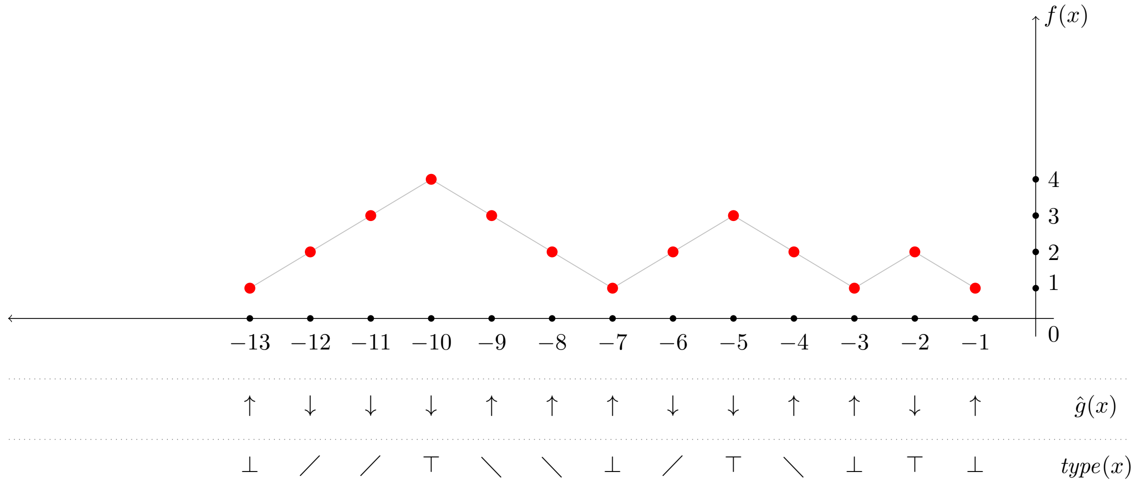


Figure 4.1: Graph of $f : \mathbb{Z}^- \to \mathbb{N} \setminus \{0\}$ and representations of $\hat{g} : \mathbb{Z}^- \to \{\uparrow, \downarrow\}$ and $type : \mathbb{Z}^- \to \{\perp, \top, \diagup, \diagdown\}$.

that $\langle \overline{O} \rangle \ntrianglelefteq$ $\mathsf{ALEDO\overline{ALBED}}$, thus making also for $\langle O \rangle$ a significant step towards the identification of the unique sound, complete, and optimal set of definabilities in Dis.

In what follows, we first define the model over which our N-bisimulation proposal is based; then, we give our proposal; finally, we give evidence supporting its correctness.

4.1 The interval model for the $\mathsf{ABD\overline{ABE}}_N$-bisimulation proposal

An N-bisimulation, for a positive natural number N, is a sequence of binary relations between two models. In our case, the two models coincide. In what follow, we define the model $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ over which our bisimulation proposal is based. To begin with, M is based on the set of integers and the corresponding ordering relation $<$, i.e., $\mathbb{D} = \mathbb{Z}$.

Its evaluation function V is defined with the help of an auxiliary surjective function $f : \mathbb{Z}^- \to \mathbb{N} \setminus \{0\}$. The following definitions are quite involved from the technical viewpoint. However, their intuitive meaning is quite easy to grasp pictorially. Thus, we provide a graphic intuition of f, and successive functions, in Figure 4.1. Formally, f uses function $\hat{g} : \mathbb{Z}^- \to \{\uparrow, \downarrow\}$ (also depicted in Figure 4.1), and is defined as follows.

Definition 6 (function f). *For every $x \in \mathbb{Z}^-$, let $K_x \geq 0$ be the unique natural number such that $K_x \cdot (K_x + 1) < -x \leq (K_x + 1) \cdot (K_x + 2)$. We define $\hat{g} : \mathbb{Z}^- \to \{\uparrow, \downarrow\}$ in the following way:*

$$\hat{g}(x) = \begin{cases} \uparrow & \text{if } -x \leq (K_x + 1)^2 \\ \downarrow & \text{otherwise,} \end{cases}$$

and $f : \mathbb{Z}^- \to \mathbb{N} \setminus \{0\}$ as:

$$f(x) = \begin{cases} 1 & \text{if } x = -1 \\ f(x+1) + 1 & \text{if } \hat{g}(x+1) = \uparrow \\ f(x+1) - 1 & \text{if } \hat{g}(x+1) = \downarrow . \end{cases}$$

Finally, we can formally define the evaluation function associated with our model as: $V(p) = \{[x, f(x)] \mid x \in \mathbb{Z}^-\}$, where f is defined in Definition 6, and p is the only proposition letter in $\mathcal{AP}$.

4.2 Our $\mathsf{ABD}\overline{\mathsf{ABE}}_N$-bisimulation proposal

In this section, we introduce some technical notions that we need to define, for every $N \in \mathbb{N} \setminus \{0\}$, an $\mathsf{ABD}\overline{\mathsf{ABE}}_N$-bisimulation violating $\langle O \rangle$. To begin with, we define function $g : \mathbb{N} \setminus \{0\} \to \mathbb{Z}^-$.

Definition 7 (function g)**.** *Function $g : \mathbb{N} \setminus \{0\} \to \mathbb{Z}^-$ is defined as follows: for every $y \in \mathbb{N} \setminus \{0\}$*

$$g(y) = max\{x \in \mathbb{Z}^- \mid f(x) = y\}.$$

Looking at figure 4.1 it should be easy to understand the meaning of g. For example, $g(1) = -1$, $g(2) = -2$, $g(3) = -5$, and $g(4) = -10$. Function g enjoys the following property, which will come in handy later.

It is easy to see that the following properties, relating f and g hold.

Proposition 1. *For every $y_1, y_2 \in \mathbb{N} \setminus \{0\}$, if $y_1 < y_2$ then $g(y_2) < g(y_1)$. Moreover, for every $x, y \in \mathbb{Z}$, if $f(x) > y$ then $x \leq g(y+1)$.*

The definition of g allows us to express a structural property of f (graphically depicted in Figure 4.2), whose importance will be clear later.

Proposition 2. *For every $y \in \mathbb{N} \setminus \{0\}$, it holds that $g(y) - g(y+1) = 2 \cdot y - 1$ (see Figure 4.2).*

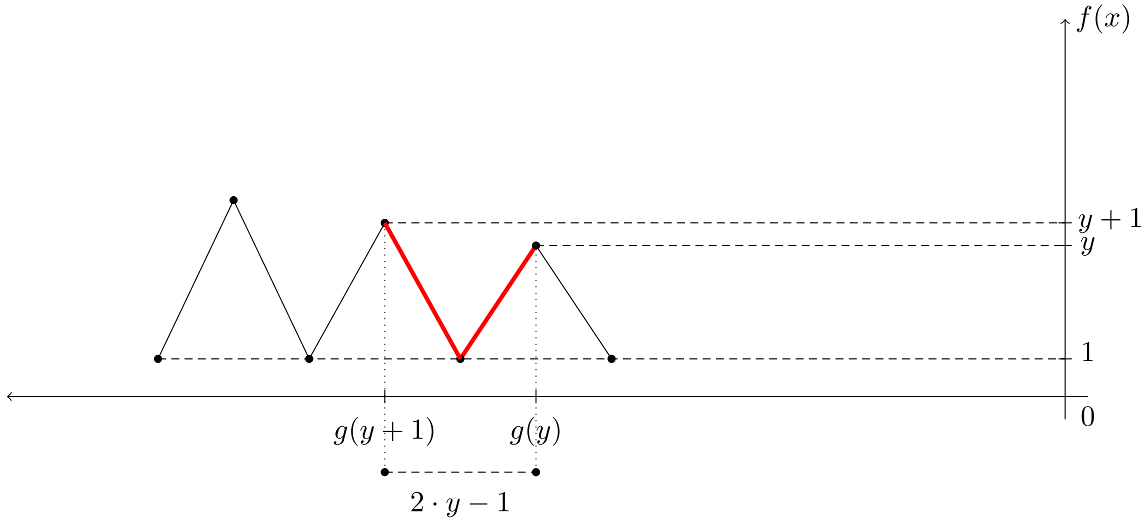


Figure 4.2: Graphical representation of the structural property of f stated in Proposition 2.

Function $type : \mathbb{Z}^- \to \{\bot, \top, \diagup, \diagdown\}$, defined in the following, partitions the domain of f according to the shape of its graph (see Figure 4.1).

Definition 8 ($type$)**.** *For every $x \in \mathbb{Z}^-$*

$$type(x) = \begin{cases} \bot & \textit{if } x = -1 \\ \bot & \textit{if } x < -1 \textit{ and } f(x) < f(x-1) = f(x+1) \\ \top & \textit{if } x < -1 \textit{ and } f(x) > f(x-1) = f(x+1) \\ \diagup & \textit{if } x < -1 \textit{ and } f(x-1) < f(x) < f(x+1) \\ \diagdown & \textit{if } x < -1 \textit{ and } f(x-1) > f(x) > f(x+1) \end{cases}$$

Let us recall that, for $N \in \mathbb{N} \setminus \{0\}$, an $\mathcal{F}_N$-bisimulation is a sequence $Z_N, \ldots, Z_1$ of N relations between intervals, guaranteeing that for every pair $([x, y], [x', y']) \in Z_h$, with $N \geq h > 1$, intervals $[x, y]$ and $[x', y']$ satisfy the same $\mathcal{F}$-formulae with modal depth up to $h - 1$. Thus, an N-bisimulation somehow defines a relation of indistinguishability between intervals, which is parametric in $h \in \{1, \ldots, N\}$. Sometimes we refer to h as the *bisimulation step*; for example, we may say that intervals $[x, y]$ and $[x', y']$ are indistinguishable by $\mathcal{F}$-formulae of modal depth up to $h - 1$ at bisimulation step h. Intuitively, it is not difficult to convince oneself that, due to the finitary nature of modal logics, two intervals that are "long enough" are indistinguishable, to some extent, by formulae, with the notion of begin "long enough" depending on the modal depth. Therefore, we need a parametric function, called *long*, allowing us to characterize such a notion at each bisimulation step $h \in \{1, \ldots, N\}$. Roughly speaking, an interval $[x, y]$ is "long enough" at bisimulation step h if it is longer than $long(h)$, i.e., $y - x > long(h)$; in this case, we say that $[x, y]$ is h-long.

Definition 9 ("*long* enough" at bisimulation step h – function *long*)**.** *We define the function long :* $\mathbb{N} \setminus \{0\} \to \mathbb{N}$ *as follows, for every* $h \in \mathbb{N} \setminus \{0\}$*:*

$$long(h) = \begin{cases} 4 & \textit{if } h = 1 \\ long(h-1) + 1 & \textit{otherwise} \end{cases}$$

In order to define our N-bisimulation, we need to define a notion of indistinguishability also among points, which will be somehow lifted to intervals. Therefore, points of the model that are "far enough" from the origin (point 0) are indistinguishable, with the notion of begin "far enough" depending again on the modal depth of formulae. Thus, we also need parametric functions to characterize the concept of being far from the origin at bisimulation step $h \in \{1, \ldots, N\}$. For technical reasons, we will use different functions for positive and negative points of the model, namely *posBig* and *negBig*, respectively. Roughly speaking, a positive integer is "far enough" from the origin at bisimulation step h if it is bigger than $posBig(h)$; similarly, a negative integer is "far enough" from the origin at bisimulation step h if its absolute value is bigger than $negBig(h)$. A (positive or negative) point that is "far enough" from the origin at bisimulation step h is said to be h-far. All the other points of the domain are considered "close to the origin" at bisimulation step h.

Definition 10 ("far enough" at bisimulation step h – functions *posBig* and *negBig*)**.** *We define functions posBig,negBig:* $\mathbb{N} \setminus \{0\} \to \mathbb{N}$*, as follows. For every* $h \in \mathbb{N} \setminus \{0\}$

$$posBig(h) = \begin{cases} 5 & \textit{if } h = 1 \\ posBig(h-1) + 2 \cdot long(h-1) + 4 & \textit{otherwise} \end{cases}$$

and

$$negBig(h) = -g(posBig(h))$$

Functions *negBig*, *posBig*, along with the above defined function *type*, are used to characterize the following notion of *equivalence between two points at bisimulation step h.*

Definition 11 (point equivalence $\simeq_h$)**.** *For every $h \in \mathbb{N} \setminus \{0\}$, we define an equivalence relation parameterized by h, denoted $x \simeq_h y$ as follows. For every pair of integer $x, y \in \mathbb{Z}$, $x \simeq_h y$ if and only if one of the following holds:*

- $x = y$;
- $x, y > 0$ *and both x and y are h-far, i.e., $x, y > posBig(h)$;*
- $x, y < 0$, *both x and y are h-far, i.e., $x, y < -negBig(h)$, $f(x) \simeq_h f(y)$, and if $f(x) \leq posBig(h)$ then $type(x) = type(y)$.*

It is possible to observe that functions *negBig*, *posBig*, and *long* satisfy the following property: $negBig(1) = -g(posBig(1)) > posBig(1) = 5 > 4 = long(1)$. As a matter of fact, the following general property holds.

Proposition 3. *For every $h \in \mathbb{N} \setminus \{0\}$, it holds that $negBig(h) > posBig(h) > long(h)$.*

Notice also that, from Proposition 2, the following holds for every $h \in \mathbb{N} \setminus \{0\}$:

$$-negBig(h) - g(posBig(h) + 1) = 2 \cdot posBig(h) - 1.$$

This, in turn, implies the following proposition.

Proposition 4. *For every $h \in \mathbb{N} \setminus \{0\}$, with $h > 1$, it holds that $negBig(h) - negBig(h-1) > posBig(h)$.*

We are now ready to formally define, for every $N \in \mathbb{N} \setminus \{0\}$, our $\mathsf{ABD\overline{ABE}}_N$-bisimulation proposal. To this end, we define a sequence $Z_N, \ldots, Z_1$ of binary relations over M.

Definition 12. *For each $N \in \mathbb{N} \setminus \{0\}$ we define a sequence of N relations $Z_N, \ldots, Z_1$ as follows. For every $h \in \{1, \ldots, N\}$, we have that $[x, y] Z_h [w, z]$ if and only if all of the following conditions hold:*

1. $x \simeq_h w$ *and* $y \simeq_h z$;
2. *either $y - x = z - w$ or they are both h-long, that is $y - x > long(h)$ and $z - w > long(h)$;*
3. *if $x < 0$ and $y > 0$, then one of the following holds:*
   (a) $|f(x) - y| \leq long(h)$ *and* $f(x) - y = f(w) - z$;
   (b) $f(x) - y > long(h)$ *and* $f(w) - z > long(h)$;
   (c) $f(x) - y < -long(h)$ *and* $f(w) - z < -long(h)$.

4.3 Correctness of the $\mathsf{ABD\overline{A}\overline{B}E}_N$-bisimulation proposal

Let N be a positive natural number. Our conjecture is that the sequence of relations $Z_N,\dots,Z_1$ defined in Definition 12, is an $\mathsf{ABD\overline{A}\overline{B}E}_N$-bisimulation over M (defined in Section 4.1) that violates $\langle O\rangle$. We first show that the sequence does violate $\langle O\rangle$, i.e., there exist two $\mathsf{ABD\overline{A}\overline{B}E}_N$-bisimilar intervals that can be distinguished by modal operator $\langle O\rangle$, as formally stated in the following theorem. Next, we give evidence that the sequence is indeed an $\mathsf{ABD\overline{A}\overline{B}E}_N$-bisimulation.

Theorem 2. *For each $N\in\mathbb{N}\backslash\{0\}$, the sequence of relations $Z_N,\dots,Z_1$, defined in Definition 12, violates $\langle O\rangle$, i.e., there exist $[x,y]$ and $[w,z]$, with $[x,y]Z_N[w,z]$, such that $M,[x,y]\Vdash\langle O\rangle p$ and $M,[w,z]\nVdash\langle O\rangle p$.*

Proof. Let $N\in\mathbb{N}\setminus\{0\}$ and let us consider intervals $[g(posBig(N+2)),posBig(N+2)]$ and $[x,f(x)]$, for some x such that $f(x)=posBig(N+1)$ and $x<g(posBig(N+2))$ (it is not difficult to see that such a point x exists). Notice that $f(g(posBig(N+2)))=posBig(N+2)$. We show that $M,[x,f(x)]\Vdash\langle O\rangle p$, $M,[g(posBig(N+2)),posBig(N+2)]\nVdash\langle O\rangle p$, and $[x,f(x)]Z_N[g(posBig(N+2)),posBig(N+2)]$. It is immediate to check that $M,[x,f(x)]\Vdash\langle O\rangle p$ (as $x<g(posBig(N+2))<posBig(N+1)<posBig(N+2)$ and $M,[g(posBig(N+2)),posBig(N+2)]\Vdash p$). In order to verify that $M,[g(posBig(N+2)),posBig(N+2)]\nVdash\langle O\rangle p$, it suffices to observe that, by Proposition 1, $g(y)<g(posBig(N+2))$ for all $y>posBig(N+2)$, which means that there is no point x such that $g(posBig(N+2))<x<posBig(N+2)$ and $f(x)>posBig(N+2)$ (recall that, according to the definition of M in Section 4.1, the only intervals satisfying p are those of the form $[x,f(x)]$ for some x). Finally, in order to show that $[x,f(x)]Z_N[g(posBig(N+2)),posBig(N+2)]$, we need to prove that the three conditions of Definition 12 hold true.

1. The first conditions of Definition 12 requires that $x\simeq_N g(posBig(N+2))$ and $f(x)\simeq_N posBig(N+2)$. First, it is easy to see that $f(x)\simeq_N posBig(N+2)$ holds, as $f(x)=posBig(N+1)$ and both $posBig(N+2)$ and $posBig(N+1)$ are larger than $posBig(N)$ (cf. Definition 11). Since $x<g(posBig(N+2))<0$, to show that $x\simeq_N g(posBig(N+2))$ holds, we need to verify the following conditions:

   - both x and $g(posBig(N+2))$ are smaller than $-negBig(N)$, which follows from the facts that $x<g(posBig(N+2))$ and, by Proposition 1, $g(posBig(N+2))<g(posBig(N))=-negBig(N)$;
   - $f(x)\simeq_N f(g(posBig(N+2)))$, which follows from $f(g(posBig(N+2)))=posBig(N+2)$ (we have shown above that $f(x)\simeq_N posBig(N+2)$ holds); and
   - $f(x)>posBig(N)$, which follows from $f(x)=posBig(N+1)>posBig(N)$.

2. The second conditions of Definition 12 requires that $f(x)-x>long(N)$ and $posBig(N+2)-g(posBig(N+2))>long(N)$. Both properties immediately follow from the observation that $f(x)=posBig(N+1)>long(N)$ and $posBig(N+2)>long(N)$ (by Proposition 3) and that $x<g(posBig(N+2))<0$ (by Definition 7).

3. The third condition of Definition 12 holds trivially since $f(g(posBig(N+2)))=posBig(N+2)$ and thus we have $f(x)-f(x)=f(g(posBig(N+2)))-posBig(N+2)=0$. □

As already pointed out, a sequence of relations $Z_N, \ldots, Z_1$, for $N \in \mathbb{N}\backslash\{0\}$, is an $\mathsf{ABD\overline{ABE}}_N$-bisimulation if it satisfies the local, forward, and backward conditions. In our case, it is not difficult to verify the local condition. Moreover, since relations $Z_N, \ldots, Z_1$ defined in Definition 12 are symmetric, the forward condition implies the backward one. Therefore, the hardest task consists in showing that forward condition is satisfied. While we claim that sequence $Z_N, \ldots, Z_1$ is an $\mathsf{ABD\overline{ABE}}_N$-bisimulation, in this work we only prove that it is a D_N-bisimulation, as formally stated in the next theorem, whose proof is presented in the appendix.

Theorem 3 (correctness)**.** *The sequence of relations $Z_N, \ldots, Z_1$ defined in Definition 12 is a D_N-bisimulation.*

The proof of Theorem 3 is quite technically involved and tedious. We are currently working on developing similar proofs for the other modal operators in $\mathsf{ABD\overline{ABE}}$, which is a time-consuming and meticulous job, towards a formal demonstration of the following claim.

Claim 1. *The set of definabilities in Table 3.1 and the top of Table 3.2 is sound, complete, and optimal for* Dis*.*

5

Conclusions

Temporal logics have been successfully used as a formalism for representing and reasoning about temporal information, both in AI as a language for encoding temporal knowledge and in theoretical computer science, e.g., as a tool for specification, formal analysis, and verification of systems. Classic temporal logics are point-based, meaning that events and properties are evaluated over points. In the last decades, temporal logics based on intervals emerged. Among them, we focus on Halpern and Shoham's Modal logic of Time Intervals (HS) [12], where events and properties are evaluated over pairs of points (rather than single points), thus giving rise to a more expressive formalism. Clearly, the drawback consists in a worsening of the computational behavior; for example, deciding satisfiability of HS formulae is undecidable. Thus, it makes sense to study expressiveness of HS fragments, in the quest for decidable and expressive ones.

HS is a modal logic, whose language features 12 modal operators, one for each of the 12 qualitatively different binary relations between intervals over linear orders, identified by Allen [3]. Therefore, each subset of modal operators gives rise to a syntactically different HS fragment. However, due to the possibility of defining modal operators in terms of others, not all HS fragments are different in terms of expressive power, and identifying the set of semantically different HS fragments, as well as obtaining a complete comparison of HS fragments with respect to their expressive power, is the goal of this dissertation. More precisely, in this thesis, we conjecture that the natural definability for the modal operator *overlaps* in terms of modal operators *finishes* and *begun-by*, already known from [12], is the only possible one over discrete linear orders.

After an overview of the field of interval temporal logics, with a focus on expressiveness results already known from the literature (Chapter 1), and the introduction of all conceptual tools (definitions, notations, ...) needed for a formal study of the expressive power of interval temporal logics (Chapter 2), we present an important classic theoretical result, known as the bisimulation invariance property for modal logics (Chapter 3), which is at the basis of undefinability results for modal logics. Our technical and original contributions are finally presented in Chapter 4, where we provide a proposal for an N-bisimulation, and give strong and convincing evidence to support its correctness. A complete formal proof is quite technically involved and tedious, and requires a meticulous case-by-case analysis. We present here a formal proof for one such cases, the one that we deemed to be the more problematic (details can be found in Appendix A).

In conclusion, the work done for this thesis paves the way towards a complete proof of the above conjecture. We are currently working to complete the proof, by addressing one by one the remaining cases. It is worth to remark that proving the conjecture would close the long-standing open issue of obtaining a complete expressiveness picture of the family of HS fragments over the class of all discrete linear orders. While complete expressiveness pictures have been obtained for other important classes of linear orders, such as the class of all [9] and the class of dense [1] linear orders, the problem is open for the classes of discrete and finite linear order.

A

Proof of correctness

Towards a proof of correctness of our D_N-bisimulation, we present the following lemmas that will be useful in the proof.

Lemma 1. *For every* $h \in \{2 \ldots, N\}$ *there exists* $x \in \mathbb{Z}$ *such that* (i) $-negBig(h) \leq x < -negBig(h-1)$, (ii) $f(x) > posBig(h-1)$, *and* (iii) *for every* $k \in \{-long(h-1)-1, \ldots, 0, \ldots, long(h-1)+1\}$ *it holds that* $posBig(h-1) < f(x)+k < posBig(h)$.

Proof. Let $h \in \{2, \ldots, N\}$. We choose x such that $x = g(\frac{posBig(h)+posBig(h-1)}{2})$ (and thus $f(x) = \frac{posBig(h)+posBig(h-1)}{2}$ – see Figure A.1). It is immediate to check that (i) $-negBig(h) < x < -negBig(h-1)$ (as it holds that $-negBig(h) = g(posBig(h)) < g(\frac{posBig(h)+posBig(h-1)}{2}) = x < g(posBig(h-1)) = -negBig(h-1)$ by Proposition 1) and (ii) $f(x) = \frac{posBig(h)+posBig(h-1)}{2} > posBig(h-1)$.

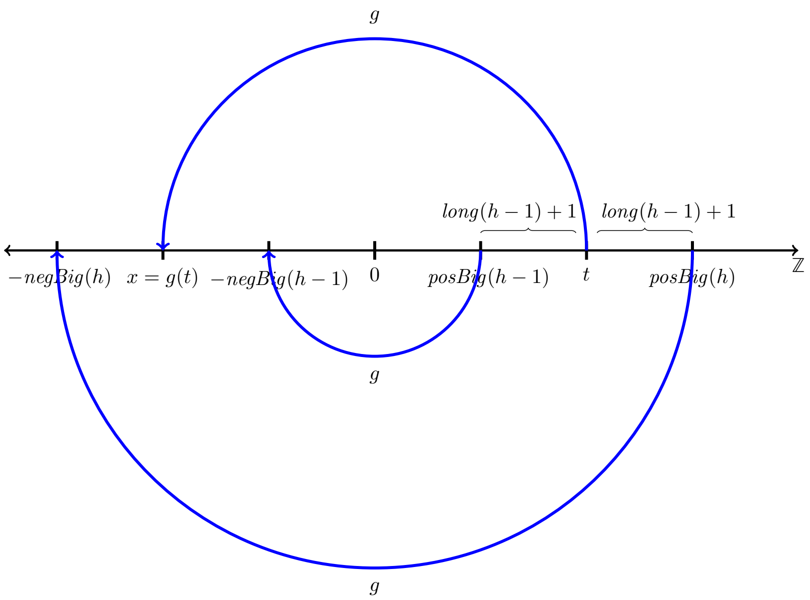
In order to prove item (iii), it is enough to show that $posBig(h-1) < f(x) - long(h-1) - 1$ and that $f(x) + long(h-1) + 1 < posBig(h)$. The first inequality follows from $f(x) = \frac{posBig(h)+posBig(h-1)}{2} = posBig(h-1) + long(h-1) + 2 > posBig(h-1) + long(h-1) + 1$. The second one follows from $posBig(h) - f(x) = posBig(h-1) + 2 \cdot long(h-1) + 4 - (posBig(h-1) + long(h-1) + 2) = long(h-1) + 2 > long(h-1) + 1$. □

Lemma 2. *For every* $y \in \mathbb{N} \setminus \{0\}$ *and* $k \in \{2, \ldots, y-1\}$, *there exist* $x_1, x_2 \in \mathbb{Z}^-$ *such that* (i) $0 < g(y) - x_1 < y - 1$ *(i.e.,* x_1 *belongs to a* left neighborhood *of* $g(y)$*) and* $0 < x_2 - g(y) < y - 1$ *(i.e.,* x_2 *belongs to a* right neighborhood *of* $g(y)$*)*, (ii) $type(x_1) = \diagup$ *and* $type(x_2) = \diagdown$, *and* (iii) $f(x_1) = f(x_2) = k$.

Proof. For every $k \in \{2, \ldots, y-1\}$, we choose $x_1 = g(y) - y + k$, $x_2 = g(y) + y - k$. Clearly, for every $k \in \{2, \ldots, y-1\}$, properties (i) and (ii) are fulfilled (see also Figure 4.2, and observe that $f(g(y)) = y$). Property (iii) follows from the observation that, for every $k \in \{2, \ldots, y-1\}$, it holds that $f(x_1) = f(g(y) - y + k) = y - y + k = k$ and $f(x_2) = f(g(y) + y - k) = y - y + k = k$. □

Corollary 2.1. *For every* $h \in \{1, \ldots, N\}$, $k \in \{2, \ldots, posBig(h)\}$, *and* $\sim \in \{\diagup, \diagdown\}$, *there exists* $x \in \mathbb{Z}$ *such that* (i) $x < -negBig(h) - long(h)$, (ii) $type(x) = \sim$, *and* (iii) $f(x) = k$.

Proof. The claim follows by Lemma 2, by instantiating y with $posBig(h)+1$. More precisely, by Lemma 2, for every $k \in \{2, \ldots, posBig(h)\}$, if $\sim = \diagup$ (resp., $\sim = \diagdown$), then there is x in the left (resp., right) neighborhood of $g(posBig(h)+1)$ such that $type(x) = \sim$ and $f(x) = k$. In order to see that property



$$\boxed{-negBig(h) = g(posBig(h))}$$

$$\boxed{-negBig(h-1) = g(posBig(h-1))}$$

Figure A.1: A graphic account of the proof of Lemma 1.

(i) is fulfilled as well, observe that, by Lemma 2, it holds that $x < g(posBig(h)+1) + posBig(h) = g(posBig(h)) - 2 \cdot posBig(h) + 1 + posBig(h) \leq -negBig(h) - long(h)$, by Propositions 2 and 3. $\square$

Corollary 2.2. *For every $h \in \{1, \ldots, N\}$, $k, k' \in \{2, \ldots, posBig(h)\}$, and $\sim, \sim' \in \{\diagup, \diagdown\}$, there exist $x_1, x_2, x_3 \in \mathbb{Z}$ such that:*

(i) $-negBig(h+1) < x_1 < x_2 < x_3 < -negBig(h) - long(h)$*;*

(ii) $x_3 - x_2 > long(h)$ *and* $x_2 - x_1 > long(h)$*;*

(iii) $type(x_1) = \sim$*,* $type(x_2) = \perp$*, and* $type(x_3) = \sim'$*;*

(iv) $f(x_1) = k$*,* $f(x_2) = 1$*, and* $f(x_3) = k'$*.*

Proof. The claim follows by Lemma 2, by instantiating y with $posBig(h)+1$. More precisely, by Lemma 2, for every $k \in \{2, \ldots, posBig(h)\}$, if $\sim = \diagup$ (resp., $\sim = \diagdown$), then there is x_1 in the left (resp., right) neighborhood of $g(posBig(h)+4)$ such that $type(x_1) = \sim$ and $f(x_1) = k$. Moreover, always by Lemma 2, for every $k' \in \{2, \ldots, posBig(h)\}$, if $\sim' = \diagup$ (resp., $\sim' = \diagdown$), then there is x_3 in the left (resp., right) neighborhood of $g(posBig(h)+1)$ such that $type(x_3) = \sim'$ and $f(x_3) = k'$. Finally, $x_2 = g(posBig(h)+3) + posBig(h)+2$ (see Figure A.3). Observe that $f(x_2) = f(g(posBig(h)+3)+posBig(h)+2) = posBig(h)+3-posBig(h)-2 = 1$,and thus $type(x_2) = \perp$. To prove that property (i) is fulfilled as well, observe that, by Lemma 2, it holds that $-negBig(h+1) < g(posBig(h)+5) = g(posBig(h)+4) - 2\cdot(posBig(h)+4)+1 < g(posBig(h)+4) - posBig(h) - 3 < x_1 < x_2$ by Propositions 2 and 3. Moreover, it holds that $x_2 < x_3 < g(posBig(h)+1) + posBig(h) = g(posBig(h)) - 2 \cdot posBig(h) + 1 + posBig(h) \leq -negBig(h) - long(h)$ by Propositions 2 and 3. In order to prove (ii), it is enough to show that for every $k \in \{2, \ldots, posBig(h)\}$, $\sim = \diagdown$, it holds that $x_2 - x_1 > long(h)$, and for every $k' \in \{2, \ldots, posBig(h)\}$, $\sim' = \diagup$, it holds that

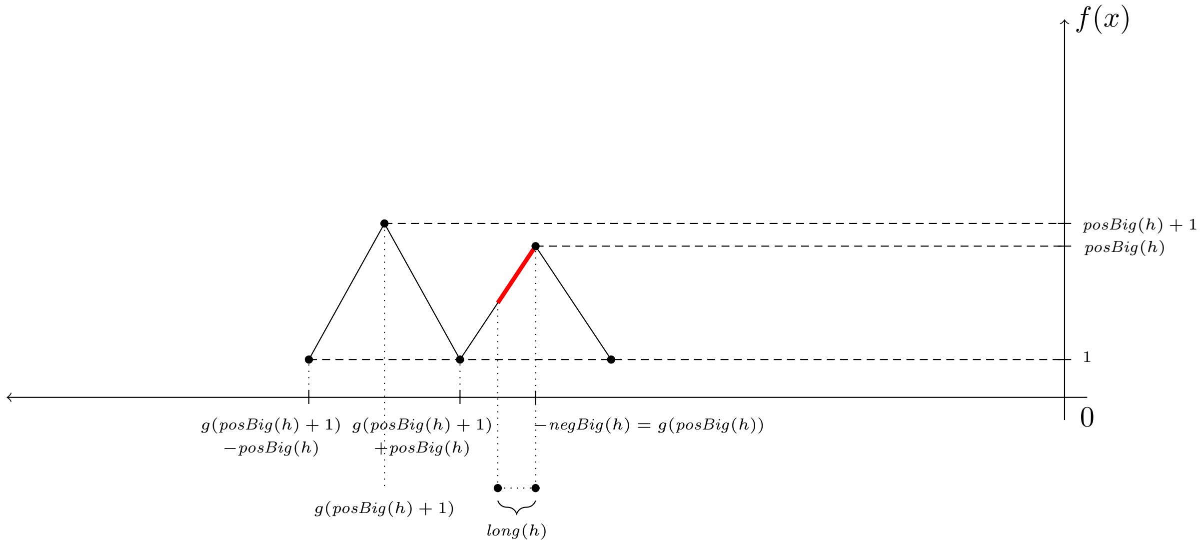


Figure A.2: A graphical account of Corollary 2.1.

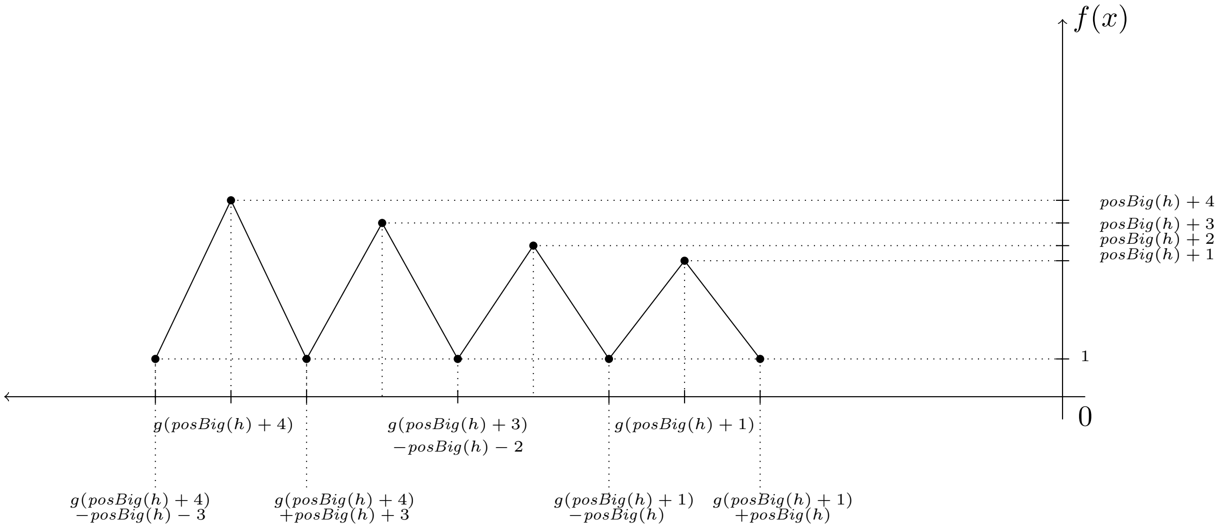


Figure A.3: A graphical account of Corollary 2.2.

$x_3 - x_2 > long(h)$. The first one follows from, $x_2 - x_1 = x_2 - g(posBig(h)+4) + posBig(h) + 4 - k > x_2 - g(posBig(h)+3) = posBig(h) + 2 > long(h)$ by Preposition 3. The second one follows from $x_3 - x_2 = g(posBig(h)+1) - posBig(h) - 1 + k' - x_2 > g(posBig(h)+2) - x_2 = posBig(h) + 1 > long(h)$ by Preposition 3.

□

Corollary 2.3. *For every $h \in \{1, \ldots, N\}$, $k \in \{posBig(h), \ldots, posBig(h+1) - 1\}$, there exists $x \in \mathbb{Z}$ such that (i) $-negBig(h+1) < x < -negBig(h) - long(h)$, (ii) $f(x) = k$.*

Proof. The claim follows by Lemma 2, by instantiating y with $posBig(h+1)$ and increasing the smallest value that k can assume. Moreover, by $k \geq posBig(h)$, we are not interested to $type(x)$. More precisely, by Lemma 2, for every $k \in \{posBig(h), \ldots, posBig(h+1) - 1\}$, there is x in the right neighborhood of $g(posBig(h+1))$ such that $f(x) = k$. It is easy convince oneself that property (i) is fulfilled as well. It holds that $-negBig(h+1) = g(posBig(h+1)) < x < g(posBig(h+1)) + posBig(h+1) - 1 =$

$g(posBig(h+1)-1)-2\cdot(posBig(h+1)-1)+1+posBig(h+1)-1 = g(posBig(h+1)-1)-posBig(h+1) < -negBig(h) - long(h)$ by Propositions 2 and 3. □

We are now ready to prove Theorem 3.

Theorem 3 (correctness). *The sequence of relations $Z_N, \ldots, Z_1$ defined in Definition 12 is a D_N-bisimulation.*

Proof. **Local and backward conditions.** To verify that the local condition is fulfilled, consider a pair of intervals such that $[x, y]\, Z_N\, [w, z]$. The following chain of equivalences holds:

$$M, [x, y] \Vdash p \Leftrightarrow y = f(x) \Leftrightarrow z = f(w) \Leftrightarrow M, [w, z] \Vdash p,$$

where the first and last equivalences hold by the definition of the valuation function of model M, while the second one holds by Definition 12, item 3a. Thus, the local condition holds.

As for the backward condition, since all relations in the sequence $Z_N, \ldots, Z_1$ are symmetric, the forward condition implies the backward one.

Forward condition. To verify that the forward condition holds, consider three intervals $[x, y]$,$[w, z]$, and $[x', y']$ such that $[x, y]\, Z_h\, [w, z]$, with $N \geq h > 1$, and $[x, y]\, R_D\, [x', y']$ (i.e., $x < x' < y' < y$). We need to exhibit an interval $[w', z']$ such that $[w, z]\, R_D\, [w', z']$ and $[x', y']\, Z_{h-1}\, [w', z']$.

For technical convenience, we distinguish three cases to be treated separately.

(*i*) $x < 0 < y$,

(*ii*) $x < y < 0$,

(*iii*) $0 < x < y$.

We provide here a full treatment of the first case. For the sake of brevity, we omit the treatment of the remaining two ones, which are easiest to deal with. Therefore, from now on, we assume $x < 0 < y$. We distinguish six cases.

1. If $-negBig(h-1) \leq x' < y' \leq posBig(h-1)$, then choose $[w', z']$=$[x', y']$. We have to show that $w < w' < z' < z$. First, let us show that $w < w'$ holds. If $w = x$, then $w = x < x' = w'$; otherwise, we have that $w, x < -negBig(h) < -negBig(h-1) \leq x' = w'$, where the first inequality follows from Definitions 11 and 12, and the facts that $x < 0$ and $[x, y]\, Z_h\, [w, z]$. Then, we show that $z < z'$ holds as well. If $z = y$, then $z' = y' < y = z$; otherwise, we know from $[x, y]\, Z_h\, [w, z]$ and from $y > 0$ that $y, z > posBig(h)$, by Definitions 11 and 12. From $y' \leq posBig(h-1)$ we have that $z, y > posBig(h) > posBig(h-1) \geq y' = z'$. Finally, it holds $w' = x' < y' = z'$.

2. If $-negBig(h-1) \leq x' \leq posBig(h-1) < y'$, then $f(x') \leq posBig(h-1)$, by Proposition 1 and Definition 10. We choose $w' = x'$ and we have to show that $w < w'$. If $w = x$, then $w = x < x' = w'$. If $w \neq x$, it holds that $w, x < -negBig(h) < -negBig(h-1) < x' = w'$. To choose z', we first assume that $y \leq posBig(h)$, which implies $z = y$. In this case, we choose $z' = y'$ and it clearly holds that $w' = x' < y' = z' < y = z$. Now, let us assume that $y > posBig(h)$, which implies $z > posBig(h)$. We distinguish the following cases:

- If $y' - x' \leq \mathit{long}(h-1)$, then choose $z' = y'$ and we have to show that $w' < z' < z$. We have that $w' = x' \leq \mathit{posBig}(h-1) < y' = z'$. To show that $z' < z$, first observe that from $w'(= x') \leq \mathit{posBig}(h-1)$ and $z' - w' = y' - x' \leq \mathit{long}(h-1)$, it follows $z' \leq \mathit{posBig}(h)$, because $\mathit{posBig}(h) - \mathit{posBig}(h-1) > \mathit{long}(h-1)$ (by Definition 10); then, $z' < z$ follows from $z > \mathit{posBig}(h)$.

- If $y' - x' > \mathit{long}(h-1)$, $x' < 0$, and $y' \leq f(x') + \mathit{long}(h-1)$, then choose $z' = y'$. We have to show that $w' < z' < z$. We know that $w' = x' < y' = z'$. To prove that $z' < z$, observe from $f(x') \leq \mathit{posBig}(h-1)$ and $y' \leq f(x') + \mathit{long}(h-1)$ it follows that $y' \leq \mathit{posBig}(h)$; moreover, recall that we have $z > \mathit{posBig}(h)$, and thus we conclude that $z' = y' \leq \mathit{posBig}(h) < z$.

- If $y' - x' > \mathit{long}(h-1)$ and either $x' \geq 0$ or $y' > f(x') + \mathit{long}(h-1)$, then choose $z' = z - 1$. Observe that from $z' = z - 1$ and $z > \mathit{posBig}(h)$ it follows that $z' \geq \mathit{posBig}(h)$. We have to show the following properties.
  - $z' > \mathit{posBig}(h-1)$. It immediately follows from $z' \geq \mathit{posBig}(h)$ and $\mathit{posBig}(h) > \mathit{posBig}(h-1)$.
  - $w' < z' < z$. It holds that $w' = x' \leq \mathit{posBig}(h-1) < \mathit{posBig}(h) \leq z' < z$.
  - $z' - w' > \mathit{long}(h-1)$. It is true by Definition 10 and the facts that $w' \leq \mathit{posBig}(h-1)$, $z' \geq \mathit{posBig}(h)$, and $\mathit{posBig}(h) - \mathit{posBig}(h-1) > \mathit{long}(h-1)$.
  - If $x' < 0$, then $z' > f(w') + \mathit{long}(h-1)$. It holds that $z' \geq \mathit{posBig}(h) > \mathit{posBig}(h-1) + \mathit{long}(h-1) \geq f(x') + \mathit{long}(h-1) = f(w') + \mathit{long}(h-1)$ by Definition 10.

3. If $x' < -\mathit{negBig}(h-1)$ and $\mathit{posBig}(h-1) < y'$, then we distinguish the following cases. (Recall that, if $x \geq -\mathit{negBig}(h)$, then $f(x) \leq \mathit{posBig}(h)$, by Definition 10.)

   - $f(x') \leq \mathit{posBig}(h-1)$. By Corollary 2.2, there exists w' such that $-\mathit{negBig}(h) < w' < -\mathit{negBig}(h-1) - \mathit{long}(h-1)$, $f(w') = f(x')$ and, $\mathit{type}(w') = \mathit{type}(x')$. We have to show that $w' > w$:
     - If $x \neq w$ then, $x, w < -\mathit{negBig}(h)$ and since $w' > -\mathit{negBig}(h)$ then $w' > w$.
     - If $x = w$ then, we take $x = w < x' = w'$.

     Now, we have to choose z'. We distinguish two cases.
     - If $y \leq \mathit{posBig}(h)$ then $y = z$ and we choose $z' = y'$.We have to show that $z' < z$. It clearly holds that $w' < 0 < z' = y' < y = z$.
     - If $y > \mathit{posBig}(h)$ then, $z > \mathit{posBig}(h)$ $(y \neq z)$ and we

$$z' = \begin{cases} y' & \text{if } y' \leq f(x') + \mathit{long}(h-1) \\ f(x') + \mathit{long}(h-1) + 1 & \text{otherwise} \end{cases}$$

       We have to show that $z' < z$. If $z' = y'$, then it is same as above. Otherwise, we have that

$w' < 0 < z' = f(x') + long(h-1) + 1 \leq posBig(h-1) + long(h-1) + 1 < posBig(h) < z$ by Definition 10.

- If $f(x') > posBig(h-1)$.
  - If $x < -negBig(h)$, then, $w < -negBig(h)$. By Lemma 1, there exist w',z' such that (i) $w < -negBig(h) \leq w' < -negBig(h-1)$, (ii) $f(w') > posBig(h-1)$, (iii) $posBig(h-1) < z' < posBig(h)$, and (iv) only one of the followings hold.

    (a) $y' - f(x') = z' - f(w')$;

    (b) $y' - f(x') > long(h-1)$ and $z' - f(w') > long(h-1)$;

    (c) $y' - f(x') < -long(h-1)$ and $z' - f(w') < -long(h-1)$.

    If $y > posBig(h)$ and $z > posBig(h)$, then, it clearly holds that $z' < posBig(h) < z$. If $y \leq posBig(h)$ and $f(x') > posBig(h)$, then, $y' < y = z \leq posBig(h)$, and Lemma 1 guarantees us that there exists w', z' such that $-negBig(h) \leq w' < -negBig(h-1) < posBig(h-1) < z' < z = y$, $f(w') > posBig(h-1)$ and they satisfy (iv). If $f(x') \leq posBig(h)$, then, Lemma 1 guarantees us that there exists w' such that $-negBig(h) \leq w' < -negBig(h-1)$, $f(w') = f(x')$, and we choose $z' = y'$. It holds that $y' - f(x') = z' - f(w')$.
  - if $x \geq -negBig(h)$ and $y \neq z$, then, $w = x$ and $y, z > posBig(h)$. It holds that $y - f(x) > long(h-1)$ and $z - f(x) > long(h-1)$ (Recall that $f(x) \leq posBig(h)$). We distinguish the following sub-cases.
    * If $y' < z$ choose $z' = y'$ and $w' = x'$. It holds that $w = x < x' = w' < z' = y' < z$.
    * If $y' \geq z$ and $y' > f(x') + long(h-1)$. We choose $w' = g(posBig(h-1)+1)$, it holds that $w' < -negBig(h-1)$ by Proposition 1. Moreover, since $w = x < x' \leq g(posBig(h-1)+1) = w'$ we choose $z' = z - 1$. It holds that:

      (i) $f(w') + long(h-1) = posBig(h-1) + 1 + long(h-1)$,

      (ii) $posBig(h-1) + 1 + long(h-1) < posBig(h)$ by Definition 10,

      (iii) $posBig(h) \leq z' < z$.

      We conclude that $z' > f(w') + long(h-1)$ and $w' < 0 < z'$.
    * If $y' \geq z$ and $y' \leq f(x') + long(h-1)$, we choose $w' = g(posBig(h-1)+1)$. It holds that $w = x < x' \leq g(posBig(h-1)+1) = w' < -negBig(h-1)$ by Proposition 1. Moreover,we choose $z' = f(w') + (y' - f(x'))$. It holds that

      (i) $f(w') + (y' - f(x')) \leq f(w') + long(h-1)$,

      (ii) $f(w') + long(h-1) = posBig(h-1) + 1 + long(h-1)$,

      (iii) $posBig(h-1) + 1 + long(h-1) < posBig(h)$ by Definition 10,

      (iv) $posBig(h) < z$.

We conclude that $z' < z$ and $w' < 0 < z'$.

4. If $x' < -negBig(h-1) \leq y' \leq posBig(h-1)$, then choose $z' = y'$ and we have to show that $w < z' < z$.

   - If $y \neq z$ then, $y, z > posBig(h)$ and $z' = y' \leq posBig(h-1) < posBig(h) < z$,
   - If $y = z$ then, it clearly holds that $z' = y' < y = z$.

   We distinguish two sub-cases.

   - If $y' - x' > long(h-1)$. If $w = x$, we choose $w' = x'$. Otherwise, we distinguish two sub-cases.
     - If $f(x') \leq posBig(h-1)$, then, by Corollary 2.2, there exists w' such that (i) $w < -negBig(h) < w' < -negBig(h-1) - long(h-1) < z' = y'$, (ii) $f(w') = f(x')$, (iii) $type(w') = type(x')$, and (iv) $f(w') - z' = f(x') - y'$.
     - If $f(x') > posBig(h-1)$, by Corollary 2.3, there exists w' such that (i) $w < -negBig(h) < w' < -negBig(h-1) - long(h-1) < z' = y'$, (ii) $f(w') > posBig(h-1)$, and
       only one of the followings hold.
       (a) $f(w') - z' = f(x') - y'$;
       (b) $f(w') - z' > long(h-1)$ and $f(x') - y' > long(h-1)$.

     Finally, it clearly holds that $[w', z'] > posBig(h-1)$.
   - If $y' - x' \leq long(h-1)$, in that $-negBig(h) < x'$ by Proposition 4, we choose $x' = w'$. It clearly holds that $w < w' = x' < y' = z'$.

5. If $posBig(h-1) < x' < y'$. We distinguish two sub-cases.

   - If $y' \leq posBig(h)$, $x' > -negBig(h)$, then, we choose $z' = y'$ and $w' = x'$. It holds that $w' = x' < y' = z'$. We have to show that $w < w'$ and $z' < z$.
     - if $w = x$ then, it clearly holds that $w = x < w' = x'$.
     - If $w \neq x$, we know that $w, x < -negBig(h)$. It clearly holds that $w = x < -negBig(h) < posBig(h-1) < w' = x'$.
     - If $z = y$ then, it clearly holds that $z = y > z' = y'$.
     - If $z \neq y$ we know that $z, y > posBig(h)$. it clearly holds that $z' \leq posBig(h) < z$.

   - if $y' > posBig(h)$ then $z > posBig(h)$ and $y > posBig(h)$
     - if $y' - x' \leq long(h-1)$, we choose $z' = z - 1$ and $w' = z' - (y' - x')$. We have that $z' - w' = y' - x'$ and $w' > posBig(h-1)$ by Definition 10. Moreover, it holds that $w < 0 < w' < z' < z$.
     - if $y' - x' > long(h-1)$, we choose $z' = z - 1$ and $w' = posBig(h-1) + 1$. It holds that $z' - w' > long(h-1)$ by Definition 10. Moreover, it holds that $w < 0 < w' < z' < z$.

6. If $x' < y' < -negBig(h-1)$, we distinguish the following cases.

- If $x' \geq -negBig(h)$, then choose $w' = x'$ and $z' = y'$. It clearly holds that $w < w' < z' < z$.

- If $x' < -negBig(h)$ and $x' > w$, then choose $w' = x'$ and $z' = y'$. It clearly holds that $w < w' < z' < z$.

- If $x' < -negBig(h)$ and $x' \leq w$: (from now on, we assume that $x, w < -negBig(h)$)
  - If $y' - x' \leq long(h-1)$. We distinguish two sub-cases.
    * If $f(x') > posBig(h-1)$ and $f(y') > posBig(h-1)$, we choose $w' = -negBig(h)$ and $z' = w' + (y' - x')$. We have to show that $z' < -negBig(h-1)$, $f(w') > posBig(h-1)$ and $f(z') > posBig(h-1)$. Observe that, $z' = w' + (y' - x') < -negBig(h) + posBig(h) - long(h-1) = g(posBig(h)-1) - posBig(h) + 3 - long(h-1) < -negBig(h-1)$ by Proposition 3 and 4. Moreover, it clearly holds that $f(w') = posBig(h) > posBig(h-1)$ and, $f(z') > posBig(h-1)$ by Preposition 3 and 4. Finally, it holds that $z' - w' = w' + (y' - x') - w' = y' - x'$ and $w < -negBig(h) = w' < z' < 0 < z$.

    * If either $f(x') \leq posBig(h-1)$ or $f(z') \leq posBig(h-1)$. By Corollary 2.2, there exists w' (resp., z') such that $w < w' < -negBig(h-1) - long(h-1)$ (resp., $w < -negBig(h) + long(h-1) < z < -negBig(h-1) - long(h-1)$) ,$f(w') = f(x')$ and $type(w') = type(x')$ (resp., $f(z') = f(y')$ and $type(z') = type(y')$). We choose $z' = w' + (y' - x')$ (resp., $w' = z' - (y' - x')$).It clearly holds that $z' - w' = w' + (y' - x') - w' = y' - x'$, $f(z') = f(y')$, $type(z') = type(y')$ (resp.,$f(w') = f(x')$, and $type(y') = type(x')$), and $w < -negBig(h) < w' < z' < -negBig(h-1) < 0 < z$.
  - if $y' - x' > long(h-1)$:
    * If $f(x') \leq posBig(h-1)$ and $f(y') \leq posBig(h-1)$. We distinguish two sub-cases.
      (a) If $type(x') = \perp$ and $type(y') = \perp$, then, we choose $w' = g(posBig(h-1)+2) - posBig(h-1) - 1$ and $z' = g(posBig(h-1)+2) + posBig(h-1) + 1$ (see Figure A.2 and observe that $w' < -negBig(h-1)$,$z' < -negBig(h-1)$ and $f(z') = f(w') = 1$). It holds that $z' - y' = g(posBig(h-1)+2) + posBig(h-1) + 1 - (g(posBig(h-1)+2) - posBig(h-1) - 1) > posBig(h-1) > long(h-1)$ by Preposition 3. Finally, it holds that $w < -negBig(h) < w' < z' < -negBig(h-1) < z$.
      (b) Otherwise, by Corollary 2.2 there exist w', z' such that
      (i) $w < -negBig(h) < w' < z' < -negBig(h-1) - long(h-1) < z$,
      (ii) $z' - w' > long(h-1)$,
      (iii) $f(w') = f(x')$,$f(z') = f(y')$, and
      (iv) $type(w') = type(x')$, $type(z') = type(y')$.

    * Otherwise, By Corollary 2.2 there exist w', z' such that
      · $w < -negBig(h) < w' < z' < -negBig(h-1) - long(h-1) < z$,

      · $z' - w' > long(h-1)$, and

      · only one of the followings hold.
      (i) $f(w') = posBig(h-1) + 1$,$f(z') = posBig(h-1) + 1$;

(ii) $f(w') = f(x')$, $type(w') = type(x')$, and $f(z') = posBig(h-1)+1$;

(iii) $f(w') = posBig(h-1)+1$, $f(z') = f(y')$, and $type(z') = type(y')$.

$\square$

Bibliography

[1] Luca Aceto, Dario Della Monica, Anna Ingólfsdóttir, Angelo Montanari, and Guido Sciavicco. A complete classification of the expressiveness of interval logics of allen's relations over dense linear orders. In *Proc. of the 20th International Symposium on Temporal Representation and Reasoning*, pages 65–72. IEEE, 2013.

[2] Luca Aceto, Dario Della Monica, Anna Ingólfsdóttir, Angelo Montanari, and Guido Sciavicco. On the expressiveness of the interval logic of Allen's relations over finite and discrete linear orders. In E. Fermé and J. Leite, editors, *Proc. of the 14th European Conference on Logics in Artificial Intelligence (JELIA)*, volume 8761 of *LNAI*, pages 267–281, 2014.

[3] James F Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843, 1983.

[4] Davide Bresolin, Valentin Goranko, Angelo Montanari, and Pietro Sala. Tableaux for logics of subinterval structures over dense orderings. *Journal of Logic and Computation*, 20(1):133–166, 2010.

[5] Davide Bresolin, Valentin Goranko, Angelo Montanari, and Guido Sciavicco. Propositional interval neighborhood logics: Expressiveness, decidability, and undecidable extensions. *Annals of Pure and Applied Logic*, 161(3):289–304, 2009.

[6] Davide Bresolin, Angelo Montanari, Pietro Sala, and Guido Sciavicco. Optimal tableau systems for propositional neighborhood logic over all, dense, and discrete linear orders. In *Automated Reasoning with Analytic Tableaux and Related Methods: 20th International Conference, TABLEAUX 2011, Bern, Switzerland, July 4-8, 2011. Proceedings 20*, pages 73–87. Springer, 2011.

[7] Zhou Chaochen and Michael R Hansen. *Duration calculus*. Springer, 2004.

[8] Zhou Chaochen, Charles Anthony Richard Hoare, and Anders P Ravn. A calculus of durations. *Information processing letters*, 40(5):269–276, 1991.

[9] Dario Della Monica, Valentin Goranko, Angelo Montanari, and Guido Sciavicco. Expressiveness of the interval logics of allen's relations on the class of all linear orders: complete classification. In *IJCAI Proceedings-International Joint Conference on Artificial Intelligence*, page 845, 2011.

[10] Valentin Goranko, Angelo Montanari, and Guido Sciavicco. A road map of interval temporal logics and duration calculi. *Journal of Applied Non-Classical Logics*, 14(1-2):9–54, 2004.

[11] Valentin Goranko and Martin Otto. Handbook of modal logic. In *Model Theory of Modal Logic*, pages 255–325. Amsterdam, The Netherlands: Elsevier, 2006.

[12] Joseph Y Halpern and Yoav Shoham. A propositional modal logic of time intervals. *Journal of the ACM (JACM)*, 38(4):935–962, 1991.

[13] Matthew Hennessy and Robin Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM (JACM)*, 32(1):137–161, 1985.

[14] Robin Milner. *A calculus of communicating systems*. Springer, 1980.

[15] Angelo Montanari, Gabriele Puppis, Pietro Sala, and Guido Sciavicco. Decidability of the interval temporal logic $AB\overline{B}$ over the natural numbers. In Jean-Yves Marion and Thomas Schwentick, editors, *27th International Symposium on Theoretical Aspects of Computer Science, STACS 2010, March 4-6, 2010, Nancy, France*, volume 5 of *LIPIcs*, pages 597–608. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2010.

[16] Ben C. Moszkowski and Zohar Manna. Reasoning in interval temporal logic. In Edmund M. Clarke and Dexter Kozen, editors, *Logics of Programs, Workshop, Carnegie Mellon University, Pittsburgh, PA, USA, June 6-8, 1983, Proceedings*, volume 164 of *Lecture Notes in Computer Science*, pages 371–382. Springer, 1983.

[17] Benjamin Charles Moszkowski. *Reasoning about digital circuits*. Stanford University, 1983.

[18] Yde Venema. Expressiveness and completeness of an interval tense logic. *Notre Dame journal of formal logic*, 31(4):529–547, 1990.