Expressiveness issues in Interval Temporal Logics

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joint work with Dario Della Monica

Logic Colloquium 2024
Gothenburg, June 24-28

Outline

Interval Temporal Logics

Halpern-Shoham's modal logic HS

Expressiveness of HS fragments over discrete/finite linear orders

Conclusions

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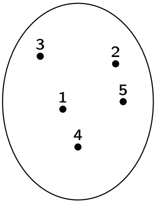
Temporal logics: origins and application fields

- Temporal logics play a major role in computer science
  - Specification and verification of reactive systems
- Temporal logics are (special case of) modal logics

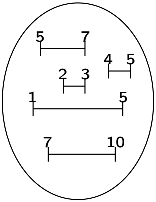


set of worlds
primitive temporal entity
time points/instants

accessibility relations
$\longrightarrow$: next
$\longrightarrow^{\star}$: finally

A different approach: from points to intervals

- worlds are intervals (time period — pairs of points)



set of worlds
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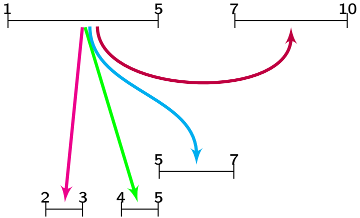
time intervals/periods



accessibility relations
all binary relations between pairs of intervals

Intervals and interval structures

$\mathbb{D} = \langle D, < \rangle$: strict partial order with

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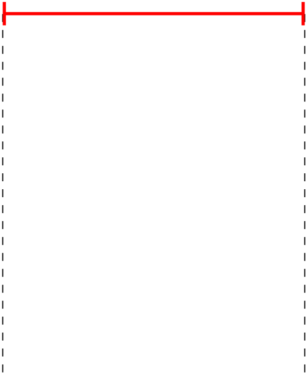
In this talk I will restrict attention to linear interval structures, i.e., interval structures over linear orders.

Binary interval relations on linear orders



J. F. Allen
Maintaining knowledge about temporal intervals
Communications of the ACM, volume 26(11), pages 832-843, 1983

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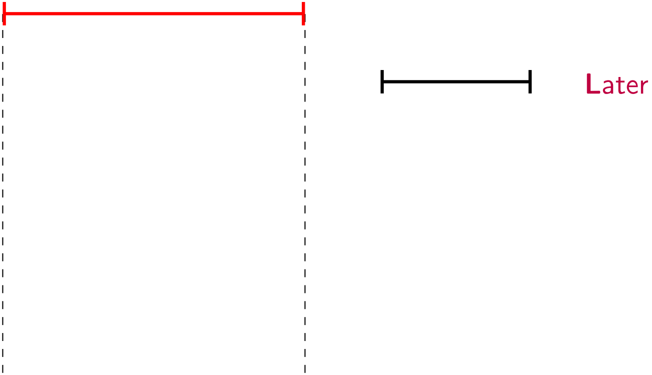

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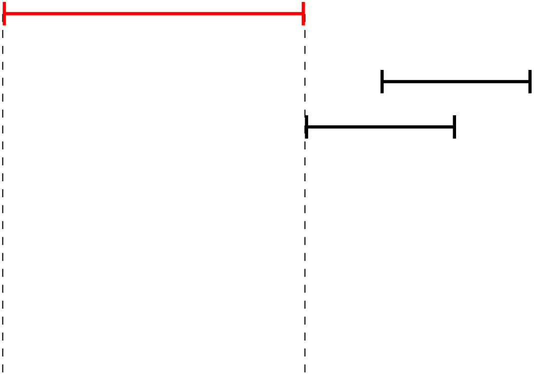

Later

After (right neighbour)

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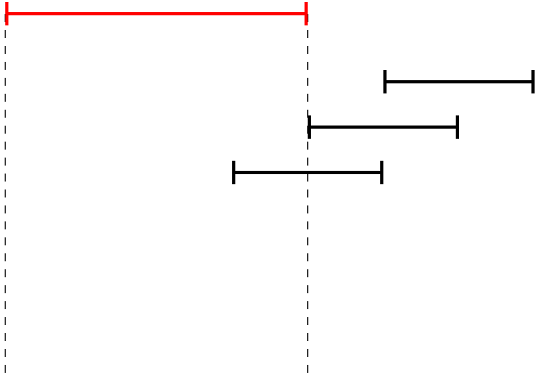

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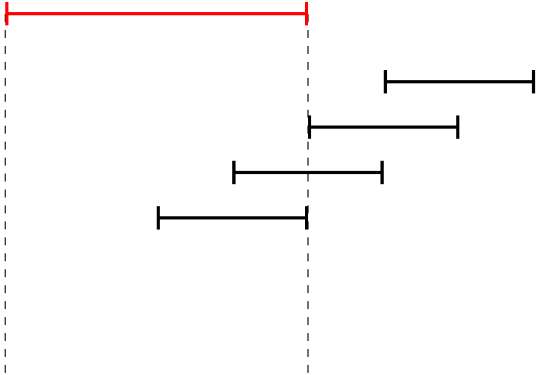

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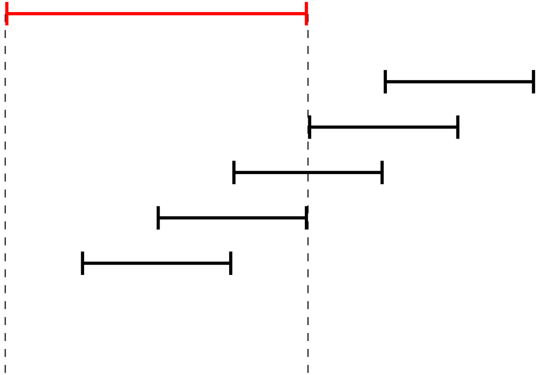

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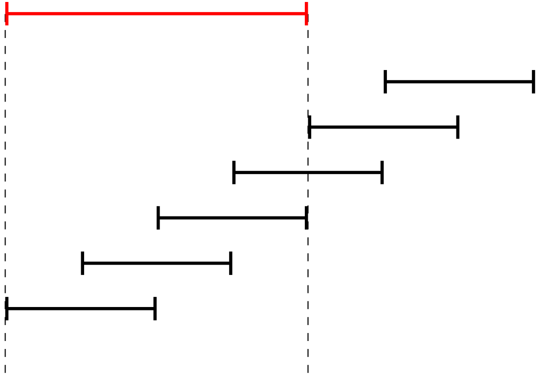

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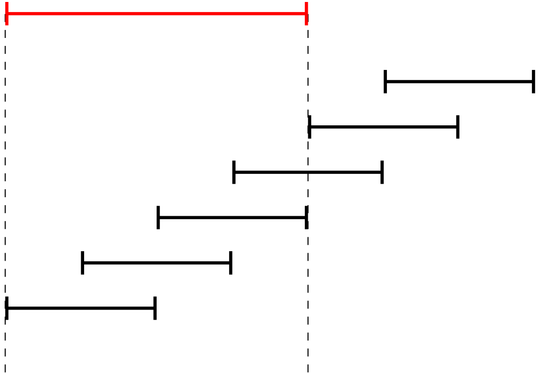

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6 relations + their inverses = 12 Allen's relations

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Halpern-Shoham's modal logic of interval relations

interval relations give rise to modal operators

HS logic

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interval relations give rise to modal operators → HS logic

HS is undecidable over all significant classes of linear orders

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Syntax:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle X \rangle \varphi$$

$$\langle X \rangle \in \{\langle A \rangle, \langle \mathrm{L} \rangle, \langle B \rangle, \langle E \rangle, \langle \mathrm{D} \rangle, \langle \mathrm{O} \rangle, \langle \overline{\mathrm{A}} \rangle, \langle \overline{\mathrm{L}} \rangle, \langle \overline{\mathrm{B}} \rangle, \langle \overline{E} \rangle, \langle \overline{\mathrm{D}} \rangle, \langle \overline{\mathrm{O}} \rangle\}$$

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Models:

$$\mathrm{M} = \langle \mathbb{I}(\mathbb{D}), V \rangle$$

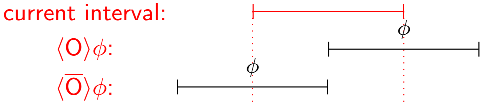
$$V : \mathbb{I}(\mathbb{D}) \mapsto 2^{\mathcal{AP}}$$

$\mathcal{AP}$ set of atomic propositions

Formal semantics of HS

$\langle O \rangle$: M, $[d_0, d_1] \Vdash \langle O \rangle \phi$ iff there exists d_2, d_3 such that $d_0 < d_2 < d_1 < d_3$ and M, $[d_2, d_3] \Vdash \phi$.

$\langle \overline{O} \rangle$: M, $[d_0, d_1] \Vdash \langle \overline{O} \rangle \phi$ iff there exists d_2, d_3 such that $d_2 < d_0 < d_3 < d_1$ and M, $[d_2, d_3] \Vdash \phi$.



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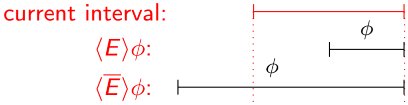

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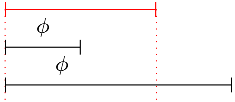
current interval:

$\langle B \rangle \phi$:

$\langle \overline{\mathrm{B}} \rangle \phi$:



Formal semantics of HS - contd'

$\langle L \rangle$: $M, [d_0, d_1] \Vdash \langle L \rangle \phi$ iff there exists d_2, d_3 such that $d_1 < d_2 < d_3$ and $M, [d_2, d_3] \Vdash \phi$.

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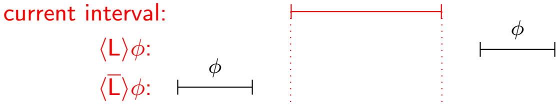

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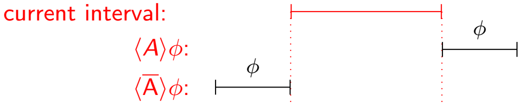

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$\langle D \rangle$: $M, [d_0, d_1] \Vdash \langle D \rangle \phi$ iff there exists d_2, d_3 such that $d_0 < d_2 < d_3 < d_1$ and $M, [d_2, d_3] \Vdash \phi$.

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current interval:

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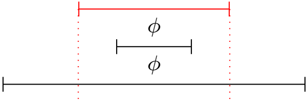
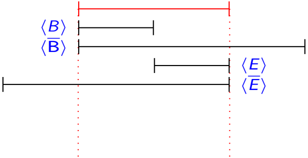


Definabilities among modalities

All modalities are definable in terms of $\langle B\rangle$, $\langle \overline{\mathrm{B}}\rangle$, $\langle E\rangle$, $\langle \overline{E}\rangle$

$HS \equiv \mathrm{B}\overline{\mathrm{B}}\mathrm{E}\overline{\mathrm{E}}$



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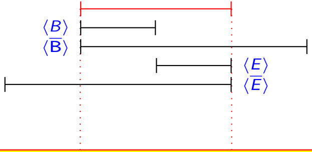

In general, it is possible defining HS modalities in terms of others

The zoo of fragments of HS

- $2^{12} = 4096$ fragments of HS (syntactic)
- Not all these fragments are expressively different
- expressiveness classification wrt. several classes of interval structures
  - all, dense, discrete, finite, ???

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Classification over all linear orders

L. Aceto, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco

Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification

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Classification over all dense linear orders

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ACTA Informatica, 2014

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We focus here on:

- finite
- discrete

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The expressiveness classification programme

Expressiveness classification problem: classify the fragments of HS with respect to their expressiveness, relative to classes of finite/discrete interval models.

Comparing expressive power of HS fragments

L_1, L_2 HS-fragments

L_1 L_2

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$$L_1 \ \{\prec, \equiv, \succ, \not\approx\} \ L_2$$

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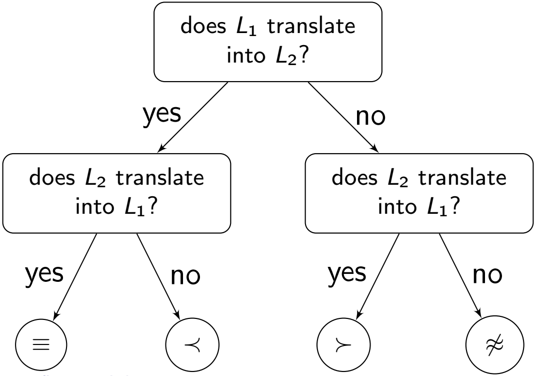
How do we decide the relation between fragments L_1 and L_2?



Truth-preserving translation

There exists a truth-preserving translation of L_1 into L_2
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L_2 is at least as expressive as L_1
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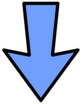

Each modality $\langle X \rangle$ of L_1 is definable in L_2 $(\langle X \rangle \lhd L_2)$
(i.e., $\exists$ a L_2-formula φ s.t. $\langle X \rangle p \equiv \varphi$)

Example: $\langle \mathrm{L} \rangle p \equiv \langle A \rangle \langle A \rangle p$

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2^{12} fragments... $\frac{2^{12} \cdot (2^{12}-1)}{2}$ comparisons

Our approach

Notation:

$$\mathsf{X_1 X_2 \ldots X_n}$$
$$=$$
HS-fragment with modalities
$\langle X_1 \rangle, \langle X_2 \rangle, \ldots, \langle X_n \rangle$

Our approach

Solution:
To find a complete set of definabilities among modalities

Notation:

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HS-fragment with modalities $\langle X_1\rangle, \langle X_2\rangle, \dots, \langle X_n\rangle$

$$\overbrace{\mathsf{X_1X_2\dots X_n}}^{\mathcal{X}} \qquad \overbrace{\mathsf{Y_1Y_2\dots Y_m}}^{\mathcal{Y}}$$

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Notation:

$\mathsf{X_1 X_2 \ldots X_n}$ = HS-fragment with modalities $\langle X_1 \rangle, \langle X_2 \rangle, \ldots, \langle X_n \rangle$

$$\overbrace{\mathsf{X_1 X_2 \ldots X_n}}^{\mathcal{X}} \quad \overset{\{\prec, \equiv, \succ, \not\approx\}}{??} \quad \overbrace{\mathsf{Y_1 Y_2 \ldots Y_m}}^{\mathcal{Y}}$$

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$\langle X_1 \rangle \lhd \mathsf{Y_1 \ldots Y_m}$??

$\ldots$??

$\langle X_n \rangle \lhd \mathsf{Y_1 \ldots Y_m}$??

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To find a complete set of definabilities among modalities

Notation:

$X_1 X_2 \ldots X_n$
$=$
HS-fragment with modalities
$\langle X_1 \rangle, \langle X_2 \rangle, \ldots, \langle X_n \rangle$

$$\overbrace{X_1 X_2 \ldots X_n}^{\mathcal{X}} \qquad \overset{\{\prec, \equiv, \succ, \not\approx\}}{??} \qquad \overbrace{Y_1 Y_2 \ldots Y_m}^{\mathcal{Y}}$$

$\langle X_1 \rangle \lhd Y_1 \ldots Y_m$	??
	$\wedge$
$\ldots$	??
	$\wedge$
$\langle X_n \rangle \lhd Y_1 \ldots Y_m$	??
	$=$
$\mathcal{X} \preceq \mathcal{Y}$	??

Our approach

Solution:
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\langle X_1 \rangle \lhd \mathsf{Y_1 \ldots Y_m} & ?? & \texttt{true} \\
 & \wedge & \wedge \\
\ldots & ?? & \texttt{true} \\
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Complete sets of definabilities among modalities

$\langle L \rangle \lhd A \qquad \langle L \rangle p \equiv \langle A \rangle \langle A \rangle p$

$\langle D \rangle \lhd BE \qquad \langle D \rangle p \equiv \langle B \rangle \langle E \rangle p$

$\langle O \rangle \lhd \overline{B}E \qquad \langle O \rangle p \equiv \langle E \rangle \langle \overline{B} \rangle p$

$\langle L \rangle \lhd \overline{B}E \qquad \langle L \rangle p \equiv \langle \overline{B} \rangle [E] \langle \overline{B} \rangle \langle E \rangle p$

complete set of definabilities for the class of all linear orders

$\langle A \rangle \lhd \overline{B}E \qquad \langle A \rangle p \equiv \varphi(p) \vee \langle E \rangle \varphi(p)^{\dagger}$

$\langle O \rangle \lhd ??? \qquad \langle O \rangle p \equiv ???$

complete (???) set of definabilities for classes of discrete/finite linear orders

investigated in my bachelor thesis

$^{\dagger}\varphi(p) := [E]\bot \wedge \langle \overline{B} \rangle ([E][E]\bot \wedge \langle E \rangle (p \vee \langle \overline{B} \rangle p))$

Complete sets of definabilities among modalities

$\langle \mathrm{L} \rangle \lhd \mathrm{A}$ $\quad \langle \mathrm{L} \rangle p \equiv \langle A \rangle \langle A \rangle p$

$\langle \mathrm{D} \rangle \lhd \mathrm{BE}$ $\quad \langle \mathrm{D} \rangle p \equiv \langle B \rangle \langle E \rangle p$

$\langle \mathrm{O} \rangle \lhd \overline{\mathrm{B}}\mathrm{E}$ $\quad \langle \mathrm{O} \rangle p \equiv \langle E \rangle \langle \overline{\mathrm{B}} \rangle p$

$\langle \mathrm{L} \rangle \lhd \overline{\mathrm{B}}\mathrm{E}$ $\quad \langle \mathrm{L} \rangle p \equiv \langle \overline{\mathrm{B}} \rangle [\mathrm{E}] \langle \overline{\mathrm{B}} \rangle \langle E \rangle p$

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complete (???) set of definabilities for classes of discrete/finite linear orders

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> Remark:
> Completeness of the set of definabilities does not necessary hold any longer if the semantics is restricted to a specific class of linear orders

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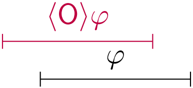
The missing piece of the puzzle: the cases $\langle \mathrm{O} \rangle$

Semantics:

$$M, [a, b] \Vdash \langle \mathrm{O} \rangle \varphi \overset{def}{\Leftrightarrow} \exists c, d \text{ such that } a < c < b < d \text{ and } M, [c, d] \Vdash \varphi$$



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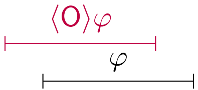

We conjecture that
there are no more inter-definability equations for $\langle O \rangle$
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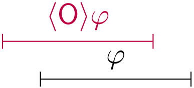

We conjecture that
there are no more inter-definability equations for $\langle \mathrm{O} \rangle$
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$\langle \mathrm{O} \rangle$ is not definable in terms of any other fragment besides $\overline{\mathrm{B}}\mathrm{E}$

Formal proof of our conjecture

Operator $\langle O \rangle$ is definable in terms of $\overline{\mathrm{B}}\mathrm{E}$ $\qquad \langle O \rangle \varphi \equiv \langle E \rangle \langle \overline{\mathrm{B}} \rangle \varphi$

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They are the two maximal fragments not defining it

Proving non-existence

Existence is easy... **...non-existence is hard**

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a new
land

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Bisimulations

Bisimulation between interval models

$Z \subseteq M_1 \times M_2$ is a bisimulation wrt the fragment $\mathsf{X}_1\mathsf{X}_2 \ldots \mathsf{X}_n$ iff

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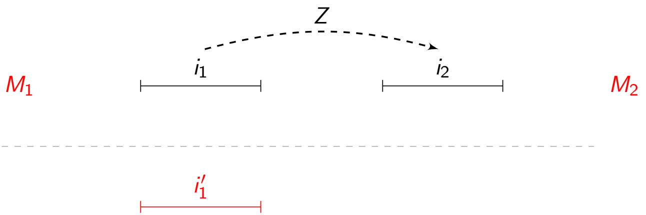

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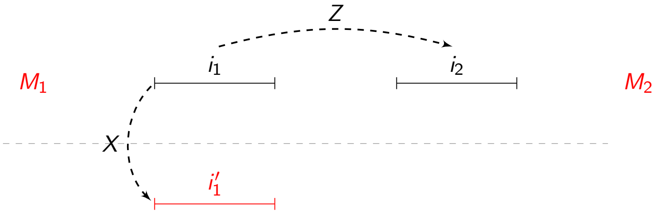
$$(i_1, i_2) \in Z$$
$$(i_1, i'_1) \in X$$



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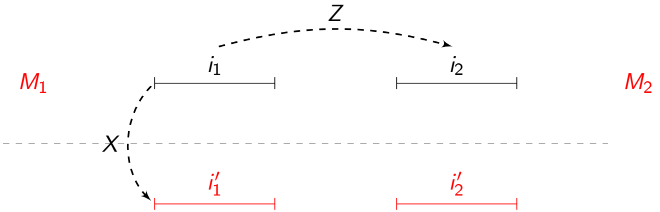

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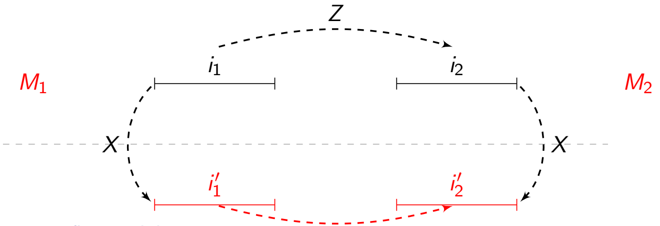

Invariance of modal formulae wrt bisimulations

Theorem A bisimulation for $\mathcal{L}$ preserves the truth of $\mathcal{L}$-formulae

$M_1, [a, b]$ and $M_2[c, d]$ are bisimilar
φ is a $\mathcal{L}$-formula

$$\Downarrow$$

φ is true in $M_1, [a, b]$ iff φ is true in $M_2, [c, d]$

Goranko Valentin and Otto Martin
Handbook of modal logic
Model Theory of Modal Logic, pages 255-325, 2006

How to use bisimulations to disprove definability

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$\langle X \rangle$ is not definable in terms of $\mathcal{L}$

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By contradiction

If $\langle X \rangle$ is definable in terms of $\mathcal{L}$ then,
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N-bisimulation

- IMPORTANT!!! $\langle O \rangle$ is definable in terms of fragment $\mathsf{ABD}\overline{\mathsf{ABE}}$ using infinitary formulas (i.e., infinite disjunction)
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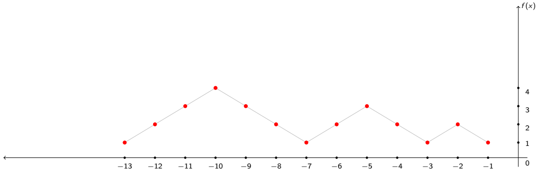
It must be used the N-bisimulation

In my thesis we presented a proposal
for an N-bisimulation wrt $\mathsf{ABD}\overline{\mathsf{ABE}}$ that violates $\langle O \rangle$

My contribution

Definition (ABD$\overline{\text{ABE}}_N$-bisimulation that violates ⟨O⟩)

- $\mathcal{AP} = \{p\}$, models: $M_1 = M_2 = \langle \mathbb{I}(\mathbb{N}), V \rangle$ where
  - $V(p) = \{[x, f(x)] \mid x \in \mathbb{Z}^-\}$ is defined with the help of the following $f : \mathbb{Z}^- \to \mathbb{N} \setminus \{0\}$



For each $N \in \mathbb{N} \setminus \{0\}$ we define a sequence of N relations $Z_N, \ldots, Z_1$ as follows. For every $h \in \{1, \ldots, N\}$, we have that $[x, y] Z_h [w, z]$ if and only if all of the following conditions hold:

1. $x \simeq_h w$ and $y \simeq_h z$;
2. either $y - x = z - w$ or they are both h-long, that is $y - x > long(h)$ and $z - w > long(h)$;
3. if $x < 0$ and $y > 0$, then one of the following holds:
   (a) $|f(x) - y| \leq long(h)$ and $f(x) - y = f(w) - z$;
   (b) $f(x) - y > long(h)$ and $f(w) - z > long(h)$;
   (c) $f(x) - y < -long(h)$ and $f(w) - z < -long(h)$.

Outline

Interval Temporal Logics

Halpern-Shoham's modal logic HS

Expressiveness of HS fragments over discrete/finite linear orders

Conclusions

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My contribution

- A proposal for an N-bisimulation wrt $\mathsf{ABD}\overline{\mathsf{ABE}}$ that violates $\langle O \rangle$
- strong and convincing evidence to support its correctness

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- A proposal for an N-bisimulation wrt $\mathsf{ABD}\overline{\mathsf{ABE}}$ that violates $\langle O \rangle$
- strong and convincing evidence to support its correctness

Future work

- to complete the formal proof
- finding an analogous N-bisimulation wrt $\mathsf{ABE}\overline{\mathsf{AED}}$ that violates $\langle O \rangle$

Thank you
Any questions?