Find the Trajectory using Dynamic Optimization

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In the report, it is important to give the results and an interpretation of those, but certainly also to describe the chosen method, its background and assumptions.

1, Trajectory of a Suspended Chain

The shape of a suspended chain can be described and solved using dynamic optimization. The equations to describe the dynamics of the chain are simplified to one dimensional unconstrained equation, and the angels are described in the Radian system. That is, For $i = 0, 1, \dots N-1$, we have:

$$\begin{aligned}\left[\begin{array}{c} z \\ y \end{array}\right]_{i+1} &= f\left(\left[\begin{array}{c} z \\ y \end{array}\right]_i, \left[\begin{array}{c} u \\ v \end{array}\right]_i\right) = \left[\begin{array}{c} z \\ y \end{array}\right]_i + \left[\begin{array}{c} u \\ v \end{array}\right]_i \\ &= f\left(\left[\begin{array}{c} z \\ y \end{array}\right]_i, \theta_i\right) = \left[\begin{array}{c} z \\ y \end{array}\right]_i + l\left[\begin{array}{c} \cos(\theta_i) \\ \sin(\theta_i) \end{array}\right]\end{aligned}$$

where the end points are:

$$\begin{aligned}\left[\begin{array}{c} z \\ y \end{array}\right]_0 &= \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \\ \left[\begin{array}{c} z \\ y \end{array}\right]_N &= \left[\begin{array}{c} h \\ 0 \end{array}\right]\end{aligned}$$

So the constraint $u_i^2 + v_i^2 = l^2$ is no longer needed. Notice that the angels are not constrained, but they can be estimated to be between -0.5π and 0.5π. With the new one dimensional unconstrained equation, the (steady state) potential energy can be used as the cost function:

$$\begin{aligned}J &= \sum_{i=0}^{N-1} \frac{1}{2} mg\left(y_i + y_{i+1}\right) \\ &= \sum_{i=0}^{N-1} \left(mgy_i + \frac{1}{2} mglsin(\theta_i)\right)\end{aligned}$$

where the scalar ϕ, L can be expressed by the following equations with N fixed:

$$\begin{aligned}\phi\left(\left[\begin{array}{c} z \\ y \end{array}\right]_N\right) &= 0 \\ L\left(\left[\begin{array}{c} z \\ y \end{array}\right]_i, \theta_i\right) &= mgy_i + \frac{1}{2} mglsin(\theta_i)\end{aligned}$$

The Hamiltonian function of the cost function can be expressed as:

$$H_i = mgy_i + \frac{1}{2}mgl\sin(\theta_i) + \lambda_{i+1}^z\left[z_i + l\cos(\theta_i)\right] + \lambda_{i+1}^y\left[y_i + l\sin(\theta_i)\right]$$

Hence the set of Euler-Lagrange equations can be expressed by the following five equations:

$$\begin{aligned} z_{i+1} &= z_i + l\cos(\theta_i) \\ y_{i+1} &= y_i + l\sin(\theta_i) \\ \lambda_i^z &= \lambda_{i+1}^z \\ \lambda_i^y &= mg + \lambda_{i+1}^y \\ 0 &= \left[\frac{1}{2}mg + \lambda_{i+1}^y\right]\cos(\theta_i) - \lambda_{i+1}^z\sin(\theta_i) \end{aligned}$$

where the boundary conditions are:

$$\begin{aligned} \begin{bmatrix} z \\ y \end{bmatrix}_0 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} z \\ y \end{bmatrix}_N &= \begin{bmatrix} h \\ 0 \end{bmatrix} \\ \begin{bmatrix} \lambda_z \\ \lambda_y \\ \theta \end{bmatrix}_N &= \begin{bmatrix} \nu_z \\ \nu_y \\ 1 \end{bmatrix} \end{aligned}$$

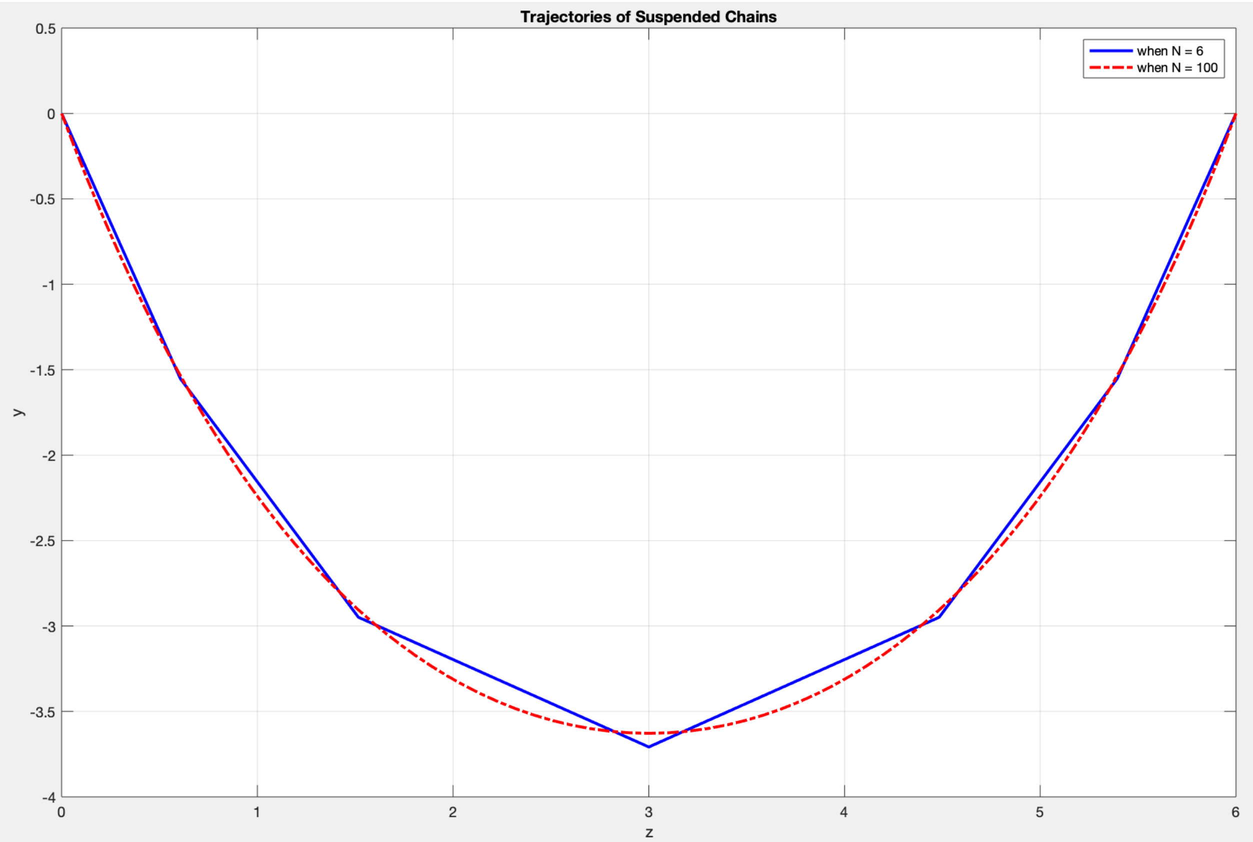
Numerical method can be used to find the optimal values. The calculate procedure for iterations can be expressed by the following five equations:

$$\begin{aligned} \lambda_{i+1}^z &= \lambda_i^z \\ \lambda_{i+1}^y &= \lambda_i^y - mg \\ \theta_i &= \arctan\left[\left(\lambda_{i+1}^y + \frac{1}{2}mg\right)/\lambda_{i+1}^z\right] \\ z_{i+1} &= z_i + l\cos(\theta_i) \\ y_{i+1} &= y_i + l\sin(\theta_i) \end{aligned}$$

When $h = 6, L = 10, M = 14, N = 6$, the value of the costate vector at 0, $[\lambda_i^z, \lambda_i^y]_0^T$, is $[-22.3645, 68.6700]$.

When $h = 6, L = 10, M = 14, N = 100$, the value of the costate vector at 0, $[\lambda_i^z, \lambda_i^y]_0^T$, is $[-22.4092, 68.6700]$.

The two results can be visualized by the figure 1:



Vertical Force and Costate Vector

Determine the vertical force in the origin (i = 0). Compare this with the costate at the origin. Discuss your observations. Give a qualified guess on the sign of horizontal force in the origin.

The vertical forces at the left end and the right end are equal, and their sum equals the weight of the chain. So The vertical forces at the left end is 68.67, which is the same value as λ_i^y.

The horizontal forces at different joints are the same. We can get the force at the end of the 2nd f_3 sections by analyzing the balance of 3rd and 4th sections of the chain:

$$2 f_3 \sin(\theta_2) = 2mg$$

So we can get $f_3 = 50.2457$, so the value of horizontal force is 44.7289, which is two times of the value of λ_0^z. When $N = 100$, $f_{50} = 44.8394$, and the value of horizontal force is 44.8183. So we can say that the vertical force at the left end equals λ_0^y, the horizontal force at the left end is two times of λ_0^z.

Pontryagins Maximum principle

If the original two-dimensional expressions are to be used, we can write the corresponding Hamiltonian function as:

$$H_i = mgy_i + \frac{1}{2}mgv_i + \lambda_{i+1}^z (z_i + u_i) + \lambda_{i+1}^y (y_i + v_i)$$

according to Pontryagins Maximum principle, if we consider :

$$\begin{aligned}
\lambda_{i+1}^z &= \lambda_i^z \\
\lambda_{i+1}^y &= \lambda_i^y - mg \\
[u_i, v_i]^T &= \arg \min_{u_i^2+v_i^2=l^2} \left\{ mgy_i + \frac{1}{2}mgv_i + \lambda_{i+1}^z (z_i + u_i) + \lambda_{i+1}^y (y_i + v_i) \right\} \\
z_{i+1} &= z_i + u_i \\
y_{i+1} &= y_i + v_i
\end{aligned}$$

The optimization algorithms and the inputed original values have large influence on the final result. If the correct values are inputed as the original values, we can get the correct values.

Two Symmetric Half Chains

For $i = 0, 1, \ldots N-1$, we have:

$$\begin{aligned}
\begin{bmatrix} z \\ y \end{bmatrix}_{i+1} &= f\left(\begin{bmatrix} z \\ y \end{bmatrix}_i, \begin{bmatrix} u \\ v \end{bmatrix}_i \right) = \begin{bmatrix} z \\ y \end{bmatrix}_i + \begin{bmatrix} u \\ v \end{bmatrix}_i \\
&= f\left(\begin{bmatrix} z \\ y \end{bmatrix}_i, \theta_i \right) = \begin{bmatrix} z \\ y \end{bmatrix}_i + l \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix}
\end{aligned}$$

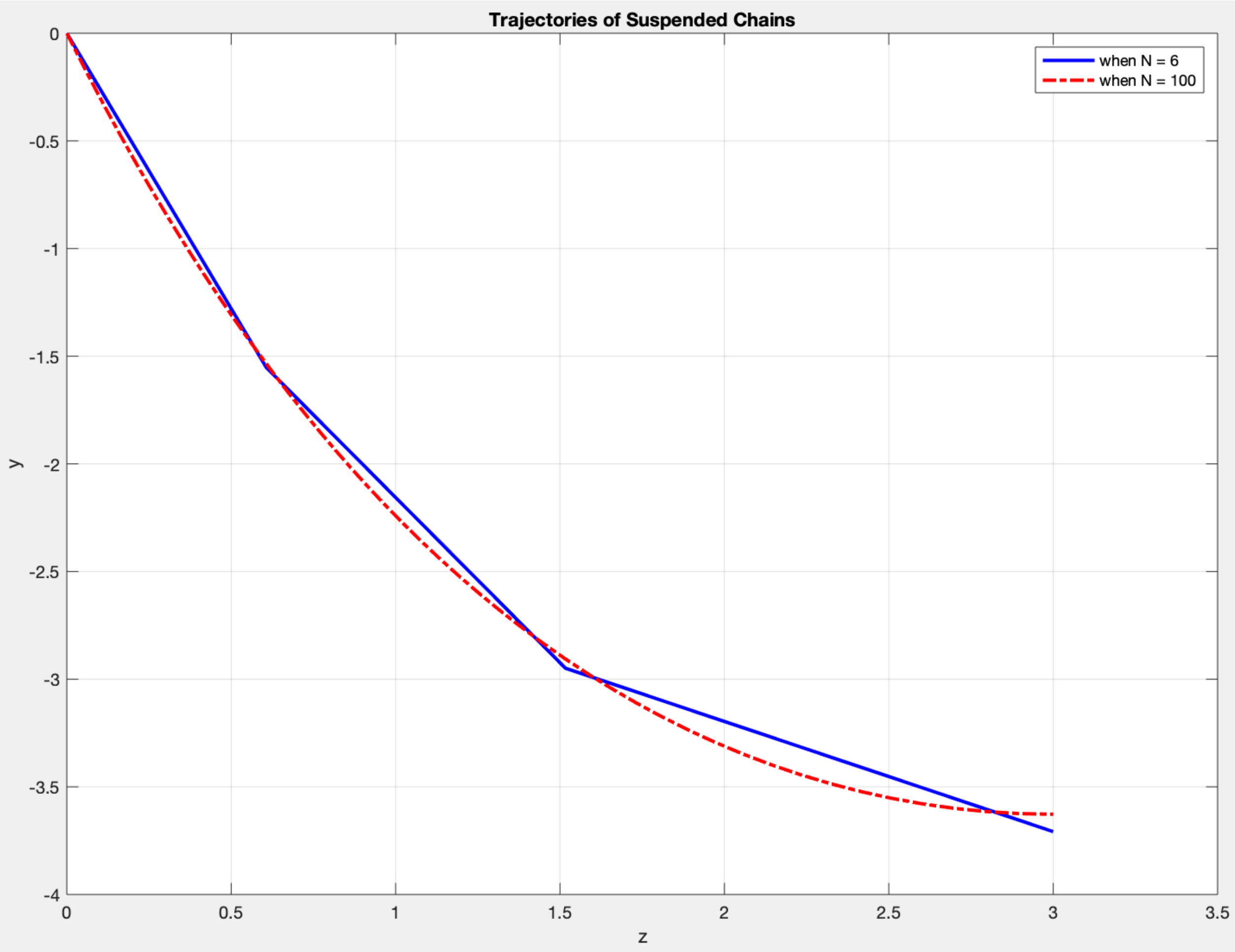
where the boundary conditions are:

$$\begin{aligned}
\begin{bmatrix} z \\ y \end{bmatrix}_0 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
z_{N/2} &= h/2
\end{aligned}$$

Numerical method is almost the same as that in the above section. The calculate procedure for iterations can be expressed by the following five equations:

$$\begin{aligned}
\lambda_{i+1}^z &= \lambda_i^z \\
\lambda_{i+1}^y &= \lambda_i^y - mg \\
\theta_i &= \arctan\left[\left(\lambda_{i+1}^y + \frac{1}{2}mg \right) / \lambda_{i+1}^z \right] \\
z_{i+1} &= z_i + l\cos(\theta_i) \\
y_{i+1} &= y_i + l\sin(\theta_i)
\end{aligned}$$

The two results when $N = 6$ and $N = 100$ respectively can be visualized by the figure 2, which is the same as figure 1:



2, Trajectory of a Suspended Wire

Now, the chain is substituted by a wire and the problem becomes a continuous problem. Let s the distance along the wire. The positions along the wire obey

$$\frac{d}{ds}\begin{bmatrix} z_s \\ y_s \end{bmatrix} = f(\theta_s) = \begin{bmatrix} \cos(\theta_s) \\ \sin(\theta_s) \end{bmatrix}$$

where the boundary conditions are:

$$\begin{bmatrix} z \\ y \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} z \\ y \end{bmatrix}_L = \begin{bmatrix} h \\ 0 \end{bmatrix}$$

So the potential energy in steady state can be expressed by:

$$J = \int_0^S \rho g y / S \mathrm{d}s$$

where

$$\phi(\theta_S) = 0$$
$$L(y_s) = \rho g y / S$$

The Hamiltonian function is:

$$H_s = \rho g y / S + \lambda^z \cos(\theta) + \lambda^y \sin(\theta)$$

Euler-Lagrange equations are:

$$\dot{z} = \cos(\theta)$$
$$\dot{y} = \sin(\theta)$$
$$-\dot{\lambda}^z = 0$$
$$-\dot{\lambda}^y = \rho g / S$$
$$0 = -\lambda^z \sin(\theta) + \lambda^y \cos(\theta)$$

The equations in `ode45` can be expressed as:

$$\theta = \arctan\left(\lambda^y/\lambda^z\right) \dot{z} = \cos\left(\theta\right)$$
$$\dot{y} = \sin\left(\theta\right)$$
$$\dot{\lambda}^z = 0$$
$$\dot{\lambda}^y = -\rho g/S$$

The results are compared with results when the number of sections of a chain is 6: