KKT – class exercise and answers

Solve the following problem:

Problem 1. $f_0(x) = e^{x_1-x_2} - x_1 - x_2 \to \min,\ f_1(x) = x_1 + x_2 - 1 \leq 0, f_2(x) = -x_1 \leq 0, f_3(x) = -x_2 \leq 0$

Solution:

1. This is a convex problem with strict convex objective function: this follows from the strict convexity of the function $t \to e^t$

2. We have inequality constraints and therefore use KKT.
   Lagrange function: $L = \lambda_0(e^{x_1-x_2} - x_1 - x_2) + \lambda_1(x_1 + x_2 - 1) + \lambda_2(-x_1) + \lambda_3(-x_2)$
   Write down the KKT conditions:

   - $L_{x_1} = L_{x_2} = 0 \to \lambda_0(e^{x_1-x_2} - 1) + \lambda_1 - \lambda_2 = \lambda_0(-e^{x_1-x_2} - 1) + \lambda_1 - \lambda_3 = 0)$
   - $\lambda_i \geq 0, i = 1, 2, 3$
   - $\lambda_1(x_1 + x_2 - 1) = \lambda_2(-x_1) = \lambda_3(-x_2) = 0$

   Note that (0,0) is a feasible Slater point, so we may assume $\lambda_0 \geq 0$, say $\imath ambda_0 = 1$

3. Now we need to check all possible cases until we find a solution that suffices all KKT conditions. The cases are that any subset of constraints could be tight (including the empty set).
   We select as the most promising case that the first two inequality constraints are tight: $x_1 + z_2 \leq 0,\ -x_1 \leq 0$. Then $x_1 = 0, x_2 = 1, \lambda_3 = 0$ and this leads to $\lambda_1 = 1 + e^{-1} \geq 0$ and $\lambda_2 = 2e^{-1} \geq 0$. That is, all KKT conditions hold. Note that as soon as we found a point for which all KKT conditions hold, we are done and do not need to check all other cases.

4. The problem has the unique solution (0,1)