DTU42136, Large Scale Optimization using Decomposition

Assignment 1, OptiGas: Optimize Gas Network using Benders Algorithm
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1 Introduction

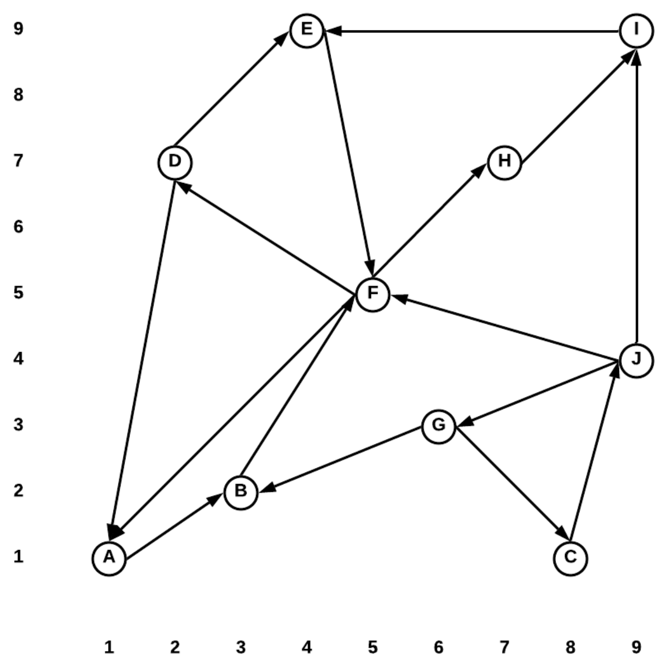

Figure 1. Existing Gas Pipe Network

The cost of sending gas between cities correspond directly to the distance.
New pipelines can be built with fixed cost being 10 times the distance.

2 Problem Formulation

Definition of parameters and variables for OptGas is shown in the following tab1:

Name	Math Notation	Code Notation
distance between node *m* and node *n*	β_n^m	mat_distance[m, n]
arc or not from node *m* to node *n*	α_n^m	mat_arcTwoNodes[m, n]
fixed cost if new arc from node *m* to node *n*	ρ_n^m	mat_fixedCost[m, n]
minimun injected gas to *n*	κ_n	vec_netInject[n]
maximum ejected gas from *m*	ω_m	vec_netEject[m]
pumped gas from *m* to *n*	x_n^m	mat_x[m, n]
arc or not from node *m* to node *n*	y_n^m	mat_y[m, n]

Table 1. Definition of Parameters and Variables in OptGas Problem

The mixed integer linear programming problem for OptGas is:

$$\min \quad \sum_{m,n} \beta_n^m x_n^m + \sum_{m,n} \rho_n^m \left(y_n^m - \alpha_n^m\right) \tag{1}$$

$$\text{s.t.} \quad x_n^m \leq y_n^m \times 170 \quad \forall m,n \tag{2}$$

$$y_n^m \geq \alpha_n^m \quad \forall m,n \tag{3}$$

$$\sum_{n'} x_m^{n'} - \sum_n x_n^m \geq \min\{\kappa_m, -\omega_m\} \quad \forall m \tag{4}$$

$$x_n^m \geq 0 \quad \forall m,n \tag{5}$$

$$y_n^m \in \{0,1\} \quad \forall m,n \tag{6}$$

The standard mixed integer linear programming (MILP) is:

$$\min \quad \mathbf{c}^T\mathbf{x} + \mathbf{f}^T\mathbf{y} \tag{7}$$

$$\text{s.t.} \quad \mathbf{Ax} + \mathbf{By} \geq \mathbf{b} \tag{8}$$

$$\mathbf{y} \in \mathbf{Y} \tag{9}$$

$$\mathbf{x} \geq 0 \tag{10}$$

where the **y** is vector of integer variables.

With the following transformation equations, the OptGas can be expressed by the standard MILP:

$$\mathbf{x} = \left[x_1^1, x_2^1, ..., x_{10}^1, x_1^2, x_2^2, ..., x_n^m, x_{10}^{10}\right]^T \tag{11}$$

$$\mathbf{y} = \left[y_1^1, y_2^1, ..., y_{10}^1, y_1^2, y_2^2, ..., y_n^m, y_{10}^{10}\right]^T \tag{12}$$

$$\mathbf{c} = \left[\beta_1^1, \beta_2^1, ..., \beta_{10}^1, \beta_1^2, \beta_2^2, ..., \beta_n^m, \beta_{10}^{10}\right]^T \tag{13}$$

$$\mathbf{f} = \left[\rho_1^1, \rho_2^1, ..., \rho_{10}^1, \rho_1^2, \rho_2^2, ..., \rho_n^m, \rho_{10}^{10}\right]^T \tag{14}$$

$$\mathbf{Y} = \{0,1\} \tag{15}$$

$$\mathbf{A_1} = -\mathbf{I}_{100} \tag{16}$$

$$\mathbf{A_2} = \mathbf{0}_{100} \tag{17}$$

$$\mathbf{A_3} = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1, & 1 & 0 & 0 & \cdots & 0, & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots & 0, & -1 & 0 & -1 & \cdots & -1, & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots & 0, & 0 & 0 & 1 & \cdots & 0, & -1 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{18}$$

$$\mathbf{B_1} = 170 \times \mathbf{I}_{100} \tag{19}$$

$$\mathbf{B_2} = \mathbf{I}_{100} \tag{20}$$

$$\mathbf{B_3} = \mathbf{0}_{100} \tag{21}$$

$$\mathbf{b} = \left[\underbrace{0, 0, ..., 0}_{100}, \underbrace{\alpha_1^1, \alpha_2^1, ..., \alpha_1^2, \alpha_2^2, ...\alpha_n^m, ..., \alpha_{10}^{10}}_{100} \quad , \underbrace{(\kappa_1 - \omega_1), (\kappa_2 - \omega_2), ..., (\kappa_m - \omega_m), ..., (\kappa_{10} - \omega_{10})}_{10}\right]^T \tag{22}$$

The master problem in Benders Decomposition of MILP is:

$$\min \quad \mathbf{f}^T\mathbf{y} + q \tag{23}$$

$$\text{s.t.} \quad \overline{\mathbf{u_j^r}} \cdot (\mathbf{b} - \mathbf{By}) \leq 0 \quad \forall j \tag{24}$$

$$\overline{\mathbf{u_i^p}} \cdot (\mathbf{b} - \mathbf{By}) \leq q \quad \forall i \tag{25}$$

$$y \in \mathbf{Y} \tag{26}$$

$$q \in R \tag{27}$$

The sub problem in Benders Decomposition of MILP is:

$$\max \quad (\mathbf{b} - \mathbf{B}\overline{\mathbf{y}})^T\mathbf{u} \tag{28}$$

$$\text{s.t.} \quad \mathbf{A}^T\mathbf{u} \leq \mathbf{c} \tag{29}$$

$$\mathbf{u} \geq \mathbf{0} \tag{30}$$

The ray problem in Benders Decomposition of MILP is:

$$\max \quad 1 \tag{31}$$

$$\text{s.t.} \quad (\mathbf{b}-\mathbf{B}\overline{\mathbf{y}})^T\mathbf{u} = 1 \tag{32}$$

$$\mathbf{A}^T\mathbf{u} \leq \mathbf{0} \tag{33}$$

$$\mathbf{u} \geq \mathbf{0} \tag{34}$$

Then, it can be solved by the "BendersMilp_EDXU.jl", and the result is:

```
boundUp: 1270.0, boundLow: 1270.0, difference: 0.0
vec_x: [0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        10, 10, 0,  10, 40, 0, 0,  20, 0,  0;
        0,  20, 20, 0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 65, 0,  15, 0]
vec_y: [0 1 0 0 0 0 0 0 0 0;
        0 0 0 0 0 1 0 0 0 0;
        0 0 0 0 0 0 0 0 0 1;
        1 0 0 0 1 0 0 0 0 0;
        0 0 0 0 0 1 0 0 0 0;
        1 1 0 1 1 0 0 1 0 0;
        0 1 1 0 0 0 0 0 0 0;
        0 0 0 0 0 0 0 0 1 0;
        0 0 0 0 1 0 0 0 0 0;
        0 0 1 0 0 1 1 1 1 0]
```

Here is a table showing the objectives from the sub-problem and master problem of each iteration.

```
 ─────  ─────────  ───────────  ───────────  ──────────  ──────────  ─────────────
  Seq   boundUp    boundLow     obj_mas      q           type_sub    obj_sub/ray
 ─────  ─────────  ───────────  ───────────  ──────────  ──────────  ─────────────
  1     Inf        20.0         20.0         0.0         ray         1.0
  2     Inf        20.0         20.0         0.0         ray         1.0
  3     Inf        30.0         30.0         0.0         ray         1.0
  4     Inf        30.0         30.0         0.0         ray         1.0
  5     Inf        30.0         30.0         0.0         ray         1.0
  6     Inf        39.86667     39.86667     −0.13333    ray         1.0
  7     Inf        40.0         40.0         0.0         ray         1.0
  8     Inf        40.0         40.0         0.0         ray         1.0
  9     Inf        40.0         40.0         0.0         ray         1.0
  10    Inf        49.82759     49.82759     −0.17241    ray         1.0
  11    Inf        49.86667     49.86667     −0.13333    ray         1.0
  12    Inf        49.86667     49.86667     −0.13333    ray         1.0
  13    Inf        49.9375      49.9375      −0.0625     ray         1.0
  14    Inf        49.9375      49.9375      −0.0625     ray         1.0
  15    Inf        50.0         50.0         −0.0        ray         1.0
  16    Inf        50.0         50.0         0.0         ray         1.0
  17    Inf        50.0         50.0         −0.0        ray         1.0
  18    Inf        59.82759     59.82759     −0.17241    ray         1.0
  19    Inf        59.86667     59.86667     −0.13333    ray         1.0
  20    Inf        59.9375      59.9375      −0.0625     ray         1.0
  21    Inf        69.69231     69.69231     −0.30769    ray         1.0
  22    Inf        69.82759     69.82759     −0.17241    ray         1.0
  23    Inf        69.86667     69.86667     −0.13333    ray         1.0
  24    Inf        79.64        79.64        −0.36       ray         1.0
  25    Inf        79.64        79.64        −0.36       ray         1.0
  26    Inf        79.69231     79.69231     −0.30769    ray         1.0
  27    Inf        79.78571     79.78571     −0.21429    ray         1.0
  28    Inf        79.78571     79.78571     −0.21429    ray         1.0
  29    Inf        79.82759     79.82759     −0.17241    ray         1.0
  30    Inf        79.82759     79.82759     −0.17241    ray         1.0
  31    Inf        79.82759     79.82759     −0.17241    ray         1.0
  32    Inf        79.86667     79.86667     −0.13333    ray         1.0
```

```
33   Inf   79.86667    79.86667    −0.13333   ray   1.0
34   Inf   79.86667    79.86667    −0.13333   ray   1.0
35   Inf   79.86667    79.86667    −0.13333   ray   1.0
36   Inf   80.0        80.0        0.0        ray   1.0
37   Inf   89.64       89.64       −0.36      ray   1.0
38   Inf   89.69231    89.69231    −0.30769   ray   1.0
39   Inf   89.69231    89.69231    −0.30769   ray   1.0
40   Inf   89.74074    89.74074    −0.25926   ray   1.0
41   Inf   89.74074    89.74074    −0.25926   ray   1.0
42   Inf   89.78571    89.78571    −0.21429   ray   1.0
43   Inf   89.78571    89.78571    −0.21429   ray   1.0
44   Inf   89.78571    89.78571    −0.21429   ray   1.0
45   Inf   89.82759    89.82759    −0.17241   ray   1.0
46   Inf   89.82759    89.82759    −0.17241   ray   1.0
47   Inf   89.86667    89.86667    −0.13333   ray   1.0
48   Inf   99.38095    99.38095    −0.61905   ray   1.0
49   Inf   99.52174    99.52174    −0.47826   ray   1.0
50   Inf   99.52174    99.52174    −0.47826   ray   1.0
51   Inf   99.58333    99.58333    −0.41667   ray   1.0
52   Inf   99.58333    99.58333    −0.41667   ray   1.0
53   Inf   99.64       99.64       −0.36      ray   1.0
54   Inf   99.64       99.64       −0.36      ray   1.0
55   Inf   99.64       99.64       −0.36      ray   1.0
56   Inf   99.64       99.64       −0.36      ray   1.0
57   Inf   99.64       99.64       −0.36      ray   1.0
58   Inf   99.64       99.64       −0.36      ray   1.0
59   Inf   99.74074    99.74074    −0.25926   ray   1.0
60   Inf   99.74074    99.74074    −0.25926   ray   1.0
61   Inf   99.74074    99.74074    −0.25926   ray   1.0
62   Inf   99.74074    99.74074    −0.25926   ray   1.0
63   Inf   99.82759    99.82759    −0.17241   ray   1.0
64   Inf   99.82759    99.82759    −0.17241   ray   1.0
65   Inf   109.45455   109.45455   −0.54545   ray   1.0
66   Inf   109.45455   109.45455   −0.54545   ray   1.0
67   Inf   109.45455   109.45455   −0.54545   ray   1.0
68   Inf   109.52174   109.52174   −0.47826   ray   1.0
69   Inf   109.52174   109.52174   −0.47826   ray   1.0
70   Inf   109.58333   109.58333   −0.41667   ray   1.0
71   Inf   109.69231   109.69231   −0.30769   ray   1.0
72   Inf   109.69231   109.69231   −0.30769   ray   1.0
73   Inf   109.74074   109.74074   −0.25926   ray   1.0
74   Inf   109.74074   109.74074   −0.25926   ray   1.0
75   Inf   109.78571   109.78571   −0.21429   ray   1.0
76   Inf   119.21053   119.21053   −0.78947   ray   1.0
77   Inf   119.38095   119.38095   −0.61905   ray   1.0
78   Inf   119.38095   119.38095   −0.61905   ray   1.0
79   Inf   119.38095   119.38095   −0.61905   ray   1.0
80   Inf   119.38095   119.38095   −0.61905   ray   1.0
81   Inf   119.38095   119.38095   −0.61905   ray   1.0
82   Inf   119.45455   119.45455   −0.54545   ray   1.0
83   Inf   119.52174   119.52174   −0.47826   ray   1.0
84   Inf   119.58333   119.58333   −0.41667   ray   1.0
85   Inf   119.64      119.64      −0.36      ray   1.0
86   Inf   119.69231   119.69231   −0.30769   ray   1.0
87   Inf   129.21053   129.21053   −0.78947   ray   1.0
88   Inf   129.21053   129.21053   −0.78947   ray   1.0
89   Inf   129.3       129.3       −0.7       ray   1.0
90   Inf   129.38095   129.38095   −0.61905   ray   1.0
91   Inf   129.38095   129.38095   −0.61905   ray   1.0
92   Inf   129.52174   129.52174   −0.47826   ray   1.0
93   Inf   129.58333   129.58333   −0.41667   ray   1.0
94   Inf   129.69231   129.69231   −0.30769   ray   1.0
95   Inf   129.69231   129.69231   −0.30769   ray   1.0
96   Inf   139.0       139.0       −1.0       ray   1.0
97   Inf   139.3       139.3       −0.7       ray   1.0
98   Inf   139.38095   139.38095   −0.61905   ray   1.0
99   Inf   139.45455   139.45455   −0.54545   ray   1.0
100  Inf   139.64      139.64      −0.36      ray   1.0
101  Inf   148.875     148.875     −1.125     ray   1.0
102  Inf   148.875     148.875     −1.125     ray   1.0
```

```
 103   Inf      149.0      149.0      -1.0      ray    1.0
 104   Inf      149.0      149.0      -1.0      ray    1.0
 105   Inf      149.0      149.0      -1.0      ray    1.0
 106   Inf      149.0      149.0      -1.0      ray    1.0
 107   Inf      149.21053  149.21053  -0.78947  ray    1.0
 108   Inf      149.21053  149.21053  -0.78947  ray    1.0
 109   Inf      149.38095  149.38095  -0.61905  ray    1.0
 110   Inf      159.0      159.0      -1.0      ray    1.0
 111   Inf      159.0      159.0      -1.0      ray    1.0
 112   Inf      168.73333  168.73333  -1.26667  ray    1.0
 113   Inf      168.73333  168.73333  -1.26667  ray    1.0
 114   Inf      169.0      169.0      -1.0      ray    1.0
 115   Inf      169.0      169.0      -1.0      ray    1.0
 116   Inf      169.0      169.0      -1.0      ray    1.0
 117   Inf      169.58333  169.58333  -0.41667  ray    1.0
 118   Inf      189.0      189.0      -1.0      ray    1.0
 119   Inf      210.0      210.0      -0.0      ray    1.0
 120   Inf      240.0      240.0      -0.0      ray    1.0
 121   Inf      270.0      270.0      -0.0      ray    1.0
 122   Inf      270.0      270.0      -0.0      ray    1.0
 123   Inf      280.0      280.0      -0.0      ray    1.0
 124   Inf      280.0      280.0      0.0       ray    1.0
 125   Inf      290.0      290.0      0.0       ray    1.0
 126   Inf      330.0      330.0      0.0       ray    1.0
 127   Inf      380.0      380.0      0.0       ray    1.0
 128   Inf      390.0      390.0      0.0       ray    1.0
 129   Inf      390.0      390.0      0.0       ray    1.0
 130   Inf      400.0      400.0      0.0       ray    1.0
 131   Inf      400.0      400.0      0.0       ray    1.0
 132   Inf      440.0      440.0      0.0       ray    1.0
 133   Inf      450.0      450.0      0.0       ray    1.0
 134   Inf      480.0      480.0      0.0       ray    1.0
 135   Inf      490.0      490.0      0.0       ray    1.0
 136   Inf      520.0      520.0      0.0       ray    1.0
 137   Inf      540.0      540.0      0.0       ray    1.0
 138   1280.0   640.0      640.0      0.0       sub    740.0
 139   1280.0   930.0      930.0      300.0     sub    640.0
 140   1270.0   1110.0     1110.0     470.0     sub    640.0
 141   1270.0   1110.0     1110.0     470.0     sub    640.0
 142   1270.0   1140.0     1140.0     470.0     sub    660.0
 143   1270.0   1210.0     1210.0     640.0     sub    600.0
 144   1270.0   1230.0     1230.0     660.0     sub    720.0
 145   1270.0   1240.0     1240.0     640.0     sub    700.0
 146   1270.0   1240.0     1240.0     640.0     sub    680.0
 147   1270.0   1240.0     1240.0     660.0     sub    680.0
 148   1270.0   1250.0     1250.0     640.0     sub    700.0
 149   1270.0   1270.0     1270.0     600.0     sub    680.0
 ─────  ─────────  ───────────  ───────────  ──────────  ──────────  ─────────────
```

There is a missing part in the objective function of the calculation, which is:

$$\text{obj} = \text{result} - \sum_{m,n} \rho_n^m \alpha_n^m \tag{35}$$

There is something wrong in the result. Because the q is actually 600 in the final iteration, the iteration causes the iteration to stop by raising the lower bound by 20. The q doesn't equal to "obj_sub", and it's 600 all the time since about 131-th iteration. Finally, it just return a result with sum equals 1270. while I can't figure out how the result can become 1270 with q being 680 instead of 600.

I think the most promising result is:

```
obj: 1280
vec_x: [0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
        10, 10, 0,  10, 40, 0, 0,  20, 0,  0;
        0,  20, 20, 0,  0,  0, 0,  0,  0,  0;
        0,  0,  0,  0,  0,  0, 0,  0,  0,  0;
```

```
          0,   0,   0,   0,   0,   0, 0,   0,   0,   0;
          0,   0,   0,   0,   0,   0, 65,  0,   15,  0]
vec_y: [0 1 0 0 0 0 0 0 0 0;
         0 0 0 0 0 1 0 0 0 0;
         0 0 0 0 0 0 0 0 0 1;
         1 0 0 0 1 0 0 0 0 0;
         0 0 0 0 0 1 0 0 0 0;
         1 1 0 1 1 0 0 1 0 0;
         0 1 1 0 0 0 0 0 0 0;
         0 0 0 0 0 0 0 0 1 0;
         0 0 0 0 1 0 0 0 0 0;
         0 0 0 0 0 1 1 0 1 0]
```

on which I base the following plots.

Using eq.35, the minimum objective from the optimization is 740.

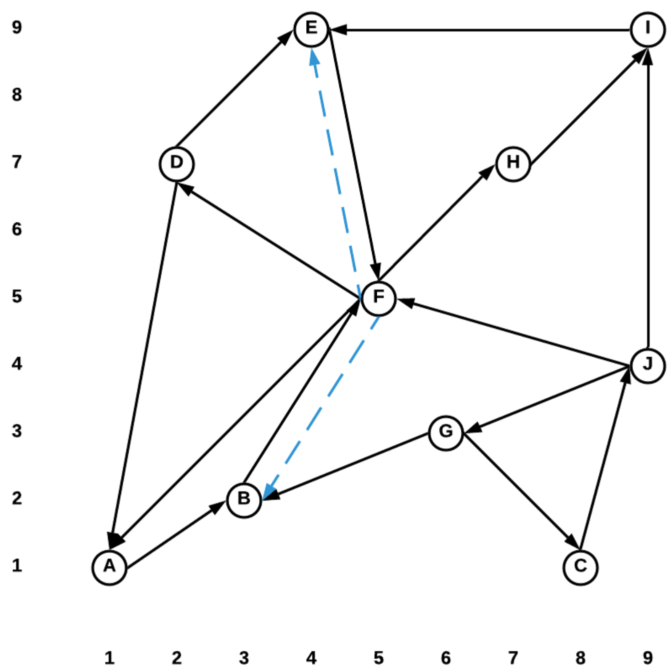

Figure 2. Existing Gas Pipe Network and Newly Built Pipes

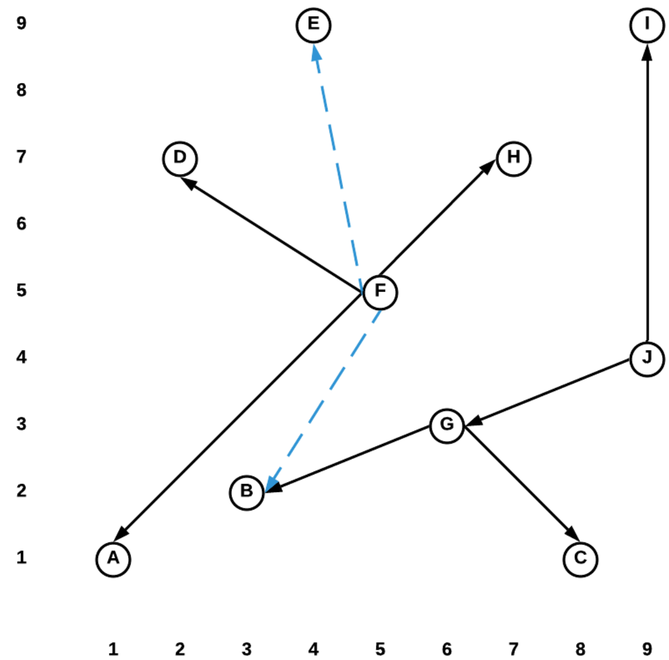

Figure 3. Operating Pipes in Existing Gas Pipe Network and Newly Built Pipes

3 Question 5 6

When OptiGas considers building their own gas-network, the mixed integer linear programming problem becomes:

$$\min \quad \sum_{m,n} \beta_n^m x_n^m + \sum_{m,n} \rho_n^m \left(y_n^m - \alpha_n^m\right) \tag{36}$$

$$\text{s.t.} \quad x_n^m \le y_n^m \times 170 \quad \forall m,n \tag{37}$$

$$\sum_{n'} x_m^{n'} - \sum_n x_n^m \ge \min\{\kappa_m, -\omega_m\} \quad \forall m \tag{38}$$

$$x_n^m \ge 0 \quad \forall m,n \tag{39}$$

$$y_n^m \in \{0,1\} \quad \forall m,n \tag{40}$$

With the following transformation equations, the OptGas can be expressed by the standard MILP:

$$\mathbf{x} = \left[x_1^1, x_2^1, \ldots, x_{10}^1, x_1^2, x_2^2, \ldots, x_n^m, x_{10}^{10}\right]^T \tag{41}$$

$$\mathbf{y} = \left[y_1^1, y_2^1, \ldots, y_{10}^1, y_1^2, y_2^2, \ldots, y_n^m, y_{10}^{10}\right]^T \tag{42}$$

$$\mathbf{c} = \left[\beta_1^1, \beta_2^1, \ldots, \beta_1^2, \beta_2^2, \ldots, \beta_n^m, \beta_{10}^{10}\right]^T \tag{43}$$

$$\mathbf{f} = \left[\rho_1^1, \rho_2^1, \ldots, \rho_1^2, \rho_2^2, \ldots, \rho_n^m, \rho_{10}^{10}\right]^T \tag{44}$$

$$\mathbf{Y} = \{0,1\} \tag{45}$$

$$\mathbf{A_1} = -\mathbf{I}_{100} \tag{46}$$

$$\mathbf{A_2} = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1, & 1 & 0 & 0 & \cdots & 0, & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots & 0, & -1 & 0 & -1 & \cdots & -1, & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots & 0, & 0 & 0 & 1 & \cdots & 0, & -1 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{47}$$

$$\mathbf{B_1} = 170 \times \mathbf{I}_{100} \tag{48}$$

$$\mathbf{B_2} = \mathbf{0}_{100} \tag{49}$$

$$\mathbf{b} = \left[\underbrace{0, 0, ..., 0}_{100}, \underbrace{(\kappa_1 - \omega_1), (\kappa_2 - \omega_2), ..., (\kappa_m - \omega_m), ..., (\kappa_{10} - \omega_{10})}_{10} \right]^T \tag{50}$$

Then, it can be solved by the "BendersMilp_EDXU.jl", and the result is:

```
boundUp: 830.0, boundLow: 830.0, difference: 0.0
vec_x: [0   0   0   0   0   0  0   0   0   0;
        10  0   0   0   0   0  0   0   0   0;
        0   0   0   0   0   0  0   0   0   0;
        0   0   0   0   0   0  0   0   0   0;
        0   0   0   0   0   0  0   0   0   0;
        0   40  0   10  40  0  0   0   0   0;
        0   0   0   0   0   0  0   0   0   0;
        0   0   0   0   0   0  0   0   15  0;
        0   0   0   0   0   0  0   0   0   0;
        0   0   20  0   0   0  25  35  0   0]
vec_y: [0  0  0  0  0  0  0  0  0  0
        1  0  0  0  0  0  0  0  0  0
        0  0  0  0  0  0  0  0  0  0
        0  0  0  0  0  0  0  0  0  0
        0  0  0  0  0  0  0  0  0  0
        0  1  0  1  1  0  0  0  0  0
        0  0  0  0  0  0  0  0  0  0
        0  0  0  0  0  0  0  0  1  0
        0  0  0  0  0  0  0  0  0  0
        0  0  1  0  0  0  1  1  0  0]
```

Here is a table showing the objectives from the sub-problem and master problem of each iteration.

```
 ─────  ─────────  ───────────  ───────────  ──────────  ──────────  ─────────────
  Seq    boundUp    boundLow     obj_mas      q           type_sub    obj_sub/ray
 ─────  ─────────  ───────────  ───────────  ──────────  ──────────  ─────────────
  1      Inf        20.0         20.0         0.0         ray         1.0
  2      Inf        20.0         20.0         0.0         ray         1.0
  3      Inf        30.0         30.0         0.0         ray         1.0
  4      Inf        30.0         30.0         0.0         ray         1.0
  5      Inf        30.0         30.0         0.0         ray         1.0
  6      Inf        39.86667     39.86667     -0.13333    ray         1.0
  7      Inf        40.0         40.0         0.0         ray         1.0
  8      Inf        40.0         40.0         0.0         ray         1.0
  9      Inf        40.0         40.0         0.0         ray         1.0
  10     Inf        49.82759     49.82759     -0.17241    ray         1.0
  11     Inf        49.86667     49.86667     -0.13333    ray         1.0
  12     Inf        49.86667     49.86667     -0.13333    ray         1.0
  13     Inf        49.9375      49.9375      -0.0625     ray         1.0
  14     Inf        49.9375      49.9375      -0.0625     ray         1.0
  15     Inf        50.0         50.0         -0.0        ray         1.0
  16     Inf        50.0         50.0         0.0         ray         1.0
  17     Inf        50.0         50.0         -0.0        ray         1.0
  18     Inf        59.82759     59.82759     -0.17241    ray         1.0
  19     Inf        59.86667     59.86667     -0.13333    ray         1.0
  20     Inf        59.9375      59.9375      -0.0625     ray         1.0
  21     Inf        69.69231     69.69231     -0.30769    ray         1.0
  22     Inf        69.82759     69.82759     -0.17241    ray         1.0
  23     Inf        69.86667     69.86667     -0.13333    ray         1.0
  24     Inf        79.64        79.64        -0.36       ray         1.0
  25     Inf        79.64        79.64        -0.36       ray         1.0
  26     Inf        79.69231     79.69231     -0.30769    ray         1.0
  27     Inf        79.78571     79.78571     -0.21429    ray         1.0
  28     Inf        79.78571     79.78571     -0.21429    ray         1.0
  29     Inf        79.82759     79.82759     -0.17241    ray         1.0
  30     Inf        79.82759     79.82759     -0.17241    ray         1.0
  31     Inf        79.82759     79.82759     -0.17241    ray         1.0
  32     Inf        79.86667     79.86667     -0.13333    ray         1.0
  33     Inf        79.86667     79.86667     -0.13333    ray         1.0
  34     Inf        79.86667     79.86667     -0.13333    ray         1.0
  35     Inf        79.86667     79.86667     -0.13333    ray         1.0
  36     Inf        80.0         80.0         0.0         ray         1.0
  37     Inf        89.64        89.64        -0.36       ray         1.0
```

38	Inf	89.69231	89.69231	−0.30769	ray	1.0
39	Inf	89.69231	89.69231	−0.30769	ray	1.0
40	Inf	89.74074	89.74074	−0.25926	ray	1.0
41	Inf	89.74074	89.74074	−0.25926	ray	1.0
42	Inf	89.78571	89.78571	−0.21429	ray	1.0
43	Inf	89.78571	89.78571	−0.21429	ray	1.0
44	Inf	89.78571	89.78571	−0.21429	ray	1.0
45	Inf	89.82759	89.82759	−0.17241	ray	1.0
46	Inf	89.82759	89.82759	−0.17241	ray	1.0
47	Inf	89.86667	89.86667	−0.13333	ray	1.0
48	Inf	99.38095	99.38095	−0.61905	ray	1.0
49	Inf	99.52174	99.52174	−0.47826	ray	1.0
50	Inf	99.52174	99.52174	−0.47826	ray	1.0
51	Inf	99.58333	99.58333	−0.41667	ray	1.0
52	Inf	99.58333	99.58333	−0.41667	ray	1.0
53	Inf	99.64	99.64	−0.36	ray	1.0
54	Inf	99.64	99.64	−0.36	ray	1.0
55	Inf	99.64	99.64	−0.36	ray	1.0
56	Inf	99.64	99.64	−0.36	ray	1.0
57	Inf	99.64	99.64	−0.36	ray	1.0
58	Inf	99.64	99.64	−0.36	ray	1.0
59	Inf	99.74074	99.74074	−0.25926	ray	1.0
60	Inf	99.74074	99.74074	−0.25926	ray	1.0
61	Inf	99.74074	99.74074	−0.25926	ray	1.0
62	Inf	99.74074	99.74074	−0.25926	ray	1.0
63	Inf	99.82759	99.82759	−0.17241	ray	1.0
64	Inf	99.82759	99.82759	−0.17241	ray	1.0
65	Inf	109.45455	109.45455	−0.54545	ray	1.0
66	Inf	109.45455	109.45455	−0.54545	ray	1.0
67	Inf	109.45455	109.45455	−0.54545	ray	1.0
68	Inf	109.52174	109.52174	−0.47826	ray	1.0
69	Inf	109.52174	109.52174	−0.47826	ray	1.0
70	Inf	109.58333	109.58333	−0.41667	ray	1.0
71	Inf	109.69231	109.69231	−0.30769	ray	1.0
72	Inf	109.69231	109.69231	−0.30769	ray	1.0
73	Inf	109.74074	109.74074	−0.25926	ray	1.0
74	Inf	109.74074	109.74074	−0.25926	ray	1.0
75	Inf	109.78571	109.78571	−0.21429	ray	1.0
76	Inf	119.21053	119.21053	−0.78947	ray	1.0
77	Inf	119.38095	119.38095	−0.61905	ray	1.0
78	Inf	119.38095	119.38095	−0.61905	ray	1.0
79	Inf	119.38095	119.38095	−0.61905	ray	1.0
80	Inf	119.38095	119.38095	−0.61905	ray	1.0
81	Inf	119.38095	119.38095	−0.61905	ray	1.0
82	Inf	119.45455	119.45455	−0.54545	ray	1.0
83	Inf	119.52174	119.52174	−0.47826	ray	1.0
84	Inf	119.58333	119.58333	−0.41667	ray	1.0
85	Inf	119.64	119.64	−0.36	ray	1.0
86	Inf	119.69231	119.69231	−0.30769	ray	1.0
87	Inf	129.21053	129.21053	−0.78947	ray	1.0
88	Inf	129.21053	129.21053	−0.78947	ray	1.0
89	Inf	129.3	129.3	−0.7	ray	1.0
90	Inf	129.38095	129.38095	−0.61905	ray	1.0
91	Inf	129.38095	129.38095	−0.61905	ray	1.0
92	Inf	129.52174	129.52174	−0.47826	ray	1.0
93	Inf	129.58333	129.58333	−0.41667	ray	1.0
94	Inf	129.69231	129.69231	−0.30769	ray	1.0
95	Inf	129.69231	129.69231	−0.30769	ray	1.0
96	Inf	139.0	139.0	−1.0	ray	1.0
97	Inf	139.3	139.3	−0.7	ray	1.0
98	Inf	139.38095	139.38095	−0.61905	ray	1.0
99	Inf	139.45455	139.45455	−0.54545	ray	1.0
100	Inf	139.64	139.64	−0.36	ray	1.0
101	Inf	148.875	148.875	−1.125	ray	1.0
102	Inf	148.875	148.875	−1.125	ray	1.0
103	Inf	149.0	149.0	−1.0	ray	1.0
104	Inf	149.0	149.0	−1.0	ray	1.0
105	Inf	149.0	149.0	−1.0	ray	1.0
106	Inf	149.0	149.0	−1.0	ray	1.0
107	Inf	149.21053	149.21053	−0.78947	ray	1.0

```
108   Inf     149.21053   149.21053   −0.78947   ray   1.0
109   Inf     149.38095   149.38095   −0.61905   ray   1.0
110   Inf     159.0       159.0       −1.0       ray   1.0
111   Inf     159.0       159.0       −1.0       ray   1.0
112   Inf     168.73333   168.73333   −1.26667   ray   1.0
113   Inf     168.73333   168.73333   −1.26667   ray   1.0
114   Inf     169.0       169.0       −1.0       ray   1.0
115   Inf     169.0       169.0       −1.0       ray   1.0
116   Inf     169.0       169.0       −1.0       ray   1.0
117   Inf     169.58333   169.58333   −0.41667   ray   1.0
118   Inf     189.0       189.0       −1.0       ray   1.0
119   Inf     189.0       189.0       −1.0       ray   1.0
120   Inf     189.0       189.0       −1.0       ray   1.0
121   Inf     197.6       197.6       −2.4       ray   1.0
122   880.0   210.0       210.0       −0.0       sub   680.0
123   880.0   210.0       210.0       −0.0       ray   1.0
124   880.0   220.0       220.0       −0.0       sub   880.0
125   880.0   220.0       220.0       −0.0       ray   1.0
126   880.0   220.0       220.0       −0.0       ray   1.0
127   880.0   260.0       260.0       −0.0       sub   720.0
128   880.0   270.0       270.0       −0.0       sub   680.0
129   880.0   310.0       310.0       −0.0       sub   740.0
130   880.0   800.0       800.0       600.0      sub   600.0
131   880.0   800.0       800.0       600.0      ray   1.0
132   880.0   810.0       810.0       600.0      sub   690.0
133   880.0   810.0       810.0       600.0      ray   1.0
134   880.0   810.0       810.0       600.0      sub   670.0
135   880.0   810.0       810.0       600.0      ray   1.0
136   850.0   810.0       810.0       600.0      sub   640.0
137   850.0   810.0       810.0       600.0      sub   720.0
138   850.0   820.0       820.0       600.0      sub   860.0
139   850.0   820.0       820.0       600.0      sub   640.0
140   850.0   820.0       820.0       600.0      sub   630.0
141   850.0   820.0       820.0       600.0      sub   650.0
142   850.0   820.0       820.0       600.0      ray   1.0
143   850.0   820.0       820.0       600.0      sub   730.0
144   850.0   820.0       820.0       600.0      sub   680.0
145   850.0   820.0       820.0       600.0      sub   690.0
146   850.0   820.0       820.0       600.0      sub   680.0
147   850.0   830.0       830.0       600.0      sub   750.0
148   850.0   830.0       830.0       600.0      sub   640.0
149   850.0   830.0       830.0       600.0      sub   670.0
150   830.0   830.0       830.0       600.0      sub   600.0
————— ————————— ——————————— ——————————— —————————— —————————— ————————————
```

The final result is this result, which is 830.

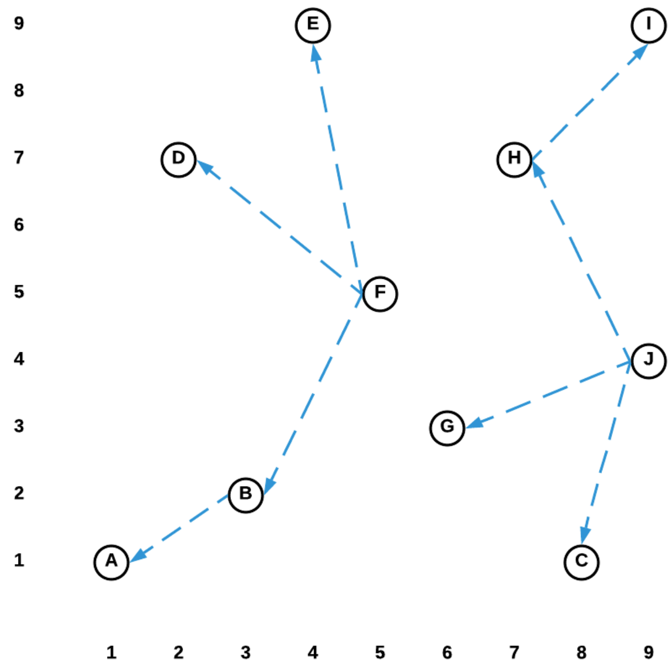

Figure 4. Brand-Newly Built Pipe Network

4 Question 7

What happens in the Benders Decomposition if the original problem is infeasible in the x variables?

Benders Decomposition can not solve the sub or ray problem.

5 Question 8

What happens in the Benders Decomposition if the original problem is infeasible in the y variables?

Benders Decomposition can not solve the master problem.

6 Question 9 & 10

Assume Benders Decomposition is applied to a minimization MIP problem. Is the upper bound guaranteed to decrease or at least not increase during the execution of the Benders Decomposition? Besides, is the lower bound guaranteed to increase or at least not decrease during the execution of the Benders Decomposition?

Both yes. Because for every iteration, the assignment of upper bound is selected the minimum between last upper bound and latest result, so the the new upper bound must be less than or equal to the last the upper bound. The same reason goes for the assignment of lower bound, except only the maximum is selected. The code to do that is shown below:

```
boundLow = max(boundLow, obj_mas)
boundUp = min(boundUp, obj_sub + (transpose(vec_f) * vec_yBar)[1])
```