NonlinearOptim-Assign

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1 Part A: Method of Lagrange Multipliers

The decision variables in the problem are summarized in the following table.

Variables	Definition
x_1	num of dinner
x_2	num of drinks

Table 1. Features of Variables

The problem can be formulated as:

$$\begin{aligned} \max \quad & 8 \ln x_1 + \ln x_2 \\ \text{s.t.} \quad & 35x_1 + 5x_2 \leq 315 \\ & x_1, x_2 \geq 1 \\ & x_1, x_2 \in \mathbb{R} \end{aligned} \tag{1}$$

There exists an optimal solution according to Weierstrass theorem, because The function is continuous, and coercive for maximization (check that objective function increases monotonically when x_1 or x_2 increase and there are constraints). Besides, the less equation sign can be replaced by equal sign.

According to the method of Lagrange multipliers, the Lagrangian function of the above problem is:

$$L(x_1, x_2, \lambda) = 8 \ln x_1 + \ln x_2 - \lambda(35x_1 + 5x_2 - 315) \tag{2}$$

which can be maximized by critical points from the following equations:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 8/x_1 - 35\lambda = 0 \\ \frac{\partial L}{\partial x_2} &= 1/x_2 - 5\lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= 35x_1 + 5x_2 - 315 = 0 \end{aligned} \tag{3}$$

By eliminating λ, we get:

$$\begin{aligned} x_1 &= 8 \\ x_2 &= 7 \\ \lambda &= -1/35 \end{aligned} \tag{4}$$

So, to have 8 dinners and 7 drinks using 315 euros can maximize the utility.

2 Part B

The integrality regrading the bottles is neglected in this part. The decision variabls in the problem are summarized in the following table.

Variables	Definition
x_1	hours spent producing IPA
x_2	hours spent producing Lager

Table 2. Features of Variables

2.1 Part B 1

The problem can be formulated as:

$$\begin{aligned} \max \quad & 15\sqrt{x_1} + 16\sqrt{x_2} \\ \text{s.t.} \quad & x_1 + x_2 \leq 120 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{5}$$

There exists an optimal solution according to Weierstrass theorem, because The function is continuous, and coercive for maximization (check that objective function increases monotonically when x_1 or x_2 increase and there are constraints). Besides, the less equation sign can be replaced by equal sign.

The constraint, $x_1, x_2 \geq 0$, is neglected for the time being. According to the method of Lagrange multipliers, the Lagrangian function of the above problem is:

$$L(x_1, x_2, \lambda) = 15\sqrt{x_1} + 16\sqrt{x_2} - \lambda(x_1 + x_2 - 120) \tag{6}$$

which can be maximized by critical points from the following equations:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 15/2\sqrt{x_1} - \lambda = 0 \\ \frac{\partial L}{\partial x_2} &= 8/\sqrt{x_2} - \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= x_1 + x_2 - 120 = 0 \end{aligned} \tag{7}$$

By eliminating λ, we get:

$$\begin{aligned} x_1 = 56.133 \quad & \text{so that } 3\sqrt{x_1} \approx 22.5 \\ x_2 = 63.867 \quad & \text{so that } 4\sqrt{x_2} \approx 32 \\ \lambda \approx 1 & \\ 15\sqrt{x_1} + 16\sqrt{x_2} = 240.250 & \end{aligned} \tag{8}$$

which respects the constraint, $x_1, x_2 \geq 0$.

So I would allocate 56.133 hours in total to produce 22 and half bottles of IPA, and 63.867 hours in total to produce 32 bottles of Lager. The obtained maximum revenue is 240.250.

2.2 Part B 2

The problem can be formulated as:

$$\begin{aligned} \max \quad & 6\ln 3\sqrt{x_1} + 4\ln 4\sqrt{x_2} \\ \text{s.t.} \quad & x_1 + x_2 \leq 120 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{9}$$

There exists an optimal solution according to Weierstrass theorem, because The function is continuous, and coercive for maximization (check that objective function increases monotonically when x_1 or x_2 increase and there are constraints). Besides, the less equation sign can be replaced by equal sign.

The constraint, $x_1, x_2 \geq 0$, is neglected for the time being. According to the method of Lagrange multipliers, the Lagrangian function of the above problem is:

$$L(x_1, x_2, \lambda) = 6\ln 3\sqrt{x_1} + 4\ln 4\sqrt{x_2} - \lambda(x_1 + x_2 - 120) \tag{10}$$

which can be maximized by critical points from the following equations:

$$
\begin{aligned}
\frac{\partial L}{\partial x_1} &= 3/x_1 - \lambda = 0 \\
\frac{\partial L}{\partial x_2} &= 2/x_2 - \lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= x_1 + x_2 - 120 = 0
\end{aligned}
\tag{11}
$$

By eliminating λ, we get:

$$
\begin{aligned}
&x_1 = 72 \quad \text{so that } 3\sqrt{x_1} = 25.45 \\
&x_2 = 48 \quad \text{so that } 4\sqrt{x_2} = 27.71 \\
&\lambda = 0.042 \\
&6\ln 3\sqrt{x_1} + 4\ln 4\sqrt{x_2} = 32.709
\end{aligned}
\tag{12}
$$

which respects the constraint, $x_1, x_2 \geq 0$.

So I would allocate 72 hours in total to produce 25.45 and half bottles of IPA, and 48 hours in total to produce 27.71 bottles of Lager. The obtained maximum utility is 32.709.

2.3 Part B 3: Production with Carl

How many hours would you ask Carl to help you out to maximize total revenue? How would you divide your own time in producing Lager and IPA in this case? The decision variables are summarized again using the following table.

Variables	Definition
x_1	hours spent producing IPA
x_2	hours spent producing Lager
x_3	Carl's hours spent producing Lager

Table 3. Features of Variables

The problem can be formulated as:

$$
\begin{aligned}
\max \quad & 15\sqrt{x_1} + 16\sqrt{x_2} + 8\sqrt{x_3} \\
\text{s.t.} \quad & x_1 + x_2 + 0.1x_3 \leq 115 \quad \text{if } x_3 > 0 \\
& x_1 + x_2 \leq 120 \quad \text{if } x_3 = 0 \\
& x_3 \leq 60 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
\tag{13}
$$

where there are two convex functional constraints. Furthermore, it can be easily verified that the objective function is convex. Hence, the corollary applies, so any solution that satisfies the KKT conditions will definitely be an optimal solution.

The Lagrangian function of the above problem is:

$$
L(x_1, x_2, x_3, u_1, u_2) = 15\sqrt{x_1} + 16\sqrt{x_2} + 8\sqrt{x_3} - u_1(x_1 + x_2 + 0.1x_3 - 115) - u_2(x_3 - 60) \tag{14}
$$

KKT conditions are listed:

$$
\begin{aligned}
15/2/\sqrt{x_1} - u_1 &\leq 0 \\
x_1(15/2/\sqrt{x_1} - u_1) &= 0 \\
8/\sqrt{x_2} - u_1 &\leq 0 \\
x_2(8/\sqrt{x_2} - u_1) &= 0 \\
4/\sqrt{x_3} - 0.1u_1 - u_2 &\leq 0 \\
x_3(4/\sqrt{x_3} - 0.1u_1 - u_2) &= 0 \\
x_1 + x_2 + 0.1x_3 - 115 &\leq 0 \\
u_1(x_1 + x_2 + 0.1x_3 - 115) &= 0 \\
x_3 - 60 &\leq 0 \\
u_2(x_3 - 60) &= 0 \\
x_1, x_2, x_3 &\geq 0 \\
u_1, u_2 &\geq 0
\end{aligned}
\tag{15}
$$

Listing 1. MATLAB code to solve the equations

```
syms x1 x2 x3 u1 u2
eq1 = 15 / 2  - u1 * sqrt(x1) <= 0;
eq2 = x1 * (15 / 2 - u1 * sqrt(x1)) == 0;
eq3 = - u1 * sqrt(x2) + 8 <= 0;
eq4 = x2 * (- u1 * sqrt(x2) + 8) == 0;
eq5 = - 0.1 * u1 * sqrt(x3) - u2 + 4 <= 0;
eq6 = x3 * (- 0.1 * u1 * sqrt(x3) - u2 + 4) == 0;
eq7 = x1 + x2 + 0.1 * x3 - 115 <= 0;
eq8 = u1 * (x1 + x2 + 0.1 * x3 - 115) == 0;
eq9 = x3 - 60 <= 0;
eq10 = u2 * (x3 - 60) == 0;
eq11 = x1 >= 0;
eq12 = x2 >= 0;
eq13 = x3 >= 0;
result = solve([eq1 eq2 eq3 eq4 eq5 eq6 eq7 eq8 eq9 eq10 eq11 eq12 eq13])
```

$$
\begin{aligned}
x_1 &= 50.987 \\
x_2 &= 58.012 \\
x_3 &= 60.000 \\
u_1 &= 1.050 \\
u_2 &= 0.411 \\
15\sqrt{x_1} + 16\sqrt{x_2} + 8\sqrt{x_3} &= 290.940
\end{aligned}
\tag{16}
$$

So I would ask Carl to work for 60 hours in total. My time is divided to three parts, of which 50.987 to produce IPA, 58.012 to produce Lager, 5 hours to train Carl, and 6.001 hours to supervise him.

2.4 Part B 4

What is the maximum amount you can offer Carl for his services, where you would obtain the same revenue for yourself as without his help (part 1), ignoring production costs?

The problem can be formulated as:

$$
\begin{aligned}
\max \quad & -x_3 \\
\text{s.t.} \quad & x_1 + x_2 + 0.1x_3 \leq 115 \\
& -(15\sqrt{x_1} + 16\sqrt{x_2} + 8\sqrt{x_3}) \leq -240.250 \\
& x_3 \leq 60 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
\tag{17}
$$

where there are two convex functional constraints. Furthermore, it can be easily verified that the objective function is convex. Hence, the corollary applies, so any solution that satisfies the KKT conditions will definitely be an optimal solution.

The Lagrangian function of the above problem is:

$$L(x_1,x_2,x_3,u_1,u_2,u_3) = -x_3 - u_1(x_1+x_2+0.1x_3-115) - u_2(x_3-60) - u_3(-15\sqrt{x_1}-16\sqrt{x_2}-8\sqrt{x_3}+240.250) \tag{18}$$

KKT conditions:

$$\begin{aligned}
-u_1+15u_3/2/\sqrt{x_1} &\leq 0\\
x_1(-u_1+15u_3/2/\sqrt{x_1}) &= 0\\
-u_1+8u_3/\sqrt{x_2} &\leq 0\\
x_2(-u_1+8u_3/\sqrt{x_2}) &= 0\\
-1-0.1u_1-u_2+4u_3/\sqrt{x_3} &\leq 0\\
x_3(-1-0.1u_1-u_2+4u_3/\sqrt{x_3}) &= 0\\
x_1+x_2+0.1x_3-115 &\leq 0\\
u_1(x_1+x_2+0.1x_3-115) &= 0\\
x_3-60 &\leq 0\\
u_2(x_3-60) &= 0\\
15\sqrt{x_1}+16\sqrt{x_2}+8\sqrt{x_3}-240.250 &\geq 0\\
u_3(15\sqrt{x_1}+16\sqrt{x_2}+8\sqrt{x_3}-240.250) &= 0\\
x_1,x_2,x_3 &\geq 0\\
u_1,u_2,u_3 &\geq 0
\end{aligned} \tag{19}$$

Listing 2. MATLAB code to solve the equations

```
syms x1 x2 x3 u1 u2 u3
eq1 = - u1 * sqrt(x1) + 15 / 2 * u3 <= 0;
eq2 = x1 * (- u1 * sqrt(x1) + 15 / 2 * u3) == 0;
eq3 = - u1 * sqrt(x2) + 8 * u3 <= 0;
eq4 = x2 * (- u1 * sqrt(x2) + 8 * u3) == 0;
eq5 = - sqrt(x3) - 0.1 * u1 * sqrt(x3) - u2 * sqrt(x3) + 4 * u3 <= 0;
eq6 = x3 * (- sqrt(x3) - 0.1 * u1 * sqrt(x3) - u2 * sqrt(x3) + 4 * u3) == 0;
eq7 = x1 + x2 + 0.1 * x3 - 115 <= 0;
eq8 = u1 * (x1 + x2 + 0.1 * x3 - 115) == 0;
eq9 = x3 - 60 <= 0;
eq10 = u2 * (x3 - 60) == 0;
eq11 = 15 * sqrt(x1) + 16 * sqrt(x2) + 8 * sqrt(x3) - 240.25 >= 0;
eq12 = u3 * (15 * sqrt(x1) + 16 * sqrt(x2) + 8 * sqrt(x3) - 240.25) == 0;
eq13 = x1 >= 0;
eq14 = x2 >= 0;
eq15 = x3 >= 0;
eq16 = u1 >= 0;
eq17 = u2 >= 0;
eq18 = u3 >= 0;
result = solve([eq1 eq2 eq3 eq4 eq5 eq6 eq7 eq8 eq9 eq10 eq11 eq12 eq13 eq14 eq15])
```

3 Part C: Method of Lagrange Multipliers

3.1 Part C 1

The problem can be formulated as:

$$\begin{aligned}
\min\quad & 100x_1^2+25x_2^2+150x_1x_2+599x_3^2+200x_2x_3\\
\text{s.t.}\quad & 0.02x_1+0.02x_2+0.05x_3 \geq 0.03\\
& x_1+x_2+x_3 = 1\\
& x_1,x_2,x_3 \geq 0
\end{aligned} \tag{20}$$

According to the method of Lagrange multipliers, the Lagrangian function, $L((x_1, x_2, x_3, \lambda_1, \lambda_2))$, of the above problem is:

$$L = 100x_1^2 + 25x_2^2 + 150x_1x_2 + 599x_3^2 + 200x_2x_3 - \lambda_1(0.02x_1 + 0.02x_2 + 0.05x_3 - 0.03) - \lambda_2(x_1 + x_2 + x_3 - 1) \tag{21}$$

which can be minimized by critical points from the following equations:

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= 200x_1 + 150x_2 - 0.02\lambda_1 - \lambda_2 = 0\\
\frac{\partial L}{\partial x_2} &= 50x_2 + 150x_1 + 200x_3 - 0.02\lambda_1 - \lambda_2 = 0\\
\frac{\partial L}{\partial x_3} &= 1198x_3 + 200x_2 - 0.05\lambda_1 - \lambda_2 = 0\\
\frac{\partial L}{\partial \lambda_1} &= 0.02x_1 + 0.02x_2 + 0.05x_3 - 0.03 = 0\\
\frac{\partial L}{\partial \lambda_2} &= x_1 + x_2 + x_3 - 1 = 0
\end{aligned} \tag{22}$$

The above systems of equations can be solved by the following code.

Listing 3. MATLAB code to solve the equations

```
syms x1 x2 x3 lambda1 lambda2
eq1 = 200 * x1 + 150 * x2 - 0.02 * lambda1 - lambda2 == 0;
eq2 = 50 * x2 + 150 * x1 + 200 * x3 - 0.02 * lambda1 - lambda2 == 0;
eq3 = 1198 * x3 + 200 * x2 - 0.05 * lambda1 - lambda2 == 0;
eq4 = 0.02 * x1 + 0.02 * x2 + 0.05 * x3 -0.03 == 0;
eq5 = x1 + x2 + x3 - 1 == 0;
eq6 = x1 >= 0;
eq7 = x2 >= 0;
eq8 = x3 >= 0;
result = solve([eq1 eq2 eq3 eq4 eq5 eq6 eq7 eq8])
```

$$\begin{aligned}
x_1 &= 0\\
x_2 &= 0.666\\
x_3 &= 0.333\\
\lambda_1 &= 14422.222\\
\lambda_2 &= -188.444
\end{aligned} \tag{23}$$

$$100x_1^2 + 25x_2^2 + 150x_1x_2 + 599x_3^2 + 200x_2x_3 = 122.111$$

So the combination is 0% fund 1, 2/3% fund 2, and 1/3% fund 3. The minimum risk is 122.111.

3.2 Part C 2

The problem can be formulated as:

$$\begin{aligned}
\min \quad & 100x_1^2 + 25x_2^2 + 150x_1x_2 + 599x_3^2 + 200x_2x_3\\
\text{s.t.} \quad & 0.02x_1 + 0.02x_2 + 0.05x_3 \geq 0.03\\
& x_1 + x_2 + x_3 = 1
\end{aligned} \tag{24}$$

The above systems of equations can be solved by the following code.

Listing 4. MATLAB code to solve the equations

```
syms x1 x2 x3 lambda1 lambda2
eq1 = 200 * x1 + 150 * x2 - 0.02 * lambda1 - lambda2 == 0;
eq2 = 50 * x2 + 150 * x1 + 200 * x3 - 0.02 * lambda1 - lambda2 == 0;
eq3 = 1198 * x3 + 200 * x2 - 0.05 * lambda1 - lambda2 == 0;
eq4 = 0.02 * x1 + 0.02 * x2 + 0.05 * x3 -0.03 == 0;
eq5 = x1 + x2 + x3 - 1 == 0;
result = solve([eq1 eq2 eq3 eq4 eq5])
```

$$
\begin{aligned}
x_1 &= 0 \\
x_2 &= 0.666 \\
x_3 &= 0.333 \\
\lambda_1 &= 14422.222 \\
\lambda_2 &= -188.444 \\
100x_1^2 + 25x_2^2 + 150x_1x_2 + 599x_3^2 + 200x_2x_3 &= 122.111
\end{aligned}
\tag{25}
$$

So we will stick to the previous combination even if we are allowed to short the funds, and the risk is the same.

4 Part D: Linear Programming and Simplex Algorithm

The decision variables in the problem are summarized in the following table.

Variables	Definition	is.integer
$x_{1,1}$	My hours on managing forests per year	Y
$x_{1,2}$	My hours on hunting per year	Y
$x_{1,3}$	My hours on managing cabin events per year	Y
$x_{2,1}$	Uncle's hours on managing forests per year	Y
$x_{2,2}$	Uncle's hours on hunting per year	Y
$x_{2,3}$	Uncle's hours on managing cabin events per year	Y
$x_{3,1}$	Cousin's hours on managing forests per year	Y
$x_{3,2}$	Cousin's hours on hunting per year	Y
$x_{3,3}$	Cousin's hours on managing cabin events per year	Y
x_4	Num of cabin events per year	Y

Table 4. Features of Variables

The constraints about forest areas are omitted. To be more specific, there is no limit on the number of harvested christmas trees. Then the problem can be formulated as:

$$
\begin{aligned}
\max \quad & 1000(x_{1,1} + x_{2,1} + x_{3,1}) + 750(x_{1,2} + x_{2,2} + x_{3,2}) + 2000x_4 \\
\text{s.t.} \quad & x_{1,1} + x_{1,2} + x_{1,3} \le 40 \\
& x_{2,1} + x_{2,2} + x_{2,3} \le 50 \\
& x_{3,1} + x_{3,2} + x_{3,3} \le 60 \\
& x_{1,1} \ge 20 \\
& x_{2,1} = 0 \\
& x_{2,2} \ge 20 \\
& x_{2,3} \le 50/3 \\
& x_{3,3} \ge 3(x_{1,3} + x_{2,3}) \\
& x_{1,2} + x_{2,2} + x_{3,2} \le 30 \\
& x_4 \le 20 \\
& x_{1,3} + x_{2,3} + x_{3,3} \ge 2x_4 \\
& x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, x_{3,1}, x_{3,2}, x_{3,3}, x_4 \in \mathbb{Z}^+
\end{aligned}
\tag{26}
$$

BV	Z	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{3,1}$	$x_{3,2}$	$x_{3,3}$	x_4	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	b
Z	1	-1000	-750	0	-1000	-750	0	-1000	-750	0	-2000	0	0	0	0	0	0	0	0	0	0	0
y_1	0	1	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	40
y_2	0	0	0	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	50
y_3	0	0	0	0	0	0	0	1	1	1	0	0	0	1	0	0	0	0	0	0	0	60
y_4	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	- 20
y_5	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
y_6	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-20
y_7	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	50 / 3
y_8	0	0	0	3	0	0	3	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0
y_9	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	30
y_{10}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	20

Table 5. Simplex Algorithm in Tabular Form

4.1 Problem D2: Simplex Algorithm

Write the problem in standard form:

$$
\begin{aligned}
\max \quad & 1000(x_{1,1}+x_{2,1}+x_{3,1})+750(x_{1,2}+x_{2,2}+x_{3,2})+2000x_4 \\
\text{s.t.} \quad & x_{1,1}+x_{1,2}+x_{1,3} \leq 40 \\
& x_{2,1}+x_{2,2}+x_{2,3} \leq 50 \\
& x_{3,1}+x_{3,2}+x_{3,3} \leq 60 \\
& -x_{1,1} \leq -20 \\
& x_{2,1} = 0 \\
& -x_{2,2} \leq -20 \\
& x_{2,3} \leq 50/3 \\
& -x_{3,3}+3x_{1,3}+3x_{2,3} \leq 0 \\
& x_{1,2}+x_{2,2}+x_{3,2} \leq 30 \\
& x_4 \leq 20 \\
& -x_{1,3}-x_{2,3}-x_{3,3}+2x_4 \leq 0 \\
& x_{1,1},x_{1,2},x_{1,3},x_{2,1},x_{2,2},x_{2,3},x_{3,1},x_{3,2},x_{3,3},x_4 \in \mathbb{Z}^+
\end{aligned}
\tag{27}
$$

The first iteration of simplex algorithm for solving the above problem is illustrated in table 5 and table 6. During the first iteration, x_4 is selected as the entering basic variable, of which the coefficient has the maximum absolute value in first row. Solve the new BF solution, and update first row.

BV	Z	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{3,1}$	$x_{3,2}$	$x_{3,3}$	x_4	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	b
Z	1	-1000	-750	0	-1000	-750	0	-1000	-750	0	0	0	0	0	0	0	0	0	0	0	2000	40000
y_1	0	1	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	40
y_2	0	0	0	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	50
y_3	0	0	0	0	0	0	0	1	1	1	0	0	0	1	0	0	0	0	0	0	0	60
y_4	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	- 20
y_5	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
y_6	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	-20
y_7	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	50 / 3
y_8	0	0	0	3	0	0	3	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0
y_9	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	30
x_4	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	20

Table 6. Simplex Algorithm in Tabular Form

4.2 Part D 3

The dual problem is:

$$
\begin{aligned}
\min \quad & 40y_1 + 50y_2 + 60y_3 + 20y_4 + 20y_6 + 50/3y_7 + 30y_9 + 20y_{10} \\
\text{s.t.} \quad & y_1 + y_4 \geq 1000 \\
& y_1 + y_9 \geq 1000 \\
& y_1 - 3y_8 + y_{11} \geq 1000 \\
& y_2 + y_5 \geq 750 \\
& y_2 + y_6 \geq 750 \\
& y_2 + y_7 - 3y_8 + y_{11} \geq 750 \\
& y_3 \geq 0 \\
& y_3 + y_9 \geq 0 \\
& y_3 + y_8 + y_{11} \geq 0 \\
& y_{10} - 2y_{11} \geq 2000 \\
& y_1 + y_2 + y_3 + y_7 + y_9 + y_{10} \geq 0 \\
& y_4 + y_6 + y_8 + y_{11} \leq 0 \\
& y_5 = 0
\end{aligned} \tag{28}
$$

4.3 Part D 4

Put the solution in all constraints. If all are respected, then it is a feasible solution. Change all the inequality to equality constraints, and find those met by the set. If two or more equality constraints are met, then the solution is one of the cross sections in the edge of the feasible region. Move the solution along the edges to see if there is a better solution. If not, the solution is the optimal one.

4.4 Part D 5

There is no sufficient coefficients.

5 Part E: Linear Programming

$$
\begin{aligned}
\max \quad & 2x_1 + 3x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \leq 16 \\
& x_1 - x_2 \leq 2 \\
& -x_1 + 2x_2 \geq 4 \\
& x_2 \leq 6 \\
& x_1, x_2 \geq 0
\end{aligned} \tag{29}
$$

So the point (4, 6) is the optimal solution.

If we add the following constraint to the problem, the optimal solution will change. The new optimal solution is (8 / 3, 10 / 3).

$$
x_1 - x_2 \geq -6 \tag{30}
$$

If the objective function is nonlinear, the optimal soluation may lay on some point in the feasible edges instead of the feasible points. I would compare the value of the points whose slope equals the feasible edges.

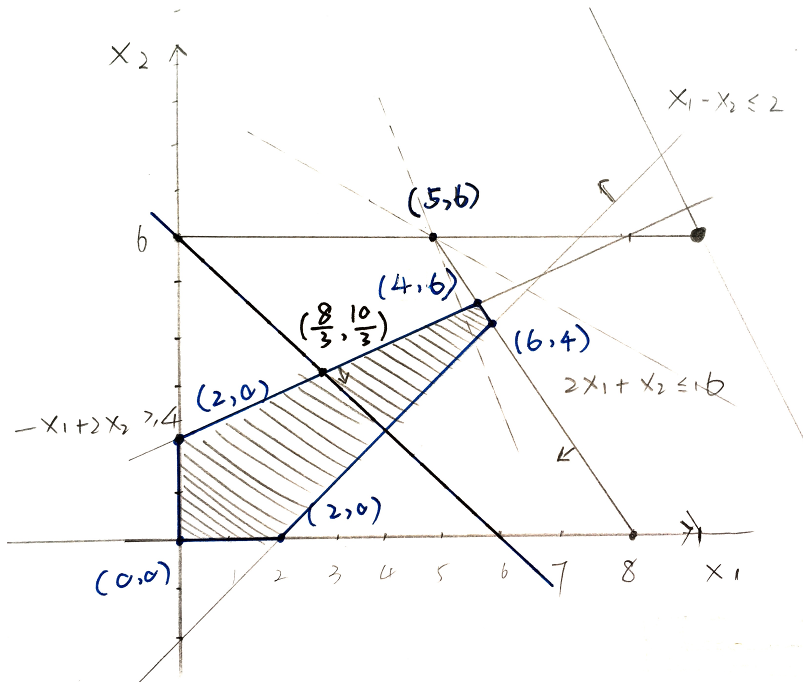

Figure 1. Illustration of Feasible Regions

6 Part F: Lagrangian Dual Problem

6.1 Part F 1

$$\begin{aligned} \max \quad & x_1^2 + 6x_2 + 4x_3^2 \\ \text{s.t.} \quad & 2x_1 + x_2 + 4x_3 \leq 10 \\ & x_1, x_3 \geq 0 \\ & x_2 \in (0,1) \end{aligned} \tag{31}$$

The Lagrangian dual function is:

$$L(x_1, x_2, x_3, \lambda) = -x_1^2 - 6x_2 - 4x_3^2 + \lambda(-10 + 2x_1 + x_2 + 4x_3) \tag{32}$$

$$\theta(\lambda) = -10\lambda + \max\{-x_1^2 + 2\lambda x_1\} + \max\{(\lambda - 6)x_2\} + \max\{-4x_3^2 + 4\lambda x_3\} \tag{33}$$

For a fixed value of $\lambda > 0$, the maximum of $L(x_1, x_2, x_3, \lambda)$ over $x \in X$ is attained at:

$$\begin{aligned} & x_1(\lambda) = \lambda \\ & x_2(\lambda) = \begin{cases} 0 & \text{when } \lambda \leq 6 \\ 1 & \text{when } \lambda > 6 \end{cases} \\ & x_3(\lambda) = \lambda/2 \end{aligned} \tag{34}$$

So, we get $L(x_1(\lambda), x_2(\lambda), x_3(\lambda), \lambda)$, when $\lambda \leq 6$:

$$\begin{aligned} L(x_1(\lambda), x_2(\lambda), x_3(\lambda), \lambda) &= -10\lambda + \lambda^2 + \lambda^2 \\ &= -10\lambda + 2\lambda^2 \end{aligned} \tag{35}$$

of which the minimum is $\lambda = 2.5$. Then, $x_1 = 2.5$, $x_2 = 0$, $x_1 = 1.25$, and objective value is 12.5.

While when $\lambda > 6$, it becomes:

$$\begin{aligned} L(x_1(\lambda), x_2(\lambda), x_3(\lambda), \lambda) &= -10\lambda + \lambda^2 + (\lambda - 6) + \lambda^2 \\ &= 2\lambda^2 - 9\lambda - 6 \end{aligned} \tag{36}$$

of which the minimum is $\lambda = 2.25$, Then, $x_1 = 2.25$, $x_2 = 1$, $x_1 = 1.125$, and objective value is 10.125.

So the maximum value is 12.5, when $x_1 = 2.5$, $x_2 = 0$, $x_1 = 1.25$.

6.2 Part F 2

It can also be solved by using KKT conditions.

7 Part G: Gradient Search Algorithm

Illustration two iterations of Gradient Search Algorithm to optimize the following question:

$$\max \quad f(\boldsymbol{x}) = 4x_1 + 2x_1x_2 - x_2 - 4x_1^2 - x_2^2 \tag{37}$$

Thus,

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 4 + 2x_2 - 8x_1 \\ \frac{\partial f}{\partial x_2} &= -1 + 2x_1 - 2x_2 \end{aligned} \tag{38}$$

Suppose that $\boldsymbol{x}' = (1,1)$ as the initial trial solution. Then the gradient is:

$$\nabla f(\boldsymbol{x}') = (-2, -1) \tag{39}$$

Substitute $\boldsymbol{x}' + t\nabla f(\boldsymbol{x}')$ in to f:

$$\begin{aligned} f(\boldsymbol{x}' + t\nabla f(\boldsymbol{x}')) &= f(1-2t, 1-t) \\ &= 4(1-2t) + 2(1-2t)(1-t) - (1-t) - 4(1-2t)^2 - (1-t)^2 \\ &= 5t - 13t^2 \end{aligned} \tag{40}$$

which can be maximized at $t = 5/26$ by the following equations:

$$\frac{\partial(5t - 13t^2)}{\partial t} = 5 - 26t \tag{41}$$

Reset $\boldsymbol{x}' = (1,1) + 5/26(-2,-1) = (0.6154, 0.8077)$. Repeat the above procedure again.
Solve $\nabla f(\boldsymbol{x}') = \mathbf{0}$, we get:

$$\begin{aligned} x_1 &= 0.5 \\ x_2 &= 1 \end{aligned} \tag{42}$$