Lagrange Multiplier rule – class exercises and answers

Solve the following problems:

Problem 1. $f_0(x) = e^{x_1 x_2} \to \max, f_1(x) = x_1^3 + x_2^3 = 1$

Solution:

1. The function $f_0(x)$ is coercive for maximum: $x_1 \sqrt[3]{1 - x_1^3} \approx -x_1^2$ for $|x_1|$ sufficiently large, and so $e^{x_1 \sqrt[3]{1-x_1^3}} \to 0$ for $|x_1| \to \infty$.

2. Lagrange function: $L = \lambda_0 e^{x_1 x_2} + \lambda_1 (x_1^3 + x_2^3 - 1)$
   Lagrange Multiplier rule: $L_x = 0^T \to$
   $\frac{\delta L}{\delta x_1} = \lambda_0 x_2 e^{x_1 x_2} + 3\lambda_1 x_1^2 = 0$
   $\frac{\delta L}{\delta x_2} = \lambda_0 x_1 e^{x_1 x_2} + 3\lambda_1 x_2^2 = 0$
   If $\lambda_0 = 0$, then by the Lagrange Multiplier rule $\lambda_1 \neq 0$ and then the Lagrange equations would give:
   $\frac{\delta L}{\delta x_1} = 0 \cdot x_2 e^{x_1 x_2} + 3\lambda_1 x_1^2 = 0$
   $\frac{\delta L}{\delta x_2} = 0 \cdot x_1 e^{x_1 x_2} + 3\lambda_1 x_2^2 = 0$
   both with $\lambda_1 > 0 \to x_1 = x_2 = 0$, contradicting the equality constraint.
   Therefore, we may put $\lambda_0 = 1$.

3. Solve the equations by eliminating $\lambda_1 : x_1^3 = x_2^3 \to x_1 = x_2$. Form the equality constraint $f_1(x) = x_1^3 + x_2^3 = 1$, by using $x_1 = x_2$, it follows that $x_1^3 = x_2^3 = \frac{1}{2}$, and so $x_1 = x_2 = \frac{1}{2}\sqrt[3]{4}$.

4. It follows that $\hat{x} = (\frac{1}{2}\sqrt[3]{4}, \frac{1}{2}\sqrt[3]{4})^T$

Problem 2. $f_0(x) = x_1^2 + 12x_1x_2 + 2x_2^2 \to \text{extr}, f_1(x) = 4x_1^2 + x_2^2 = 25$

Solution:

1. Global extrema exist by Weierstrass (continuous function on closed interval)

2. Lagrange function: $L = \lambda_0(x_1^2 + 12x_1x_2 + 2x_2^2) + \lambda_1(4x_1^2 + x_2^2 - 25)$
   Lagrange: $L_x = 0_2^T \to$
   $\frac{\delta L}{\delta x_1} = \lambda_0 (2x_1 + 12x_2) + \lambda_1 (8x_1) = 0$ & $\frac{\delta L}{\delta x_2} = \lambda_0 (12x_1 + 4x_2) + \lambda_1 (2x_2) = 0$
   We put $\lambda_0 = 1$, as we may: if $\lambda_0 = 0$, then $\lambda_1 \neq 0$ and $x_1 = x_2 = 0$, contradicting the constraint.

3. Eliminate λ_1:
   $x_1x_2 + 6x_2^2 = 24x_1^2 + 8x_1x_2$
   This can be rewritten as: $6\left(\frac{x_2}{x_1}\right)^2 - 7\left(\frac{x_2}{x_1}\right) - 24 = 0$, provided $x_1 \neq 0$. This gives $x_2 = \frac{8}{3}x_1$ or $x_2 = -\frac{3}{2}x_1$. In the first (resp. second) case we get $x_1 = \pm\frac{3}{2}$ and so $x_2 = \pm 4$ (respectively $x_1 = \pm 2$, and so with reverse sign to x_1, $x_2 = \pm 3$), using

the equality constraint. Compare: $f_0(2,-3) = f_0(-2,3) = -50$ and $f_0(\frac{3}{2},4) = f_0(-\frac{3}{2},-4) = 106\frac{1}{4}$.

4. $(2,-3)$ and $(-2,3)$ are global minima and $(\frac{3}{2},4)$ and $-\frac{3}{2},-4)$ global maxima

Problem 3. $f_0(x) = x_1x_2^2x_3^3 \to \text{extr}, f_1(x) = x_1^2 + x_2^2 + x_3^2 = 1$ **Solution:**

1. This is a continuous function evaluated on a bounded set, so, by Weierstrass global extrema exist. Note that the constraint in the first question (because of including third-power terms) was not sufficient to bind x_1, x_2 to a closed interval.

2. Lagrange function $L = \lambda_0 x_1x_2^2x_3^3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1)$.
   Lagrange multiplier rule: $L_x = 0_3^T \to$
   $\lambda_0 x_2^2 x_3^3 + 2\lambda_1 x_1 = 0$
   $2\lambda_0 x_1 x_2 x_3^3 + 2\lambda_1 x_2 = 0$
   $3\lambda_0 x_1 x_2^2 x_3^2 + 2\lambda_1 x_3 = 0$
   If $\lambda_0 = 0$, then by definition $\lambda_1 \neq 0$; and from the Lagrangian derivatives it would follow that $x_1 = x_2 = x_3 = 0$, which contradicts the equality constraint. Therefore we may put $\lambda_0 = 1$.

3. Eliminating λ_1 from above equations gives:
   $2x_1 = x_2^2 = \frac{2}{3}x_3^3$
   Note that we may ignore $x_1 = x_2 = x_3 = 0$, which is an in-admissable slution and moreover by considering f_0 from inspection cannot be global extrema.
   From the equality constraint $:= x_1^2 + x_2^2 + x_3^2 = 1 \to= x_1^2 + 2x_1^2 + 3x_1^2 = 6x_1^2 = 1 \to$ $x_1^2 = \frac{1}{6}$ and so $x_2^2 = \frac{1}{3}$,$x_3^2 = \frac{1}{2}$.
   Note that these are 8 points (two values x_1 times two values x_2 times two values x_3. Note by evaluating f_0 that the value of f_0 is the same at each of these points, apart from the sign.

4. There are eight global extrema $(\pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{2}})$. A point is a global minimum (respectively maximum) if x_1 and x_3 have different signs (respectively the same sign).

Puzzle: Re-distribute your Blu-ray collections

You and your friend decide to share your Blu-ray collections for the upcoming christmas holidays. You own 22 crime series Blu-rays, and your friend owns 13 comedy series Blu-rays. By redistributing these over the two of you, each of you will end up with an interesting set of series to watch. Your aim is to redistribute the 35 Blu-rays over the two of you in such a way that the social welfare is maximal. The social welfare here is defined as the sum of the utilities for the both of you. Your utility is $4\ln(x) + \ln(y)$, and your friends utility is $5ln(x)+20ln(y)$, where x denotes the number of crime series Blu-rays this person has, and y denotes the number of comedy series Blu-rays this person has.

Question 1 Which distribution of the Blue-rays gives maximal social welfare?
Solution

1. Maximizing social welfare means maximizing the welfare of you and your friend. Thus: max $f_0(x_1, x_2, y_1, y_2) = 4\ln(x_1)+\ln(y_1)+5\ln x_2+20\ln y_2$ such that $f_1(x_1, x_2) = x_1 + x_2 = 22$ and $f(y_1, y_2) = y_1 + y_2 = 13$, where x_1, y_1 are the number of crime series respectively comedy series you will have, and x_2, y_2 the number of crime series respectively comedy series your friend will have. The equality constraints bind the $\ln(x)$ function from above, and none of the decision values can be smaller than zero. What remains to be argued is that we can ignore the limit of $\ln(x)$ around $x \to 0$. Let's replace x_2, y_2 by $22-x_1, 13-x_2$, as we my due to the equality constraints. Now f_0 is a function in only two variables: x_1, y_1. Note also that the objective function is separable in x_1 and y_1, so we can present the bounding arguments separately for x_1 and y_1.
   Take a feasible solution $x_1 = 1$ (and thus $x_2 = 22 - 1 = 21$). For sufficiently large M, when $\bar{x}_1 \leq \frac{1}{M}$ then $f_0(\bar{x}_1) < f_0(1)$. Therefore, the maximum of the function must be attained for values $\frac{1}{M} \leq x_1$. We may thus look for maximum in the interval $\frac{1}{M} \leq x_1 \leq 22$, for sufficiently large N. Therefore, by Weierstrass, there exists a maximum. The argument for y_1 is similar. From these findings the existence of a global maximum follows for $f_0(x_1, x_2)$.

2. Lagrange function: $L = \lambda_0(4\ln(x_1) + \ln(y_1) + 5\ln x_2 + 20\ln y_2) + \lambda_1(13 - x_1 - x_2) + \lambda_2(22 - y_1 - y_2)$
   Lagrange:
   $\frac{\delta L}{\delta x_1} = \frac{4\lambda_0}{x_1} - \lambda_1 = 0$
   $\frac{\delta L}{\delta y_1} = \frac{\lambda_0}{y_1} - \lambda_2 = 0$
   $\frac{\delta L}{\delta x_2} = \frac{5\lambda_0}{x_2} - \lambda_1 = 0$
   $\frac{\delta L}{\delta y_1} = \frac{20\lambda_0}{y_2} - \lambda_2 = 0$ Now if we put $\lambda_0 = 0$, then $\lambda_1 = \lambda_2 = 0$, which contradicts the Lagrange multiplier rule condition that not all λ may be equal to zero. Therefore we may put $\lambda_0 = 1$.

3. Eliminate the Lagrange multipliers:
   $\frac{4}{x_1} = \frac{5}{x_2} \to 4x_2 = 5x_1$
   $\frac{1}{y_1} = \frac{20}{y_2} \to 20y_1 = y_2$.
   Substituting this into the equality constraints: $x_1 + \frac{5}{4}x_1 = 22$ & $y_1 + 20y_1 = 13 \to$

$(x_1, x_2, y_1, y_2) = (9\frac{7}{9}, 12\frac{2}{9}, \frac{13}{21}, 12\frac{8}{21})$

Optional:

Now we can of course not assign fractional blue rays to each party. So one could round to the nearest integer solution: $(x_1, x_2, y_1, y_2) = (10, 12, 1, 12)$

Rounding may not always lead to the optimal solution. However, in this case it does, as will be demonstrated below.

First let's consider the distribution of crime series (the argument for comedy series will be similar). We can replace $x_2 = 22 - x_1$, and consider the distribution of crime series as an unconstrained problem: $f(x_1) = 4\ln x_1 + 5\ln(22 - x_1)\,\mathrm{max}$. Solving $g'(x_1) = \frac{\delta g(x_1}{\delta x_1} = \frac{4}{x_1} - \frac{5}{22-x_1} = 0 \rightarrow \frac{4}{22-x_1} = 5x_1 \rightarrow \hat{x}_1 = \frac{88}{9} = 9\frac{7}{9}$. The derivative $g'(x_1)$ is monotonic decreasing on the feasible area $0 \leq x \leq 22$. Therefore $g'(x_1) > 0$ for $x_1 < 9\frac{7}{9}$ and $g'(x_1) < 0$ for $x_1 < 9\frac{7}{9}$. This shows that the optimal integral solution can be obtained by comparing $x_1 = 9$ and $x_1 = 10$: $4\ln 9 + 5\ln 13 \approx 21.614$ and $4\ln 10 + 5\ln 12 \approx 21.635$. Therefore $(x_1, x_2) = (10, 12)$ is the optimal integer solution to the problem.

For (y_1, y_2) we can follow the same approach. The derivative $h'(y_1) = \frac{1}{y_1} + \frac{20}{13-y_1}$, where $h'(y_1) = 0 \rightarrow \hat{y}_1 = \frac{13}{21}$. The derivative is decreasing in the feasible region $0 \leq y_1 \leq 13$. Now we compare $h(y_1, y_2) = h(0, 13)$ with $h(y_1, y_2) = h(1, 12)$. The function $\lim_{y_1 \rightarrow 0} h(y_1) = \lim_{x_1 \rightarrow 0} \ln y_1 + 20\ln(13 - y_1) = -\infty$. $h(1) = \ln 1 + 20\ln 12 \approx 49.698$. Therefore $(y_1, y_2) = (1, 12)$ is the optimal integer solution to distribute the comedy series.

Question 2 Comment on the fairness of the outcome obtained in 1. The solution may seem unfair to you: you would have to give away 12 of your crime series Blu rays (more than half), to get only one comedy series back from your friend.

Question 3 Estimate how much this maximal social welfare would increase if you would suddenly obtain another crime series Blu-ray. We can obtain an estimate in the change in objective function from calculating λ_1:$\lambda_1 = \frac{4}{x_1} = \frac{4}{10} = 0.4$ Resolving $x_1 + \frac{5}{4}x_1 = 23 \rightarrow (x_1, x_2) = 10\frac{2}{9}, 12\frac{7}{9}$. The change in objective function $4\ln(9\frac{7}{9}) + 5\ln(12\frac{2}{9}) - 4\ln(10\frac{2}{9}) + 5\ln(12\frac{7}{9}) \approx -0.40007$.

To find who receives an extra Blu ray, you can compare: You can compare $(x_1, x_2) = (10, 13)$ with $(x_1, x_2) = (11, 12)$: $4\ln 10 + 5\ln 13 \approx 22.03509$; $4\ln 11 + 5\ln 12 \approx 22.01611$. So the additional Blu Ray goes to your friend.