Lecture 4: Stochastic Programming

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Stochastic Programming

The main reason for Benders Algorithm. In this lecture Stochastic Programming is briefly introduced, and the explanation for the applicability of Benders is discussed.

Un-Certainty

Until now we have assumed that everything is deterministic. This is very often **not** the case! Then what ?

- What is optimum ?
- How can we formulate problems ?
- Are the results reliable ?

A Given a linear program ....

Min

$$\sum_j c_j \cdot x_j + \sum_k f_k \cdot y_k$$

s.t.

$$\sum_j A_{ij} \cdot x_j + \sum_k B_{ik} \cdot y_k \geq b_i \quad \forall\ i$$

$$x_j \geq 0$$

$$y_k \in Z$$

How can we include un-certainty?

Scenarios: Discretized probability

We cannot work with a continuum of possible outcomes of the future. Instead we discretize the continuum and create a number of scenarios s with different probabilities π_s. Now we have:

- c_j^s and f_k^s : One objective coefficient array for each scenario s
- A_{ij}^s and B_{ik}^s : One constraint matrix for each scenario s
- b_i^s: One right-hand-side coefficient array for each scenario s

We now assume that we have to find a solution (represented by the variables y_k which is shared for all the s scenarios) and the scenario specific variables x_j^s.

Inserting scenario's ....

Min

$$\sum_s \pi^s (\sum_j c_j^s \cdot x_j^s + \sum_k f_k^s \cdot y_k)$$

s.t.

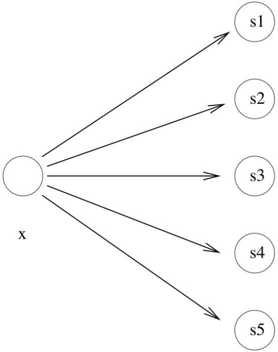
$$\sum_j A_{ij}^s \cdot x_j^s + \sum_k B_{ik}^s \cdot y_k \geq b_i^s \quad \forall\ s, i$$

$$x_j^s \geq 0$$

$$y_k \in Z$$

BUT this basically increase the problem size by S !

A Single-Stage figure

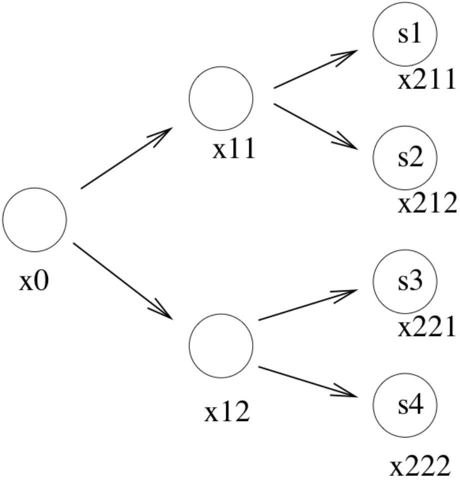


Single-Stage comments

- This stochastic programming approach is actually not so simple ...
- The precision of the results very often requires a high number of scenarios

Furthermore, the single-stage model has a major feature missing: What if we can decide something based on improved, but not complete, knowledge ?

A Multi-Stage figure



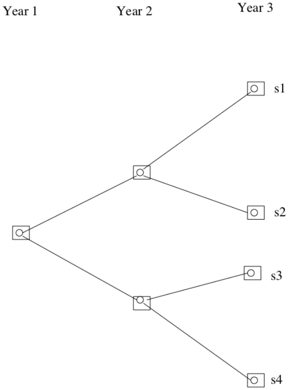
Multi-Stage decisions

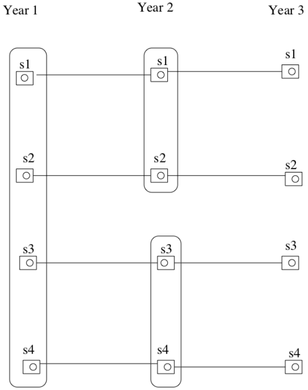
We perform the planning at time 0 and evaluate at time 2, but we can influence the performance at time 1:

- All scenarios share the variables at time 0
- Only some of the scenarios share the variables at time 1

First Formulation Using Scenarios (1)

Consider the following scenario tree:





Why is Benders algorithm interesting for this type of problems ?

If we look at the previous MIP model, what happens if the y_k variables are fixed: $\overline{y_k}$:

Min

$$\sum_s \pi^s (\sum_j c_j^s \cdot x_j^s + \sum_k f_k^s \cdot \overline{y_k})$$

s.t.

$$\sum_j A_{ij}^s \cdot x_j^s + \sum_k B_{ik}^s \cdot \overline{y_k} \geq b_i^s \quad \forall\ s, i$$

$$x_j^s \geq 0$$

$$y_k \in Z$$

NOTICE: The model dis-integrates into many different sub-problems !!!

Why is this faster ?

Because it may be faster to solve many small (sub) problems than solving one large (sub) problem because the LP solution time grows more than linearly with the number of variables.

For this to be interesting however, the time-consuming problem in the Benders algorithm should be the LP (sub) problem ! This is only the case if there is **HUGE** number of continuous variables ! When is this the case ? Primarily in stochastic programming !

Dualization I: Min to Max

Min
$$\sum_j c_j \cdot x_j$$
s.t.
$$\sum_j A_{ij} \cdot x_j \geq b_i \quad \forall\ i$$
$$x_j \geq 0$$

Through duallization we get:

Max
$$\sum_i b_i \cdot \alpha_i$$
s.t.
$$\sum_i A_{ij} \cdot \alpha_i \leq c_j \quad \forall\ j$$
$$x_j \geq 0$$

Dualization I: Comments (Min to Max)

We get one dual variable (α_i) for each constraint in the primal problem:

Constraint $\geq$	$\Rightarrow$	Dual variable $\alpha \in R^+$
Constraint $=$	$\Rightarrow$	Dual variable $\alpha \in R$
Constraint $\leq$	$\Rightarrow$	Dual variable $\alpha \in R^-$

We get one constraint for each variable x in the primal problem:

Variable $x \in R^+$	$\Rightarrow$	$\leq$
Variable $x \in R$	$\Rightarrow$	$=$
Variable $x \in R^-$	$\Rightarrow$	$\geq$

Dualization I: Max to Min

$$\text{Max} \quad \sum_j c_j \cdot x_j$$

$$\text{s.t.} \quad \sum_j A_{ij} \cdot x_j \leq b_i \quad \forall\ i$$

$$x_j \geq 0$$

Through duallization we get:

$$\text{Min} \quad \sum_i b_i \cdot \alpha_i$$

$$\text{s.t.} \quad \sum_i A_{ij} \cdot \alpha_i \geq c_j \quad \forall\ j$$

$$x_j \geq 0$$

Dualization I: Comments (Max to Min)

We get one dual variable (α_i) for each constraint in the primal problem:

Constraint $\leq$	$\Rightarrow$	Dual variable $\alpha \in R^+$
Constraint $=$	$\Rightarrow$	Dual variable $\alpha \in R$
Constraint $\geq$	$\Rightarrow$	Dual variable $\alpha \in R^-$

We get one constraint for each variable x in the primal problem:

Variable $x \in R^+$	$\Rightarrow$	$\geq$
Variable $x \in R$	$\Rightarrow$	$=$
Variable $x \in R^-$	$\Rightarrow$	$\leq$