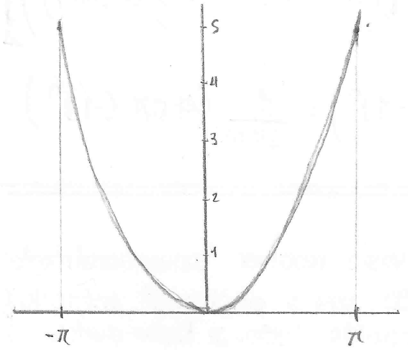
$$f(x)=\frac{x^2}{2} \quad -\pi < x < \pi$$

Tal que $f(x)=f(x+2\pi)$



$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(nwx)+b_n\,\text{sen}(nwx)\right)$$

ω = Frecuencia Fundamental

$\omega=\frac{2\pi}{T}$

T = Periodo

$$a_0=\frac{1}{T/2}\int_T f(x)\,dx \qquad a_n=\frac{1}{T/2}\int_T f(x)\cos(nwx)\,dx$$

$$b_n=\frac{1}{T/2}\int_T f(x)\,\text{sen}(nwx)\,dx$$

$T=2\pi$

$w=1$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x^2}{2}\,dx$$

$$a_0=\frac{1}{\pi}\cdot\frac{x^3}{6}\Big|_{-\pi}^{\pi}=\frac{1}{6\pi}\left(\pi^3-(-\pi)^3\right)=\frac{1}{6\pi}(2\pi^3)=\frac{\pi^3}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x^2}{2}\cos(nx)\,dx=\frac{1}{2\pi}\left[\int_{-\pi}^{\pi}x^2\cos(nx)\,dx\right]$$

Integracion por sustitución

$$\int x^2\cos(nx)\,dx=\int a^2\cos(a)\,da \qquad a=nx$$

$$=\frac{1}{n^3}\int a^2\cos(a)\,da$$

Integracion por partes

$u=a^2 \qquad du=2a\,da \qquad dv=\cos(a)\,da \qquad \int u\cdot dv=u\cdot v-\int v\cdot du$

$\frac{du}{da}=2a \qquad \int dv=v=\text{sen}(a)$

$$\frac{1}{n^3}\int a^2\cos(a)\,da=\frac{1}{n^3}\left[u^2\,\text{sen}(a)-\int\text{sen}(a)\cdot 2a\,da\right]=\frac{1}{n^3}\left[u^2\,\text{sen}(u)-2\int a\cdot\text{sen}(a)\,da\right]$$

Integracion por partes

$\int a\cdot\text{sen}(a)\,da \qquad e=a \qquad de=da \qquad dw=\text{sen}(a)$

$\frac{de}{da}=1 \qquad \int dw=w=-\cos(a)$

$$\int e\cdot dw=e\cdot w-\int w\cdot de$$

$$\int a\cdot\text{Sen}(a)\,da=a\cdot(-\cos(a))-\int(-\cos(a))\cdot da=-a\cos(a)+\int\cos(a)\,da$$

$$\int a\cdot\text{sen}(a)\,da=-a\cdot\cos(a)+\text{sen}(a)$$

$$\frac{1}{n^3}\int a^2\cos(a)\,da=\frac{1}{n^3}\left[u^2\,\text{sen}(u)-2\left(-a\cdot\cos(a)+\text{sen}(a)\right)\right]$$

$$\int x^2\cdot\cos(nx)\,dx=\frac{1}{n^3}\left[(nx)^2\,\text{sen}(nx)-2\left(-nx\cos(nx)+\text{sen}(nx)\right)\right]+C$$

$$a_n=\frac{1}{2\pi}\left[\frac{1}{n^3}\left[(nx)^2\,\text{sen}(nx)+2\left(nx\cos(nx)-\text{sen}(nx)\right)\right]\right]\Bigg|_{-\pi}^{\pi}$$

$$=\frac{1}{2\pi n^3}\left[\left((n\pi)^2\text{sen}(n\pi)+2\left(n\pi\cos(n\pi)-\text{sen}(n\pi)\right)\right)-\left((n(-\pi))^2\text{sen}(n(-\pi))+2\left(n(-\pi)\cos(n(-\pi))-\text{sen}(n(-\pi))\right)\right)\right]$$

Recordando que

$\cos(x)$ es una función par $\rightarrow$ $\cos(-x) = \cos(x)$ || $\cos(\pi n) = (-1)^n$
$\operatorname{sen}(x)$ es una función impar $\rightarrow$ $\operatorname{sen}(-x) = -\operatorname{sen}(x)$ || $\operatorname{sen}(\pi n) = 0$ $\Big\}\ n \in \mathbb{N}$

$$= \frac{1}{2\pi n^3}\left[\left((n\pi)^2 \cancelto{0}{\operatorname{sen}(n\pi)} + 2\left(n\pi \cos(n\pi) - \cancelto{0}{\operatorname{sen}(n\pi)}\right)\right) - \left((n(-\pi))^2\left(-\cancelto{0}{\operatorname{sen}(n\pi)}\right) + 2\left(n(-\pi)\cos(n\pi) + \cancelto{0}{\operatorname{sen}(n\pi)}\right)\right)\right]$$

$$= \frac{1}{2\pi n^3}\left[2n\pi(-1)^n - 2\left(n(-\pi)\,(-1)^n\right)\right] = \frac{1}{2\pi n^3}\left(2n\pi(-1)^n + 2n\pi(-1)^n\right) = \frac{1}{2\pi n^3}\left(4n\pi(-1)^n\right)$$

$$a_n = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x^2}{2}\operatorname{sen}(nx)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \operatorname{sen}(nx)\,dx$$

Nota:
Esta integral no la voy a resolver "convencionalmente" porque es muy larga, voy a emplear el método de la tabulación, facilita todo. Si tiene dudas, podría buscarlo, hay muchos ejemplos en internet

Método de la tabulación

$$\int x^2 \operatorname{sen}(nx)\,dx \qquad g(x) = x^2 \qquad h(x) = \operatorname{sen}(nx)$$

Derivadas de $g(x)$		Integrales de $h(x)$
x^2	$\boxed{+}$	$\operatorname{sen}(nx)$
$2x$	$\boxed{-}$	$-\cos(nx)/n$
2	$\boxed{+}$	$-\operatorname{sen}(nx)/n^2$
0		$\cos(nx)/n^3$

$$\int x^2 \operatorname{sen}(nx)\,dx = -\frac{x^2}{n}\cos(nx) + \frac{2x}{n^2}\operatorname{sen}(nx) + \frac{2}{n^3}\cos(nx) + C$$

$$b_n = \frac{1}{2\pi}\left[\left(-\frac{x^2}{n}\cos(nx) + \frac{2x}{n^2}\operatorname{sen}(nx) + \frac{2}{n^3}\cos(nx) + C\right)\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{2\pi}\left[\left(-\frac{\pi^2}{n}\cos(n\pi) + \frac{2\pi}{n^2}\cancelto{0}{\operatorname{sen}(n\pi)} + \frac{2}{n^3}\cos(nx)\right) - \left(-\frac{(-\pi)^2}{n}\cos(n(-\pi)) - \frac{2\pi}{n^2}\operatorname{sen}(n(-\pi)) + \frac{2}{n^3}\cos(n(-\pi))\right)\right]$$

$$b_n = \frac{1}{2\pi}\left[\left(-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}(-1)^n\right) - \left(-\frac{\pi^2}{n}\cos(nx) + \frac{2\pi}{n^2}\operatorname{sen}(n\pi) + \frac{2}{n^3}\cos(n\pi)\right)\right]$$

$$b_n = \frac{1}{2\pi}\left[\cancel{-\frac{\pi^2}{n}(-1)^2} + \cancel{\frac{2\pi}{n^3}(-1)^n} + \cancel{\frac{\pi^2}{n}(-1)^n} - \cancel{\frac{2}{n^3}(-1)^n}\right] = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(n\omega x) + b_n \operatorname{sen}(n\omega x)\right)$$

$$= \frac{\pi^3/3}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + \cancelto{0}{b_n}\operatorname{sen}(nx)\right)$$

$$= \frac{\pi^3}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos(nx)$$