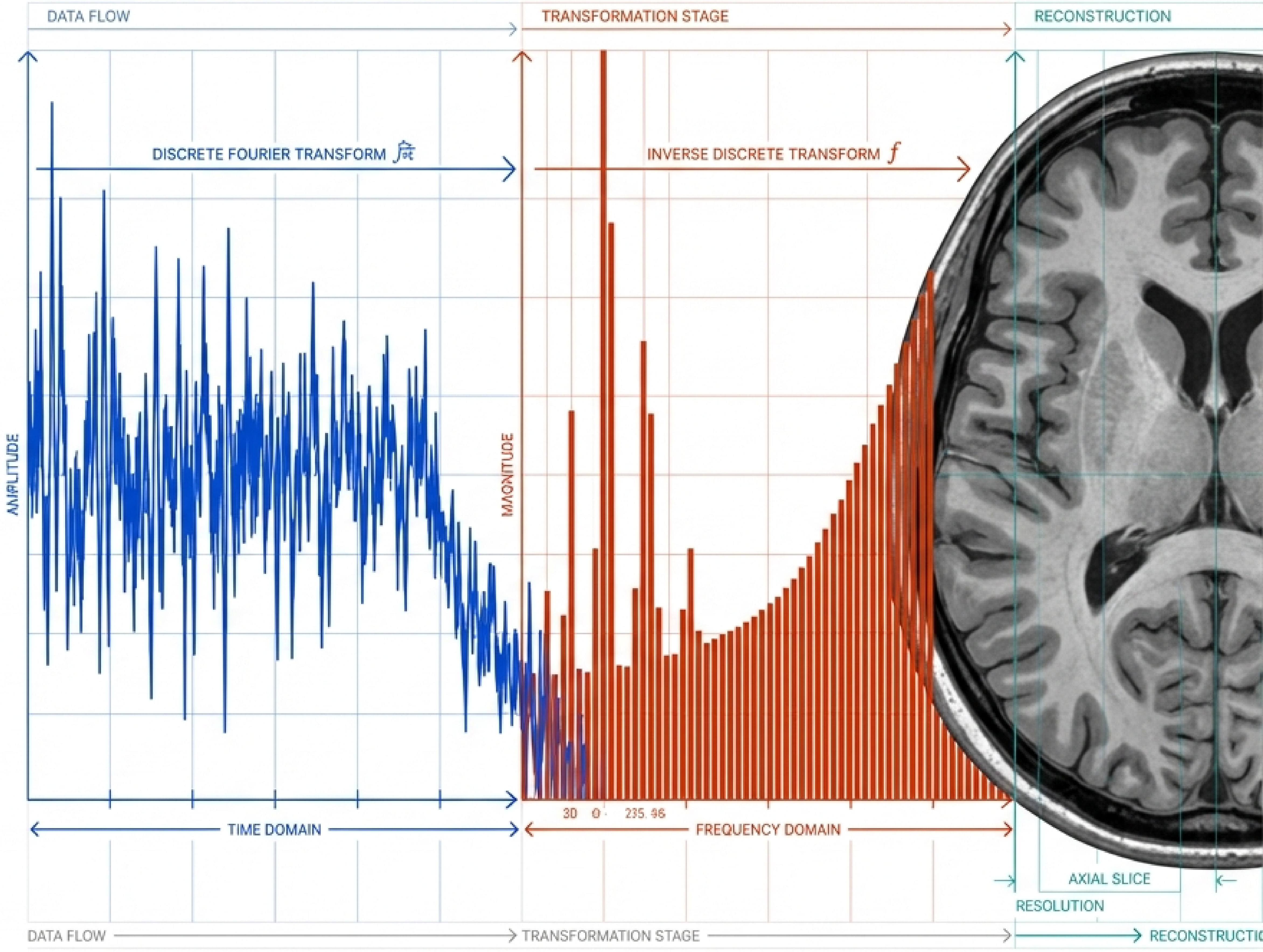
Unlocking Hidden Dimensions

From Signal Analysis to Medical Imaging: An Exploration of Discrete Transforms

Narrative: We live in the time and spatial domains. However, the mathematical engines that drive modern technology—from MP3s to MRI scans—operate in the frequency domain. This deck deconstructs the tools that allow us to translate between these worlds.



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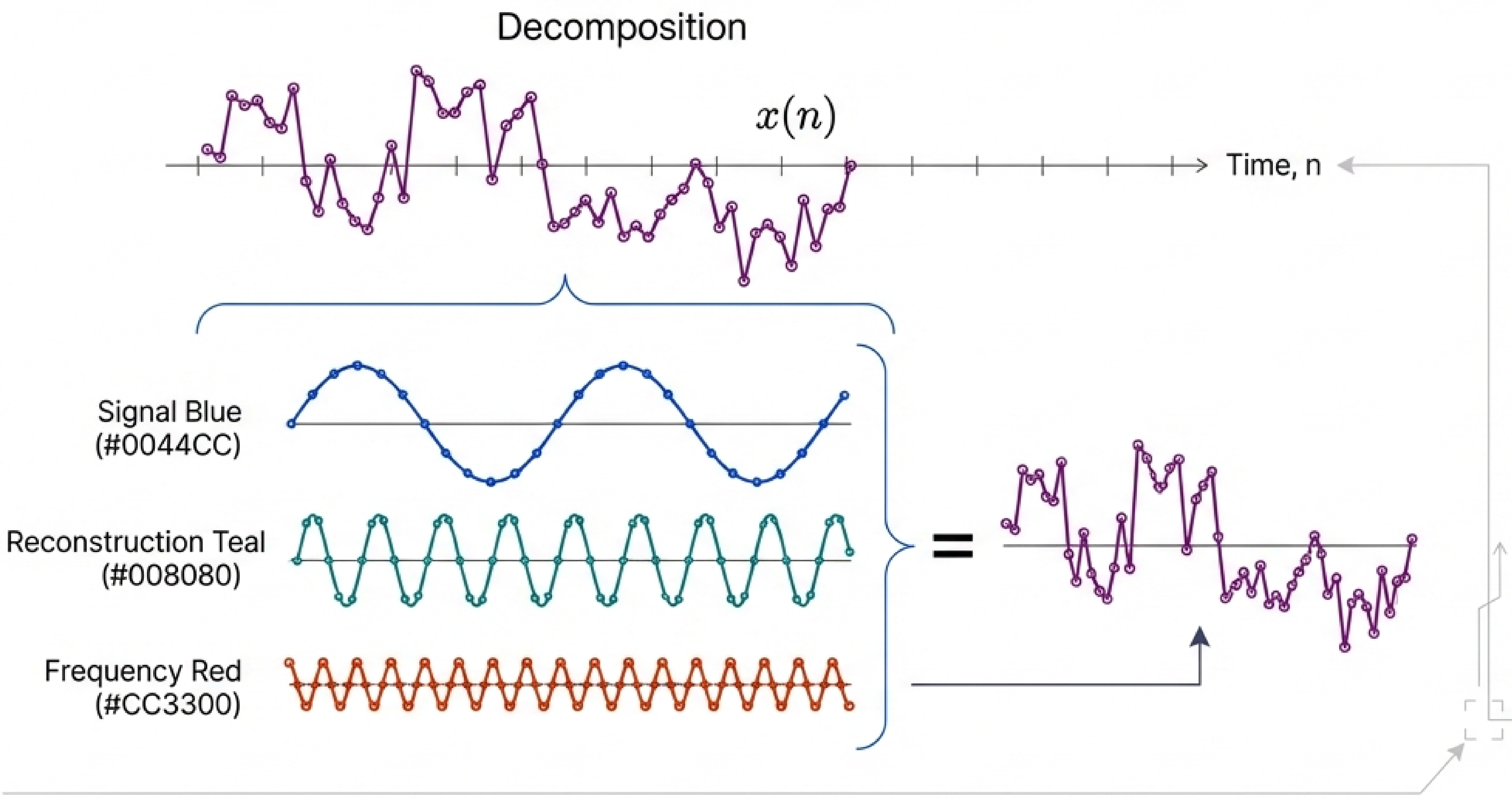
The Gateway: Discrete Fourier Transform (DFT)

Bridging the continuous physical world and the discrete digital computer.



The Mechanism:

Computers cannot process infinite continuous signals. The DFT samples the signal and assumes periodicity, solving a set of linear equations to find frequency coefficients.

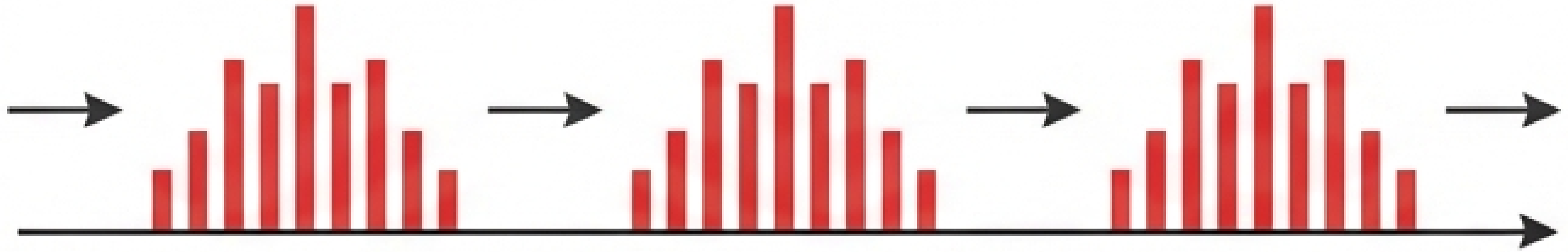
The Equation:

$$x(n) = \sum_{k=0}^{N-1} X'(k) e^{j\frac{2\pi}{N}kn}$$

Key Takeaway: The DFT calculates the 'correlation' of the signal with reference sine and cosine waves to determine how much of each frequency exists in the data.

The Laws of the Frequency Domain

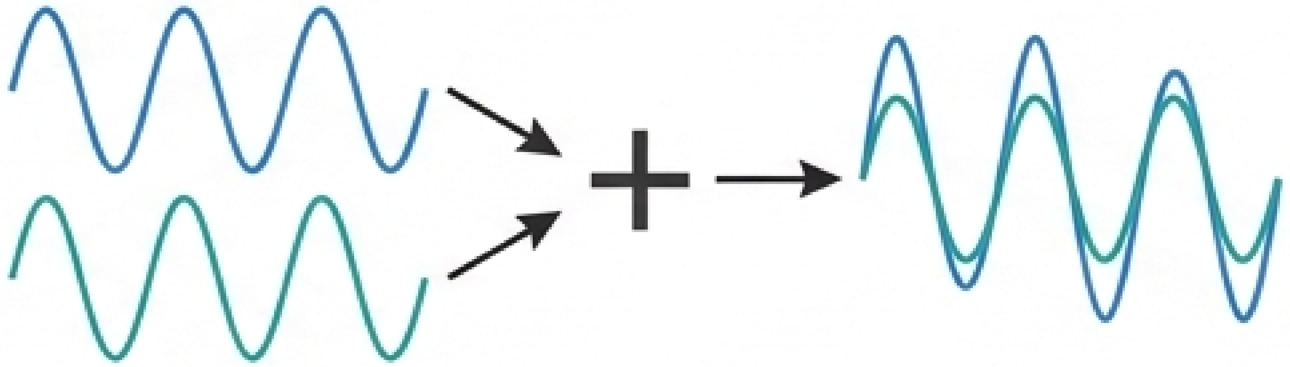
1. Periodicity



The spectrum repeats indefinitely.
It is circular, not linear.

$$X(k) = X(k+N)$$

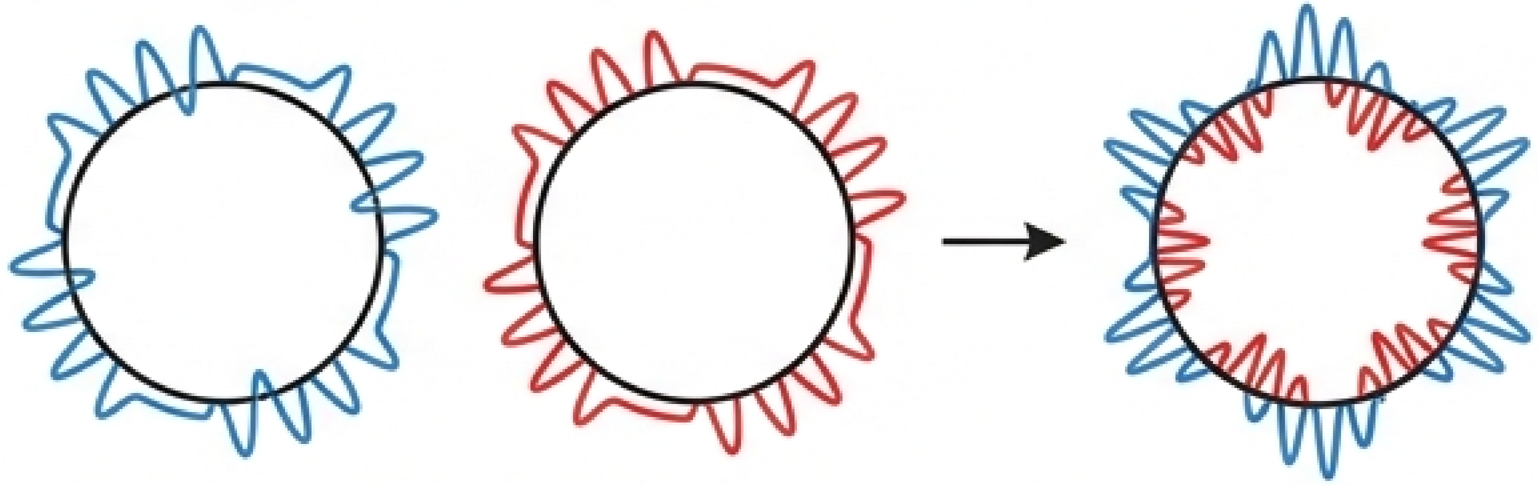
2. Linearity



The DFT of a sum is the sum of the DFTs.

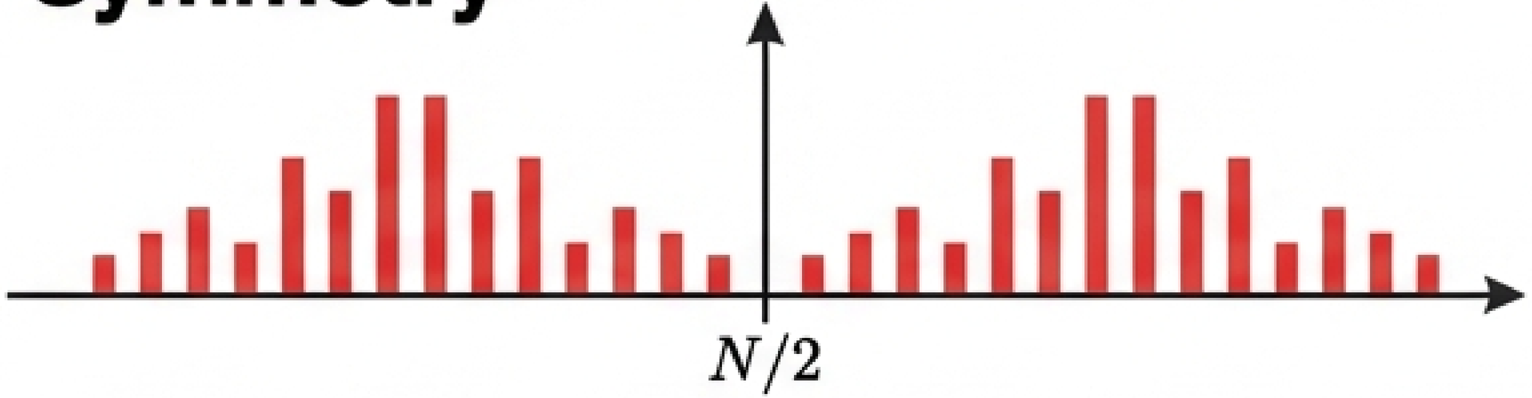
$$a_1 x_1 + a_2 x_2 \rightarrow a_1 X_1 + a_2 X_2$$

3. Circular Convolution



Multiplication in frequency equals circular convolution in time. The basis of digital filtering.

4. Symmetry



For real-valued signals, the spectrum has conjugate symmetry.

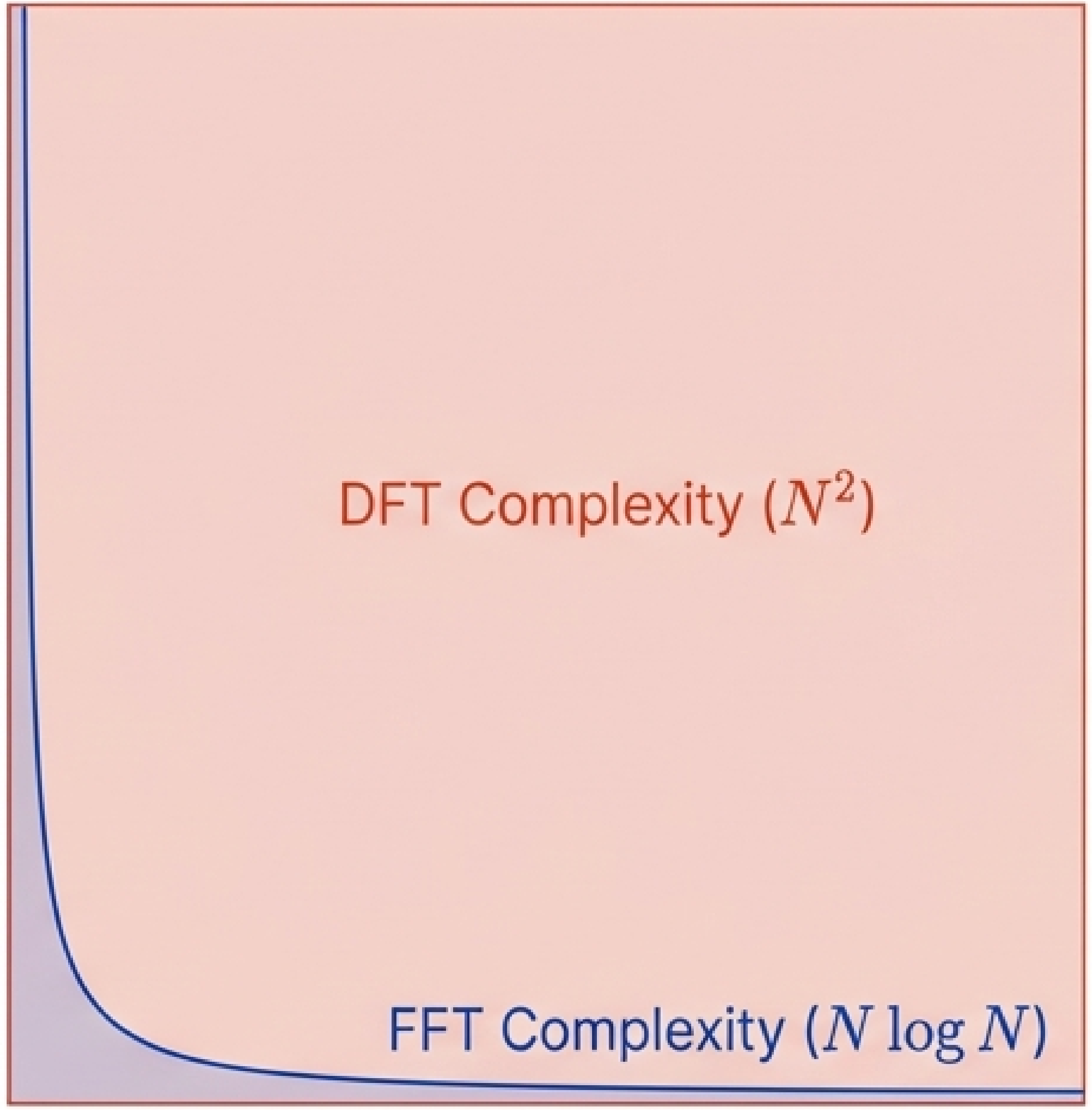
$$X(k) = X^*(N-k)$$

The Need for Speed: DFT vs. FFT

The Problem: Standard DFT requires N^2 computations. Slow for large datasets.

The Solution: Fast Fourier Transform (FFT). Exploits symmetry to reduce operations to $N \log N$.

The Constraint: Sample size must be a power of two (2^n, e.g., 512, 1024).



For N = 1024 samples:
DFT ≈ 1,000,000 ops
FFT ≈ 10,000 ops

Result: 100x Speed Increase

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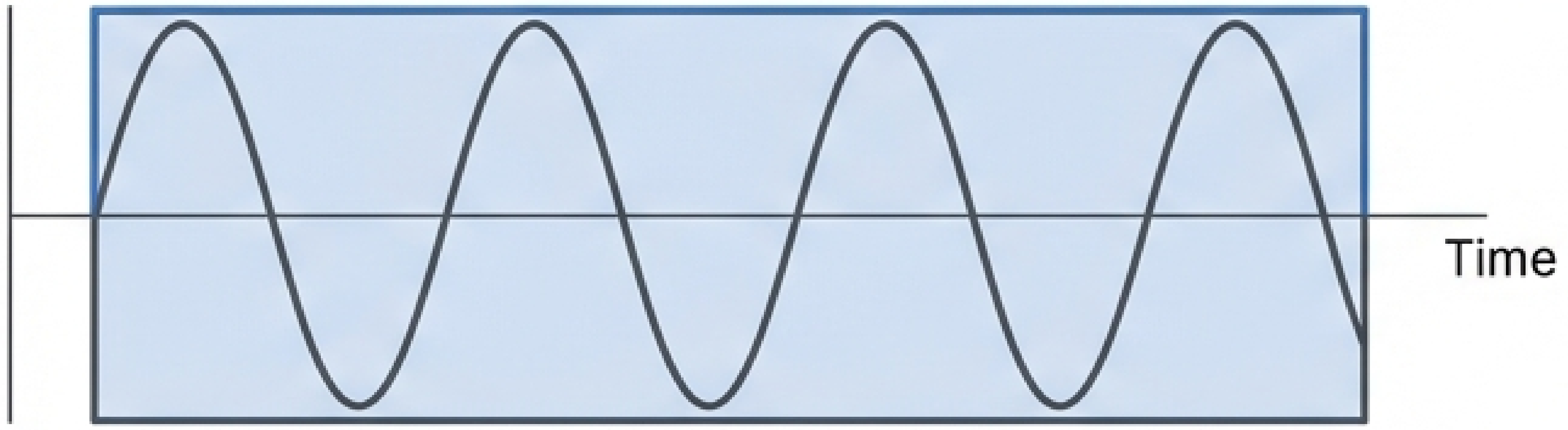
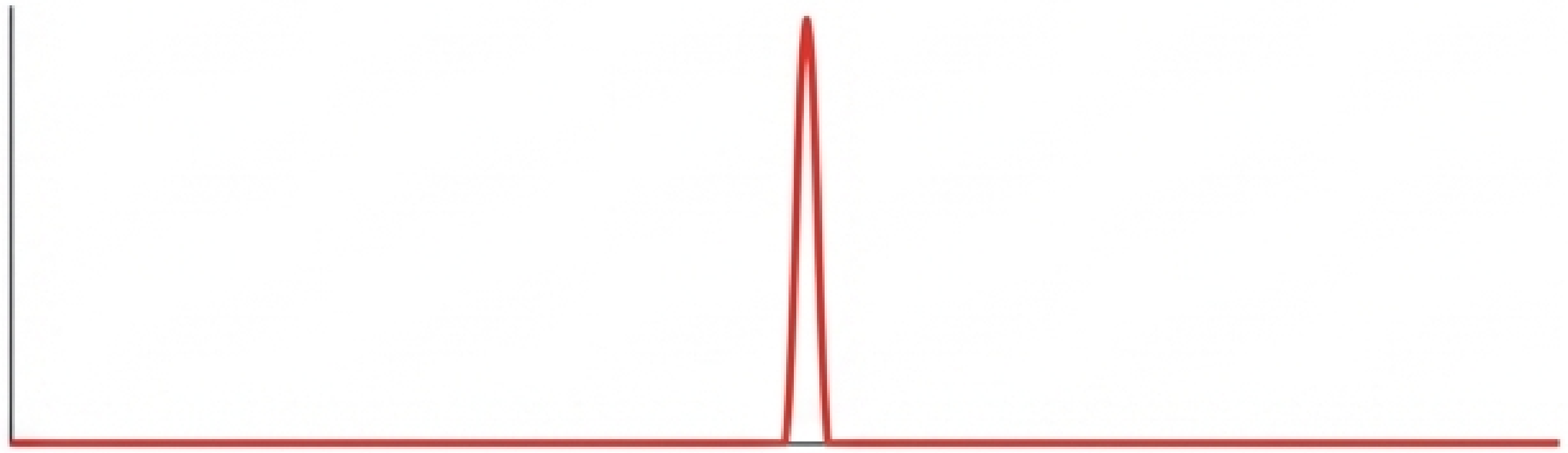
The Cost of Efficiency: Spectral Leakage

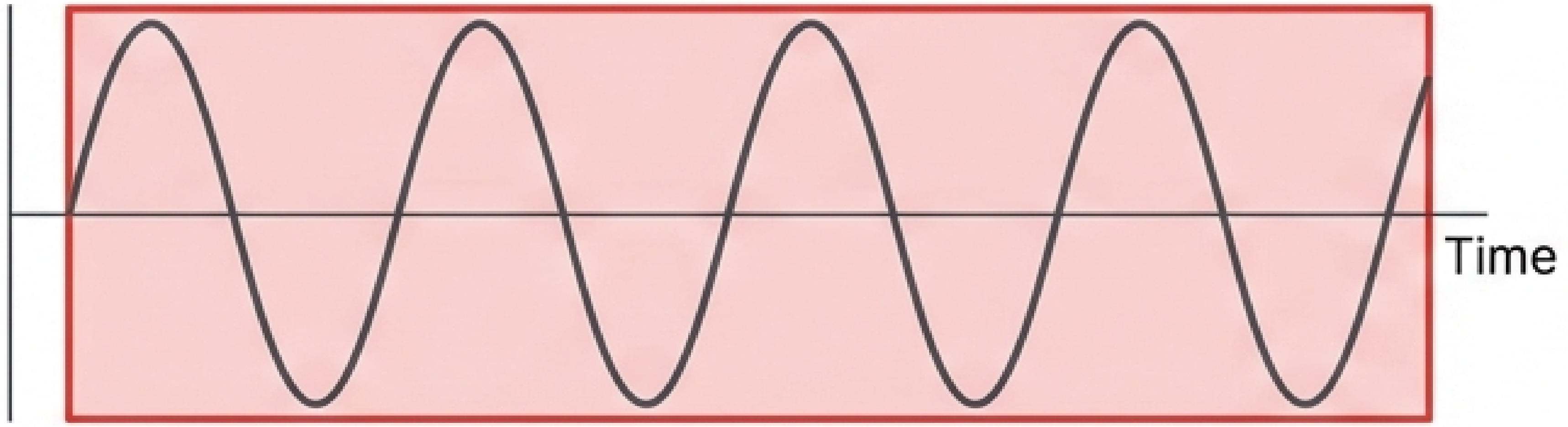
Ideal Windowing (DFT)

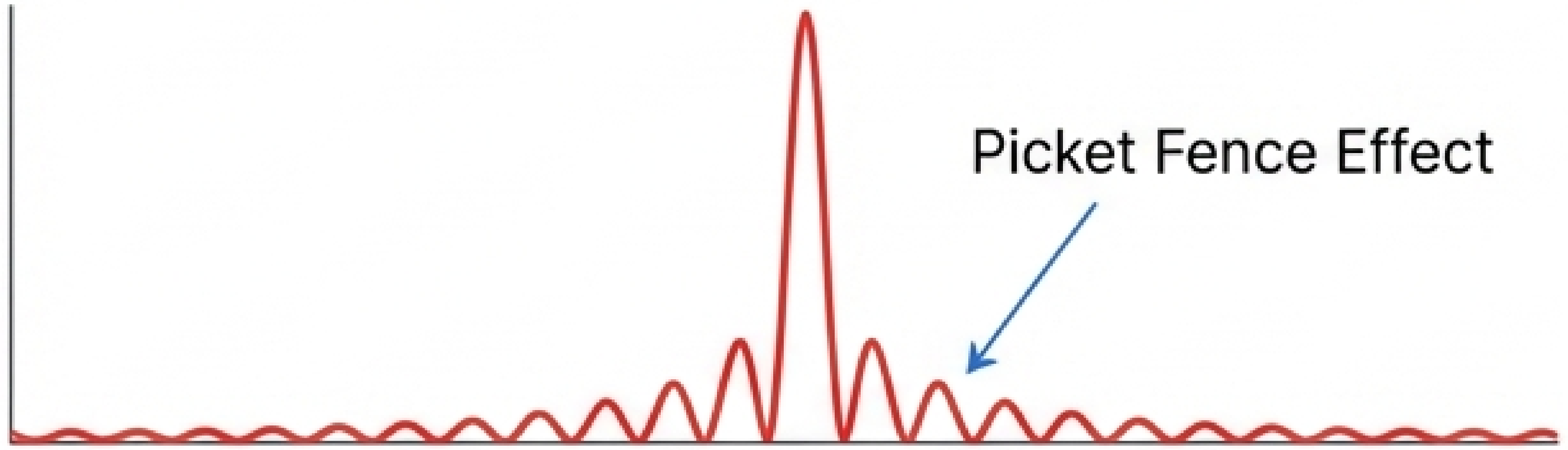




Matched Window: 833 samples. Signal cycle matches window. Error $\approx 0\%$.

Forced Binary Window (FFT)





Mismatched Window: 1024 samples (2^{10}). Discontinuity at edges creates 'Spectral Leakage'. Energy spills into adjacent bins.

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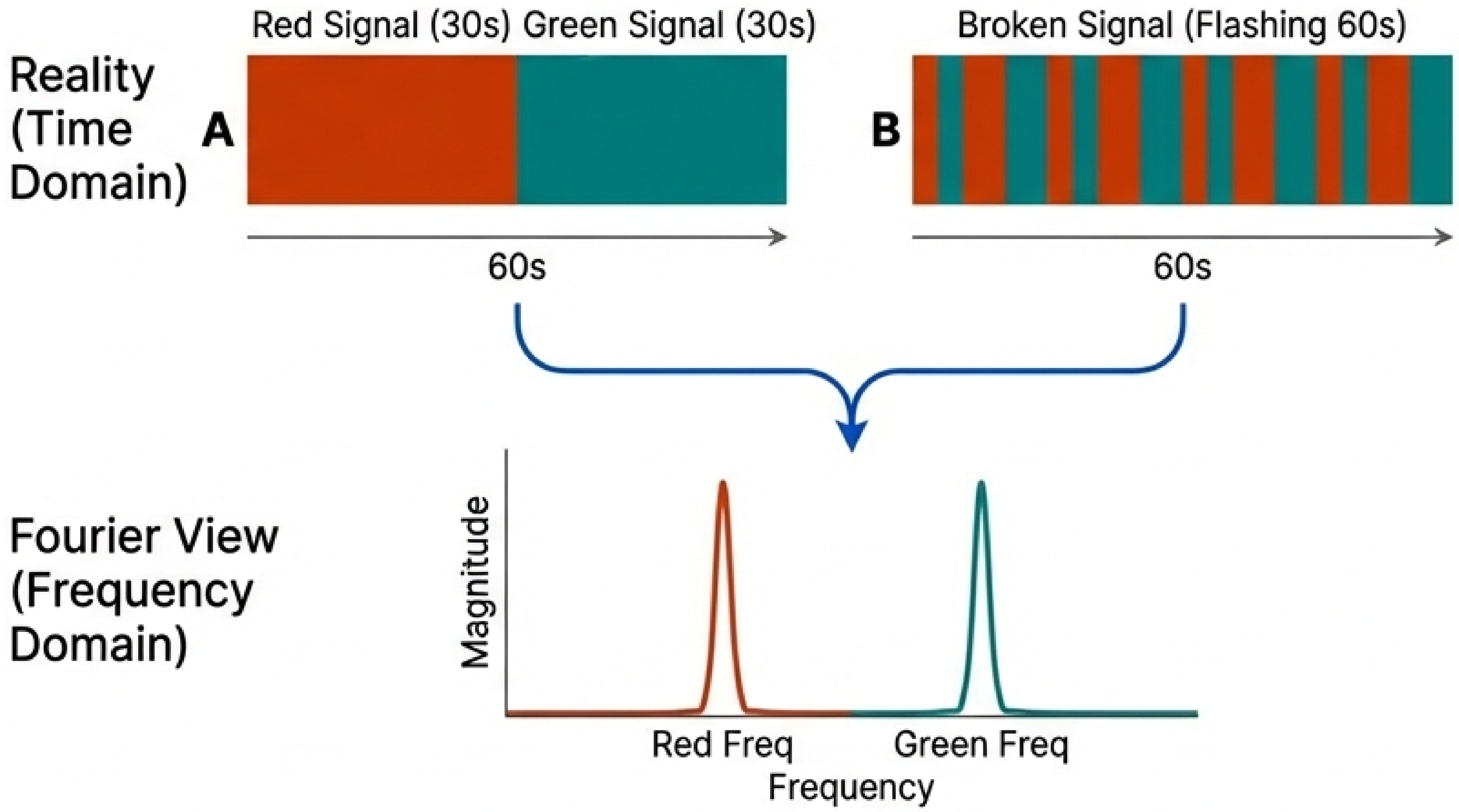
The Blind Spot: Frequency Without Time

Limitation: Fourier analysis reveals *what* frequencies are present, but not *when* they happened.

Heisenberg Uncertainty: Narrowing the time window widens the frequency bandwidth. Precision in one domain sacrifices the other.

The Traffic Light Paradox



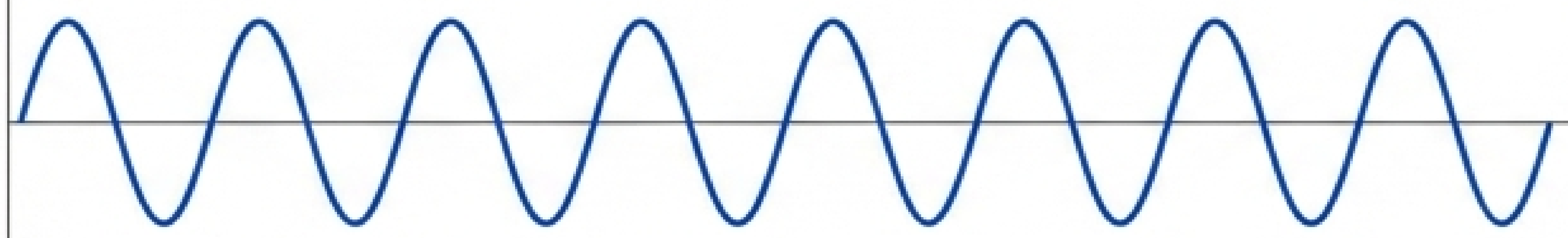
Both realities produce the exact same Fourier Spectrum. Temporal order is lost.

The Evolution: Discrete Wavelet Transform (DWT)

Solving the time-frequency dilemma with multi-resolution analysis.

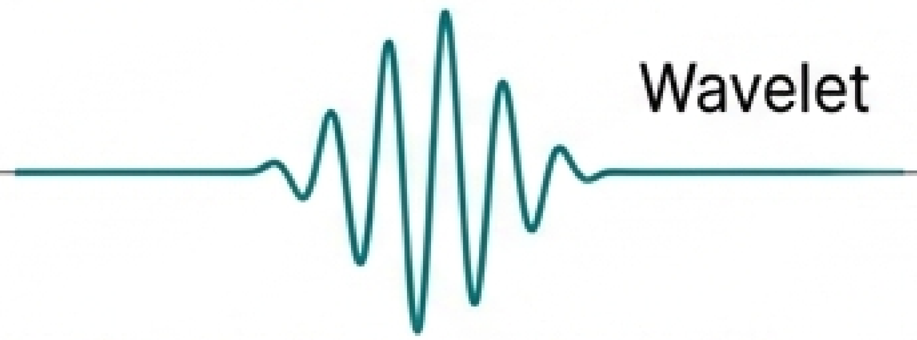
Basis Functions

Fourier Basis



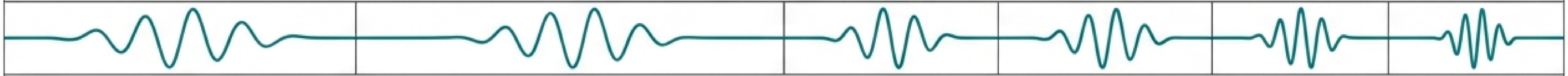
Infinite duration. Good for stationary signals. No time localization.

Wavelet Basis

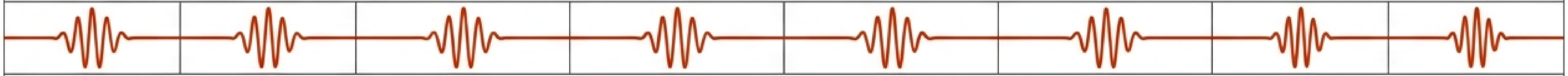


Finite duration. Localized in time and frequency.

Multi-Resolution Analysis



Low Frequency, Low Time Precision

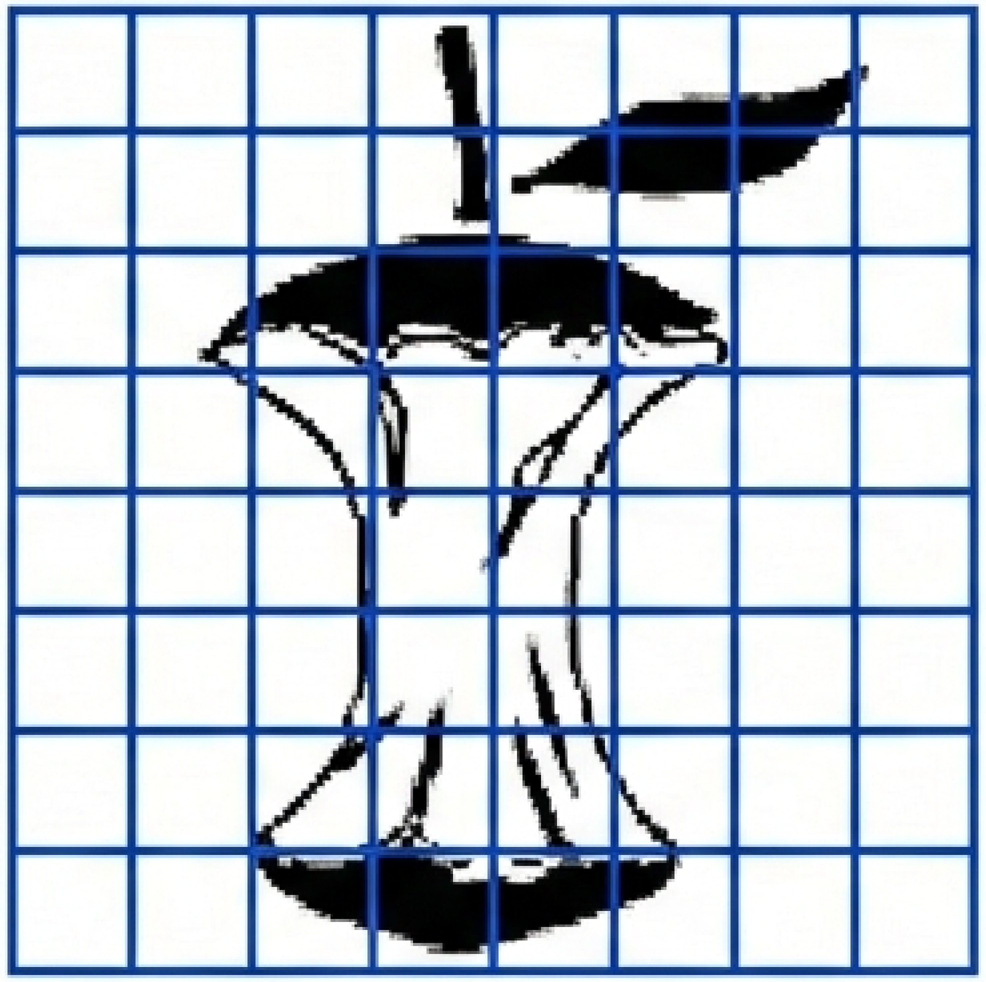


High Frequency, High Time Precision

DWT adapts the lens: Short windows for high frequencies (detail), long windows for low frequencies (trend).

Compression Wars: DCT vs. DWT

Discrete Cosine Transform (DCT)
Standard JPEG



- Processes image in 8 × 8 blocks.
- Artifact: Blockiness / Grid Effect.

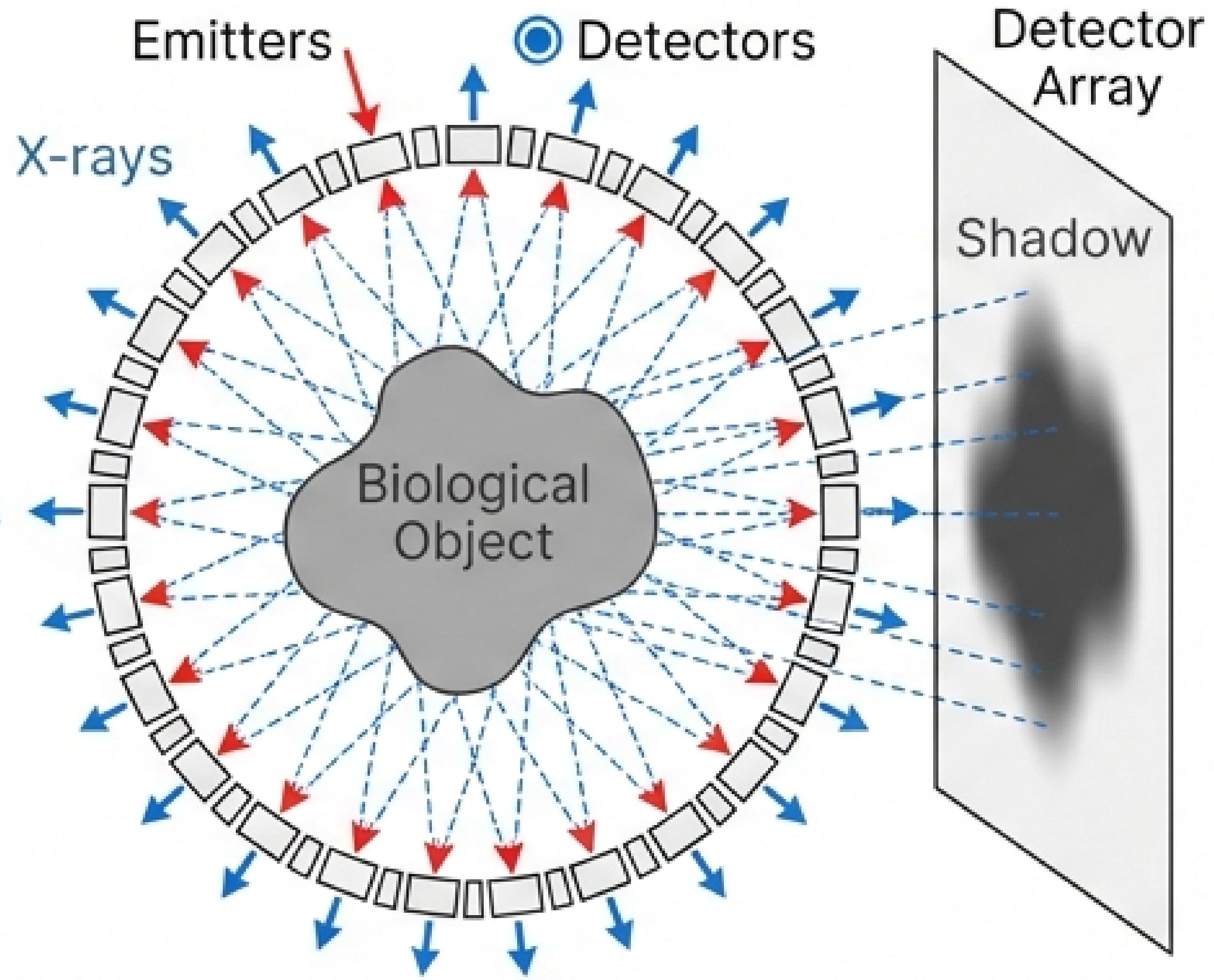
Discrete Wavelet Transform (DWT)
JPEG 2000

- Processes entire image (Global).
- Artifact: Blurring / Ringing edges.

Winner: DWT achieves higher compression ratios with lower error (MSE).

Reconstructing Reality: The Inverse Problem

Measurement:
We collect projections (Shadows/Echoes).

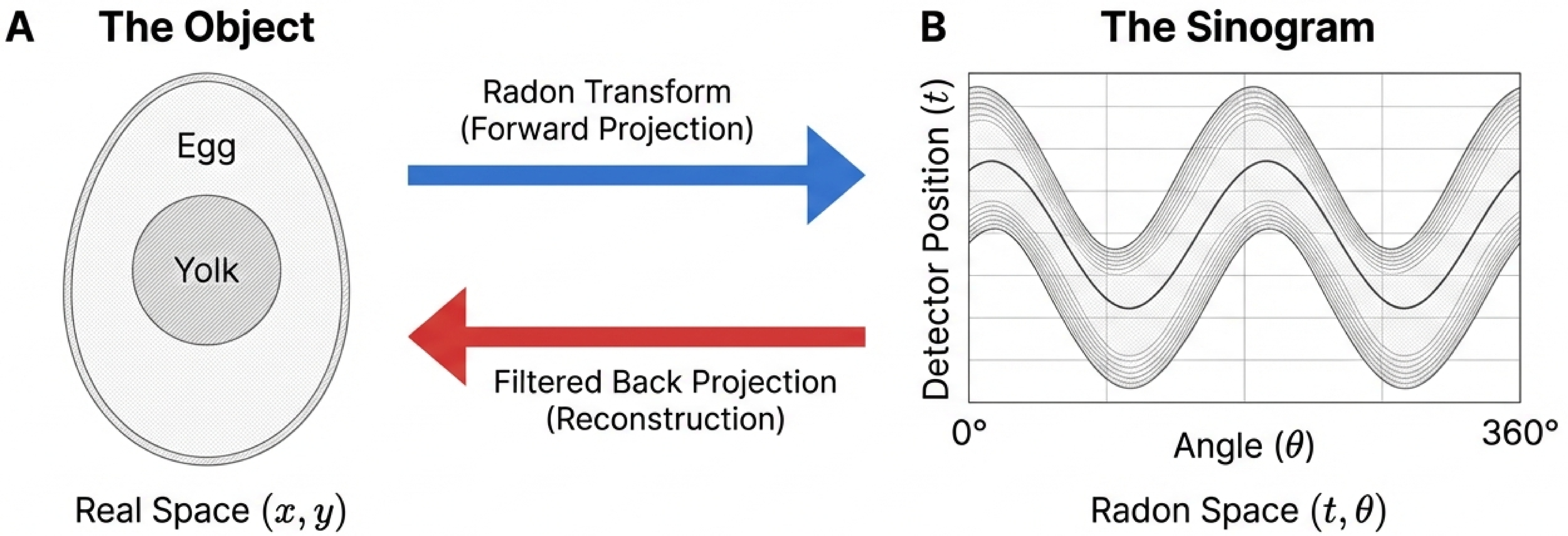


Reconstruction:
We must solve for $\mu(x, y)$ (Density) inside the object.

The Tools of Reconstruction	
CT (Computed Tomography)	**MRI (Magnetic Resonance Imaging)**
Radon Transform & Filtered Back Projection.	Inverse 2D DFT & K-Space.

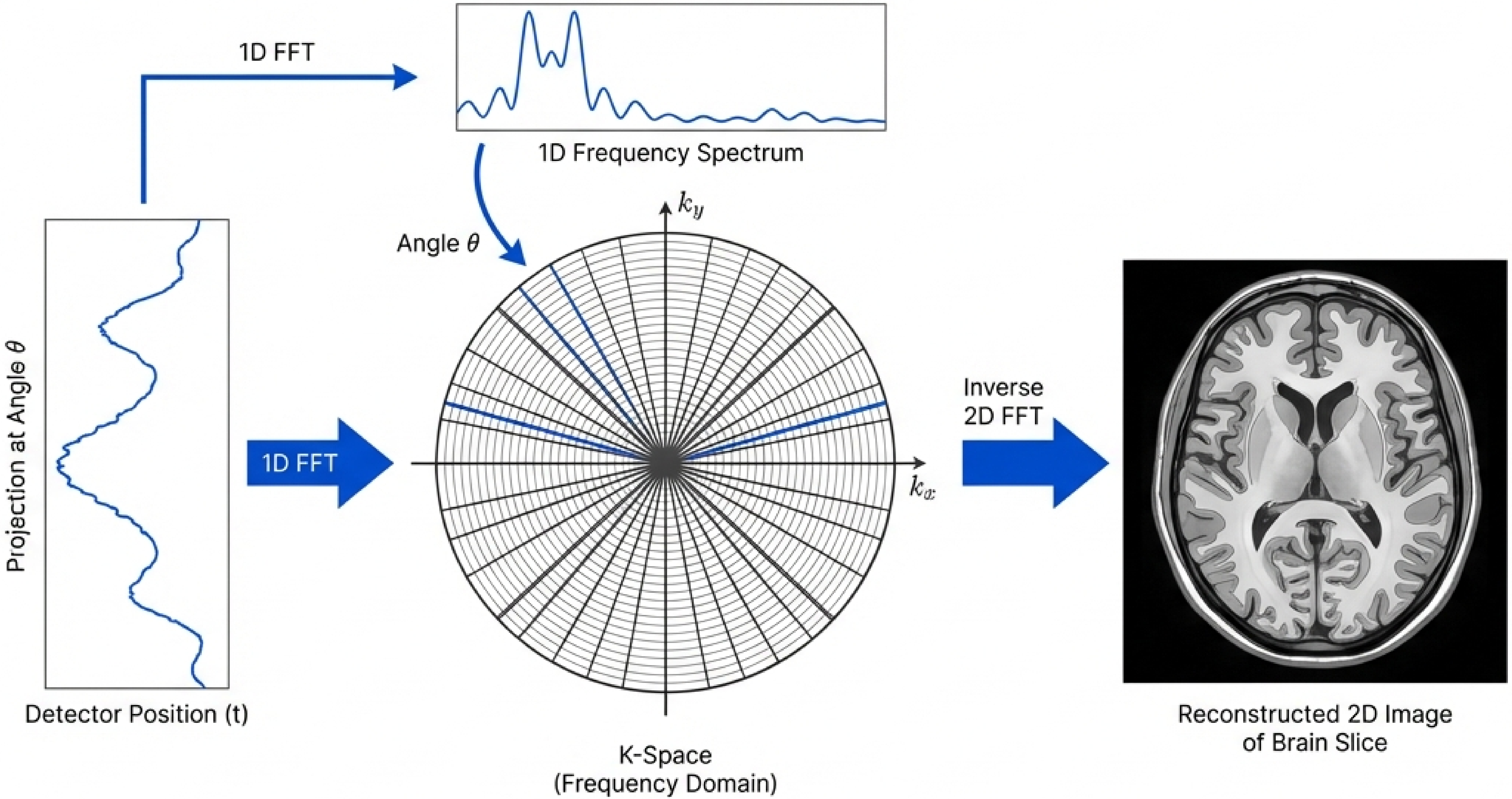
Swiss Scientific Editorial

CT Scans: Reading the Sinogram



A Sinogam maps the position of a point across all viewing angles (0°–360°).
A single point in reality traces a sine wave in data space.

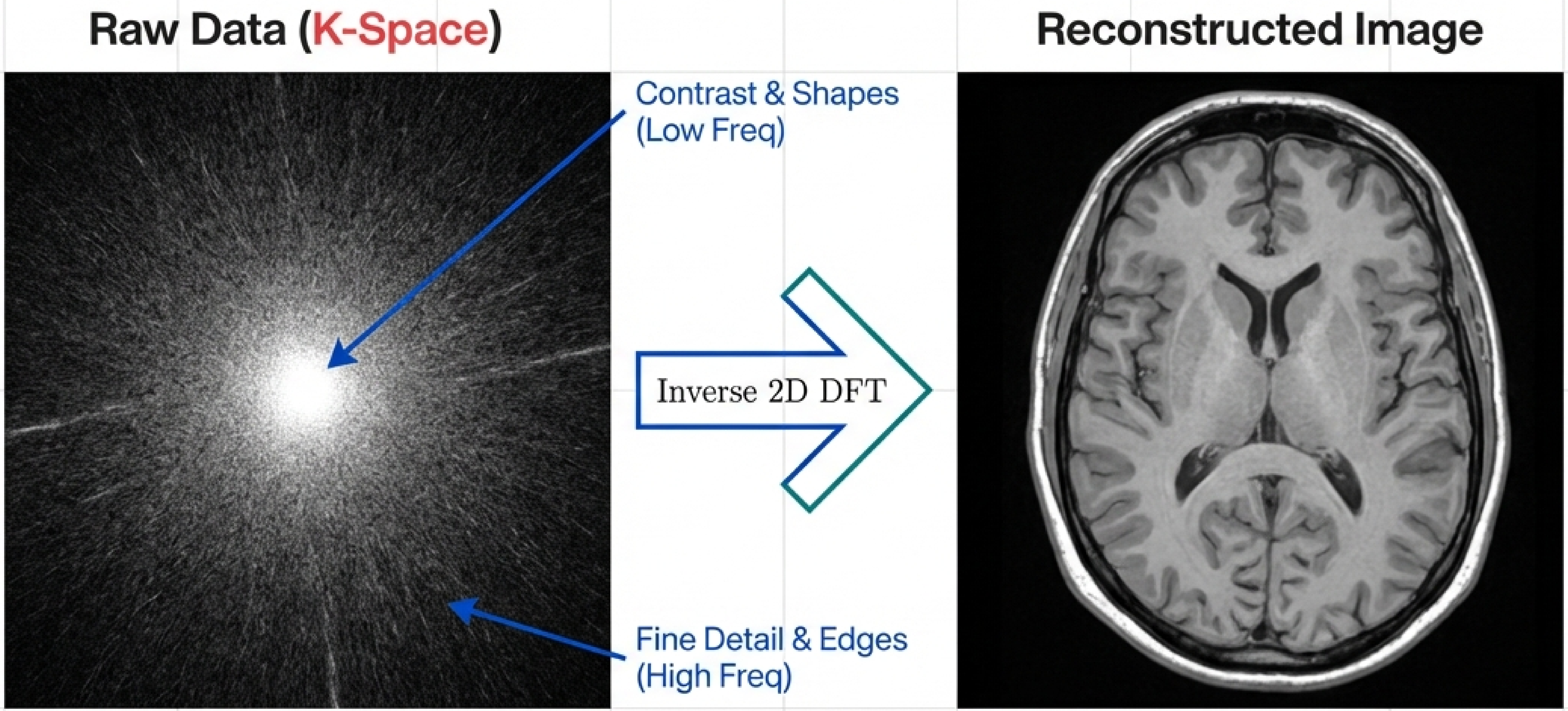
The Fourier Slice Theorem



The Theorem: The 1D Fourier transform of a projection is equal to a single “slice” of the 2D Fourier transform of the object.

Note: **Center frequencies overlap (high density), edges are sparse. A “Ramp Filter” is applied to prevent blurring.**

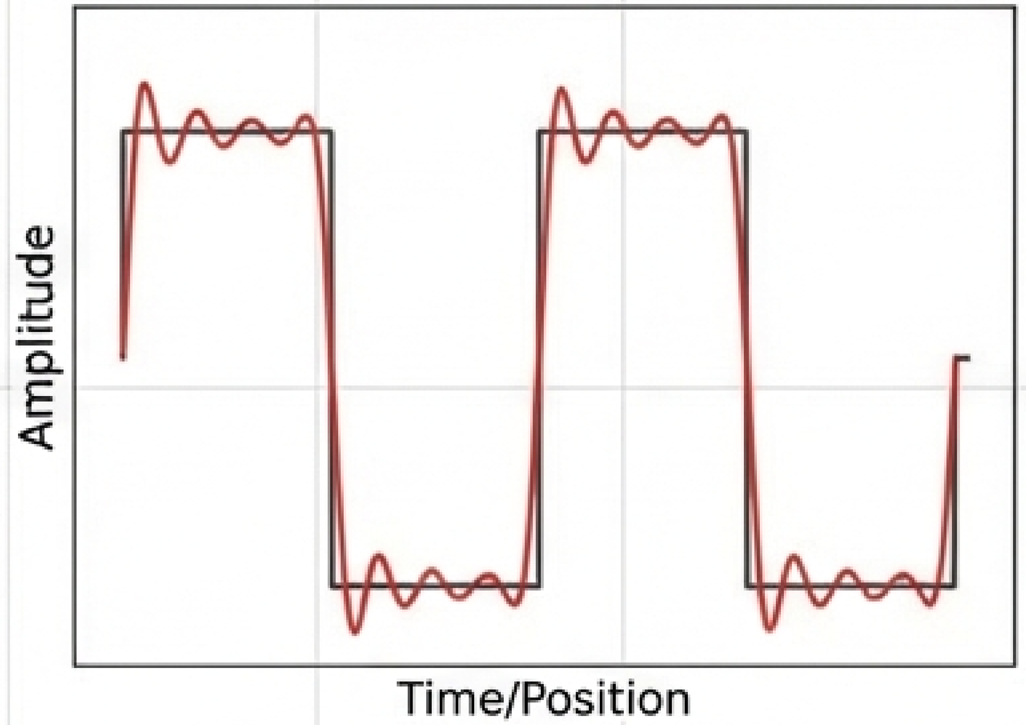
MRI: Writing in K-Space



MRI scanners do not capture pixels. They capture spatial frequencies directly into a matrix called **K-Space**.

The Scars of Approximation: Imaging Artifacts

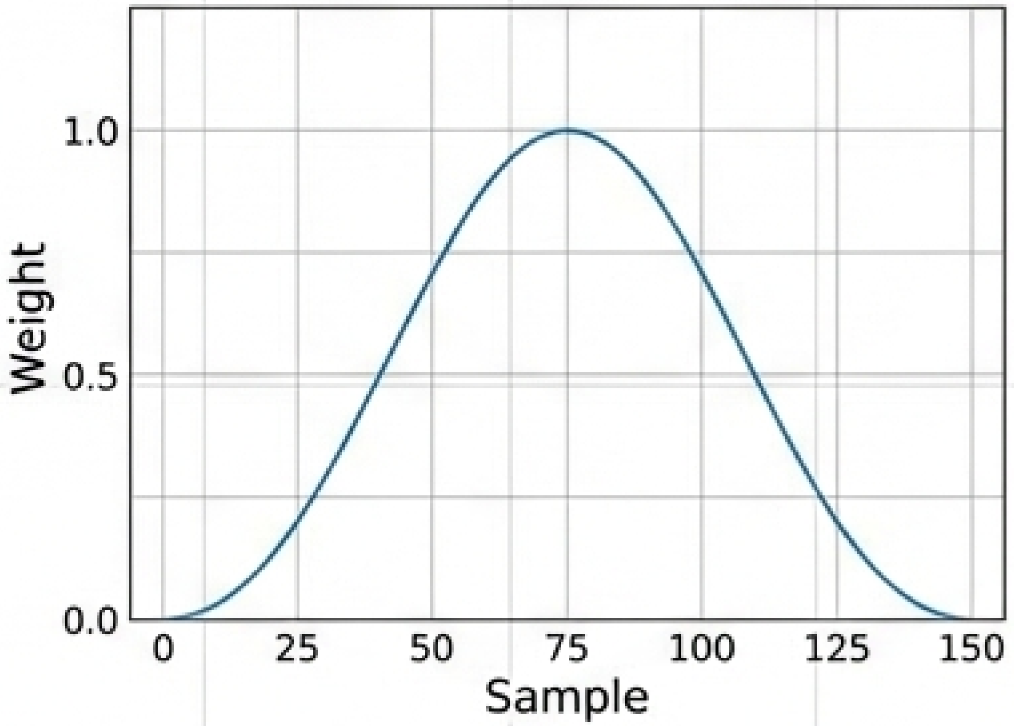
Gibbs Ringing



Caused by truncating infinite Fourier series. Sharp edges create ripples.

$$\sum_{n=1}^{N} \frac{\sin((2n-1)\omega t)}{2n-1}$$

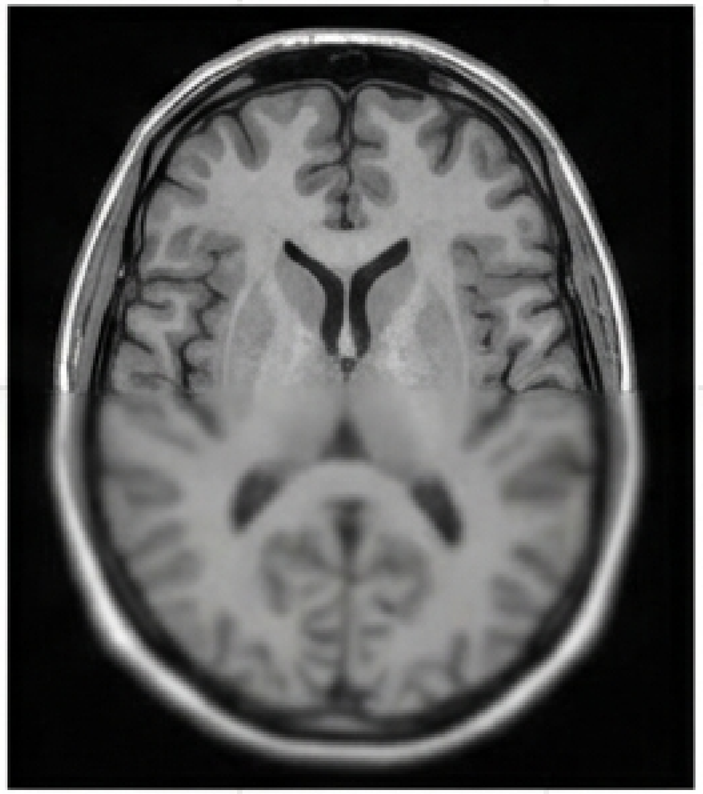
The Cure: Windowing



Applying a window function (e.g., Hanning) smooths the data edges, dampening the ringing but slightly reducing sharpness.

$$w(n) = 0.5\left(1 - \cos\left(\frac{2\pi n}{N-1}\right)\right)$$

Truncation Blur



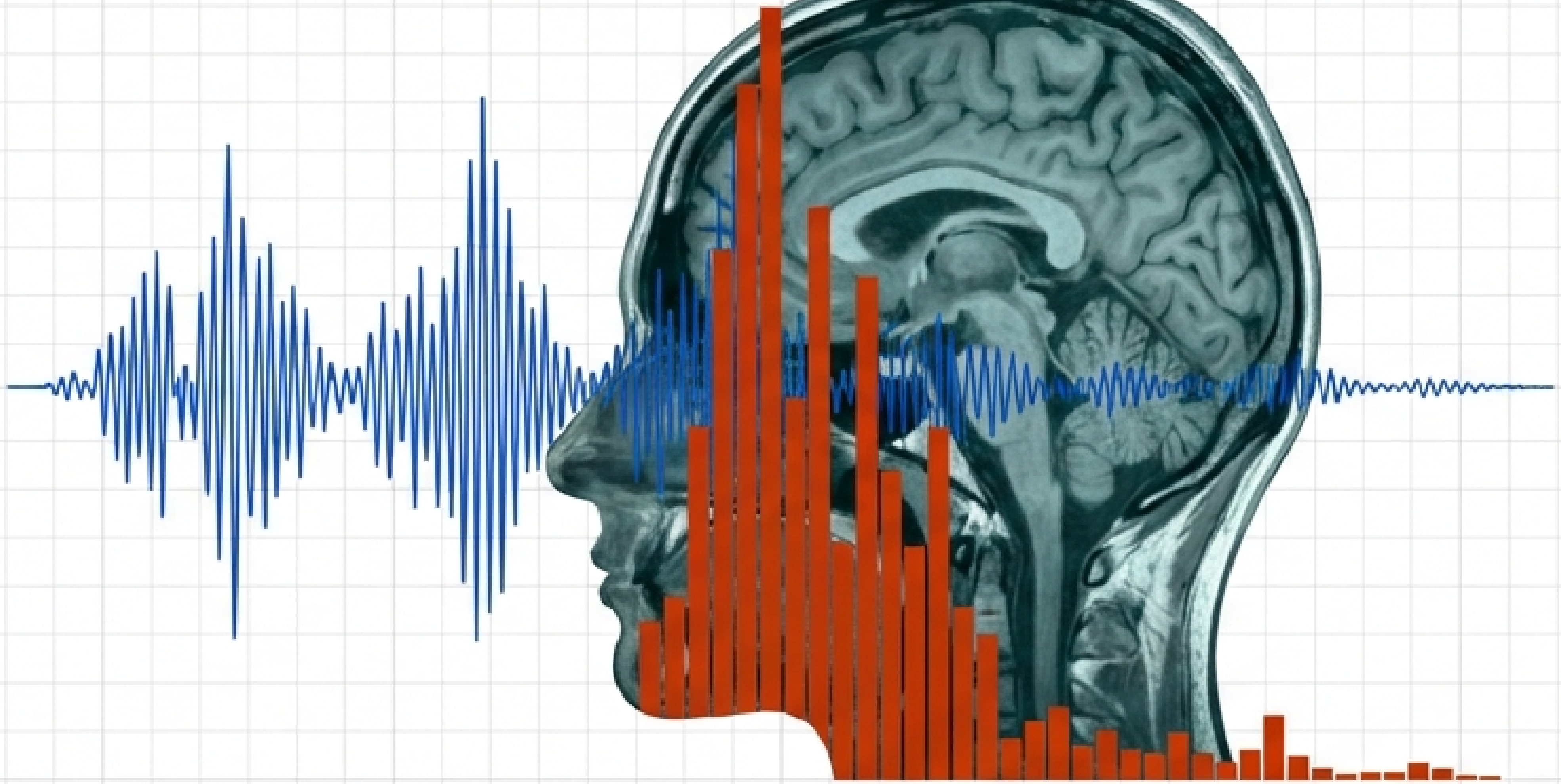
Missing high-frequency data (edges of K-space) results in loss of detail.

$$\Delta x \propto \frac{1}{k_{\max}}$$

The Toolbox: Choosing the Right Lens

Algorithm	Best Application	Pros	Cons
DFT	Harmonic Analysis	✓ Extreme Precision	✗ Slow ($O(N^2)$)
FFT	Real-Time Processing	✓ Speed ($O(N \log N)$)	✗ Rigid (2^n), Spectral Leakage
DCT	JPEG Compression	✓ Energy Compaction	✗ Block Artifacts
DWT	JPEG2000 & Medical Signals	✓ Time-Frequency Localization	✗ Computational Complexity

The Invisible Mathematics



From the music we stream to the medical diagnoses that save lives, Discrete Transforms are the invisible engines of our world. They allow us to perceive reality not just as it is in time, but as it exists in frequency—unlocking dimensions otherwise invisible to the naked eye.