Solutions Manual for Gregory F. Lawler's *Introduction to Stochastic Processes*

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Part I

Chapter 1

Finite Markov Chains

Problem 1.1.

Problem 1.2. Consider a Markov chain with state space 0,1 and transition matrix

$$\mathbf{P} = \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix}$$

Assuming that the chain starts in state 0 at time $n = 0$, what is the probability that it is in state 1 at time $n = 3$?

Solution. This is just some basic matrix multiplication. The chain starts in state 0 at time $n = 0$ so we will look at the first row of the matrix $\mathbf{P^3}$. $\square$

Problem 1.3.

Solution. (a) We have that $P = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0 & .4 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$ The matrix P^n converges pretty quickly so you only need to use a power of say, 100. Using R gives the following:

$$P^{100} = \begin{bmatrix} 0.3787879 & 0.2575758 & 0.3636364 \\ 0.3787879 & 0.2575758 & 0.3636364 \\ 0.3787879 & 0.2575758 & 0.3636364 \end{bmatrix}$$

Thus the common row vector is

$$\pi = (0.3787879, 0.2575758, 0.3636364)$$

$\square$

Problem 1.4.

Solution. $\square$

Problem 1.5.

Solution. (1) Recurrent classes: $\{0,1\},\{2,4\}$. Transient class: $\{3,5\}$
(2) To analyze large time behavior of the Markov chain on the class $R_1 = \{0,1\}$, we need only to consider its matrix

$$\mathbf{P}_{\{0,1\}} = \begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & \;\;1 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

Solving $\pi\mathbf{P}_{\{0,1\}} = \pi$ we get its invariant probability $\pi = (\frac{3}{8}, \frac{5}{8})$. Then $lim_{n\to\infty}P_n(0,0) = \frac{3}{8}$.

(3) To find $lim_{n\to\infty}P_n(5,0)$, we first find $lim_{n\to\infty}P_n(0,R_1)$, the probability that the chain will be absorbed into $R_1 = \{0,1\}$. Rearrange P we can write it as

$$\tilde{\mathbf{P}}_{\{0,1\}} = \begin{array}{c} \\ \{0,1\} \\ \{2,4\} \\ 3 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \{0,1\} & \{2,4\} & 3 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0.25 & 0 & 0.25 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{pmatrix} \end{array} = \begin{bmatrix} I & 0 \\ S & Q \end{bmatrix}$$

Then it follows from $lim_{n\to\infty}\tilde{\mathbf{P}}^{\mathbf{n}} = \begin{pmatrix} I & 0 \\ (I-Q)^{-1}S & 0 \end{pmatrix}$ (see Section 1.5) and

$$(I-Q)^{-1}S = \frac{1}{11} * \begin{pmatrix} 7 & 4 \\ 6 & 5 \end{pmatrix}$$

that $lim_{n\to\infty}P_n(5,R_1) = \frac{6}{11}$. Combining it with (2) we get $lim_{n\to\infty}P_n(5,0) = \frac{6}{11} * \frac{3}{8} = \frac{9}{44} =$ .2045 □

Problem 1.6.

Solution. □

Problem 1.7.

Solution. □

Problem 1.8.

Solution. □

Problem 1.9.

Solution. □

Problem 1.10.

Solution. □

Problem 1.11.

Solution. □

Problem 1.12.

Solution. □

Problem 1.13.

Solution. □

Problem 1.14.

Solution. □

Problem 1.15.

Solution. □

Problem 1.16.

Solution. □

Problem 1.17.

Solution. □

Problem 1.18.

Solution. □

Problem 1.19.

Solution. □

Problem 1.20.

Solution. □

Problem 1.21.

Solution. □

Chapter 2

Countable Markov Chains

Problem 2.1.

Solution. $\square$

Chapter 3

Continuous-Time Markov Chains

Problem 3.1.

Solution. $\square$

Chapter 4

Optimal Stopping

Problem 4.1.

Solution. $\square$

Chapter 5

Martingales

Problem 5.1.

Solution. $\square$

Chapter 6

Renewal Processes

Problem 6.1.

Solution. $\square$

Chapter 7

Reversible Markov Chains

Problem 7.1.

Solution. $\square$

Chapter 8

Brownian Motion

Problem 8.1.

Solution. $\square$

Chapter 9

Stochastic Integration