Report #5

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1 Show that $R(\boldsymbol{q})$ orthogonal.

To show that $R(q)$, the rotation matrix derived from a quaternion q , is orthogonal, we need to verify that:

$$R(q)^\top R(q) = I \tag{1}$$

where $R(q)^\top$ is the transpose of $R(q)$, and I is the identity matrix.

$$R(q) = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) \\ 2(xy+wz) & 1-2(x^2+z^2) & 2(yz-wx) \\ 2(xz-wy) & 2(yz+wx) & 1-2(x^2+y^2) \end{bmatrix} \tag{2}$$

where w, x, y, z are the components of the quaternion.

It's transpose is:

$$R(q)^\top = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy+wz) & 2(xz-wy) \\ 2(xy-wz) & 1-2(x^2+z^2) & 2(yz+wx) \\ 2(xz+wy) & 2(yz-wx) & 1-2(x^2+y^2) \end{bmatrix} \tag{3}$$

Now, we multiply $R(q)^\top$ by $R(q)$:

$$R(q)^\top R(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

The result of $R(q)^\top R(q)$ is the identity matrix I :

$R(q)^\top R(q) = I$.

So the rotation matrix $R(q)$ is orthogonal.

2 Show $\dot{A}\boldsymbol{q} = A\dot{\boldsymbol{q}}$, $\dot{B}\boldsymbol{q} = B\dot{\boldsymbol{q}}$, and $\dot{C}\boldsymbol{q} = C\dot{\boldsymbol{q}}$.

$$A = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix}, \quad B = \begin{bmatrix} -q_3 & q_2 & q_1 & -q_0 \\ q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \end{bmatrix}, \quad C = \begin{bmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ q_0 & -q_1 & -q_2 & q_3 \end{bmatrix}. \tag{5}$$

Each element of A depends on q_0, q_1, q_2, q_3 . Taking the derivative with respect to time:

$$\dot{A} = \begin{bmatrix} \dot{q}_0 & \dot{q}_1 & -\dot{q}_2 & -\dot{q}_3 \\ \dot{q}_3 & \dot{q}_2 & \dot{q}_1 & \dot{q}_0 \\ -\dot{q}_2 & \dot{q}_3 & -\dot{q}_0 & \dot{q}_1 \end{bmatrix}. \tag{6}$$

When $\dot{A}$ is multiplied by $\boldsymbol{q}$:

$$\dot{A}\boldsymbol{q} = \begin{bmatrix} \dot{q}_0 q_0 + \dot{q}_1 q_1 - \dot{q}_2 q_2 - \dot{q}_3 q_3 \\ \dot{q}_3 q_0 + \dot{q}_2 q_1 + \dot{q}_1 q_2 + \dot{q}_0 q_3 \\ -\dot{q}_2 q_0 + \dot{q}_3 q_1 - \dot{q}_0 q_2 + \dot{q}_1 q_3 \end{bmatrix} \tag{7}$$

Similarly, consider $A\dot{\boldsymbol{q}}$, where:

$$A\dot{\boldsymbol{q}} = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix} \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \tag{8}$$

Performing the multiplication:

$$A\dot{\boldsymbol{q}} = \begin{bmatrix} q_0\dot{q}_0 + q_1\dot{q}_1 - q_2\dot{q}_2 - q_3\dot{q}_3 \\ q_3\dot{q}_0 + q_2\dot{q}_1 + q_1\dot{q}_2 + q_0\dot{q}_3 \\ -q_2\dot{q}_0 + q_3\dot{q}_1 - q_0\dot{q}_2 + q_1\dot{q}_3 \end{bmatrix} \tag{9}$$

By inspecting the expressions for $\dot{A}\boldsymbol{q}$ and $A\dot{\boldsymbol{q}}$, we see that the two results are identical:

$$\dot{A}\boldsymbol{q} = A\dot{\boldsymbol{q}} \tag{10}$$

. Following the same approach, we can compute the derivatives of B and C , and we can proof:

$$\dot{B}\boldsymbol{q} = B\dot{\boldsymbol{q}}, \quad \dot{C}\boldsymbol{q} = C\dot{\boldsymbol{q}} \tag{11}$$

3 Show $\dot{A}\boldsymbol{q} = A\dot{\boldsymbol{q}}$, $\dot{B}\boldsymbol{q} = B\dot{\boldsymbol{q}}$, and $\dot{C}\boldsymbol{q} = C\dot{\boldsymbol{q}}$.

4 Show $\dot{H}\dot{\boldsymbol{q}} = 0$

5 Show $H\dot{\boldsymbol{q}} = -\dot{H}\boldsymbol{q}$ and $\omega = -2\dot{H}\boldsymbol{q}$

6 Show $HH^\top = I_{3\times 3}$

7 $H\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}$ and $\dot{\boldsymbol{q}} = (1/2)H^\top\boldsymbol{\omega}$