Przykład 1 Podaną funkcję wymierną rozłożyć na sumę ułamków prostych rzeczywistych!

$A, B, C \in \mathbb{R}$

a) $\dfrac{0x^2 + 3x+4}{(x+3)(x^2+x+1)} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+x+1} \overset{(*)}{=} \dfrac{A(x^2+x+1)+(Bx+C)(x+3)}{(x+3)(x^2+x+1)} =$

bo $\Delta < 0$ w $\mathbb{R}$

$= \dfrac{Ax^2+Ax+A+Bx^2+3Bx+Cx+3C}{(x+3)(x^2+x+1)} = \dfrac{(A+B)x^2+(A+3B+C)x+(A+3C)}{(x+3)(x^2+x+1)}$

$$\begin{cases} A+B = 0 \\ A+3B+C = 3 \\ A+3C = 4 \end{cases}$$

Rozwiązując układ równań dowolną metodą otrzymujemy

$$\begin{cases} A = -\frac{5}{7} \\ B = \frac{5}{7} \\ C = \frac{11}{7} \end{cases}$$

$$\overset{(*)}{=} \boxed{\dfrac{-\frac{5}{7}}{x+3} + \dfrac{\frac{5}{7}x+\frac{11}{7}}{x^2+x+1}}$$

odpowiedź!

b) $\frac{1}{x^2-x^3}=\frac{1}{x^2(1-x)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{1-x}\overset{*}{=}\frac{Ax(1-x)+B(1-x)+Cx^2}{x^2(1-x)}=$ $A,B,C\in\mathbb{R}$

$=\frac{Ax-Ax^2+B-Bx+Cx^2}{x^2(1-x)}=\frac{(-A+C)x^2+(A-B)x+B}{x^2(1-x)}$

$\begin{cases}-A+C=0\\ A-B=0\\ B=1\end{cases}$ $\begin{cases}C=1\\ A=1\\ B=1\end{cases}$ $\overset{*}{=}\frac{1}{x}+\frac{1}{x^2}+\frac{1}{1-x}$

Analogicznie: $\frac{1}{x^3-x^4}=\frac{1}{x^3(1-x)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{1-x}$

c) $$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \overset{(*)}{=} \frac{(Ax+B)(x^2+1)+(Cx+D)}{(x^2+1)^2} =$$

bo $\Delta<0$ w $\mathbb{R}$

$$= \frac{Ax^3+Ax+Bx^2+B+Cx+D}{(x^2+1)^2} = \frac{Ax^3+Bx^2+(A+C)x+(B+D)}{(x^2+1)^2}$$

$$\begin{cases} A = 1 \\ B = 0 \\ A + C = 0 \\ B + D = 0 \end{cases} \qquad \begin{cases} A = 1 \\ B = 0 \\ C = -1 \\ D = 0 \end{cases}$$

$$\overset{(*)}{=} \frac{1\cdot x+0}{x^2+1} + \frac{(-1)\cdot x+0}{(x^2+1)^2} = \frac{x}{x^2+1} + \frac{-x}{(x^2+1)^2}$$

d) $\frac{4}{x^5-x^3} = \frac{4}{x^3(x^2-1)} = \frac{4}{x^3(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{x+1} \overset{*}{=}$

$= \frac{Ax^2(x^2-1) + Bx(x^2-1) + C(x^2-1) + Dx^3(x+1) + Ex^3(x-1)}{x^3(x^2-1)} =$

$= \frac{Ax^4 - Ax^2 + Bx^3 - Bx + Cx^2 - C + Dx^4 + Dx^3 + Ex^4 - Ex^3}{x^3(x^2-1)}$

$$\begin{array}{l} x^4 \\ x^3 \\ x^2 \\ x \\ \\ \end{array} \begin{cases} A + D + E = 0 \\ B + D - E = 0 \\ -A + C = 0 \\ -B = 0 \\ -C = 4 \end{cases} \qquad \begin{cases} A = -2D \\ D = E \\ A = -4 \\ B = 0 \\ C = -4 \end{cases} \qquad \begin{cases} D = 2 \\ E = 2 \\ A = -4 \\ B = 0 \\ C = -4 \end{cases}$$

$\overset{*}{=} \frac{-4}{x} + \frac{0}{x^2} + \frac{-4}{x^3} + \frac{2}{x-1} + \frac{2}{x+1}$

e) $$\frac{x}{(x-1)^2(x^2+2)} = \frac{\frac{1}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} + \frac{-\frac{1}{9}x - \frac{4}{9}}{x^2+2}$$

sprawdzić

Uwaga

$$x^4+1 = x^4+2x^2-2x^2+1 = (x^2+1)^2-2x^2 = (x^2+1-\sqrt{2}x)(x^2+1+\sqrt{2}x)$$

$$\Delta<0 \qquad \Delta<0$$

$$= (x^2-\sqrt{2}x+1)\cdot(x^2+\sqrt{2}x+1)$$

$$\frac{1}{x^4+1} = \frac{Ax+B}{(x^2-\sqrt{2}x+1)} + \frac{Cx+D}{(x^2+\sqrt{2}x+1)}$$