2 Calculus

Objective

Upon completion of this chapter, you will be able to:

- Determine the limit of a function.
- Check for continuity and differentiability of a function at any given point.
- Determine the derivative of any differentiable function.
- Apply derivative to determine nature of function and finding maxima and minima.
- Determine indefinite and definite integrals.
- Find area under the curve using integration.
- Determine multiple integrals.
- Apply vector calculus to determine properties like gradient, divergence and curl.
- Apply vector integral theorems to determine line and surface integrals.

Introduction

Calculus is the mathematical study of change like algebra is the study of operations and their applications to solving equations. It has two major branches differential calculus and integral calculus and both are related to each other by the fundamental theorem of calculus. calculus has widespread uses in science, engineering and economics.

Function

A function exists between A → B if Vx – $\in$ A there exists a unique y $\in$ B that f(x) = y. That is for a unique input there should be a unique output. There cannot be one-to-many relationship between input and output of a function.

The functions can be classified into two broad categories:

Explicit function

If the dependent variable 'y' is directly expressed in terms of the independent variable in terms of a mathematical expression.

Example: $y = x(x - 2)$

In other words, a relation of the form $y = f(x)$ exists.

Implicit function

If the dependent and independent variables cannot be separated from each other. Then it is termed as an implicit function. It can be expressed in the form $\phi(x, y) = C$

Example: $x^2 + xy + y^2 = 1$

Composite function

If a variable is dependent on more than one variable which itself can be represented as a function of some other variable. Then, it is termed as a composite function.

In other words, $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$

Some special functions

Even function:

A function f(x) is said to be even function of x if $f(x) = f(-x)$. This means that function is symmetrical about y-axis.

Example: $|x|$

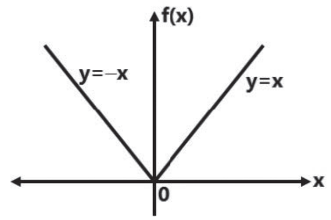

Fig. 2.1

Odd function:

A function f(x) is said to be odd function of x if $f(x) = -f(-x)$. This means that function is symmetrical in 1st and 3rd Quadrants.

Example: sin x

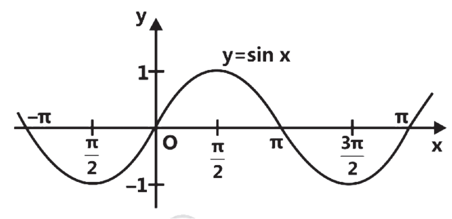

Fig. 2.2

Modulus function

Modulus of any number yields the magnitude of the number regardless of the sign.

It is defined as $f(x) = |x| = \begin{cases} x & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -x & \text{for } x < 0 \end{cases}$

The curve is not differentiable at x = 0.

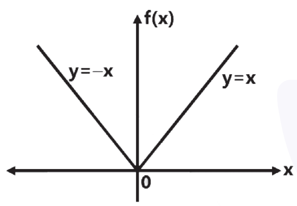

Fig. 2.3

Greatest integer function or step or bracket function

This function yields the highest integer value but less than the input real number.

It is represented as f(x) = [x] = n ∈ Z, where n ≤ x ≤ n+1

Example: [7. 2] = 7, [7.999] = 7, [7] = 7, [-1.2]=−2

It can be plotted as shown below,

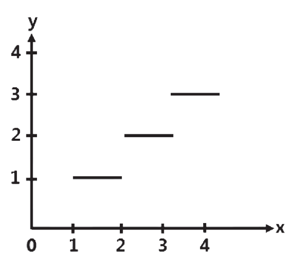

Fig. 2.4

The wave is a discontinuous function at every integer point (1, 2, ... n). So, it is not differentiable

Symmetric Properties of curves

Let f(x, y) = C be the equation of the curve. This is an implicit function in x and y.

1. If f(−x, y) = f(x, y), then it is symmetric about y axis

   Example: $x^2 - 4ay = 0$

   $f(-x, y) = x^2 - 4ay$

2. If f(x, y) = f(y, x) then the curve is symmetric about the line y=x

   Example: $x^3 + y^3 - 3axy = 0$

Limit of a function

Let f(x) be defined in deleted neighborhood of a ∈ R, then |∈ R is said to be the limit of f(x) as 'x' approach 'a' for given ∈ > 0 three exists δ > 0 such that f(x) |− |<∈| whenever |− x | a δ|

Where a ∈R, δ >0

Left and Right Hand Limit

When x tends to 'a' from the left side i.e. x<a and x → a Then, $\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$

When x tends to 'a' from the right side i.e. x>a and x→a Then, $\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$

When both the limits are equal, then limit of the function is equal to both the limits. If both the limits are unequal then the limit of the function is said does not exist.

Indeterminate Forms

If the limit of a function results in a form that is undefined, it is an indeterminate form. Some of the indeterminate forms are

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty \text{ and } \infty^0$$

- In such cases, we use L'hospital's rule if the indeterminate form is $\left[\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right]$

  $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'x}{g'(x)}$$

  where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a^+} g(x) = 0$

  Thus, limit of the function is the ratio of derivatives of both numerator and denominator.

- If the limit is of the form,
limf(x)g(x) where limf(x) = 0 and limg(x) = ∞.
x→a x→a x→a

This can be converted to previous form by taking $\phi(x)=\frac{1}{g(x)}$

$$\text{Thus,} \lim_{x\to a}(x)h(x)=\lim_{x\to a}\frac{f(x)}{\phi(x)}$$

where limf(x) = 0 and limϕ(x) =0
x→a x→a

- If the limit is of the form,

$$y=\lim_{x\to a}[[fx]]^{g(x)} \text{ where} \lim_{x\to a} f(x)=0$$
$$\text{or } 1\infty \text{ and} \lim_{x\to a} g(x)=0 \text{ or } 1 \text{ or } \infty$$

Then, the limit has an indeterminate form as, 0° , 1$^\infty$ and ∞°.

This can be converted to previous form by taking

$$\log y=\lim_{x\to a}\left(g(x)\log[[fx]]\right)$$

Standard Limits

1. $\lim_{x\to a}\frac{x^n-a^n}{X-a}=na$
2. $\lim_{x\to 0}\frac{e-1}{x}=m$
3. $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$
4. $\lim_{x\to 0}[1+ax]^x=e^a$
5. $\lim_{x\to 0}\left[1+\frac{a}{x}\right]^x=e^a$
6. $\lim_{x\to 0}\left[\frac{\sin x}{x} \text{ on } \frac{\tan x}{x}\right]=1$
7. $\lim_{x\to 0}\left[\frac{\sin mx}{x}\right]=m$
8. $\lim_{x\to\infty}\frac{\sin x}{x}=0$
9. $\lim_{x\to 0}\frac{a^x+b^x}{2}=\sqrt{ab}$
10. $\lim_{x\to 0}[\cos x+a\sin bx]^{1/x}=e^{ad}$
11. $\lim_{x\to a}\left[\frac{1-\cos-ax}{x}\right]=a\frac{2}{2}$

Solved Examples

Example: Determine $\lim_{x\to 0}\left[\frac{1-\cos 3x}{x\sin 2x}\right]$

Solution: If we substitute x = 0 then,

$$\lim_{x\to 0}\left[\frac{1-\cos 3x}{x\sin 2x}\right]=\left(\frac{0}{0}\right)\text{Form}$$

So, applying L'Hospital's rule

$$\lim_{x\to 0}\left[\frac{3\sin 3x}{\sin 2x+2x\cos 2x}\right]=\left(\frac{0}{0}\right)$$

So, applying L'Hospital's rule again

$$\lim_{x\to 0}\left[\frac{+9\cos 3x}{2\cos 2x+2\cos x-4x\sin 2x}\right]$$

$$=\left(\frac{9(1)}{2(1)+2(1)-4(0)}\right)=\frac{9}{4}$$

(OR) By using standard limit 11 and 8,

$$\lim_{x\to 0}\left[x^2\left(\frac{\sin 2}{x}\right)\right]=\left(\frac{1}{2}\right)=\frac{9}{4}$$

Example: Determine $\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}$

Solution: If we directly substitute x =1. Then,

$$\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}=\left(\frac{0}{0}\right)\text{Form}$$

Thus, applying L'Hospital's Rule,

$$\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}}=\lim_{x\to 1}\frac{-\frac{\pi}{2}\sin\frac{\pi x}{2}}{-\frac{1}{2\sqrt{x}}}-\pi$$

Example: if $\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}=$ finite, than a = _______?

Solution: If we directly substitute x = 0. Then,

$$\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}=\left(\frac{0}{0}\right)$$

Applying L'Hospital's rule,

$$\lim_{x\to 0}\frac{2\cos 2x + a\cos x}{3x^2} = \frac{2+a}{0} = \text{finite (given)}$$

Thus, 2 + a = 0 ⇒ a = – 2 which will result in indeterminate form and we can then apply L'Hospital's rule which will result in a finite limit.

Example: Determine $\lim_{x\to 0}\frac{\log \sin 2x}{\log(\sin x)} = \underline{\qquad\qquad}$?

Solution: If we directly substitute, x = 0. Then,

$$\lim_{x\to 0}\frac{\log \sin 2x}{\log(\sin x)} = \frac{\infty}{\infty}$$

Applying L'Hospital's rule

$$\lim_{x\to 0}\frac{\log \sin 2x}{\log(\sin x)} = \lim_{x\to 0}\frac{\frac{1}{\sin 2x}\times \sin 2x}{\frac{1}{\sin x}\times \cos x} = \lim_{x\to 0}\frac{2\cos x}{2\cos x} = 1$$

Example: Determine $\lim_{x\to \frac{\pi}{2}}\frac{\tan x}{\log(\cos x)}$

Solution: If we directly substitute,

$$x = \frac{\pi}{2}. \text{ Then, } \lim_{x\to \frac{\pi}{2}}\frac{\tan x}{\log(\cos x)} = \frac{\infty}{\infty}$$

Applying L'Hospital's rule

$$\lim_{x\to \frac{\pi}{2}}\frac{\tan x}{\log(\cos x)} = \lim_{x\to \frac{\pi}{2}}\frac{\sec^2 x}{\sec x(-\sin x)} =$$

$$\lim_{x\to \frac{\pi}{2}}\frac{-1}{\sin x\cos x} = \frac{-1}{1\times 0} = -\infty$$

Example: Find $\lim_{x\to 0} x^2\log x$

Solution: If we directly substitute x = 0. Then,

$$\lim_{x\to 0} x^2\log x = 0\times\infty$$

Converting the following limit to $\frac{\infty}{\infty}$ form

$$\lim_{x\to 0} x^2\log x = \lim_{x\to 0}\left[\frac{\log x}{\frac{1}{x^2}}\right] = \frac{\infty}{\infty}$$

Applying L'Hospital's rule,

$$\lim_{x\to 0}\left[\frac{\log}{\frac{1}{x^2}}\right]\lim_{x\to 0}\left(\frac{\frac{1}{x}}{\frac{-2}{x^3}}\right) = \lim_{x\to 0}\left(\frac{-x^2}{2}\right) = 0$$

Example: Determine $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$?

Solution: If we directly subs titute x = 1. Then,

$$\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2} = 0\times\infty$$

Converting the following limit to $\frac{0}{0}$ form

$$\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2} = \lim_{x\to 1}\frac{x-1}{\cot\frac{\pi x}{2}}\left(\frac{0}{0}\right)$$

Applying L'Hospital's rule

$$\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2} = \lim_{x\to 1}\frac{x-1}{-\operatorname{cosec}^2\frac{\pi}{2}\left(\frac{\pi}{2}\right)} = -\frac{\pi}{2}$$

Example: Determine $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{\tan x}\right] = ?$

Solution: $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{\tan x}\right] = \lim_{x\to 0}\left[\frac{\tan x - x}{x\tan x}\right]$

If we directly substitute x = 0. Then, this results in,

$$\lim_{x\to 0}\left[\frac{\tan x - x}{x\tan x}\right] = \frac{0}{0}$$

$$\lim_{x\to 0}\left[\frac{\tan x - x}{x\tan x}\right] = \lim_{x\to 0}\frac{\tan x - x}{\left(\frac{\tan x}{x}\right)}$$

$$\lim_{x\to 0}\frac{\tan x - x}{x^2}\times\lim_{x\to 0}\left(\frac{1}{\frac{\tan x}{x}}\right) = \lim_{x\to 0}\frac{\tan x - x}{x^2}\times 1$$

Applying L'Hospital's rule,

$$\lim_{x\to 0}\left[\frac{\tan x - x}{x\tan x}\right]\lim_{x\to 0}\left[\frac{\operatorname{sex}^2\times x - 1}{1}\right] = 0$$

Applying L'Hospital's rule,

$$\lim_{x\to 0}\left[\frac{\tan x - x}{x\tan x}\right]\lim_{x\to 0}\frac{2\sec x(\sec x\tan x)}{2} = 0$$

Example: Determine $\lim_{x\to 0}\left|\frac{x}{x}\right|$?

Solution: To find left hand limit,

$$\text{LL }\lim_{x\to 0} f(0-h) = \lim_{x\to 0}\left|\frac{h}{h}\right| = \lim_{x\to 0}\left|\frac{h}{-h}\right| = -1$$

To find right hand limit,

$$\text{RL} \lim_{x\to 0} f(0+h) = \lim_{x\to 0}\left|\frac{h}{h}\right| = \lim_{x\to 0}\left|\frac{h}{-h}\right| = 1$$

Since, both limits are unequal. Thus, $\lim_{x\to 0}\left|\frac{x}{x}\right|$ does not exist.

Example: $\lim_{x\to a}$ does not exist, when a is ?

(A) Integer

(B) Real number

(C) Rational number

(D) All of them

Solution: At any integer, we can determine the limit as, left hand limit is,

$$\text{LL} \lim_{x\to a^-}[x] = a-1$$

$$\text{Right Hand Limit is,} \quad \underset{x\to a}{\text{Lt}} f(x) = f(a)$$

$$\text{RL} \lim_{x\to a^-}[x] = a$$

Since, both limits are unequal. Hence, limit does not exist for integers.

Example: Find $\lim_{x\to 0^-}{}^{\sin x}$?

Solution: If we directly substitute x = 0. Then,

$$\lim_{x\to 0^-}{}^{\sin x} = 0^0$$

let $y=x^{\sin x}$

logy = sinx (logx)

$$\lim_{x\to 0}[\log y] = \lim_{x\to 0^-}\left[\frac{\sin x \log x}{x}\right]$$

$$\log\left[\lim_{x\to 0} y\right] = \lim_{x\to 0}\left(\frac{\log x}{\operatorname{cosec} x}\right) = \left(\frac{\infty}{\infty}\right)$$

Applying L'Hospital's rule,

$$\lim_{x\to 0} x^{\sin x} \lim_{x\to 0}\frac{1}{-\operatorname{cosec} x \cot x} \lim_{x\to 0} - \left[\frac{-\sin x}{x} \times \tan x\right]$$

$$= -1\times 0 = 0$$

$$\log\left[\lim_{x\to 0} y\right] = 0 => \lim_{x\to 0} y = 1$$

Continuity of a Function

- The function is said to be continuous at a point x=a if
- A function is said to be continuous in a interval (a, b) if it satisfied the condition

  (A) f(x) is continuous at -Vx $\in$ (a, b)

  (B) $\underset{x\to a^+}{\text{Lt}} f(x) = f(a)$

  (C) $\underset{x\to a^-}{\text{Lt}} f(x) = f(b)$

This means that function should be continuous at any point in the open interval but it must be right side continuous at lower limit and left side continuous at upper limit.

Solved Examples

Example: Check the continuity of the function

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{for } x \neq 2 \\ & \text{at } x \neq 2 \\ 0 & \text{for } x = 2 \end{cases}$$

Solution: Given $\underset{x\to 2}{\text{Lim}} f(x) = \underset{x\to 2}{\text{Lim}}\left[\dfrac{f2(2=0)}{\dfrac{x-4}{\dfrac{x-2}{x-4}}}\right] = \dfrac{0}{0}$

Applying L'Hospital's rule

$$\underset{x\to 2}{\text{Lim}}\ f(x) = \underset{x\to 2}{\text{Lim}}\left[\frac{t-4}{x-2}\right] = \underset{x\to 2}{\text{Lim}} - 4 \neq f(2)$$

$\therefore$ f(x) is discontinuous at x=2

Example: Determine the continuity of

$$f(x) = \begin{cases} 1 & \text{for } x \neq 0 \\ (1+3x)x^- & \text{at } x = 0\,? \\ e^3 & \text{for } x = 0 \end{cases}$$

Solution: Given f(0) = e^3

Determining the limit of the function at x = 0

$$\underset{x\to 2}{\text{Lim}}(1+3x)^{\frac{1}{x}} = e^3 = f(0)$$

$\therefore$ The function is continuous at x=0

Example: Let $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]} & \text{for } [x] \neq 0 \\ 0 & \text{for } [x] = 0 \end{cases}$

The reason for f(x) to be discontinuous at x=0 is?

(A) f(0) is not defined

(B) f(0) is defined but $\underset{x \to 0}{\text{Lt}} f(x)$ does not exist

(C) $\underset{x \to 0}{\text{Lt}} f(x)$ exists, f(0) is defined

(D) F(x) is discontinuous at x=0, because

$$\underset{x \to 0}{\text{Lt}} f(x) \neq f(0)$$

Solution:

$$f(x) = \begin{cases} \dfrac{\sin(-1)}{-1} & \text{for } -1 \le x \le 0 \quad [x] = -1 \\ 0 & \text{for } -1 \le x < 0 \quad [x] = 0 \end{cases}$$

Thus, f(0) = 0

The left hand limit is sin 1 and right hand limit is 0.

Since, both limits are unequal. The limit does not exist. Hence, the function is discontinuous at x = 0. Hence (D) is correct.

Differentiability

A function f(x) is said to be differentiable at x=c, if $f'(c) = \lim\limits_{x \to c}\left[\dfrac{f(x)-f(c)}{x-c}\right]$ exists and is finite.

The left hand and right hand derivatives are given as,

$$\text{LHD} = \lim_{h \to 0}\left[\frac{f(x-h)-f(c)}{-h}\right]$$

$$\text{RHD} = \lim_{h \to 0}\left[\frac{f(c+h)-f(c)}{h}\right]$$

A function f(x) is said to be differentiable if the LHD, RHD exists and are finite and equal.

1. A function f is said to be differentiable in an open interval (a, b), if it is differentiable at each point of the open interval.
2. A function $f : [[a, b]] \to R$ is said to be differentiable in closed interval [a, b] if it is
   a) Differentiable from right at a [i.e. RHD exists] and
   b) Differentiable from left at b [i.e. LHD exists] and
3. Differentiable in the open interval (a, b)

Note: If a function is differentiable at any point, then it is necessarily continuous at that point but the converse is not true.

Solved Examples

Example: Let f(x) = x |x| where x ∊ R, then f(x) at x = 0 is

(A) Continuous and differentiable

(B) Continuous but not differentiable

(C) Differentiable but not continuous

(D) Neither differentiable nor continuous

Solution: The function can be expanded as,

$$f(x) = \begin{cases} x^2 & \text{for } x > 0 \\ -x^2 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$$

The left hand limit is LL = – 0^2 = 0

Right Hand Limit is RL = 0^2 = 0

f(0) = 0

Since LL = RL = f(0), the function is continuous at x = 0.

Calculating left hand and right hand derivatives,

$$\text{LHD} = \lim_{h \to 0}\left[\frac{-(-h)^2 - 0}{-h}\right] = 0$$

$$\text{RHD} = \lim_{h \to 0}\left[\frac{(+h)^2 - 0}{h}\right] = \lim_{h \to 0} h = 0$$

Since both derivatives are equal, the function is differentiable at x = 0.

Example: Check the continuity and differentiability of $f(x) = \dfrac{1}{1+|x|}$ at $x = 0$

Solution: We can express the function as,

$$f(x) = \begin{cases} \dfrac{1}{1+x} & \text{for } x > 0 \\ \dfrac{1}{1-x} & \text{for } x > 0 \\ 1 & \text{for } x = 0 \end{cases}$$

The left hand limit is, $LL = \lim\limits_{x\to 0}\frac{1}{1-x} = \frac{1}{1-0} = 1$

The right hand limit is, $RL = \lim\limits_{x\to 0}\frac{1}{1+x} = \frac{1}{1+0} = 1$

Since LL = RL = f(0), the function is continuous at x = 0. Calculating the Left Hand Derivative,

$$LHD = \underset{h\to 0}{\text{it}}\left[\frac{\frac{1-1}{1-(-h)}}{-h}\right] = \underset{h\to 0}{\text{it}}\frac{-h}{-h(1+h)} = 1$$

Calculating Right Hand Derivative,

$$RHD = \underset{h\to 0}{\text{it}}\frac{\frac{1}{(1+h)}}{h} = \underset{h\to 0}{\text{it}}\frac{-h}{h(1+h)} = -1$$

Since RHD ≠ LHD , the function is not differentiable at x = 0.

Mean Value Theorems

Rolle's theorem

Let f(x) be defined as [a, b], such that

1. f(x) is continuous in [a, b]
2. f(x) is differentiable in (a, b)
3. f(a) = f(b)

Then there exists at least one point c ∈ (a, b) such that f'(c) = 0.

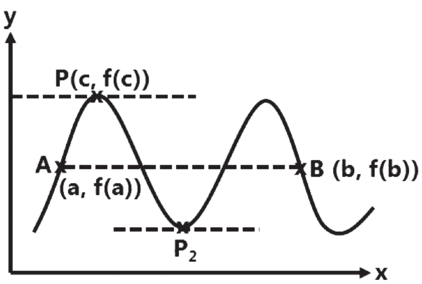

Fig. 2.5

Line joining A & B is parallel to x-axis. Since the function is differentiable so the Rolle's Theorem asserts that at some point in the interval, the derivative will go to zero.

Note: The converse of Rolle's theorem is not true that means if derivative goes to zero at some point in an interval, then it does not necessarily mean that function has same value at both end points.

Lagrange's mean value theorem

Let f(x) be defined in [a, b] such

1. f(x) is continuous in [a, b]
2. f(x) is differentiable in (a, b)

Then there exists at least one point

c ∈ (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

In graphical form it can be represented as shown below,

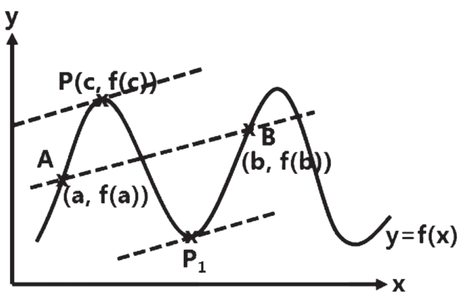

Fig. 2.6

Note: The converse of Lagrange Mean value Theorem may not be true.

Applications of LMVT

- If a function f(x) is
  - Continuous in [a, b]
  - Derivable in (a, b) and
  - f'(x) > 0 for all x in (a, b), then f(x) is strictly increasing function in [a, b]
- If a function f(x) is
  - Continuous in [a, b]
  - Derivable in (a, b) and
  - f'(x) > 0 for all x in (a, b), then f(x) is strictly decreasing function in [a, b]
- If the derivative of the function is constantly zero in the entire interval, then the function is constant in that interval.

Solved Examples

Example: The mean value c for the function

$f(x)^x\,[\sin x - \cos x]\text{in}\left[\frac{\pi}{4}, \frac{4\pi}{4}\right]$is ________ .

Solution: The derivative of the function f(x) is given by,

$$f'(x) = e^x[[\sin x - \cos x]] + e^x[[\cos x + \sin x]]$$
$$= 2e^x \sin x$$

Value of f(x) at end points of the interval is,

$$f\left(\frac{\pi}{4}\right) = 0, f\left(\frac{5\pi}{4}\right) = 0$$

By Rolle's Theorem there exists $c \in \left(\frac{\pi}{4}, \frac{5}{4}\right)$ such that f'(c) =0

i.e. $2e^c \sin c = 0$

sin c = 0

$c = 0, \pm\pi, \pm 2\pi, ...$

$c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Example: The mean value c for the function

f(x) = $f(x) = \sqrt{x^2 - 4}$ in the interval (2, 4) is?

Then, $f'(x) = \frac{1}{2\sqrt{x^2-4}} \times 2 \times \frac{x}{\sqrt{x^2-4}}$

f'(x) finite for –V (2, 4)

Computing the function values at end points of interval

$f(2) = \sqrt{2^2 - 4} = 0$

$f(4) = \sqrt{12}$

By Lagrange's theorem, there exists c ∊ (2, 4) such that

$$f'(c) = \frac{f(4) - f(2)4 - 2}{4 - 2}$$

$$\frac{c}{\sqrt{c^2 - 4}} = \frac{\sqrt{12} - 0}{2}$$

$$\frac{c}{\sqrt{C^2 - 4}} = \sqrt{3}$$

$c^2 = 3c^2 - 12$

$2c^2 = 12$

$c^2 = \sqrt{6} \in (2, 4)$

Example: Find the value of ε such that

$f(b) - f(a) = (b - a)\ f'(\varepsilon)$ for $f(x) =$
$Ax^2 + Bx + C$ in $[a, b]$

Solution: Any polynomial function is continuous and differentiable everywhere.

f(x)=Ax^2+Bx+C is continuous and differentiable in [a, b].

$f'(x) = 2Ax + B$

$$f'(\varepsilon) = \frac{f(b) - f(a)}{b - a}$$

$$2A\varepsilon + B = \frac{f(b) - f(a)}{b - a} =$$

$$\frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} =$$

$$\frac{A(b^2 - a^2) + B(b - a)}{b - a}$$

$2A\varepsilon + = (b + a)A + B$

$$\varepsilon = \frac{b + a}{2}$$

Example: The value of c ∊ (1 , e) for the function f(x) = logx in the interval [1, e] using Lagrange's mean value

theorem is ____________ ?

Solution: The derivative of function f(x) is,

$$f(x) = \frac{1}{x}$$

The value of function at interval end points is,

f(1) = 0, f(e) = 1

By Lagrange's Mean Value theorem

$$c \in (1, e), f'(c) = \frac{f(e) - f(1)}{e - 1}$$

$$\frac{1}{c} = \frac{1}{e = 1}$$

$c = e - 1$

Example: Lagrange's MVT cannot be applied

to $f(x) = 2(x - 1)^{\frac{1}{3}}$ in the interval [0,2] because?

(A) f(x) is not continuous in [0, 2]

(B) f(x) is not differentiable in (0, 2)

(C) f(0) ≠ f(2)

(D) both a ∊ b

Solution: The derivative of the function f(x) is given by,

$$f'(x)=0+\frac{2}{3}(x-1)^{\frac{1}{3}}=\frac{2}{3(x-1)^{\frac{1}{3}}}$$

$$f'(1)=\infty$$

Thus, f(x) is not differentiable in (0, 2) and it violates the condition of Lagrange Mean Value theorem. Thus, LMVT cannot be applied.

Cauchy's mean value theorem

Let f(x) and g(x) be defined in [a, b] such that

1. f(x) and g(x) are continuous in [a, b]
2. f(x) and g(x) are differentiable in (a, b)
3. $g'(x) \neq 0 \;\forall\; x \in (a, b)$

Then there exists at least a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Solved Examples

Example: The mean value 'c' for the function

$$f(x)-\frac{1}{x}c, g(x)=\frac{1}{x^2}\text{in } [1,2] \text{ is}$$

Solution: The derivative of the function f(x) is, $f'(x)=\frac{1}{x^2}$

$$g'(x)=\frac{-2}{x^3}\neq 0 \;\forall\; x\in(-1,2)$$

By Cauchy's Mean Value theorem,

$$\frac{f'(c)}{g'(c)}=\frac{f(2)-f(1)}{g(2)-g(1)}$$

$$\frac{-\frac{1}{c^2}}{\frac{2}{c^3}}=\frac{\frac{1}{2}-1}{\frac{1}{4}-1}$$

$$\frac{c}{2}=\frac{+\frac{1}{2}}{+\frac{3}{4}}$$

$$c=\frac{4}{3}$$

Example: The mean value 'c' for the functions f(x) = sinx, g(x) = cosx in $\left[-\frac{\pi}{2},0\right]$ is

Solution: Derivatives of both functions are,

$$f'(x)=\cos x, g'(x)=-\sin x\neq 0 x\in\left(\frac{\pi}{2},0\right)$$

By Cauchy's Mean Value theorem,

$$\frac{\cos}{-\sin c}=\frac{0+1}{1-0}$$

$$\frac{-\cos c}{\sin c}=1$$

$$\tan c=-1$$

$$c=-\frac{\pi}{4}\in\left(-\frac{\pi}{2},0\right)$$

Rules of Differentiation

$(f+g)'=f'+g'$ (Sum rule)

$(f-g)'=f'-g'$ (Difference rule)

$(f.g)'=fg'+gf'$ (Product rule)

$\left(\frac{f}{g}\right)=\frac{gf'-fg'}{g^2}$ (Quotient rule)

$\frac{d}{dx}\left(f\left(g(x)\right)\right)=\frac{df}{dg}\times\frac{dg}{dx}$ (Chain rule)

These rules are applicable when y is an explicit function of x.

Some of the common derivatives are,

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\log_a e.\left(\frac{1}{x}\right)$
e^x	e^x
a^x	$a^x \log_e a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec^2 x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sinh x$	$\cosh x$

$f(x)$	$f'(x)$
$\cosh x$	$\operatorname{Sinh} x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\operatorname{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\cot^{-1}$	$-\frac{1}{1+x^2}$
$\lvert x\rvert$	$\frac{x}{\lvert x\rvert}(x\neq 0)$

Differentiation by Substitution

For complicated functions, we make substitution and then apply chain rule to determine the derivative. If the

function contains an expression of the form

1. $a^2 - x^2$, put x = asint or x = acost
2. $a^2 + x^2$, put x = atant or x = acot t
3. $x^2 - a^2$, put x = asect or x = acosect
4. $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$, put $x = a\cos t$
5. $a\cos x \pm b\sin x$, put $a = r\cos\theta$ and $b = r\sin\theta$, $r > 0$

Solved Examples

Example: Differentiate the following function (by suitable substitutions) w.r.t. x.

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{put } x = \tan\theta \text{ i. e. } \theta = \tan^{-1} x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

Example: Find $\frac{dy}{dx}$ when $x^2 + xy + y^2 = 100$

Solution: Given, $x^2 + xy + y^2 = 100$

Differentiating both sides w.r.t x, we get

$$2x + \left(x\frac{dy}{dx} + y.1\right) + 2y\frac{dy}{dx} = 0$$

$$(x+2y)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$$

Example: if $X^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, find $\frac{dy}{dx}$

Solution: Given $X^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Differentiating both sides of w.r.t. x, regarding y as a function of x, we get

$$\frac{2}{3}x^{\frac{1}{3}} + \frac{2}{3}y^{\frac{1}{3}}, \frac{dy}{dx} = 0$$

$$\frac{1}{x^{\frac{1}{3}}} + \frac{1}{y^{\frac{1}{3}}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = -\sqrt[-3]{\frac{y}{x}}$$

Example:

$$\text{if } y = \sqrt{\cos x\sqrt{\cos x + \sqrt{\cos x + \text{.......to}^{\infty}}}},$$

$$\text{prove that } (1-2y)\frac{dy}{dx} = \sin x$$

Solution:

$$\text{Given, } y = \sqrt{\cos x + y}$$

$$y^2 = \cos x + y$$

$$y^2 - y = \cos x$$

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} - \frac{dy}{dx} = -\sin x$$

$$(1-2y)\frac{dy}{dx} = \sin x$$

Example: Differentiate the function $f(x) = x^x$

Solution: Let x y = x^2

Taking logarithm of both sides, we get

logy = xlogx

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\frac{1}{x} + \log x.1$$

$$\frac{dy}{dx} = y\left(1+\log x\right) = x^x\left(1+\log x\right)$$

Example: if $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{\left(1+\log x\right)^2}$

Solution: $x^y = e^{x-y}$ taking logarithm of both sides, we get

$$y\log x = \left(x - y\log e = x - y\right)$$

$$y + y\log x = x$$

$$\left(1+\log x\right)y = x$$

$$y = \frac{x}{1+\log x} \text{ differentiating w.r.t. x, we get}$$

$$\frac{dy}{dx} = \frac{\left(1+\log\right).1 - x\left(0+\frac{1}{x}\right)}{\left(1+\log x\right)} = \frac{1+\log x - 1}{\left(1+\log x\right)^2}$$

$$= \frac{\log x}{\left(1+\log x\right)^2}$$

Parametric Differentiation

If x and y are two variables such that both are explicitly expressed in terms of a third variable, say t, i.e., if x=f (t) and y =g (t) then such functions are called parametric functions and the third variable is called the parameter.

In order to find the derivative of a function in parametric form, we use chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \left(\text{provide } \frac{dx}{dt} \neq 0\right)$$

Solved Examples

Example: If x = a (t + sint), y = a (1 – cost),

find $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$

Solution: Given, x = a (t + sint), y = a (1 -cost)

Differentiating both w.r.t 't', we get

$$\frac{dx}{dt} = a\left(a + \cos t\right)$$

$$\text{And } \frac{dy}{dt} = a\left(0 - \left(-\sin t\right)\right) = a\sin t$$

$$\text{We know that} \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{a\sin t}{a\left(1+\cos t\right)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2}$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}} = \tan\frac{\pi}{4} = 1$$

Example: Differentiate $\frac{x^3}{1-x^3}$ w.r.t. x^3

Solution: Let w.r.t.x^3

$$\frac{dy}{dx} = \frac{\left(1-x^3\right)3x^2 - x^3.\left(0-3x^2\right)}{\left(1-x^3\right)} = \frac{3x^2}{\left(1-x^3\right)^2}$$

$$\text{And } \frac{dz}{dx} = 3x^2$$

$$\text{Therefore, } \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{1}{\left(1-x^3\right)^2}$$

Increasing and Decreasing functions

Let f be a real valued function defined in an interval D (a subset of R), then f is called an increasing function in an

interval D_1 (a subset of D) if

For all $x_1, x_2 \in D_1$

$$X_1 < X_2 \Rightarrow f\left(x_1\right) \leq f\left(x_2\right)$$

In other words, for an increasing function the function value f(x) increases.

And f is called a strict increasing function (or monotonically increasing function) in D_1 if for all

$x_1, x_2 \in D_1$

$$X_1 < X_2 \Rightarrow f\left(x_1\right) < f\left(x_2\right)$$

Analogously, f is called a decreasing function in an interval D_2 (a sunset of D) if for all

$x_1, x_2 \in D_2$

$X_1 < X_2 \Rightarrow f(x_1) \geq f(x_2)$

And f is called a strict decreasing function (or monotonically decreasing function) in 2 D if for all

$x_1, x_2 \in D_2$

$X_1 < X_2 \Rightarrow f(x_1) > f(x_2)$

Conditions for an Increasing or a Decreasing function

Applying Lagrange's Mean Value theorem,

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 = x_1} \quad \text{where } x \in (x_1, x_2)$$

For an increasing function,

$f(X_1) \leq f(x_2)$. Thus, $f'(x) \geq 0$ for the domain D_1

For a strictly increasing function,

$f(X_1) < f(x_2)$. Thus, $f'(x) > 0$ for the domain D_1

For a decreasing function,

$f(X_1) \geq f(x_2)$. Thus, $f'(x) \leq 0$ for the domain D_1

For a strictly decreasing function

$f(X_1) > f(x_2)$. Thus, $f'(x) < 0$ for the domain D_1

This can be summarized as,

Theorem 1: If a function f is continuous is [a, b], and derivable in (a, b) and

1. $f'(X) \geq 0$ for all $x \in$ (a,b) , then f is increasing in [a, b]
2. $f'(X) > 0$ for all $x \in$ (a,b), then f is strict increasing in [a, b]

Theorem 2: If a function f is continuous in [a, b] and derivable in (a, b) and

1. $f'(X) \leq 0$ for all x in (a,b), then f(x) is decreasing in [a, b]
2. $f'(X) < 0$ for all $x \in$ (a,b) then f(x) is strict decreasing in [a, b]

Solved Examples

Example: Determine the nature of the function $f(x) = \frac{3}{x} + 8$?

Solution: Let $f(x) = \frac{3}{x} + 8, D_f = R - [0]$

Diff. it w.r.t. x, we get

$$f'(x) = 3.(-1.x^{-2}) + 0 = -\frac{3}{X^2}$$

Since $x^2 > 0$ for all $X \in R$, $x \neq 0$.

Therefore, $f'(x) < 0$ for all $x \in R$, $x \neq 0$, i.e. for all $X \in D_1$

Thus, the given function is strictly decreasing.

Example: Determine the nature of the function $f(x) = \frac{e^x}{1 + e^x}$

Solution: Differentiating the function we get,

$$f'(x) = \frac{e_x (1 + ex) - e^{2x}}{(1 + e^x)} = \frac{e^x}{(1 + e^x)^2}$$

Since x e is positive for all values of x, f' (x) is positive for all values of x and hence f(x) monotonically increases.

Local Maxima and Minima

A function f(x) is said to be a local or relative maximum at x=a, if these exist positive number δ such that $f(a + \delta) < f(a)$ and $f(a - \delta) < f(a)$. In other words, function values on either side of maxima point must be less than the function value at that point.

A function f(x) is said to be a local or relative minimum at x = a, if there exists a positive number δ such that $f(a + \delta) > f(a)$ and $f(a - \delta) > f(a)$. In other words, function values on either side of minima point must be more than the function value at that point.

Properties of Relative Maxima and Minima

1. At least one maximum or one minimum must be lie between two equal values of a function.

2. Maximum and minimum values must occur alternatively.
3. There may be several maximum or minimum values of same function.
4. A function y = f(x) is maximum at x=a, if $\frac{dy}{dx}$ changes sign from +ve to –ve as x passes through a.
5. A function y = f(x) is maximum at x=a, if $\frac{dy}{dx}$ changes sign from -ve to +ve as x passes through a.
6. If the sign of $\frac{dy}{dx}$ does not change while x passes through a, then y is neither maximum nor minimum at x = a

Conditions from Maximum or Minimum Values

The necessary condition that f(x) should have a maximum or a minimum at x=a is that f'(a) = 0

There is a maximum of f(x) at x=a if f'(a) =0 and f"(a) is negative

Similarly there is a minimum of f(x) at x=a if f'(a) = 0 and f"(a) is positive.

Inflection point

An inflection point is a point on a curve at which the sign of the curvature (i.e., the concavity) changes. Inflection points may be stationary points, but are not local maxima or local minima. A necessary condition for 'x' to be an inflection point is f"(x)=0. A sufficient condition requires f"(x + ε) and f'' (x-ε) to have opposite signs in the neighborhood of x.

Note: If f"(a) is also equal to zero, then we can show that for a maximum or a minimum of f(x) at x = a . We must have f''(a) = 0. Then, if f'''(a) is negative, there will be a maximum at x = a and if f '''(a) is positive there will be minimum at x = a

In general if, f'(a)=f"(a)=f'''(a)=f^{n-1} (a)=0 and f^n (a) ≠ 0 then n must an even integer for maximum or minimum. Also for a maximum f" (a) must be negative and for a minimum f" (a) must be positive.

Absolute Maximum and Minimum in Range [a, b]

Absolute Maxima and Minima refers to a single maximum and minimum value of a function in a given range. If a function f is differentiable in [a, b] except (possibly) at infinitely many points, then to find (absolute) maximum and minimum values adopt the following procedure:

1. Evaluate f(x) at the points where f' (x) = 0
2. Evaluate f(x) at the points where derivative fails to exist.
3. Find f(a) and f(b)

Then the maximum of these values is the absolute maximum of the given function f and the minimum of these values is the absolute minimum of the given function f.

Solved Examples

Example: Find the absolute maximum and minimum values of:

$$f(x) = 2x^3 - 9x^2 + 12x - 5 \text{ in} [0,3]$$

Solution: Given $f(x) = 2x^3 - 9x^2 + 12x - 5$

It is differentiable for all x in [0, 3], since it is a polynomial

Differentiating (i) w.r.t. x, we get

$$f'(x) = 2 \times 3x^2 - 9 \times 2x + 12 = 6(x^2 - 3x + 2)$$

Now, for extrema points f'(x) = 0

$$6(x^2 - 3x + 2) = 0$$

$$x^2 - 3x + 2 = 0$$

$$(x - 1)(x - 2) = 0$$

$$x = 1, 2$$

Also 1, 2 both are in [0, 3], therefore 1 and 2 both are stationary point Checking Function Values at Stationery Points and end points of interval Further,

$f(1) = 2.1^3 - 9.1^2 + 12.1 - 5 = 22 - 9 + 12 - 5 = 0$

$f(2) = 2.2^3 - 9.2^2 + 12.2 - 5 = 16 - 36 + 24 - 5$

$= -1$

$f(0) = -5$

And f

$(3) = 2.3^3 - 9.3^3 + 12.3 - 5 = 54 - 81 + 36 - 5 = 4$

Thus, maximum value is 4 which occurs at x = 3 and minimum value is -5 which occurs at x = 0.

Example: The maximum value of

$f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6] is

Solution: We need absolute maximum of in the interval [1, 6]

$f(x) = x^3 - 9x^2 + 24x + 5$

First find local maximum if any by putting f'(x) = 0

i.e $f'(x) = 3x^2 - 18x + 24 = 0$

i.e. $x^2 - 6x + 8 = 0$

$x = 2, 4$

Now, $f''(x) = 6x - 18$

$f''(2) = 12 - 18 = -6 < 0$

$(\text{so } x = 2 \text{ is a point of local maximum})$

And $f''(4) = 24 - 18 = 6 > 0$

$(\text{so } x = 4 \text{ is a point of local minimum})$

Now tabulate the values of f at end point of interval and at local maximum point, to find absolute maximum in given range, as shown below:

x	f(x)
1	21
2	25
6	41

Clearly the absolute maxima is at x = 6

And absolute maximum value is 41.

$y = a\log|x| + bx^2 + x$ has extreme values at $x = +2$ and $x = -\frac{3}{4}$ then the values of a & b are ________.

Solution: $\frac{dy}{dx} = a \times \frac{1}{|x|} \times \frac{|x|}{x} + 2bx + 1 = 0$

$\frac{a}{x} + 2bx + 1 = 0$

$2bx^2 + x + a = 0$ ...(1)

$\left(x + \frac{3}{4}\right)(x - 2) = 0$

$4x^2 - 5x - 6 = 0$

$\frac{4}{5}x^2 + x + \frac{6}{5} = 0$ ...(2)

Equating (1) & (2)

$2b = \frac{4}{5} \Rightarrow b = \frac{2}{5}$

$a = \frac{6}{5}$

Example: Maximum of

$f(x) = \frac{e^{\sin x}}{e^{\cos x}}, x \in R$ is ________

Solution: $f(x) = e^{\sin - \cos x}$

Since, exponential is a monotonically increasing function. So maximum of sinx-cosx gives maximum value of function.

Let $g(x) = \sin x - \cos x$

$g'(x) = \cos x + \sin x = 0$

$\cos x = -\sin x$

$x = -\frac{\pi}{4}, \frac{3\pi}{4}$ ....

$g''\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} > 0 \to \min$

$g''\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} < 0 \to \max$

Maximum value $= f\left(\frac{3\pi}{4}\right) = e^{\frac{1}{\sqrt{2}} - \left(\frac{-1}{\sqrt{2}}\right)} = e^{\sqrt{2}}$

Taylor's theorem (Generalized MVT)

Let f(x) be defined in [a, a+h] such that it satisfies

1. f(x), f'(x), f''(x) .... $f^{n-1}(x)$ are continuous in [a, a+h]

2. f(x), f'(x), f"(x) .... fn-1(x) are differentiable in (a, a+h) then there is at least one number $\theta \in (0,1)$ such that

$$f\left(a+h\right)=f\left(a\right)+hf'\left(a\right)=\frac{h^2}{2!}f''\left(a\right)+............+\frac{h^{n-1}}{(n-a)!}f^{n-1}\left(a\right)+R_n$$

Where $R_n=\frac{h^n}{(n-1)!\times P}(1-\theta)^{n-p}f^n\left(a+\theta h\right)$

Case 1: when P = n $R_n=\frac{h^n}{n!}f^n\left(a+h\theta\right)$

is called as Lagrange's form of remainder.

Case 2: when P = 1

$$R_n=\frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n\left(a+\theta h\right)$$

is called as Cauchy's form of remainder.

As $n\to\infty, R_n\to 0$ then,

$$f\left(a+h\right)=f\left(a\right)+hf'\left(a\right)+\frac{h^2}{2!}f''\left(a\right)+............\text{is}$$

$$f\left(x\right)=f\left(a\right)+\left(x-a\right)f'\left(a\right)+\frac{(x-a)^2}{2!}f''\left(a\right)+.......\infty$$

is a Taylor series expansion of f(x) about x=a

$$f\left(x\right)f\left(0\right)+xf'\left(a\right)+\frac{x^2}{2!}f''\left(0\right)+....................\infty \text{ is a}$$

Taylor Series expansion of f(x) about x=0 which is also called as

Maclaurin's Series.

Some common Taylor Series expansions are,

1. $e^x=1+x+\frac{x^2}{2!}+.......\infty$
2. $\sin x=x\frac{x^3}{3!}+\frac{x^5}{5!}............\infty$
3. $\cos x=1\frac{x^2}{2!}+\frac{x^4}{4!}............\infty$
4. $\log\left(1+x\right)=x-\frac{x^2}{2}+\frac{x^3}{3}-............\infty$
5. $\log(1-x)=-\left[x+\frac{x^2}{2}+\frac{x^3}{3}+..........\infty\right]$
6. $\tan x=x\frac{x^3}{3}+\frac{2x^2}{15}+............\infty$
7. $(1+x)^{-1}=1-x+x^2-x^3+...........\infty$

Solved Examples

Example: The coefficient of $(x-2)^4$ in the Taylor's Series expansion of logx about x = 2 is

Solution: Coefficient of (x –2)

$$f'\left(x\right)=\frac{1}{x},\ f''\left(x\right)=-\frac{1}{x^2},\ f'''\left(x\right)=\frac{2}{x^3}$$

$$f''''\left(x\right)=-\frac{6}{x^4}$$

$$f''''\left(2\right)=\frac{-6}{16}=-\frac{3}{8}$$

Coefficient of

$$\left(x-2\right)^4=\frac{f^{(iv)}\left(2\right)}{4!}=\frac{-3/8}{4!}=\frac{-3}{8\times 24}=\frac{-1}{64}$$

Example: The Taylor Series expansion of tanx about $x=\frac{\pi}{4}$ is

Solution: $f\left(x\right)=f\left(\frac{\pi}{4}\right)+\left(x-\frac{\pi}{4}\right)$

$$f'\left(\frac{\pi}{4}\right)+\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right)$$

.............∞

$$f\left(x\right)=\tan x$$

$$f'\left(x\right)=\sec^2 x$$

$$f''\left(x\right)=2\sec^2 x\tan x$$

$$f'\left(\frac{\pi}{4}\right)=2$$

$$f''\left(\frac{\pi}{4}\right)=4$$

$$f\left(x\right)=1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+..............\infty$$

Example: Determine the first 3 non zero term in the expansion of $e^x\tan x$ is?

Solution: $f'\left(0\right)=f\left(0\right)+xf'\left(0\right)+\frac{x^2}{2!}f''\left(0\right)+.....\infty$

$$f'\left(0\right)=0$$

$$f'\left(x\right)=e^x\left[\tan x+\sec^2 x\tan x\right]$$

$$f'\left(x\right)=e^x\left[\sec^2 x+\tan x+\sec^2 x+2\sec^2 x\tan x\right]$$

$$\Rightarrow f''\left(0\right)=2$$

$$f'''\left(x\right)=e^x$$

$$\left[\tan x+2\sec^2 x+2\sec^2 x\tan x+\sec^2 x\right.$$
$$\left.+4\sec^2 x\tan x+2\sec^4 x+\sec^2 x\tan^2 x\right]f'''(0)=5$$

$$f(x)=0+x(1)+\frac{x^2}{3!}(5)+......+\infty=x+$$
$$\frac{2x^3}{3}+x^2+\frac{x^3}{2}+............\infty$$

or

$$e^x\tan x=x+x^2+\frac{5}{6}x^3+................\infty$$

Example: The Taylor Series Expansion of $\tan^{-1}x$ is

Solution:

$$f'(x)=\frac{1}{1+x^2}=\left(1+x^2\right)^{-1}=1-x^2+x^4-x^6+...$$

on integration

$$f(x)+c=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+............\infty$$

Put $x=0$

$0+c=0\Rightarrow c=0$

$$\therefore f(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...........\infty$$

Partial Differentiation

When a variable depends on more than one independent variable, then it is called as Multi-Variable Function.

When the derivative of such a variable is computed with respect to any of the independent variable then it is

called as Partial Derivative.

If z = f(x, y) then

Derivative of z w.r.t. x is given by,

$$z_x=\frac{\partial z}{\partial x}-\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

Derivative of z w.r.t. y is given by,

$$z_x=\frac{\partial z}{\partial y}-\lim_{h\to 0}\frac{f(x,+h+k)-f(x,y)}{k}$$

Similarly $\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial y^2},\frac{\partial^2 z}{\partial x\partial y},$ can be computed

To calculate any partial derivative w.r.t. any one variable then other independent variables are considered as constant.

The terms $\frac{\partial^2 z}{\partial y\partial y},$ are called as mixed partials and for a continuous function mixed partials are equal i.e.

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$$

Homogeneous function

If all the terms of a function have the same degree then it is called as homogenous function and degree of each term is called as order of the function.

Example:

1. 2x + 3y : Homogenous Function of order 1.
2. $x^3z+4x^2y^2-2xyz^2$: Homogenous Function of order 4.
3. $\frac{x^3y+z^2y^2}{4x-3y}, n=4-1=3$: Homogenous

   Function of order 3.

   n = order of the function = order of Numerator – Order of Denominator
4. $u=\cos^{-1}\left(\frac{x^2+y^2}{2x-3y}\right)\to$ Non homogenous function
5. $z=\log\left(\frac{x}{y}\right)\to$ Non homogenous function
6. Sinx → Non homogeneous function
   a) If the function of trigonometric, exponential and logarithmic function are homogeneous with degree 0 then the whole function will be homogenous.

Note:

1. If f(kx, ky) = kn f(x, y) then f(x, y) is a homogenous function with degree 'n'.
2. If f(x, y) is a homogeneous function with degree 'n' then $f(x,y)=\begin{cases}x^n\phi(y/x)\\y^n\phi(x/y)\end{cases}$

Euler's Theorem

If f(x, y) is a homogeneous function with degree 'n' the

a. $x\frac{\partial f}{\partial x}+yx\frac{\partial f}{\partial x}=nf$

b. $x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2x\frac{\partial^2 f}{\partial y^2}=n(n-1)f$

Note: If u(x,y) = f(x,y) +g(x,y) where f and g are homogeneous function with degree m and n respectively, then

a) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=mf+ng$

b) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2x\frac{\partial^2 u}{\partial y^2}=m(m-1)f+n(n-1)g$

Note: If f(u) is a homogeneous function in two variables x an y with degree 'n' then

a) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=F(u)$

b) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=F(u)[F'(u)-1]$

Total Differentiation

If z=f(x, y) where x = g(t) and y = h(t) then the total derivative of 'z' w.r.t. $\frac{dz}{dt}=\frac{\partial f}{dt}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$

The total differential of z = f(x, y) is

$$dz=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

Note: If f(x, y) = c is an implicit function then

$$\frac{dy}{dx}=\frac{\partial f/\partial x}{\partial f/\partial y}$$

If z = f(x, y) where x = g(u,v) and y = h(u,v) then

$$\frac{\partial z}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Solved Examples

Example: If w = x^2 + y^2 where

$x=\frac{t^2-1}{t}$ and $y=\frac{t}{t^2+1}$

then $\frac{dw}{dt}\Big|_{t=1}=$ __________?

Solution: $\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial y}$

$$\frac{dw}{dt}=2x\left[1+\frac{1}{t^2}\right]+2y\left[\frac{(t^2+1)1-t(2t)}{(t^2+t)^2}\right]$$

At $t=1$

$$\frac{dw}{dt}=2\times(0)(1+1)+\left(\frac{1}{2}\right)\left(\frac{2-2}{4}\right)-0+0=0$$

Example:

If $v=x^2+y^2+z^2$ where $x=e^{2t}, y=e^{2t}$ $\sin 3t$ and $z=e^{2t}\cos 3t$, then $\frac{dv}{dt}=$ ____?

Solution:

$v=e^{4t}+e^{4t}\sin^2 3t+e^{4t}\cos^2 3t=e^{4t}[1+1]$

$v=e^{4t}$

$\frac{dc}{dt}=8e^{4t}$

Example: The total derivative of x^3 y^2 w.r.t. x where x and y are connected by the relation x^3 + y^3 – 3xy = 0 is

__________ ?

Solution: Let u = x^3 y^2

Since, x^3 + y^3 – 3xy = 0

Differentiate both sides w.r.t. x

$$3x^2+3y^2\frac{dy}{dx}-3y-3x\frac{dy}{dx}=0$$

Then, $\frac{dy}{dx}=\frac{3x^2-3y}{3x-3y^2}=\frac{x^2-y}{x-y^2}$

$$\frac{du}{dx}=\frac{\partial u}{\partial x},\frac{dx}{dx}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=3x^2y^2(1)$$

$$+2x^3y\frac{dy}{dx}=3x^2y^2+2x^3y\left[\frac{x^2-y}{x-y^2}\right]$$

Example: If u = f (2x −3y, 3y − 4z,4z − 2x) then $6u_x + 4u_y$ = __________ ?

Solution: Let r = 2x −3y; s = 3y − 4z; t = 4z − 2x

Then, u = f(r,s,t)

$$u_x = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x} =$$

$$f,\times + fs \times 0 + f_t \times (-2)$$

$$6u_x = 12f, -12f_t$$

$$u_y = f_t \times (-3) + f_s \times 3 + f_t \times 0$$

$$4u_y = -12f_r + 12f_s$$

$$\therefore 6u_x + 4u_y = 12f_s - 12f_t$$

$$u_z = f_r(0) + f_s(-4) + f_t(4)$$

$$-3u_z = 12f_s - 12f_t$$

$$\text{Thus}, 6u_x + 4u_y = -3u_z$$

Example: $\text{If } V = r^n, r = \sqrt{X^2 + y^2 + z^2}$ then $V_{xx} + V_{yy} + V_{zz} =$ ______

Solution: $\dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \dfrac{x}{r}$

$$V_x = \frac{\partial V}{\partial r} \times \frac{\partial r}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}, \frac{x}{r} = nr^{n-2}.x$$

$$V_{xx} = n\left[r^{n-2}.(1)\right] + xn(n-2)r^{n-3}.\frac{\partial r}{\partial x} = nr^{n-2} +$$

$$nx(n-2)r^{n-3}.\frac{x}{r} = nr^{n-2} + n(n-2)r^{n-4}x^2$$

Similarly

$$V_{yy} = nr^{n-2} + n(n-2)r^{n-4}.y^2$$

$$V_{zz} = nr^{n-2} + n(n-2)r^{n-4}.z^2$$

$$V_{xx} + V_{yy} + V_{zz} = 3nr^{n-2} + n(n-2)r^{n-4}\left(x^2 + y^2 + z^2\right)$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}\left(r^2\right)$$

$$V_{xx} + Vyy + V_{zz} = r^{n-2}\left[3n + n^2 - 2n\right] = r^{n-2}\left[n^2 + n\right]$$

$$= n(n+1)r^{n-2}$$

Maxima & Minima for functions of two variables

$$\text{Let } z = f(x,y), p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2},$$

$$s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

Method

1. Find p, q, r, s & t
2. Equate p & q to zero for obtaining the stationary points.
3. At each stationary point find r, s, t
   a) If $rt - s^2 > 0$ and $r > 0 \Rightarrow$ Minimum at the stationary point.
   b) If $rt - s^2 > 0$ and $r < 0 \Rightarrow$ Maximum at the stationary point.
   c) If $rt - s^2 < 0$ then f(x, y) has no extreme at that stationary point and each points are SADDLE points.

Solved Examples

Example: The function $f(x,y) = 1-x^2-y^2$ has maximum at point?

Solution: p = −2x, q = −2y

r = −2, s = 0, t = −2

$$\text{For stationery point s,} \left.\begin{matrix} p = 0 \\ q = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix}$$

$$\text{At } (0,0)\, rt - s^2$$

$$\therefore \text{maximum at } (0,0)$$

Example: The minimum value of the function $x^3-3x^2+4y^2 -10$ occurs at

Solution: $p=3x^2-6x$

q=8y

p=q=0 ⇒ (0,0) & (2,0) are the stationary point.

Determining double derivatives

r = 6x − 6

s = 0,t = 8

At (0, 0), r = −6, s = 0, t = 8

$rt - s^2 = -48 < 0$ → Saddle point

At (2, 0), r = 6,s = 0,t = 8

$rt - s^2 = 48 > 0$ and r = 6 > 0 min at (2, 0)

Example: A rectangular box open at the top is to have a volume of 32 cubic feet. Then the dimensions of the box

requiring least material for its construction are ________?

Solution: Let x, y, z be the dimensions. Since, it is open at the top,

$s = xy + 2yz + 2xz$

$v = xyz = 32$

$f = xy + \frac{64}{x} + \frac{64}{y}$

Calculating the partial derivatives

$p = y - \frac{64}{x^2} \text{ and } q = x - \frac{64}{y^2}$

Equating Partial Derivatives to zero to determine stationery points,

$p = 0 => y = \frac{64}{x^2}$

$q = 0 => x = \frac{64}{y^2} = \frac{x^4}{64}$

$y = 0, y = 4$

$x = 0.x - 4$

If x = 0 and y = 0 then z = inf

But dimension cannot be infinite so x =4 and y =4. Thus, z =2.

Example: The distance between origin and a point nearest to it on the surface $z^2 = (1 + xy)$ is __________ ?

Solution: Let P(x, y, z) be a point on the surface z2=(1+xy)

$d = OP = \sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + y^2 + 1 + xy}$

$\text{Let } f = x^2 + y^2 + 1 + xy$

Calculating partial derivatives,

p = 2x + y and q = 2y + x

Equating both to zero we get x = 0 and y=0 as the stationery point

Calculating double partial derivative r =2 and t=2 and s=1

At (0, 0) 2 rt – s^2 =3 > 0

r=2 > 0 so minima exists.

Minimum distance $= \sqrt{0 + 0 + 1 + 0} = 1$

Constrained Maxima & Minima

A constraint is a certain condition on independent variables that must be satisfied while optimizing a multivariable function.

To determine Constrained Maxima and Minima we use the method of Lagrange's Multipliers.

Suppose, we have to optimize the function f(x, y, z)

where $\phi(x, y, z) = c$ is the constraint

Let $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

Here, λ is called as Lagrange Multiplier

Now, we equate all partial derivatives of F to zero.

$$\text{i,e } \left.\begin{aligned} \frac{\partial F}{\partial x} + \lambda \frac{d\phi}{\partial x} = 0 \\ \frac{\partial F}{\partial y} + \lambda \frac{d\phi}{\partial y} = 0 \\ \frac{\partial F}{\partial z} + \lambda \frac{d\phi}{\partial z} = 0 \end{aligned}\right\} \text{ Lagrange's equation}$$

Solving equations (1) .... (4) we obtain x, y, z & λ

x, y, z → stationary point

f(x, y, z) → extreme value

Solved Examples

Example: The max-value of volume of parallel piped that can be inscribed in an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution: Constraint is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Function to be optimized is, v=8xyz

$$\text{Thus, } F = 8xyz, \phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\text{Thus, } F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

Calculating the partial derivatives,

$$8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \Rightarrow \frac{-\lambda}{4} = \frac{a^2yz}{x}$$

$$8xz + \lambda\left(\frac{\partial y}{b^2}\right) = 0 \Rightarrow \frac{-\lambda}{4} = \frac{b^2xz}{y}$$

$$8xz + \lambda\left(\frac{\partial y}{c^2}\right) = 0 \Rightarrow \frac{-\lambda}{4} = \frac{c^2xz}{z}$$

$$\frac{a^2yz}{x}=\frac{b^2xz}{z}\Rightarrow\frac{y^2}{b^2}=\frac{x^2}{a^2}$$

$$\text{And}\ \frac{b^2xz}{y}=\frac{c^2xy}{z}\Rightarrow\frac{z^2}{c^2}=\frac{y^2}{b^2}$$

$$\therefore\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\Rightarrow\frac{3x^2}{a^2}=1$$

$$x=\frac{a}{\sqrt{3}},$$

$$\text{similarly}\ y=\frac{b}{\sqrt{3}},z=\frac{c}{\sqrt{3}}$$

$$\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)\rightarrow\text{Stationary point}$$

$$\text{Extreme value } f=\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)=\frac{8abc}{3\sqrt{3}}$$

Indefinite Integration

Indefinite Integration is also called as Anti-derivative and it is the reverse process of differentiation.

$$\int f(x)dx=F(x)+c$$

Indefinite Integration is the process of finding F(x) for function f(x) such that

F'(x) = f (x)

The result of indefinite integration represents a family of curves. Here, constant 'c' is called as constant of Integration.

Basic Integration Formulas

$$\int x^n dx=\frac{x^{n+1}}{n+1}$$

$$\int \frac{1}{x}dx=\log x$$

$$\int \sin x dx - \cos x$$

$$\int \cos x dx=\sin x$$

$$\int \sec^2 x dx=\tan x$$

$$\int \operatorname{cosec}^2 x dx=-\cot x$$

$$\int \sec x\tan x=\sec x$$

$$\int \operatorname{cosec}\cot x=-\operatorname{cosec}x$$

$$\int \cot x dx=\log|\sin x|$$

$$\int \tan x dx=\log|\sec x|$$

$$\int \operatorname{cosec}x=\log|\operatorname{cosec}x-\cot x|$$

$$\int \cosh x=\sinh x$$

$$\int\frac{1}{1+x^2}dx=\tan^{-1}x$$

$$\int \sinh x dx=\cosh x$$

$$\int\frac{1}{1-x^2}dx=\sin^{-1}x$$

$$\int \sec x=\log|(\sec x+\tan x)|$$

$$\int\frac{1}{x\sqrt{x^2-1}}dx=\sin^{-1}x$$

$$\int\frac{1}{\sqrt{a^2+x^2}}dx=\sinh^{-1}\left(\frac{x}{a}\right)=\log\left[x+\sqrt{x^2-a^2}\right]$$

$$\int\frac{dx}{a^2+x^2}=\sin^{-1}\left(\frac{x}{a}\right)$$

$$\int\frac{dx}{\sqrt{x^2-a^2}}=\cosh^{-1}\left(\frac{x}{a}\right)=\log\left[x+\sqrt{x^2-a^2}\right]$$

$$\int\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)\text{or}\ \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left\{x+\sqrt{x^2+a^2}\right\}$$

$$\int f'(ax+b)dx=\frac{f(ax+b)}{a}$$

$$\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\int\left[f(x)^n f'(x)dx=\frac{[f(x)]^{n+1}}{(n+1)}\ \text{when } n\neq 1\right.$$

Solved Examples

Example: Evaluate $\int \frac{4x^3}{1+x^4}dx$

Solution:

Substitute $\left(1+x^4\right) = t \Rightarrow 4x^3dx = dt$

The integral reduces to

$$\int \frac{dt}{t} = \log t = \log\left(1+x^4\right)$$

Integration by Parts

Let u and v be two functions of x. then we have from differential calculus.

$$\frac{d}{dx}(uv) = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

Integrating both sides of (1) with respect to x, we have

$$uv = \int u.\frac{dv}{dx}dx + \int v.\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}dx = uv - \int v.\frac{du}{dx}.dx$$

$$\text{i.e.,} \int udv = uv - \int v.du$$

This can also be written as

$$\int uvdx = u\int vdx = -\int\left[\frac{du}{dx}\int vdx\right]dx$$

This choice of which function will be u and which function will be v is very important in solving by integration by

parts.

The ILATE method helps to decide this.

ILATE stands for

I : Inverse trigonometric functions ($\sin^{-1} x$, $\cos^{-1}x$ etc)

L : Logarithmic functions (logx, In x etc)

A : Algebraic functions (x^2 , $x^3 + x^2 + 2$, etc)

T : Trigonometric functions (sinx, cosx etc)

E : Exponential function (e^x , a^x etc)

Whichever of the two functions comes first in ILATE, get designated as u and other function gets designated as v.

Based on above method we can define,

$$\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$$

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx - b\sin bx\right)$$

Integration by Partial Fractions

$$I = \int \frac{1}{x^2-a^2}dx \quad (x > a)$$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x-a}\right)$$

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\left\{\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right\} =$$

$$\frac{1}{2a}\log(x-a) - \log(x+a) = \frac{1}{2a}\log\frac{x-a}{x+a}$$

$$\text{Thus,} \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\frac{x-a}{x+a}, \ (x > a)$$

Some common integrals are,

(a) $\int \frac{1}{x^2-x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

(b) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\frac{a+x}{a-x}$

(c) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\frac{x-a}{x+a}$

(d) $\int \frac{1}{\sqrt{a^2+x^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right)$

$= \log\left[x+\sqrt{x^2+a^2}\right]$

(e) $\int \frac{1}{a^2-x^2}dx = \sin^{-1}\left(\frac{x}{a}\right)$

(f) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right)$

$$= \log\left[x+\sqrt{x^2+a^2}\right]$$

(g) $\int \frac{1}{x\sqrt{a^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left(\left|\frac{x}{a}\right|\right)$

(h) $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

(i) $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

A few other useful integration formulae

$$\int_0^{\pi/2} \sin^m x\cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left[\frac{m+n+2}{2}\right]}$$

Where Γ(n+1) is called the gamma function which satisfies the following properties

$\Gamma(n+1)=n\Gamma n$ if n is a positive integer

$\Gamma(n+1) = n!, \Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Walle's formula

$$\int_0^{\pi/2} \sin^n x = \int_0^{\pi/2} \cos^n x =$$

$$\begin{cases} \frac{(n-n)(n-3)(n-5)}{(n)(n-2)(n-4)} \ldots\ldots\ldots\ldots \frac{2}{3}, \text{when n is odd} \\ \frac{\pi}{2} x \frac{(n-1)(n-3)(n-5)}{(n)(n-2)(n-4)} \ldots\ldots \frac{3}{4}, \frac{1}{2} \text{when n is even} \end{cases}$$

Note: The result of indefinite integration can be verified in exam by differentiating the options.

Definite Integral

Let f(x) be continuous in [a, b] and F be the antiderivative of the given function f(x) then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note: $\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(x)dx\right] = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$

Properties of Definite Integral

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$

  If $c \in (a,b)$ then $\int_a^b f(x)dx = \int_a^b f(x)dx + \int_c^d f(x)dx$
- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_0^a \frac{f(x)}{f(x)+f(a+b-x)}dx = \frac{b-a}{2}$
- $\int_{-a}^a f(x)dx = \begin{cases} \int_0^a f(x)dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$
- $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$
- $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$ if $f(x+a) = f(x)$
- $\int_0^a xf(x)dx = \frac{a}{2}\int_0^a f(x)dx$ if $(a-x) = f(x)$
- $\int_0^{\pi b} \sin^n xdx = \int_0^{\pi/2} \cos^n xdx =$

  $\left[\frac{n-1}{n} \times \frac{n-3}{n-2} \times \ldots\ldots\ldots\ldots \times \frac{2}{3}\left(\text{or } \frac{1}{2}\right)\right]K$

  where $K = \begin{cases} 1 & \text{if n is odd} \\ \frac{\pi}{2} & \text{if n is even} \end{cases}$
- $\int_0^{\pi/2} \sin^m x\cos^n dx$

  $\frac{[(m-a)(m-3)\ldots\ldots\ldots\ldots 2][(n-1)(n-3)\ldots\ldots\ldots 2(\text{or})1xK]}{[(m+n)(m+n-2)\ldots\ldots\ldots\ldots 2(\text{or}1)]}$

  where $K = \begin{cases} 1 & \text{if n is odd} \\ \frac{\pi}{2} & \text{if n is even} \end{cases}$

Solved Examples

Example: Evaluate $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x} dx$

Solution: Let f(x) = tanx

$$f\left(0+\frac{\pi}{2}-x\right) = \cot x$$

$$\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x} dx = \int_0^{\pi/2} \frac{f(x)}{f(x)+f\left(0+\frac{\pi}{2}-x\right)} dx$$

$$= \frac{b-a}{2} = \frac{\pi}{4}$$

Example: Evaluate $\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot}} dx$

Solution:

$$\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot}} dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} = \sqrt{\cos x}} dx$$

$$\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}} dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{0+\frac{\pi}{2}-x}} dx \frac{\frac{\pi}{2}-0}{2} = \frac{\pi}{4}$$

Example: Determine the value of $\int_0^{\pi} |\cos x| dx$

Solution:

$$|\cos x| = \begin{cases} \cos x \le x \le \frac{\pi}{2} \\ -\cos x \frac{\pi}{2} \le x \le \pi \end{cases}$$

$$\int_0^{\pi} |\cos x| dx = \int_0^{2\pi} \cos x dx + \int_{\pi/0}^{\pi} -\cos x dx = [\sin x]_0^{\pi/2}$$

$$-[\sin x]_{\pi/2}^{\pi} = (1-0)-(0-1) = 2$$

Example:

Determine the value of $\int_0^2 |x^2+2x-3| dx$

Solution:

$$I = \int_0^2 |x^2+2x-3| dx = \int_0^2 |(x+3)(x-1)| dx$$

$$|\cos| x = \begin{cases} -(x+3)(x-1) & 0 \le x \le 1 \\ (x+3)(x-1) & 1 \le x \le 2 \end{cases}$$

$$I = \int_0^1 (-x^2-2x+3) dx + \int_1^2 (x^2+2x-3) dx$$

$$= \left[\frac{-x^3}{3} - \frac{2x^2}{2} + 3x\right]_0^1 + \left[\frac{x^3}{3} + \frac{2x^2}{2} - 3x\right]_1^2$$

$$I = \left(\frac{-3}{3} - 1 + 3\right) + \left(\frac{8}{3} + 4 - 6\right) - \left(\frac{1}{3} + 1 - 3\right) = 4$$

Example: Find the value of $\int_0^n [x] dx$

Solution: [x] = step function

$$I = \int_0^n [x] dx = \int_0^1 0 dx + \int_1^2 1 dx + \int_2^3 2 dx +$$

$$\int_3^4 3 dx + \ldots\ldots\ldots + \int_{n-1}^n (n-1) dx$$

$$I = 1+2+3\ldots\ldots\ldots+(n-1) = \frac{n(n-1)}{2}$$

Example:

Determine the integral $I = \int_0^{\pi/2} \log(\tan x) dx$?

Solution: $I = \int_0^{\pi/2} \log(\cot x) dx$

$$\left[\because \int_0^{\pi/2} \log(\tan x) = \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2}-x\right)\right)\right]$$

$$I + I = \int_0^{\pi} [\log(\tan x) + \log(\cot x) dx] = \int_0^{\pi/2} \log(1) dx = 0$$

Example:

Determine the integral $\int_0^{\pi} \frac{\sec^2 x dx}{a^2+b^2\tan^2 x} dx$

Solution: In this case, $2a = \pi \text{ and } a = \frac{\pi}{2}$

Since, $\frac{\sec^2 x}{a^2+b^2\tan^2 x} = \frac{\sec^2(\pi-x)}{a^2+b^2\tan^2(\pi-x)}$

$$\int_0^{\pi} \frac{\sec^2 xdx}{a^2+b^2\tan^2 x}dx = 2\int_0^{\pi/2} \frac{\sec^2 xdx}{a^2+b^2\tan^2 x}dx$$

Assume t = tanx and dt = $\sec^2$ xdx

$$\int_0^{\pi} \frac{\sec^2 xdx}{a^2+b^2\tan^2 x}dx = 2\int_0^{\infty} \frac{1}{a^2+b^2t^2}dt$$

$$= 2\times\frac{1}{a}\tan^{-1}\left[\frac{bt}{a}\right]\times\frac{1}{b}\bigg|_0^{\infty} = \frac{2}{ab}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{ab}$$

Example: Find the value of $\int_0^{\pi/2} \sin^8 xdx$

Solution: Since, $\int_0^{\pi b} \sin^n xdx = \int_0^{\pi/2} \cos^n xdx$

$$= \left[\frac{n-1}{n}\times\frac{n-3}{n-2}\times\ldots\ldots\ldots\ldots\frac{2}{3}\left(\text{ or }\frac{1}{2}\right)\right]K$$

$$\text{where } K = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

In this case, n = 8 (even)

$$\int_0^{\pi/2} \sin^8 xdx = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{35\pi}{256}$$

Example: Determine the value of $\int_0^{\pi/2} \cos^7 xdx$

Solution: Since n = 7 (odd)

$$\int_0^{\pi/2} \sin^7 xdx = \int_0^{\pi/2}\sin^7\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_0^{\pi/2} \cos^7 xdx = \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times 1 = \frac{16}{35}$$

Example: Find the value of integral

$$\int_0^{\pi/2} \sin^7 x \cos^9 x\, dx$$

Solution: $\int_0^{\pi/2} \sin^m x\cos^n xdx$

$$= \frac{[(m-1)(m-3)\ldots 2(\text{ or})1][(n-1)(n-3)\ldots 2(\text{ or})xK]}{[(m+n)(m+n-2)\ldots 2(\text{or})1]}$$

Here, m = 7 and n = 9 which are odd

$$I = \frac{(6\cdot4\cdot2)(8\cdot6\cdot4\cdot2)}{16\cdot14\cdot12\cdot10\cdot8\cdot6\cdot4\cdot2}\times 1 = \frac{1}{448}$$

Example: Calculate the value of

$$\int_0^{\pi/2} \sqrt{\sin x}\cos^3 xdx$$

Solution: Let sin x = t

cosx dt = dt

$$I = \int_0^1 \sqrt{t}\left(1-t^2\right)dt = \int_0^1\left(t^{1/2}-t^{5/2}\right)dt\Bigg]_0^1$$

$$= \frac{t^{3/2}}{3/2} = \frac{t^{7/2}}{7/2}$$

$$I = \frac{2}{3}-\frac{2}{7} = \frac{8}{21}$$

Improper Integrals:

Certain integrals cannot be computed directly using the rules of definite integrals.
First kind:

$$\int\int_a^b f(x)dx \text{ if } a=-\infty \text{ (or) } b=\infty \text{ (or) both}$$

$$\text{i.e.} \int\int_{-\infty}^{b} f(x)dx \text{ (or) } \int_a^{\infty} f(x)dx \text{ (or) } \int_{-\infty}^{\infty} f(x)dx$$

Second kind: $\int_a^b f(x)dx$

When a and b are finite, but f(x) is infinite for some $x \in [a,b]$

Example:

1. $\int_0^1 \log(1-x)dx \to$ infinite for $x=1$
2. $\int_{-1}^1 \sqrt{\frac{1-x}{1+x}}dx \to$ infinite for $x=-1$
3. $\int_0^2 \frac{1}{1-x}dx = \int_0^1 \frac{1}{1-x}dx + \int_1^2 \frac{1}{1-x}dx$

$\to$ infinite for $x=1$

Convergence:

1. $\int_a^b f(x)dx =$ finite, then it is a convergent improper integral

2. $\int_a^b f(x)dx =$ infinite, then it is a divergent improper integral

Solved Examples

Example: Find the convergence of following integrals.

1. $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}dx$

2. $\int_0^{\infty} x\sin x dx$

Solution:

1. $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x\Big]_1^{\infty} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

   Since, the integral is finite, it is convergent

2. $\int_0^{\infty} x\sin x dx = [x[-\cos x] + (1)[\sin x]]_0^{\infty} = \infty$

   Since, the integral becomes infinite, it is divergent improper integral

Example:

The integral $\int_{-\infty}^{0} \frac{1}{(1-3x)^2}dx$ converges to_____

Solution:

$$\int_{-\infty}^{0} \frac{1}{(1-3x)^2}dx = \frac{-1}{1-3x} \times \frac{-1}{3}\Big]_{-\infty}^{0} = \frac{1}{3} - 0 = \frac{1}{3}$$

Example: Check the convergence of the following integral $\int_1^{\infty} \log\left(\frac{1}{x}\right)dx$

Solution:

$$\int_1^{\infty} \log\left(\frac{1}{x}\right)dx = \int_1^{\infty} -\log x dx = -[x(\log x - 1)]_0^{\infty} = \infty$$

Thus, the following integral is divergent improper integral.

Example: Determine the integral

$$\int_{-1}^{1} \sqrt{\frac{1 + x}{1 - x}}\, dx = \_\_\_\_\_\_$$

Solution: Multiply both numerator and denominator by $\sqrt{1+x}$

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}dx = \int_{-1}^{1} \frac{1+x}{\sqrt{1-x^2}}dx$$

$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx + \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}dx$$

Since, the function in second integral is odd.

Thus, $\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}dx = 0$

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}dx = 2$$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = 2\left(\sin^{-1} x\right)_0^1 = 2 \times \left[\frac{\pi}{2} - 0\right] = \pi$$

Example: Calculate the value of integral $\int_{-1}^{1} \frac{1}{x^2}dx$

Solution:

$$\int_{-1}^{1} \frac{1}{x^2}dx = \int_{-1}^{0} \frac{1}{x^2}dx + \int_{0}^{1} \frac{1}{x^2}dx$$

$$\int_{-1}^{1} \frac{1}{x^2}dx = \left[\frac{-1}{x}\right]_{-1}^{0} + \left[\frac{-1}{x}\right]_{0}^{1} = \infty$$

Thus, it is a divergent improper integral.

Example: Determine the integral

$$\int_0^3 \frac{1}{x^2 - 3x + 2}dx$$

Solution:

$$\int_0^3 \frac{1}{x^2-3x+2}dx = \int_0^3 \frac{1}{(x-1)(x-2)}dx$$

$$\int_0^3 \frac{1}{x^2-3x+2}dx = \int_0^1 \frac{dx}{(x-1)(x-2)}$$

$$+\int_1^2 \frac{dx}{(x-1)(x-2)} + \int_2^3 \frac{dx}{(x-1)(x-2)}$$

$$I = \log\left(\frac{x-2}{x-1}\right)\Bigg]_0^1 + \log\left(\frac{x-2}{x-1}\right)\Bigg]_1^2 + \log\left(\frac{x-2}{x-1}\right)\Bigg]_2^3$$

$$= \infty$$

The first integral goes to infinity as x goes to 1. Hence, this integral if divergent improper integral.

Comparison Test

For first kind of improper integrals, we use this test,

1st Method:

Let $0 \le f(x) \le g(x)$ then

1. $\int_a^b f(x)dx$ converges if $\int_a^b g(x)dx$ is convergent.

2. $\int_a^b g(x)dx$ diverges if $\int_a^b f(x)dx$ is divergent.

Note: The following comparisons hold good for independent variable x

- x < finite ⇒ x is finite
- x > infinite ⇒ x is infinite
- x > finite ⇒ x may be finite (or) infinite
- x < infinite ⇒ x may be finite or infinite

2nd Method [Limit form]:

Let f(x) and g(x) be two positive function such that $\lim_{x\to\infty}\frac{f(x)}{g(x)} = l$ [non zero, finite]

Then $\int_a^b f(x)dx$ and $\int \int_a^b g(x)dx$ both diverge (or) converge together.

Solved Examples

Example: Find the convergence of following improper integrals

1. $\int_1^\infty e^{-x^2}dx$

2. $\int_2^\infty \frac{1}{\log x}dx$

3. $\int_1^\infty \frac{1}{x^2\left(e^{-x}+1\right)}dx$

4. $\int_1^\infty \frac{x\tan^{-1}x}{\sqrt{4+x^3}}dx$

Solution:

1. $e^{-x^2} \le e^{-x} \forall x \ge 1$

Thus, we can check for the convergence of

$$\int_1^\infty e^{-x}dx = \frac{e^{-x}}{-1}\Bigg]_1^\infty = \frac{1}{e}$$

Since, this integral is convergent, the given integral is convergent.

2. $\int_2^\infty \frac{1}{\log x}dx$

Since, $\log x < x \ \forall\ x \ge 2$

Thus, $\frac{1}{\log x} > \frac{1}{x}$

Checking the convergence of the integral,

$$\int_2^\infty \frac{1}{x}dx = \log x\Bigg]_2^\infty = \infty$$

Since, this integral is divergent. The integral $\int_2^\infty \frac{1}{\log x}dx$ is divergent.

3. $\int_1^\infty \frac{1}{x^2\left(e^{-x}+1\right)}dx$

The function $x^2\left(1+e^{-x}\right) > (0+1)x^2$

Thus, $\frac{1}{x^2\left(1+e^{-x}\right)} \leq \frac{1}{x^2}$

$\frac{f(x)}{g(x)} = \frac{1}{e^{-x}+1} \Rightarrow \lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{1}{0+1} = 1$

$g(x) = \frac{1}{x^2}$

$\int_1^\infty \frac{1}{x^2}dx = \left[\frac{-1}{x}\right]_1^\infty = 1$

Both integrals will converge together. Thus, the integral $\int_1^\infty \frac{1}{x^2\left(e^{-x}+1\right)}dx$ is convergent.

4. $\int_1^\infty \frac{x\tan^{-1}x}{\sqrt{4+x^3}}dx =$ ______

$f(x) = \frac{x\tan^{-1}x}{x\sqrt{x}\sqrt{\frac{4}{x^3}+1}}$

Let $g(x) = \frac{1}{\sqrt{x}}$

$\frac{f(x)}{g(x)} = \frac{\tan^{-1}x}{\sqrt{\frac{4}{x^3}+1}} \to \lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{\pi/2}{\sqrt{0+1}} = \frac{\pi}{2}$

Thus, the integrals $\int_1^\infty \frac{x\tan^{-1}x}{\sqrt{4+x^3}}dx$ and $\int_1^\infty \frac{1}{\sqrt{x}}dx$ t converge and diverge together

$\int_1^\infty \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_1^\infty = \infty$

Since, one integral diverges. Both the integrals will diverge.

Comparison Test for Second Kind of Improper Integrals

1st method remains same for both kind of improper integrals. For the second method i.e. the limit form,

Let f(x) and g(x) be two +ve functions such that

1. 'a' is a point of discontinuity and

   $\lim_{x\to a}\frac{f(x)}{g(x)} = l_1$ [non zero, finite] or

2. 'b' is a point of disontunity and

   and $\lim_{x\to a}\frac{f(x)}{g(x)} = l_2$ [Non zero, finite then

   $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both converge

   (or) diverge together.

Solved Examples

Example: Check the convergence of the following integral $\int_0^{\pi/2}\frac{\sin x}{x\sqrt{x}}dx$

Solution: Since, $\frac{\sin x}{x} \leq 1$

Thus, $\frac{\sin x}{x}\cdot\frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$

Checking the convergence of the integral

$\int_0^{\pi/2}\frac{1}{\sqrt{x}}dx$

$\int_0^{\pi/2}\frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big]_0^{\pi/2} = 2\sqrt{\frac{\pi}{2}}$

Thus, this integral is convergent.

$\therefore \int_0^{\pi/2}\frac{\sin x}{x\sqrt{x}}dx$ is convergent

Example: Check the convergence of the integral $\int_1^2 \frac{\sqrt{x}}{\log x}dx$

Solution: Since, logx < x

Thus, $\frac{1}{\log x} > \frac{1}{x} \Rightarrow \frac{\sqrt{x}}{\log x} > \frac{\sqrt{x}}{x} > \frac{1}{\sqrt{x}}$

Checking the convergence of the integral

$\int_1^2 \frac{1}{\sqrt{x}}dx$

$$\int_1^2 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big]_1^2 = 2\sqrt{2}-2$$

Since, this integral is finite so it converges.

Thus, $\int_1^2 \frac{\sqrt{x}}{\log x}dx > \int_1^2 \frac{1}{\sqrt{x}}dx$ may be convergent or divergent.

Applying Method II

Now let $g(x) = \frac{1}{x\log x} \Rightarrow \frac{f(x)}{g(x)} = x\sqrt{x}$

$$\lim_{x\to 1^+}\frac{f(x)}{g(x)} = 1$$

Checking the convergence of $\int_1^2 \frac{1}{x\log x}dx$

Let log x = t

$$\frac{1}{x}dx = dt$$

$$\therefore \int_0^1 \frac{1}{x\log x}dx = \int_0^{\log 2}\frac{1}{t}dt = \log t\Big]_0^{\log 2} = \infty$$

Thus, $\int_1^2 \frac{\sqrt{x}}{\log x}dx$ also diverges.

Example: Calculate the integral $\int_0^1 x\log x\,dx$

Solution: Applying Integration by parts

$$\int_0^1 x\log x dx = \log x\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}dx = \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1$$

$$I = \frac{-1}{4} - \lim_{x\to 0}\frac{x^2}{2}\log x = \frac{-1}{4} - \lim_{x\to 0}\frac{\log x}{2/x^2}$$

Using L' Hospital's Rule $\lim_{x\to 0}\frac{\log x}{2/x^2} = \lim_{x\to 0}\frac{1/x}{-4/x^3}$

$$I = -\frac{1}{4} - \left[\lim_{x\to 0}\frac{-x^2}{4}\right] = -\frac{1}{4} - 0 = -\frac{1}{4}$$

Gamma Function

$$\overline{|n} = \int_0^\infty e^{-x}x^{n-1}dx$$

Note:

1. $\overline{|1} = 1$
2. $\overline{|\tfrac{1}{2}} = \sqrt{\pi}$
3. $\overline{|n+1} = n\overline{|n} \quad \forall n > 0$
4. $\overline{|n+1} = n! \quad \forall n \in Z^+$
5. $\int_0^\infty e^{-\alpha x}x^{n-1}dx = \frac{\overline{|n}}{\alpha^n}$

Solved Examples

Example: Calculate the integral $\int_0^\infty e^{-x^2}dx$?

Solution: Let $x^2 = t \Rightarrow 2xdx = dt$

$$dx = \frac{1}{2}t^{-1/2}dt$$

$$\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-t}\cdot\frac{1}{2}t^{-1/2}dt = \frac{1}{2}\int_0^\infty e^{-t}t^{\frac{1}{2}-1}dt = \frac{1}{2}\overline{|\tfrac{1}{2}} = \frac{\sqrt{\pi}}{2}$$

Example: Calculate the integral $\int_0^\infty e^{-y^3}y^{1/2}dy$

Solution: Let $y^3 = t \Rightarrow 3y^2dy = dt$

$$dy = \frac{1}{3}t^{-2/3}dt$$

Thus, $I = \int_0^\infty e^{-t}t^{1/6}\frac{1}{3}t^{-2/3} = \frac{1}{3}\int_0^\infty e^{-t}t^{-1/2}dt$

$$I = \frac{1}{3}\overline{|\tfrac{1}{2}} = \frac{\sqrt{\pi}}{3}$$

Example: Determine the value of integral

$$\int_0^1 (x\log x)^4 dx$$

Solution: Let $\log x = -t \Rightarrow x = e^{-t}$

$$dx = -e^{-t}dt$$

$$I = \int_\infty^0 \left(e^{-t}(-t)\right)^4\left(-e^{-t}\right)dt = \int_\infty^0 e^{-5t}t^{5-1} = \frac{\sqrt{5}}{5^5} = \frac{4!}{5^5}$$

Example: Determine the integral $\int_0^\infty 5^{-4x^2}dx$

Solution: Let $5^{-x^2} = e^{-t}$

$$-4x^2\log 5 = -t$$

$$x = \frac{1}{2\sqrt{\log 5}}\sqrt{t}$$

$$dx = \frac{1}{2\sqrt{\log 5}} \times \frac{1}{2\sqrt{t}}dt$$

$$I = \int_0^\infty e^{-t}\frac{1}{4\sqrt{\log 5}}t^{-1/2}dt = \frac{1}{4\sqrt{\log 5}}\left\lceil\frac{1}{2}\right. = \frac{\sqrt{\pi}}{4\sqrt{\log 5}}$$

Beta Function:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad (m>0, n>0)$$

Note:

1. $\beta(m,n) = \beta(n,m)$
2. $\beta(m,n) = \dfrac{\lceil m \lceil n}{\lceil m+n}$
3. $\beta(m,n) = \int_0^\infty \dfrac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \dfrac{x^{n-1}}{(1+x)^{m+n}}dx$
4. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$ is

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

Solved Examples

Example: Determine the integral

$$\int_0^2 x^7\left(16-x^4\right)^{10}dx\,?$$

Solution: Let $x^4 = 16t \Rightarrow 4x^3dt = 16dt$

$$\mathrm{I} = \int_0^1 16t(16-16t)^{10}4\mathrm{dt} = 4\times 16^{11}\int_0^1 t^{2-1}(1-\mathrm{t})^{11-1}\mathrm{dt}$$

$$\mathrm{I} = 4\times 16^{11}\times\beta(2,11)$$

$$\mathrm{I} = 4\times 16^{11}\times\frac{\lceil 2\times\lceil 11}{\lceil 13} = 4\times 16^{11}\times\frac{1\times 10!}{12!}$$

Example: Calculate the integral $\int_0^\infty \dfrac{x^3\left(1+x^3\right)}{(1+x)^{13}}dx$?

Solution: $I = \int_0^\infty \dfrac{x^3}{(1+x)^{13}}dx + \int_0^\infty \dfrac{x^8}{(1+x)^{13}}dx$

$$I = \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+9}}dx + \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+4}}dx$$

$$I = \beta(4,9)+\beta(9,4)$$

$$I = 2\beta(4,9) = 2\left(\frac{\lceil 4\cdot\lceil 9}{\lceil 13}\right)$$

$$I = 2\left[\frac{3!\times 8!}{12!}\right]$$

Example: Calculate the value of $\int_0^\infty\left[\dfrac{x}{1+x^2}\right]^3dx$

Solution: Let $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$$I = \int_0^{\pi/2}\left[\frac{\tan\theta}{\sec^2\theta}\right]^3\sec^2\theta d\theta = \int_0^{\pi/2}\frac{\tan^3\theta}{\sec^4\theta}d\theta$$

$$= \int_0^{\pi/2}\sin^3\theta\cos\theta d\theta$$

Let $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$$I = \int_0^1 t^3dt = \frac{1}{4}$$

Multiple Integrals

For multi-variable function, integral is defined in more than one dimensions and then it is called as Multiple Integral.

Double Integral

Consider the function f (x, y) of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas δA_1, δA_2 ,........., δA_n. Let (x_r, y_r) be any point in the r^{th} elementary area δA_r. Consider the sum

$$f(x_1,y_1)\delta A_1 + f(x_2,y_2)\delta A_2 + \ldots\ldots + f(x_n,y_n)\delta A_n$$

i.e., $\sum_{r=1}^{n} f(x_r,y_r)\delta A_r$

the limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the double integral of f(x, y) over the region R and is written as $\iint_R f(x,y)dA$

Thus, $\iint_R f(x,y)dA = \underset{\substack{n\to\infty \\ \delta A\to 0}}{\mathrm{Lt}}\sum_{r=1}^{n} f(x_r,y_r)\delta A_r$

Thus, double integral can be evaluated as the limit of a sum but the process in such a case becomes tedious and hence we try to evaluate double integral in terms of single integrals.

Evaluating a Double Integral

Case 1: When the limits are y = ϕ_1 (x) to y = ϕ_2 (x) and x = c_1 to x = c_2

$$\therefore \iint_{P_1} f(x,y)dxdy = \int_{c_1}^{c_2}\left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy\right]dx$$

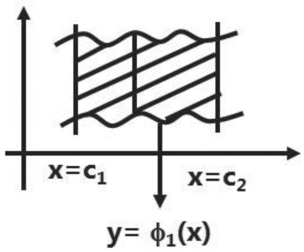

Fig. 2.7

In this case, we first keep x-constant and evaluate the integral with respect to y but the limits of y are a function of 'x' and hence we obtain a function of x which we again integrate with constant limits with respect to 'x'.

This can be understood graphically as we are determining area by summing the area of vertical strips.

Case 2 When the limits are x = ϕ_1 (y) to x = ϕ_2 (y) and y = d_1 to y = d_2

$$\therefore \iint_{P_1} f(x,y)dxdy = \int_{y=d_1}^{d_2}\left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dx\right]dy$$

In this case, we first keep y-constant and evaluate the integral with respect to x but the limits of x are a function of 'y' and hence we obtain a function of y which we again integrate with constant limits with respect to 'y'.

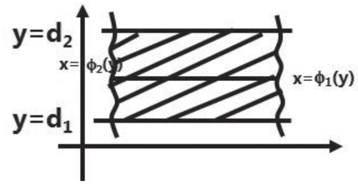

Fig. 2.8

This can be understood graphically as we are determining area by summing the area of horizontal strips.

Case 3: When the limits are x = c_1 to x = c_2 and y = d_1 to y = d_2

$$\iint_{P_1} f(x,y)dxdy$$

$$= \int_{x=c_1}^{c1}\left[\int_{y=d_1}^{d_1} f(x,y)dx\right]dy = \int_{y=d_1}^{d_1}\left[\int_{y=c_1}^{c_1} f(x,y)dx\right]dy$$

Here, since both limits are independent of each other, the order of integration does not matter.

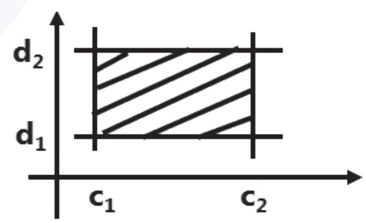

Fig. 2.9

The domain in this case is rectangular. So we can determine the area in terms of horizontal as well as vertical strips.

Solved Examples

Example: Evaluate the following integral

$$\int_1^2\int_3^4 \frac{1}{(x+y)^2}\,dxdy$$

Solution: Since both limits are independent of each other, the order of integration does not matter. Integrating first w.r.t. x

$$I = \int_1^2\left[\frac{-1}{x+y}\right]_3^4 dy = \int_1^2\left[\frac{1}{3+y} - \frac{1}{4+y}\right]dy$$

$$I = \left[\log\left(\frac{3+y}{4+y}\right)\right]_1^2 = \log\frac{5}{6} - \log\frac{4}{5} = \log\frac{25}{24}$$

Example: Evaluate the following integral

$$\int_0^3\int_0^x (6-x-y)dydx$$

Solution: Here, the limit of y depends on x. So, we first have to integrate w.r.t. y and then integrate w.r.t. x

$$\int_0^3\int_0^x (6-x-y)dydx = \int_0^3\left(6y - xy - \frac{y^2}{2}\right)_0^x dx$$

$$= \int_0^3\left(6x - x^2 - \frac{x^2}{2}\right)_0^x dx$$

$$I = \left[\frac{6x^2}{2} - \frac{3x^3}{6}\right]_0^3 = 27 - \frac{27}{2} = \frac{27}{2}$$

Example: Evaluate the following integral $\int\limits_0^4 \int\limits_0^{y^2} e^{x/y} dxdy$

Solution: Here, the limit of x is dependent on y. So, we first have to integrate w.r.t. x

$$\int\limits_0^4 \int\limits_0^{y^2} e^{x/y} dxdy = \int\limits_0^4 \left[\frac{e^{x/y}}{1/y}\right]_0^{y^2} dy = \int\limits_0^4 y\left[e^y - 1\right] dy$$

$$I = e^y(y-1) - \frac{y^2}{2}\Bigg]_0^4 = \left(3e^4 - 8\right) + 1 = 3e^4 - 7$$

Example: The value of $\iint\limits_R xy \, dxdy$ where R is the region of the 1st quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is______________.

Solution: In first quadrant both x and y are positive. Evaluating the limits of x in terms of y

x can go to a maximum value of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ i.e. } x = \frac{a}{b}\sqrt{b^2 - y^2}$$

$$\int\limits_0^b \int\limits_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy \, dxdyy = \int\limits_0^b \left[y\frac{x^2}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$I = \int\limits_0^b \left[\frac{y}{2}\frac{a^2}{b^2}\left(b^2 - y^2\right)\right] dy$$

$$I = \frac{a^2}{2b^2}\left[\frac{b^2y^2}{2} - \frac{y^4}{4}\right]_0^b = \frac{a^2}{2b^2}\left[\frac{b^4}{2} - \frac{b^4}{4}\right] = \frac{a^2b^2}{8}$$

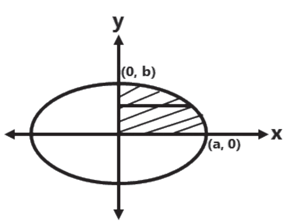

Fig. 2.10

Example: The value of $\iint\limits_R y \, dxdy$ where R is the area bounded by x = 0, y = x^2 , x + y = 2 in the 1st quadrant is?

Solution: The region bounded by these curves is shown below, For a fixed x, y varies from x^2 to (2 − x)

$$\iint\limits_R y \, dxdy = \int\limits_0^1 \int\limits_{x^2}^{2-x} y \, dydx = \int\limits_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$I = \int\limits_0^1 \left[\frac{(2-x)^2}{2} - \frac{x^4}{2}\right] dx = \frac{1}{2}\int\limits_0^1 \left[4 - x^2 - 4x - x^4\right] dx$$

$$I = \frac{1}{2}\left[4x + \frac{x^3}{3} - \frac{4x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left[4 + \frac{1}{3} - 2 - \frac{1}{5}\right]_0^1$$

$$= \frac{16}{15}$$

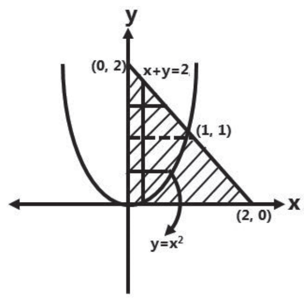

Fig. 2.11

Example: The value of $\iint\limits_R r^2 \sin\theta \, dr d\theta$ for the curve r = 2a cosθ in the first quadrant.

Solution: In the first quadrant,

$r = 0$ to $2a\cos\theta$

$\theta = 0$ to $\frac{\pi}{2}$

$$\iint\limits_R r^2 \sin\theta \, dr d\theta = \int\limits_0^{\pi/2} \int\limits_0^{2\cos\theta} r^2 \sin\theta \, dr d\theta$$

$$= \int\limits_0^{\pi/2} \sin\theta \left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$I = \frac{1}{3}\int\limits_0^{\pi/2} 8a^3 \cos^3\theta \sin\theta \, d\theta$$

$$= \frac{8a^3}{3}\left[\frac{-\cos^4\theta}{4}\right]_0^{\pi/2} = \frac{2a^3}{3}$$

Note: The plots for the some of the regions in x-y plane are,

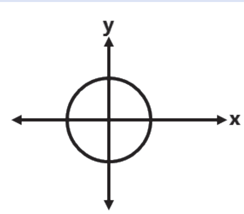

Fig. 2.12

r = a

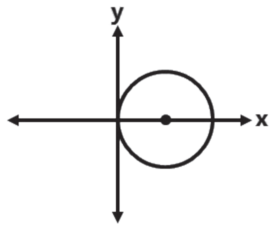

Fig. 2.13

r = acosθ

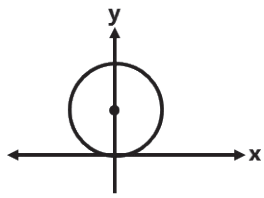

Fig. 2.14

r = asinθ

Area of the Region

The area of the region bounded by y = f(x) & y=g(x) between x = c_1 and x = c_2 is

$$A = \int_{x=c_1}^{c_2} [g(x) - f(x)]dx$$

It can also be computed in terms of Double Integral as,

$$A = \int_{c_1}^{c_2} \int_{f(x)}^{g(x)} 1\,dy\,dx$$

In polar form,

$$\text{Area} = \int_{\theta_1}^{\theta_2} \int_{f=f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta$$

Solved Examples

Example: The area bounded by the curve y = x^2 & y = x is ?

Solution: The point of intersection of these two curves have been marked in the figure below,

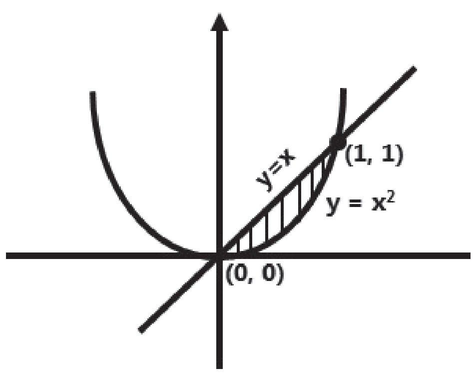

Fig. 2.15

The shaded area marks the area bounded by the two curves.

$$A = \int_0^1 \left[x - x^2\right] dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Example: The area bounded by 2y = x^2 and x = y – 4 is ?

Solution: To determine the points of intersection of these two curves,

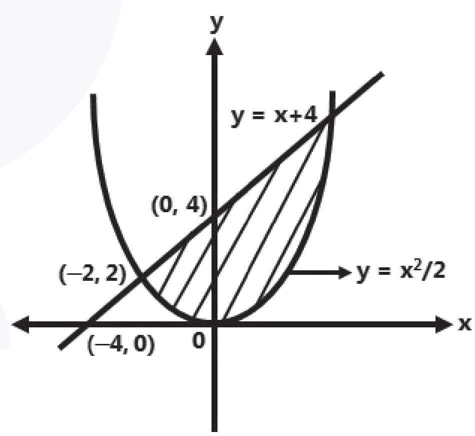

Fig. 2.16

$$x = \frac{x^2}{2} - 4 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow (x-4)(x+2) = 0$$

x = 4, –2 and y = 8, 2

$$A = \int_{-2}^{4} \left(x + 4 - \frac{x^2}{2}\right) dx = \left[\frac{x^2}{2} + 4x - \frac{x^3}{6}\right]_{-2}^{4}$$

$$A = \left(8 + 16 - \frac{64}{6}\right) - \left(2 - 8 + \frac{8}{6}\right) = 18$$

Example: Area in between the circle r = 2sinθ and r = 4sinθ is?

Solution: The given two curves are r = 2sinθ to r = 4sinθ

$\theta = 0$ to π

$$A = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi} 12\sin^2\theta d\theta$$

$$A = 3\left[\int_0^{\pi}(1-\cos 2\theta)d\theta\right] = 3\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = 3\pi$$

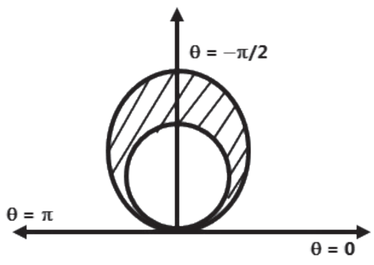

Fig. 2.17

or

Comparing these equations with equation of circle r = 2asinθ

where, a is the radius of the circle.

$r_1 = 1$, $r_2 = 2$

Since area of any circle is $A = \pi r^2$

$A_1 = \pi$, $A_2 = 4\pi$

$\therefore A_2 - A_1 = 3\pi$

Changing of Order of Integration

If we wish to reverse the order of integration i.e. if the integration is given first w.r.t. x and then w.r.t. y and we need to change the order i.e. first w.r.t. y and then w.r.t. x. This is called as change in order.

Essentially, we are converting the problem from finding the area in terms of Vertical Strips to a problem where we need to calculate the area in terms of Horizontal Strips.

This method is illustrated in the problems below,

Solved Examples

Example: Determine the value of integral

$$\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}dydx$$

Solution: The domain of above integration is shown in the figure below,

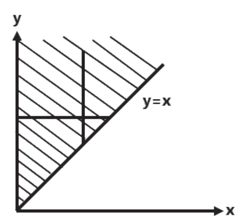

Fig. 2.18

y=x to y=∞

x=0 to x=∞

From the figure, same area can be expressed as,

x=0 to x=y

y=0 to y=∞

$$I = \int_0^{\infty}\int_0^{y}\frac{e^{-y}}{y}dxdy$$

$$I = \int_0^{\infty}\frac{e^{-y}}{y}[x]_0^y dy = \int_0^{\infty}\frac{e^{-y}}{y}ydy = \left.\frac{e^{-y}}{-1}\right]_0^{\infty} = 1$$

Here, by changing the order of integration the integration is simplified.

Example: By reversing the order of integration, the integral $I = \int_0^8\int_{x/4}^2 f(x,y)dydx$ reduces to $I = \int_r^s\int_p^q f(x,y)dydx$ then q =___.

Solution: Given limits are $y = \frac{x}{4}$ to $y = 2$ and x = 0 to x = 8

If we reverse the order, the limits can be written as,

x = 0 to x = 4y

y = 0 to y = 2

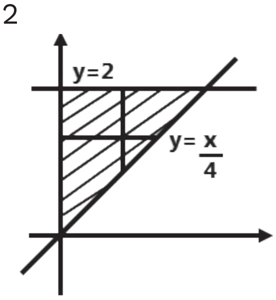

Fig. 2.19

$$I=\int_0^2\int_0^{4y} f(x,y)dxdy$$

Thus, q = 4y

Triple Integral

Consider a function f(x, y, z) defined at every point the 3-dimentional finite region V. Divide V into n elementary volumes δV_1, δV_2,........., δV_n

Let (x_r, y_r, z_r) be any point within the r_{th} sub-division of the volume

Consider the sum

$$\sum_{r=1}^{\infty} f\left(x_r, y_r, z_r\right)\delta V_r$$

The limit of this sum, if it exists, as n → ∞ and $\delta V_r \to 0$ is called the triple integral of f(x, y, z) over the region V and is denoted by

$$\iiint_V f(x,y,z)dV$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f\left(x_1 y_1 z\right)dxdydz$$

If x_1, x_2 are constant, y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y.

Thus, $\iiint_R f(x,y,z)dxdydz$

$$=\int_{x-c_1}^{c_2}\left[\int_{y=g_1(x)}^{g_1(x)}\left[\int_{g=h(x,y)}^{h(x,y)} f(x,y,z)dz\right]dy\right]dx$$

Solved Examples

Example: Determine the integral $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}$ dzdydx

Solution: Integrate first w.r.t. z keeping x and y as constant

$$I=\int_0^a\int_0^x\left(e^{x+y+z}\right)_0^{x+y}dydx$$

$$=\int_0^a\int_0^x\left[e^{2(x+y)}-e^{(x+y)}\right]dydx$$

Now, integrate w.r.t. y keeping x as constant

$$I=\int_0^a\left[\frac{e^{2(x+y)}}{2}-e^{x+y}\right]_0^x dx$$

$$=\int_0^a\left[\frac{e^{4x}}{2}-e^{2x}-\frac{e^{2x}}{2}+e^x\right]dx$$

$$I=\left[\frac{e^{4x}}{8}-3\frac{e^{2x}}{4}+e^x\right]_0^a dx$$

$$=\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^a-\left(\frac{1}{8}-\frac{3}{4}+1\right)$$

$$=\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^a-\frac{3}{8}$$

Example: Evaluate the integral over region $R\iiint_R ydxdydz$ where the region R is defined by, x = 0, y = 0, z = 0, x + y + z = 1 ?

Solution: Expressing the limits of z in terms of x and y

z goes from 0 to (1-x-y);

Similarly, maximum value of y occurs when z = 0 and then y = 1-x.

Thus, x goes from 0 to 1

$$I=\int_0^1\int_0^{1-x}\int_0^{1-x-y} ydzdydx=\int_0^1\int_0^{1-x} y[z]_0^{1-x-y}dydx$$

$$I=\int_0^1\int_0^{1-x} y(1-x-y)dydx=\int_0^1\left[(1-x)\frac{y^2}{2}-\frac{y^3}{3}\right]_b^{1-x}dx$$

$$I=\int_0^1\left[\frac{(1-x)^3}{2}-\frac{(1-x)^3}{3}\right]dx$$

$$=\frac{1}{6}\int_0^1(1-x)^3dx=\frac{1}{6}\left[\frac{(1-x)^4}{-4}\right]_0^1$$

$$I=\frac{1}{6}\left[0+\frac{1}{4}\right]=\frac{1}{24}$$

Change of Variables

In double integral

Let x = f(u, v), y = g(u, v)

Suppose, we wish to change the integral in terms of u,v variables from x,y variables.

$$\iint_R \phi(x,y)dxdy=\iint_R \phi(u,v)\,|\,\mu\,|\,dudv$$

|J| = Jacobian of transformation

$$|J| = J\left[\frac{x, y}{u, v}\right] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Case 1: Cartesian Form → Polar Form

(x, y) → (r, θ)

x = rcosθ, y = rsinθ

$x^2 + y^2 = r^2$

$$|J| = J\left[\frac{x, y}{r, \theta}\right] = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore \iint_R \phi(x, y)dxdy = \iint_R \phi\left(r\cos\theta, r\sin\theta\right) rdrd\theta$$

If we wish to change the variables in triple integral,

Case 2: Cartesian → cylindrical

(x, y, z) → (r, θ, z)

x = rcosθ, y = rsinθ, z = z

$x^2 + y^2 = r^2$, z = z

$$|J| = J\left[\frac{x, y, z}{r, \theta, z}\right] = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\therefore \int\iint_V \phi(x, y, z)dxdydz$$

$$= \int\iint_V \phi(r\cos\theta, r\sin\theta, z)rdrd\theta d$$

Case 3: Cartesian → spherical polar form

(x, y, z) → (r, θ, ϕ)

x = r sinθ cosϕ, y = r sinθ sinϕ, z = r cosθ

$x^2 + y^2 + z^2 = r^2$

$$|\mu| = J\left[\frac{x, y, z}{r, \theta, \phi}\right] = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

Different variables have the following ranges for Cylindrical and Spherical Systems:

Cylinder	Sphere
r : 0 to r	r : 0 to r
θ : 0 to 2π	Φ : 0 to 2π
z : z_1 to z_2	θ : 0 to π

Solved Examples

Example: Evaluate the integral $\int_0^\infty\int_0^\infty e^{-(x^2+h^2)}dxdy$

Solution: Changing to Polar Co-ordinates,

x = r cosθ and y = r sinθ

|J| = r

Limits: r : 0 to ∞

$\theta : 0$ to $\frac{\pi}{2}$ (Since, x and y lie only in first quadrant)

$$I = \int_0^{\pi/2}\int_0^\infty e^{-r^2} rdrd\theta$$

$$\text{Let } r^2 = t \Rightarrow rdr = \frac{dt}{2}$$

$$I = \int_0^{\pi/2}\int_0^\infty e^{-t}\frac{dt}{2}d\theta = \frac{1}{2}\int_0^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^\infty d\theta = \frac{1}{2}\times 1\times\frac{\pi}{2} = \frac{\pi}{4}$$

Example: Express the following integral by change of variables to Spherical Polar Coordinates,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}} ?$$

Solution: Cartesian Coordinates in terms of Spherical Polar Coordinates are given as,

Let $y = r \sin\theta \sin\phi$, $x = r \sin\theta \cos\phi$, $z = r \cos\theta$

Thus, $x^2 + y^2 + z^2 = r^2$ and $|J| = r^2 \sin\theta$

Since, z varies from

$$z = 0 \text{ to } z = \sqrt{1-x^2-y^2} \Rightarrow x^2 + y^2 + z^2 = 1$$

Thus, the limits of Spherical Polar Co-ordinates are,

$r : 0$ to 1

$\phi : 0$ to $\frac{\pi}{2}$

$\theta : 0$ to $\frac{\pi}{2}$

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{r^2 \sin\theta}{\sqrt{1-r^2}} dr d\theta d\phi$$

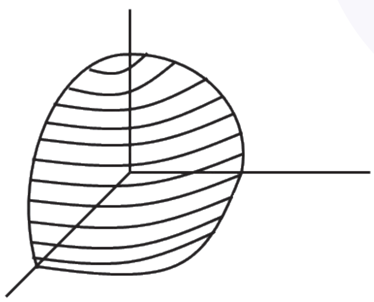

Fig. 2.20

Example: If the following relation exists between (x,y) and (u,v) variables

$$x(u,v) = uv, y(u,v) = \frac{v}{u}.$$

Then, In a double integral f(x, y) changes to

$$f\left(uv, \frac{v}{u}\right)\phi(u,v) \text{ then } \phi(u,v) = \_\_\_\_\_\_\_\_\_\_?$$

Solution:

$$\phi(u,v) = |J| = J\left[\frac{x,y}{u,v}\right] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$= \begin{vmatrix} v & u \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{v}{u} + \frac{v}{u} = \frac{2v}{u}$$

Length of a Curve

Suppose we have to determine the length of curve y = f (x) between $x = x_1$ and $x = x_2$

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

In polar form, $V = \int_{x1}^{x_2} \pi y^2 dx$

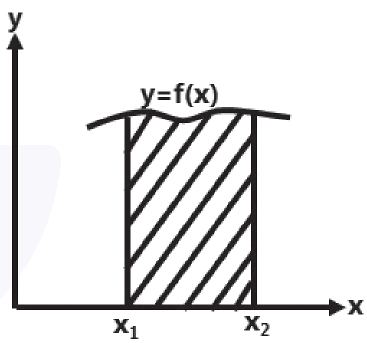

Fig. 2.21

Volume Of Solid Revolution

Volume generated by revolving the area enclosed by y = f(x) between $x = x_1$ and $x = x_2$ about x or u.

$$V = \int_{x1}^{x_2} \pi y^2 dx$$

Similarly volume generated by revolution about y or v

$$V = \int_{x_1}^{x_2} \pi x^2 dy$$

In polar form,

About initial line, $V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta d\theta$

About the line $\theta = \frac{\pi}{2}, V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \cos\theta d\theta$

Solved Examples

Example: The length of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ between x = 0 and x = 1 is_____.

Solution: $\frac{dy}{dx} = \frac{2}{3} \times \frac{3}{2} x^{\frac{1}{2}} = \sqrt{x}$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \left. \frac{(1+x)^{3/2}}{3/2} \right]_0^1 = \frac{2}{3}(2\sqrt{2} - 1)$$

Example: Find the length of the curve defined by Cardioid r = a(1 + cosθ) where 0 < θ < π?

Solution:

$$L = \int_0^{\pi} \sqrt{r^2 + (-a\sin\theta)^2} d\theta$$

$$= \int_0^{\pi} \sqrt{a^2 \left(1 + \cos^2\theta + 2\cos\theta\right) + a^2 \sin^2\theta d\theta}$$

$$L = a\int_0^{\pi} \sqrt{2(1+\cos\theta)} d\theta$$

$$L = a\int_0^{\pi} 2\cos\frac{\theta}{2} d\theta = 2a \left[\frac{\sin\frac{\theta}{2}}{\frac{1}{2}} \right]_0^{\pi} = 4a[1-0] = 4a$$

Example: Determine the volume of revolution generated by revolving the curve $\frac{x^2}{4} + \frac{y^2}{4} = 1$ about x-axis.

Solution: $v = \int_{x_1}^{x_2} \pi y^2 dx = \int_{-2}^{2} \pi\left(4 - x^2\right) dx$

$$V = 2\pi \times \left[4x - \frac{x^3}{3}\right]_0^2$$

$$V = 2\pi\left[\frac{16}{3}\right] = \frac{32\pi}{3}$$

Example: The volume obtained by moving the area bounded by the parabola $y^2 = 8x$ and x = 2 and y axis is ____.

Solution: The volume generated by rotating the straight line x = 2 between y = –4 to +4 about y

$$V_1 = \int_{-4}^{4} \pi x^2 dy = \int_{-4}^{4} \pi(2)^2 dy = 4\pi[y]_{-4}^{4} = 32\pi$$

The volume generated by rotating the parabola $x = \frac{y^2}{8}$ between x = –4 to 4 about y axis is

$$V_2 = \int_{-4}^{4} \pi x^2 dy = \int_{-4}^{4} \pi\left(\frac{y^4}{64}\right) dy = \frac{\pi}{64}\left[\frac{y^5}{5}\right]_{-4}^{4} = \frac{32\pi}{5}$$

$\therefore$ required volume $= V_1 - V_2 = 32\pi - \frac{32\pi}{5} = \frac{128\pi}{5}$

Note: Some of the equations of different curves are given as follows.

Circle: Cartesian Form

1. $x^2 + y^2 = a^2$: Circle with centre (0, 0) and radius a.
2. $(x – h)^2 + (y – k)2 = a^2$: Circle with centre (h, k) and radius a.

Polar Form:

1. r = a : Circle with centre (0, 0) and radius a.
2. r = a sinθ : Circle with centre $\left(0, \frac{a}{2}\right)$ and radius $a/2$
3. r = a cosθ : Circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $a/2$

Parabola:

1. $x^2 = 4ay$: Parabola with vertex at (0, 0) and focus at (0, a) and latus rectum = 4a
2. $x^2 = –4ay$: Parabola with vertex at (0, 0) and focus at (0, -a) and latus rectum = 4a
3. $y^2 = 4ax$: Parabola with vertex at (0, 0) and focus at (a, 0) and latus rectum = 4a
4. $y^2 = –4ax$: Parabola with vertex at (0, 0) and focus at (-a, 0) and latus rectum = 4a
5. $(x – h)^2 = 4a(y – k)^2$: Parabola with centre at (h, k) and focus at (0 + h, a + k) and latus rectum = 4a

Ellipse:

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: Ellipse with centre at (0, 0) and major axis = 2a and minor axis = 2b .
2. $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$: Ellipse with centre at (h, k) and major axis = 2a and minor axis = 2b .

Hyperbola:

1. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$: Hyperbola with vertex at (a, 0) and (-a, 0) and centre at (0, 0).

2. $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$: Hyperbola with vertex at (0, b) and (0, -b) and centre at (0, 0).

Vectors

There are two types of quantities: scalars and vectors. Scalars are those quantities which only have only magnitude like length, temperature and voltage. Vectors are those quantities which have magnitude as well as direction like Force is a vector which has a magnitude and direction of application and velocity which has a magnitude i.e. speed and the direction of motion.

A vector has a tail i.e. initial point and a tip or terminal point. The length (or magnitude) of a vector a (length of the arrow) is also called the norm (or Euclidean norm) of and is denoted by |a|.

A vector of length 1 is called a unit vector.

Components of a Vector

We choose an xyz Cartesian coordinate system in space, that is, a usual rectangular coordinate axes. Then if a given vector 'a' has initial point P : (x_1, y_1, z_1) and terminal point Q : (x_2, y_2, z_2) the three numbers.

$a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, $a_3 = z_2 - z_1$; are called the components of the vector a with respect to that coordinate system, and we write simply a = $[a_1, a_2, a_3]$

By definition, the length |a| of a vector a is the distance between its initial point P and terminal point Q. Then, $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

If the initial point of the vector is taken as origin, then it is called as Position Vector. The position vector of point A(x, y, z) is a vector with initial point (0, 0, 0) and terminal point (x, y, z) thus, it has the components $a_1 = x$, $a_2 = y$, $a_3 = z$

Vector Addition

The sum a + b of two vectors a = $[a_1, a_2, a_3]$ and b = $[b_1, b_2, b_3]$ is obtained by adding a + b = $[a_1 + b_1, a_2 + b_2, a_3 + b_3]$

Geometrically, the two vectors can be added using Triangle Law or Parallelogram Law. By Parallelogram Law, the resultant vector has a magnitude of,

$$|a+b| = \sqrt{|a|^2 + |b|^2 + 2|a||b|\cos\phi}$$

where, Φ is the angle between the two vectors.

The angle of resultant vector with respect to A is given by,

$$\theta = \tan^{-1}\left(\frac{b\sin\phi}{a + b\cos\phi}\right)$$

Properties:

(a) $\dot{a} + \dot{b} = \dot{b} + \dot{a}$ (commutativity)

(b) $(\dot{U} + \dot{V}) + \dot{W} = \dot{U} + (\dot{V} + \dot{W})$ (associativity).

(c) $\dot{a} + 0 = 0 + \dot{a} = a$

(d) $\dot{a} + (-\dot{a}) = 0$

where −a denotes the vector having the length |a| and the direction opposite to that of a.

Vector Subtraction

To subtract one vector from the other we first reverse the direction of one vector and then add it to another vector.

To perform a − b , we first determine −b which has the same length as the b vector but has opposite direction as compared to b.

By component method, a −b = $[a_1 - b1, a_2 - b_2, a_3 - b_3]$

By parallelogram law, the resultant vector has a magnitude of

$$|a-b| = \sqrt{|a|^2 + |b|^2 - 2|a||b|\cos\phi}$$

The angle of resultant vector with respect to A is given by, $\theta = \tan^{-1}\left(\frac{-b\sin\phi}{a - b\cos\phi}\right)$

Note: Vector Subtraction is not commutative i.e. a − b ≠ b − a

Both these vectors have the same magnitude but have opposite directions and for two vectors to be equal they must have same magnitude as well as direction.

Scalar Multiplication

The product of any vector 1 2 3 a = [a_1, a_2, a_3] and any scalar c (real number c) is the vector obtained by multiplying each component of a by c

ca = [ca_1, ca_2, ca_3]

Geometrically, if a ≠ 0, the ca with c > 0 has the direction a and with c < 0 the direction opposite to a. In any case, the length of ca is ca | ca | = | c | | a |, and ca = 0 if a = 0 or c = 0 (or both).

Unit Vectors

Any vector whose length is 1 is a unit vector i, j, k are example of special unit vectors, which are along x, y and z coordinate axes.

| i | = | j | = | k | = 1

In two dimensions, u = cosθi + sinθj

In three dimensions, u = sinθcosθi + sinθsinθ j + cosθk

The unit vector in the direction of any vector has the same direction as the original vector but has a magnitude of 1.

The unit vector in the direction of a, $a = \dfrac{a}{|a|}$.

Inner Product of two Vectors

The inner product or dot product a.b (read "a dot b") of two vectors a and b is the product of the lengths times the cosine of their angle.

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\phi$ ...(i)

The angle ϕ, 0 ≤ ϕ ≤ π, between a and b is measured when the vectors have their initial points coinciding, as in fig. below.

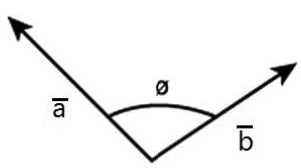

Fig. 2.22

In terms of components, a = [a_1, a_2, a_3], b = [b_1, b_2, b_3], and $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ Since the cosine in (i) may be positive, zero or negative, so may be the inner product.

If the angle between two vectors is 90° i.e. both vectors are orthogonal to each other, then the inner product or dot product of the vectors will be zero.

If the vector is taken as inner product with itself, then Φ = 0.

Then, $\vec{a} \cdot \vec{a} = |\vec{a}|\,|\vec{a}|\cos 0 = |\vec{a}|^2$

Thus, length of a vector can be determined from the dot product,

$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$

Similarly, the angle between the two vectors can be calculated by means of inner product,

$$\cos\phi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Vector product (cross product)

The vector product (cross product) a × b of two vectors a = [a_1, a_2, a_3] and b = [b_1, b_2, b_3] is a vector.

v = a × b | a | | b | sinϕ ň where Φ is the angle between a and b.

Here, ň is a unit vector in a direction perpendicular to both a and b.

The direction of cross product can be determined by Right Hand Thumb Rule. We keep the fingers in the direction of 'a' and then curl them in the direction of 'b', then the direction of thumb represents the direction of cross product.

If a and b have the same or opposite direction or if one of these vectors is the zero vectors, then v = a × b = 0. In any other case, v = a × b has the length.

$|\vec{v}| = |\vec{a}||\vec{b}|\sin\phi$

In components, 1 2 3 v = [v_1, v_2, v_3] = a × b is

$v_1 = a_2b_3 - a_3b_2$

$v_2 = a_3b_1 - a_1b_3$

$v_3 = a_1b_2 - a_2b_1$

In terms of determinants,

$$v_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix};\ v_2 = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix};\ v_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Hence v = [v_1, v_2, v_3] = v_1i, v_2i, v_3k is the expansion of the symbolical third-order determinant

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

A unit vector perpendicular to two given vectors a and b is given by

$$n = \frac{a \times b}{|a| b \# \sin\gamma} = \frac{a \times b}{|a \times b|}$$

Properties

Vector Product has the property that for every scalar l,

(la) × b = l(a × b) = a × l(b)

It is distributive with respect to vector addition, that is,

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{c}\right)$$

$$\left(\vec{a} + \vec{b}\right) \times \vec{c} = \left(\vec{a} \times \vec{c}\right) + \left(\vec{b} \times \vec{c}\right)$$

It is not commutative but anti-commutative, that is,

b × a = –(a × b)

It is not associative, that is,

a × (b × c) ≠ (a × b) × c

So that the parentheses cannot be omitted

Applications of Cross Product

- Area of triangle OAB = $\frac{1}{2}\left|\overline{OA} \times \overline{OB}\right| = \frac{1}{2}\left|\bar{a} \times \bar{b}\right|$

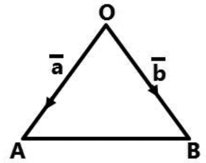

Fig. 2.23

- Area of triangle ABC =

$$\frac{1}{2}\left|\overline{AB} \times \overline{AC}\right| = \frac{1}{2}\left|\left(\bar{b} \cdot \bar{a}\right) \times \left(\bar{c} \cdot \bar{a}\right)\right|$$

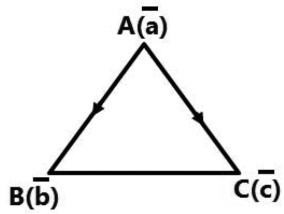

Fig. 2.24

- Area of parallelogram = $|\bar{a} \times \bar{b}|^2$

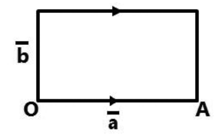

Fig. 2.25

Vector Products of the Standard Basis Vectors

Since i, j, k are orthogonal (mutually perpendicular) unit vectors. the definition of vector product gives some useful formulas for simplifying vector products in right – handed coordinates these are

i × j = k   j × k = i   k × i = j

j × i = –k   k × j = –i   i × k = –j

Scalar Triple Product

The scalar triple product or mixed triple product of three vectors.

a = [a_1, a_2, a_3]; b = [b_1, b_2, b_3]; c = [c_1, c_2, c_3]

Is denoted by (a, b, c) and is defined by (a b c) and is defined by (a b c) = a(b × c)

We can write this as a third – order determinant

For this we set b × c = v = [v_1, v_2, v_3]. Then from the dot product in components we obtain

$$a \cdot (b \times c) = a \cdot v = a_1 v_1 + a_2 v_2 + a_3 v_3 =$$

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \left(- \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} \right) + a_3 \begin{vmatrix} b_1 & b_2 \\ c_2 & c_2 \end{vmatrix}$$

The expression on the right is the expansion of a third-order determinant by its first row.

Thus (a b c) = a · (b × c) = $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

The absolute value of the scalar triple product is the volume of the parallelepiped with a, b, c as edge vectors Also, a · (b × c) = (a × b) · c = c × (a × b) i.e. the value of triple product depends upon the cycle order of the vectors, but is independent of the position of dot and cross.

However if the order is non –cycle, then value changes. i.e., a · (b × c) ≠ b · (a × c)

Note: Three vectors form a linearly independent set if and only if their scalar triple product is not zero. The scalar triple product is the most important "repeated product". Other repeated product exist, but are used only occasionally.

Vector Triple Product

If a, b and c are three vectors then the vector triple product is written as a × (b × c)

It can be proved that $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

Similarly, $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{c}\cdot\vec{a})\,\vec{b}-(\vec{c}\cdot\vec{b})\,\vec{a}$

Gradient of a Scalar Field

Gradient is an operator by which we can derive vector fields from the scalar fields.

The gradient grad f of a given scalar function f(x, y, z) is the vector function defined by

$$\text{grad } f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$$

Here we must assume that f is differentiable. It has become popular, particularly with physicists and engineers, to introduce the differential operator.

$$\nabla=\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k$$

(read nabla or del) and to write

$$\text{grad } f=\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$$

In multiple dimensions, the derivative can be defined in multiple directions and thus the concept of directional derivative arises.

Directional Derivative

The rate of change of f at any point P in any fixed direction given by a vector b is defined as in calculus. We denote it by $D_b f$ or df/ds, call it the directional derivative of f at P in the direction of b. To determine the directional derivative we take the component of gradient in the direction where directional derivative is to be computed.

Thus, $D_b f=\dfrac{b}{|b|}\cdot\text{gradf}$

where, $\dfrac{\vec{b}}{|b|}$ represents the unit vector in the direction of b.

Note: The gradient of a surface at a point represents normal to the surface at that point.

Gradient also represents the direction of maximum rate of increase of the scalar field.

Solved Examples

Example: The unit vector normal to the surface $xy^3z^2 = 4$ at (–1, –1, 2) is ______________.

Solution: Let $\phi = xy^3z^2$

$$\nabla\phi=\hat{i}\left[y^3z^2\right]+\hat{j}\left[3xy^2z^2\right]+\hat{k}\left[2xy^3z\right]$$

At $(-1,-1,2)$

$$\nabla\phi=\hat{i}\left[-1^3\times 4\right]+\hat{j}\left[3\times -1\times 1\times 4\right]+\hat{k}\left[2\times -1\times -1^3\times 2\right]$$

$$=-4\hat{i}-12\hat{j}+4\hat{k}$$

$$\nabla\phi=4\hat{i}-12\hat{j}+4\hat{k}$$

$$\frac{\nabla\phi}{|\nabla\phi|}=\frac{\hat{i}-3\hat{j}+\hat{k}}{\sqrt{1+9+1}}=\frac{i-3j+k}{\sqrt{11}}$$

Example: The sphere of unit radius is centered at origin. Then a unit vector normal to the surface of the sphere at any point P(x, y, z) is the vector.

Solution: The given equation of the sphere is $x^2 + y^2 + z^2 = 1$

Let $\phi = x^2 + y^2 + z^2$

$$\nabla\phi = 2xi + 2yj + 2zk$$

$$\frac{\nabla\phi}{|\nabla\phi|}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\nabla\phi}{|\nabla\phi|}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{1}$$

Note: Normal to the sphere with center at origin at any point P(x, y, z) is the position vector of the point itself.

Example: The directional derivative of $f = xy^2z$ at (1, –1, 1) in the direction of the vector $\bar{a}=\hat{i}+\hat{j}-2k$ is ______________.

Solution: $\nabla f=\hat{i}\left(y^2z\right)+\hat{j}\left(2xyz\right)+\hat{k}\left(xy^2\right)$

$$\nabla f\big|_{(1,-1,1)}=\hat{i}-2\hat{j}+k$$

$$\therefore D_{fa}=\nabla f\cdot\frac{\bar{a}}{|\bar{a}|}=\left(\hat{i}-2\bar{j}+\hat{k}\right)\cdot\frac{\left(\hat{i}+\hat{j}-2k\right)}{\sqrt{1+1+4}}$$

$$=\frac{1-2-2}{\sqrt{6}}=-\frac{3}{\sqrt{6}}$$

Example: The directional derivative of $\phi = xy^2 + yz^2 + zx^2$ at (1, 1, 1) along the direction of tangent to the curve $x = t$, $y = t^2$, $z = t^3$ is ______________.

Solution: The position vector of any point on the curve is,

$$r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

$$\frac{dr}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

At (1, 1, 1) it $t = 1$

The tangent is given by,

$$\frac{dr}{dt} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\nabla\phi = \hat{i}\left(y^2 + 2xz\right) + \hat{j}\left(z^2 + 2xy\right) + \hat{k}\left(x^2 + 2yz\right)$$

$$\nabla\phi\big]_{(1,1,1)} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Thus, directional derivative in the direction of tangent is,

$$\therefore D = \nabla\phi \cdot \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|} = \frac{3+6+9}{\sqrt{1+4+9}} = \frac{18}{\sqrt{14}}$$

Example: Let $f = x^{2/3} + y^{2/3}$ be a scalar point function. Then the derivative of f, along the line $y = x$ directed away from the origin at the point (8, 8) is ______________.

Solution: The given scalar field is,

$$f = x^{2/3} + y^{2/3}$$

$$\bar{r} = x\hat{i} + y\hat{j} = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}$$

$$\frac{\bar{r}}{|r|} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

Here $\theta = \frac{\pi}{4}$ since the given line is $y = x$

$$\hat{e} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

Gradient of scalar field is,

$$\nabla f = \hat{i}\left(\frac{2}{3}x^{-1/3}\right) + \hat{j}\left(\frac{2}{3}y^{-1/3}\right)$$

$$\nabla f\Big|_{(8,8)} = \frac{1}{3}\hat{i} + \frac{1}{3}\hat{j}$$

$$\therefore D = \nabla f.\hat{e} = \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$$

Example: The angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at (2, –1, 2).

Solution: Let $\phi_1 = x^2 + y^2 + z^2$

$$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla\phi_1\big]_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

Let $\phi_2 = x^2 + y^2 - z$

$$\nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla\phi_2\big]_{(2,-1,2)} = 4\hat{i} + 2\hat{j} - \hat{k}$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{16+4-4}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Divergence

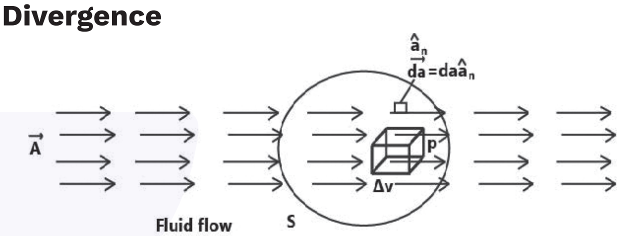

Fig. 2.26

Fluid flow=inlet outward flux per unit volume when volume shrinks to a point.

Closed surface always encloses a volume

$$\underset{\Delta Lt \to 0}{Lt} \frac{\oiint_s \vec{A} \cdot \vec{da}}{\Delta V} = \text{Div}\,\vec{A} = \nabla \cdot \vec{A}$$

The different cases of outflow and inflow have been shown below,

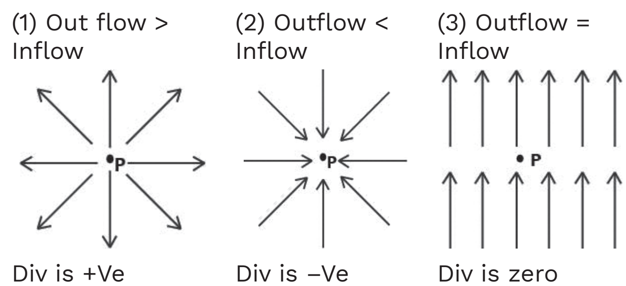

Fig. 2.27

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{d}{dx}\hat{a}_x + \frac{d}{dy}\hat{a}_y + \frac{d}{dz}\hat{a}_z\right) \cdot \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right)$$

$$\vec{\nabla} \cdot \vec{A} = \frac{dA_x}{dx} + \frac{dA_y}{dy} + \frac{dA_z}{dz}$$

Always a scalar quantity

If $\nabla.\vec{A} = 0 \Rightarrow \vec{A}$ is known as a "solenoidal fields"

Curl of a Vector Field

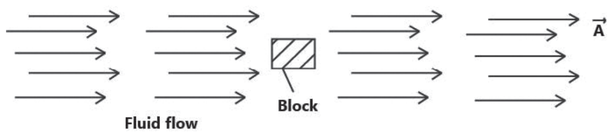

Fig. 2.28

If net rotational effect on block is present then field is solid to have curl.

$$\underset{\Delta S\to 0}{\text{Lt}}\left(\frac{\oint_c \vec{A}\cdot\vec{dl}}{\Delta S}\right) \overset{\Delta}{=} \nabla\times\vec{A} \overset{\Delta}{=} \text{Curl of } \vec{A}$$

Net circulation per unit area when area shrinks to zero.

The curl in terms of graphical representation of different fields is shown below

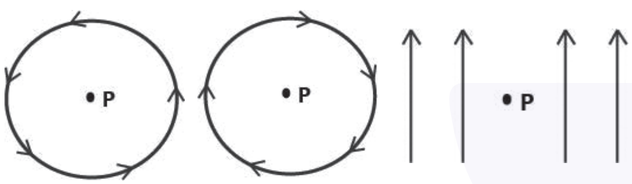

Fig. 2.29

The curl can be computed by the determinant shown below,

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ d/dx & d/dx & d/dx \\ A_x & A_y & A_z \end{vmatrix}$$

If $\nabla\times\vec{A} = 0 \Rightarrow \vec{A}$ is known as a "Irrotational field" or "Conservative field"

Note: If $\vec{v} \to$ Linear velocity

$\vec{\omega} \to$ Angular velocity

Then $\vec{V} = \vec{\omega}\times\vec{r}$

curl $\vec{v} = \nabla\times(\vec{\omega}\times\vec{r}) = 2\vec{\omega}$

Thus, $\vec{\omega} = \frac{1}{2}$ curl $\vec{v}$

Important Relations

1. div grad $f = \nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
2. curl grad $f = \nabla\times\nabla f = 0$
3. div curl $f = \nabla\cdot(\nabla\times F) = 0$
4. curl curl f = grad div $F - \nabla^2 F = \nabla(\nabla.F) - \nabla^2 F$
5. grad div f = curl curl $F + \nabla^2 F = \nabla\times\nabla\times F + \nabla^2\times F$

Solved Examples

Example: Suppose $\bar{F} = 4x^2 z\hat{i} - (7y^2 + 2xz)\hat{j} + 3yz^2\hat{k}$ represents a velocity vector

(i) Then Div. of $\bar{F}$ at (1, –1, 2) is ____________

(ii) Corresponding angular velocity at (2, 1, –2) is ______________.

Solution: The divergence of velocity is given by,

$$\text{Div }\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 8xz + (-14y) + 6yz$$

At (1, –1, 2)

Div $\vec{F} = 16 + 14 - 12 = 18$

Angular Velocity is given by, $\vec{\omega} = \frac{1}{2}$ Curl $\vec{F}$

$$\text{Curl }\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x^2 z & -(7y^2 + 2xz) & 3yz^2 \end{vmatrix}$$

$$= i[3z^2 + 2x] - j[0 - 4x^2] + \bar{k}[-2z - 0]$$

At (2, 1, –2)

Curl $\bar{F} = 16\hat{i} + 16\hat{j} + 4\hat{k}$

$\therefore \omega = \frac{1}{2}$ Curl $\bar{F} = 8\hat{i} + 8\hat{j} + 2\hat{k}$

Example: The value of λ such that the vector function $\bar{F} = (\lambda x^2 y - yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz + y^2x^2)\hat{k}$ is solenoid is ___________.

Solution: For solenoidal fields the divergence is zero

Thus, div $\bar{F} = 0$

$(2\lambda xy - 0) + (2xy - 0) + (2xy + 0) = 0 \quad \Rightarrow \lambda = -2$

Example: If $\bar{F} = (x - 2y + az)\hat{i} + (bx - 3y + 4z)\hat{j} + (2x + cy - 5z)\hat{k}$ is irrotational then a, b, c is?

Solution: For irrotational fields, Curl $\bar{F} = 0$

$$\text{Curl }\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - 2y + az & bx - 3y + 4z & 2x + cy - 5z \end{vmatrix} = 0$$

$i(c-4) - j(2-a) + k(b+2) = 0$

$c = 4,\ a = 2,\ b = -2$

Example: The directional derivative of $\operatorname{div}\bar{F}$ in the direction of outer normal to the surface of the sphere with curve at origin and radius = 3, where $\bar{F} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$ is

Solution: The outward normal to a surface is the gradient to the surface at that particular point.

$\operatorname{div}\bar{F} = 4x^3 + 4y^3 + 4z^3 = f$

Now, we need to determine the gradient of f.

$\nabla f = 12x^{2\hat{i}} + 12y^2\hat{j} + 12z^2\hat{k}$

Let's take any point on the surface of the sphere P(1,2,2)

$[\nabla f]_{(1,2,2)} = 12\hat{i} + 48\hat{j} + 48\hat{k}$

$\nabla f = 12(\hat{i} + 4\hat{j} + 4\hat{k})$

Example: If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, and $r = |\bar{r}|$ then $\operatorname{div}(r^n\bar{r}) =$ ________________

Solution: $r^n\bar{r} = \underset{F_1}{r^n x}\hat{i} + \underset{F_2}{r^n y}\hat{j} + \underset{F_3}{r^n z}\hat{k}$

$$\operatorname{div}\left(r^n\bar{r}\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}\left(r^n x\right) = r^n(1) + xnr^{n-1}\frac{\partial r}{\partial x}$$

Since, $r = \sqrt{x^2 + y^2 + z^2}$

Thus, $\dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \dfrac{x}{r}$

Thus, $\dfrac{\partial F_1}{\partial x} = r^n + xn^{n-1} \times \dfrac{x}{r}$

$$\frac{\partial F_1}{\partial x} = r^n + xn^{n-1} \times \frac{x}{r}$$

Similarly $\dfrac{\partial F_x}{\partial y} = r^n + nr^{n-2}y^2$ and $\dfrac{\partial F_x}{\partial z} = r^n + nr^{n-2}z^2$

$$\therefore \operatorname{div}\left(r^n r\right) = 3r^n + nr^{n-2}\underbrace{(x^2 + y^2 + z^2)}_{r^2}$$

$$\therefore \operatorname{div}\left(r^n r\right) = 3r^n + nr^n = (n+3)r^n$$

If $n = -3 \operatorname{div}\left(r^n\bar{r}\right) = 0$

$\dfrac{\bar{r}}{r^3}$ is solenoidal

Line Integral of Vector functions

Let $\bar{F}(x, y, z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a different vector function defined along the curve 'c' then its line integral is

$$\int_c \bar{F} \cdot d\bar{r}$$

In Cartesian form, $\int_c \bar{F}.d\bar{r} \int_c \left(F_1dx + F_2dy + F_3dz\right)$

> **Note:** If c is a closed curve then the line integral of $\bar{F}$ along C is called circulation of $\bar{F}$ divided by $\int_c \bar{F}.\overline{dr}$

We represent the curve C by a parametric representation.

$r(t) = [x(t), y(t), z(t)] = x(t)i + y(t)j + z(t)k$

$(a \le t \le b)$

We call C the path of integrating A : r(a) its initial point, and B : r(b) its terminal point. The direction from A to B, in which t increases, is called the positive direction on C. The points A and B may coincide. Then C is called a closed path as initial and final points are identical.

We call C a smooth curve if C has a unique tangent at each of its points whose direction varies continuously as we move along C.

Work done by a force

The total work done by force $\bar{F}$ in moving some particle along the curve c is W.D. = $\int_c \bar{F}.\overline{dr}$

> **Note:** The line integral of an irrotational vector function is independent of the path of the curve.

If $\bar{F}$ is irrotational then $\bar{F} = \nabla\phi$ where ϕ is a scalar potential function. Then $\int_A^B \bar{F}.dr = \phi_B - \phi_A$

For irrotational fields, always there exists a scalar potential function such that $\bar{F} = \nabla\phi$.

This can easily be proved as for irrotational field, $\nabla \times \vec{F} = 0$

By Vector identity, $\nabla \times \nabla\phi = 0$

Thus, $\bar{F} = \nabla\phi$

Solved Examples

Example: The value of $\int_c \overline{F}.d\overline{r}$ where $\overline{F} = 2x^2\hat{y} - y^2xj$

c is the curve $y = 2x^2$ joining the points (0,0) and (1,2) is ?

Solution: $\int_c \overline{F}\,d\overline{r} = \int_c 2x^2ydx - y^2xdy$

On curve C, $y = 2x^2 \Rightarrow dy = 4x\,dx$

$$\int_c \overline{F}.d\overline{r} = \int_0^1 2x^2\left(2x^2\right)dx - \left(4x^4\right)x.4x\,dx$$

$$= \int_0^1 \left[4x^4 - 16x^6\right]dx$$

$$\int_c \overline{F}.d\overline{r} = \frac{4}{5} - \frac{16}{7} = \frac{28-80}{85} = -\frac{52}{85}$$

Example: The value of $\int_c^{\overline{F}} dr$ where $\overline{F} = 3xy\hat{i} - y^2\hat{j}$ and c is the curve bounded by $y = x^2$ and $y = x$ is?

Solution: The curve C can be resolved into two parts C_1 and C_2

$$\int_c \overline{F}\cdot dr = \int_{c_1}(\ \ ) + C\int_{c2}(\ \ )$$

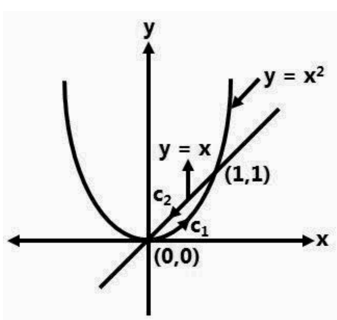

Fig. 2.30

Along c_1: $y = x^2$

$dy = 2x\,dx$

$$\int_{c_1} \overline{F}\,d\overline{r} = \int_c 3xydx - y^2dy = \int_0^1 3x\cdot x^2dx - x^4 \times 2xdx$$

$$\int_{c_1} \overline{F}\,d\overline{r} = \left.\frac{3x^4}{4} - \frac{2x^5}{6}\right]_0^1 = \frac{3}{4} - \frac{2}{6} = \frac{5}{12}$$

Along c_2: $y = x$ and x goes from 1 to 0 as we traverse C_2

$dy = dx$

$$\int_{c_1} \overline{F}\,d\overline{r} = \int_c 3xydx - y^2dy = \int_1^0 3x^2dx - x^2dx$$

$$\int_{c_2} \overline{F}\,d\overline{r} = \left.\frac{3x^3}{3} - \frac{x^3}{3}\right]_1^0 = -\frac{2}{3}$$

$$\therefore \int_c \overline{F}\cdot d\overline{r} = \frac{5}{12} - \frac{2}{3} = \frac{5-8}{12} = \frac{-1}{4}$$

Note: The line integral of an irrotational vector function over any closed curve is zero because line integral is independent of path and depends on end points. But if end points are identical, then the line integral is zero.

Example: The value of $\int_c (3x+4y)dx + (2x-3y)$ dy where c is a circle with center at the origin and radius as 2 in x - y plane is?

Solution: The equation of circle in x-y plane with center at origin and radius 2 is,

$x^2 + y^2 = 4$

Parameterizing the curve C,

Let $x = 2\cos t$ & $y = 2\sin t$

$dx = -2\sin t dt$ and $dy = 2\cos t dt$

$$\int_c (3x+4y)dx + (2x-3y)dy = \int_0^{2\pi} (6\cos t + 8\sin t)$$

$$(-2\sin t dt) + (4\cos t - 6\sin t)(2\cos t dt)$$

$$I = \int_0^{2\pi} (-24\sin t\cos t - 16\sin^2 t + 8\cos^2 t)\,dtI$$

$$= \int_0^{2\pi} (-12\sin 2t - 8(1-\cos 2t) + 4(1+\cos 2t))dt$$

$$= \int_0^{2\pi} [-12\sin 2t - 4 + 12\cos 2t]dt$$

$$I = \left.\frac{12\cos 2t}{2} - 4t + \frac{12\sin 2t}{2}\right]_0^{2\pi}$$

$$I = -4 \times 2\pi = -8\pi$$

Example: Determine the value of

$$\int_c \bar{F}\cdot d\bar{r}, \bar{F} = (2y+3)\hat{i} + xz\hat{j} - 3y^2 z\hat{k}$$ and c is the line joining the points.

i) (0, 0, 1) to (0, 1, 1)

ii) (0, 1, 1) to (2, 1, 1)

Solution: (i) In this line, the values of x and z are constant x = 0,z = 1

dx = 0,dz = 0

But, y : 0 to 1

$$\int_c \bar{F}\cdot \overline{dr} = \int_c \bar{F}_1 dx + \bar{F}_2 dy + \bar{F}_3 dz = \int_0^1 xzdy = \int_0^1 0dy = 0$$

(ii) On this line, the values of y and z are constant y = 1, z = 1, dy = 0,dz = 0, x : 0 to 2

$$\therefore \int_c^{\bar{F}} \cdot d\bar{r} = \int_0^2 F_1 dx = \int_0^2 (2y+3)dx = \int_0^2 5dx = 10$$

Example: The total work done by the force $\bar{F} = \left(3x^2+6y\right)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ in moving a particle along the straight line joining the set of points (0, 0, 0) & (1, 1, 2) is ________.

Solution: The equation of the line joining two points in three dimensions is given by,

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$$

Substituting the points we get,

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{2-0} = t$$

$$x = t, y = t, z = 2t$$

$$dx = dt, dy = dt, dz = 2dt$$

$$\text{W.D.} = \int_c \bar{F}d\bar{s} = \int_c \left(3x^2+6y\right)dx - 14yzdy + 20xz^2dz$$

$$= \int_0^1 \left(3t^2+6t\right)dt - 28t^2dt + 160t^3dt$$

$$\text{W.D.} = \left.\frac{3t^3}{3} + \frac{6t^2}{2} - \frac{28t^3}{3} + 160\frac{t^4}{4}\right]_0^1$$

$$= \frac{-25}{3} + \frac{6}{2} + \frac{160}{4} = 43 - \frac{25}{3}$$

$$\text{W.D.} = \frac{104}{3}$$

Example: Find

$\int \bar{v}\cdot dr$ where $\bar{v} = y\hat{i} + (xz+1)\hat{j} + xy\hat{k}$ from (0, 1, 0) to (2, 1, 4).

(a) 7

(b) 8

(c) 9

(d) Cannot be determined without specifying the path.

Solution: Here path is not specified check whether the function is irrotational. If it is irrotational proceed to line integral as line integral is independent of the path.

If not irrotational, the line integral cannot be determined without the specified path.

$$\text{Curl } \bar{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = i(x-x) - j(y-y) + k(z-z) = 0$$

Thus, $\bar{V}$ is irrotational. Hence, it can be expressed as gradient of scalar potential function, $\bar{v} = \nabla\phi$

If we traverse from (a, b, c) to (x, y, z)

$$\phi = \int_a^x F_1(x,y,z)dx + \int_b^y F_2(a,y,z)dy + \int_c^z F_3(a,b,z)dz$$

This integral represents that we first travel in x-direction keeping y and z constant and then in y-direction keeping x and z constant and lastly in z-direction keeping x and y as constant.

$$\phi = \int_a^x yzdx + \int_b^y (az+1)dy + \int_c^z abdz = yz[x]_a^x +$$

$$\left.(az+1)y\right]_b^y + \left.abz\right]_c^z$$

$$\phi = xyz - ayz + ayz + y - a\,|\,bz - b\,| + abz - abc$$

$$\phi(x,y,z) = xyz + y + k$$

$$\therefore \int \bar{v}\cdot dr = \phi(2,1,4) - \phi(0,1,0) = (8+1+k) - (0+1+k) = 8$$

Green's Theorem in a Plane

Let M(x, y) and N(x, y) be the continuous function having constant first order partial derivatives defined in a closed region 'R' bounded by closed curve 'c' then

$$\int_C Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]dxdy$$

Solved Examples

Example: Determine the value of $\int_c e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$ where c is a rectangle with vertices $(0,0),(\pi,0),\left(\pi,\frac{\pi}{2}\right),\left(0,\frac{\pi}{2}\right)$.

Solution: The curve C and the region bounded by the curve are shown in the figure below,

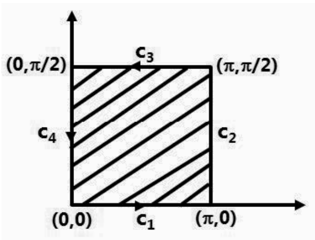

Fig. 2.31

$$M = e^{-x}\sin y \Rightarrow \frac{\partial M}{\partial y} = e^{-x}\cos y$$

$$N = e^{-x}\cos y \Rightarrow \frac{\partial N}{\partial x} = -e^{-x}\cos y$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -2e^{-x}\cos y$$

$$\therefore \int M dx + N dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dxdy$$

$$I = \int_0^{\pi/2}\int_0^{\pi} -2e^{-x}\cos y\,dxdy = \int_0^{\pi/2} -2\cos y\left[\frac{e^{-x}}{-1}\right]_{00}^{\pi} dy$$

$$I = 2\left(e^{-\pi}-1\right)[\sin y]_0^{\pi/2} = 2\left[e^{-\pi}-1\right]$$

Example: Find $\int_c (y-\sin x)dx + \cos x\,dy$ where c is the curve bounded by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$

Solution: Considering the vertical strip.

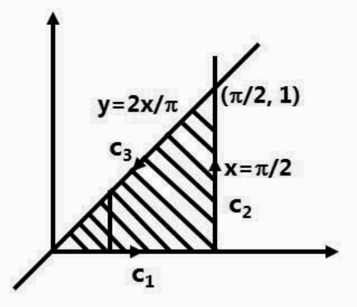

Fig. 2.32

$$y = 0 \text{ to } y = \frac{2x}{\pi}$$

$$x = 0 \text{ to } x = \frac{\pi}{2}$$

$$\frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = -\sin x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -\sin x - 1$$

$$I = \int_0^{\pi/2}\int_0^{2x/\pi} -(1+\sin x)dydx = \int_0^{\pi/2} -(1+\sin x)\frac{2x}{\pi}dx$$

Apply Integration by Parts,

$$I = \frac{-2}{\pi}\left[\frac{x^2}{2} + x(-\cos x) - (1)(-\sin x)\right]_0^{\pi/2} = \frac{-2}{\pi}\left[\frac{\pi^2}{8}+1\right]$$

Example: Determine the value of $\int_c \left(3x^2 - 8y^2\right)dx + (4y - 6xy)dy$ where C is the curve bounded by $y = \sqrt{x}$ and $y = x^2$

Solution: The closed curve and region bounded by the two curves is shown in the figure below, Consider the vertical strip.

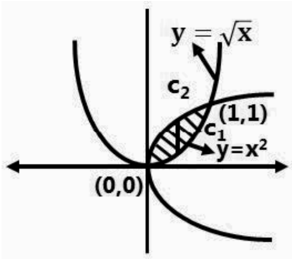

Fig. 2.33

$$y = x^2 \text{ to } y = \sqrt{x}$$

$$x = 0 \text{ to } x = 1$$

$$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$$

$$\int_0^1\int_{x^2}^{\sqrt{x}} 10y\,dydx = 10\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = 5\int_0^1\left(x - x^2\right)dx =$$

$$5\left[\frac{1}{2}-\frac{1}{3}\right] = \frac{5}{6}$$

Surface Integral

Let 1 2 3 $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a differentiable vector function defined over a surface s, then its surface integral is $\int_s \bar{F}\cdot\overline{ds}$ or $\int_s \bar{F}\cdot\bar{n}\,ds$.

Where $\bar{n} \to$ unit outward drawn normal to the surface S.

In Cartesian form

$$\int_s \bar{F}\cdot\bar{n}\,ds = \int\int_s \left[F_1 dydz + F_2 dxdz + F_3 dxdy\right]$$

To calculate the value of Surface Integral, we have to take the projection of region R onto xy plane or yz plane or xz plane.

If R_1 is the projection of 's' onto xy plane then

$$\int_s \bar{F}\,\bar{n}\,ds = \int\int_{R_1} \bar{F}\cdot\bar{n}\frac{dxdy}{|\bar{n}\cdot\bar{k}|}$$

If R_2 is the projection of 's' onto yz plane then

$$\int_s \bar{F}\,\bar{n}\,ds = \int\int_{R_2} \bar{F}.\bar{n}\frac{dydz}{|\bar{n}.\bar{i}|}$$

If R^3 is the projection of 's' onto xz plane then

$$\int_s \bar{F}.\bar{n}\,ds = \int\int_{R_2} \bar{F}.\bar{n}\frac{dxdz}{|\bar{n}.\bar{j}|}$$

Solved Examples

Example: The value of $\int_s \bar{F}.\bar{n}$ ds, where $\bar{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S = surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5 is ________________.

Solution: The cylinder in the first octant is shown in the figure below,

Let $\phi = x^2 + y^2$

$\nabla\phi = 2x\hat{i} + 2y\hat{j}$

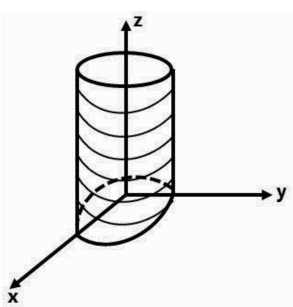

Fig. 2.34

The normal vector to the surface is unit vector in the direction of gradient

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\bar{n} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j}}{4}$$

$$\bar{F}.\bar{n} = \frac{xz}{4} + \frac{xy}{4} = \frac{x}{4}(z + y)$$

Let R be the projection of S onto yz plane.

$$\int_s \bar{F}.\,\bar{n}ds = \iint_R \bar{F}\cdot\bar{n}\frac{dydz}{|\bar{n}\cdot\hat{i}|} = \iint_R \frac{x}{4}(y+z)\frac{dydz}{x/4}$$

$$I = \int_{z=0}^{5}\int_{y=0}^{4}(y+z)dydz = \int_{z=0}^{5}\left[\frac{y^2}{2} + zy\right]_0^4 dz$$

$$I = \int_0^5 [8 + 4z]dz = 8z + \frac{4z^2}{2}\Bigg]_0^4 = 40 + 50$$

$$I = 90$$

Example: The value of $\int_s \bar{F}.\,\bar{n}ds$ where $\bar{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is a surface of the cube bounded by x = 0, x = 1, y = 0, y = 1 and z = 0, z = 1 is ______________.

Solution: To integrate over the entire surface of cube, we can resolve the integral into 6 parts one over each surface of the cube.

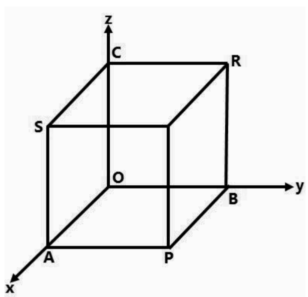

Fig. 2.35

$$\int_s \bar{F}.\,\bar{n}ds = \int_{s_1} + \int_{s_2} \ldots\ldots\ldots\ldots\ldots + \int_{s_1}$$

Over S_1: In xy plane (OAPB)

$z = 0, \bar{n} = -\bar{k}$

$\bar{F}.\bar{n} = -yz = 0$(Since, z = 0 for OAPB)

$\int_{s_1} \bar{F}.\bar{N}\,ds = 0$

Over S_2: parallel to xy plane (SQRC)

$z = 1, \bar{n} = \bar{K}, \bar{F}.\bar{n} = yz = y$

$\int_s \bar{F}.\bar{n}\,ds = \int_s y\,ds = \iint_R y\frac{dxdy}{|\bar{n}\cdot\bar{K}|} = \int_0^1\int_0^1 y\,dydx = \frac{1}{2}$

Over S_3: In yz plane (OBRC)

$x = 0, \bar{n} = -\bar{i}, \bar{F}.\bar{n} = -4xz = 0$

$\int_s \bar{F}.\bar{n}\,ds = 0$

Over S_4: Parallel to yz plane (APQS)

$x = 1, \bar{n} = \bar{i}, \bar{F}.\bar{n} = 4xz = 4z$

$\int_{s_4} \bar{F}.\bar{n}\,ds = \int_0^1\int_0^1 4z\,dzdy = 2$

Over S_5: In xz plane (OCSA)

$y = 0, \bar{n} = -\bar{j}, \bar{F}.\bar{n} = y^2 = 0$

$\int_{s_5} \bar{F}.\bar{n}\,ds = 0$

Over S_6: Parallel to xz plane (BRQP)

$y = 1, \bar{n} = \bar{j}\ \bar{F}.\bar{n} = -y^2 = -1$

$\int_{s_6} \bar{F}.\bar{n}\,ds = \int_{s_6} -1\,ds = -1$

$\therefore \int_s \bar{F}.\bar{n}\,ds = 0 + \frac{1}{2} + 0 + 2 + 0 - 1 = \frac{3}{2}$

Or if we use Gauss's Divergence Theorem (to be discussed later)

$\int_S \bar{F}.\bar{n}\,ds = \int_V \text{div}\,\bar{F}\,dv = \int_V (4z - 2y + y)\,dv =$

$\int_0^1\int_0^1\int_0^1 (4z - y)\,dzdydxz$

$\int_S \bar{F}.\bar{n}\,ds = \frac{3}{2}$

Volume Integral

Let ϕ (x, y, z) be a differentiable scalar to and let $\bar{F}$ (x, y, z) be a differentiable vector function defined over a region whose volume bounded is V.

Then the volume integration are $\int_V \phi(x,y,z)dv$

Or, $\int_V \bar{F}(x,y,z)dv = \int_V F_1 dv + j\int_V F_2 dv + k\int_V F_3 dv$

Solved Examples

Example: Find $\int_V (2x + y)$ dv where V is the volume bounded by x = 0,x = 2, y = 0, y = 2, z = x ,z = 4

Solution:

$$\int_V (2x+y)\,dv = \int_{y=0}^{2}\int_{x=0}^{2}\int_{z=x^2}^{4} (2x+y)\,dzdxdy$$

$$I = \int_0^2\int_0^2 (2x+y)\left(4-x^2\right)dxdy = \int_0^2\int_0^2 \left[8x - 2x^3 + 4y - yx^2\right]dxdy$$

$$I = \int_0^2 \left[8.\frac{x^2}{2} - \frac{2x^4}{4} + 4yx - y.\frac{x^3}{3}\right]_0^2 dy = \int_0^2 \left[16 - 8 + 8y - \frac{8y}{3}\right]dy$$

$$I = \int_0^2 \left[8 + \frac{16y}{3}\right]dy = 8y + \left.\frac{16y^2}{6}\right]_0^2$$

$$I = 16 + \frac{32}{3} = \frac{80}{3}$$

Example: The volume of an object expressed in spherical coordinates is given by

$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2 \sin\theta\,drd\phi d\theta$ then the value of V is _____________.

Solution: Integrating with respect to r

$$I = \int_0^{2\pi}\int_0^{\pi/3}\left[\sin\phi\frac{r^3}{3}\right]_0^1 d\phi d\theta = \int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}\sin\phi d\phi d\theta$$

$$I = \frac{1}{3}\int_0^{2\pi}[-\cos\phi]_0^{\pi/3}\,d\theta = \frac{1}{3}\int_0^{2\pi}\left(-\frac{1}{2}+1\right)d\theta$$

$$I = \frac{1}{3}\times\frac{1}{2}\times 2\pi = \frac{\pi}{3}$$

Gauss Divergence Theorem

Let S be a closed surface enclosing a volume 'V' and $\bar{F}(x,y,z)$ be a differentiable vector function defined over the surface S.

Then, $\int_s \bar{r}.\bar{n}\,ds = \int_s \operatorname{div}.\bar{F}\,dv$

In Cartesian form.

$$\int_s F_1 dydz + F_2 dxdz + F_3 dxdy = \int_s \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] dv$$

Solved Examples

Example: The value of $\int_s \bar{r}.\bar{N}\,ds$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and S is the closed surface enclosing a volume V is ?

Solution: To apply Gauss's Divergence Theorem,

$$\int_s \bar{F}.\bar{n}\,ds = \int_s \operatorname{div}\bar{F}\,dv$$

$$\operatorname{div}\bar{r} = 1+1+1 = 3$$

$$\int_s \bar{r}\,\bar{N}ds = \int_v \operatorname{div}\bar{r}dv = \int_v 3dv = 3V$$

Example: The value of $\int_s xdydz + ydxdz + zdxdy$ where S is the surface of

i. Cylinder $x^2 + y^2 = 9$, $y = 0$, $y = 4$

ii. Sphere $x^2 + y^2 + z^2 = 16$

Solution: The given vector function is,

$\vec{F} = xi + yj + zk$

Divergence of given vector function is,

$$\nabla \cdot \vec{F} = 1+1+1 = 3$$

$$\int_v (1+1+1)dv = 3V$$

i) For the case of cylinder,

$3V = 3\times\pi r^2 h = 3\times\pi\times 3^2 \times 4 = 108$

ii) For the case of sphere,

$\pi 3V = 3\times\frac{4}{3}\times\pi r^3 = 4\pi\times(4)^3 = 256\pi$

Example: The value of $\int_s \left(x^2 + 2y + 3z^2\right)dS$ where S is the surface of a unit sphere with center at the origin.

Solution: The equation of sphere with center at origin and radius 1.

$$x^2 + y^2 + z^2 = 1$$

$$\bar{N} = \frac{\nabla\phi}{|\nabla\phi|} = \hat{i} + y\hat{j} + z\hat{k}$$

Let $\bar{F} = \bar{F}_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\therefore \bar{F}\cdot\bar{n} = \bar{F}_1\hat{i} + \bar{F}_2\hat{j} + \bar{F}_3\hat{k} = x^2 + 2y^2 + 3z^2$

$\bar{F}_1 = x, \bar{F}_2 = 2y, \bar{F}_3 = 3z$

$\therefore \bar{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$

$\operatorname{div}\bar{F} = 1+2+3 = 6$

$$\therefore \int_s \left(x^2 + 2y^2 + 3z^2\right)ds = \int_V \operatorname{div}\bar{F}dV = \int_V 6dV$$

$$\therefore \int_s \left(x^2 + 2y^2 + 3z^2\right)ds = 6V = 6\times\frac{4}{3}\pi r^3 = 8\pi(1)^3 = 8\pi$$

Example: Value of $\int_s \bar{F}.\bar{n}ds$ where $\bar{F} = 8x^2z\hat{i} - y^2\hat{j} + 2xz^2\hat{k}$ and S is the surface bounded by $0\le x\le 1$, $0\le y\le 2$, $0\le z\le 3$ is ______________.

Solution: $\operatorname{div}\bar{F} = 16xz - 2y + 4xz = 20xz - 2y$

$$\int_s \bar{F}.\bar{n}ds = \int_V \operatorname{div}\bar{F}\,dV = \int_V (20xz - 2y)dV$$

$$\int_s \bar{F}.\bar{n}ds = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} (20xz - 2y)dzdydx$$

$$= \int_0^1\int_0^2 \left[20x.\frac{9}{2} - 2y\times 3\right]dydx$$

$$\int_s \bar{F}.\bar{n}ds = \int_0^1 [90x\times 2 - 6.2]dx = \int_0^1 (180x - 12)dx$$

$$\int_s \bar{F}.\bar{n}ds = 90 - 12 = 78$$

Example: The value of $\int_S \bar{F}\cdot\bar{n}ds$ where $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ where S is the surface bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$ is ______________.

Solution: The divergence of given vector field,

$\operatorname{div}\bar{F} = 4 - 4y + 2z$

$$\int_S \bar{F}.\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv = \int_V (4 - 4y + 2z)\,dV$$

$$\int_S \bar{F}.\bar{n}\,ds = \int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - 4y + 2z)\,dydxdz$$

Let $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$|J| = r, r = 0 \text{ to } 2$

$\theta = 0 \text{ to } 2\pi$

$z = 0 \text{ to } 3$

$$\int_S \bar{F}.\bar{n}ds = \int_0^3\int_0^{2\pi}\int_0^2 (4 - 4r\sin\theta + 2z)rdrd\theta dz$$

$$\int_S \bar{F}.\bar{n}ds = \int_0^3\int_0^{2\pi}\left[4\times\frac{r^2}{2} - 4\sin\theta\cdot\frac{r^3}{3} + 2z\cdot\frac{r^2}{2}\right]_0^2 d\theta dz$$

$$= \int_0^3\int_0^{2\pi} 8 - \frac{32}{3}\sin\theta + 4z \; d\theta dz$$

$$\int_S \bar{F}.\bar{n}ds = \int_0^3 \left[(8+4z)\theta - \frac{32}{3}(-\cos\theta)\right]_0^{2\pi} dz$$

$$\int_S \bar{F}.\bar{n}ds = \int_0^3 (8+4z)2\pi dz = 2\pi\left[8z + 2z^2\right]_0^3$$

$$= 2\pi[24 + 18] = 84\pi$$

Stokes Theorem

This theorem is used to convert Line Integral to Surface Integral and vice versa.

Let S be an open surface bounded by a closed curve 'c' and $\bar{F}(x, y, z)$ be a differentiable vector function defined along the curve 'c' then $\int_c \bar{F}\cdot d\bar{r} = \int_S \text{Curl}\,\bar{F}\cdot\bar{n}\,ds$

$$\int_c F_1dx + F_2dy + F_3dz = \int_S (\nabla\times\bar{F}).\bar{n}\,ds$$

In two dimensions, $\text{Curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

Thus, this theorem will reduce to Green's Theorem.

Solved Examples

Example: The value of $\int \bar{F}d\bar{r}$ where $\bar{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$ and C is the curve bounded by x + y = 2, x = 0, y = 0 in xy plane is ____________.

Solution: The curl of given vector field is,

$$\text{Curl}\;\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$\nabla\times\bar{F} = \hat{i}(x - x) - \hat{j}(y - y) + \hat{k}(z - z)$$

Curl $\bar{F} = 0 \Rightarrow \bar{F}$ is irrotational

$$\int \bar{F}d\bar{r} = \int_s Curl\bar{F}\cdot\bar{n}ds = \int_s \bar{0}\cdot\bar{n}ds = 0$$

Example: The value of $\int \bar{F}d\bar{r}$, where $\bar{F} = -y^3\hat{i} + x^3\hat{j}$ and C is the boundary of circular disc $x^2 + y^2 \le 1$, $z = 0$

Solution: The curl of given vector field is,

$$\text{Curl}\;\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}\left(3x^2 + 3y^2\right)$$

$$\therefore\; \text{Curl}\;\bar{F} = 3\left(x^2 + y^2\right)\bar{k}$$

The normal vector to the given surface area is,

$\bar{n} = \bar{k}$

$$\text{Curl}\;\bar{F}\cdot\bar{n} = 3\left(x^2 + y^2\right)$$

$$\int_c \bar{F}\cdot d\bar{r} = \int_s Curl\bar{F}\cdot\bar{n}ds = \int_3 3\left(x^2 + y^2\right)ds$$

$$= \iint_R 3\left(x^2 + y^2\right)dxdy$$

Changing the co-ordinate system to polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$

$|J| = r, x^2 + y^2 = r^2$

$$\int_c \bar{F}.d\bar{r} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 3r^3drd\theta = 3\times\frac{1}{4}\times 2\pi = \frac{3\pi}{2}$$