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Graph Theory

Introduction

Graph

A graph G is defined by $G = (V, E)$ where V is set of all vertices in G and E is set of all edges in G.

- **Null Graph:** A graph with no edges is called null graph.
- **Directed Graph:** In a digraph an edge (u, v) is said to be from u to v.
- **Undirected Graph:** In a undirected graph an edge $\{u, v\}$ is said to join u and v or to be between u and v.
- **Isolated Vertex**: A vertex with degree zero is called as Isolated vertex or lone vertex.
- [redacted]: A vertex with degree one is called as Pendent vertex.
- **Pendent Edge:** It is an edge which incident with pendent vertex.
- **Path:** It is the sequence of edges, without vertex repetition.
- [redacted]: It is a graph with only one source and one sink.
- [redacted]): It is the sequence of edges without edge repetition (vertex may repeat).
- **Independence Number:** Number of vertices in largest maximal independent set.
- **Diameter of a Graph:** Maximum distance between any two vertices in a graph.
- **Loop:** An edge drawn from a vertex to itself.
- [redacted]: A graph with one vertex and no edges.
- [redacted]: A graph with only isolated vertices and no edges.
- **Pseudo Graph:** A graph in which self loops are allowed as well as parallel or multiple edges are allowed.
- **Simple Graph:** A graph with no loops and no parallel edges is called a simple graph.
- [redacted]  [redacted], [redacted] = 10

- Girth = size of shortest cycle
- **Hand Shaking Theorem:** Indegree = Outdegree
- $\underset{\text{min degree}}{\delta} \le \left\lfloor \frac{2e}{n} \right\rfloor \le \underset{\text{max degree}}{\Delta}$
- **Complete Bipartite Graph:** (m, n)
  Diameter = 2, Chromatic = 2,
  Number of vertex = $m + n$, Number of edges = $m \times n$

Note:

- Maximum n[illegible]er of edges possible in a simple graph with n-vertices = $n(n-1)/2 = {}^nC_2$.
- [illegible].
- [illegible].
- Number of edges disjoint Hamiltonian cycle = $\frac{n-1}{2}$ i.e. for even no edge disjoint Hamiltonian cycle.
- [illegible]s = [illegible].
- **Hand Shaking Theorem:** Let $G = (V, E)$ be a non-directed graph with $V = \{V_1, V_2, \ldots V_n\}$. Then $\sum_{i=1}^{n} \deg(V_i) = 2|E|$.
- In any graph the number of vertices with odd degree is always even.
- If degree of each vertex is k then such a graph is called k-regular graph and in such a graph $|E| = \frac{k|V|}{2} = \frac{nk}{2}$ (where $|V| = n$).
- If degree of each vertex is atleast k i.e. the minimum degree = k, then $|E| \ge \frac{k|V|}{2}$.

- **Regular Graph:** A graph in which all vertices have same degree. If degree of each vertex is 'k' then it is "***k*-regular Graph**".
- **Complete Graph:** A simple graph with "***n*-mutually adjacent vertices**" is complete graph, represented by K_n.

Note:

- Each vertex in K_n has degree $n-1$
- Number of edges in $K_n = |E(K_n)| = n(n-1)/2$
- Every complete graph is regular but converse need not be true.

- **Cycle Graph:** A simple graph with n-vertices ($n \geq 3$) and n-edges is called a cycle graph if all the edges form a cycle of length n. 2-chromatic = even cycle, 3-chromatic = odd cycle.
- **Wheel Graph:** A wheel graph (w_n) with n-vertices ($n \geq 4$) is a simple graph obtained from c_{n-1}, by adding a new vertex which is adjacent to all vertices of c_{n-1}. Diameter = 2, Hamiltonian graph, 3 color = n = even, 4 color = n = odd, $n+1$ vertex.

  Number of edges $= |E(w_n)| = 2n$
- **Cut Vertex (Articulation Point):** A vertex whose removal makes the [illegible].
- [illegible]d.
- **Cut Set:** It is [illegible], provided no proper subset of these edges disconnects the graph.
- ***n* cube:** Number of vertex $= 2^n$, number of edges $= n \times 2^{n-1}$. 2 colors needed.

Note:

- One or more cut sets may exist in the connected graph.
- Whenever a cut edge exists, cut vertex also exist in graph because atleast one vertex of the cut edge is a cut vertex but the vice versa is not true.
- [illegible].

Edge Connectivity

Number of edges in a smallest cutset of G is called edge connectivity of G. It is also the minimum number of edges whose removal disconnects the graph.

[illegible]

A [illegible]

Weakly Connected

A digraph is weakly connected if the underlying undirected graph (obtained by removing all the arrows in directed graph) is connected.

Vertex Connectivity

Minimum number of vertices whose removal results in a disconnected graph or reduces it to a trivial graph.

k-connected Graph

On removal of k-vertices, the connected graph becomes disconnected.

[illegible]

On removal of k-edges, the connected graph becomes disconnected.

Non-separable Graph

Graph with no cut vertices and hence no cut edges (bridges).

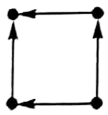
Strongly Connected

In digraph, if a path exist between any vertex to any other vertex i.e. for two given vertices u and v $\exists$ a path from u to v as well as from v to u.

[illegible]

In digraph, if for every two vertices u and v there is a path from u to v or from v to u (not necessarily both)

Weak Graph



Some vertex has indegree but not out degree so vertex not reach to each other.

Note: ....

- [illegible] $\delta \leq \left\lfloor \frac{2e}{n} \right\rfloor \leq \Delta$

  [illegible] ee   Max [illegible]

- [illegible] if

  $$|E| > \frac{(n-1)(n-2)}{2}$$

- A simple graph with n-vertices and k component has atleast $(n-k)$ [illegible].

- A simple graph G with n vertices, k components has atmost [(n−k [illegible] es.

Tree

A connected graph with no cycle is called a tree.

Spanning Tree

It is a tree and subgraph to a graph 'G' which includes all vertices of 'G'.

- A tree with 'n' vertices has $n-1$ edges.
- [illegible] $\frac{^{2n}C_n}{n+1} = \frac{2n!}{n!(n+1)!}$.
- k-trees (forest) with total n-vertices have $(n-k)$ edges.
  Number of spanning trees for $k_n = n^{n-2}$.
- "Number of edges that must be removed" from connected graph with n-vertices and e-edges to produce a spanning tree is called 'circuit rank of graph'; Circuit Rank or nullity or cyclomatic complexity $= e-(n-1)$ edges.
- [illegible] of graph with 'n' vertices, 'e' edges and [illegible] edges.
- [illegible] 1.
- A finite tree (with atleast one edge) has atleast two vertices of degree '1'.
- [illegible]

$$= \frac{(n-2)!}{(d_1-1)!(d_2-1)!\ldots(d_n-1)!}.$$

Bipartite Graph

In bipartite graph, Vertex set V of a graph is divided into two vertex sets V_1 and V_2, such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$.

- [illegible] $\left\lfloor \frac{n^2}{4} \right\rfloor$.
- It is either acyclic or contains only even length cycles.

Complete Bipartite Graph

A bipartite graph $G = (V, E)$ with vertex partition $V = \{V_1, V_2\}$ is complete bipartite graph, if every vertex in V_1 is adjacent to every vertex in V_2.

- A complete bipartite graph ($K_{m,n}$) has ($m + n$) vertices and mn edges.
- A complete bipartite graph $K_{m,m}$ is a regular graph of degree m.

Planar Graph

Planar graph is a graph or a multigraph that can be drawn in a plane or sphere such that it's edges do not cross.

- $\sum_{i=1}^{n}$ [illegible]
- $\sum_{i=1}^{n}$ [illegible] where r_i are the regions.
- If degree of each region is K then $K \cdot |R| = 2|E|$
- If degree of each region is atleast 3 then $3|R| \le 2|E|$
- For simple planar graph :

  (*i*) **Euler's formula:**

  $\boxed{|R| = |E| - |V| + 2}$ if graph is connected

  $\boxed{|R| = |E| - |V| + (k+1)}$ with 'k' components

  (*ii*) **For connected planar simple graph:** $\boxed{|E| \le \{3|V| - 6\}}$

  (*iii*) **For connected planar simple graph with no triangles:** $\boxed{|R| \le \{2|V| - 4\}}$

- If $K_{3,3}$ and K_5 homomorphic fusion (degree = 1 vertex) subgraph then not planner = Kuratowski's.
- **For disconnected graph:** $n - k \le e \le \dfrac{(n-k)(n-k+1)}{2}$, $k \ge n - e$
- **For connected graph:** $n - 1 \le e \le \dfrac{n(n-1)}{2}$
- There exists atleast one vertex $v \in G$ such that degree $(v) \le 5$.
- $K_{m,n}$ is planner $\Leftrightarrow$ ($m \le 2$ or $n \le 2$)
- K_n is planner $\Leftrightarrow n \le 4$

- A non planar graph with minimum number of vertices is K_5.
- A non planar graph with minimum number of edges is $K_{3,3}$.

Polyhedral Graph

A simple connected planar graph in which every interior region is a polygon of same degree and degree of every vertex deg $(V) \geq 3 \quad \forall V \in G$.

- $\boxed{3|V| \leq 2|E|}$
- $\boxed{3|R| \leq 2|E|}$

Complementary Graph

Complement of a graph G denoted by $\bar{G}$ is also a simple graph with same vertices as of G, and two vertices are adjacent in $\bar{G}$ iff the two vertices are not adjacent in G.

- $\boxed{G \cup \bar{G} = K_n}$
- $\boxed{|E(G)| + |E(\bar{G})| = |E(K_n)| = \frac{n(n-1)}{2}}$

Isomorphic Graphs

Two graphs G and G^* are isomorphic, if there is a function $f: V(G) \rightarrow V(G^*)$ such that f is bijection and "for each pair of vertices u and v of G: $\{u, v\} \in E(G)$ iff $\{f(u), f(v)\} \in E(G^*)$".

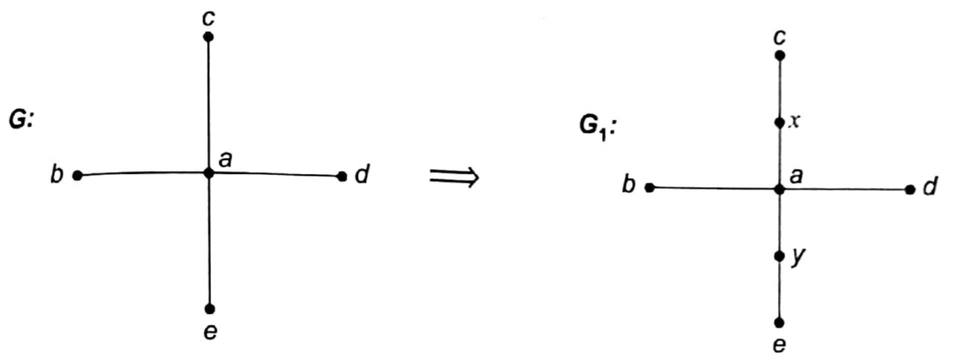
- Two graphs are isomorphic, iff their complements are isomorphic.
- If G and $\bar{G}$ are isomorphic then
  (*i*) The number of vertices in G and G' are same.
  (*ii*) The number of edges in G and G' are same.
  (*iii*) The degree sequence of G and G' are same.
  (*iv*) The number of cycles of every length in G and G' are same.
- If G is a simple graph such that $G \cong \bar{G}$ then G is said to be "self complementary".
- **In a self-complementary graph:**

$\boxed{|E(G)| = \frac{n(n-1)}{4}\text{; where } n \text{ is number of vertices in } G}$

Homomorphism

A graph G_1 is said to be homomorphic to G if G_1 can be obtained by dividing some edge(s) of G.

Example: G_1 is homomorphic to G is shown in the following:



{a, c} of G is divided into {c, x} and {x, a} edges.
{e, a} of G is divided into {e, y} and {y, a} edges.

Coloring or Proper Coloring

Vertices of a graph G are colored such that no two adjacent vertices [illegible] r.

Chromatic Number $\chi(G)$

Minimum number of colors needed for vertex coloring of graph G is called *chromatic number.*

- Chromatic number of $K_n = \chi(K_n) = n$
- [illegible]. i.e. a non-empty graph is bichromatic iff it is Bipartite.
- [illegible]
- [illegible] e
- Equivalence relation between vertices of the same color in a connected graph gives the chromatic partition.
- [illegible] even length cycle
- [illegible] s odd length cycle

Matching

- [illegible].
- [illegible].

$$\boxed{\text{degree}(v) \le 1, \quad \forall v \in G}$$

Maximal Matching

Maximal matching is a maximal matching in which no edge of the graph can be added to it.

Maximum Matching

Maximum matching is a matching with maximum number of edges.

Matching Number

The number of edges in maximum matching of the graph.

Perfect Matching

Every vertex of the matching contain exactly one degree. i.e., every vertex is incident with exactly one edge. i.e. A matching which is also a covering is called perfect matching.

$$\boxed{\begin{array}{l}\text{degree}(v) = 1,\ \forall v \in G \\[6pt] \text{Number of perfect matchings for } K_{2n} = \dfrac{(2n)!}{2^n \times n!}\end{array}}$$

K [illegible].

Complete Matching

In a bipartite graph having a vertex partition V_1 and V_2. A complete matching of vertices in a set V_1 into those of V_2 is a matching such that every vertex in V_1 is matched against some certain vertex in V_2, such that no two vertices of V_1 is matched against a single vertex in V_2.

Covering

It is set of edges where every vertex of graph incident with atleast one edge in 'G' [deg $(v_i) \geq 1$]; $\forall v_i \in G$.

Note:

- A line covering of a graph with n-vertices has atleast $n/2$ edges.
- No minimal line covering can contain a cycle and the components of a minimal cover are always stargraphs and from a minimal cover no edge can be deleted.

Minimal Covering

It is covering in which no deletion of an edge is possible while still covering the vertices.

Minimum Covering

Smallest (less number of edges) minimal covering is minimum covering.

Covering Number

The number of edges in minimum covering is covering number.

Traversable Multigraph

If there is a path in graph which includes all the vertices and uses each edge exactly once (i.e. the graph has either Euler cycle or Euler trail) then such graph is traversable.

Eulerian Graph

If a graph contains "closed traversable trial or Euler circuit" (it may repeat vertices), then it is Eulerian Graph. When all vertex of even degree.

Note:

- A graph G is traversable, if number of vertices with odd degree in the graph is exactly zero or two.
- Euler path exists but Euler Circuit doesn't exist iff the number of vertices with odd degree is exactly two.
- Euler Circuit exists but Euler path does not exist iff number of vertices with odd degree is 0.

Hamiltonian Path

If there exists a path which contains each vertex of the graph exactly once, then such a path is called as Hamiltonian path.

Hamiltonian Cycle

It is Hamiltonian path where first and last vertices are sam ... t

edge then not Hamiltonian.

DIRAC

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ORES

... ent vertex.

■■■