# Mathematical Logic

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## Introduction

- **Proposition:** It is a declarative statement either TRUE or FALSE.
- **Compound Proposition:** It is a proposition formed using the logical operators (Negation (¬), Conjunction (∧), Disjunction (∨), etc.) with the existing propositions.
- Logical Operators:

  (*i*) Negation of $p$ : $\neg p$ or $\bar{p}$ or $\sim p$

  (*ii*) Conjunction of $p$ and $q$ : $p \wedge q$

  (*iii*) Disjunction of $p$ and $q$ : $p \vee q$

  (*iv*) Implication/Conditional : $p \rightarrow q$ (if $p$, then $q$)

  (*v*) Bi-conditional : $p \leftrightarrow q$
- Precedence order of logical operators from high to low: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- $P \oplus R = PR' + P'R$, $P \leftrightarrow R = P'R' + PR$
- Number of distinct boolean expression with $n$ variable = $2^{2^n}$.
- **Normal form:** PCNF (∨) = (POS = 0), PDFL (∧) = (SOP = 1)

  Total size = $2^n$ with $n$ variable.

**Note:** ..........................................................................................

- Converse of $p \rightarrow q$ is : $q \rightarrow p$
- [illegible] $\neg q$
- ( [illegible]

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## Tautology

If compound proposition is always true then it is tautology.

*Example:* $p \vee \neg p$

## Contradiction

[illegible]on.

*Example:* $p \wedge \neg p$

## Contingency

[illegible]n.

*Example:* $p$

## Logical Equivalence

$P \Leftrightarrow Q$ is tautology iff $P$ and $Q$ are logically equivalent.

## Functionally Complete

If any formula can be written as an equivalent formula containing only the connectives in a set of operators, then such a set of operators is called as functionally complete.

*Example:*

$\{\uparrow\}, \{\downarrow\}, \{\neg, \vee\}, \{\neg, \wedge\}, \{\neg, \vee, \wedge\}$ are functionally complete (NAND).

## Consistent

If $H_1 \wedge H_2 \wedge H_3 \wedge \ldots \wedge H_n$ is satisfiable then $H_1, H_2, \ldots$ and $H_n$ are consistent (Tautology, contingency but not contradiction).

## Inconsistent

If $H_1 \wedge H_2 \wedge H_3 \wedge \ldots \wedge H_n$ is unsatisfiable then $H_1, H_2, \ldots$ and $H_n$ are inconsistent (only contradiction).

- [redacted]
- Sufficient ($\rightarrow$), necessary ($\leftarrow$), [redacted]
- $p \rightarrow q \equiv q$ unless $\neg p =$ [redacted] either $p$ is not true or $q$ is true.
- [redacted] $p \wedge q$.

## Equivalences

$$P \vee (P \wedge Q) \equiv P$$

[redacted]

$$P \wedge (P \vee Q) \equiv P$$

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$$

$$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

$$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$$

[redacted]

$$\neg(P \leftrightarrow Q) \equiv P \leftrightarrow (\sim Q) \equiv (\sim P) \leftrightarrow Q \equiv P \oplus Q$$

[redacted]

$$P \vee Q \equiv \neg P \rightarrow Q$$

$$P \wedge Q \equiv \neg(P \rightarrow \neg Q)$$

$$\neg(P \rightarrow Q) \equiv (P \wedge \neg Q)$$

Identity Laws : (*i*) $P \wedge T = P$, (*ii*) $P \vee F = P$

Domination Laws : (*i*) $P \vee T = T$, (*ii*) $P \wedge F = F$

Idempotent Laws : (*i*) $P \wedge P = P$, (*ii*) $P \vee P = P$

Commutative Laws :

(*i*) $P \vee Q = Q \vee P$

(*ii*) $P \wedge Q = Q \wedge P$

Associative Laws :

(*i*) $(P \vee Q) \vee R = P \vee (Q \vee R)$

(*ii*) $(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$

Distributive Laws :

(*i*) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

(*ii*) $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

Demorgan's Laws :

(*i*) $\neg(P \wedge Q) = \neg P \vee \neg Q$

(*ii*) $\neg(P \vee Q) = \neg P \wedge \neg Q$

Absorption Laws :

(*i*) $P \vee (P \wedge Q) = P$

(*ii*) $P \wedge (P \vee Q) = P$

Negation Laws :

(*i*) $P \vee \neg P = T$

(*ii*) $P \wedge \neg P = F$

Double Negation Laws : $\neg(\neg P) = P$

## Rules of Inference (Tautological Implications)

Simplification :

$$(P \wedge Q) \Rightarrow P$$

$$(P \wedge Q) \Rightarrow Q$$

Addition :

$$P \Rightarrow (P \vee Q)$$

$$Q \Rightarrow (P \vee Q)$$

Disjunctive Syllogism : $(\sim P, P \vee Q) \Rightarrow Q$

Modus Ponens : $(P, P \rightarrow Q) \Rightarrow Q$

Modus Tollens : $(\sim Q, P \rightarrow Q) \Rightarrow \sim P$

Hypothetical Syllogism : $(P \rightarrow Q, Q \rightarrow R) \Rightarrow (P \rightarrow R)$

Conjunctive Syllogism : $((P \vee Q), P) \Rightarrow \sim Q$

Dilemma : $(P \vee Q, P \rightarrow R, Q \rightarrow R) \Rightarrow R$

Constructive Dilemma : $(P \vee Q, P \rightarrow R, Q \rightarrow S) \Rightarrow R \vee S$

Destructive Dilemma : $(\sim R \vee \sim S, P \rightarrow R, Q \rightarrow S) \Rightarrow \sim P \vee \sim Q$

Other rules :

$$\sim P \Rightarrow (P \rightarrow Q)$$

$$Q \Rightarrow (P \rightarrow Q)$$

$$\sim(P \rightarrow Q) \Rightarrow P$$

$$\sim(P \rightarrow Q) \Rightarrow \sim Q$$

Exactly one = $\exists!$ or $\exists x\,[P(x) \wedge P(y) \Rightarrow y = x]$, $\forall x[\exists y\,(B(x, y) \wedge (B(x, z) \rightarrow y = z)$

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**Principle Conjunctive Normal Form (PCNF)**

Product of sums (max term)

PCNF: $[P(x_1) \vee P(x_2)] \wedge [P(x_3) \vee P(x_4)]$

**Principle Disjunctive Normal Form (PDNF)**

Sums of products (min term)

PDNF: $[P(x_1) \wedge P(x_2)] \vee [P(x_3) \wedge P(x_4)]$

Number of non equivalent propositional functions with $n$-propositional variables are = $2^{2^n}$.

- $\forall x\,(\alpha \rightarrow \beta) \Rightarrow (\forall x\,\alpha \Rightarrow \forall x\,\beta)$ true only with properties always use and but not $\rightarrow$.

## Predicate Logic

### Quantifiers

- **Universal ($\forall$)** : "for all" or "for every"
- **Existential ($\exists$)** : "there exist"

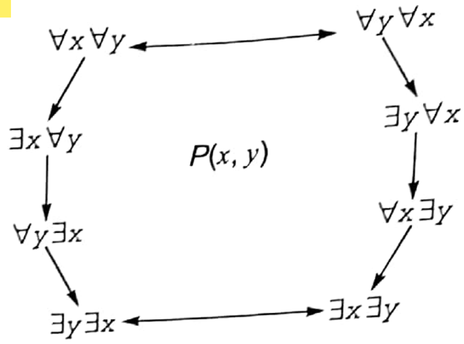
### Predicates

- $P(x)$: Propositional statement with one variable.
- $Q(x, y)$: Propositional statement with two variables.

**Note:**

- $\neg \exists x\, P(x) = \forall x\, \neg P(x)$
- $\neg \forall x\, P(x) = \exists x\, \neg P(x)$

## Logical Equivalences




1. $\forall x\,[P(x) \wedge Q(x)] \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$
2. $\exists x[P(x) \vee Q(x)] \equiv \exists x\, P(x) \vee \exists x\, Q(x)$
3. $\forall x\,(P(x) \vee Q) \equiv \forall x\, P(x) \vee Q$
4. $\forall x\,(P(x) \wedge Q) \equiv \forall x\, P(x) \wedge Q$
5. $\exists x\,(P(x) \vee Q) \equiv \exists x\, P(x) \vee Q$
6. $\exists x\,(P(x) \wedge Q) \equiv \exists x\, P(x) \wedge Q$
7. $\forall x\, P(x) \wedge \exists y Q(y) \equiv \forall x \exists y[P(x) \wedge Q(y)]$
8. $\forall x\, P(x) \vee \exists y Q(y) \equiv \forall x \exists y[P(x) \vee Q(y)]$

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