Numerical Methods

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Linear Algebraic Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1}$$

LU Decomposition or LU Factorization or Triangularization (Doolittle's or Crout's triangularisation Method)

- Principle minors of A are non-singular.

$$AX = B \Rightarrow LUX = B \quad [\text{since } A = LU]$$

- $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \text{ (Doolittle's method)}$$

- $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \text{ (Crout's method)}$$

Order of Computation in Doolittle's Method:

In Doolittle's method the equations are set-up row-wise for A's elements.

(a) First row of $A \Rightarrow u_{11} = a_{11}; \; u_{12} = a_{12}; \; u_{13} = a_{13}$

(b) Second row of $A \Rightarrow l_{21} = \frac{a_{21}}{a_{11}}; \; u_{22} = a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12}; \; u_{23} = a_{23} - l_{21}u_{13}$

(c) Third row of $A \Rightarrow$

$$l_{31} = \frac{a_{31}}{a_{11}}; \; l_{32} = \frac{1}{u_{22}}\left[a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{12}\right]; \; u_{33} = a_{33} - l_{31} \cdot u_{13} - l_{33}u_{23}$$

Similarly, in Crout's method the equations are set-up column-wise for A's elements.

(v) Error is order of h^2 and error term is h^3

(vi) If for the same integration limits, tabulation (step size) is halved then error is reduced by factor '4'.

2. **Simpson's Rule:**

(i) It gives exact result when polynomial of degree ≤ 3.

(ii) If we are evaluating $I = \int_a^b f(x)dx$ by Simpson's rule then

$$\boxed{\text{Error} = -\frac{h^5}{90} \cdot f^{iv}(\xi) \times n_i} \quad (a \leq \xi \leq b)$$

(iii) Simpson's rule is more accurate as compare to trapezoidal rule since it is a fourth order method as compared to trapezoidal rule which is second order.

(iv) Simpson's 1/3rd Rule:

$$\boxed{\int_{x_0}^{x_n} ydx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + ... + y_{n-2}) + 4(y_1 + y_3 + ... + y_{n-1})]}$$

(v) Simpson's 3/8 Rule:

$$\boxed{\int_{x_0}^{x_n} ydx = \frac{3h}{8}[(y_0 + y_n) + 2(y_3 + y_6 + ... + y_{n-3}) + 3(y_1 + y_2 + ... + y_{n-1})]}$$

(vi) If tabulation (step size) is halved, the error in Simpson's rule evaluation of the integral is reduced by a factor of '16'.

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