# Linear Algebra

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## Matrix

**Principal Diagonal:** In a square matrix all elements $a_{ij}$ for which $i = j$ are elements of principal diagonal.

## Matrices

1. **Upper Triangular matrix:** A square matrix in which all the elements below the principle diagonal are zero.
2. **Lower Triangular Matrix:** A square matrix in which all the elements above the principle diagonal are zero.
3. **Diagonal Matrix:** A square matrix in which all the elements other than the elements of principle diagonal are zero.
4. **Scalar Matrix:** *A* diagonal matrix with all elements of principle diagonal being same.
5. **Idempotent Matrix:** '$A$' is square matrix $\ni A^2 = A$.
6. **Involutary Matrix:** '$A$' is square matrix $\ni A^2 = I$.
7. **Nilpotent Matrix:** '$A$' is square matrix $\ni A^m = 0$ where $m$ is the least positive integer and $m$ is also called as Index of class of Nilpotent matrix $A$.
8. **Transpose Matrix:** $A^T$ is transpose matrix of matrix $A$. $A^T$ can be obtained by switching the rows as columns and columns as rows of $A$.
9. **Symmetric Matrix:** '$A$' is a square matrix $\ni A^T = A$.
10. **Skew-Symmetric Matrix:** '$A$' is a square matrix $\ni A^T = -A$.
11. **Orthogonal Matrix:** '$A$' is a orthogonal matrix $\ni A^T = A^{-1}$ or $AA^T = I = A^TA$.
12. **Conjugate Matrix of $A$ ($\bar{A}$) or ($\sim A$) :** '$A$' is any matrix, by replacing the elements by corresponding conjugate complex numbers the matrix obtained is conjugate of '$A$'.

    ***Example:***

$$A = \begin{bmatrix} 2+3i & 4+7i & 5 \\ 2i & 3 & 9-i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 2-3i & 4-7i & 5 \\ -2i & 3 & 9+i \end{bmatrix}$$

13. **Transpose Conjugate Matrix ($A^\theta$) or ($A^*$):** $(\bar{A})^T$.

14. **Hermitian Matrix:** '$A$' is a square matrix $\ni$ $A^\theta = A$

All diagonal elements of hermitian matrix are real number and all off-diagonal elements above and below the principle diagonal must be conjugate of each other. i.e. $a_{ij} = \overline{a_{ji}}$.

Example: $\begin{bmatrix} 2 & 3-4i \\ 3+4i & 5 \end{bmatrix}$

15. **Skew-Hermitian Matrix:** '$A$' is a square matrix $\ni$ $A^\theta = -A$

All diagonal elements of Skew-Hermitian matrix are purely imaginary or zero and all off-diagonal elements above and below the principle diagonal must be conjugate of each other with opposite sign. i.e. $a_{ij} = -\overline{a_{ji}}$.

Example: $\begin{bmatrix} 2i & 3-4i \\ -3-4i & 5i \end{bmatrix}$

16. **Unitary Matrix:** '$A$' is a square matrix $\ni$ $A^\theta = A^{-1}$ or $A\,A^\theta = I = A^\theta A$
17. **Boolean Matrix:** Any matrix with only elements '0' or '1'
18. **Sparse Matrix:** A matrix '$A$' in which more number of elements are zeros.
19. **Dense Matrix:** A matrix which is not sparse.
20. **Singular and Non-singular Matrix:** A square matrix '$A$' is singular if $|A| = 0$, and non singular if $|A| \neq 0$. Only non-singular matrices have inverse.
21. **Adjoint Matrix:** Transpose of cofactors matrix. i.e. Adj $(A)$ = (Cof $(A))^t$

## Properties of Matrices

- $A + B = B + A$ (Commutative)
- $(A + B) + C = A + (B + C)$ (Associative)
- $AB \neq BA$ (Not commutative)
- $(AB)\,C = A(BC)$ (Associative)
- $A(B + C) = AB + AC$ (Distributive)
- $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$
- $A(Adj\,A) = (Adj\,A)\,A = |A|I_n$
- $Adj\,(AB) = (Adj\,B).\,(Adj\,A)$

- $A^{-1} = \dfrac{Adj\ A}{|A|}; |A| \neq 0$
- $(A^{-1})^{-1} = A$ and $(A^{-1})^T = (A^T)^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$.
- If $A$ is a square matrix of order $n$ then $|Adj\ A| = (\det A)^{n-1} = |A|^{n-1}$ and

  $|Adj(Adj\ A)| = |A|^{(n-1)^2}$
- If $|A| \neq 0$ then $|A^{-1}| = \dfrac{1}{|A|}$
- If $A_{n \times n}$ matrix then $|KA| = K^n|A|$
- If $A$ is square matrix then
  (*i*) $A + A^T$ is always symmetric
  (*ii*) $A - A^T$ always skew-symmetric
- If $A$ and $B$ are symmetric then
  (*i*) $A + B$ is also symmetric.
  (*ii*) $A - B$ is also symmetric.
  (*iii*) $AB + BA$ is symmetric
  (*iv*) $AB - BA$ is skew-symmetric
  (*v*) $A^n$ and $B^n$ are symmetric
- If $A$ and $B$ are skew-symmetric then,
  (*i*) $A + B$ is also skew-symmetric.
  (*ii*) $A - B$ is also skew-symmetric.
  (*i*) $A^n$ and $B^n$ are symmetric, if '$n$' is even
  (*ii*) $A^n$ and $B^n$ are skew-symmetric, if '$n$' is odd
- The determinant of orthogonal matrix and unitary matrix $A$ has absolute value '1'.
- If $A_{m \times n}$ and $B_{n \times p}$ then product of $AB$ requires
  (*i*) $mnp$ multiplications
  (*ii*) $m(n-1)p$ additions
  (*iii*) for each entry, $n$ multiplications and $(n-1)$ additions.
- $(A^T)^T = A$, $(kA)^T = k(A^T)$
- $(A + B)^T = A^T + B^T$, $(AB)^T = B^TA^T$

- **Rank of Matrix ($r(A)$):** It is the order of its largest non-vanishing (non-zero) minor of the matrix.
- Rank is equal to the number of linearly independent rows or columns in the matrix.
- The system of linear equation $AX = B$ has a solution (consistent) iff rank of $A$ = Rank of $(A|B)$
- The system $AX = B$ has
  (*i*) A unique solution iff Rank $(A)$ = Rank $(A|B)$ = Number of variables
  (*ii*) Infinitely many solutions $\Leftrightarrow$ Rank $(A)$ = Rank $(A|B)$ < number of variables
  (*iii*) No solution if Rank $(A) \neq$ Rank $(A|B)$ i.e. Rank $(A)$ < Rank $(A|B)$
- The system $AX = 0$ has
  (*i*) Unique solution (zero solution or trivial solution) if Rank $(A)$ = number of variables
  (*ii*) Infinitely many number of solutions (non-trivial solutions) if Rank $(A)$ < number of variables
- If Rank $(A) = r$, and number of variables $= n$ then, the number of linearly independent infinite solutions of $AX = 0$ is $(n - r)$
- In the system of homogenous linear equation $AX = 0$
  (*i*) If $A$ is singular then the system possesses non-trivial solution (i.e. infinite solution)
  (*ii*) If $A$ is non-singular then the system possesses trivial (zero) solution (i.e. unique solution)
- Rank of a diagonal matrix = Number of non-zero elements in diagonal.
- If $A$ and $B$ are two matrices
  (*i*) $r(A + B) \leq r(A) + r(B)$
  (*ii*) $r(A - B) \geq r(A) - r(B)$
  (*iii*) $r(AB) \leq \min\{r(A), r(B)\}$
- If a matrix $A$ has rank '$R$', then $A$ contains '$R$' linearly independent vectors (row/column)
- The system of homogeneous linear equations such that number of unknowns (or variables) exceeds the number of equations necessarily possesses a non-zero solution.

## Eigen Value

Let '$A$' be a square matrix of order $n$ and $\lambda$ be a scalar then $|A - \lambda I| = 0$ is the characteristic equation of $A$. The roots of characteristic equation are called eigen values/lantent roots/Characteristic roots.

- The set of eigen values of matrix is called "spectrum of matrix"
- A matrix of order $n$ will have $n$ latent roots. not necessarily distinct.

## Eigen Vector

Corresponding to each eigen value $\lambda$, there exists a non-zero solution $X$ such that $(A - \lambda I)X = 0$ then $X$ is eigen vector/latent/vector/characteristic vector of $A$.

### Properties of Eigen Values

- Sum of eigen values of a matrix = sum of elements of principal diagonal (trace).

$$\Sigma\lambda_i = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = \text{Trace of } A$$

- Product of eigen values = Determinant of matrix.

$$\Pi\lambda_i = \lambda_1 \cdot \lambda_2 \cdot \ldots \lambda_n = |A|$$

- If $\lambda$ is eigen value of $A$ then $\frac{1}{\lambda}$ is eigen value of $A^{-1}$. (provided $\lambda \neq 0$ i.e. $A$ is non-singular).
- Eigen values of $A$ and $A^T$ are same
- If $\lambda$ is eigen value of orthogonal matrix then $\frac{1}{\lambda}$ is also its eigen value $[\because \; A^T = A^{-1}]$
- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of matrix '$A$', then
  - (*i*) $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ are eigen values of matrix $A^m$.
  - (*ii*) $\lambda_1 + K, \lambda_2 + K, \ldots, \lambda_n + K$ are eigen values of $A + KI$
  - (*iii*) $(\lambda_1 - K)^2, (\lambda_2 - K)^2, \ldots, (\lambda_n - K)^2$ are eigen values of $(A - KI)^2$
  - (*iv*) $K\lambda_1, K\lambda_2, \ldots, K\lambda_n$ are eigen values of $KA$.
- The eigen values of symmetric matrix are real.
- The eigen values of skew-symmetric matrix are either purely imaginary or zero.
- The modulus of the eigen values of orthogonal and unitary matrices = 1.
- If a matrix is either lower or upper triangular or diagonal then the principal diagonal elements themselves are the eigen values.
- Zero is eigen value of a matrix iff the matrix is singular.

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