4 Complex Variables

Objective

Upon completion of this chapter you will be able to:

- Analyze the function of complex variable.
- Understand the basics of analytic functions
- Determine the line integrals of complex functions
- Determine the line integrals by Residue Method
- Determine the singularities of complex functions

Introduction

Complex analysis, traditionally known as the theory of complex variables, is the branch of mathematical analysis that investigates the functions of complex numbers. Complex numbers are ordered pairs of real numbers (x, y). Two complex numbers are said to be equal if they are exactly same i.e. (x, y) = (u, v) which implies x = u and y = v. A complex function is one in which the dependent as well as independent variables are complex numbers or we can say that the domain and range of complex functions is the subset of the complex plane.

Complex Number

If x, y are two real numbers and 'i' is an imaginary unit such that $i^2 = -1$ or $i = \sqrt{-1}$ then the number of the form z = x + iy is called complex number.

Therefore, $z = x + iy$ where $x = \text{Re}(z)$ & $y = \text{Im}(z)$
If $z = x + iy$ then $z = x - iy$.

If $z = x + iy$ is a complex number then
$|z| = |x + iy| = \sqrt{x^2 + y^2}.\sqrt{\cdot}$ This is called as the magnitude of a complex number.

Complex exponential can be represented as, $e^{i\theta -}= \cos\theta + i\sin\theta$. The magnitude of this exponential is always 1. To represent any general complex number, amplitude and phase terms can be combined together as,

$$z = x + iy = re^{i\theta} = r\left\{\cos\theta\right\}, \text{ where } r\left|z\right.$$

$$= \sqrt{x^2 + y^2}, \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

θ = argument of $z = x + iy$.

If $z = x + iy$ and $z_0 = x_0 + iy_0$ are two complex numbers then the distance between z and z_0 is given by $|z - z_0|$ or $|z_0 - z|$

$$\therefore |z - z_0| = |x + iy - (x_0 + iy_0)| = |$$

$$(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

The equation of a circle in rectangular co-ordinates is,

$x^2 + y^2 = r^2$ or $r = \sqrt{x^2 + y^2}$

Here, 'r' represents distance of (x + iy) from origin.

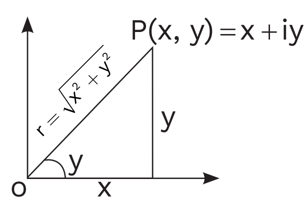
This equation in complex form can be expressed as, |z| = r.

This equation represents locus of all the points which are at a constant distance from the origin. Thus it is an equation of circle with center at origin and radius r.

The equation of a circle with center at (x0 , y0) and radius 'r' is,

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

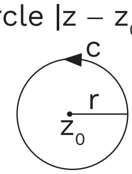
$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$$



$|z - z_0| = r$ is the equation of a circle with center at z_0 and radius r.

$|z - z_0| < r$ represents a set of all points lying within the circle $|z - z_0| = r$.

$|z - z_0| > r$ represents a set of all points lying outside the circle $|z - z_0| = r$.



Complex Function

If A and B are two sets of complex numbers and every element of the form z=x+iy in a set A is associated with the unique element of the form w=u+iv in a set B, then w=u+iv is called complex function of a complex variable z=x+iy and it is denoted by w=f(z) where, z=x+iy and w=u+iv.

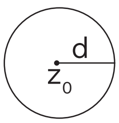
Therefore $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$

$w = f(z) = f(rei) = u(r,\theta) + iv(r,\theta)$

Neighborhood of a z_0

The set of all points within the circle having a center at z_0 but not on the circle is called neighborhood of a point z_0 and it is also called open circular disc (region).

Therefore $N_d(Z_0) = N(d, Z_0) = \{Z : |z - z|_0 < d\}$



Analytic Function

If a complex function f(z) is differentiable at a point z_0 and also differentiable at every point in some neighborhood of a point z_0 then the function f(z) is called Analytic function at a point z_0 and the point z_0 is called Analytic point of f(z).

Singular Point

If a function f(z) is not defined or not differentiable or not analytic, at a point z_0 then z_0 is called singular point of f(z).

Suppose a complex function is given as,

$$f(z) = \frac{z+4}{z-2}$$

$\therefore z = 2$ is a singular point of f(z).

Let us take another function, $f(z) = \sqrt{z-4}$

This function is defined for all values of 'z'. Consider derivative of the function

$$f'(z) = \frac{1}{2}\sqrt{z-4}a_n = (3+4i)^n$$

$$r = \frac{1}{\underset{n\to\infty}{\text{lt}} |a_n|^{1/n}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{|(3+4i)|^{1/n}}$$

1 1

z = 4 is a singular point of f(z) as f(z) is not analytic at z = 4.

Entire Function

If a complex function f(z) is differentiable or analytic at every point throughout a complex plane, then the function f(z) is called an entire function and it is also called integrable function.

Euler's Theorem

The trigonometric functions of real variables can be defined as,

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}; \cos y = \frac{e^{iy} + e^{-iy}}{2}$$

Similarly, trigonometric functions of complex variables can be defined,

$$\sinh z = \frac{e^{z} - e^{-z}}{2}; \cosh y = \frac{e^{z} + e^{-z}}{2}$$

Euler's Theorem states that,

$\cos z + i\sin z = e^{iz}$

Exponential Function

Exponential Function of real variable is very well defined and similarly we can define the exponential function of a complex variable.

$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$

De Moivre's Theorem

De Moivre's Theorem states that,

$(\cos z + i\sin z)^n = (e^{iz})^n = e^{inz} = \cos nz + i\sin nz$

Hyperbolic Functions

Different hyperbolic functions are defined as,

$$\sinh z = \frac{e^{z} - e^{-z}}{2}; \cosh y = \frac{e^{z} + e^{-z}}{2}$$

We can also define,

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

$$\coth z = \frac{\cosh z}{\sinh z} = \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^{z} + e^{-z}}$$

$$\operatorname{cosech} z = \frac{1}{\sinh z} = \frac{2}{e^{z} - e^{-z}}$$

Logarithmic Function of a Complex Variable

If z= (x + iy) and w=(u + iv) be so related that $e^w = z$, then w is said to be a logarithm of z to the base e and is written as $w = \log_e z$.

Also, $e^{w+2in\pi} = e^w e^{2in\pi} = z$

Therefore, log z = w + 2inπ

Thus, the logarithm of a complex number has an infinite number of values and is therefore a multivalued function.

The general value of logarithm of z is represented as log z to distinguish it from principal value which is written as log z.

In Cartesian coordinates log (x + iy) = 2inπ + log (x + iy) = 2inπ + $\log\left[\left[r(\cos\theta + i\sin\theta)\right]\right]$

$$\mathrm{Log}(x+iy) = 2in\pi + \log\left[\left[re^{i\theta}\right]\right]$$

$$= 2in\pi + \log r + i\theta$$

$$= \log\sqrt{x^2+y^2} + i\left[\left[2n\pi + \tan^{-1}(y/x)\right]\right]$$

The logarithm of a negative quantity is complex and can be evaluated as,

$$\mathrm{Log}_e(-x) = \log_e x + \log_e(-1) + \log x + i\pi$$

Analyticity of a Complex Function

If f (z) = u(x, y) = iv(x, y) is analytic function at a point z_0, then u_x ,u_y , v_x , v_y exists and satisfies the Cauchy Riemann equations.

$u_x = v_y$ & $v_x = -u_y$

At every point in some neighborhood of z_0.

$$u_x = \frac{\partial u}{\partial x}; u_y = \frac{\partial u}{\partial y}; v_x =; v_y = \frac{\partial v}{\partial y}$$

Sufficient condition for a function f(z) to be analytic. If

1. f (z) = u(x, y) + iv (x, y) is defined at every point in some neighborhood of z_0.
2. u and v satisfy the C-R equations at every point in some neighborhood z_0.
3. u, v, u_x, u_y, v_x, v_y are continuous at every point in some neighborhood z_0.

Then the function f(z) = u +iv is analytic at z_0 and $f'(z) = u_x + iv_x$

Note: ex, sinx, cosx, sinhx, coshx and every polynomial of the form $a_0 + a_1 x + a_2 x^2 + ...$ $a_n x^n$ ($a \neq 0$ & $n \in N$) are every where defined, continuous, differentiable and also integrable.

If f, g are two continuous functions, then

a) $f \pm g$ is also continuous

b) f.g is also continuous

c) f/g (g≠0) is also continuous

Solved Examples

Example: Test the analyticity of the function: $f(z) = x + e^x \cos y + iy + ie^x \sin y$

Solution: $u + iv = f(z) = (x + e^x \cos y) + i(y + e^x \sin y)$

$u = x + e^x \cos y$

$v = y + e^x \sin y$

$$u_x = \frac{\partial u}{dx} = 1 + e^x\cos y;\quad v_x = \frac{\partial v}{\partial y} = e^x \sin y$$

$$u_y = \frac{du}{dy} = -\left(e^x \sin y\right);\quad v_y = \frac{\partial v}{\partial y} = 1 + e^x \cos y$$

Here $u_x = v_y$ and $v_x = -u_y$ at entry point and u, v, u_x, u_y, v_x, v_y are continuous at every point.

∴ f(z) is not an analytic function.

Example: Determine whether the following function is analytic: f (z) = z^-

Solution: $u + iv = f(z) = x - iy$

$u = x, v = -y$

$u_x = 1, u_y = 0$ & $v_x = 0, v_y = -1$

$u_x \neq v_y$ and $v_x = -u_y$

∴ f(z) is not analytic function.

Note:

1. $f(z) = |z|^2$ is differentiable only at the origin but not analytic at any point.
2. $f(x) = z^-$ is not differentiable and not analytic at any point.

Example: If $x = \sqrt{-1}$ then $x^x = ?$

Solution: $x = \sqrt{-1} = i$

$$x = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{-p}{2}}$$

$$x^x = \left(e^{\frac{-p}{2}}\right) = e^{\frac{-p}{2}}$$

Note: C-R equations in polar form are given by $u_r = \frac{1}{r} v_\theta$ and $v_r = \frac{-1}{r} u_\theta$

The derivative formula in polar form is given by $f'(z) = (u_r + |iv_r|)e^{-i\theta}$

Determining an analytic Function

1. Given v (or u). Find v_x, v_y (or u_x, u_y)
2. Consider $f'(z) = u_x + iv_x = u_y + iv_y$
3. Replace x by z and y by '0' i.e. $f'(z) = g(z)$
4. $f(x) = \int g(x)dx + c$ where $c = c_1 + ic_2$

Solved Examples

Example: If $v(r,\theta) = 3r^2 \sin|2\theta| + 2r\sin\theta + 7$ then find analytic function f (z) = u + iv where v is an imaginary part of analytic function f(z).

Solution: $v_r = 6r\sin 2\theta + 2\sin\theta$ & $v_\theta = 6r^2 \cos 2\theta + 2r\cos\theta$

Consider $f'(z) = (u_r + iv_r)e^{i\theta} = \left(\frac{1}{r}v_\theta + iv\right)e^{-i\theta}$

$$f'(z) = \left\{\begin{matrix}(6r\cos 2\theta + 2\cos\theta) \\ +i[6r\sin 2\theta + 2\sin\theta]\end{matrix}\right\} e^{-i\theta}$$

Replace 'r' by 'z' and 'θ' by '0'.

$f'(z) = 6z + 2$

$\therefore f(z) = 3z^2 + 2z + c$ where $c = c_1 + ic_2$

$f(z) = 3z^2 + 2z + c_1 + ic_2 = 3(re^{i\theta})^2 + 2(re^{i\theta}) + (c_1 + ic_2)$

$$f(z) = 3r^2[\cos 2\theta + \sin 2\theta] + 2r[\cos\theta + \sin\theta] + c_1 + ic_2$$

$$f(z) = (3r^2 \cos 2\theta + 2r\cos\theta + c_1) + i(3r^2 \sin 2\theta + 2r\sin\theta + c_2)$$

Harmonic Conjugate Function

If u_x, u_y, u_{xx} & u_{yy} are continuous functions and $u_{xx} + u = 0$ or $\nabla^2 u = 0$ then u(x, y) is called harmonic function.

$u_{xx} + u_{yy} = 0$ or $\nabla^2 u = 0$ is called Laplace equation

Note:

1. If f(x) = u + iv is analytic function, then u and v satisfy Laplace equations.
2. If u and v are harmonic functions then u+iv may or may not be an analytic function.

If u and v are harmonic function as u+iv is also analytic function, then v is called harmonic conjugate function of 'u'. Similarly, '-u' is the harmonic conjugate function of 'v'.

Method

Step 1: If v(x, y) is given to find u(x, y) the consider

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Step 2: $du = u_x dx + u_y dy = (v_y)dx + (-v_x)\,dy$

$(\because u_x = v_y \;\&\; v_x = -uy)$

Step 3:

$$u = \int (v_y)dx + \int [\text{terms not containing x in } (-v_x)dy] + k$$

Treating y as constant, K → real integral constant.

Solved Examples

Example: If $v(r,\theta) = 3r^4 \sin(4\theta) + 4$, then find its harmonic conjugate function?

Solution: Consider

$$du = (u_r)dr + (u\theta)d_\theta = \left(\frac{1}{r}v_\theta\right)dr + (-rv_r)d\theta$$

$$\left(\because u_r = \frac{1}{r}v_\theta \;\&\; v_r = \frac{-1}{r}u_e\right)$$

$$du = [12r^3 \cos(4\theta)]dr + [-12_r^4 \sin(4\theta)]d\theta$$

$$u = \int [12r^3 \cos(4\theta)]dr + \int [-12_r^4 \sin(4\theta)]d\theta + k$$

$$\therefore u_{(r,\theta)} = 3r^4 \cos(4\theta) + k$$

Note:

f (z) = u(x,y) + iv (x,y) is analytic function.

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complex potential function | velocity potential function | stream function

↑ ↑ ↑

$f(z) = \Phi(x, y) + i\psi(x, y)$

Example:

If $f(z) = x^3 - 3xy^2 + i\varphi(x,y)$ where $i = \sqrt{-1}$ and $f(x+iy)$ is analytic function, then find the stream function ψ.

Solution: Given $u = x^3 = 3xy^2$, $v = \varphi$

$u_x = 3x^2 - 3y^2$, $u_y = 6xy$

Consider $dv = v_x\, dx + v_y\, dy$

$dv = -u_y\, dx + u_x\, dy$

$dv = 6xydx + (3x^2 - 3y^2)\, dy$

$v = \int (v_x)\, dx + \int$[terms not containing x in (v_y) dy] + K

$v = \int (6y\, xdx + \int (-3y^2)\, dy + K$

$v = 3x^2 y - y^3 + K$

Complex Integration

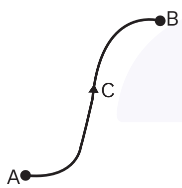
If a complex function f(z) is defined at every point on the curve C from a point A to B, then the evaluation of integral of complex function f(z), is called line integral of a complex function f(z) and is denoted by $\int_c f(z)\, dz$ where C is called path of integratio.

The relation between real line integral and complex line integral.

If f(z) is given by $f(z) = u + iv$ and $dz = dx + idy$ where $z = x + iy$ then

$$\int_C f(z)\, dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$



Parameterization of the Curve

The curve for line integral can be represented by a parametric representation as $z(t) = x(t) + iy(t)$.

The sense of increasing 't' is called as positive sense on C, and we assume that C is a smooth curve i.e. C has acontinuous and non-zero derivative $\dot{z} = \dfrac{dz(t)}{dt}$ at each point.

Solved Examples

Example: Evaluate $\int_0^{1+i} zdz$ along a curve C where C is the curve y = x

Solution: Assume x =t and y = t

$$\int_0^{1+i} zdz = \int_{(x,y)=(0,0)}^{(1,1)} (x+iy)(dx+idy)$$

$$= \int_0^1 (t+it)(dt+idt)$$

Example: Evaluate $\int_0^{1+i} zdz\,(x^2 - iy)dz$ along the curve C where C is (i) y = x (ii) y = x²?

Solution:

1. Assume x = t and y= t

$$\int_{0,0}^{(1,1)} (x^2 - iy)(dx+idy) = \int_0^1 (t^2 - it)(dt+idt)$$

$$\int_0^{1,i} (x^2 - iy)dz = \int_0^1 t^2dt + it^2dt - itdt + tdt$$

$$= \left[\frac{t^2}{3} + i\frac{t^3}{3} - i\frac{t^2}{2} + \frac{t^2}{2}\right]_0^1 = \frac{1}{3} + \frac{i}{3} - \frac{i}{2} + \frac{1}{2} = \frac{5-i}{6}$$

2. $$\int_{(0,0)}^{(1,1)} (x^2 - iy)(dx+idy) = \int_0^1 (t^2 - it^2)(dt + i2tdt)$$

$x = t;\ y = t^2$

$dx = dt$

$dy = 2tdt$

$$\int_{(0,0)}^{(1,1)} (x^2 - iy)(dx+idy) = \int_0^1 (1-i)t^2 \times dt(1 + i \times 2t)$$

$$\int_{(0,0)}^{(1,1)} (x^2 - iy)(dx+idy) = (1-i)\int_0^1 t^2dt + i2t^3dt$$

$$= (1-i)\left[\frac{t^3}{3} + i\frac{2t^4}{4}\right]_0^1$$

$$\int_{(0,0)}^{(1,1)} (x^2 - iy)(dx+idy) = (1-i)\left[\frac{1}{3} + \frac{i}{2}\right] = \frac{5+i}{6}$$

Note: If the integrant function is analytic, then the value of the integral depends only on the end points of the paths not on the path.

i.e. $\int_C f(z)\, dz = \int_A^B f(z)\, dz$

Example: $I = \int_{z=0}^{1+i} zdz = \left.\frac{z^2}{2}\right]_0^{1+i} = \frac{(1+i)^2}{2} = \frac{2i}{2} = i$

Parameterisation of Circle

$$x^2 + y^2 = r^2 \Rightarrow |z| = r\sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$x = r\cos\theta \quad x = x_0 + r\cos\theta$

$y = r\sin \quad y = y_0 + r\cos\theta$

$r = |z - z_0| \quad z = z_0 + re^{i\theta}$

Note: The parameter equation of a circle $|z - z_0| = r$ is $z = z_0 + re^{i\theta}$ where, → 0 to 2π for total circular path.

Solved Examples

Example: Evaluate $\int \frac{2z+3}{z} dz$ along a curve C, where C is $|z| = 3$?

Solution: $C = |z| = 3$

$z = 3e^{i\theta}$

$dz = 3e^{i\theta}d\theta$

Here $\theta = 0$ to 2π

$$I = \int_C \frac{2z+3}{z} dz = \int_{\theta=0}^{2\pi} \frac{2 \times 3e^{i\theta} + 3}{3e^{i\theta}} \times i3e^{i\theta} d\theta$$

$$I = \int_0^{2\pi} \left(6ie^{i\theta} + 3i\right) d\theta = \left[\frac{6ie^{i\theta}}{i} + 3i\theta\right]_0^{2\pi}$$

$$I = \left(6e^{i2\pi} + 6i\pi\right) - \left(6e^0 + 0\right) = 6i\pi$$

Simple Connected Domain

Let f(z) be analytic in a simple connected domain D. A domain D is called simple connected if every closed curve without self-intersections encloses points only in D. Then there exists an indefinite integral of f(z) in the domain D, that is, an analytic function F(z) such that F'(z) = f(z) in D and for all the paths in D joining two points z_0 and z_1 in D we have,

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$$

Example: $\int_{z_0}^{1+i} z^2 dz = \frac{1}{3}z^3 \Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$

Cauchy's Integral Theorem

If a function f(z) is analytic at every point within and on a simple closed curve C, then integral over C.

$$\oint_C f(z)dz = 0$$

Where $\oint_C f(z)dz$ represents integral of f(z) over a closed curve C.

This theorem is actually an extension of the fact that integral of an analytic function depends only on the endpoints. In case of closed curve the initial and end points are same so integral is zero.

If a function f(z) is analytic everywhere within and on a triply connected region R bounded by 3 simply closed curve C_1, C_2, C_3 but not analytic within C_1, C_2 and analytic only in C_3, then

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz$$

Since, f(z) is analytic in C_3,

$$\oint_{C_3} f(z)dz = 0$$

Thus, $\oint_C (z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$

Cauchy's Integral Formula

If f(z) is analytic at every point within and on a simple closed curve C and z_0 is any point within C then

1. $\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$

2. $\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

Thus, if a function is analytic on the simple closed curve C then the values of function and all its derivatives can be found at any point of C.

Method to use Cauchy's integral formula,

- Let $\oint f(z)dz = I$, first find the singular points of f(z) i.e. points where f(z) goes to infinity.
- Check which of these points lie inside the closed curve given.
- Apply CIF only at those points and if none of the points lie inside the given curve, $\int f(z)dz = 0$

Solved Examples

Example: Evaluate

$$\int_c \frac{2z+\sin z+e^x}{(z-4)^{10}(z-6)^{100}}dz \text{ where C is } |z|=\frac{3}{2}?$$

Solution: Let $f(z)=\dfrac{2z+\sin z+e^z}{(z-4)^{10}(z-6)^{100}}$

Singular points: z = 4, 6

The given curve is, $|z|=\dfrac{3}{2}$

Since the curve is a circle centered at origin and radius 3/2. Both the points lie outside the circle.

$$\therefore \int_c f(z)dz=0$$

Example: Evaluate $\oint \dfrac{2z+3}{z}dz$ along C where C : |z| = 3 ?

Solution: $g(z)=\dfrac{2z+3}{2}$

Singular Point of this function is, z = 0

The given curve is |z| = 3 . Since z = 0 lies inside the given curve.

$$\therefore \text{By CIF } \oint \frac{2z+3}{z-0}dz = \oint \frac{f(z)}{z}dz = \frac{2\pi_i}{1}\times f(0)$$

$$= 2\pi i\times(2\times 0+3)=6\pi i$$

Example: Evaluate $\oint_c \dfrac{z}{(z-1)(z-2)}dz$ along curve C where $C:|Z-2|=\dfrac{1}{2}$?

Solution: $g(z)=\dfrac{z}{(z-1)(z-2)^2}$

Singular point: z= 1, 2

$C:|Z-2|=\dfrac{1}{2}$?

If z = 1

$|-1|=1>\dfrac{1}{2}$, so it lies outside the given circle.

z = 2

$|0|=1<\dfrac{1}{2}$,

$$\frac{\frac{z}{(z-1)}}{(z-2)^2}=\frac{f(z)}{(z-z_0)^{n+1}} \text{ where } f(z)=\frac{z}{z-1}$$

Example: Evaluate $\int_c \dfrac{e^z+\cos z}{(z-3)(z-2)}dz,\ C:|z|=5$

Solution: $f(z)=\dfrac{e^z+\cos z}{(z-3)(z-2)}$

$$\frac{1}{(z-a)(z-v)}=\frac{1}{(a-b)(z-a)}-\frac{1}{(a-b)(z-b)}$$

Singular point: z = 2,3

$$f(z)=\frac{e^z+\cos z}{z-3}-\frac{e^z+\cos z}{z-2}$$

C : |z| < 5 so both singularities lie inside the given curve.

$$\therefore \oint_c \Phi(z)dz=\int_c \frac{e^z+\cos z}{z-3}dz-\int_c \frac{e^z+\cos z}{z-2}$$

$$=2\pi i[f(3)-f(2)]$$

Example: Evaluate $\int_c \dfrac{\bar{z}}{z}dz$ along a unit circle?

Solution: Let $\Phi(z)=\dfrac{\bar{z}}{z}$

C : |z| = 1

The singular point of the function is z=0

$$g(z)=\frac{\bar{z}}{z-0}=\frac{f(z)}{z-z_0}$$

Since, $\bar{z}$ is not analytic anywhere so we have to calculate this integral as given below,

Method 1:

$z.\,\bar{z}=|z|^2$

$$\bar{z}=\frac{|z|^2}{z}\Rightarrow\frac{\bar{z}}{z}=\frac{|z|^2}{z^2}=\frac{1}{z^2}\quad\therefore |z|=1$$

$$\bar{z}=\frac{1}{(z-0)^2}=\frac{f(z)}{(z-2_0)^{n+1}}$$

$\therefore$ By Cauchy integral formula we have

$$\int_c \frac{\bar{z}}{z}dz=\frac{2\pi i}{1!}f'(o)=2\pi i\left[\frac{d}{dz}f(z)\right]=0$$

Example: Evaluate $\int_c \dfrac{z}{z-2}dz$

along a circle |z| = 2

Solution: Let $\Phi(z)=\dfrac{z}{z-2}=\dfrac{f(z)}{2-z_0}$

Singular point of this function is, z = 2

The given curve is, C : |z| = 2

$z = 2e^{i\theta}$

$dz = 2e^{i\theta}d\theta$

$\theta = 0$ to 2π

$$I = \int_c \frac{z}{z-2}dz = \int_0^{2\pi} \frac{2e^{i\theta}}{2e^{i\theta}-2} 2ie^{i\theta}d\theta = 2i\int_0^{2\pi} \frac{e^{2i\theta}}{e^{2i\theta}-1}d\theta$$

The Singular point lies on the curve C. So the function cannot be evaluated.

Complex Power Series

An infinite series of the form

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \text{..........}$$
$$+a_0(z-z_0)^n + \text{...........}$$
$$(or)\, f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

In powers of $(z - z_0)$ z or about a point z_0. In the above power series a_n is a real or complex constant which is called coefficient of power series, z is a complex variable and z_0 is a fixed complex constant which is called center of the power series.

$$\text{for } a_n = 1, f(z) = \sum_{n=0}^{\infty} (z-z_0)^n$$

Region of convergence (ROC)

The set of all values of z for which the power series converges is called region of convergence.

$$\text{Eg. } 1+z+z^2+\text{.........} = (1-z)^{-1}; |z| < 1$$

Here $|z| < 1$ is an ROC, $|z| = 1$ is a circle of convergence (COC) and radius r = 1 is a radius of convergence of the power series.

Similarly, $1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\text{......} = e^z, \forall z \in C$

Here an entire complex plane is an ROC of power series.

If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ then|

The radius of convergence of the above power series is given by

$$r = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{1/n}} \,(or)\, r = \lim_{n\to\infty} \left|\frac{a_n}{a^{n+1}}\right| x$$

The circle of convergence of above series is given by $|z - z_0| = r$

The region of convergence ROC of above power series is given by $|z - z_0| < r$

Solved Examples

Example: Find the radius of convergence, COC and ROC of the given power series.

$$\sum_{n=0}^{\infty} n!z^n$$

Solution: Compare the given series with

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Here $a_n = n!$ & $z_0 = 0$

Radius of convergence

$$\therefore r = \underset{n\to\infty}{Lt} \left|\frac{n!}{(n+1)!}\right| = \underset{n\to\infty}{Lt} \left|\frac{1}{n+1}\right| = 0$$

ROC $|z - 0| < 0$

Circle of convergence $|z - 0| = 0$

Example: Find the radius of convergence, COC and ROC of the given power series.

$$\sum_{n=0}^{\infty} (3+4i)(z+2i)^n$$

Solution: $\sin y = \frac{e^{iy} - e^{-iy}}{2i}; \cos y = \frac{e^{iy} + e^{-iy}}{2}$

Example: Find the radius of convergence, COC and ROC of the given power series.

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}(z-3)^n$$

Solution: Compare given power series with

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Here $a_n = \dfrac{2^n}{n!}$ & $z_0 = 3$

$$r = \underset{n\to\infty}{Lt}\left|\frac{a_n}{a_{n+1}}\right| = \underset{n\to\infty}{Lt}\left|\frac{2^n}{n!}\cdot\frac{(n+1!)}{2^{n+1}}\right|$$

$$= \underset{n\to\infty}{Lt}\left|\frac{2^n}{n!}\cdot\frac{(n+1)n!}{2^n\times 2}\right| = \underset{n\to\infty}{Lt}\left|\frac{n+1}{2}\right| = \infty$$

Circle of convergence: $|z-3| = \infty$

Region of convergence: $|z-3| < \infty$

Note:

(1) If f(z) is an analytic function at z0 (not a Singular Point) $\Rightarrow$ Taylor series.

(2) If f(z) is not an analytic function at z0 (Singular Point) $\Rightarrow$ Laurent series.

Taylor's Theorem

If a function f(z) is analytic at every point within a circle C, then for every point z within the circle C, the function f(z) can be expressed as a power series in +ve powers of $(z - z_0)$ or about $z = z_0$.

$$\text{i.e. } f(z) = f(z_0) + (z-z_0)f'(z) + \frac{(z-z_0)^2}{2!}$$

$$f''(z_0) + \dots\dots\dots + \frac{(z-z_0)^n}{n!}f^n(z_0) + \dots\dots\dots$$

$$f(z) = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{n!}f^n(z_0) = \sum_{n=0}^{\infty}a_n(z-z_0)^n$$

$$\text{Where } a_n = \frac{f^n(z_0)}{n!}$$

The RHS of above is called Taylors series about 0 $z = z_0$. The ROC of Taylor series is given by $|z-z_0| = r$. Where the radius of convergence r is a distance from a center of the power series z0 to its nearest singular point of the same function f(z).

Example: Find the Taylor series expansion of $f(z) = \dfrac{1}{z-2}$ about a point z=1. Hence find radius of convergence, COC and ROC

Solution: Given $f(z) = \dfrac{1}{z-2}, z = 1$

Here $z_0 = 1$ and singular point, z = 2

$r = |S.pt. - z_0| = |2-1| = 1$

COC : $|z - z_0| = r$

i.e. $|z-1| = 1$

ROC : $|z-z_0| < r \Rightarrow |z-1| < 1$

Expansion :

$$f(z) = \frac{1}{z-2}, z = 1$$

Let $z - 1 = t$ then $z = 1 + t$

$$f(z) = \frac{1}{t-1} = -(1-t)^{-1} \quad |t| < 1$$

$$f(z) = \left[1 + t + t^2 + t^3 \dots\dots\dots + t^n + \dots\dots\dots\right]$$

$$f(z) = (-1)\left[1 + (z-1) + (z-1)^2 + (z-1)^3 + \dots\dots\dots\right]$$

Laurent's Theorem

If a function f(z) is analytic at every point within a ring shaped region R bounded by two concentric circles C_1, C_2 having center at z_0, with radii r_1, r_2 such that $r_2 < r_1$, then for every point z within R, the function f(z) can be represented by a power series in both +ve and –ve powered of 0 $z - z_0$ or about $z = z_0$.

$$\text{i.e. } f(z) = \sum_{n=0}^{\infty}a_n(z-z_0)^n + \sum_{n=1}^{\infty}b_n(z-z_0)^{-n}$$

Here second summation is known as principle part of Laurent's series

$$\text{Where } a_n = \frac{1}{2\pi i}\oint_{c_1}\frac{f(z)}{(z-z_0)^{n+1}dz}$$

$$\text{and } b_n = \frac{1}{2\pi i}\oint_{c_2}\frac{f(z)}{(z-z_0)^{-n+1}dz}$$

The RHS of the above is called Laurent's series about $z = z_0$ and the ROC of a Laurent's series is given by $r_2 < |z - z_0| < r_1$.

Solved Examples

Example: Expand $f(z) = \dfrac{e^{2z}}{(z-1)^2}$ as an infinite series about z = 1 and also find ROC.

Solution: Let $z - 1 = t$ then $z = 1 + t$

$$f(z)=\frac{e^{2(1+t)}}{t^2}=e^2\times\frac{e^{2t}}{t^2}\left[1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\ldots\ldots\ldots\right]$$

$$f(z)=e^2\left[\frac{1}{t^2}+\frac{2}{t}+2+\frac{4t}{3}+\ldots\ldots\ldots\right]$$

$$f(z)=e^2\left[\frac{1}{(z-1)^2}+\frac{2}{(z-1)}+\frac{2^2}{2!}+\frac{2^3}{3!}(z-1)+\ldots\ldots\ldots\right]$$

Therefore the above series is a Laurent's series about $z=1$ and entire complex plane is an ROC except $z=1$.

Example: Expand $f(z)=(z-3)\sin\left(\frac{1}{z+2}\right)$ as an infinite series about z = –2 and also find ROC.

Solution: Let z – (–2) = t

Then z = t – 2

$$f(z)=(t-5)\sin\frac{1}{t}$$

$$\sin x=x-\frac{x^3}{3!}+\frac{X^5}{5!}+\ldots\ldots$$

$$f(x)=(t-5)\left[\frac{1}{t}-\frac{1}{t^3}\times\frac{1}{3!}+\left(\frac{1}{t}\right)^5\frac{1}{5!}+\ldots\ldots\ldots\right]$$

$$=1-\frac{5}{t}-\frac{1}{3!t^2}+\frac{5}{3!t^3}+\ldots\ldots\ldots$$

$$f(x)=1-\frac{5}{z+2}-\frac{1}{3!(z+2)^2}+\frac{5}{3!(z+2)^3}+\ldots\ldots\ldots$$

Therefore the above series is Laurent's serie. Entire complex plane is an ROC except $z=-2$.

Example: Expand $f(z)=\frac{z}{(z+1)(z+2)}$ as an infinite series about z = –2

Solution: Let z – (–2) = t ⇒ z = t – 2

$$f(z)=\frac{t-2}{(t-1)t}=\frac{-1}{t-1}+\frac{2}{t}$$

$$\frac{2}{t}+\frac{1}{1-t}=\frac{2}{t}+(1-t)^{-1}=\frac{2}{t}+1+t+t^2+t^3+\ldots\ldots\ldots\ldots[|t|<1]$$

$$\text{i.e. } f(z)=\frac{2}{z+2}+\left[1+(z+2)+(z+2)^2+(z+2)^3+\ldots\ldots\ldots\right]$$

Therefore, the ROC of above power series is $0<|z+2|<1$

Zeros and Singularities of Complex Function

If f(z) is analytic at a point z_0 and f (z_0) = 0, then the point z_0 is called zero of the function f(z).

Suppose, $f(z)=(z-3)^4$. Here, the function is analytic at z = 3 and f(3) = 0, z = 3 is a zero of f(z).

If f(z) is an analytic function at z_0 and $f(z_0)=0, f'(z_0)=0, f''(z_0)=0$ and so on...........

$f^{(m-1)}(z_0)=0$ but $f^m(z_0)\neq 0$ then the point z_0 is called zero of order m.

Solved Examples

Example: Find the order of zero z = 2 of the function f(z) = $(z-2)^3$

Solution:

$f'(z)=3(z-2)^2;\ f'(2)=0$

$f''(z)=6(z-2);\ f''(2)=0$

$f'''(2)=6 \qquad f^{iii}(2)\neq 0$

Therefore z = 2 is a zero of order 3.

Example: Determine the order $z=n\pi$, $n\in I$ for the function f (z) = sinz

Solution: $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$

$\cos z=\frac{e^{iz}-e^{-iz}}{2i}$

$f(z)=\sin z,\ z=n\pi, n\in I$

$\therefore z=n\pi, n\in I$ are first order zeros.

Example: Determine the order of zero $z=(2n+1)\frac{\pi}{2}, n\in I$ for the function f (z) = cosz

Solution: $f'(z)=\sin z\neq 0$ at $z=(2n+1)\frac{\pi}{2}, n\in I$

$z=(2n+1)\frac{\pi}{2}, n\in I$ are simple zeros of f(z)

Similarly, f(z) = sinhz, z = $n\pi i$, $n\in I$ (simple zeros)

$s(z)=\cosh z, z=i(2n+1)\frac{\pi}{2}, n\in I$(simple zeros)

Types of Singularities

Isolated Singular Point

If z_0 is a singular point of f(z) and f(z) is analytic at every point except z_0 or in at least 1 neighborhood of z_0, then the point z_0 is called isolated singular point of f(z).

Example : $f(z) = \frac{(z+4)^3}{z-2}$

$z = 2 \rightarrow$ Isolated singular point

Similarly, $f(z) = \frac{(z-2)^3 (z-4)}{(z-5)^2 (z-6)^3}$

singular point is z = 5, 6

At least one region exists, so 5, 6 are isolated singular points.

$f(z) = \frac{1}{\sin z}$

$\text{Singular Point}: z = n\pi, n \in I \Rightarrow \underbrace{z = 0, \pm\pi, \pm 2\pi}_{\text{Isolated point}} \ldots\ldots$

We can't find any other singular point in this region. So isolated singular point.

Removable Singular Point

If the principal part of Laurent's series expansion of f(z) about z – z0 does not exist then the singular point z0 is

called removable Singular Point of f(z).

Example : $f(z) = \frac{\sin z}{z}$

Pole (of order m)

If the principal part of Laurent's expansion of f(z) about $z - z_0$ contains finite number of –ve powers of $(z - z_0)$ then

the singular point 'z_0' is called pole of order m i.e. say m terms.

Example : $f(z) = \frac{e^z}{z}$

z = 0 is the singular point.

Now expand about this point.

$$f(z) = \frac{e^2}{z} = \frac{1}{2}\left\{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots\ldots\ldots\ldots\right\}$$

$$= \frac{1}{z-0} + 1 + \frac{z-0}{2!} + \frac{(z-0)^2}{3!} + \ldots\ldots\ldots\ldots$$

$$f(z) = 0 \times \frac{1}{(z-0)^2} + 0 \times \frac{1}{(z-0)} + \frac{1}{(z-0)} \ldots\ldots\ldots$$

$\therefore$ z = 0 is a pole of order 1 (simple pole)

Essential Singular Points

If the principal part of Lourent's series expansion of f(z) about $z - z_0$ contains infinite number of negative powersof $(z - z_0)$ then the regular point z_0 is called essential singular point of f(z).

Example : $f(z) = (z-4)\sin\left(\frac{1}{z-4}\right)$

$$= (z-4)\left\{\frac{1}{z-4} - \frac{1}{(z-4)^3 3!} + \frac{1}{(z-4)^5 5!} + \ldots\ldots\ldots\right\}$$

$$f(z) = 1 - \frac{1}{(z-4)^2 3!} + \frac{1}{(z-4)^4 5!} + \ldots\ldots\ldots\ldots$$

Classification of Singular Points

1. Suppose $f(z) = \frac{N^r}{D^r}$
2. Find the singular points of f(z) [i.e. zeros of Dr function].
3. N^r (at singular points)
   $\begin{cases} \neq 0 \Rightarrow \text{Poles} \\ = 0 \Rightarrow \text{pole or removable singuar point} \end{cases}$
4. If m > n ⇒ pole order = m – n
   Else $m \leq n \Rightarrow$ removable singular point
   where n → order of zero of Nr and m → order of zero of Dr

Solved Examples

Example: Determine the type of singular points for the function

$$f(z) = \frac{(z-4)^3 (z-6)^2}{(z-5)^{10} (z-7)^5}$$

Solution: Singular points: z = 5 (order = 10) and z = 7 (order = 5)

Since, numerator is non-zero at both these singular points, these are poles.

Example: Determine the type of singular points for the function $f(z) = \frac{\sin x}{z - \frac{\pi}{2}}$

Solution: Singular point $z = \frac{\pi}{2}[\text{order } 1]$

Since, numerator is non-zero at $z = \frac{\pi}{2}$ pole of order 1 (simple pole)

Example: Determine the type of singular points for the function f(z) = tanz

Solution: $f(z) = \frac{\sin z}{\cos z}$

Singular point: $z = \frac{\pi}{2}(2n+1) \quad n \in I$

Thus, there are infinite number of singular points.

Since, Numerator is non-zero at all these points $\therefore\ z - (2n+1)\frac{\pi}{2}$ is a pole of order 1.

Example: Determine the type of singular points for the function $f(z) = \frac{\cos z}{z - \frac{\pi}{2}}$

Solution: $f(z) = \frac{\cos z}{z - \frac{\pi}{2}} = \frac{\phi(z)}{(z - z_0)^m} m = 1$

Singular Point = $\frac{\pi}{2}$

Since, numerator is zero at $z = \frac{\pi}{2}$ this is a removable singularity or a pole. To determine the order of zero of N^r

f(z) cosz 0 at $z = \frac{\pi}{2}$

f'(z) cosz 0 at $z = \frac{\pi}{2}$

Therefore $z = \frac{\pi}{2}$ is a pole of order 1.

$\therefore$ n = 1

$\therefore\ m = n = 1 \Rightarrow z\ z = \frac{\pi}{2}$ is a removable Singular Point

Example: Determine the type of singular points for the function $f(z) = \frac{1 - \cos z}{z}$

Solution: singular point: z = 0, Since Nr = 0 at z = 0

$$f(z) = \frac{1 - \cos z}{z} = \frac{g(z)}{(z - z_0)^m} m = 1$$

$$g(z) = 1 - \cos z = 0 \text{ at } z = 0$$

$$g'(z) = \sin z = 0 \text{ at } z = 0$$

$$g''(z) = \cos z \neq 0 \text{ at } z = 0$$

Thus, n = 2 i.e. z = 0 is a zero of numerator of order 2

Since n > m $\therefore$ z = 0 is a removable singular point

Example: Determine the type of singular points for the function $f(z) = (z-4)\sin\left(\frac{2}{z-4}\right)$

Solution: $f(z) = (z-4)\left[\frac{2}{z-4} - \frac{2^3}{3!(z-4)^3} + \frac{2^5}{5!(z-4)^5} \ldots\ldots\ldots\right]$

$$f(z) = 2 - \frac{2^3}{3!(z-4)^2} + \frac{2^5}{(z-4)^4\, 4!} \ldots\ldots\ldots\ldots$$

Singular Point z = 4 $\Rightarrow$ Essential singular point.

Residue of a Complex Function

If z0 is an isolated singular point of f(z), then the coefficient of $\frac{1}{z - z_0}$ in Laurent's series of f(z) about z = z_0 is called residue of f(z) and it is denoted by Res [f(z) : z = z_0]

Therefore, Res[f(z) : z = z0] = The coefficient of $\frac{1}{z - z_0}$ in Laurent's series.

The coefficient is given by, $b_1 = \frac{1}{2\pi i}\oint f(z)dz$

Cauchy's Residue Theorem

If f(z) is analytic at every point within and on a simple closed curve X except at a finite number of isolated singular

points z, z_2, ... z_n within C, then

$$\oint_c f(z)dz = 2\pi i\left(\sum_{j-1}^{n} R_j\right)$$

where $R_j = \text{Res}[f(z) : z = z]$

This theorem is useful in computing line integrals of complex functions over a closed curve.

Methods to find Residues

Removable Singular Point

If z_0 is a removable Singular Point of f(z) then Res(f(z) : z = z_0) = b1 = 0

Essential Singular Point

If the point z_0 is an essential Singular Point of f(z) then expand f(z) as a Laurent's series, about $z = z_0$ and collectthe coefficient of $\frac{1}{z-z_0}$ in the Laurent's series which gives the residue of f(z).

Pole

(A) If $f(z)=\frac{P(z)}{Q(z)}$ has simple pole at z_0, then.

$$\left[\operatorname{Res}\left[f(z):z=z_0\right]=\underset{z\to z_0}{\text{Lt}}\left[(z-z_0)f(z)\right]\right]$$

(B) If $f(z)=\frac{\phi(z)}{\psi(z)}$ has simple pole at z_0

then $\left[\operatorname{Res}\left(f(z):z=z_0\right)=\frac{\phi(z_0)}{\psi(z_0)}\right]$

where $\phi(z_0)\neq 0$ and $\psi'(z_0)\neq 0$

(C) $f(z)=\frac{g(z)}{z-z_0}$ has simple pole at z_0, then

$\left[\operatorname{Res}\left[f(z):z=z_0\right]=g(z_0)\right]$ where $g(z_0)\neq 0$

(D) If f(z) has pole at z0 of order m, then

$$\operatorname{Res}\left[f(z):z=z_0\right]=\frac{1}{(m-1)!}$$

$$\underset{z\to z_0}{\text{Lt}}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-z_0)^m f(z)\right\}\right]$$

Solved Examples

Example: Find the residue of following function. $f(z)=\frac{z}{z^2+4}$

Solution: To determine the singular point

$z^2 + 4 = 0$

$z = +2i \rightarrow$ singular points, Simple poles.

$R_1:\operatorname{Res}\left\{f(z):z=2i\right\}=\underset{z\to 2i}{\text{Lt}}\left[(z-2i)f(z)\right]$

$=\underset{z=2i}{\text{Lt}}\frac{(z-2i)\times z}{(z-2i)(z+2i)}=\frac{2i}{4i}=\frac{1}{2}$

$R_2:\operatorname{Res}\left\{f(z):z=-2i\right\}=\underset{z\to 2i}{\text{Lt}}\left[(z-2i)f(z)\right]$

$=\underset{z=2i}{\text{Lt}}\frac{(z+2i)\times z}{(z+2i)(z-2i)}=\frac{2i}{-2i-2i}=\frac{1}{2}$

Example: Find the residue of following function.

Solution: Singular Point $z = \pi \rightarrow$ simple pole.

$\therefore$ Residue = $g(\pi)$ = -1

Example: Find the residue of following function. f (z) = cot z

Solution: $f(z)=\cot z=\frac{\cos z}{\sin z}=\frac{\phi(z)}{\psi(z)}$

Singular Point $z = n\pi$ $n \in I$

$\operatorname{Res}\left\{f(z):z=z_0\right\}=\frac{\phi(n\pi)}{\psi'(n\pi)}=\frac{\cos n\pi}{\cos n\pi}=1$

Example: Find the residue of following function. $f(z)=\frac{\cos z}{z-\frac{\pi}{2}}$

Solution:

$f(z)=\frac{\cos z}{z=\frac{\pi}{2}}=\frac{g(z)}{(z-z_0)^m}$ $m=1$

$z-\frac{\pi}{2}\rightarrow$ pole of order 1.

$N^r=0$ at $z=\frac{\pi}{2}$

$g(z)=\cos z=0$

$g'(z)=-\sin z\neq 0 \therefore n=1$

$m=n=1\Rightarrow z=\frac{\pi}{2}\rightarrow$ Removable Singular Point

$\therefore R_1=\operatorname{Res}\left[f(z):z=\frac{\pi}{2}\right]=0$

Example: Find the residue of following function. f(z) = $(z-2)^{3/z-2}$

Solution: $f(z)=(z-2)\left[1+\frac{3}{z-2}+\frac{3^2}{2!(z-2)}+\frac{3^3}{3!(z-2)}+\ldots\ldots\ldots\right]$

z = 2 essential singularity point. (∞ number of negative powers of z)

$\therefore R_1=\operatorname{Res}\left\{f(z):z=2\right\}=b_1=\frac{3^2}{2!}=\frac{9}{2}$

Method to calculate Complex Integrals by Residue Method

1. Let f(z) = Integrand function
2. Find singular point of f(z)
3. Consider the region R enclosed by curve C
4. Check whether the singular point is within the region or not.

No singular point within the region $\Rightarrow \text{CIT} \oint f(z)dz = 0$

One or more singular point within the region,

1. Classify the singular points
2. Find the residues
3. Substitute residue in Cauchy integral theorem $\oint f(z)dz = 2\pi i(R_1 + R_2 + \ldots\ldots\ldots + R_n)$

Solved Examples

Example: Evaluate $\int_c \frac{z}{(z-1)(z-2)^2}dz, c : |z-2| = \frac{1}{2}$

Solution: Let $f(z) = \frac{z}{(z-1)(z-2)^2}$

Singular point : z = 1, 2

$c : |z-2| = \frac{1}{2}$

$z = 1$ outside $c > \frac{1}{2}$

$z = 1$ inside $c > \frac{1}{2}$

$z = 2$ pole of order 2

$N^r \neq 0$ at $z = 2$

$$R_1 = \text{Res}\{f(z) : z = 2\} = \frac{1}{(2-1)!}$$

$$\underset{z \to 2}{\text{Lt}} \left[\frac{d}{dz}(z-2)^2 \times \frac{z}{(z-1)(z-2)^2}\right]$$

Example: Evaluate $\oint z^2 e^{\frac{1}{z}} dz$ along a unit circle $|z| = 1$

Solution:

$$f(z) = z^2 e^{\frac{1}{z}}$$

$$f(z) = z^2\left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots\ldots\ldots\ldots\right]$$

$$f(z) = z^2 + z + \frac{1}{2} + \frac{1}{3!z^2} + \frac{1}{4!z^3} + \ldots\ldots\ldots\ldots$$

Singular point = z = 0 $\rightarrow$ Essential singular point

$C : |z| = 1$

z = 0 lies inside C

Coefficient of $\frac{1}{z - z_0} = \frac{1}{3!}$

$$\therefore R_1 = \text{Res}\{f(z) : z = 0\} = \frac{1}{3!} = \frac{1}{6}$$

$$\therefore \text{by c.r.t we have} \oint_c f(z)dz = 2\pi i \times R_1 = \frac{\pi i}{3}$$