3 Graph Theory

Graph Theory

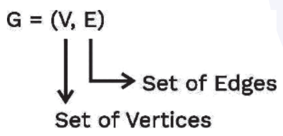
Graph theory helps in solving real time problems. In graphs theory, we will study mainly about undirected graphs.

Undirected graph and graph models:

Definition

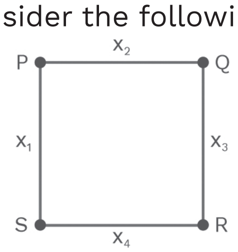
A graph G = (V, E) consists of V, a non-empty set of vertices (or nodes) and E, a set of edges and every edge is associated with unordered pair of vertices.

Each edge is connected to one or two vertices, which are referred to as its endpoints.



In an undirected graph, an edge from vertex v to vertex u will be same as an edge from vertex u to vertex v.

Example: Consider the following graph:



V = {P, Q, R, S}

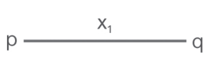
E = {x_1, x_2, x_3, x_4}

Terminologies:

There are some terminologies in graph theory:

Adjacent vertices:

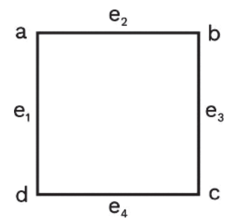
Two vertices v and u, belonging to vertex set V of an undirected graph G(V,E), are adjacent if there exists an edge e, such that e belongs to set E and connects the vertices u and v.



In given figure, p and q are adjacent vertices, and x_1 is the common edge.

Adjacent edges:

Adjacent edges are those edges which are fain to the common vertex.



In the above shown figure, e_1 is adjacent to both e_2 (having common vertex a) and e_4 (having common vertex d).

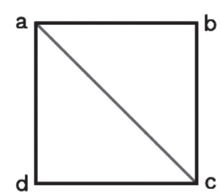
Order of a graph:

Order of a graph G, is the total number of vertices present in the graph.

It is denoted by O(G).

Size of graph:

Size of a graph G, is the total number of edges present in the graph.



O(G) = 4

Size(G) = 5

Self loop:

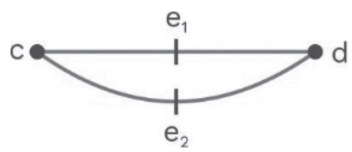
It is an edge in the graph which has same end points, i.e., it connects a vertex to itself.

Parallel edges (multiple edges):

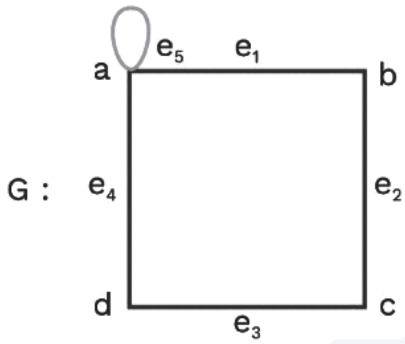
If in a graph between two vertices, if more than one edge is there, then those edges are known as parallel edges.

The graph, which contains parallel edges is known as multigraph.

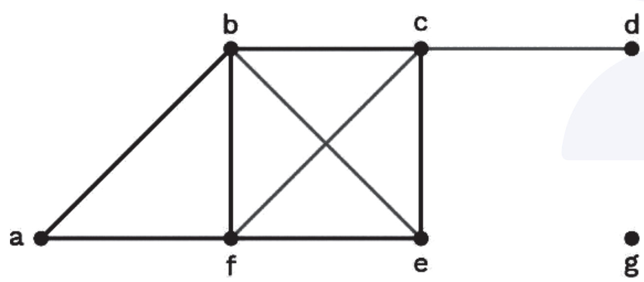
Multiple edges from c to d are shown by e_1 and e_2 in the diagram.



Edge-labelled graphs:
A graph with labelled edges is called edge labelled graph.

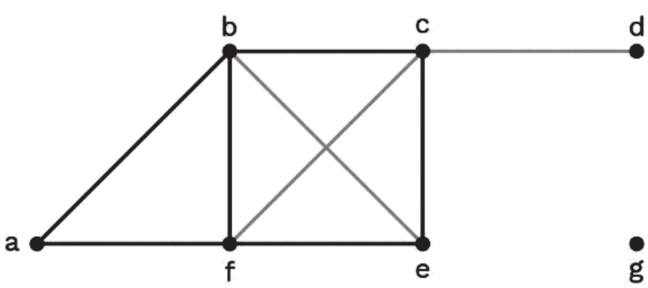


Isolated vertex:
An isolated vertex does not have any vertex adjacent to it, i.e., it is a vertex with degree zero.
Example: Consider the following figure:



In the above figure, the isolated vertex is 'g'

Pendant vertex:
In a graph, if any vertex has degree one, then that vertex is known as pendent vertex.
Example:



In the given figure, the pendent vertex is 'd'.

Simple graph:
Simple graph can be defined as an undirected, unweighted graph having no self loops and multiple/parallel edges.

Note:
- A simple graph can be either connected or disconnected.
- For directed graph: E ~ V x V

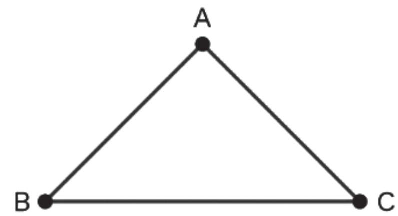


Fig. 3.1 A simple Graph

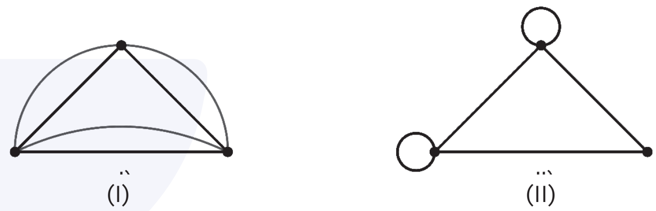


Fig. 3.2 Non-Simple Graphs (I) with Multiple Edges (II) with Self Loops

Properties of simple graph:
- For a simple complete graph consisting of 'n' number of vertices, the maximum possible edges are:

$$= \frac{n(n-1)}{2}$$

- Number of simple graph possible with n vertices = $2^{n(n-1)/2}$
- With 'n' vertices and 'm' edges, the number of simple graphs that are possible:

$$= C\left(\frac{n(n-1)}{2}, m\right)$$

Example: Find the maximum number of simple graphs possible with five vertices and two edges.
Solution:
Maximum edges possible = C(5, 2) = 10

$$\left\{\because \frac{n(n-1)}{2} = \frac{5 \times 4}{2} = 10\right\}$$

Maximum number of graphs = C(10, 2) = 45

Multigraph:
Graphs that may have multiple/parallel edges and no self loops are called multigraphs.

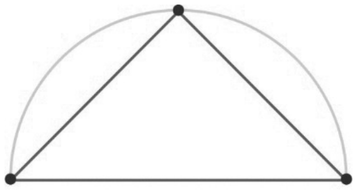


Fig. 3.3 Example of Multigraph

Pseudograph:
A graph G consisting of self loops and parallel/multiple edges can be defined as pseudograph.

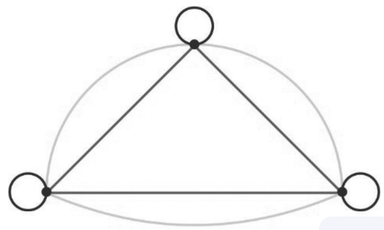
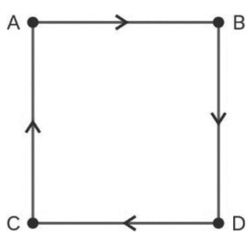


Fig. 3.4 Pseudograph

Directed graph:
If in a graph all the edges are ordered edges, i.e., edges with direction then the graph is known as directed graph, i.e., edge from vertex v to u is not same as edge from vertex u to v.



Here, V = {A, B, C, D}
E = {(A, B), (B, D), (D, C), (C, A)}

Simple directed graph:

Definition

"A directed graph or (d igraph) G = (V, E) consists of a non-empty set of vertices V and a set of directed edge E associated with the ordered pair (u, v) is said to start at u and end at v. When a directed graph has no Loops and has no multiple directed edges, it is called a simple directed graph."

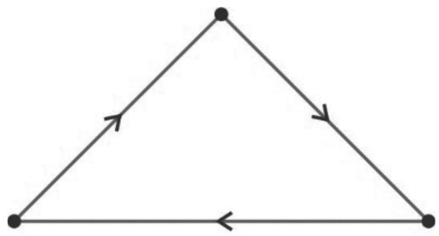


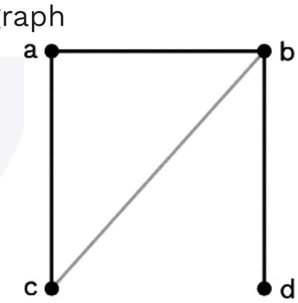
Fig. 3.5 Simple Directed Graph

Directed Multigraph:
A directed graph having more than one edge between two vertices, is known as directed multigraph.

Note:

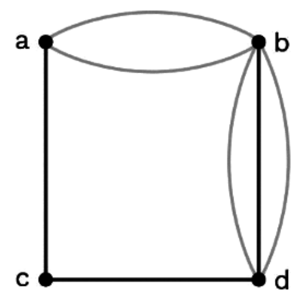
A graph with both directed and undirected edges is called mixed graph.

1. Simple graph



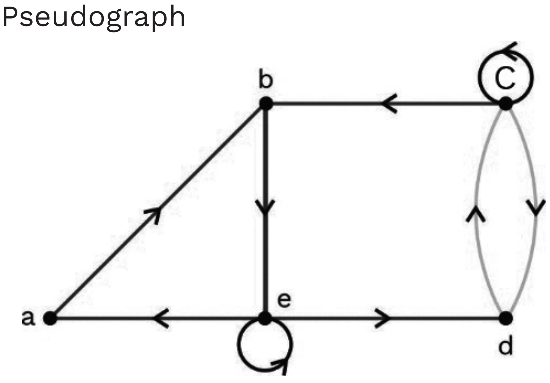
⇒ It has undirected edges, no multiple edges, no self loops.

2. Multigraph



⇒ It has undirected edges, multiple edges between a and b and no self loops.

3. Pseudograph



a) It has directed edges.
b) No multiple edges, first is from c to d and second is from d to c.
c) It has 2 self loops (on c and e).

Rack Your Brain

What kind of graph can be used to model a highway system between major cities where:
1. There is an edge between the vertices, representing cities, if there is an interstate highway between them?
2. There is an edge between the vertices, representing cities, for each interstate highway between them.

Rack Your Brain

Describe a graph model that can be used to represent all forms of electronic communication between two people in a single graph. What kind of graph is needed?

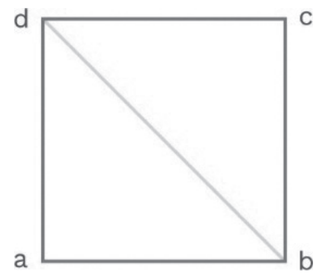
Degree of vertex:
The number of edges that are incident onto a vertex is termed as the degree of that vertex.

For each self loop the degree is counted as 2.

Min-degree: Least among the degrees of all given vertices, represented by δ.

Max-degree: Maximum among the degrees of all the vertices in G. It is represented by Δ.

Example:



Solution:

	deg(v)
a	2
b	3
c	2
d	3

$\delta(G) = 2$ and $\Delta(G) = 3$.

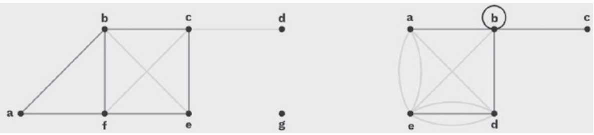
Note:

Let G be a simple undirected graph with v-vertices and e-edges.

$$\frac{x}{(1-x)^2}$$

Solved Examples

1. What are the degree of vertices in the graphs G and H displayed in given figure:



(G) (H)

Solution:

In G:
deg (a) = 2
deg (b) = deg (c) = deg (f) = 4
deg (d) = 1
deg (e) = 3
deg (g) = 0
In H:
deg (a) = 4
deg (b) = 6
deg (e) = 6
deg (c) = 1
deg (d) = 5

In and out degree of a vertex in directed graph:

In degree:

In-degree of a vertex V in a directed graph G(V, E) is the total number of incoming edges to the vertex.

Out degree:

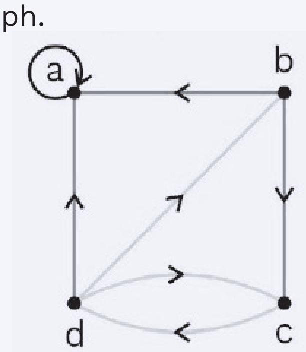
Out-degree of a vertex V in a directed graph G(V, E) is the total number of outgoing edges from a vertex.

Note:

- A loop at a vertex contributes one to both the in-degree and the out degree of this vertex.
- As each edge has an initial vertex and a terminal vertex, the sum of the in-degree and the sum of the out-degree of all vertices in a graph with directed edges are the same. Both of these sums are the number of edges in the graph.

Rack Your Brain

Determine number of vertices and edges and find the in-degree and out-degree of each vertex for given directed multigraph.



Handshaking theorem: Further consequent theorems

Handshaking theorem states that the total number of edges contained in a graph is half the degree sum of all the vertices.

$$\sum_{u \in v} \deg(u) = 2|E|$$

Note:

This theorem is applicable even if multiple edges and self loops are present (pseudograph) in the graph.

Corollary 1: Let G = (V,E) be a directed graph with V = {V_1, V_2 , ..., V_n} then

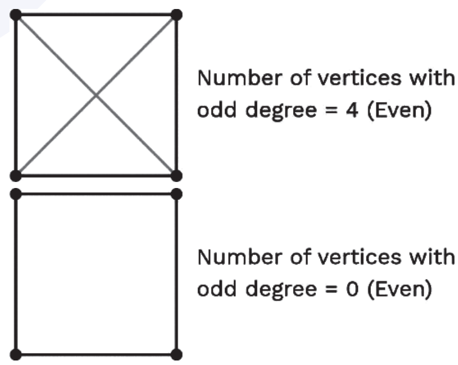
$$\sum_{i=1}^{n} \deg^{+}(V_i) = \sum_{i=1}^{n} \deg^{-}(V_i) = |E|$$

i.e., for a directed graph sum of all the out degrees of vertices is equal to sum of all in degrees of the vertices, which is equal to number of edges in the graph because in directed graph every edge will be counted as in degree for some vertex and out degree for some other vertex.

where e = number of edges in the graph.

deg(V) = in-degree(V) + out-degree(V)

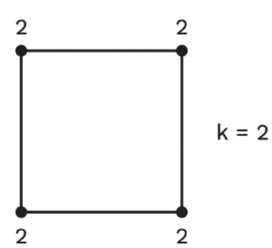
Corollary 2: In an undirected graph, the number of odd degree vertices are even.



Corollary 3: If G is an undirected graph with degree of each vertex k, then k×|v| = 2|E|.

Corollary 4: If G is an undirected graph with degree of each vertex atleast k (≥ k) then k×|v| ≤ 2|E|.

Example:

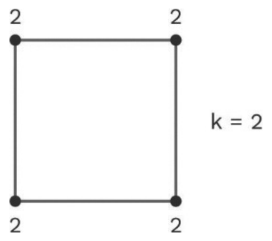


2 × 4 ≤ 2 × 4

8 ≤ 8

Corollary 5: If G is an undirected graph with degree of each vertex atmost k ($\leq$ k) then k*|v| $\geq$ 2|E|.

Example:



2 × 4 $\geq$ 2 × 4

Degree sequence:

- Arrangement of degree of all the vertices either in ascending or descending order, the sequence then obtained is called degree sequence.
- In any undirected graph, if degree of each vertex is distinct, then simple graph does not exist.

Havell-Hakimi theorem:

The purpose of this theorem is to check if a given degree sequence is a degree sequence of a simple graph or not.

Consider the degree sequences S_1 and S_2, and assume that S_1 is in descending order.

S_1: $\{S, t_1, t_2, \ldots t_S, d_1, d_2, \ldots, d_n\}$

S_2: $\{t_1 - 1, t_2 - 1, \ldots, t_S - 1, d_1, d_2, \ldots d_n\}$

S_1 is graphic & S_2 is graphic.

Steps involved in this theorem:

- Sort given degree sequence in descending order.
- Pick the largest element and remove it from the list.
- Construct a new list.
- Take number of elements = largest element chosen.
- Subtract 1 from all the taken elements.
- Check if obtained degree sequence is valid or not (say this S_2).
- S_1 is a valid graphical degree sequence if S_2 is a valid graphical sequence.
- Repeat steps 1-7.

Points to remember:

- If we get atleast one negative term in degree sequence, then simple graph does not exist for the given degree sequence.
- When there are not enough degree present in the degree sequence, then also simple graph does not exist.

Solved Examples

2. <6, 6, 6, 6, 3, 3, 2, 2>, determine whether the given degree sequence represents a simple undirected graph.

Solution:

<6, 6, 6, 6, 3, 3, 2, 2>

<5 5 5 2 2 1 2>

<5 5 5 2 2 2 1>

<4 4 1 1 1 1>

<3 0 0 0 1>

<3 1 0 0 0>

<0 –1 –1 0> As, it is visible that there are two negative integers present in degree sequence. Therefore it can not represent a simple undirected graph.

3. <7, 6, 6, 4, 4, 3, 2, 2>, determine whether the given degree sequence represents a simple undirected graph.

Solution:

<7, 6, 6, 4, 4, 3, 2, 2>

<5 5 3 3 2 1 1>

<4 2 2 1 0 1>

<4 2 2 1 1 0>

<1 1 0 0 0>

<0 0 0 0>- All zeros present, therefore, it represents a simple undirected graph.

4. <8, 7, 7, 6, 4, 2, 1, 1>, determine whether the given degree sequence represents a simple undirected graph.

Solution:

In the above given degree sequence, after choosing 8 we need to subtract 1 from 8 elements out of remaining elements but there are only 7 elements left. Therefore, given degree sequence is not valid sequence for simple undirected graph.

OR

If we have 8 vertex maximum degree is 7, not 8. So simple graph does not exist.

5. How many edges are there in a graph with 10 vertices each of degree six?

Solution:

2e = 60(Using handshaking theorem).

Therefore e = 30.

> **Note:**
>
> The undirected graph that results from ignoring directions of edges is called the underlying undirected graph.

Previous Years' Question

The degree sequence of a simple graph is the sequence of the degrees of the nodes in the graph in decreasing order. Which of the following sequences can not be the degree sequence of any graph? **[GATE CSE 2010]**

I. 7, 6, 5, 4, 4, 3, 2, 1

II. 6, 6, 6, 6, 3, 3, 2, 2

III. 7, 6, 6, 4, 4, 3, 2 ,2

IV. 8, 7, 7, 6, 4, 2, 1, 1

(A) I and II (B) III and IV

(C) IV only (D) II and IV

Solution: (D)

Some special simple graphs:

Null graph:

A graph having 'V' vertices and zero edges is known as null graph.

Trivial graph:

Trivial graph consists of one vertex and zero edges.

Regular graph:

A graph in which all the vertices have same degree is known as regular graph.

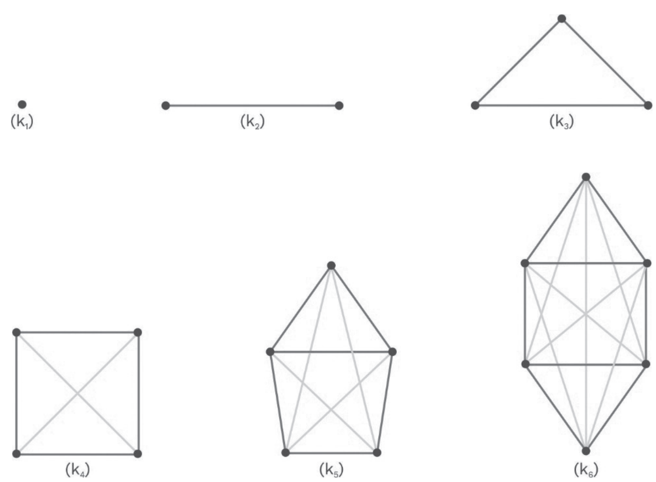
> **Note:**
>
> Every polygon is a 2-regular graph.

2-regular 3-regular

> **Note:**
>
> In a simple graph with n vertices at least two vertices should have same degree.

Complete graph:

A simple graph with V vertices where every vertex has an edge with all the other vertices is known as complete graph. The graph k_n, for n = 1, 2, 3, 4, 5, 6 as shown in the given figure.



Note:

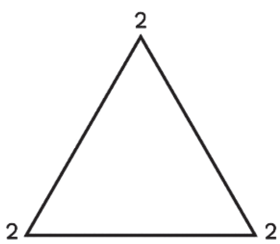
- A complete graph is a simple graph with maximum number of edges.
- A complete graph is a simple graph with every vertex having maximum degree.

Properties of complete graph:

- Every complete graph is a regular graph, but every regular graph need not be complete.
- A complete graph is a simple graph with maximum number of edges.
- Number of edges in $k_n = \frac{n(n-1)}{2}$
- Degree of each vertex = $(n - 1)$

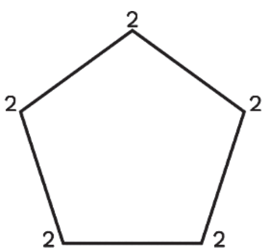
Cycle graph:

The cycle graph C_n, for $n \geq 3$, is a simple connected graph with degree of every vertex as 2.

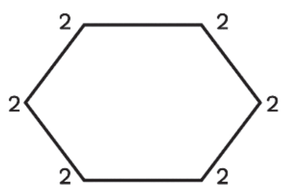


C_3

C_4



C_5



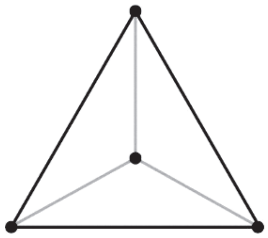
C_6

Note:

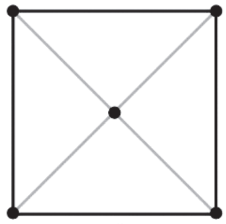
If a graph is a cycle graph then number of vertices is equal to number of edges but not vice versa.

Wheel graph:

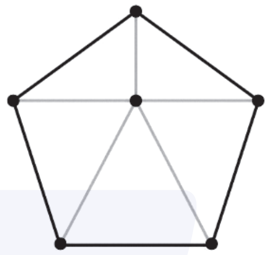
Wheel graph represented by w_n, where n is greater than or equal to four, can be obtained by adding a vertex to the cycle graph such that the new vertex is adjacent to all vertices.



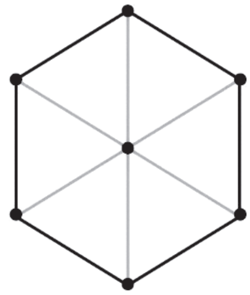
W_4



W_5



W_6



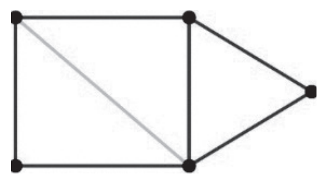
W_7

Note:

Number of edges in $W_n = 2(n - 1)$

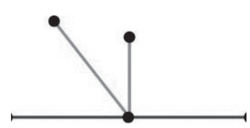
Cyclic graph:

A cyclic graph is a simple graph which contains atleast one cycle.



Acyclic graph:

Simple graph having no cycle is known as acyclic graph.



Bipartite graph:

Definition

"A simple graph G is called a bipartite graph if its vertex set V can be partitioned into two disjoint sets V_1 & V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G."

Example:

Consider the following cyclic graph C_6. The vertex set of C_6 can be partitioned into two sets V_1 = {1,3,5} and V_2 = {2,4,6}, such that there is no edge between vertices of same set: therefore C_6 is a bipartite graph.

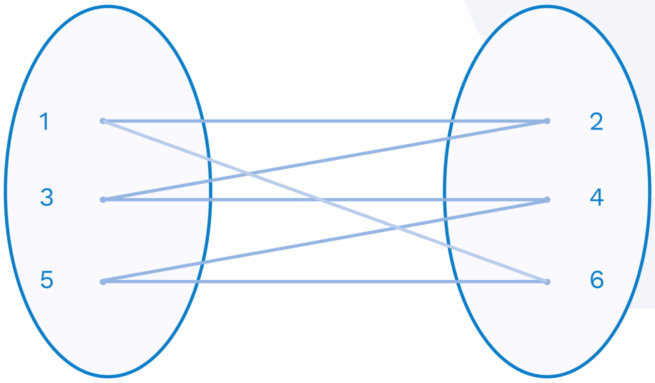
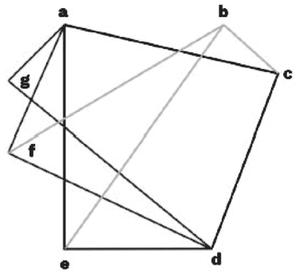


Fig. 3.6 C_6 is Bipartite

6. Determine if given graph is bipartite?



Solution:

The vertex set of the following graph can be partitioned into two sets, V_1 = {a,b,d} and V_2 = {c,e,f,g}.

Since, there is no edge between the vertices of same set, so the given graph is bipartite graph.

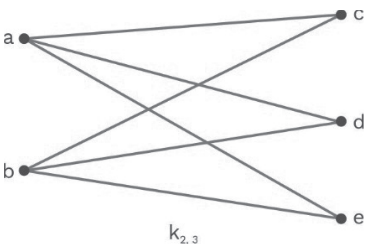
Note:

- A simple graph is bipartite if and only if it is possible to assign one of two different colours to each vertex of the graph so that no two adjacent vertices are assigned he same colour.
- A graph is bipartite iff it has no odd length cycle.

Complete bipartite graph:

The complete bipartite graph $K_{m,n}$ is a graph in which vertex set V is divided into two subsets, P having m vertices and Q having n vertices respectively, such that every vertex from set P is connected to every other vertex in set Q and there is no edge between the vertices from same subset.

Example:

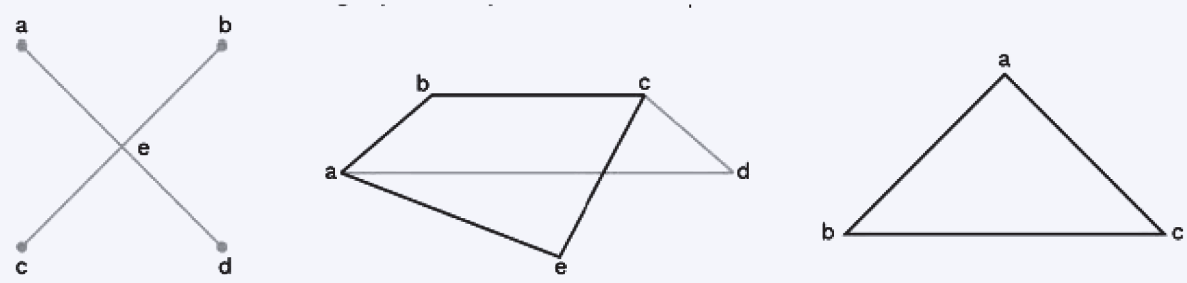


Note:

$K_{m,n}$ has (m + n) vertices and m*n edges.

Rack Your Brain

Determine whether the graph is bipartite or not.



(I) (II) (III)

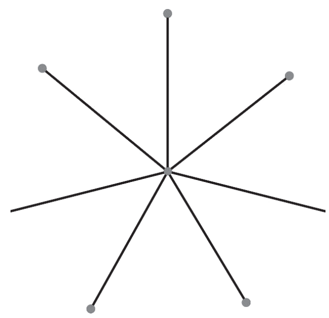
Star graph:

Star graph is a special type of graph in which a single vertex is connected to all the other vertices.

For a graph having n vertices, n – 1 vertices will have degree 1 and one vertex will have degree n – 1.

It is represented as: $k(1, n-1)$.

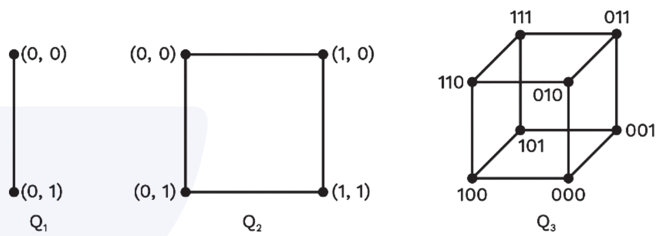
- Used in client server architecture.



Hypercube graph/n dimensional cube:

In hypercube graph, there is an edge between two vertices only if the vertices differ by single bit position.

- The hypercube graph is represented by Q_n.
- Order of hypercube graph $O(Q_n)$ = number of vertices = 2^n
- Q_n is a n degree regular graph.
- Number of edges in $Q_n = n \times 2^{n-1}$ (using handshaking lemma i.e., in every finite undirected graph, the number of vertices that touch an odd number of edges is even).



Connected graph:

- A graph having V vertices and E edges, is said to be connected if there is a path between all the pairs of vertices in the graph.
- Maximally connected subgraph of a graph is a component.
- Every connected graph has exactly one component.
- If G is a simple graph with n vertices, e edges, and k components.

$$n-k \le e \le \frac{(n-k)(n-k+1)}{2}$$

Examples of connected graph:

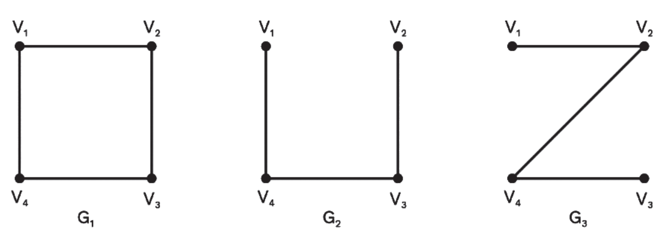


Fig. 3.6 Connected Graphs

Disconnected graph:

For a disconnected graph ($k > 1$ or $k \geq 2$), the path is not available between atleast one pair of vertices.

Examples of disconnected graph:

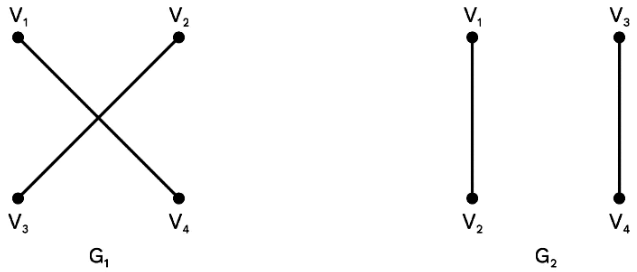


Fig. 3.7 Disconnected Graphs

> **Note:**
>
> - If G is a simple graph with n-vertices and IEI edges, $|E| > \frac{(n-1)(n-2)}{2}$, then G is connected (sufficient condition)
> - in simple connected graph with n vertices, $n-1 \leq e \leq \frac{n(n-1)}{2}$

Rack Your Brain

Which of the following graph is always connected?

(A) G with 5 vertex and 5 edges

(B) G with 5 vertex and 12 edges

(C) G with 5 vertex and 10 edges

(D) G with 10 vertex and 18 edges

Complement of a graph:

The complement of an undirected graph having v vertices is denoted by $\bar{G}$, which have the same number of vertices as that of G and an edge $\{u, v\} \in \bar{G}$ iff $\{u, v\} \notin G$.

> **Note:**
>
> - $G \cup G = kn$, where n = number of vertices.
> - $|E(G)| + |E(G)| = |E(kn)|$, where $n = |v(G)|$

Example: Consider a simple graph G having 8 vertices and 15 edges. Calculate the number of edges in $\bar{G}$ ($|E(\bar{G})|$).

Solution:

We have

$|E(G)| + |E(\bar{G})| = |E(k_n)|$

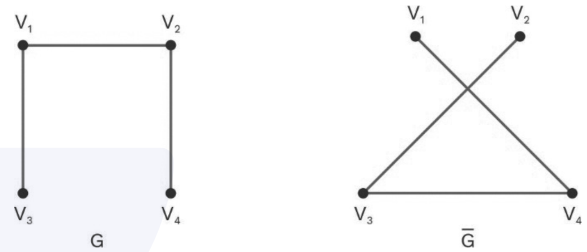
$15 + |E(\bar{G})| = \frac{n(n-1)}{2} = \frac{8 \times 7}{2} = 28$

$|E(\bar{G})| = 28 - 15$

$|E(\bar{G})| = 13$

Self complementary:

When a graph is isomorphic to its complement is called self complementary graph.



$\boxed{G \cong \bar{G}}$

> **Note:**
>
> - The number of edges in a self complementary graph is $= \frac{n(n-1)}{2}$
> - The number of vertices in a self complementary graph is of the form 4k or 4k + 1, k is an any positive integer.

New graph from old graph:

Sometimes only a part of the graph is needed to solve a problem. In this case, we can remove some vertices, and all edges incident on that vertex are removed. This will help in obtaining a smaller graph, called as a subgraph of the original graph.

Definition

A subgraph of a graph G (V, E) is a graph H (W, F) where, $W \subseteq V$ and $F \subseteq E$.

A subgraph H of G is a proper subgraph of G if $H \neq G$.

The following graph is subgraph of K_5.

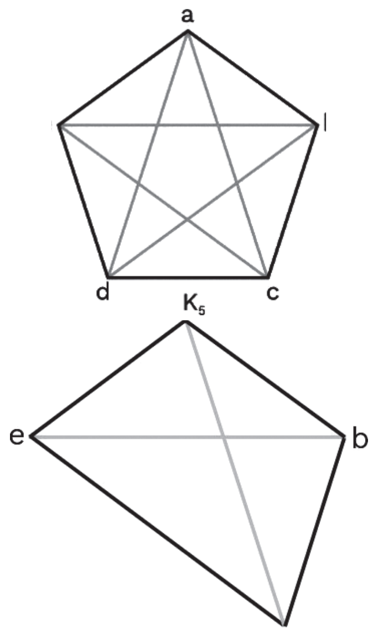


Fig. 3.8 Subgraph of K_5

Union of graphs:

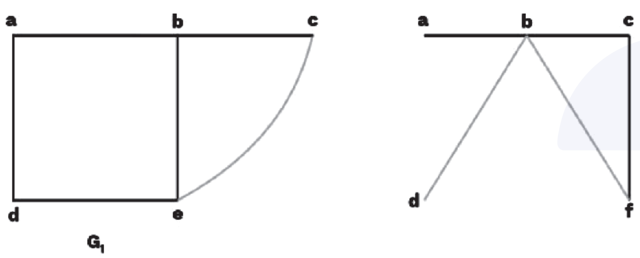
Two or more graphs can be combined in various ways; the new obtained graph after combining all the vertices and edges is called the union of graph(s).

Definition

The union of two simple graphs G1 = (V_1, E_1) and G2 = (V_2, E_2) is a simple graph with vertex set V_1 u V_2 and edge set E_1 u E_2• The union of G_1 and G2 is denoted by G_1 u G_2.

Solved Examples

7. Determine the union of the following graphs?



Solution:

The vertex set of $G_1 \cup G_2$ is the union of the two vertex sets, namely, {a, b, c, d, e, f}. The edge set of $G_1 \cup G_2$ is the union of the two edge sets.

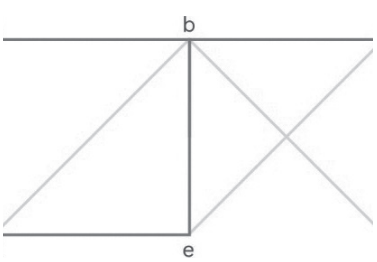


Fig. 3.9 $G_1 \cup G_2$

Representing graphs:

A graph can be represented in many ways:

- One way to represent a graph without multiple edges is to list all the edges of the given graph. Another way is to use an adjacency list.

Example:

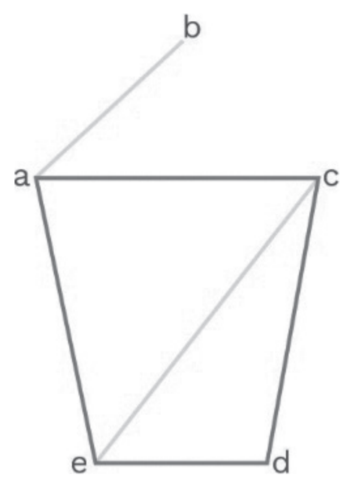


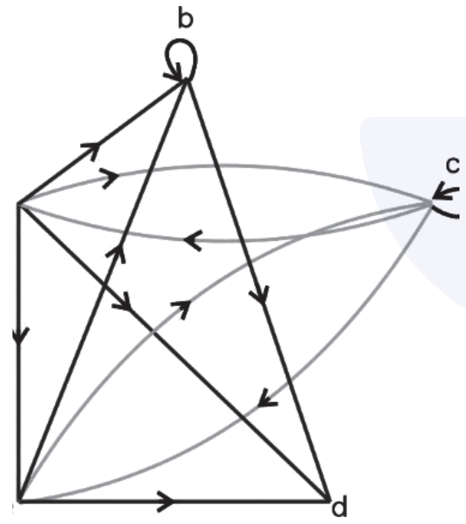
Fig. 3.10 A Simple Graph

The adjacency list for the following simple graph will be.

Initial Vertex	Terminal Vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Fig. 3.11 An Adjacency List

Example: Consider the following graph. What will be its adjacency list representation?



Solution:
For the given directed graph, the adjacency list representation will be as follows:

Initial Vertex	Terminal Vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

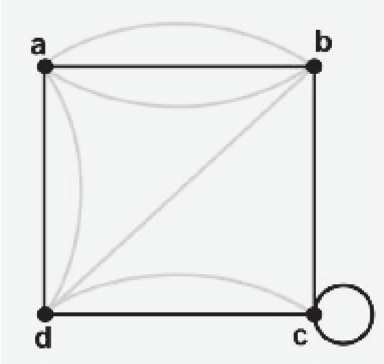
- Another method used for representing graphs is the adjacency matrix.

  For a simple graph whose vertices are listed as 1, 2, ..., n. The adjacency matrix A with respect to the listing of vertices is nxn zero-one matrix.

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

Solved Examples

8. Consider the following pseudo graph. What will be the adjacency matrix representation of the following graph?

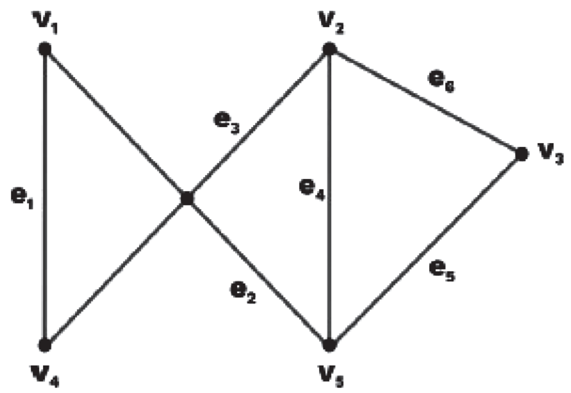


Solution:
The adjacency matrix for the following graph will be:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

When a graph contains relatively fewer edges, i.e., when it is sparse, it is preferable to use an adjacency list rather than an adjacency matrix to represent a graph.

9. Represent the following graph with an incidence matrix.



Solution:

The incidence matrix for above graph is:

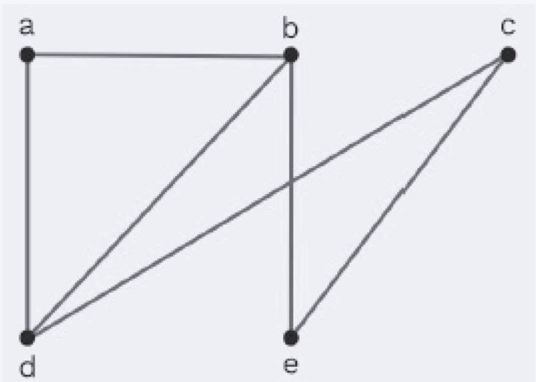
$$
\begin{array}{c|cccccc|}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 1 & 0 & 1 \\
3 & 0 & 0 & 0 & 0 & 1 & 1 \\
4 & 1 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

Note:

Incidence matrix can also be used to represent multiple edges and self loops. Multiple edges are represented in the incidence matrix using columns with identical entries. Loops are represented using a column with exactly one entry = 1, corresponding to the vertex that is incident with this loop.

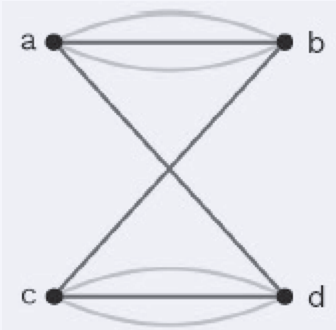
Rack Your Brain

What will be the adjacency list representation of the following graph?



Rack Your Brain

What will be the adjacency matrix representation of the following graph?



Walk, trail, circuit, path, and cycle:

Walk:

A walk is an alternative sequence of vertices and edges, which begins and end with a vertex. Both edges and vertices can be repeated in a walk.

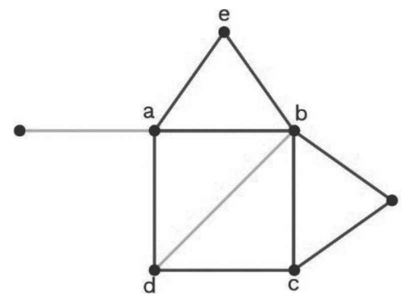
Trail:
A walk is said to be a trail when there is no repetition of edges, but vertex can be repeated.

Closed trail:
A trail starting and ending at the same vertex is known as a closed trail.
A closed trail is called a circuit.

Path:
It is a walk in which repetition of vertices and edges is not allowed.

Closed path:
A path that starts at a vertex and ends in the same vertex is called a closed path.

It is also called cycles.



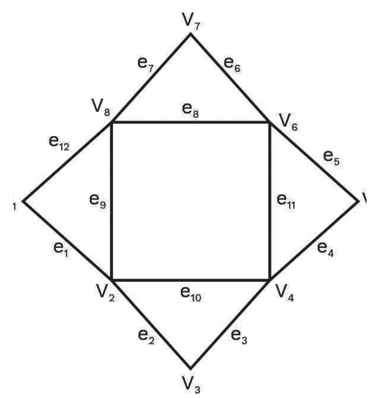
Walk (u, v) = u – a – b – d – a – b – v
Trail (u, v) = u – a – b – d – c – b – v
Circuit (a, a) = a – b – v – c – b – d – a
Path (u, v) = u – a – b – v
Closed path (d, d) = d – a – e – b – d

Graph	Repetition of Edge	Repetition of Vertex
Walk	Allowed	Allowed
Trail	Not allowed	Allowed
Path	Not allowed	Not allowed

Note:

A cycle is a part of circuit but a circuit can't always be a cycle.

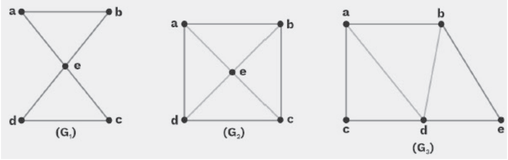
Euler graph:
If a graph contains Euler circuit, then it is known as an Euler graph.
Or
A graph is an Euler graph if it has a closed trail containing all edges.



The above graph has Euler circuit $V_1 - e_1 - V_2 - e_9 - V_8 - e_8 - V_6 - e_{11} - V_4 - e_{10} - V_2 - e_2 - V_3 - e_3 - V_4 - e_4 - V_5 - e_5 - V_6 - e_6 - V_7 - e_7 - V_8 - e_{12} - V_1$

Solved Examples

10. Which of the following are Euler graph?



Solution:

Graph G_1 has an Euler circuit (a – e – d – c – e – b – a), i.e., there is a closed trail that covers all the edges. Therefore, it is an Euler graph. But G_2 and G_3 do not contain Euler circuit. So G_2 an G_3 are not Euler graph.

Note:

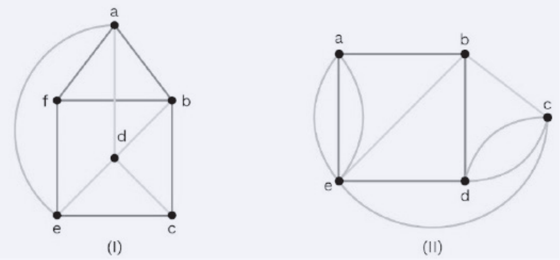
- Euler circuits are also applicable to the graph with loops (unless it is a loop in a component with one vertex).
- Euler circuits are also applicable to the Multigraph.

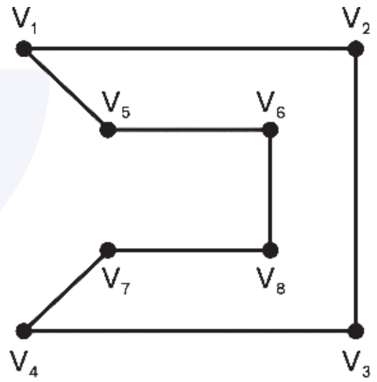
Result:

A graph is an Euler graph if and only if it is connected and $\forall$ V $\in$ G, degree (V) is even.

Rack Your Brain

Determine whether the given graph is Euler graph?

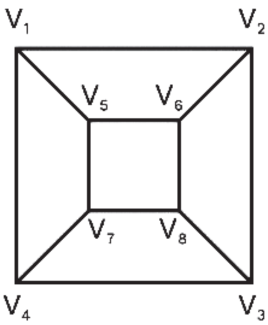




The given graph is a Hamiltonian graph because of Hamiltonian cycle
$V_1 - V_5 - V_6 - V_8 - V_7 - V_4 - V_3 - V_2 - V_1$.

Hamiltonian graph:

A graph is said to be a Hamiltonian graph, if it contains a Hamiltonian cycle.
Hamiltonian cycle, in a connected graph, is a cycle that covers all the vertices.
If any edge is removed from the Hamiltonian cycle, then it gets converted into Hamiltonian path.



Sufficient condition for hamiltonian graph:

DIRAC's theorem:

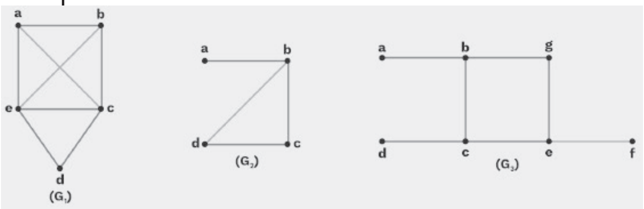
A simple connected graph G having 'v' vertices ($v \geq 3$), is said to have a Hamiltonian cycle if the minimum degree of every vertex in the graph is n/2.

ORE's theorem:

If G is a simple connected graph with n vertices ($n \geq 3$), such that $deg(u) + deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamiltonian cycle.

Solved Examples

11. Which of the following simple graphs have a Hamiltonian cycle or Hamiltonian path?

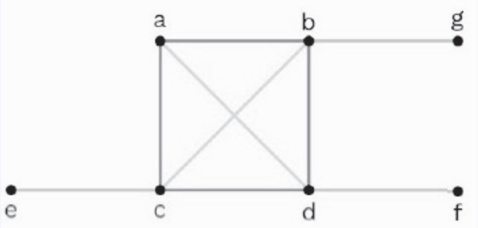


Solution:

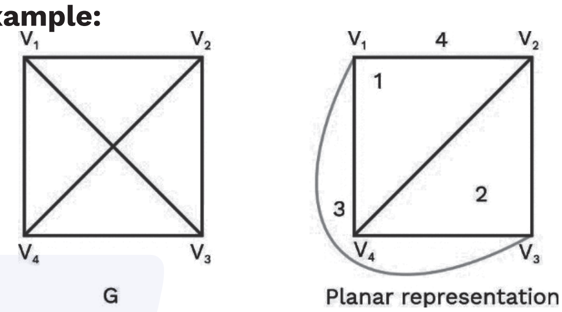
G_1 has a Hamiltonian cycle (a – b – c – d – e – a). There is no Hamiltonian cycle in G_2 but G_2 has a Hamiltonian path (a – b – c – d). G_3, neither has a Hamiltonian cycle nor a Hamiltonian path because edges {a – b}, {e – f}, and {e – d} are covered more than one time.

Rack Your Brain

Determine whether the following graph is Hamiltonian graph?



Example:



The given graph G is a planar graph because in planar representation, no edges intersect each other.

Planar graph:

- A graph having a planar representation is called a planar graph.
- Planar representation means drawing a graph on a plane without crossing the edges.
- The planar representation of a planar graph divides the entire plane into regions or faces.

Definition

"A graph is called planar, if it can be drawn on the plane without any edges crossing (where a crossing of edges is the intersection of the Lines or arcs representing them at a point other than their common end point). Such a drawing is called planar representation of a graph."

Euler's formula:

As we have already seen, a planar graph splits the plane into regions, including an unbounded region. According to Euler, for any simple connected graph having 'V' vertices and 'e' edges, the number of regions 'r' in the planar representation of graph is given by:

$$V - e + r = 2$$

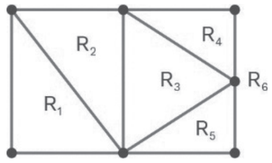


Fig. 3.12 The Regions of Planar Representation of Graph

Results for planar graph:

Minimum degree for region k

- $V - e + r = 2$
- $kr \leq 2e$
- $e \leq \dfrac{k(V-2)}{k-2}$

Polyhedral graph:

A planar graph in which every interior region is polygon is called polyhedral graph.

- In a polyhedral graph degree of every vertex, i.e., $\deg(V) \geq 3\ \forall\ V \in G$
- For a polyhedral graph, the following inequality must hold:
  - $V - e + r = 2$
  - $3r \leq 2e$
  - $e \leq 3V - 6$

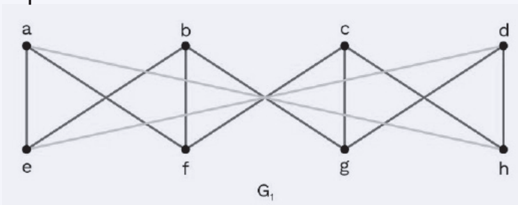
Previous Years' Question

Let G be a connected planar graph with 10 vertices. If the number of edges on each face is three, then the number of edges in G is **[2015 Set 1]**

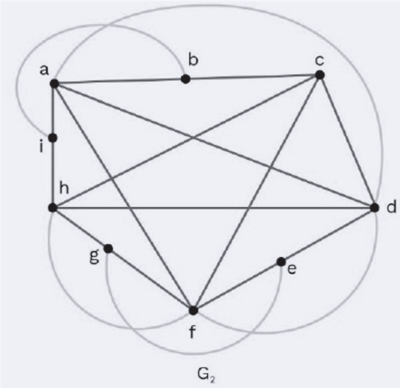
Solution: 24

Rack Your Brain

Determine which of the following graphs are planar?



G_1



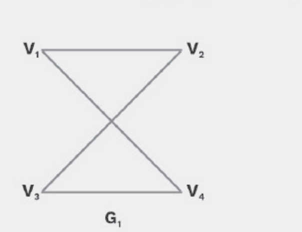
G_2

Note:

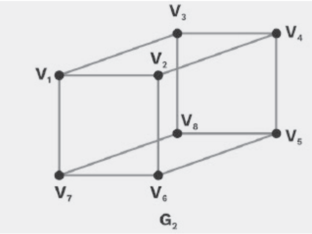
- K_5 is a non planar graph with minimum number of vertices.
- $K_{3.3}$ is non planar graph with minimum number of edges.

Solved Examples

12. Which of the following graphs are planar?



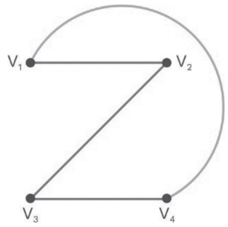
G_1



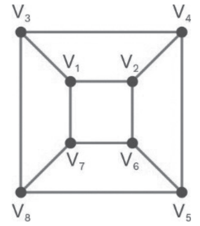
G_2

Solution:

G_1 is planar because it can be represented without edge crossings.



G_2 is also planar because it can also be drawn without any edge crossings.



13. Suppose that a connected planar simple graph has 10 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Solution:

Using Handshaking theorem, Σ degree (V) = 2 × number of edges

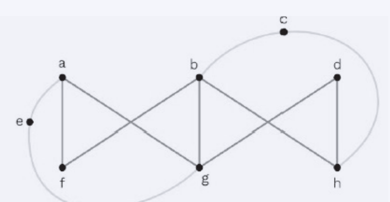
$3 * 10 = 2 \times e$
$e = 15$
From Euler's formula the number of region is
$r = e - v + 2$
$r = 15 - 10 + 2$
$r = 7$

Kuratowski's theorem:
"If a graph is planar, so will be any graph obtained by removing an edge {u, v} and adding a new vertex w together with edges {u, w} and {w, u}. Such an operation is called an elementary subdivision. The graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision."

A graph is planar if it does not contain any graph homeomorphic to K_5 or $K_{3,3}$.

Rack Your Brain

Determine whether the given graph is homeomorphic to $K_{3, 3}$.



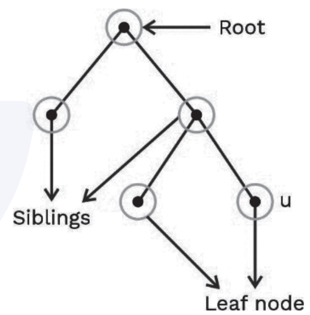
Trees:
Basically, this is the topic of data structure and algorithms but here, we will study this topic in terms of Discrete Mathematics.

Tree can be defined as a minimally connected acyclic graph.
For every (u, v), there exists exactly one path between u and v or it is a 1-connected graph.

Every tree is bi-chromatic and bipartite.

Fundamental cycle:
A cycle obtained by adding a single edge is called a fundamental cycle.
Now, number of fundamental cycles = ${}^nc_2$ when n = number of vertices.

Rooted tree:
A tree with a specific vertex is chosen as a root.



Root, siblings (having a common parent), leaf node can be only defined for a rooted tree.

depth (u) = distance (u, root), where u is node

height (T) = Maximum of depth (u), where u is node and T is a tree

level (u) = 1 + depth (u)

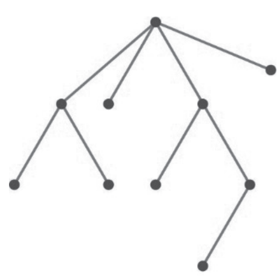
OR

level (T) = 1 + height (T), where u is node and T is a tree

K-ary tree:

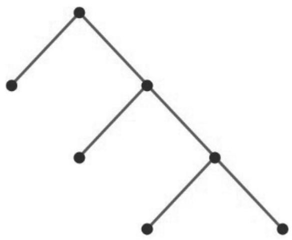
It is a rooted tree.

$0 \leq$ number of children (u) $\leq k$



4-ary tree

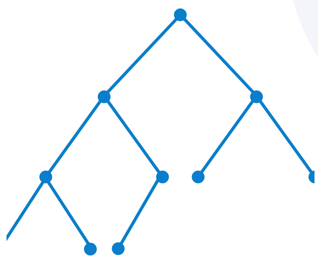
Full k-ary tree:
- It is a rooted tree.
- Number of children (u) = 0 or k



full 2-ary tree

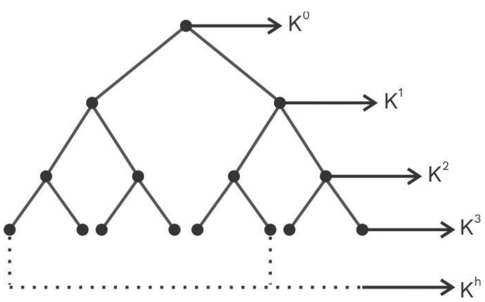
Complete k-ary tree:
- It is a rooted tree.
- All levels except the last one are completely filled.
- Last level is left adjusted.

Left adjusted:
When we start filling from left to right, then it is called left adjusted.



> **Note:**
>
> Complete k-ary tree does not implies a full k-ary tree and vice versa.

Properties of k-ary tree:
Considering, l = number of leaves
n = Number of nodes
h = Height of tree
i = Number of internal nodes
For maximum number:
1. $l \leq k^h$



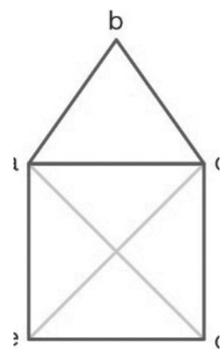
2. $n \leq \dfrac{k^{h+1}-1}{k-1}$
3. $i \leq \dfrac{k^h-1}{k-1}$

For minimum number:
1. $n \geq h+1$
2. $i \geq h$
3. $\log_k l \leq h$

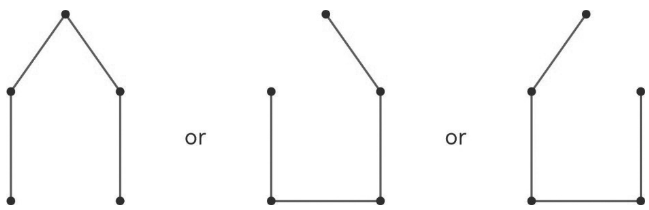
Spanning tree:
A spanning tree of G is a subgraph of G which includes every vertex of G.

> **Note:**
>
> Every connected graph has a spanning tree.

Example:



Solution:
Possible spanning trees:

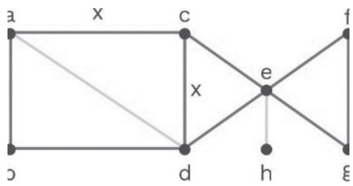


A subgraph H of G is called a spanning tree of G, if

H is a tree | H contains all vertices of G

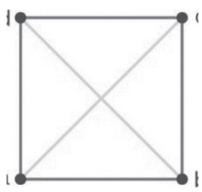
Circuit rank: The circuit rank of any given graph can be defined as the number of edges that are required to remove in order to obtain a spanning tree that is equal to [m-(n-1)], where m is the number of edges, and n equals to the number of vertices.

Example:



Circuit rank (G) $= m - (n - 1)$
$= 4$

Example: For the given graph, calculate the total number of spanning trees possible.



(A) 6
(B) 12
(C) 8
(D) 16

Solution:

Number of spanning trees in a complete graph

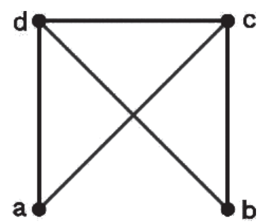
$= n^{n-2}$ (Cayley's formula)
$= 16$

Kirchoff's theorem:

For a connected graph G, let A be its adjacency matrix representation.
Consider a matrix M obtained by replacing all 1's in matrix A with -1 and all 0's in principle diagonal of matrix A with the degree of the corresponding vertex.
Cofactor of any element of M is equal to the number of spanning trees in G.

Example: Consider the following matrix representation of a graph. Calculate the number of possible spanning trees for the graph.

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad M = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$



Cofactor of M_{11} $= (-1)^{1+1} \begin{vmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}$

$= 2(9 - 1) + 1(-3-1) - 1(1 + 3)$
$= 16 - 4 - 4 = 8$

Graph colouring:

Vertex colouring: In vertex colouring, every vertex of the graph is given a colour such that the vertices adjacent to each other must have different colours.

Chromatic number: Chromatic number is a number that tell minimum how many colours will be needed to colours all the vertices of the graph such that no two vertices have the same colour. It is denoted as $\lambda(G)$

- $\lambda(G) = 1$ if G is a null graph
- If G is not a null graph, then $\lambda(G) \geq 2$
- A graph G is said to be n-colourable, if there exists a vertex colouring that uses atmost n colours, i.e., $\lambda(G) \leq n$

Four colour theorem:

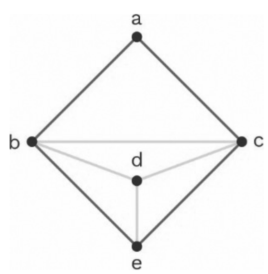
Every planar graph G is 4-colourable, i.e., $\lambda(G) \leq 4$.

Welsh-Powell algorithm:

Welsh-Powell Algorithm

1. Find the degree of each vertex.
2. Arrange vertices of G in the descending order of their degrees.
3. Color the first vertex with color C_1 in sequential order, assign C_1 to each vertex which is not adjacent to previous vertex which was assign C_1.
4. Repeat step (3) with a second color C_2 and the subsequence of non-color vertices.
5. Repeat step (4) with a third color C_3, then a fourth color C_4 and so on until all vertices are colored.
6. Exit.

Example: Consider the following graph, what will be its chromatic number?



(A) 2

(B) 3

(C) 4

(D) 5

Answer: (C)

Solution:

Vertex	a	b	c	d	e
Color	C_1	C_2	C_3	C_1	C_4

As the graph is planar, we will apply four-colour theorem chromatic number

$\lambda(G) \leq 4$ ... (i)

Further, we have 4 mutually adjacent vertices {b, c, d, e}

$\therefore \lambda(G) = 4$ ... (ii)

From (i) and (ii)

$\lambda(G) = 4$

Connectivity:

Vertex connectivity:

In a connected graph G, the minimum number of vertices whose deletion makes graph disconnected or reduces G into a trivial graph is called vertex connectivity of a connected graph G. It is denoted by k(G)

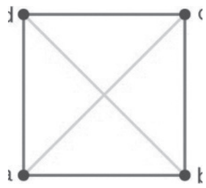
- If G has a cut vertex, then k(G) = 1

Edge connectivity:

- In a connected graph G, the minimum number of edges whose deletion makes the graph disconnected is called edge connectivity of G.
  It is denoted by $\lambda(G)$.
- If G has a cut edge, then edge connectivity $\lambda(G) = 1$ (also known as bridge).

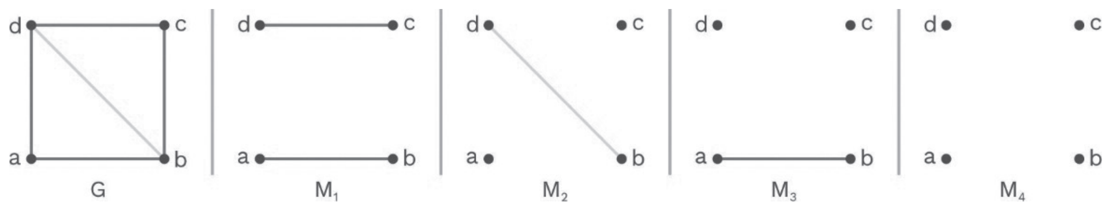
- The number of edges in the smallest cut set of G is said to be edge connectivity of G.

Example: For complete graph k_n
Vertex connectivity of $k_n = n - 1$
Edge connectivity of $k_n = n - 1$



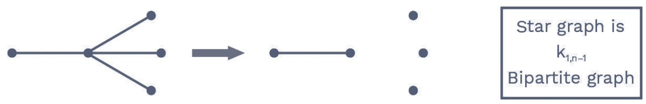
Cut set:
It is the minimum set of edges whose removal will disconnect the graph.

Matching and covering:
- **Matching:** If every vertex of given graph G is incident with at the most one vertex, i.e., $\deg(V) \leq 1$, then that subgraph is called matching of the given graph.

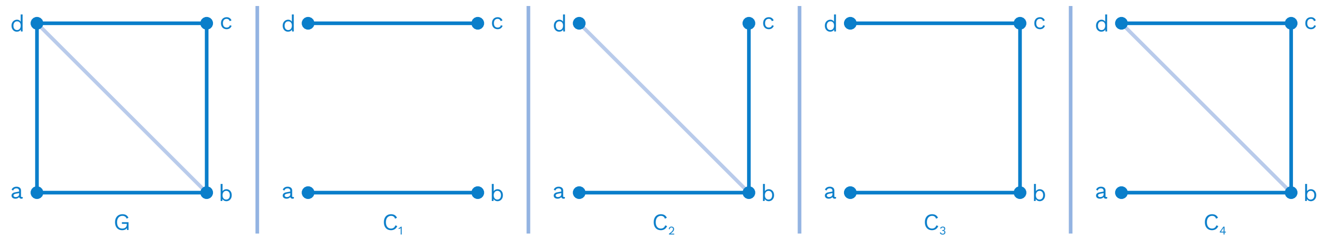


In a matching, no two edges are adjacent, so $\deg(V) \leq 1$
- **Maximal matching:** If in any matching M of graph G, further no more edges can be included then M is said to be maximal matching.
  M_1 and M_2 are the maximal matching of G in the above example.
- **Maximum matching:** For any matching M of Graph G, if M consists of the maximum number of edges.
- **Matching number:** It is the number of edges included in a maximum matching of G.
  Matching number of $M_1 = 2$
- **Perfect matching:** A matching of a graph in which every vertex is matched is called perfect matching.

Example: What is the matching number of star graph with n vertices ($n \geq 2$) is:



Solution: 1



- K_n has a perfect matching iff n is even.

Previous Years' Questions

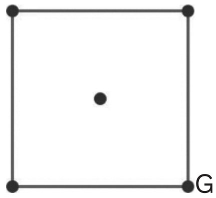
How many perfect matching are there in a complete graph of 6 vertices?
[GATE CSE 2003]

(A) 15
(B) 24
(C) 30
(D) 60

Solution: (A)

Covering:
Line covering:
A subset S(E), of graph G(V, E), will be a line covering of G if all the vertices of G will incident with a minimum one edge in S, i.e., degree of each vertex will be atleast 1.

If, in any case, graph G has an isolated vertex, line covering will not exist.

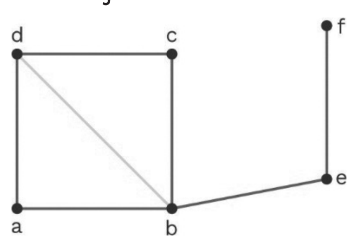


Minimal line covering: Line covering is minimal if the deletion of any edge from the line cover is not possible.

Minimum line covering:

- The number of edges present in minimum line covering is called the line covering number of a graph G = α_1
  The graph above example has α_1 = 2.
  (C_1 graph is minimum line covering).
- Line covering of graph with n vertices contains atleast $\left\lceil \frac{n}{2} \right\rceil$ edges.
- No minimal line covering contains a cycle.
- In a line covering, if there is no path of length 3 or more, then C is minimal.
- In the line covering, if there are no paths of length 3 or more, then all components of C are star graphs. Then from those star graphs, no edge can be deleted.

Independent line set:
Let, G = (V, E) be a graph, then a subset L of E is called an independent line set if no two edges in L are adjacent.



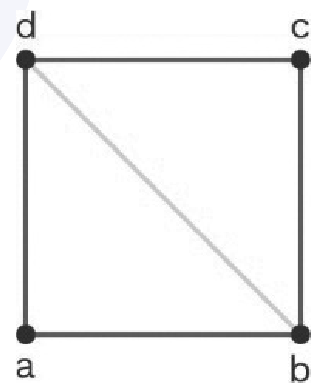
L_1 = {b, d}
L_2 = {(b, d), (e, f)}
L_3 = {(a, d), (b, c), (e, f)}
L_4 = {(a, b), (e, f)}

Maximal independent line set: For a graph G having an independent line set L, if no more edges of G can be included in L, then L is said to be a maximal independent line set.

Maximum independent line set: The largest maximal independent line set L of graph G which contains a maximum number of edges, is known as the maximum independent line set.

- Number of edges present in the maximum independent line set of a graph is known as the line independent number of G. It is denoted by β_1
- Line independent number = matching number of G_1
  For the graph in above example L_3 is a maximum independent line set and β_1 = 3.
- For any graph $G_1 \alpha_1 + \beta_1 = |V|$ where α_1 is the line covering number.

Vertex covering: Let G = (V, E) be graph then a subset k of V is called a vertex covering of G, if every edge of G is incident with a vertex in k.



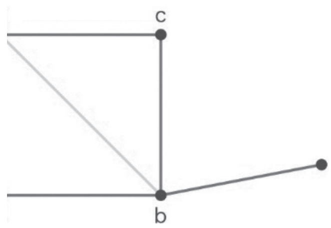
k_1 = {b, d}
k_2 = {a, b, c}
k_3 = {b, c, d}

Minimal vertex covering: The vertex covering k is said to be minimal vertex covering if no vertex can be eliminated from it.
k_1 and k_2 are minimal vertex covering.

Minimum vertex covering: The minimum number of vertices in the vertex covering of a graph is called minimum vertex covering.

- The vertex covering a number of graph G is denoted by α_2, which is defined as the total number of vertices present in minimum vertex covering.
  In the above graph G for k, α_2 = 2

Independent vertex set:
Let G = (V, E) be a graph, then the subset S of V is called an independent set if no two vertices in S are adjacent.



S_1 = {b}
S_2 = {d, e}
S_3 = {a, c}

Maximal independent vertex set: An independent vertex set is said to be maximal, if no other vertex of G can be added to the set.

Example: S_1 = {b}
S_2 = {d, e}
S_3 = {a, c, e}

Maximum independent vertex set:

- Vertex independent number tells total how many vertices are there in the maximal independent vertex set. Vertex independent number of G denoted by β_2.

Example: S_3 = {a, c, e}
$\therefore \beta_2 = 3$

- For any graph $\alpha_2 + \beta_2 = |v|$
- For any graph if S is independent set of G then V – S = A vertex covering of G.

Example: For the star graph with n vertices ($n \geq 2$).
Solution: $\alpha_2 = 1$
$\beta_2 = n - 1$
$\alpha_2 + \beta_2 = n$

Example: For the cycle graph C_n ($n \geq 3$)
Solution: $\alpha_2 = \lceil n/2 \rceil$
$\beta_2 = \lfloor n/2 \rfloor$

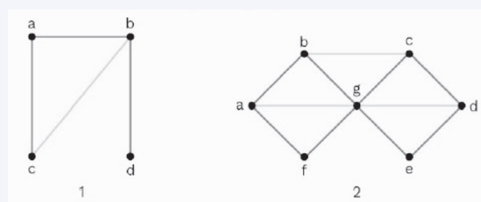
Example: Wheel graph w_n ($n \geq G$).

Solution: $\alpha_2 = \left\lceil \frac{n+1}{2} \right\rceil$

$\beta_2 = \left\lfloor \frac{n-1}{2} \right\rfloor$

Rack Your Brain

Find the chromatic number of the given graph.



Shortest path algorithms:
In order to find the shortest path between any two vertices in a graph, Dijkstra algorithm is used.

Dijkstra Algorithm

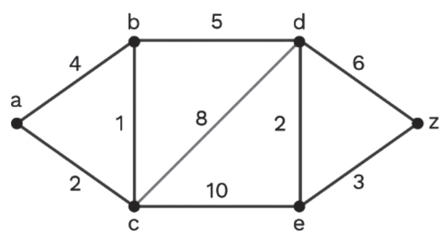
Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph. (without negative weight cycle).

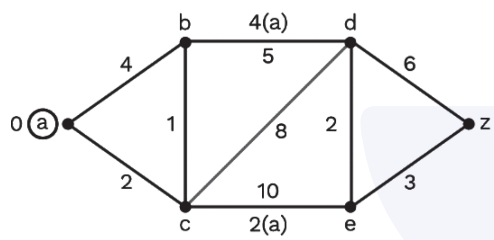
Dijkstra's algorithm uses $0(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph (without negative edge weight cycle) with n vertices.

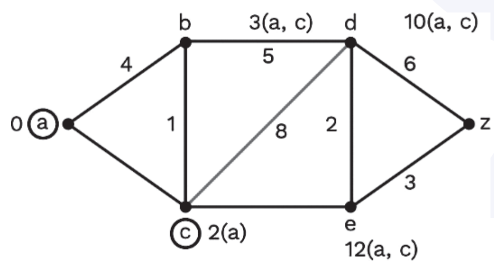
Solved Examples

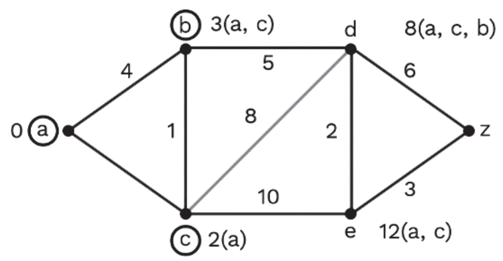
14. Use Dijkstra's algorithm to find the length of the shortest path between the vertices a and z in a weighted graph given below:

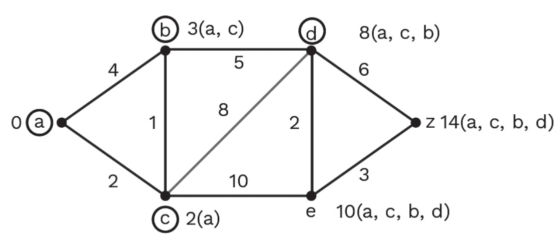
Solution:

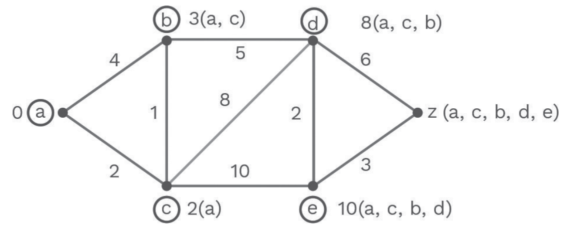
I) 

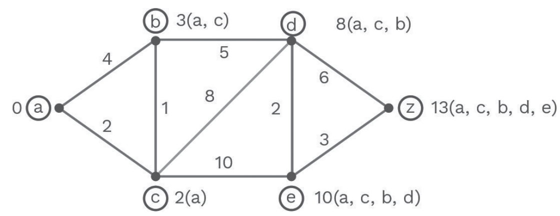
II) 

III) 

IV) 

V) 

VI) 

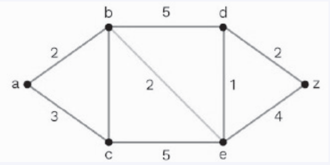
VII) 

The algorithm terminates at step (VII) when z is circled.
Therefore, the shortest path from a to z is a, c, b, d, e, z with length 13.

- The travelling salesman problem requires a circuit in a weighted, complete, undirected graph that visits each vertex exactly once and returns to its starting place with the least total weight. Because each vertex is visited exactly once in the circuit, this is equal to requesting a Hamiltonian circuit with the lowest total weight in the whole graph.

Rack Your Brain

Find the shortest path between a and z in the given weighted graph.



Grey Matter Alert!

Floyd Warshell's Algorithm
This algorithm can also be used to find the length of the shortest path between all pairs of vertices in a weighted connected simple graph. However, this algorithm cannot be used to construct shortest paths.

Previous Years' Questions

Let G be a finite group on 84 elements. The size of a largest possible proper sub-group of G is _______. **[GATE CSE 2018]**
Solution: 42

Previous Years' Questions

Let G be an undirected complete graph on n vertices, where n > 2. Then, the number of different Hamiltonian cycles in G is equal to: **[GATE CSE 2019]**
(A) n!
(B) 1
(C) (n - 1)!
(D) $\frac{(n-1)!}{2}$
Solution: (D)

Previous Years' Questions

Let G = (V, G) be a weighted undirected graph and Let T be a Minimum Spanning Tree (MST) of G maintained using adjacency Lists. Suppose a new weighed edge (u, v) E VxV is added to G. The worst case time complexity of determining if Tis still an MST of the resultant graph is: **[GATE CSE 2020]**
(A) $\Theta(|E| + |V|)$
(B) $\Theta(|E|.|V|)$
(C) $\Theta(|E| \text{ Log } |V|)$
(D) $\Theta(|V|)$
Solution: (D)

Chapter Summary

- An undirected graph G = (V,E) consists of V, a non-empty set of vertices (or nodes) and E, a set of edges, and every edge is associated with unordered pair of vertices.
- Types of graphs.

Types of Graphs
- Null
- Trivial
- Non-directed
- Directed
- Connected
- Disconnected
- Regular
- Complete cycle
- Cyclic
- Acyclic
- Finite
- Infinite
- Bipartite
- Planar
- Simple
- Multi
- Pseudo
- Euler
- Hamiltonian

- Adjacent vertices: If two vertices have an edge in between.
- Adjacent Edges: If two edges have common vertices in between.
- A vertex with 0 edges is called the isolated vertex.
- A vertex with one edge is called a pendant vertex.
- A complete graph is a simple graph with a maximum number of edges possible.

$$|E| \text{ in } K_n = \frac{n(n-1)}{2}$$

Graph	Multiple Edges	Loops
Simple	×	×
Multigraph	✓	×
Pseudo graph	✓	✓

- If G is a simple graph $G \cup \bar{G} = K_n$
- Isomorphic graphs must have:
  - Same number of vertices.
  - Same number of edges.
  - Same degrees of corresponding vertices.
- If closed trail, every edge is coverable

  $\downarrow$

  Euler circuit

  $\downarrow$

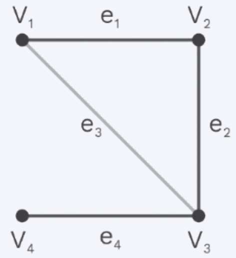
  Euler graph
- If closed path, every vertex is coverable

  $\downarrow$

  Hamiltonian cycle

  $\downarrow$

  Hamiltonian Graph
- K_5 and $K_{3,3}$ are known as Kuratowski's two graphs. These two graphs are special since, K_5 is a non–planar graph with minimum number of vertices and $K_{3,3}$ is the non–planar graph with minimum number of edges.
- Every tree with 2 or more vertices is 2-chromatic.
- Every bipartite graph is 2 colourable and vice versa.



- Cut vertex: V_3
- Bridge: e_4
- Vertex Cut Set: $\{V_3\}$, $\{V_3, V_1\}$, $\{V_3, V_2\}$
- Edge Cut Set: $\{e_1, e_3\}$, $\{e_1, e_2\}$, $\{e_4\}$, $\{e_1, e_2, e_3, e_4\}$