2 Set Theory

Set

Sets are used to group objects together having similar properties.

Definition

Any collection of well defined and distinct object is called a set.

Example: Collection of even positive natural number less than 12

A = {2,4,6,8,10}

Set notations:

Sets are usually denoted by capital letters of the english alphabet A, B, C, D, ..., etc.

Elements are denoted by small letters a, b, c, d, ..., etc.

Symbol '$\in$' (read as 'belong to' or 'is a member of').

Symbol $\notin$ (read as 'does not belong' or 'is not a member of').

Methods of depiction of sets:

A set can be represented by two methods:

1. Roster form or Tabular form
2. Set builder form or Rule method

Roster form or tabular form:

In this representation, all the elements/ members of a set are listed, separated by a comma and then enclosed within the pair of curly brackets {}.

Set builder form or rule method:

In this form, we choose a variable say 'x', which represents each element of the set satisfying a particular property.

Example: Set of prime number less than 7

A = {x | x is prime number less than 7}

Rack Your Brain

Which of the following is set?
1. Collection of intelligent students in Prepladder.
2. The collection of rich persons in India.
3. Collection of beautiful women in India.
4. Collection of solution of the equation $x^2-9x+18=0$

Note:

Order of elements are immaterial (The order in which elements of a set are listed does not matter).
No element is repeated (It does not matter if an element of a set is listed more than once).
Example: {1, 1, 3, 3, 3, 5} is same as that of {1, 3, 5} because they have same elements.

Some sets in discrete mathematics are denoted using specific letters.

N	Set of natural numbers
Z	Set of integers
Z^+	Set of positive integers
Q	Set of rational numbers

R	Set of real numbers
R^+	Set of positive real numbers
C	Set of complex numbers

Types of sets:

1. **Null set/empty set/void set:**
The set with 0 elements is called an empty set and is represented by ϕ or {}.

2. **Singleton set:**
A set having a single element is called a singleton set. E.g., {1}.

3. **Finite set:**
A finite set is one with a countable number of elements.
Example: A = {1, 3, 5, 7, 9};
A = {x | x is between 1 and 5}

4. **Infinite set:**
The term "infinite set" refers to a set with an infinite number of elements.
Example: A = {1, 2, 3 ...}

5. **Equivalent sets:**
Two sets having an equal number of elements.
Example: X = {11, 12, 13}, Y = {p, y, q} here X and Y are equivalent.

6. **Equal sets:**
When two sets A and B have the same elements, they are said to be equal sets. And every equal sets are also equivalent sets.
Example: A = {11, 12, 13}, B = {13, 12, 11} here A and B are equal sets as well as equivalent sets.

Subsets, Universal Sets, Power Sets:

Subset:

Definition

The set A is a subset of B if and only if every element of A is also an element of B.
We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.
Using quantifier subset can be defined as $\forall x(x \in A \rightarrow x \in B)$.

Example:

A = {a, e, i, o, u}

B = {x | x is a letter of english alphabets}

A is a subset of B (because every element of A belongs to B).

Properties:

For every set S,

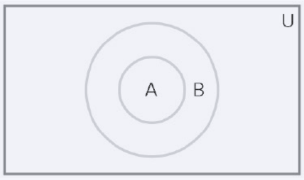
- $\varnothing \subseteq S$
- $S \subseteq S$
- $S \subseteq U$

Rack Your Brain

Is null set is a finite set?
Is equal sets are equivalent sets?

Previous Years' Questions

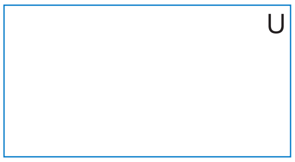
Let S be a set consisting of 10 elements. The number of tuples of the form (A,B) such that A and B are subsets of S, and $A \subseteq B$ is:



Solution: 59049

Universal set:

Set of all elements that we all care about in a given context.
Universal set is represented by: U
Venn Diagram is represented by "rectangle"



Power set:

The power set of A, indicated by P(A), is the set of all possible subsets of set A.

Example: If A = {1}

P(A) = {ϕ, {1}}

Cardinality of a finite set:

The cardinlity of a set A is the number of different elements in a finite set A. and it denoted by n(A) or |A|.

Example 1: Set of even numbers less than 10.
A = {2,4,6,8}
Then |A| = 4

Solution: 4

Definition

Let S be a set. If there are exactly n distinct elements in S where n is a non negative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by ISi.

Note:

The term cardinality comes from the common usage of the term cardinal numbers as the size of finite set.

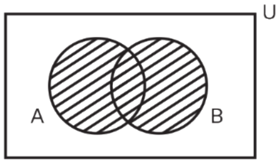
Previous Years' Questions

The cardinality of the power set of {0,1,2, ... ,10} is ________.
Solution: 2048

Set operations:

Various operations can be performed on two or more than two sets. Some of operations we will be discussing here:

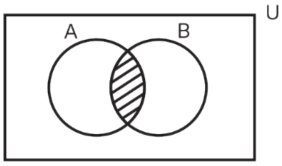
1. **Union:**
   The set denoted by A∪B contains elements that are either in A or B or in both.



A ∪ B is shaded
Fig. 2.1

2. **Intersection:**

   The set containing those elements in both sets A and B is denoted by A∩B, which is the intersection of A and B.

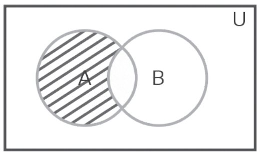


A ∩ B is shaded
Fig. 2.2

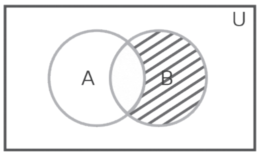
Note:

Two sets are called disjoints if their intersection is empty set.

3. **Set–difference:**
   If and only if x∈A and x∉B, an element x belongs to the difference of A and B.
   $A - B = \{x \mid x \in A \land x \notin B\}$



A–B is shaded



B–A is shaded

Definition

Let A and B be sets, the difference of A and B, denoted by A-B, is the set containing those elements that are in A but not in B. The difference of A and B is also known as complement of B with respect to A.

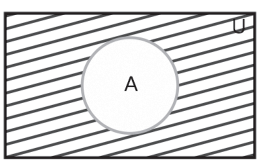
Rack Your Brain

The difference of sets A = {1, 2, 3, 4, 5, 6, 7, 8} and B = {1, 3, 5, 6, 7, 8, 9} is?

4. Complement:

An element belongs to $\bar{A}$ if and only if $x \notin A$.

$\bar{A} = \{x \mid x \notin A\}$



$\bar{A}$ is shaded

Definition

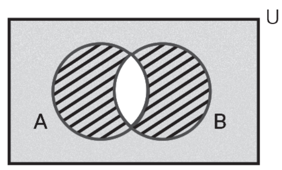
Let U be the universal set. The complement of the set A, denoted by $\bar{A}$ is the complement of A with respect U. In other words, the complement of the set A is U-A.

Example 1:

Let A be a set of all positive integers greater than 5.

Then $\bar{A} = \{1, 2, 3, 4, 5\}$

5. Symmetric difference:



A ⊕ B is shaded

Fig. 2.3

$A \Delta B = A \oplus B = \{x \mid x \in A \text{ or } x \in B \text{ and } x \notin A \cap B\}$

Rack Your Brain

If A, B, C and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?

Grey Matter Alert!

$|A| + |B|$ counts each element in A but not in B or in B but not in A exactly once, and each element that is in both A and B exactly twice. Thus, if the number of elements that are in both A and B is subtracted from $|A| + |B|$, elements in $A \cap B$ will be counted only once. Hence, $|A \cup B| = |A| + |B| - |A \cap B|$

Note:

The union of arbitrary number of set is called principle of inclusion - exclusion.

1. Let X and V are two sets. The symmetric difference is defined as $A \oplus B = (A-B) \cup (B-A)$. Which of the following is false?

(A) $A \oplus B = B \oplus A$

(B) $A \oplus \varnothing = A$

(C) $A \oplus A = \varnothing$

(D) $A \oplus B = (A \cap B') \cap (B \cap A)$

Solution: (D)

Let us understand this using Venn diagram.

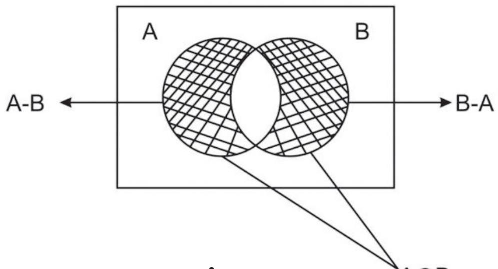


Fig. 2.4

(A) Given: $A \oplus B = (A-B) \cup (B-A)$

$B \oplus A = (B-A) \cup (A-B)$ therefore, it is true.

(B) $A \oplus \varnothing$ $= (A - \varnothing) \cup (\varnothing - A)$

$= (A) \cup (\varnothing)$

$= A$

Which is true.

(C) $A \oplus A$ $= (A-A) \cup (A-A)$

$= \varnothing \cup \varnothing$

$= \varnothing$

Which is true

(D) Option (D) is ruled out.

Set identities:

Identity	**Name**
$Z \cup \phi = Z$ $Z \cap U = Z$	Identity laws
$Z \cup U = U$ $Z \cap \phi = \phi$	Domination laws
$Z \cup Z = Z$ $Z \cap Z = Z$	Idempotent laws

Identity	**Name**
$\overline{(\overline{Z})} = Z$	Double complement
$(Z \cup K) = (K \cup Z)$ $(Z \cap K) = (K \cap Z)$	Commutative laws
$Z \cup (K \cup C) = (Z \cup K) \cup C$ $Z \cap (K \cap C) = (Z \cap K) \cap C$	Associative laws
$Z \cap (K \cup C) = (Z \cap K) \cup (Z \cap C)$ $Z \cup (K \cap C) = (Z \cup K) \cap (Z \cup C)$	Distributive laws
$\overline{Z \cup K} = \overline{Z} \cap \overline{K}$ $\overline{Z \cap K} = \overline{Z} \cup \overline{K}$	De Morgan's laws
$Z \cup (Z \cap K) = Z$ $Z \cap (Z \cup K) = Z$	Absorption laws
$Z \cup \overline{Z} = U$ $Z \cap \overline{Z} = \phi$	Complement laws

Table 2.1 Important Laws

Previous Years' Questions

If P, Q, and R are subsets of the universal set U, then $(P \cap Q \cap R) \cup (P' \cap Q \cap R) \cup Q' \cup R'$ is:

[GATE CSE 2008]

(A) $Q' \cup R'$
(B) $P \cup Q' \cup R'$
(C) $P' \cup Q' \cup R'$
(D) U

Solution: (D)

Solved Examples

2. Use set builder method and logical equivalences to prove $A \cap B = (A \cup B)$.

Solution: $\overline{A \cap B} = \{x \mid x \notin A \cap B\}$

$$= \{x \mid x \in (\overline{A \cap B})\}$$

{By demorgan's theorem}

$= \{x \mid x \notin A \vee x \notin B)\}$

$= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$

{By demorgan's theorem}

$= \bar{A} \cup \bar{B}$

3. Let A, B and C be sets. Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

Solution:

We have,

$\Rightarrow \overline{A \cup (B \cap C)} = \overline{A} \cap \left(\overline{B \cap C}\right)$ [Using first De Morgan's law]

$\Rightarrow \overline{A} \cap \left(\overline{B} \cup \overline{C}\right)$ [Using second De Morgan's law]

$\Rightarrow \left(\overline{B} \cup \overline{C}\right) \cap \overline{A}$ [Using Commutative law of intersection]

$\Rightarrow \left(\overline{C} \cup \overline{B}\right) \cap \overline{A}$ [Using commutative law of unions]

$\Rightarrow$ Hence Proved

Generalised union and intersection:

Let's call these two sets A and B. Now, A ∪ B contains items that belong to at least one of the sets A or B, while A ∩ B contains elements that belong to both sets A and B.

Definition

The union of a collection of sets is the set that contains those elements that are the members at least one set in the collection. The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

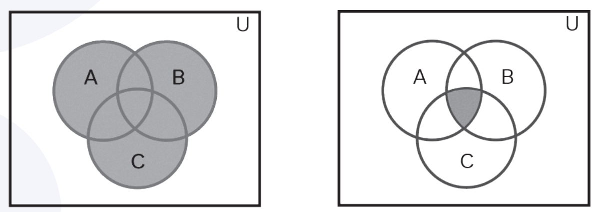
To denote the union of sets, we use the notation below.

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

We use the following notation to represent the intersection of sets

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Below is a Venn diagram depicting the intersection and union of sets.



A∪B∪C is shaded   A∩B∩C is shaded

Fig. 2.5

4. Let A= {1, 2, 3} and B = {1, 5, 7, 8}. What are A∪Band A∩B?

Solution:

A ∪ B = {1, 2, 3, 5, 7, 8}

A ∩ B = {1}

Grey Matter Alert!

Fuzzy sets are used in artificial intelligence. Each element in the universal set U has a degree of membership, which is a real number between O and 1 (including O and 1), in a fuzzy set S. The fuzzy set S is denoted by listing the elements with their degrees of membership (elements with O degree of membership are not listed). For instance, we write {0.6 Alice, 0.9 Brian, 0.4 Fred, 0.1 Oscar, 0.5 Rita} for the set F (of famous people) to indicate that Alice has a 0.6 degree of membership in F, Brian has a 0.9 degree of membership in F, Fred has a 0.4 degree of membership in F, Oscar has a 0.1 degree of membership in F, and Rita has a 0.5 degree of membership in F (so that Brian is the most famous and Oscar is the least famous of these people). Also suppose that R is the set of rich people with R = {0.4 Alice, 0.8 Brian, 0.2 Fred, 0.9 Oscar, 0.7 Rita}.
The complement of a fuzzy set S is the set s, with the degree of the membership of an element in S equal to 1 minus the degree of membership of this element in s.
The union of two fuzzy sets S and T is the fuzzy set S u T, where the degree of membership of an element in S u Tis the maximum of the degrees of membership of this element in S and in T.
The intersection of two fuzzy sets S and T is the fuzzy set S n T , where the degree of membership of an element in S n Tis the minimum of the degrees of membership of this element in S and in T.

Cartesian product:

Definition

The ordered n tuple $(a_1, a_2, \ldots a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element and an being the n^{th} element.
Two ordered n tuples are equal if $a_i = b_i$ for i = 1, 2, 3 ... n.
Let A and B be sets. the Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,
A x B = {(a, b) |a E A $\wedge$ b $\in$ B}
The Cartesian product of the sets $A_1, A_2, \ldots A_n$ denoted by $A_1 \times A_2 \times \ldots A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots a_n)$ where a_i belongs to A_i for i = 1, 2, ... n.
In other words:
$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots a_n) \mid a_i \in A_i\}$

Solved Examples

5. What is Cartesian product of A×B×C, where A= {O, 1}, B = {1}, C = {1, 2}?

Solution:

A × B × C = {(0, 1, 1), (0, 1, 2), (1, 1, 1), (1, 1, 2)}

1. **Relations:**
The first thing we should understand is that relations is built on sets. For example, we have two sets A and B, a relation between A and B is written as A R B.

Definition

Let A and B are two non-empty sets.
A relation R from set A to set B is a subset of $A \times B$.
In other words, If $R \subseteq A \times B$, we call R is relation from set A to Set B.

Graph representation of relations:

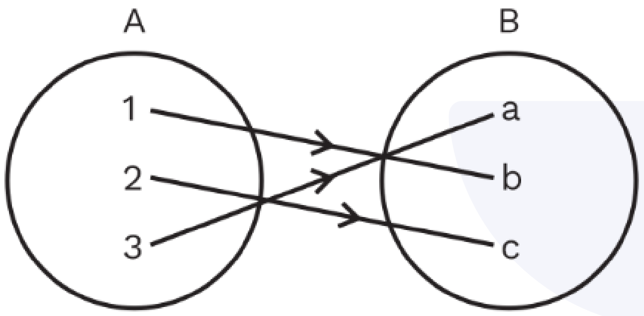


Fig. 2.6

Properties of relations:

There are many properties, some of which we will discuss here.

1. **Reflexive relation:**
If a relation is reflexive, then $(a, a) \in R\ \forall\ a \in A$.
It can be expressed in Quantifier as:
$\forall \mathbf{a}\ ((\mathbf{a}, \mathbf{a}) \in \mathbf{R})$

2. **Irreflexive relation:**

If a relation is irreflexive, then $(a, a) \notin R\ \forall\ a \in A$.
It can be expressed in Quantifier as:
$\forall \mathbf{a}\ ((\mathbf{a}, \mathbf{a}) \notin \mathbf{R})$

3. **Symmetric:**

If a relation is symmetric, then $(a, b) \in R$ then $(b, a) \in R,\ \forall\ a, b \in A$.
It can be expressed in Quantifier as:
$\forall \mathbf{a}\ \forall \mathbf{b}((\mathbf{a}, \mathbf{b}) \in \mathbf{R} \rightarrow (\mathbf{b}, \mathbf{a}) \in \mathbf{R})$

4. **Asymmetric:**

If a relation is asymmetric, then $(a, b) \in R$ then $(b, a) \notin R,\ \forall\ a, b \in A$.
It can be expressed in Quantifier as:
$\forall \mathbf{a}\ \forall \mathbf{b}((\mathbf{a}, \mathbf{b}) \in \mathbf{R} \rightarrow (\mathbf{b}, \mathbf{a}) \notin \mathbf{R})$

5. **Antisymmetric:**

If a relation is antisymmetric, then $(a, b) \in R$ and $(b, a) \in R$, then $a = b,\ \forall\ a, b \in A$.
It can be expressed in Quantifier as:
$\forall \mathbf{a}\ \forall \mathbf{b}(((\mathbf{a}, \mathbf{b}) \in \mathbf{R} \wedge (\mathbf{b}, \mathbf{a}) \in \mathbf{R})) \rightarrow (\mathbf{a} = \mathbf{b})$

6. **Transitive property:**

If a relation is transitive, then $(z, k) \in R$ and $(k, c) \in R$ then $(z, c) \in R\ \forall\ z, k, c \in A$.
It can be expressed in Quantifier as:
$\forall \mathbf{a}\ \forall \mathbf{b}\ \forall \mathbf{c}\ (((\mathbf{a}, \mathbf{b}) \in \mathbf{R} \wedge (\mathbf{b}, \mathbf{c}) \in \mathbf{R}) \rightarrow (\mathbf{a}, \mathbf{c}) \in \mathbf{R})$

Rack Your Brain

Consider the following relations on {1, 2, 3, 4}:

R, = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}

R2 = {(1, 1), (1,2), (2, 1)}

R3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}

R4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}

R5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4)}

Rs = {(3, 4)}

Which of these relations are reflexive, irreflexive, symmetric, asymmetric, antisymmetric, and transitive?

Note:

- An empty relation is always irreflexive, symmetric, antisymmetric, asymmetric, transitive but not reflexive.
- A relation can be both symmetric as well as antisymmetric.
- Every asymmetric relation is antisymmetric.

Rack Your Brain

Which is the following is true?

1. A relation is reflexive will not be irreflexive.
2. A relation is irreflexive will not be reflexive.
3. A relation is not reflexive will be irreflexive.
4. A relation is not irreflexive will be reflexive.

Solved Examples

6. Is the "divides" relation on the set of positive integers transitive?

Solution:

Assume that a divides b and that b divides c. The positive integers k and l are then such that b = ak and c = bl. As a result, c = a(bl), and a divides c.

As a result, this relationship is transitive.

Combining relations:

Two relations can be combined in the same way, two sets are combined.

Definition

Composition of Relation/Composite Relation

Let R be a relation from a set A to set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where a $\in$ A, c $\in$ C, and for which there exists an element b $\in$ B such that (a, b) $\in$ R and (b, c) $\in$ S. We denote the composite of R and S by S o R.

Note:

The relation R on a set A is transitive if $R^n \subseteq R$ for n = 1, 2, 3, ...

Rack Your Brain

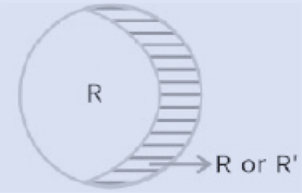
Let R be a relation R = {(a, b) | a divides b} on the set of positive integers, find R^{-1}.

Note:

Complement of a relation (R or R'):

R = A → B

R or R': (A × B) - R



Solved Examples

7. Find the matrix representation the relation R2 , where the matrix representing R is

$$M_{R^2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution:

The matrix for R^2 is:

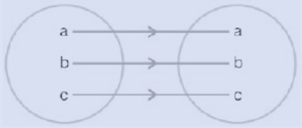
$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Note:

Diagonal relation

Δ_A: A→ A

Δ_A = {(x, x) I x ∈ A}



Representation of relations:

There are various ways to represent relations:

1. To list its ordered pairs
2. Using a table
3. Using matrices (Most appropriate one)
4. Using directed graphs

Representation using matrices:

A matrix can be used to represent a relation. Let R be a relation between A and B.

The matrix $M_R = [m_{ij}]$ can be used to describe the relation R.

$$m_{ij} = \begin{cases} 1; & \text{if } (a_i, b_j) \in R, \\ 0; & \text{if } (a_i, b_j) \notin R \end{cases}$$

Representation using digraphs:

This is another way of representing relations. Each element is represented by a point, and each ordered pair is represented using arc with its directions indicated by an arrow.

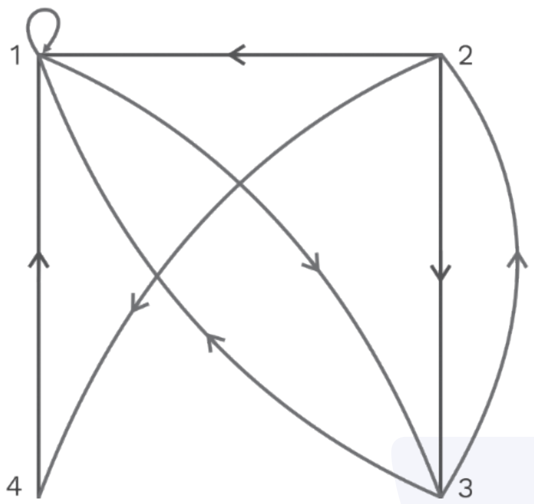
Definition

A directed graph, or digraph, consist of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

Solved Examples

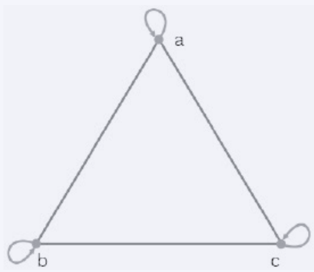
8. Draw the directed graph for the given set of ordered pairs.
R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)} on the set {1, 2, 3, 4}

Solution:



Rack Your Brain

1. List the ordered pairs in the relations on {1, 2, 3}, corresponding to this matrix
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$
2. List the ordered pairs in the relation represented by the following directed graphs



Closures of relations:

Let us learn this concept with a simple family tree example. You and your mother are directly related, but you and your maternal cousin are not directly related, you are related to your cousin through your mother. Let R be a relation containing (a, b), if a is directly related to b. Now the question is, how to determine indirect link? Answer is right here! using "closure of relations" "concepts, we can find all pairs of relations that have a link by constructing a transitive relation S containing R, such that S is subset of every transitive relation containing R. This relation is called transitive closure of R.

General definition of closure:

Let R be a relation on set A. R may or may not have some property P but if there exists another relation S with property P containing R, such that S is subset of every relation with property P containing R, then S is called closure of R with respect to P.

Closures:

1. Reflexive closure:
Smallest reflexive relation containing relation R.
For e.g., Let relation R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1 2, 3}. As it can be clearly seen that the relation is not reflexive, but we can make it reflexive by including (2, 2) and (3, 3) to R. The new relation obtained contains R. Now, every relation that contains R must also contain (2, 2), (3, 3) because this is reflexive relation and is contained within every reflexive relation that contains R.

Note:

Reflexive closure of R_r = |R ∪ Δ, where Δ = {(a, a) | ∀ a ∈ A}.

Solved Examples

9. What is reflexive closure of the relation R = {(a, b)| a< b} on the set of integers?

Solution:

The reflexive closure of R is

$R \cup \Delta = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in z\}$

$= \{(a,b) \mid a \le b\}$

2. Symmetric closure:

Smallest symmetric relation containing relation R.

For example, relation {(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)} on {1, 2, 3} is not symmetric, but we can make it symmetric by adding (2, 1) and (1, 3), then the relation obtained will be symmetric.

The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse i.e., $R \cup R^{-1}$ is the symmetric closure of R, where R^{-1} = {(b, a), | (a, b) ∈ R}

10. What is the symmetric closure of the relation R = {(a, b) | a> b} on the set of positive integers?

Solution:

The symmetric closure of R is the relation

$R \cup R^{-1}$ = {(a, b) | a > b} ∪ {(b, a) | b > a}
= {(a, b) | a ≠ b}

3. Transitive closure:

We all know the definition of transitivity: "A relation is said to be transitive if transitivity (a, b) ∈ R and (b, c) ∈ R, then (a, c) ∈ R"

To transform a non-transitive relation on a given set into a transitive closure, simply make the provided relation transitive as defined.

For e.g., relation {(79,84), (84,36), (94,36)} on given set {36,79,84,94} is not transitive. We can convert it to transitive, but it is a little on the tougher side than reflexive and symmetric.

A relation's transitive closure can be obtained by adding newer ordered pairs that must be present and continuing the process until no more ordered pairs are required.

11. Find the zero-one matrix of the transitivity closure of the relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

There is a theorem which states that; zero–one matrix of R* is:

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}$$

Now,

$$M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\therefore M_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Rack Your Brain

Let R be relation on the set {1, 2, 3, 4} containing ordered pairs {(1, 2), (2, 2), (2, 3), (3, 1), (3, 3) and (4, 1)}.
Find the
(A) Reflexive closure of R
(B) Symmetric closure of R
(C) Transitive closure of R

Equivalence relations:

- If a given relation R on a given set S is: a) Reflexive b) Symmetric c) Transitive, it is called an equivalence relation.
- When a and b are equivalent elements in terms of a particular equivalence relation, the notation a ~ b is frequently employed.

Solved Examples

12. Is "divide" relation on the set of positive integers is an equivalence relation?

Solution:

The divide relation is reflexive and transitive, but not symmetric, on the set of all positive integers. ($2 \in z$, $4 \in z$ 2 divides 4, but 4 does not divide 2)

$\therefore$ It is not an equivalence relation.

13. Let R be a relation on the set of integers such that aRb iff a = b or a = -b. Is this an equivalence relation.

Solution:

The given relation is:

- Reflexive
- Transitive
- Symmetric

Therefore, relation is equivalence relation.

14. Let R is a relation on the set of english letters such that aRb iff l(a) = l(b), where l(x) is length of the string x. Is R an equivalence relation?

Solution:

Because l(a) = l(a), wherever a is a string, it follows that aRa, and R is reflexive. Assume that aRb, and that l(a) = l(b). Because l(b) = l, the result is bRa. As a result, R is symmetric. Finally, consider aRb and bRc. As a result, l(a) = l(c), resulting in aRc. As a result, R is a transitive. R is an equivalence relation since it is reflexive, symmetric, and transitive.

Equivalence classes:

Definition

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write [a] for this equivalence class.

OR

If R is an equivalence relation on a set A, the equivalence class of the element a is $[a]_R = \{s \mid (a, s) \in R\}$.

Note:

If $b \in [a]_R$ then b is called representative of this equivalence class.

Solved Examples

15. What is the equivalence class of an integer for the equivalence relation on the set of integers such that aRb if a = b or a = -b.

Solution:

An integer is equivalent to itself, and its negative as provided.

As a result, $[a] = \{-a, a\}$ for $[0] = \{0\}$.

16. What are the equivalence class of 0 and 1 for congruence modulo 4?

Solution:

All numbers a such that $a \equiv 0$, 0 are included in the equivalence class of 0. (mod 4). This class contains integers that are divisible by four. As a result, this relationship has an equivalence class of 0.

$$[0] = \{\ldots -8, -4, 0, 4, 8, \ldots\}$$

$$[1] = \{\ldots -7, -3, 1, 5, 9, \ldots\}$$

Rack Your Brain

What is the congruence class [4Jm] when m is:
(A) 2
(B) 8

Equivalence classes and partitions

We can divide R into disjoint subsets, each of which contains a certain major, if we consider R to be an equivalence. R equivalence classes make up these subsets.

An equivalence relation's equivalence class divides a set into disjoint, non-empty sets in this way. There are two theorems that show that the equivalence classes of two components of A are either identical or disjoint, and that relations and partitions are connected.

Let R be a set A equivalence relation. These are similar statements for A's elements a and b.

1. aRb
2. $[a] = [b]$
3. $[a] \cap [b] = \phi$

Solved Examples

17. What are the sets in the partition of the integers arising from congruence modulo 4.

Solution:

$[0]_4 = \{\ldots -8, -4, 0, 4, 8, \ldots\}$

$[1]_4 = \{\ldots -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots -6, -2, 2, 6, 10, \ldots\}$

$[3]_4 = \{\ldots -5, -1, 3, 7, 11, \ldots\}$

These congruence classes are disjointed, and each integer belongs to one of them.

Rack Your Brain

Which of the following collection of subsets are partitions of {1, 2, 3, 4, 5, 6}
(A) {1, 2}, {2, 3, 4}, {4, 5, 6}
(B) {1}, {2, 3, 6}, {4}, {5}
(C) {2, 4, 6}, {1, 3, 5}
(D) {1, 4, 5}, {2, 6}

Grey Matter Alert!

If every set in P_1 is a subset of one of the sets in P_2, the partition P_1 is called a refinement of the partition P_2.

Partial orderings:

Definition

The relation R on set S is called a partial ordering or partial order if it is reflexive, antisymmetric and transitive. A set S together with partial ordering relation R is called a partially ordered set, or poset, and is denoted by (S, R).
Members of S are called elements of the poset.

Solved Examples

18. Show that the inclusion relation~ is a partial ordering on the power set of a set S.

Solution:

Any set is a subset of itself, hence $\subseteq$ is reflexive, as we all know.

- It's antisymmetric because if $A \subseteq B$ and $B \subseteq A$, then A = B.
- Because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$, $\subseteq$ is transitive.

As a result, (P (S), $\subseteq$) is a poset and is a partial ordering on P(s).

Solved Examples

19. In the post (Z^+, I), are the integers 3 and 9 comparable?

Solution:

Integers 3 and 9 comparable, because 3/9.

Note:

Partial word is used because pair of elements may be incomparable.

Totally ordered set:

Definition

If (S, $\preceq$) is a poset and every two elements of S are comparable, S is called totally ordered or linearly ordered set, and $\preceq$ is called total order or linear order. A totally ordered set is also called a chain.

The poset (Z, $\preceq$) is totally ordered, because a $\preceq$ b or b $\preceq$ a whenever a and b are integers.

Comparable and incomparable posets

Definition

The elements a and b of a poset (S, $\preceq$) is called comparable if either a $\preceq$ b or b $\preceq$ a.

When a and b are elements of S such that neither a $\preceq$ b nor b $\preceq$ a, a and b are called incomparable.

Well ordered set:

A set $(S, \prec)$ is well ordered if it is a poset with $\preceq$ is a total ordering and a least element in every non–empty subset of S.

Lexicographic order:

A lexicographic ordering $\preceq$ on $A_1 \times A_2$ is defined as one pair being less than another if the first entry of the first pair is less than the first entry of the second pair, or if the first entries are equal, but the second entry of this pair is less than the second entry of the second pair, i.e., $(a_1, a_2) \prec (b_1, b_2)$ either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ and $a_2 \prec_2 b_2$.

Hasse diagrams:

Hasse diagram represents partial order relations in simpler forms.

1. Omit all loops.
2. Omit all arrows that can inferred transitivity.
3. Draw arrows without heads.
4. Make sure all arrows point upwards.
   The diagram is called the Hasse diagram, named after the twentieth century German mathematician Helmut Hasse.

Solved Examples

20. Determine whether (3, 5) $\prec$ (4, 8), whether (3, 8) $\prec$ (4, 5), and whether (4, 9) $\prec$ (4, 11), in the poset (Z x Z, $\preceq$), where $\preceq$ is the lexicographic ordering constructed from the usual $\preceq$ relation on Z.

Solution:

As 3 < 4, it therefore (3, 5) $\preceq$ (4, 8) and (3, 8) $\preceq$ (4, 5). We have (4, 9) $\preceq$ (4, 11) as the first entries of (4, 9) and (4, 11) are the same but 9 < 11.

Solved Examples

21. Draw the Hasse diagram representing the partial ordering {(a, b) I a divides b} on {1, 2, 3, 4, 6, 8, 12}.

Solution:

- Diagraph for this partial order.

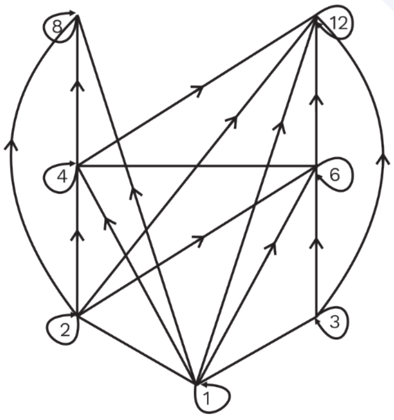


Fig. 2.7

- Remove all loops.

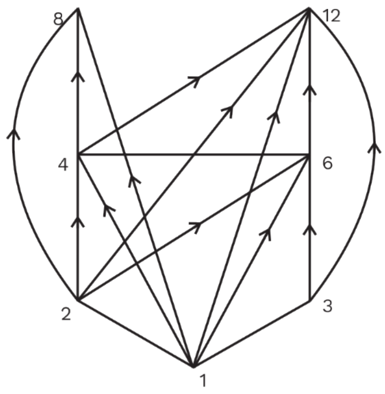


Fig. 2.8

- Remove all edges implied by the transitive property and reposition all edges upwards.

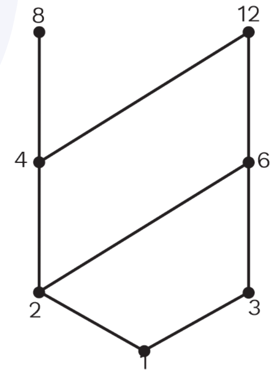


Fig. 2.9

This is the resulting Hasse diagram

22. Draw the Hasse Diagram for partial ordering {(A, B) | A $\subseteq$ B} on the power set P(S) where S = {a, b, c}.

Solution:

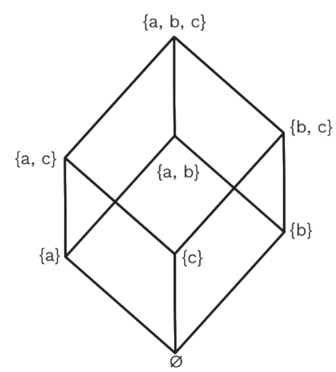


Fig. 2.10

Solved Examples

Maximal and minimal elements:

A poset element is maximal if it is not less than any other poset element, i.e., a is maximal in the poset (S, $\preceq$) if there is no b$\in$S such that a$\prec$b

Similarly, if there is no element b$\in$S such that b$\prec$a, then a is minimum.

The "top" and "bottom" elements in the diagram are the maximal and minimal elements.

23. Which elements of the poset ({2, 4, 5, 10, 12, 20, 25}, I) are maximal and which are minimal.

Solution:

The Hasse diagram will be:

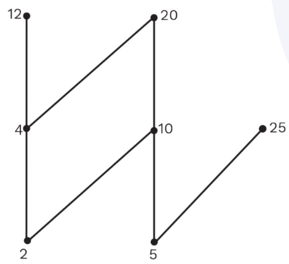


Fig. 2.11

Maximal elements = 12, 20, 25 minimal elements = 2 and 5

Upper bound and lower bound:

Let us understand this concept with the help of an example.

Rack Your Brain

Draw the Hasse diagram for the "greater than or equal to" relation on {0, 1, 2, 3, 4, 5}.

Note:

Maximal: an element is not related to any other element.

Minimal: no element is related to an element.

Greatest and least elements:

Definition

An element in a poset that is greater than every other element. such an element is called the greatest element. That is, a is the greatest element of the poset (S, $\preceq$) if b $\preceq$ a for all b $\in$ S. The greatest element is unique when it exists. Likewise, an element is called the least element if it is less than all the other elements in the poset. That is, a is the least element of (S, $\preceq$) if a $\preceq$ b for all b $\in$ S. The least element is unique when it exists.

Let <p, $\preceq$ > be a poset.

An element g $\in$ p is greatest, if x $\leq$ g $\forall$ x $\in$ p

An element l $\in$ p is least, if l $\leq$ y $\forall$ y $\in$ p.

Example 1:

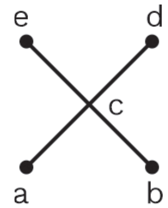


Fig. 2.12

greater $\to$ no greatest

least $\to$ no least

Example 2:

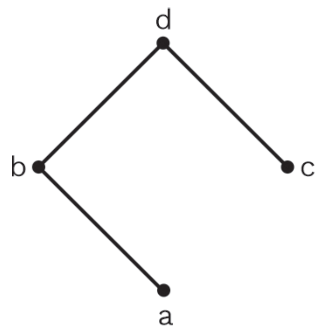


Fig. 2.13

greatest $\to$ d

least $\to$ No least

Solved Examples

24. Find the greatest lower bound and the least upper bound of {b, d, g}, if they exist, in

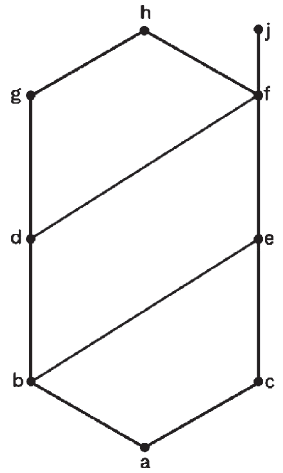


Fig. 2.14

Solution:

The upper bounds of b, d, and g are g and h, respectively. G is the least upper bound since $g \prec h$. a and b are the lower bounds of b, d, and g. Because $a \prec b$ is the greatest lower bound, b is the greatest upper bound.

Lattice:

"A poset in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice. Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra."

Note:

	$\leq$	$\mid$	$\subseteq$
$a \vee b$	max {a, b}	LCM {a, b}	$a \cup b$
$a \wedge b$	min {a, b}	GCD {a, b}	$a \cap b$

Previous Years' Questions

Consider the following Hasse diagrams: **[GATE IT 2008]**

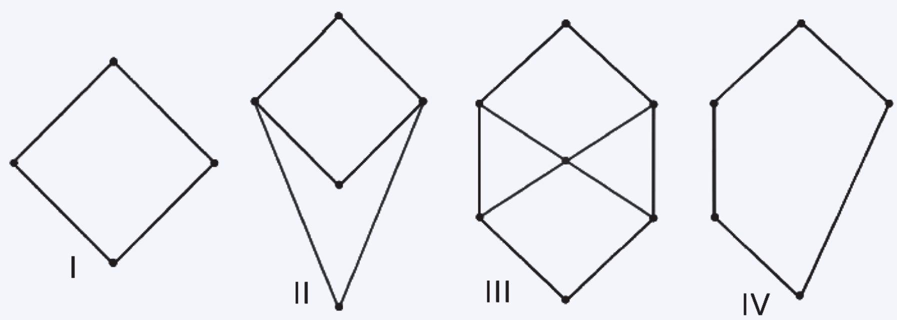


Fig. 2.15

Which of the following represent a lattice?

(A) I and IV only
(B) II and III only
(C) III only
(D) I, II and IV only

Solution: (A)

Previous Years' Questions

The inclusion of which of the following sets into
S = {{1, 2}, {1, 2, 3}, {1, 3, 5}, {1, 2, 4}, {1, 2, 3, 4, 5}}
is necessary and sufficient to make S a complete lattice under the partial order defined by set containment?

[GATE CSE 2004]

(A) {1}
(B) {1}, {2, 3}
(C) {1}, {1, 3}
(D) {1}, {1, 3}, {1, 2, 3, 4}, {1, 2, 3, 5}

Solution: (A)

Note:

Join - The join of two elements is their least upper bound. It is denoted by $\vee$, not to be confused with disjunction.

Meet - The meet of two elements is their greatest lower bound. It is denoted by $\wedge$, not to be confused with conjunction.

Properties of lattice:

Let < L, $\preceq$ > be a lattice, then the following properties holds.

1. Idempotent:
   $a \vee a = a$
   $a \wedge a = a$

2. Commutative:
   $a \vee b = b \vee a$
   $a \wedge b = b \wedge a$

3. Associative:
   $a \vee (b \vee c) = (a \vee b) \vee c$
   $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. Absorption:
   $a \vee (a \wedge b) = a$
   $a \wedge (a \vee b) = a$

Distributive lattice:

Distributive lattice holds following laws:

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Non distributive lattice: There are two famous non distributive lattice:

1. Kite or Diamond lattice
2. Pentagon lattice

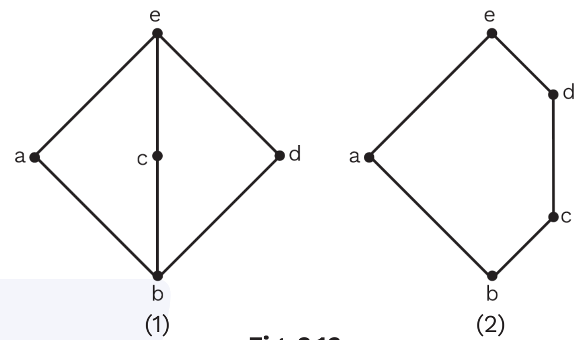


Fig. 2.16

Bounded lattice:

If a lattice L has the greatest element 1 and the least element 0, it is said to be bounded.

Example:

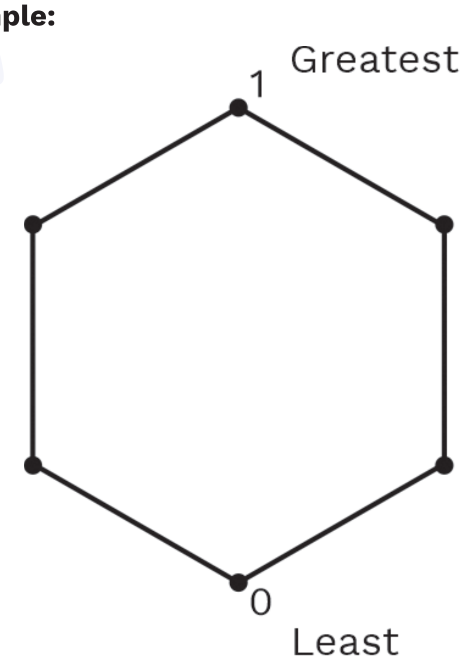


Fig. 2.17

Greatest element $\rightarrow$ 1

Least element $\rightarrow$ 0

Note:

Every finite lattice is always bounded.

Complement of an element in a lattice:

Let <L, $\preceq$> be a bounded lattice. An element $b \in L$ is complement of $a \in L$ when $a \vee b = 1$ (greatest element)

$a \wedge b = 0$ (Least element)

b is the complement of a;

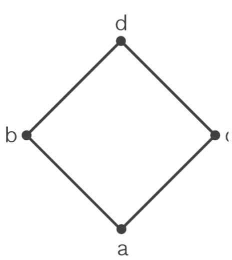
then a is the complement of b.

Note:

- 1 v 0=1
- 1 Λ 0=0

greatest and least element are complement of each other.

Example:



Element	Complement
a	d
d	a
b	c
c	b

Note:

Complement, if exist need not to be unique.

Complemented lattice:

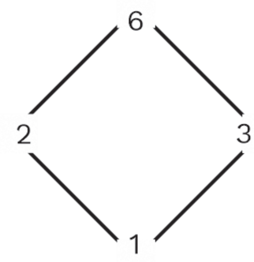
A bounded lattice in which complement of every element exist is called complemented lattice.

Note:

In a distributive lattice complement if exist are unique.

Example:

D_6 {1, 2, 3, 6} is complemented lattice because it is bounded and every element of this lattice has a complement.



Boolean algebra:

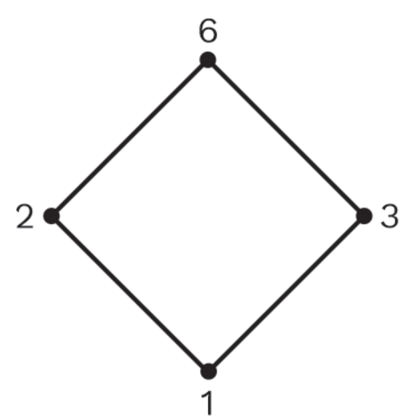
A bounded distributive and complemented lattice is called Boolean Algebra.

Example 1:

<p(A), $\subseteq$ > is Boolean Algebra.

$n = p_1, p_2 \dots p_k$, where $p_1, p_2 \dots p_k$ are distinct prime numbers then. D_n is Boolean Algebra.

$D_6 = 6 = 2 \times 3$



Example 2:

$p^2|n$, p is prime. Then D_n is not Boolean algebra

D_4 where 2 is prime

$2^2 \mid 4$ it is not Boolean algebra.

Not Boolean algebra

Some points to remember:

In any lattice, the semidistributive laws
$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Topological sorting:

Assume a project consists of 20 separate tasks. Some tasks can only be done after the completion of others. What is the best way to find an order for these tasks? To describe this problem, we create a partial order on the set of tasks such that $a \prec b$ means, b cannot be started if a is not completed. To create a project schedule, we must first create an order for all 20 jobs that is compatible with this partial order. We'll demonstrate how to do it.

"A total ordering $\preceq$ is said to be compatible with the partial ordering R if $a \preceq b$ whenever aRb. Constructing a compatible total ordering from a partial ordering is called topological sorting."

Solved Examples

25. Find a compatible total ordering for the poset ({1, 2, 4, 5, 12, 20}, I).

Solution:

Step 1: Choose minimal element, i.e., 1.

Step 2: Select minimal element from ({2,4,5,12,20} , |), i.e., 2 and 5(let's choose 5).

Step 3: Choose minimal element from the remaining elements, i.e., 4.

Step 4: Again choose minimal out of the remaining elements, i.e., 12.

$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$

Rack Your Brain

Answer these questions for the poset ({2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}).

1. Find the maximal elements.
2. Find the minimal elements.
3. Is there a greatest element?
4. Is there a least element?
5. Find all upper bounds of {2, 9}.
6. Find the least upper bound of {2, 9}, if it exists.
7. Find all lower bounds of {60, 72}.
8. Find the greatest lower bound of {60, 72}, if it exists.

Rack Your Brain

Which of these pairs of elements are comparable in the poset (Z', I)?

(A) 5, 15 (B) 6, 9
(C) 8, 16 (D) 7, 7

Functions

Consider the following illustration:

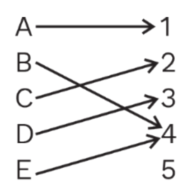


Fig. 2.18

Definition

Let A and B be two non-empty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write f(a) = b if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write f: A $\to$ B

Note:

Functions are sometimes also called mappings/transformation.

Definition

While defining a function we have to specify its domain, co-domain and mapping of elements of the domain to elements in the co-domain. Two functions are equal when they have same domain, co-domain and map elements of their common domain to the same elements in their common co-domain.

Note:

If we change either the domain or co-domain of a function, then we obtain a different function. If we change mapping of elements, then we also obtain a different f.

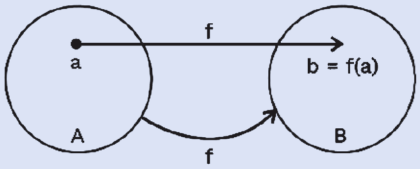


Fig. 2.19 The Function of Maps A and B

Rack Your Brain

Find the domain and range of the function

$$f(x) = \frac{\sqrt{x+2}}{x^2-9}$$

Definition

- If f is a function from A to B, we say that A is the domain off and B is the co-domain off. If f(a) = b, we say that b is the image of a and a is pre-image of b. The range of the f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.
- Let f_1 and f_2 be functions from A to R. Then $f_1 + f_2$ and f_1f_2 are also functions from A to R.

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x)\, f_2(x)$$

- Let f be a function from set A to set B and let S be a subset of A. The image of S under the function f is the subset of B that consist of the images of the elements of S. We denote the image of S by f(s), so

$$f(s) = \{t \mid \exists \mid s \in S\ (t = |f(s))\}$$

We can also use the shorthand $\{f(s) \mid s \in S\}$to denote this set.

Solved Examples

26. Let f_1 and f_2 be two functions from R to R such that $f_1(x) = x^2$ and $f_2(x) = x-x^2$ • What are the functions f_1+f_2 and f_1f_2?

Solution:

From the definition of the sum and products of functions, if follows that

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x-x^2) = x$

and

$(f^1f^2)(x) = x^2(x-x^2) = x^3-x^4$

Note:

When f is a function from a set A to set B, the image of a subset of A can also be defined.

Example:

Let Z = {p,m,k,f,t} and B = {1, 2, 3, 4} with f(p) = 2, f(m) = 1, f(k) = 4, f(f) = 1 and f(t) = 1.

Solution:

The image of the subset S = {m,k,f} is the set f(s) = {1, 4}

Rack Your Brain

Let S = {-1, 0, 2, 4, 7}. Find f(s) if
f(x) = 2x+1
f(x) = [x/5]

Previous Years' Question

Suppose that f: TR → TR is a continuous function on the interval [-3, 3] and a differentiable function in the interval (-3, 3) such that for every x in the interval, $f'(x) \leq 2$. If f(-3) = 7, then f(3) is at most ______.

Solution: 19

One to one and onto functions:

A function is said to be one-to-one when it does not assign same value to two different domain elements.

Definition

A function is said to be one-to-one, or injective, if and only if f(a) = f(b) implies that a = b for all a and b in the domain off. A function is said to be an injection if it is one-to-one.

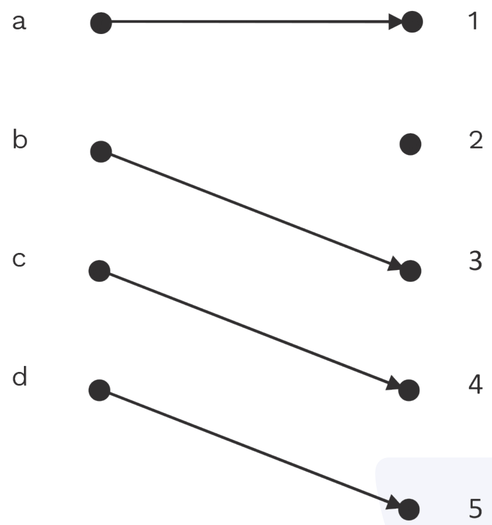
Note:

Function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$.

Solved Examples

27. Determine whether the function f from {a, b, c, d} { 1, 2, 3, 4, 5} with f(a) = 1, f(b) = 3, f(c) = 4, f(d) = 5 is one-to-one.

Solution:



The given function is one-to-one because f is taking different values for all four elements of its domain.

Definition

A function ffrom A to Bis called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with f(a) = b. A function f is called a surjection if it is onto.

Definition

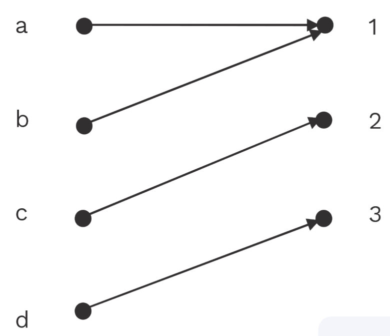
A function f whose domain and co-domain are subset of the set of real numbers is called increasing if f(x) _s_ f(y), and strictly increasing if f(x) < f(y), whenever x < y and x and y are in domain off. similarly, f is called decreasing if f(x) > f(y), and strictly decreasing iff(x) ~ f(y) whenever x < y and x and y are in domain off. (The word strictly in this definition indicates a strict inequality.)
Now, we can see that some of the functions are strictly increasing and some of them are strictly decreasing and one-to-one. However, some function which are increasing but not strictly increasing and decreasing but not strictly decreasing need not necessarily one-to-one.
Now there are some functions whose range and co-domain are equal i.e every member of co-domain is the image of some element of the domain. Functions with this property are called onto functions.

Solved Examples

28. Let f be a function from {a, b, c, d} to {1, 2, 3} defined by f(a) = 1, f(b) = 1, f(c) = 2, f(d) = 3. Is f an onto function?

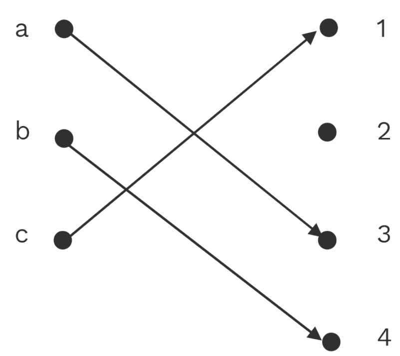
Solution:

We can argue that f is onto since all of the components of the co-domain are images of items in the domain.

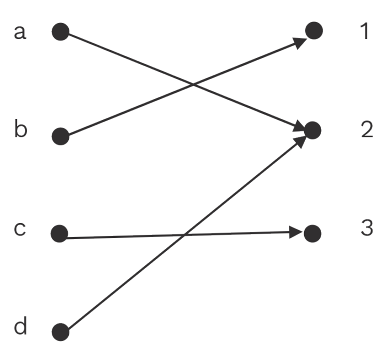


> **Note:**
>
> A function is said to be one-to-one correspondence, or a bijection if it is both one-to-one and onto.

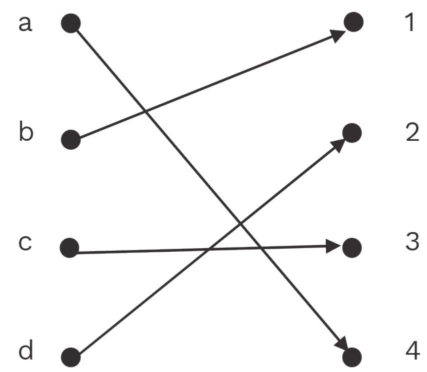
1. One-to-one, not onto



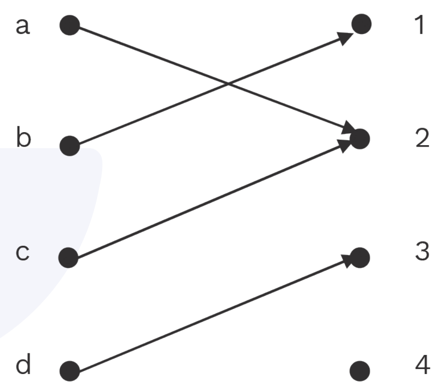
2. Onto, not one-to-one



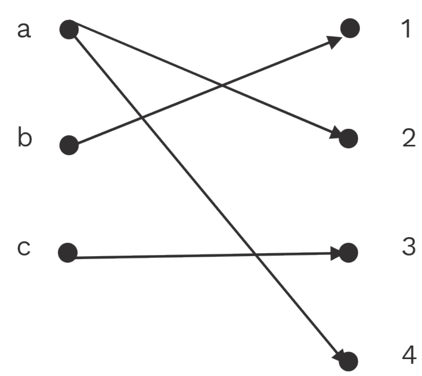
3. One-to-one and onto



4. Neither one-to-one nor onto



5. Not a function



Examples of different type of correspondences:

Different types of functions can be seen in the diagram above. i) is one-to-one but not onto, ii) is onto but not one-to-one, ii) is both one-to-one and onto, iii) is neither one-to-one nor onto, and iv) is neither one-to-one nor onto, and v) is neither one-to-one nor onto, and v) is not even a function because it sends an element to two different elements.

Let's pretend f is a function from A to A. (itself). When A is finite, f is one-to-one only if and only if it is onto. If A is infinite, this isn't always the case.

Definition

Let one-to-one correspondence from set to the set B. The inverse function off is the function that assigns to an element b belonging to B the unique element a in A such that f(a) = b.
The increase function off is deviated by f^{-1}. Hence $f^{-1}(b) = a$ when $f(a) = b$

Note:

Do not confuse with f 1 and 1/f.

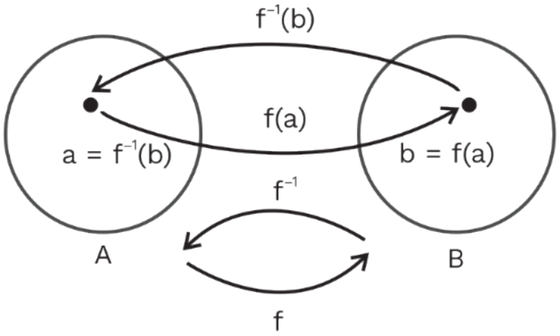


Fig. 2.20 Inverse of a Function

We can not determine the inversion of any function if that function is not a bijective function. If function is not bijection, then there are two possibilities, one is either function is not one–to–one and second is, function is not onto.

We can define inverse of a bijective function. Therefore, it is called invertible. A function is not invertible if it is not one–to–one correspondence.

Solved Examples

29. Let f is the function from {G,A,T,E} to {2,0,4,3} such that f(G) = 2, f(A) = 4, f(T) = 3, f(E) = O, is f invertible? If it is, what is its inverse?

Solution:

As the given function is one–to–one correspondence therefore, it is invertible.

Now,

$f^{-1}(2) = G$

$f^{-1}(0) = E$

$f^{-1}(4) = A$

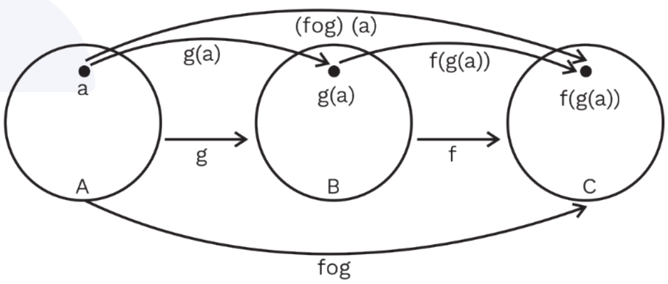
$f^{-1}(3) = T$

30. Consider a function R to R with $f(x) = x^2$. Is f invertible?

Solution:

As $f(-4) = f(4) = 16$, f is one–to–one. If an inverse of function is defined, it will assign two values to 16. Hence, f is not invertible.

Composite of functions:



Definition

Let g be a function from the set A to the set B and let f be a function from set B to the set C. The composition of the functions f and g, denoted by f o g, is defined by (f o g) (a) = f(g(a)).

Note:

- Commutative law does not hold for composite functions, i.e., $f \circ g \neq g \circ f$.
- The composition of a function and its inverse gives identity function.

Solved Examples

31. Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition off and g?

Solution:

$$(f \circ g)(x) = f(g(x)) = f(3x + 2)$$
$$= 2(3x + 2) + 3$$
$$= 6x + 7$$

The graph of functions:

The set of pairs in A×B to each function from A to B.

This set of pair is called graph of a function.

Definition

If f be a function from a set A to set B. The graph of the function f is the set of ordered pairs.
$\{(a, b) | a \in A \text{ and } f(a) = b\}$

Note:

- Floor function
  It is denoted by $\lfloor x \rfloor$
  It means value less than or equal to x.
- Ceiling function
  It is denoted by $\lceil x \rceil$
  It means value greater than equal to x.

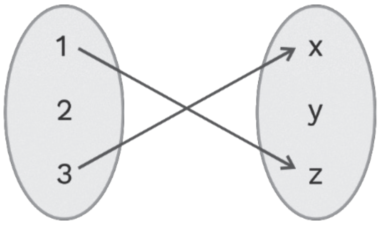
Rack Your Brain

1. Determine whether the function $f: Z \times Z \to Z$ is onto if:
a) $f(m, n) = |n|$
b) $f(m, n) = m-n$

2. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from R to R.

Partial functions:

When there is one element in the domain which is not defined, then that function is called partial function.



Previous Years' Questions

If $g(x) = 1 - x$ and $h(x) = \frac{x}{x-1}$ then $\frac{g(h(x))}{h(g(x))}$ is:

(A) $\frac{h(x)}{h(x)}$ (B) $\frac{-1}{x}$ (C) $\frac{g(x)}{h(x)}$ (D) $\frac{x}{(1-x)^2}$

Solution: (A)

Previous Years' Questions

The number of functions from an m element set to an n element set is:

[GATE CSE 1998]

(A) m + n
(B) m^n
(C) n^m
(D) m * n

Solution: (C)

Previous Years' Questions

Let N be the set of natural numbers. Consider the following sets, P: Set of Rational numbers (positive and negative)
Q: Set of functions from {0,1} to N
R: Set of functions from N to {0,1}
S: Set of finite subsets of N
Which of the above sets are countable?

[GATE CSE 2018]

(A) Q and S only
(B) P and S only
(C) P and R only
(D) P, Q, and S only

Solution: (D)

Chapter Summary

- A set is a collection of objects and is represented in roster form or set builder form.

  Roster Notation: V = {a, e, i, o, u }

  Set Builder Notation

  Here V is a set of all vowels in the english alphabet.

  : 0 = {$x \in z^+$ I x is odd and x < 10}

  or

  0 = {x I x is an odd positive integer less than 10}

  Both represent 0 is a set of odd positive integers less than 0.

  Empty set (0) : Set with no elements, also known as null set

  Singleton set : Set with only one element

  - Subset: The set A is a subset of B if and only if every element of A is also one element.
  - Proper Subset : $A \subset B$

    Subset : $A \subseteq B$

    Power Set : The power set of a set S is the set of all subsets of the set S.

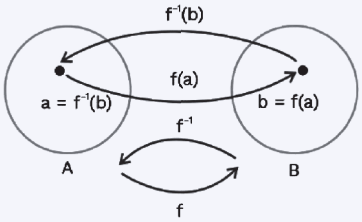
    It is denoted by P(S)
  - If a set has n-elements, then its power set has 2^n elements.

Set identities:

Identity	**Name**
$A \cup \phi = A$ $A \cap U = A$	Identity laws
$A \cap U = U$ $A \cap \phi = \phi$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$(\overline{A}) = A$	Complementation laws
$(A \cup B) = (B \cup A)$ $(A \cap B) = (B \cap A)$	Commutative laws

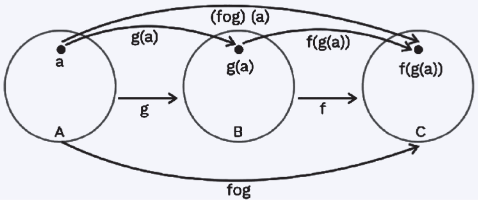
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B)$ $\cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B)$ $\cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \phi$	Complement laws

- Functions are sometimes called as mappings or transformations.
- One-to-one or injection: A function is said to be one to one if and only if f(a) = f(b) implies that a = b for all a and b in the domain f.
- Onto function or surjection: A function f from A to B is called onto if and only if for every element b $\in$ B. There is an element a E A with f(a) = b.
- Functions: Functions are also called mappings/transformations.
- A function can be one-to-one or onto.
- Inverse of a function:



Inverse of a Function

- Composite of function:



- Graph of a function: The set of pair in A x B to each function from A to B. This set of pair is called graph of a function.
- A relation is subset of cartesian product.
- A relation can be:
    - Symmetric
    - Antisymmetric
    - Asymmetric
    - Reflexive
    - Irreflexive
    - Transitive
- A relation can be represented using:
    - Diagraphs
    - Matrices
- A relation is said to be equivalence if it is:
    - Reflexive
    - Symmetric
    - Transitive
- Lattice:
    - Dual lattice is a lattice
    - Product of two lattice is a lattice
    - Every chain is a lattice
- POSET: An non-empty set P, forms a poset if it is:
    - Reflexive
    - Anti-symmetric
    - Transitive
- Hasse diagram: To obtain Hasse diagram follow these three simple steps:
    - Draw diagraph for given POSET
    - Remove all loops
    - Delete all edges implied by transitive property and arrange all edges pointing upwards