7 Transform Theory

Objective

Upon completion of this chapter you will be able to:

- Analyze a function in terms of Laplace Transform.
- Determine the Z-Transform of a discrete function.
- Determine Fourier Transform of a continuous function.

Introduction

The transforms are used to transform the complicated differential or difference equation into a simple algebraic equation. This is called as operational calculus. In the case of Laplace and Z-Transforms especially we have the advantage that we can consider initial conditions directly without too much of an effort. Fourier Transform is a special case of Laplace Transform and it converts a function from time domain to frequency domain. It is helpful in analyzing the frequency spectrum of a function i.e., the strength of different frequency components in a function.

Laplace Transform

Response of a LTI system with impulse response h(t) for an input $x(t) = e^{st}$ is $y(t) = H(s)\, e^{st}$

where, H(s) is the Laplace Transform of Impulse response h(t).

Analysis equation: $H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$

Where, $s = (\sigma + j\,\omega)$

$$X(\sigma + j\omega) = \int_{-\infty} x(t) e^{(-\sigma + j\omega t)} dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt$$

If the real part of s, $\sigma = 0$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \text{Fourier Transform}$$

So, for imaginary values of 's' Laplace Transform converges to Fourier Transform.

Region of convergence (ROC)

ROC is a range of values of s for which Laplace Transform converges is known as region of convergence. Region of Convergence makes the Laplace Transform of a signal unique.

Types of Laplace Transform

1. Unilateral Laplace Transform
2. Bilateral Laplace Transform

Unilateral Laplace Transform (ULT)

The unilateral Laplace transform is defined by the analysis equation

$$H(s) = \int_{0}^{\infty} h(t) e^{-st} dt$$

- The unilateral Laplace transform is restricted to causal time functions and take initial condition into account in the solution of differential equation and in the analysis of systems.
- ULT is obtained for only right sided signal as ULT of left sided signal is always '0'
- ULT is obtained for signal where BLT cannot be obtained

Example: $x(t) = e^{-2t}u(t)$

$$X(s) = \int_{0}^{\infty} e^{-2t} e^{-st} dt = \frac{1}{(s+2)}$$

Bilateral Laplace Transform (BLT)

The bilateral Laplace transform is defined by the analysis equation

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

- The bilateral Laplace transform can represent both causal and non-causal time functions. Initial conditions are accounted by including additional inputs. It is also used to describe frequency response and stability.
- It is obtained for right sided, left sided, 2 sided signals.
- BLT & ULT are the same for right sided signal.

Inverse Laplace Transform

To recover a signal from its Laplace Transform we have to use the following synthesis equation which is also called as inverse Laplace Transform.

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

Solved Examples

Example: Find the Laplace transform of given signals & indicate the region of convergence for the following signals.

(A) $x(t) = e^{-at}u(t)$

Solution:

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt$$

If $s + a > 0;\ e^{-\infty} = 0$

If $s + a < 0;\ e^{-(\infty)} = \infty$

Hence, s + a should be greater than

Region of convergence $R_e\ \{s + a\} > 0 => R_e\ (s) > -a$

$$X(s)=\frac{1}{-(s+a)}e^{-(s+a)t}\Big|_0^{\infty}$$

$$X(s)=\frac{1}{(s+a)}$$

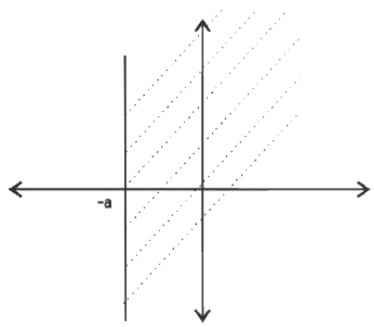

Fig. 7.1

(B) $x(t) = -e^{-at}u(-t)$

Solution:

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

Region of convergence

$R_e\ (s + a) < 0 => R_e\ (s) < -a$

$$X(s)=\frac{1}{(s+a)}e^{-(s+a)}\Big|_{-\infty} = \frac{1}{(s+a)}$$

Note: For the above two signals the expression for the Laplace Transform is same but their ROCs are different. So ROC makes the Laplace Transform of a signal unique.

Example: Find the ROC of the continuous time signal $x(t) = 3e^{-2t}u(t) + 4e^{t}\ u(-t)$.

Solution:

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt\Big| = \int_{-\infty}^{\infty}\left[3e^{-2t}u(t)+4e^{t}u(-t)\right]e^{-st}dt$$

$$X(s) = 3\int_{0}^{\infty} e^{-2t}e^{-st}dt+4\int_{-\infty}^{} e^{t}e^{-st}dt$$

$$= 3\int_{0}^{\infty} e^{-(s+2)t}dt + 4\int_{-\infty}^{0} e^{-(s-1)}dt$$

$$= 3\underbrace{\left[\frac{e^{-(s+2)t}}{-(s+2)}\right]_0^{\infty}}_{\text{I}} + 4\underbrace{\left[\frac{e^{-(s-1)}}{-(s-1)}\right]_{-\infty}^{0}}_{\text{II}}$$

$$X(s) = \frac{3}{(s+2)} - \frac{4}{(s-1)}$$

For I integral to be converge Re(s + 2)>0 or Re(s)>-2

For II integral to be converge Re (s – 1)<0 or Re(s)<1

Therefore, ROC: -2<Re(s)<1

Poles & Zeroes

If the Laplace Transform of a signal can be expressed in rational form as shown below

$$G(s) = \frac{K(s-s_1)(s-s_2).....(s-s_n)}{(s-s_a)(s-s_b)...(s-s_m)}.$$ Where k is constant.

- If in the Laplace Transform we put $s = s_a,\ s_b, ..., s_m$ the value of Laplace Transform becomes infinity & thus these are called as poles of Laplace Transform.
- If in Laplace Transform we put $s = s_1, s_2, ..., s_n$, the value of Laplace Transform is zero & these are called as zeroes of Laplace Transform.

Properties of ROC

1. ROC of Laplace Transform consists of strips parallel to the imaginary axis
2. ROC should not contains any poles & ROC doesn't depend on zeroes.
3. If x(t) is of finite duration then the ROC is the entire s-plane except possibly $s = \infty$ or $s = 0$

Example: $x(t) = \delta(t+1)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t+1) e^{-st} dt = e^{-s(-1)}$$

$$= e^{s} = 1 + s + \frac{s^2}{2!} + \ldots$$

∴ x(t) is defined by only when 's' is finite. If $s = \infty$ then $x(s) = \infty$

∴ ROC is the entire s–plane except, if x(t) is finite duration.

Example:

$$x(t) = \begin{Bmatrix} e^{-at} & ; 0 < t < T \\ 0 & ; \text{elsewhere} \end{Bmatrix}$$

$$x(s) = \lim_{0-2} e^{-at} e^{-st} dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^T = \frac{1 - e^{-(s+a)T}}{s+a}$$

$$\lim_{s-a} X(s) = \lim_{s-a} \frac{Te^{-(s+a)T}}{1} = T$$

ROC is the entire s-plane.

4. If x(t) is a right sided signal then all values of 's' for which $Re\{s\} = \sigma_0$ is include the ROC. Then the right side signal is $Re\{s\} > \sigma_0$ and if the signal is left side ROC is $Re\{s\} < \sigma_0$

Example: $e^{-at}u(t) \xleftrightarrow{\text{L.T.}} \frac{1}{s+a}; Re(s) > -a$

or $-e^{-at}u(-t) \xleftrightarrow{\text{L.T.}} \frac{1}{s+a}; Re(s) < -a$

5. If x(t) is two sided signal then we will consider common ROC or If signal x(t) is infinite 2 sides signal then ROC is a strip between 2 poles.
6. If x(s) is rational and if the signal is right sided then the ROC is right of the right most pole and if the signal is left sided ROC is left of the left most pole.

Example:

$$x(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Poles are $s = -1$, $s = -2$

If x(t) is right sided then

$Re\{s\} > -1$

$x(t) = e^{-t}u(t) - e^{-2t}u(t)$

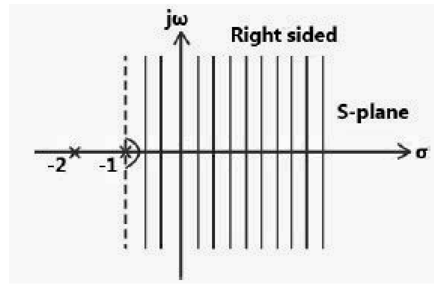

Fig. 7.2

If x(t) is left side then

$Re\{s\} < -2$

$x(t) = -e^{-t} u(-t) - e^{-2t} u(-t)$

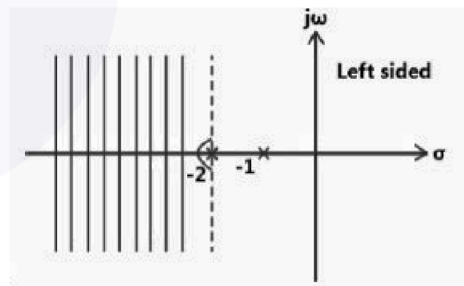

Fig. 7.3

If x(t) two sided then

$Re\{s\} > -2$ but $Re\{s\} < -1$

$-2 < Re\{s\} < -1$

$\therefore x(t) = -e^{-t} u(-t) - e^{-2t} u(t)$

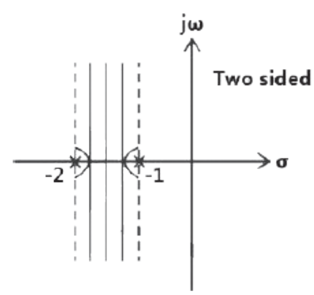

Fig. 7.4

7. If x(t) is band limited signal then ROC is the entire s-plane except 0 & ∞.

Note: If there is no common ROC between the poles for an infinite 2 sided signal then Laplace transform does not exist.

Gamma Function

$$L\{t^n\} = \begin{cases} \dfrac{n!}{s^{n+1}} & ;n \in Z^+ \\ \dfrac{n+1}{s^{n+1}} & ;n \notin Z^+ \end{cases} \quad \text{For } s > 0$$

Gamma Function $\lceil\overline{n+1}$, use only when n> –1

$\lceil\overline{n+1}\, n \lceil\overline{n} = n\,(n-1)(n-2)\ \ldots$ for +ve values of n.

$\lceil\overline{n+1} = n!,\ n \in Z^+$

$\lceil\overline{1} = 1\ \&\ \lceil\overline{\tfrac{1}{2}} = \sqrt{\pi}$

Some common Laplace Transform with ROC

Signal	Laplace Transform	ROC
$\delta(t)$	1	Entire s-plane
$u(t)$	$\dfrac{1}{s}$	$R_e(s) > 0$
$tu(t)$	$\dfrac{1}{s^2}$	$R_e(s) > 0$
$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$	$R_e(s) > 0$
$-u(-t)$	$\dfrac{1}{s}$	$R_e(s) < 0$
$-tu(-t)$	$\dfrac{1}{s^2}$	$R_e(s) < 0$
$-t^n u(-t)$	$\dfrac{n!}{s^{n+1}}$	$R_e(s) < 0$
$e^{-at}u(t)$	$\dfrac{1}{s+a}$	$R_e(s) > -a$
$te^{-at}u(t)$	$\dfrac{n!}{(s+a)^{n+1}}$	$R_e(s) > -a$
$t^n e^{-at}u(t)$	$\dfrac{1}{s+a}$	$R_e(s) > -a$
$-e^{-at}u(t)$	$\dfrac{n!}{s^2+a^2}$	$R_e(s) < -a$
$-t^n e^{-at}u(-t)$	$\dfrac{n!}{(s+a)^{n+1}}$	$R_e(s) > 0$
$\cos(at)\ u(t)$	$\dfrac{s}{s^2+a^2}$	$R_e(s) > 0$
$\sin(at)\ u(t)$	$\dfrac{a}{s^2-a^2}$	$R_e(s) > a$
$\cosh(at)\ u(t)$	$\dfrac{s}{s^2-a^2}$	$R_e(s) > a$
$\sinh(at)\ u(t)$	$\dfrac{a}{s^2-a^2}$	$R_e(s) > a$

$e^{-at}\cos(bt)\,u(t)$	$\frac{s+a}{(s+a)^2+b^2}$	$R_e(s) > -a$
$e^{-at}\sin(bt)\,u(t)$	$\frac{b}{(s+a)^2+b^2}$	$R_e(s) > -a$

Table 7.1

Properties of Laplace Transform

1. Linearity:

If $x_1(t) \xleftrightarrow{L.T.} e_0^{-st}\ X_1(s)$ with ROC $= R_1$

If $x_2(t) \xleftrightarrow{L.T.} e_0^{-st}\ X_2(s)$ with ROC $= R_2$

Then $ax_1(t) + bx_2(t) \xleftrightarrow{L.T.} e_0^{-st}\ aX_1(s) + bX_2(s)$

ROC $= R_1 \cap R_2$

2. Time scaling:

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s)$ with ROC $= R$

Then $x(at) \xleftrightarrow{L.T.} e_0^{-st}\ \frac{1}{|a|} \times \left(\frac{s}{a}\right)$ with ROC $= \frac{R}{a}$

3. Time shifting:

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s)$ with ROC $= R$

Then $x(t - t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s)$ with ROC $= R$

4. Shifting in s-domain:

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s)$ with ROC $= R$

Then $e^{s_0 t}x(t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s-s_0)$ with ROC $=$ $R + \text{Re}\{s_0\}$

5. Time reversal:

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s)$ with ROC $= R$

Then $x(-t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s)$ with ROC $= -R$

6. Convolution:

If $x_1(t) \xleftrightarrow{L.T.} e_0^{-st}\ X_1(s)$ with ROC $= R_1$

If $x_2(t) \xleftrightarrow{L.T.} e_0^{-st}\ X_2(s)$ with ROC $= R_2$

Then In time: $x_1(t) * x_2(t) \xleftrightarrow{L.T.} e_0^{-st}\ X_1(s) \times X_2(s)$ with ROC $= R_1 \cap R_2$

And In s-domain:

$x_1(t)x_2(t) \xleftrightarrow{L.T.} e_0^{-st}\ \frac{1}{2\pi}\left[X_1(s) * X_2(s)\right]$ with

ROC $= R_1 \cap R_2$

7. Differentiation in s-domain:

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s)$ with ROC $= R$

If $tx(t) \xleftrightarrow{L.T.} e_0^{-st}\ \frac{d}{ds}X(s)$ with ROC $= R$

In general $(-t)^n\, x(t) \xleftrightarrow{L.T.} e_0^{-st}\ \frac{d^n}{ds^n}X(s)$

8. Differentiation in time domain:

Valid only for ULT

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s)$

$\frac{d}{dt}x(t) \xleftrightarrow{L.T.} e_0^{-st}\ sX(s) - x(0^-)$

In general $\frac{d^n}{dt}x(t) \xleftrightarrow{L.T.} e_0^{-st}\ s^nX(s) + s^{n-1}X(0^-)$ $- s^{n-2}x(0^-) \ldots - X^{n-1}(0^-)$

9. Integration in time:

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s)$

$\underset{0-1}{\text{limit}}\ x(\tau)\, d\tau \xleftrightarrow{L.T.} X(s)/s \ldots$ (i)

$\underset{\propto - t}{\text{limit}}\ x(\tau)\, d\tau \xleftrightarrow{L.T.} \left(\frac{X^s s}{s}\right) + \frac{\int_{-\infty}^{0^-} x(\tau)\,d\tau}{s}$ ....(ii)

10. Integration in s-domain:

If $x(t) \xleftrightarrow{L.T.} e_0^{-st}\ X(s)$

If $\frac{x(t)}{t} \xleftrightarrow{L.T.} \underset{s-\infty}{\text{limit}}\ X(\lambda)\, d\lambda$

11. Initial value theorem:

Valid only for ULT

$x(0^+) = \lim_{s\to\infty} sX(s)$

12. Final value theorem:

Valid only for ULT

$x(\infty) = \lim_{s\to 0} sX(s)$

Basic Types of Functions

1. Unit Impulse or Kronecker delta or Dirac delta function $\left[\delta(t) \,\&\, \delta[n]\right]$

An ideal impulse function is a function that is zero everywhere but at the origin, it is infinitely high. However, the area of the impulse is finite. $[\delta n]$ is also known as Kronecker delta or sample function.

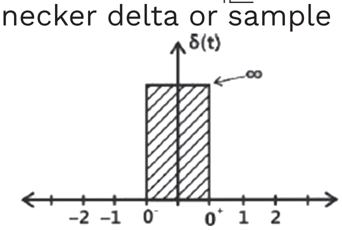

Fig. 7.5

$$\int_{0+}^{0-} \delta(t)\,dt = 1;\ \delta(t) = 0\ \forall\ t \neq 0$$

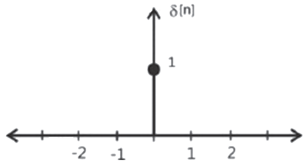

Fig. 7.6

$$\delta[n] = \begin{Bmatrix} 1, n = 0 \\ \hline 0, n \neq 0 \end{Bmatrix}$$

Properties of δ (t)

a) x(t) δ (t) = x(0) δ (t)

b) $\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = x(0)$

c) $x(t)\delta(t)dt = x(t_0)\delta(t - t_0)$

d) $\int_{-\infty}^{\infty} x(t)\delta(t - t)dt = x(t_0)$

e) $\int_{-\infty}^{\infty} x(t)\frac{d^k}{dt^k}(\delta(t - t_0))dt^0$

$$= (-1)^k \frac{d^k}{dt^k}(x(t))\Big|_{t=t_0}$$

Time – Scaling

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Similarly $\delta(at + b) = \frac{1}{|a|}\delta\left(t + \frac{b}{a}\right)$

Time – Scaling

a) $\delta(an) = \delta(n)$ (Time scaling is unaffected)

$$\delta[n] = \begin{cases} 1; & n = 0 \\ 0; & n \neq 0 \end{cases}$$

$$\text{But}\ \delta(Kn) = \begin{cases} 1; & Kn = 0 \\ 0; & Kn \neq 0 \end{cases}$$

$$\delta(Kn) = \delta(n)$$

2. Unit or Heaviside step function

u (t) & u (n)

The unit step function is just a piecewise function with a jump discontinuity at t = a

$$Mu(t - a) = \begin{cases} M, & t > a \\ 0, & t < a \end{cases}$$

Where M in the function represents the height of the jump and a is the number of units shifted to the right.

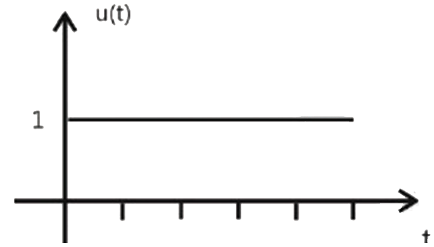

Fig. 7.7 Uniform probability distribution

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

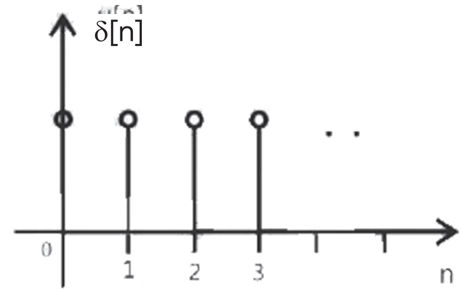

Fig. 7.8

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Heaviside Unit Step function u(t)

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \\ \frac{1}{2}, & t = 0 \end{cases}$$

Relation between unit step and impulse function:

a) $\frac{d}{dt}u(t) = \delta(t)$

b) $\delta(n) = (n) - u(n - 1)$

c) $u(t) = \int_{-\infty}^{t} \delta(t)\,dt$

d) $u(n) = \sum_{K=0}^{\infty} \delta\, n - K$

3. Ramp Function r (t) or r (n)

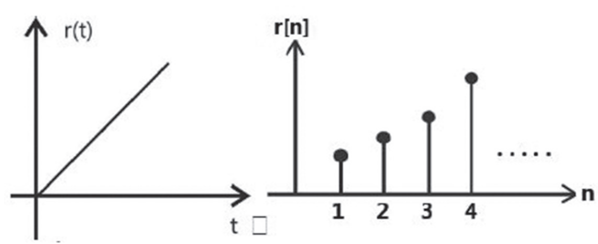

Fig. 7.9

$$r(t)\begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases} \qquad r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n \leq 0 \end{cases}$$

a) $\frac{d}{dt} r(t) = u(t)$

b) $\frac{d^2}{dt} r(t) = \delta(t)$

c) $\int_{-\infty}^{t} u(t)\, dt = r(t)$

d) $u(n) = r(n+1) - r(n)$

e) $\sum_{K=\infty}^{n-1} |u(K) \sum_{K=0}^{\infty} u(n-K-1) = r(n)$

4. Parabola p [t]

$$p(t)\begin{cases} t^2, & t > 0 \\ 0, & t > 0 \end{cases}$$

a) $\frac{d}{dt} p(t) = r(t)$

b) $\int_{-\infty}^{t} r(t) dt = p(t)$

c) $\frac{d^2}{dt} p(t) = u(t)$

d) $\frac{d^3}{dt} p(t) = \delta(t)$

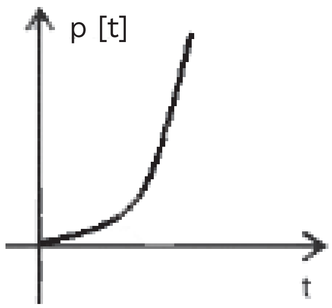

Fig. 7.10

5. Gate Function or Rectangular function

$$A\prod\left(\frac{t}{T}\right) \text{or } A\text{rect}\left(\frac{t}{T}\right)$$

$$A\text{rect}\left(\frac{t}{T}\right) = \begin{cases} A & ; \frac{T}{2} < t < \frac{T}{2}, \\ 0 & ; \text{elsewhere} \end{cases}$$

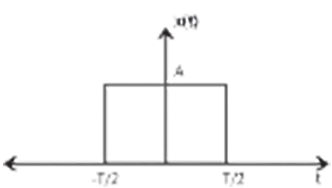

Fig. 7.11

6. Triangular Function

$$A\Delta\left(\frac{t}{T}\right) = \begin{cases} A\left(1+\frac{t}{T}\right) & ; -T < t < 0 \\ A\left(1-\frac{t}{T}\right) & ;\ 0 < t < T \\ 0 & ;\ \text{elsewhere} \end{cases}$$

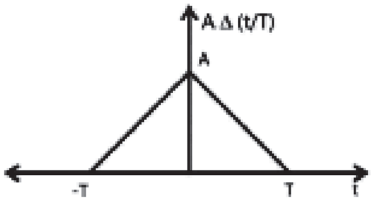

Fig. 7.12

7. Periodic Function

Cycle: A complete set of values of an alternating quantity is called as cycle T=indicate the fundamental period

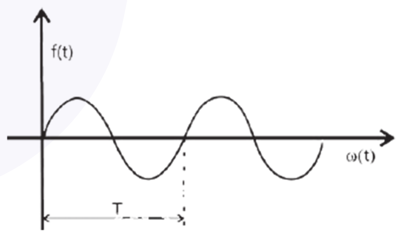

Fig. 7.13

The time taken to complete one cycle of an alternating quantity is known as fundamental period.

f (t) = Asin (ωt + θ) Where $\omega = 2\pi f = \frac{2\pi}{T}$

For discrete time signals,

f [n] = Asin (ωn + θ); $\frac{\omega}{2\pi} = \frac{m}{N}$

where N is fundamental period and 'm' and 'N' have no common factors

8. Continuous exponential

x(t) = Ae^{-at}

Case 1: a > 0 exponential decay

a = 1 x(t) = Ae^{-t}

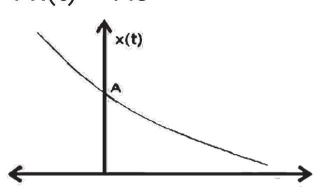

Fig. 7.14

Case 2: a < 0, exponential rising

a=-1 $x(t) = Ae^{t}$

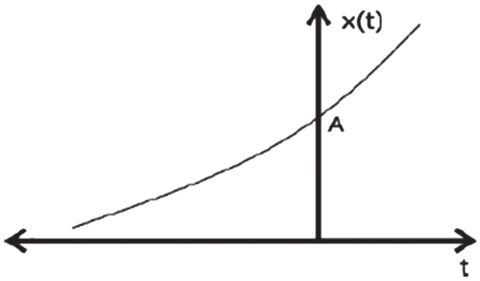

Fig. 7.15

9. **Discrete exponential**

$X(n) = a^{n}$

Case 1: 0 < a < 1

a = 0.2 $\quad (n) = (0.2)^{n}$

For $|a| < 1$, the signal is exponentially decaying.

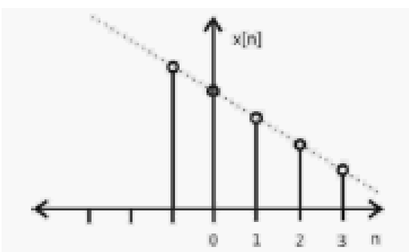

Fig. 7.16

Case 2: 0 < a < 1

a = 2 $\qquad [n]= 2^{n}$

For $|a| > 1$, the signal is exponentially increasing.

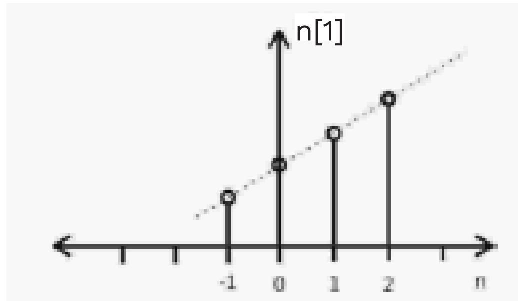

Fig. 7.17

10. **Signum Function**

$$\text{sgn}(t) = \begin{cases} 1 & ,t>0 \quad \text{or} \quad 2u(t)-1 \\ -1 & ,t<0 \end{cases}$$

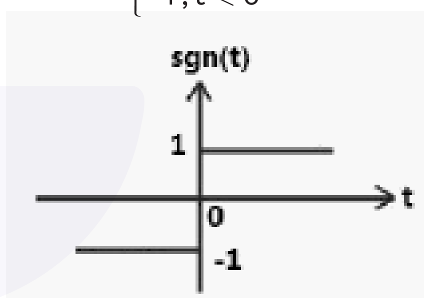

Fig. 7.18

Solved Examples

Example: Determine the Laplace Transform of a^{t}; a > 0

Solution: Laplace Transform of e^{-at} will be $\frac{1}{s+a}$

$$F(s) = L\left\{e^{\log a^{t}}\right\} = \left\{e^{t\log a}\right\} = \frac{1}{s-\log a}$$

Example: Determine Laplace Transform of $\left\{t^{1/2}+t^{1/2}+t^{3}\right\}$

$$F(s) = L\left\{t^{7/2}+t^{-1/2}+t^{3}\right\} = \frac{\left\lceil\frac{7}{2}+1\right.}{\frac{-1}{s^{2}}+1} + \frac{\left\lceil\frac{-1}{2}+1\right.}{\frac{-1}{s^{2}}+1} + \frac{3!}{s^{4}}$$

$$F(s) = \frac{7/2.\,5/2.\,1/2.\left\lceil 1/2\right.}{s^{9/2}} + \frac{\left\lceil 1/2\right.}{s^{1/2}} + \frac{6}{s^{4}}$$

$$= \frac{105\sqrt{\pi}}{16s^{9/2}} + \frac{\sqrt{\pi}}{s^{1/2}} + \frac{6}{s^{4}}$$

Example: Determine Laplace Transform of $\sin\sqrt{t}$.

Solution:

$$F(s) = L\left\{\sin\sqrt{t}\right\} = \left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \ldots\right\}$$

$$F(s) = \frac{\left\lceil\frac{1}{2}+1\right.}{s^{3/2}} - \frac{\left\lceil\frac{3}{2}+1\right.}{3!\,s^{5/2}} + \frac{\left\lceil\frac{5}{2}+1\right.}{5!\,s^{7/2}} \ldots\ldots$$

$$= \frac{1/2\sqrt{\pi}}{s^{3/2}} - \frac{3/2.\,1/2\sqrt{\pi}}{3!\,s^{3/2}} + \frac{5/2.\,1/2\sqrt{\pi}}{5!\,s^{7/2}} + \ldots$$

$$F(s) = \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^{2/3}} - \frac{1}{3!(4s)}\right] + \ldots.$$

$$= \frac{\sqrt{\pi}}{2s^{e-1/4s}}(s>0)$$

Example: Determine Laplace Transform of et^2.

Solution:

$$F(s) = L\left\{e^{t^2}\right\} = \left\{1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \ldots\ldots\right\}$$

Does not exist or integral is not absolutely converging

$$F(s) = \frac{1}{s} + \frac{2!}{s^3} + \frac{4!}{2!s^5} + \frac{6!}{3!s^7} + \ldots\ldots.$$

$$\sum_{n=0}^{\infty} \frac{(2n)!}{n!s^{2n+1}} \to \text{Divergent series}$$

Since all are increasing terms. These transformations do not exist.

Example: Determine the Laplace Transform of $\sin 3t \times \cos 4t$

Solution:

$$F(s) = L\{\sin 3t \cos 4t\} = \frac{1}{2} L\{\sin 7t + \sin(-t)\}$$

$$F(s) = \frac{1}{2}\left[\frac{7}{s^2+49} - \frac{1}{s^2+1}\right] = \frac{1}{2}$$

$$\left\{\frac{6s^2-42}{(s^2+49)(s^2+1)}\right\} = \frac{3s^2-21}{(s^2+49)(s^2+1)}$$

Example: Determine the Laplace Transform of {cost cos2t cos8t}

Solution:

$$F(s) = L\{\cos t \ \ \cos 2t \ \ \cos 8t\}$$

$$= \frac{1}{2} L\{(\cos 3t + \cos(-t))\cos 3t\}$$

$$F(s) = \frac{1}{2} L\{\cos^2 3t + \cos t \ \ \cos 3t\}$$

$$= \frac{1}{4} L\{1 + \cos 6t + \cos 4t + \cos(-2t)\}$$

$$F(s) = \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2} + \frac{s}{2} + \frac{s}{2}\right]$$

Example: If

$$L\{f(t)\} = \frac{e^{-1/s}}{s} \text{ then } L\left\{e^{3t}\int_0 f(3t)\,dt\right\} = \_\_\_\_\_\_?$$

Solution:

$$L\{f(3t)\} = \frac{1}{3}\left(\frac{s}{3}\right) = \frac{3e^{-3/5}}{3s} = \frac{e^{-3/5}}{s}$$

$$L\left\{\int_0^t f(3t)\,dt\right\} = \frac{1e^{-3/5}}{s} = \frac{e^{-3/5}}{s^2}$$

$$L\left\{e\int_0^{-3t} f(3t)\,dt\right\} = \frac{e^{-3/(s+3)}}{(s+3)^2}$$

Example: Determine Laplace Transform of

$$f(t) = \begin{cases} (t-2)^2 & ; t \geq 2 \\ 0 & ; t < 2 \end{cases} ?$$

Solution: $f(t) = (t-2)^2\, u(t-2)$

$$L\{t^2\} = \frac{2}{s^3}$$

$$L\left\{(t-2)^2 u(t-2)\right\} = e^{-2s} \times \frac{2}{s^3}$$

Example: Determine Laplace Transform of given wave.

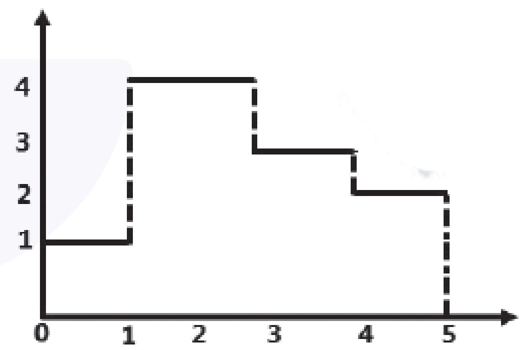

Fig. 7.19

Solution:

$$f(t)\left[\left[u(t) - u(t-1)\right]\right] + 4\left[\left[u(t-1) - u(t-3)\right]\right]$$
$$+3\left[\left[u(t-3) - u(t-4)\right]\right] + 2\left[\left[u(t-4) - u(t-5)\right]\right]$$

$$f(t) = u(t) + 3(u(t-1)) - u(t-3) - 2u(t-5)$$

$$F(S) = \frac{1}{s}\left[\left[1 + 3e^{-s} - e^{-3s} - e^{-4s} - 2e^{-5s}\right]\right]$$

Staircase/Step/Integral Function

$[[x]] = n \; ; n \leq x < n+1 \qquad n \in Z$

$f(t) = u(t-1) + u(t-2) + u(t-3) + \ldots\ldots\ldots$

$$L\{f(t)\} = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s} + \ldots\ldots\ldots.$$

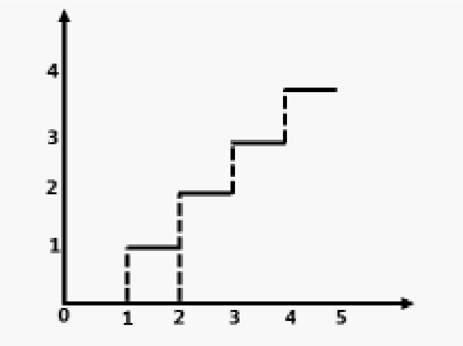

Fig. 7.20

Laplace Transform of periodic signals

- Bilateral LT doesn't exist for periodic signals rather unilateral LT exits as there is no common ROC between poles.
- $F(s) = \frac{X(s)}{1-e^{-st}}$ Where X(s) is Laplace transform of one period of the signal. Where T represents the time period of the signal

Solved Examples

Example: Determine the Laplace Transform of the following periodic function

$$f(t)\begin{cases} \sin\omega t & ;0 \le t < \frac{\pi}{\omega} \\ 0 & ;\frac{\pi}{\omega} \le t < \frac{2\pi}{\omega} \end{cases}$$

Solution: Since this function is periodic with a period of $\frac{2\pi}{\omega}$

$$f\left(t+\frac{2\pi}{\omega}\right) = f(t)\,v\,t|$$

$$f(t) = \sin\omega t\left(u(t) - ut - \frac{\pi}{\omega}\right)$$

$$L\{f(t)\} = \left(\frac{1}{1-e\frac{s2\pi}{\omega}}\right)L\left\{\sin\omega t\, u(t) - u\left(t-\frac{\pi}{\omega}\right)\right]$$

$$L\{f(t)\} = \left(\frac{1}{1-e^{\frac{s2\pi}{\omega}}}\right)\begin{cases}\sin\omega t\times u(t) + \sin\omega\left[t-\frac{\pi}{\omega}\right] \\ \times u\left[t-\frac{\pi}{\omega}\right]\end{cases}$$

$$L\{s(t)\} = \left(\frac{1}{1-e^{\frac{-s2\pi}{\omega}}}\right)\left\{\frac{\omega}{s_2+\omega_2}\left(1+e^{-\frac{s\pi}{\omega}}\right)\right\}$$

$$= \frac{1}{\left(1+e^{\frac{-s\pi}{\omega}}\right)\left(1-e^{\frac{-s\pi}{\omega}}\right)}\left\{\frac{\omega}{s_2+\omega_2}\left(1+e^{-\frac{s\pi}{\omega}}\right)\right\}$$

$$L\{f(t)\} = \left\{\frac{1}{1-e^{-\frac{s\pi}{\omega}}}\frac{\omega}{s^2+\omega^2}\right\}$$

Example: If

$$L\{\sin\sqrt{t}\} = \frac{\sqrt{\neq e}-^{1/4s}}{2s^{3/2}}\text{find } L\left\{\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]\right\}$$

Solution: $f(t) = \sin\sqrt{t} \Rightarrow f(0) = 0$

$$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = \frac{s\sqrt{\pi}e^{-1/4s}}{2s^{3/2}} = \frac{1}{2}s\sqrt{\frac{\pi}{s}}e^{-1/4s}$$

$$L\left\{\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]\right\} = \sqrt{\frac{\pi}{s}}e^{-1/4s}$$

Example: Determine inverse Laplace Transform of $\frac{1}{\sqrt{4s-3}}$

Solution:

$$L^{-1}\left\{\frac{1}{\sqrt{4s-3}}\right\} = \frac{1}{\sqrt{4}}L^{-1}\left\{\left(\frac{1_{1/2}}{s\frac{3}{4}}\right)\right\} = \frac{1}{\sqrt{4}}e^{4^{t^3}}$$

$$L^{-1}\left\{\left(\frac{1_1}{s^2}\right)\right\} = \frac{1}{2}e^{\frac{3}{4}t}\frac{t^{2^{1-1}}}{|1/2} = \frac{e^{\frac{3}{4}t}}{2\sqrt{\neq t}}$$

Therefore, $L^{-1}\left\{\frac{1}{(as+b)}\right\} = \frac{1}{a^n} - \frac{b}{e^{at}}\frac{t^{n-1}}{\sqrt{n}}$

Example: Determine the inverse Laplace Transform of $\frac{1}{s}\cot^{-1}(s)$

Solution: Let F(s) = $\cot^{-1}$ (s)

$$\frac{d}{ds}F(s)=\frac{-1}{s^2+1}$$

$$L^{-1}\left\{\frac{d}{ds}\right\}F(s)=-t\,L^{-1}\{F(s)\}$$

$$L^{-1}\left\{\frac{1}{s^2}\right\}=tL^{-1}\{\cot^{-1}(s)\}\text{ i.e., }\frac{\sin t}{t}=L^{-1}\{\cot^{-1}(s)\}$$

$$\therefore\ L_1\left[\frac{1}{s}\cot^{-1}s\right]=\int_0^t\frac{\sin t}{t}dt$$

Example: $L\{f(t)\}=\frac{2s+3}{s^2-2s+2}$.

Then determine $\underset{t\to\infty}{Lt}\ f(t)=?$

Solution: $\frac{2s+3}{s^2-2s+2}=\frac{2s+3}{(s-(1+i)\{s-(1-i)\})}$

Final value theorem is not applicable as poles are on right side. So the system is unstable. Hence, final value is undefined.

Example: Determine the solution of following differential equation using Laplace Transform.

$$\frac{dy}{dt^2}-3\frac{dy}{dt}+2y=8t;\ \ y(0)=0,\ \ \frac{dy}{dt}$$
When $t=0=1$.

Solution: Take Laplace Transform on both sides

$$s^2y(s)-sy(0)-y'(0)-3[[sy(s)-y(0)]]$$
$$+2y(s)=1$$
$$(s^2-3s+2)y(s2)=1+y'(0)=1+1=2$$

Fourier Transform

For continuous time signals, Fourier Transform is called as Continuous-Time Fourier Transform (CTFT) and for discrete signals it is called as Discrete-Time Fourier Transform (DTFT).

Note:

- The spectrum of Fourier Transform is always continuous i.e., non-periodic signals are converted into continuous frequency.
- In spite of having a number of application of Fourier Transform, Fourier Transform could analyse only the bounded signal & stable systems. So Laplace and Z-transform are used to analyse both bounded & unbounded signals; stable & unstable systems.
- Laplace transform is used for continuous time signals & Z-transform for discrete time signals.

Continuous time fourier transform:

The Fourier transform is a reversible, linear transform with many important properties. For any function f(x) (which is usually real-valued, but f(x) may be complex), the Fourier transform can be denoted F(s), where the product of x and s is dimensionless. Often x is a measure of time t (i.e., the time-domain signal) and so s corresponds to inverse time or frequency f (i.e., the frequency-domain signal).

These equations are used to find the Fourier Transform of a signal and analyse the frequency components of a signal and hence these are termed as Analysis equation.

$$\underset{\text{Continuous}}{x(t)}\ \xleftrightarrow{LT}\ \left.\begin{matrix}X(j\omega)\\X(\omega)\\X(f)\end{matrix}\right\}\text{ Continous}$$

$$X(\omega)=\int_{-\infty}^{\infty}X(t)e^{-j\omega t}dt$$

$$X(f)=\int_{-\infty}^{\infty}X(t)e^{-j2\pi ft}dt$$

Inverse Fourier Transform

These equations are used to derive signal in time domain from the frequency domain and hence these are termed as Synthesis Equation.

$$x(t)=\frac{1}{2}{}_{-\infty}^{\infty}\times(\omega)e^{j\omega t}d\omega$$

$$x(t)=\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}df$$

Dirichlet's Conditions

Like in Fourier Series these conditions must be satisfied for Fourier Transform to converge.

1. x(t) must be absolutely integrable

$$\int_{-\infty}^{\infty} X|t|dt < \infty$$

2. x(t) must have a finite number of discontinuities.
3. X(t) must have a finite number of maxima and minima.

So, here expect the periodicity condition on x(t), all other conditions must be satisfied for Fourier Transform to exist.

Note:
1. The spectrum of Fourier Transform is always continuous frequency response
2. Its magnitude response is symmetrical and phase response is anti-symmetrical

Fourier Sine and Cosine Transformations

For odd functions only the sine transform is defined and its cosine transform is zero.

$$F_s\{f(t)\} = F_s\{\omega\} = 2\int_0^{\infty} f(t)\sin\omega t\, dt$$

(Fourier Sine Transformation)

$$f(t) = F_s^{-1}\{F_3(\omega)\} = \frac{1}{\pi}\int_0^{\infty} F_s(\omega)\sin\omega t\, d\omega$$

(Inverse Fourier Sine Transformation)

For even functions only cosine transform is defined and its sine transform is zero.

$$F_c\{f(t)\} = F_c(\omega) = 2\int_0^{\infty} f(t)\cos\omega t\, dt$$

(Fourier Cosine Transformation)

$$f(t) = (F_c(\omega)) = \frac{1}{\pi}\int_0^{\infty} F_c(\omega)\cos\omega\, t d\omega$$

(Inverse Cosine Transformation)

Note:
- $A\operatorname{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\text{F.T.}} AT \times Sa\left(\frac{\omega T}{2}\right)$

  or $At \times \operatorname{Since}(fT)$
- $\operatorname{Sinc}(X) = \frac{\sin \pi.x}{\pi.x}$
- $Sa(x) = \frac{\sin x}{x}$
- $\Delta\left(\frac{t}{\tau}\right) \xleftrightarrow{\text{LT}} \tau \operatorname{sinc}^2\left(\frac{\omega\tau}{2}\right)$

Solved Examples

Example: Find Fourier Transform of f (t)

$$= \begin{cases} 1-t^2 & ; |t| \le 1 \\ 0 & ; |t| > 1 \end{cases}$$

Solution:

$$F\{f(t)\} = \int_{-1}^{1}(1-t^2)e^{-j\omega t}dt = \left[\frac{e^{-j\omega t}}{-j\omega}(1-t)^2 - (-2t)\right]$$

$$\left[\left[\frac{e^{-j\omega t}}{(-j\omega)^2}\right] + (-2)\frac{e^{-j\omega t}}{(-j\omega)^3}\right]_{-1}^{1}$$

$$F\{f(t)\} = \left|\begin{matrix} \frac{2e^{-j\omega}}{2} - \frac{2e^{-j\omega}}{-j\omega^3} + \frac{2e^{j\omega}}{-j\omega^3} = \frac{-2e}{-j\omega^3} \\ = \frac{-2e^{-j\omega}}{\omega^2}\frac{2e^{j\omega}}{-j\omega^3} + \frac{2e - 2e^{j\omega}}{+j\omega^3} \end{matrix}\right|$$

$$F\{f(t)\} = (e^{\omega} + e^{-\omega}) + \frac{2}{j\omega^3}(e^{\omega} - e^{-\omega})$$

$$F\{f(t)\} = \left[\frac{\omega}{\omega^2}(2\cos\omega) + \frac{2}{j\omega^3}(2j\sin\omega)\right]$$

$$= \frac{4}{\omega^3}[\sin\omega - \cos\omega]$$

Example: Find Fourier Transform of e^{-at}; (a>0)

Solution:

$$F(\omega) = 1\int_{-\infty}^{\infty} e^{at}e^{j\omega t}dt + \int_0^{\infty} e^{-at}e^{j\omega t}dt$$

$$= \left[\frac{e^{(a+j\omega t)} + e^{-(a+j\omega t)}}{(a+j\omega)_{-\infty} - a(a-j\omega)_{-0}}\right]$$

$$F(\omega) = \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2}$$

Example: Find Fourier Sine Transform of e^{-ax}?

Solution:

$$F(\omega) = 2\int_0^{\infty} e^{-ax}\sin\omega x dx = 2\int_0^{\infty} e^{-ax}$$

$$\left(\frac{e^{j\omega x} - e^{-j\omega x}}{2j}\right)dx$$

$$F_s(\omega)=2\int_0^{\infty}\frac{e^{-(a-j\omega x)}-e^{-(a+j\omega x)}}{2j}$$

$$dx=-j\left[\frac{e^{-(a-j\omega x)}}{a-j\omega}+\frac{e^{-(a+j\omega x)}}{a+j\omega}\right]_{-0}^{\infty}$$

$$=-j\left[\frac{1}{a-j\omega}-\frac{1}{a+j\omega}\right]=\left[\frac{2\omega}{a^2+\omega^2}\right]$$

Example: Find Fourier Sine Transform of xe^{-ax}

Solution: $F(\omega) = 2\ xe^{-ax}\ \sin x dx$

Integrate w.r.t

$$(\omega)d\omega=2\int_0^{\infty}xe\left(\frac{-\cos\omega x}{x}\right)dx$$

$$F(\omega)d\omega=-2\int_0^{\infty}e^{-ax}\cos\omega x\,dx$$

$$\int_s^F(\omega)d\omega=-2\int_0^{\infty}e\Big|^{-ax}\frac{e^{j\omega x}+e^{-j\omega x}}{2}dx$$

$$\int_s^F(\omega)d\omega=-2\int_0^{\infty}e\Big|\frac{e^{-(-aj\omega)x}+e^{-(a+j\omega)x}}{2}dx$$

$$\int F_s(\omega)d\omega=\left[-\frac{e^{-(-aj\omega)x}}{a-j\omega}-\frac{e^{-(a+j\omega)x}}{a+j\omega}\right]_{-0}^{\infty}$$

$$=-\left[\frac{1}{a-j\omega}+\frac{1}{a+j\omega}\right]=-\left[\frac{2a}{a^2+\omega^2}\right]$$

Diff. w.r.t. ω

$$F_s(\omega)\left|\frac{4a\omega}{(\omega^2+a^2)}\right|$$

Self-Reciprocal Functions

If the transformation of a function is the function itself then it is called self-reciprocal.

$$F_s\left(\frac{1}{\sqrt{x}}\right)=\frac{1}{\sqrt{\omega}}$$

Gauss Function is also self-reciprocal.

$$F\left(e\frac{x^2}{-2}\right)=e-\frac{\omega^2}{2}$$

Example: If

$$\int_0^{\infty}(x)\sin tx dx=\begin{cases}1\,;\,0\le t\le 1\\ 2\,;\,1\le t\le 2\text{ Then find }f(x)?\\ 0\,;\,t\le 2\end{cases}$$

$$f(x)=F_s^{-1}(F_s(t))=\frac{1}{\pi}\int_0^{\infty}F_s(t)\frac{1}{\pi}$$

$$\left[\int_0^1 1\sin tx dt+\int_1^2 2\sin tx dx+\int_2^{\infty}0dt\right]$$

$$=\frac{1}{\pi}\left[\left(\frac{-\cos tx}{x}\right)^1+\left(\frac{2\cos tx}{x}\right)^2\right]$$

$$=\frac{1}{\pi}\left[\left(\frac{-\cos x}{x}+\frac{1}{x}^{t-0}-\frac{2\cos 2x}{x}+\frac{2\cos 2x}{x}\right)\right]$$

$$=\frac{1}{\pi}\left[\left(\frac{\cos x}{x}+\frac{1}{x}-\frac{2\cos 2x}{x}\right)\right]$$

Properties of Fourier Transform

1. **Linearity**

   if $x_1(t)\xleftrightarrow{F.T.}x_1(\omega)$

   Ifif $x_2(t)\xleftrightarrow{F.T.}x_2(\omega)$

   Then $ax_1(t)+bx_2(t)\text{if }x_1(t)\xleftrightarrow{F.T.}ax_1(\omega)$

   $+bx_1(\omega)\text{or }ax_1(f)+bx_1(f)$

2. **Time scaling**

   If $x(t)\xleftrightarrow{F.T.}X(\omega)$

   Then $x(at)\xleftrightarrow{F.T.}\frac{1}{|a|}\times\left(\frac{\omega}{a}\right)$

Example: Find the Fourier transform of

$$\text{(i)}\ x_1(t)=A\text{rect}\left(\frac{t}{2T}\right)\qquad\text{(ii)}\ x_2(t)=A\text{rect}\left(\frac{2t}{T}\right)$$

Solution:

$$\text{(i)}\ A\text{rect}(t/T)\xleftrightarrow{F.T.}AT\sin c\left(\frac{\omega T}{2}\right)\text{or }AT\sin c(fT)$$

$$x_1(t)=x(t/2)\Rightarrow a=\frac{1}{2}$$

$$\therefore x_1(\omega)=\frac{1}{\left|\frac{1}{2}\right|}x(2\omega)=2AT\times Sa(\omega T)$$

$$\text{(ii)}\ x_2(t)=x(2t)\Rightarrow a=2$$

$$\therefore x_2(\omega)=\frac{1}{|2|}X\left(\omega/_2\right)=\frac{1}{|2|}AT\times Sa\left(\frac{\omega/_2.T}{2}\right)$$

$$x_2(\omega)=\frac{1}{2}AT\times Sa\left(\frac{\omega T}{4}\right)$$

3. **Time shifting**

   If $x(t)\xleftrightarrow{F.T.}X(\omega)$

   Then $X(t-t_0)\xleftrightarrow{F.T.}e^{-j\omega t_0}X(\omega)$

4. Symmetry or duality

If $x(t) \xleftrightarrow{\text{F.T.}} X(\omega)$

Then $X(t) \xleftrightarrow{\text{F.T.}} 2\pi x(-\omega)$ or $X(t) \xleftrightarrow{\text{F.T.}} x(-f)$

5. Shifting in frequency

If $x(t) \xleftrightarrow{\text{F.T.}} X(\omega)$

Then $e^{j\omega_0 t}X(t) \xleftrightarrow{\text{F.T.}} X(\omega - \omega_0)$

6. Differentiation in time

If $x(t) \xleftrightarrow{\text{F.T.}} X(\omega)$

Then $\frac{d}{dt}X(t) \xleftrightarrow{\text{F.T.}} j\omega X(\omega)$ or $j2\pi f x(f)$

$\frac{d}{dt}X(t) \xleftrightarrow{\text{F.T.}} (j\omega)^n X(\omega)$

or $(j2\pi f)^n X(f)$

7. Differentiation in frequency

If $x(t) \xleftrightarrow{\text{F.T.}} X(\omega)$

Then $-jtx(t) \xleftrightarrow{\text{F.T.}} \frac{d}{d\omega}X(\omega)$

or $tx(t) \xleftrightarrow{\text{F.T.}} \frac{d}{d\omega}X(\omega)$

$(-jt)X(t) \xleftrightarrow{\text{F.T.}} \frac{d^n X(\omega)}{d\omega^n}$

8. Integration in time

If $x(t) \xleftrightarrow{\text{F.T.}} X(\omega)$

$$\text{Then} \int_{-\infty}^{t} X(\tau)d\tau \xleftrightarrow{\text{F.T.}} \begin{cases} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega) & \text{if } X(0) \\ \frac{X(f)}{j2\pi f} + \frac{X(0)\delta(f)}{2} & \text{if } X(0) \neq 0 \end{cases}$$

9. Convolution

In Time:

If $x_1(t) \xleftrightarrow{\text{F.T.}} X_1(\omega)$

and $x_2(t) \xleftrightarrow{\text{F.T.}} X_2(\omega)$ then

$x_1(t) * x_2(t) \xleftrightarrow{\text{F.T.}} X_1(\omega)X_2(\omega)$

$x_1(t) * x_2(t) \xleftrightarrow{\text{F.T.}} X_1(f)X_2(f)$

10. Conjugation

If $x(t) \xleftrightarrow{\text{F.T.}} X_1(j\omega)$

Then $x^*(t) \xleftrightarrow{\text{F.T.}} X^*(-j\omega)$ or $X^*(-f)$

then $x^*(t) \xleftrightarrow{\text{F.T.}} X^*(-j\omega) X^*(-f)$

Where '*' denotes complex conjugate

Note: If x(t) is a real valued function, then its Fourier transform will be even conjugate or conjugate symmetric.

$X^*(j\omega) = X(-j\omega)$ or $X(j\omega) = X^*(-j\omega)$

Here, the magnitude of Fourier transform has even symmetry while the phase has odd symmetry. If x(t) is real valued, then Fourier transform $X(j\omega)$ is generally complex.

Solved Examples

Example: The Fourier transform of a conjugate symmetric function is always______________.

Solution: According to duality property, if the Fourier transform of real signal is conjugate symmetric, then Fourier transform of conjugate symmetric will also be real.

11. Time Reversal

If $x(t) \xleftrightarrow{\text{F.T.}} X(j\omega)$

Then $x(-t) \xleftrightarrow{\text{F.T.}} X(-j\omega)$

12. Parseval's Power Theorem

If $x(t) \xleftrightarrow{\text{F.T.}} X(\omega)$ Then

$$\int_{-\infty}^{\infty} |x(t)^2| dt = \int_{-\infty}^{\infty} \left|\frac{1}{2\pi}X(\omega)\right|^2 d\omega$$

$$\text{or} \int_{-\infty}^{\infty} |X(f)|^2 df$$

Some common Fourier Transform Pairs

Signal	Fourier Transform
$\sum_{K=-\infty}^{\infty} a_k e^{jk\omega 0t}$	$2\pi \sum_{K=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$
$e^{jk\omega 0t}$	$2\pi\delta(\omega-\omega_0)$

$\cos \omega_0 t$	$\pi\left[\delta\left(\omega+\omega_0\right)+\delta\left(\omega-\omega_0\right)\right]$
$\sin \omega_0 t$	$\frac{\pi}{j}\left[\delta\left(\omega-\omega_0\right)+\delta\left(\omega+\omega_0\right)\right]$
1	$2\pi\delta(\omega)$
$\sum_{K=-\infty}^{\infty} \delta\left(t-nT\right)$	$\frac{2\pi}{T}\sum_{K=-\infty}^{\infty} \delta\left(\omega-\frac{2\pi k}{T}\right)$
$x\left(t\right)=\left\{\begin{vmatrix}1\\0\end{vmatrix},\begin{vmatrix}t<T_1\\t>T_1\end{vmatrix}\right.$	$\frac{2\sin\omega T_1}{\omega}$
$\frac{\sin\omega_0 t}{\pi t}$	$x\left(\omega\right)=\left\{\begin{vmatrix}1\\0\end{vmatrix},\begin{vmatrix}\omega<\omega_0\\\omega>\omega_0\end{vmatrix}\right.$
δ (t)	1
u (t)	$\frac{1}{j\omega}+\pi\delta\left(\omega\right)$
$\delta\ (t-t_0)$	$e^{-j\omega t0}$
$e^{-at}u(t)$, Re(a) > 0	$\frac{1}{a+j\omega}$

Table 7.2

Fourier Transform of periodic signal

$$f\left(t\right)=\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$F\left(\omega\right)=\sum_{n=-\infty}^{\infty} 2\pi C_n\delta\left(\omega-n\omega_0\right)$$

$$F\left(\omega\right)=2\pi\sum_{n=-\infty}^{\infty} C_n\delta\left(\omega-n\omega_0\right)$$

Z Transform

Z transform is defined for unstable systems for which we cannot define DTFT. The Z-transform is simply a power series representation of a discrete-time sequence.

The response of a LTI Discrete Time System with an impulse response h[n] to a complex exponential input zn is given as

$y[n] = H(z)z^n$

Where H (z) is z-Transform of impulse response h[n].

$$H\left(z\right)=\sum_{n=-\infty}^{\infty} h\left[n\right]z^{-n}$$

Let $Z = re^{j\omega}$

$$X\left(re^{j\omega}\right)=\sum_{n=-\infty}^{\infty} x[n]\left(re^{-j\omega}\right)^n$$

$$=\sum_{n=-\infty}^{\infty} x[n]\left(re^{-j\omega}\right)^n$$

$$X\left(e^{j\omega}\right)^n \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n}$$

If r = 1, which means |z| = 1

$$X\left(e^{j\omega}\right)^n=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}=\text{Fourier Transform}$$

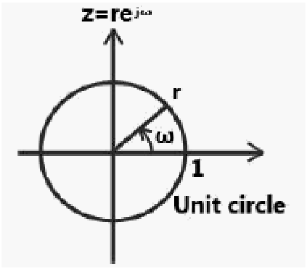

Fig. 7.21

- So, for all 'z' lying on the unit circle z-Transform converges to Fourier Transform.

Region of convergence (ROC)

ROC is a range of values of 'z' for which z-Transform converges is known as region of convergence. Region of Convergence makes the z-Transform of a signal unique.

Types of Z Transform

1. Unilateral Z Transform
2. Bilateral Z Transform

Unilateral Z Transform (UZT)

The unilateral Z transform is defined by the analysis equation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n|}$$

The unilateral Z transform is restricted to causal functions and take the initial condition into account in the solution of difference equation and in the analysis of systems.

Bilateral Z Transform (BZT)

The bilateral Z transform is defined by the analysis equation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

The bilateral Z transform can represent both causal and non-causal time functions. Initial conditions are accounted by including additional inputs. It is also used to describe frequency response and stability.

Inverse z-Transform

To recover a discrete time signal from its Z-Transform the following Synthesis Equation must be used which is also called as Inverse z-Transform.

Inverse ZT $\{X(z)\} = x[n]$

$$x[n] = \frac{1}{2\pi}\int x(z) z^{n-1} dz$$

Properties of ROC

1. ROC of Z transform consists of a ring in the Z-plane centred about origin
2. ROC does not contain any poles.
3. If x[n] is of finite duration then the ROC is the entire Z-plane, except z = 0 or $z = \infty$

Example: $\delta[n+1] \xleftrightarrow{Z.T.} Z$

ROC : entire Z-plane except at $z = \infty$

4. If x[n] is a right side sequence and if $|z| = r_0$ is in the ROC then, all values of 'z' for which $|z| > r0$ will also be in the ROC and if x[n] is left sided then ROC is $|z| < r0$
5. If X(z) is rational and if the signal is right sided, to define ROC consider the largest pole in magnitude and if it is left sided ROC is defined with the smallest pole in magnitude and if two-sided consider the common ROC.

Solved Examples

Example: Find Z-transform & ROC of

$x[n] = a^n u[n]$

Solution:

$$X(z) = \sum_{-\infty}^{\infty} a^n z^{-n} u[n] = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$X(z) = \frac{1}{1-\frac{a}{z}}; |az^{-1}| < 1 => |z| > |a|$$

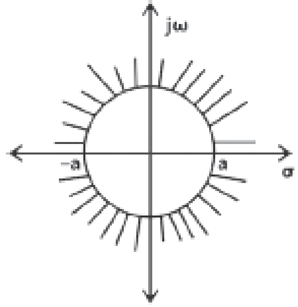

Fig. 7.22

Example: Find Z-transform & ROC of

$x[n] = a^n u[-n-1]$

$$X(z) = \sum_{-\infty}^{\infty} a^n z^{-n} u[-n-1] z^n$$

$$= \sum_{n=\infty}^{-1} -a^n z^n = -\sum_{n=\infty}^{-1} (az^{-1})^n$$

$$X(z) = \sum_{n=1}^{\infty} (az^{-1})^{-n}$$

$$X(z) = \sum_{n=1}^{\infty} (a^{-1}z)^n; |a^{-1}z| < 1$$

$$=> |z| < |a|$$

$$X(z) = \frac{z/a}{1-z/a} = \left(\frac{z}{z-a}\right)$$

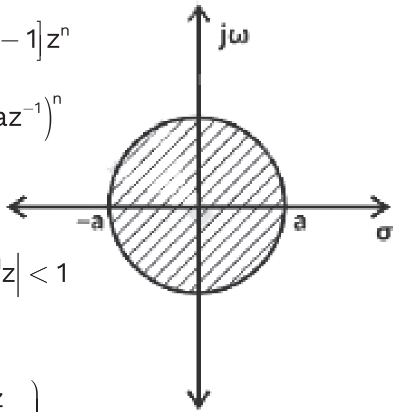

Fig. 7.23

Note: For the signals mentioned in the above examples the z-Transform turns out to be same but their ROCs are different. So, ROC makes a z-Transform unique.

	Domain	Tranform	ROC
$z\left[a^n u(n)\right]$	$n \geq 0$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$z\left[a^{n-1} u(n-1)\right]$	$n > 0$	$\frac{1}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$z\left[a^n u(-n)\right]$	$n \leq 0$	$\frac{a}{a-z}$	$\lvert z\rvert < \lvert a\rvert$
$z\left[a^n u(-n-1)\right]$	$n < 0$	$\frac{z}{a-z}$	$\lvert z\rvert < \lvert a\rvert$

Table 7.3

Example: Determine the Z-Transform of the following signal $\frac{1}{n!}$

$$Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{z^{-n}} = 1 + \frac{1}{z} + \frac{1}{2!} + \frac{1}{3!} .....n!$$

Similarly,

$$Z\left\{\frac{n!}{(n+1)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)} z^{-n} \, |z| > 0 \text{ or } |z| \neq 0$$

$$= 1 + \frac{1}{2!z^1} + \frac{1}{3!z^2} + e^z$$

$$z\left\{\frac{1}{(n+1)!}\right\} = z\left\{\frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + ...\right\}$$

$$\text{Similarly } z\left\{\frac{1}{(n-1)!}\right\} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{-n} = \frac{1}{z} e^{1/z}$$

Z-Transform of Trigonometric Functions

Combining Trigonometric Function into a complex exponential

$$Z\{\cos n\theta + i \sin n\theta\} = Z\{e^{in\theta}\}$$

$$Z\left\{\left(e^{i\theta}\right)^n\right\} = \frac{z}{\left(z - e^{i\theta}\right)} = \frac{z\left(z - e^{-i\theta}\right)}{z\left(z - e^{-i\theta}\right) z\left(z - e^{-i\theta}\right)}$$

$$\text{for } |z| > \left|e^{-i\theta}\right| = 1$$

$$Z\left\{e^{-in\theta}\right\} = \frac{z^2 - z(\cos\theta - i \sin\theta)}{z^2 - z\left(e^{\theta} + e^{\theta}\right) + e^{i\theta} e^{-i\theta}}$$

$$= \frac{z^2 (z\cos\theta + iz\sin\theta)}{z^2 - 2z\cos\theta + 1}$$

Comparing Real and Imaginary parts.

$$Z\{\cos n\theta\} = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\text{similarly} z\{\sinh n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z\{\cosh n\theta\} = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cos\theta + 1}$$

Solved Examples

Example: Determine Z-Transform of $\sin n\theta \cos n\theta$.

Solution:

$$Z\{\sin\theta \cos n\theta\} = \frac{1}{2} z(\sin(2\theta)) = \frac{1}{2}\left\{\frac{z\sin 2\theta}{z^2}\right\}$$

Example: Determine ROC for Z-Transform of the following function $f(n) = \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n u(n)$

Solution: $f(n) = \begin{cases} \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n & n \geq 0 \\ \left(\frac{1}{3}\right)^{-n} & n < 0 \end{cases}$

$$F(z) = \frac{z}{z - \frac{1}{3}} + \frac{z}{z - \frac{1}{2}} + \frac{z}{3 - z}$$

The ROC for each term in the above expression is,

$$|z| > \frac{1}{3} \;\&\; |z| > \frac{1}{2} \text{ and } |z| < 3$$

The intersection of these regions is,

$$\frac{1}{2} < |z| < 3$$

Some Common Z-Transform Pairs

Signal	z-Transform	ROC
Unit Impulse, $\delta[n]$	1	Entire z-plane
Unit step, $u[n]$	$\frac{z}{z-1}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{z}{z-1}$	$\|z\| < 1$
$nu[n]$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$-nu[-n-1]$	$\frac{z}{(z-1)^2}$	$\|z\| < 1$
$a^n u[n]$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{z}{z-a}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az}{(z-a)^2}$	$\|z\| < \|a\|$
$a^n, 0 \le n \le N-1$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos\omega_0 n u[n]$	$\frac{z(z-\cos\omega_0)}{Z^2 - 2z\cos\omega_0 + 1}$	$\|z\| > 1$
$\sin\omega_0 n u[n]$	$\frac{z\sin\omega_0}{Z^2 - 2z\cos\omega_0 + 1}$	$\|z\| > 1$

Table 7.4

Properties of Z-transform

1. **Linearity:**

$x_1[n] \xleftrightarrow{\text{Z.T.}} X_1[z]$; ROC : R_1

$x_2[n] \xleftrightarrow{\text{Z.T.}} X_2[z]$; ROC : R_2

Then $ax_1 + x_1[n] + bx_2[n] \xleftrightarrow{\text{Z.T.}}$

$aX_1[z] + bx_2$; ROC, $R_1 \cap R_2$

2. **Time shifting:**

If $x[n] \xleftrightarrow{\text{Z.T.}} x(z)$; ROC : R

Then $x[n] \xleftrightarrow{\text{Z.T.}} z^{-n=0} x(z)$; ROC : R except deletion or addition of origin or inf infinity

3. **Multiplication with an exponential:**

If $x[n] \xleftrightarrow{\text{Z.T.}} x(z)$ with ROC $=$ R

Then $a^n x[n] \xleftrightarrow{Z.T.} x[a^{-1}z] = x\left[\frac{z}{a}\right]$

with ROC $= |aR|$

4. Time reversal:

If $x[n] \xleftrightarrow{Z.T.} x(z)$ with ROC $= R$

Then $x[-n] \xleftrightarrow{Z.T.} X(z^{-1})$

with ROC $= \frac{1}{R}$

5. Differentiation in z-domain:

If $x[n] \xleftrightarrow{Z.T.} x(z)$ with ROC $= R$

Then $nx[n] \xleftrightarrow{Z.T.} z\frac{d}{dz}X(z)$

with ROC $= R$

In general $(n)^k x[n] \xleftrightarrow{Z.T.} (-z)^k \frac{d^k}{dz^k}X(z)$

with ROC $= R$

6. Convolution in time domain:

$x_1[n] \xleftrightarrow{Z.T.} x_1(z)$ with ROC $= R_1$

$x_2[n] \xleftrightarrow{Z.T.} x_2(z)$ with ROC $= R_2$

Then $x_1[n] * x_2[n] \xleftrightarrow{Z.T.} X_1(z)X_2(z)$

with ROC $= R_1 \cap R_2$

7. Conjugate property:

If $x[n] \xleftrightarrow{Z.T.} x(z)$ with ROC $= R$

Then $x^*[n] \xleftrightarrow{Z.T.} x^*(z^*)$ with ROC $= R$

8. Time Accumulation:

Only valid for UZT

If $x[n] \xleftrightarrow{Z.T.} x(z)$ with ROC $= R$

Then $\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{Z.T.} \frac{x(z)}{1-z^{-1}}$

with ROC $= R \cap |z| > 1$

9. Right Shift:

Only valid for UZT

If $x[n] \xleftrightarrow{Z.T.} x(z)$

Then $x[n-1] \xleftrightarrow{Z.T.} z^{-n}X(z) + x(-1)$

10. Left Shift:

Only valid for UZT

If $x[n] \xleftrightarrow{Z.T.} x(z)$

Then $x[n+1] \xleftrightarrow{Z.T.} zx(z) - zx(0)$

11. Initial Value Theorem:

Only valid for UZT

$x(0) = \lim_{z\to\infty} x(z)$

12. Final Value Theorem:

Only valid for UZT

$x(\infty) = \lim_{z\to 1}(1-z^{-1})x(z)$

or

$\lim_{z\to 1}(z-1)x(z)$

Note:

1. The z-transform X(z) must be a proper order i.e. order of numerator less than or equal to the order of denominator.
2. For system response H(z) to be stable all the poles must lie inside the unit circle & a simple pole is acceptable on unit circle, then final value theorem can be applied.
3. x (0) → initial value for causal

   ROC: |z| > 0

   x (∞) → final value for causal

   ROC: |z| > 0

   x (–∞) → initial value for anti-causal

   ROC: |z| < 0

   x (0) → final value for anti-causal

   ROC: |z| < 0

Example: Let X(z) be the z-transform of a DT signal x[n] given as X(z) = $\frac{0.5z^2}{(z-1)(z-0.5)}$. The initial value of x[n] is ________________.

Solution: From initial value theorem

$$x[0] = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{0.5z}{(z-1)(z-0.5)} = \lim_{z\to\infty} \frac{0.5}{\left(1-\frac{1}{z}\right)\left(1-\frac{0.5}{z}\right)} = 0.5$$

Solved Examples

Example: Find z-Transform of the function

$$f(n)=\begin{cases}4^n & ; n<0\\ 5^n & ; n<0\end{cases}$$

Solution: The ROC for the first part of the signal i.e. defined for |z| < 4 n < 0 is The ROC for second part of the signal i.e. defined for |n| > 0 is z > 5

The intersection of these two sets is a null set and thus Z-Transform does not converge for any value of z.

Example: Determine the Z-Transform of ^nC_k where k is a scalar.

Solution: Z-Transform of a given function can be determined from the equation, $\sum_{n=k}^{\infty} n\, C_k Z^{-n}$

Since, ^nC_r = $^nC_{n-r}$

$$F(z)=\sum_{n=k}^{\infty} {}^nC_{n-k} Z^{-n} C_0 Z^{-k} + {}^{(k+1)} + {}^{(k+2)}C_2 z^{-(k+2)} + ...$$

$$f(z) = z^{-k}\left\{{}^kC_0 + {}^{(k+1)}C_1 z^{-1} + {}^{(k+2)}C_2 z^{-2} + ...\right\}$$

$$= z^{-k}\left\{\left(1-\frac{1}{z}\right)^{-(k+1)}\right\} \text{ for } |z| > 1$$

Note:

$$(1-x)^n = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ...$$

$$(1-x)^{-(k+1)} = 1 + k + x + \frac{(k+1)}{2!}x^2 + \frac{(k+1)(k+2)(k+3)}{3!}x^3 + ...$$

Example: Find Z transform of $a^n nu(n)$

Solution: $z\{u(n)\} = \frac{z}{(z-1)}$

Then by differentiation in z-domain property

$$z\{nu(n)\} = -z\frac{d}{dz}\left(\frac{z}{(z-1)}\right) = \frac{z}{(z-1)^2}$$

By Scaling Property,

$$z[a^n nu(n)] = \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{az}{(z-a)^2}$$

Example:

If $z\{x(n)\} = \frac{z}{z-1} + \frac{z}{z^2+1}$ then $Z(x(n-2)) = ?$

Solution:

$$Z\{x(n-2)\} = z^{-2}X(z) = z^{-2}\left\{\frac{z}{z-1} + \frac{z}{z^2+1}\right\}$$

Example: If

$$Z(x(n)) = \frac{z^2-3z+4}{(z-3)}; \text{for } |z| > 3 \text{ then } x(3) = ?$$

Solution:

$$X(z) = \frac{z^2-3z+4}{(z-3)^2}\left|\frac{3}{z}\right| < 1$$

$$X(z) = \frac{Z^2-3z+4}{z^3\left(1-\frac{3}{2}\right)^3} = \frac{z^2-3z+4}{z^3}\left[1-\frac{3}{z}\right]^{-3}$$

$$X(z) = \frac{z^2-3z+4}{z^3}\begin{bmatrix}1+3\times\frac{3}{z}+6\times\left(\frac{3}{z}\right)^2\\ +10\left(\frac{3}{2}\right)^3 + ...\end{bmatrix}$$

$$X(z) = x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + ...$$

$$x(3) = \text{coefficient of } \frac{1}{z^3} = 54 - 27 + 4 = 31$$

Solved Examples

Example: Let X(z) be the z-transform of a DT signal x[n] given as

$$X(z) = \frac{0.5z^2}{(z-1)(z-0.5)}$$

The initial value of x[n] is ______________

Solution: From the ainitial value theorem

$$x[0] = \lim_{z\to\infty} X(z) = \lim_{z\to\infty}\frac{0.5z^2}{(z-1)(z-0.5)}$$

$$\lim_{z\to\infty}\frac{0.5}{\left(1-\frac{1}{z}\right)\left(1-\frac{0.5}{z}\right)} = 0.5$$

Inverse Z-transform by Partial Function Method

To find inverse z-Transform by Partial Fraction Method we first break the term in terms of Partial Fraction and then find Inverse z-Transform of each term. This can be seen in the example below:

Example:

$$x[n]=\frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

$$=\frac{2}{\left(1-\frac{1}{3}z^{-1}\right)}+\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)}$$

Solution:

$$x[n]=2\left(\frac{1}{3}\right)^n u(n)+\left(\frac{1}{4}\right)^n u(n);|z|>\frac{1}{3}$$

$$\text{or } x[n]=2\left(\frac{1}{3}\right)^n u[-n-1]+\left(\frac{1}{4}\right)^n u(n);\frac{1}{4}<|z|<\frac{1}{3}$$

$$\text{or } x[n]=2\left(\frac{1}{3}\right)^n u[-n-1]+\left(\frac{1}{4}\right)^n u[-n-1];$$

$$|z|<\frac{1}{4}$$

Inverse Z-Transform by Cauchy Residue Method

$$X[n]=\frac{1}{2\pi j}\int_c^0 x(z)z^{n-1}dz$$

Where C is a closed contour in the counter clockwise direction enclosing all the singularities of function $X(z)z^{n-1}$.

$$x[n]=\sum(\text{residue})$$

For a pole at $z=\beta$ of order 'm'

Residue of

$$x(z)=\frac{1}{(m-1)!}\lim_{z\to\beta}\frac{d^{m-1}}{dz^{m-1}}\left[x(z)z^{n-1}(z-\beta)^m\right]$$

Solved Examples

Example: Find inverse transform of

$$X(z)=\frac{z}{(z-2)^3}$$

Solution: Pole at $z=2$, order $=3$

$$x[n]=\frac{1}{(3-1)!}\lim_{z\to2}\frac{d^2}{dz^2}\left[z^n\right]=\frac{1}{2!}\lim_{z\to2}(n-1)z^{n-2}$$

$$=n(n-1)z^{(n-3)}u[n];\ |z|>2$$

$$\text{If } z<2;[n]=-n(n-1)2^{(n-3)}u(-n-1)$$

Example: Find Inverse Z-Transform of $e^{3/z}$

Solution:

$$Z^{-1}\left(e^{3/z}\right)=z^{-1}\left\{1+\frac{3}{z}+\frac{1}{2!}\left(\frac{3}{z}\right)^2+\frac{1}{3!}\left(\frac{3}{z}\right)^3.....\right\}$$

$$x(n)=z^{-1}\left\{\sum_{n-0}^{\infty}\frac{3^n}{n!}\right\}=3^n\frac{1}{n!}$$

Example: Determine Inverse Z-Transform of

$$\frac{z}{(z-5)(z+4)}$$

Solution: $\frac{X(z)}{z}=\frac{1}{(z-5)(z+4)}=\frac{1/9}{z-5}-\frac{1/9}{z+4}$

$$=\frac{(5)^n}{9}u(n)-\frac{(-4)^n}{9}u(n)$$

This can also be solved by Residue Theorem We have to calculate the residue of the function

$$\left[\left(z-5^z\right)-z\left({}^nz^{-1}+4\right)\right]=(z-5)z\left({}^nz+4\right)$$

Residue at z = 5

$$\lim_{z\to5}(z-5)\frac{z^n}{(z-5)(z+4)}=\frac{5^n}{9}$$

Residue at z = −4

$$\lim_{z\to4}(z+4)\frac{z^n}{(z-5)(z+4)}=\frac{(-4)^n}{9}$$

Thus, the signal is,

$$x(n)=\left[\frac{5^n}{9}+\frac{(-4)^n}{9}\right]u(n)$$

Example: Determine the Inverse Z-Transform of $\frac{z}{(z-2)^2(z+2)}$

Solution: By Residue Method,

The function whose residue is to be calculated is,

$$\frac{z \times z^{n-1}}{(z-2)^2(z+2)} = \frac{z^n}{(z-2)^2(z+2)}$$

Residue at $z=-2$

$$\lim_{z\to 2}(z+2)\frac{z^n}{(z-2)^2(z+2)} = \frac{(-2)^n}{16}$$

Residue at $z=2$

$$\lim_{z\to 2}\frac{d}{dz}(z-2)^2\frac{z^n}{(z-2)^2-(z+2)}$$

$$= \lim_{z\to 2}\left[\frac{n(z+2)z^{n-1}-z^n(1)}{(z+2)}\right] = \frac{n2^{n+1}-2^n}{16}$$

$$x(n) = \frac{n2^{n+1}-2^n}{16} + \frac{(+2)^n}{16}$$

Example: If $\{f(n)\} = \left(\frac{1}{z-2}\right)\frac{z}{(z+8)(z-2)}$ in $|z| = 1$ then $f(2) = \_\_\_\_\_\_\_\_\_\_$?

Solution: By Residue Method,

The function whose residue is to be calculated is,

$$\frac{z \times z^{n-1}}{\left(z-\frac{1}{2}\right)(z+3)(z-2)} = \frac{z^n}{\left(z-\frac{1}{2}\right)(z+3)(z-2)}$$

Since, ROC: $|z| = 1$

Only z = ½ lies inside ROC so residue will only be computed at this pole

$$x(n) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{(z)^n}{\left(z-\frac{1}{2}\right)(3+z)(z-2)} = \frac{-4}{21}\left(\frac{1}{2}\right)^n$$

$$f(2) = \frac{-4}{21}\left(\frac{1}{2}\right)^2 = -\frac{1}{21}$$

Practice Questions

1. The Laplace Transform of the function f(t) = e^{at} when t > 0 and where a is a constant is

   (A) $\frac{1}{(s-a)}$ (B) $\frac{1}{(s+a)}$

   (C) $\frac{1}{(s-a)^{-1}}$ (D) $\frac{1}{(s+a)^{-1}}$

2. The Laplace transform of f(t) is F(s) .

   Given F(s) = $\frac{\omega}{s^2+\omega^2}$, the final value of f(t) is

   (A) infinite (B) zero

   (C) one (D) none

3. The inverse Laplace transform of $\frac{s+9}{s^2+6s+13}$ is

   (A) cos 2 t + 9 sin 2t

   (B) e^{-3t} cos 2t - 3e^{-3t} sin 2t

   (C) e^{-3t} sin 2t + 3e^{-3t} cos 2t

   (D) e^{-3t} cos 2t + 3e^{-3t} sin 2t

4. If L{f(c)} = $\frac{2(s+1)}{s^2+2s+1}$ then f(0+) and f(∝) given by ......

   (A) 0, 2 respectively

   (B) 2, 0 respectively

   (C) 0, 1 respectively

   (D) 2/5, 0 respectively

5. The inverse Laplace transform of the function $\frac{s+5}{(s+1)(s+3)}$ is ....

   (A) $2e^{-t} - e^{-3t}$

   (B) $2e^{-t} + e^{-3t}$

   (C) $e^{-t} - 2e^{-3t}$

   (D) $e^{-t} + 2e^{-3t}$

6. Laplace transform L(f’), where f’ is the derivative of function f, is given by

   (A) L(f) – f(0) (B) sL(f) – f(0)

   (C) s^2L(f) – f(0) (D) L(f)/s – f(0)

7. The Laplace transform of $e^{\propto t}\cos \propto t$ is equal to

(A) $\frac{s-\alpha}{(s-\alpha)^2+\alpha^2}$

(B) $\frac{s+\alpha}{(s+\alpha)^2+\alpha^2}$

(C) $\frac{1}{(s-\alpha)^2}$

(D) none

8. The Laplace transform of $(t^2 - 2t)$ u (t–1) is

(A) $\frac{2}{s^3}e^{-s}-\frac{2}{s^2}e^{-s}$

(B) $\frac{2}{s^3}e^{-2s}-\frac{2}{s^2}e^{-s}$

(C) $\frac{2}{s^3}e^{-s}-\frac{2}{s}e^{-s}$

(D) none

9. The Laplace transform of a unit step function $Ua(t)$, defined as

$U_a(t)=\{\frac{0 \text{ for } t<a}{1 \text{ for } t<a}$ is

(A) $\frac{e^{-as}}{s}$ (B) Se^{-as}

(C) s–u(0) (D) $se^{-as}-1$

10. $(S+1)^{-2}$ is the Laplace transform of

(A) t^2 (B) t^3

(C) e^{-2t} (D) te^{-t}

Answer Key

1 – A	2 – D	3 – D	4 – B	5 – A
6 – B	7 – A	8 – D	9 – A	10 – D