A Handbook on
Computer Science

1

Discrete Mathematics & Engineering Mathematics

CONTENTS

1. Mathematical Logic 8
2. Combinatorics 13
3. Set Theory & Algebra 19
4. Graph Theory 38
5. Probability 48
6. Linear Algebra 54
7. Numerical Methods 59
8. Calculus 63

Mathematical Logic

1

Introduction

- **Proposition:** It is a declarative statement either TRUE or FALSE.
- **Compound Proposition:** It is a proposition formed using the logical operators (Negation ($\neg$), Conjunction ($\wedge$), Disjunction ($\vee$), etc.) with the existing propositions.
- **Logical Operators:**

  (*i*) Negation of p : $\neg p$ or $\bar{p}$ or $\sim p$

  (*ii*) Conjunction of p and q : $p \wedge q$

  (*iii*) Disjunction of p and q : $p \vee q$

  (*iv*) Implication/Conditional : $p \rightarrow q$ (if p, then q)

  (*v*) Bi-conditional : $p \leftrightarrow q$
- Precedence order of logical operators from high to low: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- $P \oplus R = PR' + P'R$, $P \leftrightarrow R = P'R' + PR$
- Number of distinct boolean expression with n variable $= 2^{2^n}$.
- **Normal form:** PCNF ($\vee$) = (POS = 0), PDFL ($\wedge$) = (SOP = 1)
  Total size = 2^n with n variable.

Note: ....

- Converse of $p \rightarrow q$ is : $q \rightarrow p$
- Inverse of $p \rightarrow q$ is : $\neg p \rightarrow \neg q$
- Contrapositive of $p \rightarrow q$ is : $\neg q \rightarrow \neg p$

....

Tautology

If compound proposition is always true then it is tautology.

Example: $p \vee \neg p$

Contradiction

If Compound proposition is always false then it is contradiction.

Example: $p \wedge \neg p$

Contingency

Neither tautology nor contradiction.

Example: p

Logical Equivalence

$P \Leftrightarrow Q$ is tautology iff P and Q are logically equivalent.

Functionally Complete

If any formula can be written as an equivalent formula containing only the connectives in a set of operators, then such a set of operators is called as functionally complete.

Example:

$\{\uparrow\}, \{\downarrow\}, \{\neg, \vee\}, \{\neg, \wedge\}, \{\neg, \vee, \wedge\}$ are functionally complete (NAND).

Consistent

If $H_1 \wedge H_2 \wedge H_3 \wedge. \ldots \wedge H_n$ is satisfiable then $H_1, H_2, \ldots$ and H_n are consistent (Tautology, contingency but not contradiction).

Inconsistent

If $H_1 \wedge H_2 \wedge H_3 \wedge. \ldots \wedge H_n$ is unsatisfiable then $H_1, H_2, \ldots$ and H_n are inconsistent (only contradiction).

- **Valid**: tautology, **Satisfiable:** tautology + contingency, Invalid: contradiction + contingency, **Unsatisfiable:** contradiction
- Sufficient ($\rightarrow$), necessary ($\leftarrow$), but = and, if = when = whenever, is = will = would = are = p unless q.
- $p \rightarrow q \equiv q$ unless $\neg p$ = "q is true unless p is false" either p is not true or q is true.
- p is necessary but not sufficient for $q = (q \rightarrow p) \wedge (p \rightarrow q)' = p \wedge q'$.

Equivalences

$$P \vee (P \wedge Q) \equiv P$$

$$P \rightarrow Q \equiv \neg P \vee Q \equiv \neg Q \rightarrow \neg P$$

$$P \wedge (P \vee Q) \equiv P$$

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$$

$$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

$$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$$

$$P \rightarrow (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$$

$$\neg(P \leftrightarrow Q) \equiv P \leftrightarrow (\sim Q) \equiv (\sim P) \leftrightarrow Q \equiv P \oplus Q$$

$$(P \rightarrow Q) \wedge (P \rightarrow R) \equiv P \rightarrow (Q \wedge R)$$

$$(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$$

$$(P \rightarrow Q) \vee (P \rightarrow R) \equiv P \rightarrow (Q \vee R)$$

$$(P \rightarrow R) \vee (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$$

$$P \vee Q \equiv \neg P \rightarrow Q$$

$$P \wedge Q \equiv \neg(P \rightarrow \neg Q)$$

$$\neg(P \rightarrow Q) \equiv (P \wedge \neg Q)$$

Identity Laws : (*i*) $P \wedge T = P$, (*ii*) $P \vee F = P$

Domination Laws : (*i*) $P \vee T = T$, (*ii*) $P \wedge F = F$

Idempotent Laws : (*i*) $P \wedge P = P$, (*ii*) $P \vee P = P$

Commutative Laws :

(*i*) $P \vee Q = Q \vee P$

(*ii*) $P \wedge Q = Q \wedge P$

Associative Laws :

(*i*) $(P \vee Q) \vee R = P \vee (Q \vee R)$

(*ii*) $(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$

Distributive Laws :

(*i*) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

(*ii*) $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

Demorgan's Laws :

(*i*) $\neg(P \wedge Q) = \neg P \vee \neg Q$

(*ii*) $\neg(P \vee Q) = \neg P \wedge \neg Q$

Absorption Laws :

(*i*) $P \vee (P \wedge Q) = P$

(*ii*) $P \wedge (P \vee Q) = P$

Negation Laws :

(*i*) $P \vee \neg P = T$

(*ii*) $P \wedge \neg P = F$

Double Negation Laws : $\neg(\neg P) = P$

Rules of Inference (Tautological Implications)

Simplification :

$$(P \wedge Q) \Rightarrow P$$

$$(P \wedge Q) \Rightarrow Q$$

Addition :

$$P \Rightarrow (P \vee Q)$$

$$Q \Rightarrow (P \vee Q)$$

Disjunctive Syllogism : $(\sim P, P \vee Q) \Rightarrow Q$

Modus Ponens : $(P, P \rightarrow Q) \Rightarrow Q$

Modus Tollens : $(\sim Q, P \rightarrow Q) \Rightarrow \sim P$

Hypothetical Syllogism : $(P \rightarrow Q, Q \rightarrow R) \Rightarrow (P \rightarrow R)$

Conjunctive Syllogism : $((P \vee Q), P) \Rightarrow \sim Q$

Dilemma : $(P \vee Q, P \rightarrow R, Q \rightarrow R) \Rightarrow R$

Constructive Dilemma : $(P \vee Q, P \rightarrow R, Q \rightarrow S) \Rightarrow R \vee S$

Destructive Dilemma : $(\sim R \vee \sim S, P \rightarrow R, Q \rightarrow S) \Rightarrow \sim P \vee \sim Q$

Other rules :

$$\sim P \Rightarrow (P \rightarrow Q)$$
$$Q \Rightarrow (P \rightarrow Q)$$
$$\sim(P \rightarrow Q) \Rightarrow P$$
$$\sim(P \rightarrow Q) \Rightarrow \sim Q$$

Exactly one = $\exists!$ or $\exists x\,[P(x) \wedge P(y) \Rightarrow y = x]$, $\forall x[\exists y\,(B(x, y) \wedge (B(x, z) \rightarrow y = z)$

$p \Rightarrow q \Rightarrow r \equiv (p \wedge q) \rightarrow r = q \Rightarrow (p \Rightarrow r)$

Principle Conjunctive Normal Form (PCNF)

Product of sums (max term)

PCNF: $[P(x_1) \vee P(x_2)] \wedge [P(x_3) \vee P(x_4)]$

Principle Disjunctive Normal Form (PDNF)

Sums of products (min term)

PDNF: $[P(x_1) \wedge P(x_2)] \vee [P(x_3) \wedge P(x_4)]$

Number of non equivalent propositional functions with n-propositional variables are = 2^{2^n}.

- $\forall x\,(\alpha \rightarrow \beta) \Rightarrow (\forall x\,\alpha \Rightarrow \forall x\,\beta)$ true only with properties always use and but not $\rightarrow$.

Predicate Logic

Quantifiers

- **Universal ($\forall$) :** "for all" or "for every"
- **Existential ($\exists$) :** "there exist"

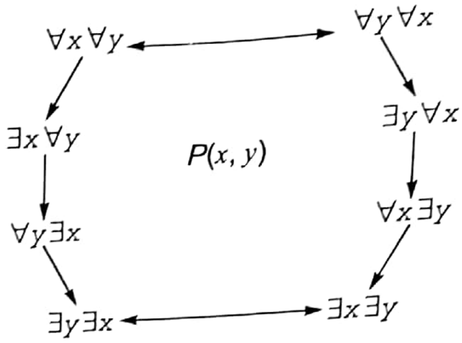
Predicates

- $P(x)$: Propositional statement with one variable.
- $Q(x, y)$: Propositional statement with two variables.

Note:

- $\neg\exists x P(x) = \forall x \neg P(x)$
- $\neg\forall x P(x) = \exists x \neg P(x)$

Logical Equivalences



1. $\forall x\,[P(x) \wedge Q(x)] \equiv \forall x\,P(x) \wedge \forall x\,Q(x)$
2. $\exists x[P(x) \vee Q(x)] \equiv \exists x\,P(x) \vee \exists x\,Q(x)$
3. $\forall x\,(P(x) \vee Q) \equiv \forall x\,P(x) \vee Q$
4. $\forall x\,(P(x) \wedge Q) \equiv \forall x\,P(x) \wedge Q$
5. $\exists x\,(P(x) \vee Q) \equiv \exists x\,P(x) \vee Q$
6. $\exists x\,(P(x) \wedge Q) \equiv \exists x\,P(x) \wedge Q$
7. $\forall x\,P(x) \wedge \exists y Q(y) \equiv \forall x \exists y[P(x) \wedge Q(y)]$
8. $\forall x\,P(x) \vee \exists y Q(y) \equiv \forall x \exists y[P(x) \vee Q(y)]$

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Combinatorics

2

Permutations (Ordered Selection/Arrangement)

- The number of permutations of n-objects :

1. taken 'r' at a time = ${}^nP_r = P(n, r) = (n)_r$ (arrangement).
2. taken 'r' at a time = n^r (with repetition) (arrangement).
3. taken all at a time = $n!$
4. taken not more than 'r' = $\dfrac{(n^{r+1}-1)}{n-1}$ (with repetition).
5. taken 'r' (atleast one repeated) = All – none = $n^r - {}^nP_r$
6. taken all at a time, in which 'r' of them are alike (identical) = $\dfrac{n!}{r!}$.
7. taken all at a time, in which n_1 are alike, n_2 are alike, ..., n_r are alike

   $= \dfrac{n!}{n_1!\,n_2!\ldots n_r!}$
8. taken 'r' at a time, in which m-particular objects are:

   (a) Never included = ${}^{(n-m)}P_r$.

   (b) Always included ($n \geq m$ and $r \geq m$) = ${}^{(n-m)}P_{(r-m)} \cdot {}^rP_m$.
9. ${}^nP_r = P(n-1, r) + r \cdot P(n-1, r-1)$
10. ${}^nP_r = r! \cdot {}^nC_r = n \times {}^{(n-1)}P_{(r-1)}$
11. n types of objects (each of infinity in number)

    $\underbrace{n_1 + n_2 + n_3 + n_4 + \ldots.. + n_n}_{n} = r = {}^{n-1+r}C_r$
12. m boys and n girls are arrange in straight line = $(m + n - 1)!$

- **The Number of Circular Permutations of n object :**

1. Taken all at a time = $(n-1)!$; (If not directed)

   $= \dfrac{(n-1)!}{2}$; (If direction (clockwise/anticlockwise) is given)

2. Taken 'r' at a time $= \frac{{}^nP_r}{r}$; (If not directed), and $\frac{1}{2} \cdot \frac{{}^nP_r}{r}$; (if directed)

3. Necklace with m identical beads, n identical beads in circular number of way $= \frac{(m+n-1)}{m! \times n! \times 2!}$.

Distinct → Distinct → Ordered partition	Distinct → Indistinct → Unordered	Indistinct → Distinct → Ball in box
$\frac{n!}{n_1!\, n_2! \dots n_n!}$	$\frac{n!}{(G!)^P}$	$\frac{n!}{(G!)^P \times P!}$

Ordered partition

Fully specified	Fully Unspecified	Partially specified
$\frac{n!}{n_1!\, n_2! \dots n_m!}$	m^n	Manually

Derangements

- Number of derangements for n-objects $= D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$

$$\left(D_n \cong \frac{n!}{e} \text{ for large value of } n\right) = \left[\frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} \dots \frac{n!}{n!}\right]$$

(The prob. of a given permutation being a derangement $= \frac{D_n}{n!} = \sum_{i=0}^{n} \frac{(-1)^i}{i!} \cong \frac{1}{e}$ for large n)

- **Binomial theorem:**

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b + \dots + {}^nC_n\, a^0 b^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$$

- **Multi-nomial coefficients:**

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1!\, n_2! \dots n_r!}$$

Note: ....

- Number of ways to distribute 'r' apples to n-persons $= {}^{(n+r-1)}C_r$
- The number of permutations of (n_1+n_2) objects, taken all at a time, in which n_1 objects are alike and n_2 objects are alike $= \frac{(n_1+n_2)!}{n_1!\, n_2!}$.

....

Combinations (Unordered Selection)

$$\boxed{{}^nC_r = C(n,r) = \frac{n!}{(n-r)!\,r!}}$$

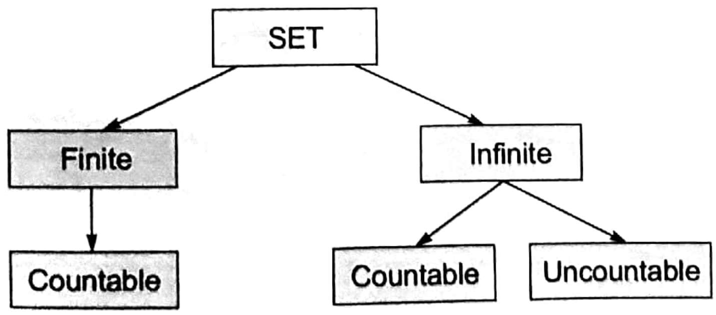
$$\boxed{C(n,r) = C(n-1,\ r) + C(n-1, r-1)}$$

- The number of combinations of n-objects :
  - (*i*) Taken (selecting) 'r' at a time = ${}^nC_r$
  - (*ii*) Taken (selecting zero or more at a time) = 2^n
  - (*iii*) Taken one or more at a time = $2^n - 1$
  - (*iv*) Where the objects are divided into p-objects of one type, q-objects of second type and so on; A non-empty selection from this can be made in $[(p + 1)(q + 1)......] -1$ ways.
  - (*v*) Taken 'r' at a time, in which m-particular objects:
    - (a) Always included = $C(n-m, r-m)$
    - (b) Never included = $C(n-m, r)$
- Number of subsets of a set X containing n-elements

  $= 2^n = \left\{{}^nC_0 + {}^nC_1 + \ldots\ldots + {}^nC_n\right\}$
- Number of diagonals for a regular polygon with n-sides

  $= {}^nC_2 - n = \dfrac{n^2 - 3n}{2}$ $\left\{\begin{array}{l} {}^nC_2 = \text{Number of diagonals and sides both,} \\ n = \text{Number of sides only = number of points} \end{array}\right\}$
- Number of triangles formed by vertices of a polygon with n-sides = ${}^nC_3$.

Counting



- Union of two countable sets in countable.
- Union of countable number of countable sets is countable.

- The set of real numbers that are solutions to quadratic equations $ax^2 + bx + c = 0$, where a, b and c are integers is '**countable**'.
- The set of lines through the origin is '**uncountable**'.

Counting Principle [Inclusion-Exclusion]

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
- Let $A_1, A_2, \ldots, A_n$ are finite sets. $n(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{1 \le i \le n} n(A_i) -$

$$\sum_{1 \le i < j \le n} n(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} n(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} n(A_1 \cap A_2 \ldots \cap A_n)$$

- If A and B are disjoint, $n(A \cup B) = n(A) + n(B)$.
- The set of functions "from" the positive integers to $\{0,1,2,3,4,5,6,7,8,9\}$ is '**uncountable**' since the power set of positive integers is uncountable.

Pigeonhole Principle

If n pigeons occupy m pigeon holes, then at least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lceil \frac{n}{m} \right\rceil$ pigeons share the same pigeon hole.

Fundamental Principle of Counting

If some event can occur in 'n_1' different ways, and if following this event a second event occur in 'n_2' different ways,... then the number of ways the event can occur in the order $= n_1 \cdot n_2 \cdot n_3 \ldots$

Summation

- $\Sigma n = \dfrac{n(n+1)}{2}$
- $\Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$
- $\Sigma n^3 = \dfrac{n^2(n+1)^2}{4} = [\Sigma n]^2$
- $\sum_{k=0}^{n} a \cdot r^k = \dfrac{a[r^{n+1} - 1]}{r-1};\ r \ne 1$
- $S_n = \sum_{k=1}^{n} f(k) = S_{n-1} + f(n)$
- $\sum_{k=n_1}^{n_2} f(k) = f(n_1) + f(n_1+1) + f(n_1+2) + \ldots + f(n_2)$; for $n_1 \le n_2$

- $\sum_{k=1}^{n}[f(k)+g(k)] = \sum_{k=1}^{n} f(k)+\sum_{k=1}^{n} g(k)$

Generating Functions

- $G(x) = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r + \ldots$
- ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$
- ${}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 \ldots.. + {}^nC_n^2 = {}^{2n}C_n$
- $(1 + ax)^n = \sum_{r=0}^{n} {}^nC_r \cdot a^r \cdot x^r = \sum_{r=0}^{n} {}^nC_r (ax)^r$
- $(1 + x^k)^n = \sum_{r=0}^{n} {}^nC_r \cdot x^{kr}$
- $\frac{1-x^{n+1}}{1-x} = \sum_{r=0}^{n} x^r$
- $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$
- $\log(1 + x) = \sum_{r=0}^{\infty} \frac{(-1)^{r+1} \cdot x^r}{r!} = -1+\frac{x}{1!}-\frac{x^2}{2!}+\frac{x^3}{3!}\ldots$
- $\frac{1}{1-ax} = \sum_{r=0}^{\infty} (ax)^r = \sum_{r=0}^{\infty} a^r \cdot x^r$
- $\frac{1}{1-x^n} = \sum_{r=0}^{\infty} (x^n)^r = \sum_{r=0}^{\infty} x^{nr}$
- $\frac{1}{(1-ax)^n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \cdot (ax)^r$
- $\frac{1}{(1+ax)^n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \cdot (-1)^r \cdot (ax)^r$
- $\sin x = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\infty$
- $\sinh x = \sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{4!} + \ldots\infty$

- $\cos x = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{x^{2r}}{(2r)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\infty$
- $\cosh x = \sum_{r=0}^{\infty} \frac{x^{2r}}{(2r)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\infty$

Recurrence Relation

$$\boxed{a_n = a_n^H + a_n^P}$$

where a_n^H is complementary solution and a_n^P is a particular solution.

- **Complementary function $[a_n^H]$ (Solution to homogenous part)**

Root	a_n^H
1. $t_1, t_2, t_3, \ldots t_k$ are distinct real numbers	$C_1 \cdot t_1^n + C_2 t_2^n + \ldots + C_k t_k^n$
2. $t_1 = t_2 = t_3$	$(C_1 + C_2 n + C_3 n^2)$
3. Pair of roots are complex $(\alpha \pm i\beta)$	$r^n[C_1 \cos n\theta + C_2 \sin n\theta]$ where $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}\frac{\beta}{\alpha}$

- Particular function $[a_n^P]$:

R.H.S	a_n^P
1. Constant	d_0
2. $C_0 + C_2 n$	$d_0 + d_1 n$
3. $C_0 + C_1 n + C_2 n^2$	$d_0 + d_1 n + d_2 n^2$
4. a^n	$d_0 a^n$
5. $n \cdot a^n$	$(d_0 + d_1 n) a^n$

d_1, d_2 and d_3 values are obtained by substituting a_n^P in given reccurrence relation.

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Set Theory & Algebra

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Set

A set is an unordered collection of objects.

The objects in a set are called the elements, or members of the set.

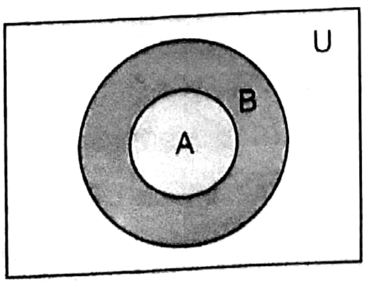
- $\mathbb{N}$ be set of natural numbers : {1, 2, 3, ...}
- $\mathbb{Z}$ be set of integers : {..., –2, –1, 0, 1, 2,...}
- $\mathbb{Q}$ be set of rational numbers
- $\mathbb{R}$ be set of real numbers
- $\mathbb{C}$ be set of complex numbers

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

Types of Set

1. **Universal set U:** A set which contains all objects under consideration (the universal set varies depending on which objects are of interest)
2. **Equal sets:** Two sets are equal iff they have the same elements i.e., if A and B are sets, then A and B are equal iff $\forall x\ (x \in A \leftrightarrow x \in B)$; denoted by $A = B$.
3. **Equivalent sets:** Two sets A and B are equivalent if they have same number of elements. i.e. $|A| = |B|$.
4. **Empty set or Null set:** A special set that has no elements. Null set can be denoted by ϕ or { }.

   Example: The set of all positive integers that are greater than their squares is the null set.
5. **Singleton set:** A set with one element is called a singleton set.
6. **Subset:** The set A is said to be a subset of B iff every element of A is also an element of B. $A \subseteq B$ indicates that A is a subset of the set B. i.e. $A \subseteq B$ iff $x \in A \Rightarrow x \in B$



Venn Diagram Showing that A is a subset of B

Note:

- For every set S : $\phi \subseteq S$ and $S \subseteq S$

Comparable: If $A \subseteq B$ or $B \subseteq A$ then A and B are comparable.

7. **Proper subset:** A set A is a subset of the set B but also $A \neq B$, we write $A \subset B$ and say that A is a proper subset of B i.e. A is a proper subset of B iff $\forall x\, (x \in A \rightarrow x \in B) \wedge \exists x\, (x \in B \wedge x \notin A)$
8. **Finite set:** A set in which number of elements are finite and hence countable i.e., cardinality of set can be obtained as a number.
9. **Infinite set:** A set is said to be infinite if it is not finite.

   Example: The set of positive integer is infinite.
10. **Power Set:** The power set of a set S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

Note:

- If a set has n-elements, then its power set has 2^n elements.
- *Example:* Power set of the set {0, 1, 2} is

  $\mathcal{P}(\{0, 1, 2\}) = \{\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

Cartesian Product of Sets

Let A and B be sets. The cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

Hence, $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Note:

- If $|A| = m$ and $|B| = n$ then $|A \times B| = mn$
- In general $A \times B \neq B \times A$ but $A \times B = B \times A$ iff $A = B$ or $A = \phi$ or $B = \phi$. However, $|A \times B| = |B \times A|$ always.

Set Operations

1. **Union:** The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

   $$A \cup B = \{x \mid x \in A \vee x \in B\}$$

   Example: The union of the sets {1, 3, 5} and {1, 2, 3} is : {1, 2, 3, 5}

Remember:

- $\text{Max}(|A|, |B|) \leq |A \cup B| \leq (|A| + |B|)$
- $|A \cup B| = |A| + |B| - |A \cap B|$

2. **Intersection:** The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

3. **Disjoint:** Two sets are called disjoint if their intersection is the empty set.
4. **Difference:** The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B.

   The difference of A and B is also called the complement of B with respect to A or the relative complement of B in A.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Inclusion Exclusion Principle

$$n(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{1\le i\le n} n(A_i) - \sum_{1\le i<j\le n} n(A_i \cap A_j) +$$

$$\sum_{1\le i<j<k\le n} n(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} n(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Properties of Sets

Let A, B and C are sets, U is universal set and ϕ is an empty set.

Identity	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
$A \cup \phi = A$ $A \cap U = A$	Identity Laws
$A \cup \bar{A} = U$ $A \cap \bar{A} = \phi$	Complement Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws
$\overline{(\bar{A})} = A$	Double Complement Law
$A \cup U = U$ $A \cap \phi = \phi$	Domination Laws
$\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$	De Morgans Laws

Multiset

A collection of objects in which an element can appear more than once is called a multiset.

Example: $\{a, a, b, b, b, c, c, c, c, c, d\} = \{2 \cdot a, 3 \cdot b, 5 \cdot c, 1 \cdot d\}$

Let $A = \{m_1 \cdot a_1, m_2 \cdot a_2, \ldots\ldots, m_k \cdot a_k\}$ where m_i = multiplicity of a_i

$B = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots\ldots, n_k \cdot a_k\}$ where n_i = multiplicity of a_i

Then $A \cup B$ = a multiset, in which multiplicity of a_i is max $\{m_i, n_i\}$

$A \cap B$ = a multiset, in which multiplicity of a_i is min $\{m_i, n_i\}$

$A + B$ = a multiset, in which multiplicity of a_i is $(m_i + n_i)$

$A - B$ = a multiset, in which multiplicity of

$$a_i = \begin{cases} m_i - n_i & \text{if} \quad m_i > n_i \\ 0 & , \quad \text{otherwise} \end{cases}$$

Functions

Definition

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

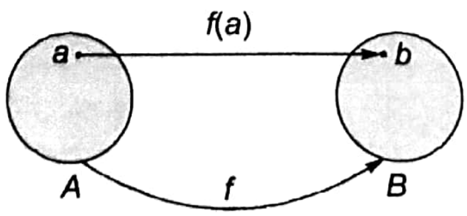
Note:

- Functions are sometimes also called mappings or transformations.

Domain and Codomain

If f is a function from A to B, we say that A is the domain of f and B is the codomain of f.

If $f(a) = b$, we say that "b is the image of a" and "a is the preimage of b".



The function f maps A to B.

- If number of elements $|A| = m$ and $|B| = n$, then number of functions possible from A to $B = n^m$.

- A function $f: A \rightarrow A$ is called a function on the set A.
  If $|A| = n$ then number of functions possible on $A = n^n$.

Types of Functions

1. **One-to-one function (Injection):** A function f is said to be one-to-one, or injective, iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.
   - If A and B are finite sets then a one-to-one from A to B is possible iff $|A| \leq |B|$.
   - If $|A| = m$ and $|B| = n$ then $(m \leq n)$ then number of one-to-one function from A to B is $P(n, m) = n(n-1)(n-2)\ldots..(n-(m-1))$.
   - If $|A| = |B| = n$ then number of one-to-one functions from A to B is $P(n, n) = n(n-1)(n-2)\ldots. 1 = n!$

2. **Onto Function (Surjection):** A function f from A to B is called onto, or surjective, iff for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$
   i.e. range(f) = co-domain(f) = B.
   - If A and B are finite sets then an onto function from A to B is possible only when $|B| \leq |A|$.
   - If $|A| = |B|$ then every one-to-one function from $A \rightarrow B$ is onto and vice-versa.
   - If $|A| = |B| = n$ then number of onto functions possible from A to B $= n!$
   - If $|A| = m$ and $|B| = n$ $(n < m)$ then number of onto functions from $A \rightarrow B = n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m \ldots + (-1)^{n-1}\, {}^nC_{n-1}(1^m)$.

3. **Bijection:** A function which is one-to-one and onto is called a bijection. If A, B are finite sets, then a bijection from A to B is possible only when $|A| = |B|$.
   - If $|A| = |B|$ then number of bijections = number of one-to-one = number of onto possible from A to $B = n!$

4. **Inverse Function:** Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$. Inverse of function f exists iff f is a bijection.
5. **Identity function:** Identity function on A is denoted by I_A. Inverse of identity function is the function itself. Every identity function is bijection, if $f(a) = a$; $\forall a \in A$.

6. **Constant function:** A function $f: A \to B$ is said to be constant function if $f(x) = c; \forall x \in A$ i.e., all the elements of domain are mapped to only one element of codomain. Therefore the range of constant function contains only one element.

Function Composition

Let f and g are two functions defined on set A:

$(f \circ g): A \to A$ defined by $(f \circ g)x = f(g(x))$

$(g \circ f): A \to A$ defined by $(g \circ f)x = g(f(x))$

Note:

- In general $(f \circ g)x \neq (g \circ f)x$
- Let $f: A \to B$ and $g: B \to C$ then $(g \circ f): A \to C$ but $(f \circ g)$ may not be defined $(f \circ g)$ is defined if range of $g(x)$ is a subset of A.
- If $f: A \to A$ is a bijection then $f \circ f^{-1} = f^{-1} \circ f = I$ where I is identity function on A.
- If $f: A \to B$ is a bijection then $f \circ f^{-1} = I_B$, $f^{-1} \circ f = I_A$, $f^{-1}: B \to A$
- If $f: A \to B$ and $g: B \to C$ are injective (one-one) then $g \circ f: A \to C$ is also injective.
- If $f: A \to B$ and $g: B \to C$ are surjective (onto) then $g \circ f: A \to C$ is also surjective.
- If $f: A \to B$ and $g: B \to C$ are functions, and $g \circ f: A \to C$ is injective then f is also injective.
- If $f: A \to B$ and $g: B \to C$ are functions, and $g \circ f: A \to C$ is surjective then g is also surjective (onto)

Relation

Definition

Let A and B be two sets. Then a binary relation from A to B is a subset of $A \times B$.

Relations on a Set

A relation on the set A is a relation from $A \times A$ i.e., a relation on a set A is a subset of $A \times A$.

- If $|A| = m$ and $|B| = n$ then number of relations possible on $A = 2^{mn}$.
- If $|A| = n$ and $|B| = n$ then number of relations possible on $A = 2^{(n^2)}$.

Types of Relation

1. **Inverse Relation:** Let R be a relation from a set A to B. The inverse of R, denoted by R^{-1} is the relation from B to A which consists of those ordered pairs, which when reversed belongs to R i.e.,
$R^{-1} = \{(b, a) \mid (a, b) \in R\}$

2. **Complementary Relation:** If R is a relation from A to B then
$R^c = \bar{R} = \{(a, b) \mid (a, b) \notin R\} = (A \times B) - R.$

3. **Diagonal Relation:** A relation R on a set A is called diagonal relation or identity relation if $R = \{(a, a) \mid a \in A\} = \Delta_A$.

4. **Reflexive Relation:** A relation R on a set A is said to be reflexive if aRa $\forall a \in A$ i.e. $(a, a) \in R$, $\forall a \in A$.
   - If $|A| = n$ then number of reflexive relations possible on $A = 2^{n(n-1)}$.
   - A diagonal relation on a set A is reflexive and any superset of diagonal relation is also reflexive.
   - Smallest reflexive relation on $A = \Delta_A$ (diagonal relation)
   - Largest reflexive relation on $A = A \times A$.

5. **Irreflexive Relation:** A relation R on a set A is said to be irreflexive, if $a \not{R} a$ i.e., $(a, a) \notin R$, $\forall a \in A$.
   - If $|A| = n$ then number of irreflexive relations possible on $A = 2^{n(n-1)}$.
   - Smallest irreflexive relation on $A = \phi$
   - Largest irreflexive relation on $A = (A \times A) - \Delta_A$.

6. **Symmetric Relation:** A relation R on a set A, is said to be symmetric if aRb then bRa, $\forall a, b \in A$.
   - If $|A| = n$ then number of symmetric relations possible on
   $$A = 2^n \times 2^{\frac{n^2-n}{2}} = 2^{n(n+1)/2}.$$
   - Number of symmetric relations possible with only diagonal pairs $= 2^n$.
   - Number of symmetric relations possible with only non-diagonal pairs $= 2^{(n^2-n)/2}$.
   - Smallest symmetric relation on $A = \phi$
   - Largest symmetric relation on $A = A \times A$.

7. **Antisymmetric Relation:** A relation R on a set A is said to be antisymmetric, if aRb and bRa then $a = b$, $\forall a, b \in A$.

- Smallest antisymmetric relation is ϕ.
- Largest antisymmetric relation on A is not unique. Number of elements in largest antisymmetric relation includes all diagonal pairs and half of non-diagonal pairs.
  i.e., $n + (n^2 - n)/2$ elements $= (n^2 + n)/2$ elements.
- Any subset of antisymmetric relation is also antisymmetric relation.
- If $A = \{1, 2, ..., n\}$ then number of antisymmetric relations possible on $A = 2^n \times 3^{n(n-1)/2}$.
  With n diagonal pairs, 2^n choices.

  With $\dfrac{n(n-1)}{2}$ non-diagonal pairs. $3^{n(n-1)/2}$ choices.
- A relation R is antisymmetric iff $R \cap R^{-1} \subseteq \Delta_A$.

8. **Asymmetric Relation:** A relation R on a set A is called asymmetric, if $(b, a) \notin R$, whenever $(a, b) \in R$, $\forall a, b \in A$.
   - Relation R is asymmetric iff it is both antisymmetric and irreflexive.
   - If $A = \{1, 2, ..., n\}$ then number of asymmetric relations $= 3^{n(n-1)/2}$.

Note: ....

- Number of reflexive and symmetric relations with n-elements $= 2^{n(n-1)/2}$.
- Number of neither reflexive nor irreflexive relations $= 2^{n^2} - 2.2^{n(n-1)}$.
- ϕ is not reflexive [empty relation]

....

9. **Partial Ordering Relation:** A relation R on a set A is partial ordered if R is reflexive, antisymmetric and transitive.
   Poset: A set A with a partial ordered relation R defined on A is called a poset. Poset is partially ordered set.
   Totally ordered set: A poset $[A; R]$ is totally ordered set, if every pair of elements in A are comparable i.e., either aRb or bRa $\forall a, b \in A$.

Note: ....

- **A relation R on a set A is:**
  (*i*) Symmetric $\Leftrightarrow R = R^{-1}$
  (*ii*) Antisymmetric $\Leftrightarrow (R \cap R^{-1}) \subseteq \Delta_A$
  (*iii*) Reflexive $\Leftrightarrow R^{-1}$ is also reflexive
  (*iv*) Reflexive $\Leftrightarrow R^C$ or $\bar{R}$ is irreflexive

- If R and S are antisymmetric, then $(R \cap S)$ is also antisymmetric for any relation S on A.
- If R is antisymmetric then every subset of R is also antisymmetric.
- If R is relation on a set A then $R \cup R^{-1}$ is always symmetric and $R \cup \Delta$ is always reflexive.
- **If R and S on set A are any two:**
  - (*i*) Reflexive relations then $(R \cup S)$ and $(R \cap S)$ are also reflexive.
  - (*ii*) Symmetric relations then $(R \cup S)$ and $(R \cap S)$ are also symmetric
  - (*iii*) Antisymmetric relations the $(R \cap S)$ is always antisymmetric
  - (*iv*) Transitive relations then $(R \cap S)$ is always transitive.
  - (*v*) Equivalence relations then $(R \cap S)$ is always equivalence relation.

Closures of Relations

1. **Transitive Closure :** Transitive closure of $R = R^*$ = smallest transitive relation on set A which contains R.

   Example: Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3)\}$.
   $$R^* = \{(1, 2), (2, 3), (1, 3)\}$$

2. **Reflexive Closure :** Reflexive closure of $R = R^+$ = smallest reflexive relation on set A which contains $R = (R \cup \Delta_A)$.

   Example: Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3)\}$.
   $$R^+ = \{(1, 2), (2, 3)\ (1, 1), (2, 2), (3, 3)\}$$

3. **Symmetric Closure :** Symmetric closure of $R = R^\#$ = smallest symmetric relation on set A which contains $R = (R \cup R^{-1})$

   Example: Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3)\}$.
   $$R^\# = \{(1, 2), (2, 3), (2, 1), (3, 2)\}$$

Partition of a Set

Let A be a set with 'n' elements dividing the set A into subsets $\{A_1, A_2, \ldots, A_n\}$ is called partition of A, if

(*i*) every subset is a non-empty set and

(*ii*) $\forall_{i,j}\ A_i \cap A_j = \phi$; $(i \neq j)$ and

(*iii*) $(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = A$.

Example: Let $A = \{1, 2, 3\}$. Then there are 5 partitions possible on A.

$P_1 = \{\{1, 2, 3\}\}$, $P_2 = \{\{1\}, \{2, 3\}\}$, $P_3 = \{\{2\}, \{1, 3\}\}$, $P_4 = \{\{3\}, \{1, 2\}\}$, and $P_5 = \{\{1\}, \{2\}, \{3\}\}$

Groups

Closure

Binary operator $*$ is said to be closed on a non-empty set A, if $a * b \in A$ for all $a, b \in A$

$$\boxed{\text{Number of binary operations on set } 'G' = |G|^{|G| \times |G|}}$$

Associativity

$(a * b) * c = a * (b * c); \quad \forall a, b, c \in G$

Identity

$\exists e \in G \forall a \in G \quad (a * e = e * a = a)$ where 'e' is identity

Note:

- If there exist an identity element in G then it must be unique.

Inverse

$\forall a \in G \exists b \in G \quad (a * b = b * a = e)$.

This also means $a^{-1} = b$ and $b^{-1} = a$

Commutative

$\forall a, b \in G \quad (a * b = b * a)$

Groupoid

An algebraic system $(G, *)$ is groupoid if it is closed operation on G.

Semigroup

An algebraic system which is groupoid and associative.

Monoid

An algebraic system $(G, *)$ which is semigroup and there is an identity in G.

Group

An algebraic system $(G, *)$ which is monoid and every element in G has inverse.

Abelian Group

An algebraic system $(G, *)$ which is a group and it is also commutative i.e., $a * b = b * a; \forall a, b \in G$.

Order of an Element of a Group

1. Let G be a group and let $g \in G$ be an element of G. Then the order of g is the smallest positive number k, such that $ak = e$.
2. Let G be a finite group and let $g \in G$. Then the order of g divides the order of G.
3. Any group of even order contains an element of order two.

Cyclic Group

- Let $G = \langle a \rangle$ be a cyclic group $G = \{a^i \mid i \in \mathbb{Z}\}$.
- Let G be a group. We say that G is cyclic if it is generated by one element.
- Let G be a cyclic group, generated by a. Then
  1. G is abelian
  2. If G is infinite, the elements of G are precisely $\ldots a^{-3}, a^{-2}, a^{-1}, e, a, a^2, a^3, \ldots$
  3. If G is finite, of order n, then the elements of G are precisely $e, a, a^2, \ldots, an^{-2}, an^{-1}$ and $an = e$.
  4. Let G be a group of prime order. Then G is cyclic.
  5. A finite group is cyclic iff there exists an element $g \in G$ whose order is same as the order of the group. Also such an element g will be the generator of that cyclic group.
  6. Let G be a finite cyclic group of order n, say $G = \langle g \rangle$. For every positive integer $d \mid n$ there is exactly one subgroup of G of order d. These are all the subgroups of G.
  7. Let G be a cyclic group. Every subgroup of G is cyclic.

Subgroup

- H is a subgroup of G iff
  (*i*) H is subset of G ($H \subseteq G$)
  (*ii*) Closure: $ab \in H$ for $a, b \in H$.
  (*iii*) Identity: The identity element of G is contained in H.
  (*iv*) Inverse: For all $a \in H$ we have $a^{-1} \in H$.
- Let G be a group and let H_i, $i \in I$ be a collection of subgroups of G. Then the intersection

$$H = \bigcap_{i \in I} H_i \text{, is a subgroup of } G.$$

- **Lagrange's theorem:** If H is a subgroup of a finite group G, then $|H|$ divides $|G|$.

Coset

- **Left Coset:** Let G be a group H is subgroup of G. A right H-coset in G is a set of the form $aH := \{ah \mid h \in H\}$.
- **Right Coset:** Let G be a group H is subgroup of G. A right H-coset in G is a set of the form $Ha := \{ha \mid h \in H\}$.
- The number of distinct right cosets (equivalently left cosets) of G is called the index of H in G and is denoted $[G : H]$.
- A left coset of a subgroup $H < G$ is is a subset of G of the form $gH = (gh : h \in H)$.
- Two left cosets are either equal or disjoint; $gH = g'H \Leftrightarrow g^{-1} g' \in H$
- A right coset of H in H is a subset of the form $Hg = (hg : h \in H)$. Two right cosets are either equal or disjoint; we have $Hg = Hg' \Leftrightarrow g^{-1}g' \in H$.
- A coset is a left or right coset. Any element of a coset is called a representative of that coset.
- If H is finite, all cosets have cardinality $|H|$.
- There are equal number of left and right cosets in group G.

Group Theory Classification

Groupoid	Semigroup	Monoid	Group	Abelian
Closure	closure + Associative	closure + Associative + Identity	closure + Associative + Identity + Inverse	Group + Commutative
Example:	*Example:*	*Example:*	*Example:*	*Example:*
$(N, +, *)$ $(Z, +, -, *)$ $(R, +, -)$ $(R-\{0\}, *, /)$	$(N, +, *)$ $(Z, +, *)$ $(R, +)$ $(R-\{0\}, *)$ [– and + are always not associative]	$(N, *)$ $(Z, +, \times)$ $(\{0, 1\}, \times)$ $(\{a, b\}^*, +)$	Non-singular matrices closed under '$*$' (multiplication)	$(\{0, 1, 2, 3\}, +_4)$ $(Z, +)$ $(R, +)$ $(R-\{0\}, *)$ $(Q, +)$ $(Q-\{0\}, *)$ $(\{1, -1, i, -i\}, *)$ $(\{1, \omega, \omega^2\}, *)$ $(\{1, -1\}, *)$

Note: ....

- $O(G) \le 5$ is always "Abelian group".
- Order of a group is equal to the number of elements in the group
- Every group of prime order is a cyclic group and every cyclic group is an Abelian group.

- $(\{0, 1, 2, \ldots, m-1\}, +_m)$ Addition modulo is Abelian group.
- If G is a finite group, and $g \in G$, then $g^{|G|} = e$, and $|g|$ always divides $|G|$ (where $|g|$ means order of element g).
- $(\{1, 2, 3, \ldots, q-1\}, \times_q)$ Multiplication modulo is Abelian group.
- If $O(G) = 2n$, then there exist atleast one element other than identity element which is "Self Invertible".
- Set of all non-singular matrices is a group under matrix multiplication, but not abelian.
- The set {0, 1, 2 . . . m – 1} with $\oplus_m$ is always a group, where $\oplus_m$ is also called addition modulo m define as follows:

$$a \oplus_m b = r\left(\frac{a+b}{m}\right);$$

Identity of this group is $e = 0$

- The set {1, 2 . . . p – 1} with $\otimes_p$ is always a group, where $\otimes_p$ is also called multiplication modulo p defined as follows:

$$a \otimes_p b = r\left(\frac{a \times b}{p}\right);$$

Identity of this group is $e = 1$

- Order of an element $O(a) = n$ and $O(a) = O(a^{-1})$ where $a \in G$ and $a^n = e$.

Lattice Theory

- **First (Least) Element:** Let A be an ordered set, the element 'a' in 'A' is first element of A if for every element 'x' in A, $a \le x$.
- **Last (Greatest) Element:** Let A be an ordered set. The element 'b' in 'A' is last element of A if for every element 'x' in A, $x \le b$.

Example:

1. Let N be the set of natural numbers, then first element of $N = 1$ and there is no last element.
2. Let 'A' be any set and let $\mathcal{P}(A)$ be the power set of A. Then first element of $\mathcal{P}(A) = \phi$ and last element of $\mathcal{P}(A) = A$

- **Minimal Element:** Elements which do not have predecessors.
- **Maximal Element:** Elements which do not have successors.

Note:

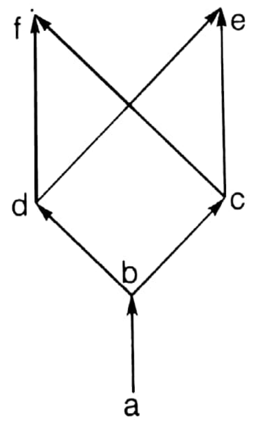
- Many minimals and maximals may exist.

- **Least Element:** 'a' is a least element of poset P; if $a \leq x;\ \forall x \in P$.
- **Greatest Element:** 'b' is a greatest element of P; if $x \leq b;\ \forall x \in P$.
- **Lower Bound:** Let $A \subseteq P$. Element a is a lower bound of A, if $a \in P$ and $a \leq x, \forall x \in A$.
- **Upper Bound:** Let $A \subseteq P$. Element b is an upper bound of A, if $b \in P$ and $x \leq b, \forall x \in A$.
- **Greatest Lower Bound [Infimum]:** 'y' is infimum of A, if y is a lower bound of 'A' and if 'z' is any other lower bound of A then $z \leq y, \forall z \in P$.
- **Least Upper Bound [Supremum]:** 'x' is supremum of A, if x is an upper bound of 'A' and if 'z' is any other upper bound, then $x \leq z, \forall z \in P$.

Note:

- If only one minimal exist then it is always "least"
- If only one maximal exist then it is always "greatest"
- Immediate successors of lower bound are called "atoms"

Example: Consider the following hasse diagram for a poset P.
Given P = {a, b, c, d, e, f}. Let S = {b, c, d} and $S \subseteq$ P.



Then

Lower bound of S = b, a

Upper bound of S = e, f

Infimum = b

No supremum element exist.

Minimal = a

Maximal = e, f

Least = a

No greatest element exist.

Lattice

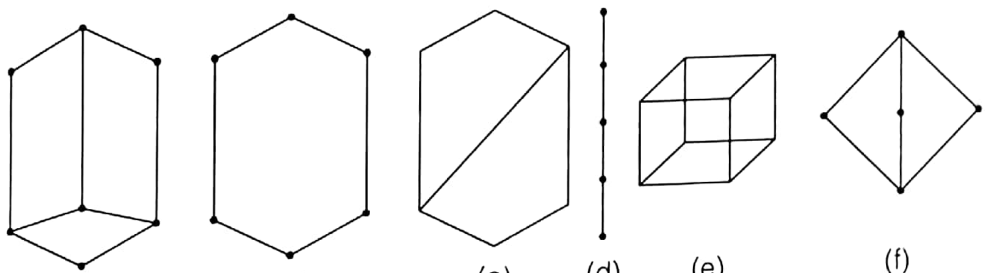
- Let (P, ≤) is a poset, in which for every two elements there exist infimum or greatest lower bound or meet (∧) and supremum or least upper bound or Join (∨) then such poset is called a "lattice".

OR

- Let 'L' be a non-empty set closed under two binary operations called meet ($\wedge$) and join ($\vee$), then 'L' is a "lattice" if for any element a, b and c of 'L' the following axioms hold.

  1. **Commutative Laws:**
     (*i*) $a \wedge b = b \wedge a$
     (*ii*) $a \vee b = b \vee a$
  2. **Associative Laws:**
     (*i*) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
     (*ii*) $(a \vee b) \vee c = a \vee (b \vee c)$
  3. **Absorption Laws:**
     (*i*) $a \wedge (a \vee b) = a$
     (*ii*) $a \vee (a \wedge b) = a$

- Following hasse diagrams are "lattices".



Note:
- Every chain is a lattice (i.e., linearly ordered set is a lattice).
- Let 'L' be a lattice, then $a \wedge b = a$ iff $a \vee b = b$.
- $x \wedge y =$ infimum (x, y) and $x \vee y =$ supremum (x, y).
- In lattice every 2-element subset has infimum and supremum.

Types of Lattices

Bounded Lattice

If there exist least element (0) and greatest element (1) for a lattice, such lattice is called "Bounded Lattice" i.e., if $0 \in L$ and $1 \in L$ then $0 \le x \le 1$, $\forall x \in L$.

- $(L, \le, \wedge, \vee)$ and $(P(S), \subseteq, \cap, \cup)$ are bounded lattices.
- $(N, \le, \text{Min}, \text{Max})$ and $(N, /, \text{gcd}, lcm)$ are not bounded.
- Every finite lattice is "Bounded Lattice".

Complemented Lattice

In a bounded lattice, if there exist atleast one complement for every element then such a bounded lattice is "complemented lattice".

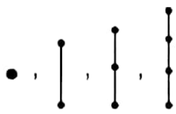
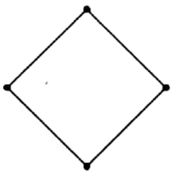
- If $x \vee y = 1$ and $x \wedge y = 0$, then x and y are complements to each other.
- Every element of complemented lattice can contain one or more complements.

Distributive Lattice

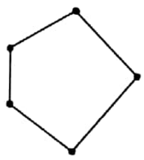
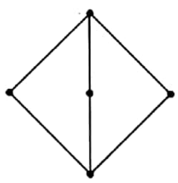
A distributive lattice 'L' satisfies:

$$\left.\begin{array}{ll}(i) & a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ (ii) & a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\end{array}\right\} \forall\, a, b, c \in L$$

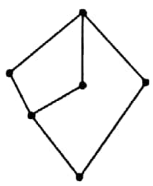
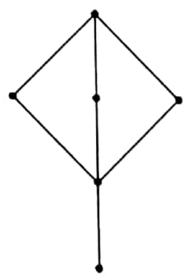
- If a distributive lattice is complemented, then every element has a unique complement.
- A lattice with less than 5-elements is always 'distributive'.

 and  are distributive lattices

- A lattice 'L' is non-distributive iff it contains a sublattice isomorphic to the following lattices:

 and  [Non-distributive Lattices]

- The following lattices are non-distributive lattices since they contain a sublattice isomorphic to one of the above lattice

 and 

Modular Lattice

A modular lattice 'L' satisfies: $a \vee (b \wedge c) = (a \vee b) \wedge c$; $\forall a, b, c \in L$ and $a \leq c$.

Note:

- Every distributive lattice is modular

Sublattice

A lattice 'L' is called "sublattice", if has the same meet ($\wedge$) and same join ($\vee$) as the parent lattice.

Example: (D_{12}, /, gcd, lcm) is sublattice of (N, /, gcd, lcm), where '/' represents the 'divides' relation.

Dual Poset

If (P, $\leq$) is poset then (P, $\geq$) is also a poset, such posets are called "dual posets".

Dual Lattice

If (L, $\leq$, $\wedge$, $\vee$) is a lattice, (L, $\geq$, $\vee$, $\wedge$) is also a lattice, such lattices are called "Dual Lattices".

Complete Lattice

A lattice 'L' is said to be complete if every subset of 'L' has infimum and supremum in L.

Lexicographical Order (Dictionary Order)

Let A_1 and A_2 be partial ordered sets, the lexicographical ordering ($\leq$) on $A_1 \times A_2$ is defined as:

$(a_1, a_2) < (b_1, b_2)$; either "if $a_1 < b_1$" or "both $a_1 = b_1$ and $a_2 \leq b_2$".

Well-ordered Set

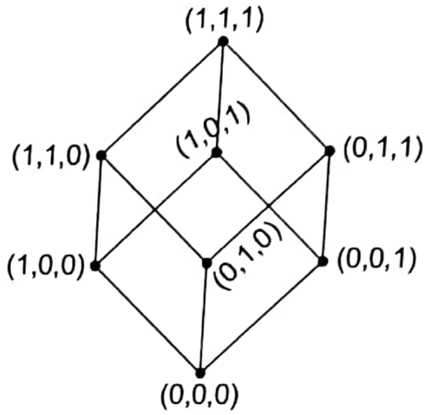
An ordered set 'A' is well-ordered if it is a chain (linearly ordered) and if it is a discrete set and every subset of 'A' contains "first element" (least element).

- Every "Finite Linearly Ordered Set" is well-ordered.
- Every well-ordered set must be linearly ordered (chain).

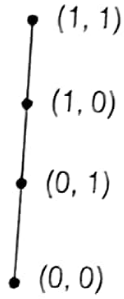
Boolean Algebra (B, $\leq$, $\wedge$, $\vee$)

- If a lattice is complemented and distributive, it is boolean algebra.

  Example: ($\mathcal{P}$(S), $\subseteq$, $\cap$, $\cup$)



Boolean algebra
(complemented & distributive)



Not boolean algebra
(not complemented)

- Boolean algebra satisfies: "Lattice [Poset, meet, join], Bounded [lower, upper], distributed and complemented lattices".
- Let *B* be a finite boolean algebra having n-atoms. Then *B* has 2^n elements and "every non-zero element of *B* is the sum of unique set of atoms".

  Example: *B* is boolean algebra with less than 100 elements, then *B* can have 2^1, 2^2, 2^3, 2^4, 2^5 or 2^6 elements.
- Let *a*, *b*, *c* be any elements in a boolean algebra '*B*' (*B*, +, ∗, ′, 0, 1)

1. **Commutative Laws:**
$$a + b = b + a$$
$$a * b = b * a$$
2. **Associative Laws:**
$$(a + b) + c = a + (b + c)$$
$$(a * b) * c = a * (b * c)$$
3. **Distributive Laws:**
$$a + (b * c) = (a + b) * (a + c)$$
$$a * (b + c) = (a * b) + (a * c)$$
4. **Identity Laws:**
$$a + 0 = a$$
$$a * 1 = a$$
5. **Complement Laws:**
$$a + a' = 1$$
$$a * a' = 0$$
6. **Idempotent Laws:**
$$a + a = a$$
$$a * a = a$$

7. **Absorption Laws:**

$$a + (a * b) = a$$
$$a * (a + b) = a$$

8. **Involution Law or Double Complement Law:**

$$[(a')' = a]$$

$$\begin{matrix} 0' = 1 \\ 1' = 0 \end{matrix} \Rightarrow (0')' = 0$$

9. **DeMorgan's Law**

$$(a + b)' = a' * b'$$
$$(a * b)' = a' + b'$$

10. **Domination Law:**

$$a + 1 = 1$$
$$a * 0 = 0$$

■■■

4

Graph Theory

Introduction

Graph

A graph G is defined by $G = (V, E)$ where V is set of all vertices in G and E is set of all edges in G.

- **Null Graph:** A graph with no edges is called null graph.
- **Directed Graph:** In a digraph an edge (u, v) is said to be from u to v.
- **Undirected Graph:** In a undirected graph an edge $\{u, v\}$ is said to join u and v or to be between u and v.
- **Isolated Vertex:** A vertex with degree zero is called as Isolated vertex or lone vertex.
- **Pendent Vertex:** A vertex with degree one is called as Pendent vertex.
- **Pendent Edge:** It is an edge which incident with pendent vertex.
- **Path:** It is the sequence of edges, without vertex repetition.
- **Network:** It is a graph with only one source and one sink.
- **Trial (Tour or Walk):** It is the sequence of edges without edge repetition (vertex may repeat).
- **Independence Number:** Number of vertices in largest maximal independent set.
- **Diameter of a Graph:** Maximum distance between any two vertices in a graph.
- **Loop:** An edge drawn from a vertex to itself.
- **Trivial Graph:** A graph with one vertex and no edges.
- **Discrete Graph or Null Graph:** A graph with only isolated vertices and no edges.
- **Pseudo Graph:** A graph in which self loops are allowed as well as parallel or multiple edges are allowed.
- **Simple Graph:** A graph with no loops and no parallel edges is called a simple graph.
- **Peterson graph**  number of edges = 15, number of vertex = 10

- Girth = size of shortest cycle
- **Hand Shaking Theorem:** Indegree = Outdegree
- $\underset{\text{min degree}}{\delta} \leq \left\lfloor \frac{2e}{n} \right\rfloor \leq \underset{\text{max degree}}{\Delta}$
- **Complete Bipartite Graph:** (m, n)
  Diameter = 2, Chromatic = 2,
  Number of vertex = $m + n$, Number of edges = $m \times n$

Note:

- Maximum number of edges possible in a simple graph with n-vertices = $n(n-1)/2 = {}^nC_2$.
- Complete graph has $n!$ Hamiltonian cycle.
- Cycle graph has $(n-1)!$ Hamiltonian cycle.
- Number of edges disjoint Hamiltonian cycle = $\frac{n-1}{2}$ i.e. for even no edge disjoint Hamiltonian cycle.
- Number of simple graphs possible with n labelled vertices $= 2^{n(n-1)/2}$.
- **Hand Shaking Theorem:** Let $G = (V, E)$ be a non-directed graph with $V = \{V_1, V_2, \ldots V_n\}$. Then $\sum_{i=1}^{n} \deg(V_i) = 2|E|$.
- In any graph the number of vertices with odd degree is always even.
- If degree of each vertex is k then such a graph is called k-regular graph and in such a graph $|E| = \frac{k|V|}{2} = \frac{nk}{2}$ (where $|V| = n$).
- If degree of each vertex is atleast k i.e. the minimum degree = k, then $|E| \geq \frac{k|V|}{2}$.

- **Regular Graph:** A graph in which all vertices have same degree. If degree of each vertex is 'K' then it is "***k*-regular Graph**".
- **Complete Graph:** A simple graph with "***n*-mutually adjacent vertices**" is complete graph, represented by K_n.

Note:

- Each vertex in K_n has degree $n-1$
- Number of edges in $K_n = |E(K_n)| = n(n-1)/2$
- Every complete graph is regular but converse need not be true.

- **Cycle Graph:** A simple graph with n-vertices ($n \geq 3$) and n-edges is called a cycle graph if all the edges form a cycle of length n. 2-chromatic = even cycle, 3-chromatic = odd cycle.
- **Wheel Graph:** A wheel graph (w_n) with n-vertices ($n \geq 4$) is a simple graph obtained from c_{n-1}, by adding a new vertex which is adjacent to all vertices of c_{n-1}. Diameter = 2, Hamiltonian graph, 3 color = n = even, 4 color = n = odd, $n + 1$ vertex.

  Number of edges = $|E(w_n)| = 2n$
- **Cut Vertex (Articulation Point):** A vertex whose removal makes the graph disconnected.
- **Cut Edge (Bridge):** An edge whose removal makes the graph disconnected.
- **Cut Set:** It is a set of edges, whose removal from graph makes the graph disconnected, provided no proper subset of these edges disconnects the graph.
- ***n* cube:** Number of vertex = 2^n, number of edges = $n \times 2^{n-1}$. 2 colors needed.

Note:

- One or more cut sets may exist in the connected graph.
- Whenever a cut edge exists, cut vertex also exist in graph because atleast one vertex of the cut edge is a cut vertex but the vice versa is not true.
- In a connected graph G an edge of graph G is a cut edge iff the edge is not part of any cycle in G.

Edge Connectivity

Number of edges in a smallest cutset of G is called edge connectivity of G. It is also the minimum number of edges whose removal disconnects the graph.

Biconnected Graph

A graph with no cut vertices and no bridges.

Weakly Connected

A digraph is weakly connected if the underlying undirected graph (obtained by removing all the arrows in directed graph) is connected.

Vertex Connectivity

Minimum number of vertices whose removal results in a disconnected graph or reduces it to a trivial graph.

k-connected Graph

On removal of k-vertices, the connected graph becomes disconnected.

k-line-connected Graph

On removal of k-edges, the connected graph becomes disconnected.

Non-separable Graph

Graph with no cut vertices and hence no cut edges (bridges).

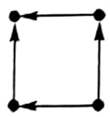
Strongly Connected

In digraph, if a path exist between any vertex to any other vertex i.e. for two given vertices u and v $\exists$ a path from u to v as well as from v to u.

Unilaterally Connected

In digraph, if for every two vertices u and v there is a path from u to v or from v to u (not necessarily both)

Weak Graph



Some vertex has indegree but not out degree so vertex not reach to each other.

Note: ....

- $$\underset{\substack{\downarrow \\ \text{Vertex} \\ \text{connect}}}{K(G)} \le \underset{\substack{\downarrow \\ \text{Edge} \\ \text{connect}}}{\lambda(G)} \le \underset{\substack{\downarrow \\ \text{Min} \\ \text{degree}}}{\delta} \le \left\lfloor \frac{2e}{n} \right\rfloor \le \underset{\substack{\downarrow \\ \text{Max} \\ \text{degree}}}{\Delta}$$
- A simple graph G with n-vertices is necessarily connected if $$|E| > \frac{(n-1)(n-2)}{2}$$
- A simple graph with n-vertices and k component has atleast $(n-k)$ edges i.e., $|E| \ge (n-k)$.

- A simple graph G with n vertices, k components has atmost $[(n-k)(n-k+1)]/2$ edges.

Tree

A connected graph with no cycle is called a tree.

Spanning Tree

It is a tree and subgraph to a graph 'G' which includes all vertices of 'G'.

- A tree with 'n' vertices has $n-1$ edges.
- Number of binary trees with n-vertices $= \frac{{}^{2n}C_n}{n+1} = \frac{2n!}{n!(n+1)!}$.
- k-trees (forest) with total n-vertices have $(n-k)$ edges.
  Number of spanning trees for $k_n = n^{n-2}$.
- "Number of edges that must be removed" from connected graph with n-vertices and e-edges to produce a spanning tree is called 'circuit rank of graph'; Circuit Rank or nullity or cyclomatic complexity $= e-(n-1)$ edges.
- Number of edges that must be removed to produce a spanning forest of graph with 'n' vertices, 'e' edges and 'k' components $= e-(n-k)$ edges.
- Number of bridges possible, for spanning tree with n-edges $= n-1$.
- A finite tree (with atleast one edge) has atleast two vertices of degree '1'.
- The number of different trees with vertex set $\{v_1, v_2, \ldots, v_n\}$ in which the vertices have degree $d_1, d_2, \ldots, d_n$ respectively

$$= \frac{(n-2)!}{(d_1-1)!(d_2-1)!\ldots.(d_n-1)!}.$$

Bipartite Graph

In bipartite graph, Vertex set V of a graph is divided into two vertex sets V_1 and V_2, such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$.

- Number of edges with 'n' vertices cannot exceed $\left\lfloor \frac{n^2}{4} \right\rfloor$.
- It is either acyclic or contains only even length cycles.

Complete Bipartite Graph

A bipartite graph $G = (V, E)$ with vertex partition $V = \{V_1, V_2\}$ is complete bipartite graph, if every vertex in V_1 is adjacent to every vertex in V_2.

- A complete bipartite graph ($K_{m,n}$) has $(m + n)$ vertices and mn edges.
- A complete bipartite graph $K_{m,m}$ is a regular graph of degree m.

Planar Graph

Planar graph is a graph or a multigraph that can be drawn in a plane or sphere such that it's edges do not cross.

- $\boxed{\sum_{i=1}^{n} \deg(V_i) = 2|E|}$
- $\boxed{\sum_{i=1}^{n} \deg(r_i) = 2|E|}$ where r_i are the regions.
- If degree of each region is K then $K \cdot |R| = 2\,|E|$
- If degree of each region is atleast 3 then $3|R| \le 2|E|$
- For simple planar graph :

  (*i*) **Euler's formula:**

  $\boxed{|R| = |E| - |V| + 2}$ if graph is connected

  $\boxed{|R| = |E| - |V| + (k+1)}$ with 'k' components

  (*ii*) **For connected planar simple graph:** $\boxed{|E| \le \{3|V| - 6\}}$

  (*iii*) **For connected planar simple graph with no triangles:** $\boxed{|R| \le \{2|V| - 4\}}$

- If $K_{3,3}$ and K_5 homomorphic fusion (degree = 1 vertex) subgraph then not planner = Kuratowski's.
- **For disconnected graph:** $n - k \le e \le \dfrac{(n-k)(n-k+1)}{2}$, $k \ge n - e$
- **For connected graph:** $n - 1 \le e \le \dfrac{n(n-1)}{2}$
- There exists atleast one vertex $v \in G$ such that degree $(v) \le 5$.
- $K_{m,n}$ is planner $\Leftrightarrow$ ($m \le 2$ or $n \le 2$)
- K_n is planner $\Leftrightarrow$ $n \le 4$

- A non planar graph with minimum number of vertices is K_5.
- A non planar graph with minimum number of edges is $K_{3,3}$.

Polyhedral Graph

A simple connected planar graph in which every interior region is a polygon of same degree and degree of every vertex $\deg(V) \geq 3 \quad \forall V \in G$.

- $\boxed{3|V| \leq 2|E|}$
- $\boxed{3|R| \leq 2|E|}$

Complementary Graph

Complement of a graph G denoted by $\bar{G}$ is also a simple graph with same vertices as of G, and two vertices are adjacent in $\bar{G}$ iff the two vertices are not adjacent in G.

- $\boxed{G \cup \bar{G} = K_n}$
- $\boxed{|E(G)| + |E(\bar{G})| = |E(K_n)| = \frac{n(n-1)}{2}}$

Isomorphic Graphs

Two graphs G and G^* are isomorphic, if there is a function $f: V(G) \to V(G^*)$ such that f is bijection and "for each pair of vertices u and v of G: $\{u, v\} \in E(G)$ iff $\{f(u), f(v)\} \in E(G^*)$".

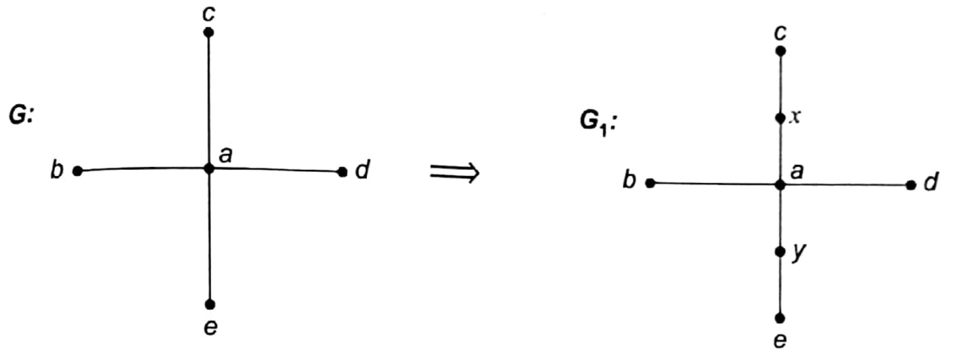
- Two graphs are isomorphic, iff their complements are isomorphic.
- If G and $\bar{G}$ are isomorphic then
  - *(i)* The number of vertices in G and G' are same.
  - *(ii)* The number of edges in G and G' are same.
  - *(iii)* The degree sequence of G and G' are same.
  - *(iv)* The number of cycles of every length in G and G' are same.
- If G is a simple graph such that $G \cong \bar{G}$ then G is said to be "self complementary".
- **In a self-complementary graph:**

$$\boxed{|E(G)| = \frac{n(n-1)}{4}\text{; where } n \text{ is number of vertices in } G}$$

Homomorphism

A graph G_1 is said to be homomorphic to G if G_1 can be obtained by dividing some edge(s) of G.

Example: G_1 is homomorphic to G is shown in the following:



$\{a, c\}$ of G is divided into $\{c, x\}$ and $\{x, a\}$ edges.
$\{e, a\}$ of G is divided into $\{e, y\}$ and $\{y, a\}$ edges.

Coloring or Proper Coloring

Vertices of a graph G are colored such that no two adjacent vertices have same color.

Chromatic Number $\chi(G)$

Minimum number of colors needed for vertex coloring of graph G is called *chromatic number.*

- Chromatic number of $K_n = \chi(K_n) = n$
- Bipartite graph is 2-colorable. i.e. a non-empty graph is bichromatic iff it is Bipartite.
- Any planar graph is 4-colorable. i.e. χ(planar graph) ≤ 4.
- Tree is 2-colorable
- Equivalence relation between vertices of the same color in a connected graph gives the chromatic partition.
- k-cycle graph is 2-colorable; k is even length cycle
- k-cycle graph is 3-colorable; k is odd length cycle

Matching

- $\beta \geq$ independent number $\geq \dfrac{n}{\text{Chromatic}}$.
- It is a set of edges with atmost one edge incident on every vertex.

$$\boxed{\text{degree}(v) \leq 1, \quad \forall v \in G}$$

Maximal Matching

Maximal matching is a maximal matching in which no edge of the graph can be added to it.

Maximum Matching

Maximum matching is a matching with maximum number of edges.

Matching Number

The number of edges in maximum matching of the graph.

Perfect Matching

Every vertex of the matching contain exactly one degree. i.e., every vertex is incident with exactly one edge. i.e. A matching which is also a covering is called perfect matching.

$$\boxed{\begin{array}{l}\text{degree}(v) = 1,\ \forall v \in G \\ \text{Number of perfect matchings for } K_{2n} = \dfrac{(2n)!}{2^n \times n!}\end{array}}$$

$\boxed{K_n \text{ has a perfect matching iff } n \text{ is even.}}$

Complete Matching

In a bipartite graph having a vertex partition V_1 and V_2. A complete matching of vertices in a set V_1 into those of V_2 is a matching such that every vertex in V_1 is matched against some certain vertex in V_2, such that no two vertices of V_1 is matched against a single vertex in V_2.

Covering

It is set of edges where every vertex of graph incident with atleast one edge in 'G' [deg $(v_i) \geq 1$]; $\forall v_i \in G$.

Note:

- A line covering of a graph with n-vertices has atleast $n/2$ edges.
- No minimal line covering can contain a cycle and the components of a minimal cover are always stargraphs and from a minimal cover no edge can be deleted.

Minimal Covering

It is covering in which no deletion of an edge is possible while still covering the vertices.

Minimum Covering

Smallest (less number of edges) minimal covering is minimum covering.

Covering Number

The number of edges in minimum covering is covering number.

Traversable Multigraph

If there is a path in graph which includes all the vertices and uses each edge exactly once (i.e. the graph has either Euler cycle or Euler trail) then such graph is traversable.

Eulerian Graph

If a graph contains "closed traversable trial or Euler circuit" (it may repeat vertices), then it is Eulerian Graph. When all vertex of even degree.

Note:

- A graph G is traversable, if number of vertices with odd degree in the graph is exactly zero or two.
- Euler path exists but Euler Circuit doesn't exist iff the number of vertices with odd degree is exactly two.
- Euler Circuit exists but Euler path does not exist iff number of vertices with odd degree is 0.

Hamiltonian Path

If there exists a path which contains each vertex of the graph exactly once, then such a path is called as Hamiltonian path.

Hamiltonian Cycle

It is Hamiltonian path where first and last vertices are same. If pendant edge then not Hamiltonian.

DIRAC

Sufficient condition degree of all vertex $\geq \frac{n}{2}$ then Hamiltonian.

ORES

$d(u) + d(v) \geq n$: u and v are non adjacent vertex.

■■■

Probability

5

Mean, Median and Mode

- $$\text{Mean } (\bar{X}) = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum f_i}$$

- $$\text{Median} = \frac{x_{n/2} + (x_{n/2} + 1)}{2}; \ n \text{ is even}$$
$$= x_{n+1/2}\ ; \ n \text{ is odd}$$
$$= L_{\text{med}} + \left(\frac{N/2 - F}{f}\right) \times w$$

where L_{med} = lower limit of median class

N = Σf_i

F = Cumulative frequency upto the median class (cumulative frequency of the preceding class)

f = Frequency of the median class

Mode : Value of 'x' corresponding to maximum frequency.

$$L_{\text{mode}} + \frac{f_m - f_1}{(2f_m - f_1 - f_2)} \times h$$

where L_{mode} = lower limit of modal class

f_m = frequency of modal class

f_1 = preceding frequency of modal class

f_2 = Following frequency of modal class

Note:

- Mode = 3 Median – 2 Mean [for Asymmetric distribution]
- Mean = Mode = Median [for Symmetric distribution]

Axioms of Probability

Let A and B be two events. Then

1. $P(\bar{A}) = 1 - P(A)$
2. $P(\phi) = 0;$
3. $P(A - B) = P(A) - P(A \cap B)$
4. $P(A - B) = P(A \cap \bar{B})$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
6. $P(A \cup B) = P(A) + P(B)$; mutually exclusive events.
7. $P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$
8. $P(A \cap B) = P(A) \cdot P(B)$; independent events.
9. $P(A \cap B) = \phi$; mutually exclusive events.
10. $P(S) = 1$; S is sample space.
11. $\max(0, P(A) + P(B) - 1) \leq P(A \cap B) \leq \min(P(A), P(B))$
12. $\max(P(A), P(B)) \leq P(A \cup B) \leq \min(1, P(A) + P(B))$
13. $\max(0, P(A_1) + P(A_2) \ldots + P(A_n) - (n-1)) \leq P(A_1 \cap A_2 \ldots \cap A_n)$
    $\leq \min(P(A_1), P(A_2) \ldots, P(A_n))$
14. $\max(P(A_1), P(A_2) \ldots, P(A_n)) \leq P(A_1 \cup A_2 \ldots \cup A_n)$
    $\leq \min(1, P(A_1) + P(A_2) \ldots + P(A_n))$
15. $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$
16. $P(A|B) = P(A)$; independent events.
17. $P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1) \cdot P(E_2) \ldots P(E_n)$; independent events.
18. Rule of total probability: $P(X) = \sum_{i=1}^{n} P(E_i)\, P(X|E_i)$
19. Baye's Theorem:

$$P(E_1|X) = \frac{P(E_1) \cdot P(X|E_1)}{P(E_1) \cdot P(X|E_1) + P(E_2) \cdot P(X|E_2) + P(E_3) \cdot P(X|E_3)}$$

In general,

$$P(E_i|X) = \frac{P(E_i) \cdot P(X|E_i)}{\sum_{j=1}^{n} P(E_j) \cdot P(X|E_j)}$$

Random Variable (Stochastic Variable)

Random variable assigns a real number to each possible outcome.

Let X be a discrete random variable, then

1. $F(x) = P(X \leq x)$ is called distribution function $\sum_{i=0}^{n} P(i)$ of X.

2. Mean or Expectation of $X = \mu = E(X) = \sum_{i=1}^{n} x_i \; P(x_i)$

3. Variance of $X = \sigma^2 = E(X^2) - [E(X)]^2 = \sum_{i=1}^{n} (x_i - \mu)^2 P(x_i)$

4. Standard deviation of $X = \sigma = \sqrt{\text{Variance}}$

5. $\sum_{i=1}^{n} P(x_i) = 1$

Types of Random Variables

1. **Discrete Random Variable:** "Finite set of values" or "Countably infinite".
2. **Continous Random Variable (Non-discrete):** "Infinite number of uncountable values".

Discrete Distributions

1. **Binomial Distribution:** The probability that the event will happen exactly r times in n trials i.e. r successes and $n-r$ failures will occur.

   $P(X = r) = P(r) = \Sigma\, {}^nC_r p^r q^{n-r}$

   Mean $= E(x) = np$

   Variance $(\sigma^2) = V(x) = npq = np(1-p)$

   S.D $(\sigma) = \sqrt{npq} = \sqrt{np(1-p)}$

   Where $r = 0, 1, \ldots, n$

   $q = 1 - p$

   $n =$ fixed number of trials

   $p =$ probability of success

2. **Poisson Distribution:**

$$\boxed{P(X = x) = \sum \frac{e^{-\lambda} \cdot \lambda^x}{x!}; \; x = 0, 1, 2, \ldots, \infty}$$

Where X = Discrete random variable

λ = Parameter of distribution (positive constant)

- Mean (μ) = Variance (σ^2) = λ
- $S.D = \sqrt{\lambda}$

Poisson distribution is a limiting case of binomial distribution as $n \to \infty$ and $p \to 0$.

X	X Counts	p(x)	Value of X	E(x)	V(x)
Discrete uniform	Outcomes that are equally likes (finite)	$\frac{1}{b-a+1}$	$a \le x \le b$	$\frac{b+a}{2}$	$\frac{(b-a+2)(b-a)}{12}$
Binomial	Number of successes in n fixed trials	$\binom{n}{x}p^x(1-p)^{n-x}$	$x = 0, 1, \ldots, n$	np	np(1 − p)
Poisson	Number of arrivals in a fixed time period	$\frac{e^{-\lambda}\lambda^x}{x!}$	$x = 0, 1, 2, \ldots$	λ	λ
Geometric	Number of trials up through 1st success	$(1-p)^{x-1}p$	$x = 1, 2, 3, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative Binomial	Number of trials up through kth success	$\binom{x-1}{k-1}(1-p)^{x-k}p^k$	$x = k, k+1, \ldots$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2}$
Hyper-geometric	Number of marked individuals in sample taken without replacement	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$	$\max(0, M+n-N) \le x \le \min(M, n)$	$n \cdot \frac{M}{N}$	$\frac{nM(N-M)(N-n)}{N^2(N-1)}$

Continuous Distribution

Let X be continous random variable. Then

(*i*) **Density functions:**

$$\text{(a) } P(X \le a) = \int_{-\infty}^{a} f(x)\cdot dx, \text{ (b) } P(a \le X \le b) = \int_{a}^{b} f(x)\cdot dx$$

(*ii*) **Mean** = $E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$

(*iii*) **Variance of** X = $V(X) = \int_{-\infty}^{\infty} [x - E(x)]^2 \cdot f(x)dx = E(x^2) - (E(x))^2$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - \left[\int_{-\infty}^{\infty} xf(x)dx\right]^2$$

(*iv*) $\int_{-\infty}^{\infty} f(x)dx = 1$

1. Uniform Distribution (Rectangular Distribution)

(i) Density function:

$$f(x) = \frac{1}{b-a}; \quad a \le x \le b$$

$$= 0 \text{ ; otherwise}$$

(ii) Cumulative function:

$$P(X \le x) = \int_{-\infty}^{x} f(x) \cdot dx = \begin{cases} 0 & ; \quad \text{if } x \le a \\ \dfrac{x-a}{b-a} & ; \quad \text{if } a \le x \le b \\ 1 & ; \quad \text{if } x > b \end{cases}$$

(iii) **Mean (μ)** = $(a + b)/2 = E(X)$

(iv) **Variance (σ^2)** = $(b - a)^2/12$

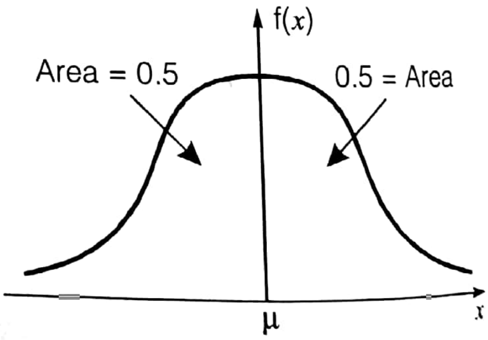
2. Normal Distribution

(i) Density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; -\infty \le x \le \infty, \sigma > 0, -\infty < \mu < \infty$

(ii) Normal distribution is symmetrical

(iii) Mean = μ; Variance = σ^2

(iv) $f(x) \ge 0$ for all x



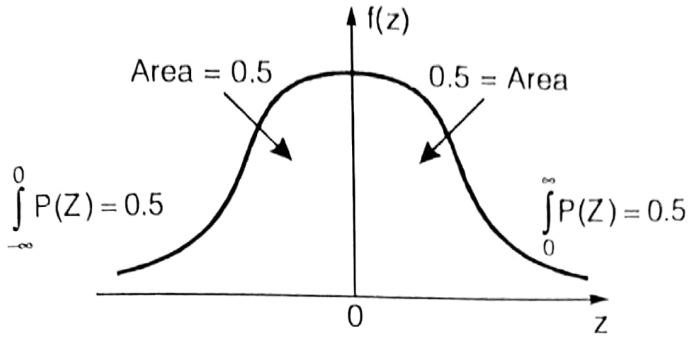
(v) $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

(vi) $P(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}}; -\infty \le Z \le \infty$

and $Z = \frac{x-\mu}{\sigma}$

$Z = \frac{x-np}{\sqrt{npq}}$ (when approximating binomial by normal)

Z = Standard normal variate



3. Exponential Distribution

(*i*) Density function:

$$f(x) = \lambda \cdot e^{-\lambda x} \quad ; x > 0$$
$$= 0 \quad ; \text{ Otherwise}$$

(*ii*) Mean (μ) = $\frac{1}{\lambda} = S.D(\sigma)$

(*iii*) Variance (σ^2) = $\frac{1}{\lambda^2}$

X	X Measures	f(x)	Value of X	E(x)	V(x)
Continuous uniform	Outcomes with equal density (continuous)	$\frac{1}{b-a}$	$a \le x \le b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
Exponential	Time between events time until an event	$\lambda e^{-\lambda x}$	$x \ge 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal	Values with a bell-shaped distribution(continuous)	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$-\infty < x < \infty$	μ	σ
Standard normal (Z)	Standard scores	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$	$Z = \frac{x-\mu}{\sigma}$	0	1
Binomial approximation	Number of successes in a fixed time period (large average)	Approx, normal if $np \ge 5$ and $n(1-p) \ge 5$ by CLT	$Z = \frac{x-np}{\sqrt{np(1-p)}}$	np	np(1 – p)
Poisson approximation	Number of occurrences in a fixed time period (large average)	Approx normal if $\lambda > 30$	$Z = \frac{x-\lambda}{\sqrt{\lambda}}$	λ	λ

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Linear Algebra

6

Matrix

Principal Diagonal: In a square matrix all elements a_{ij} for which $i = j$ are elements of principal diagonal.

Matrices

1. **Upper Triangular matrix:** A square matrix in which all the elements below the principle diagonal are zero.
2. **Lower Triangular Matrix:** A square matrix in which all the elements above the principle diagonal are zero.
3. **Diagonal Matrix:** A square matrix in which all the elements other than the elements of principle diagonal are zero.
4. **Scalar Matrix:** *A* diagonal matrix with all elements of principle diagonal being same.
5. **Idempotent Matrix:** 'A' is square matrix $\ni A^2 = A$.
6. **Involutary Matrix:** 'A' is square matrix $\ni A^2 = I$.
7. **Nilpotent Matrix:** 'A' is square matrix $\ni A^m = 0$ where m is the least positive integer and m is also called as Index of class of Nilpotent matrix A.
8. **Transpose Matrix:** A^T is transpose matrix of matrix A. A^T can be obtained by switching the rows as columns and columns as rows of A.
9. **Symmetric Matrix:** 'A' is a square matrix $\ni A^T = A$.
10. **Skew-Symmetric Matrix:** 'A' is a square matrix $\ni A^T = -A$.
11. **Orthogonal Matrix:** 'A' is a orthogonal matrix $\ni A^T = A^{-1}$ or $AA^T = I = A^TA$.
12. **Conjugate Matrix of A ($\bar{A}$) or ($\sim A$) :** 'A' is any matrix, by replacing the elements by corresponding conjugate complex numbers the matrix obtained is conjugate of 'A'.

    Example:

$$A = \begin{bmatrix} 2+3i & 4+7i & 5 \\ 2i & 3 & 9-i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 2-3i & 4-7i & 5 \\ -2i & 3 & 9+i \end{bmatrix}$$

13. **Transpose Conjugate Matrix (A^θ) or (A^*):** $(\bar{A})^T$.

14. **Hermitian Matrix:** 'A' is a square matrix $\ni$ $A^\theta = A$
All diagonal elements of hermitian matrix are real number and all off-diagonal elements above and below the principle diagonal must be conjugate of each other. i.e. $a_{ij} = \overline{a_{ji}}$.

Example: $\begin{bmatrix} 2 & 3-4i \\ 3+4i & 5 \end{bmatrix}$

15. **Skew-Hermitian Matrix:** 'A' is a square matrix $\ni$ $A^\theta = -A$
All diagonal elements of Skew-Hermitian matrix are purely imaginary or zero and all off-diagonal elements above and below the principle diagonal must be conjugate of each other with opposite sign. i.e. $a_{ij} = -\overline{a_{ji}}$.

Example: $\begin{bmatrix} 2i & 3-4i \\ -3-4i & 5i \end{bmatrix}$

16. **Unitary Matrix:** 'A' is a square matrix $\ni$ $A^\theta = A^{-1}$ or $A\,A^\theta = I = A^\theta A$
17. **Boolean Matrix:** Any matrix with only elements '0' or '1'
18. **Sparse Matrix:** A matrix 'A' in which more number of elements are zeros.
19. **Dense Matrix:** A matrix which is not sparse.
20. **Singular and Non-singular Matrix:** A square matrix 'A' is singular if $|A| = 0$, and non singular if $|A| \neq 0$. Only non-singular matrices have inverse.
21. **Adjoint Matrix:** Transpose of cofactors matrix. i.e. Adj (A) = (Cof $(A))^t$

Properties of Matrices

- $A + B = B + A$ (Commutative)
- $(A + B) + C = A + (B + C)$ (Associative)
- $AB \neq BA$ (Not commutative)
- $(AB)\,C = A(BC)$ (Associative)
- $A(B + C) = AB + AC$ (Distributive)
- $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$
- $A(Adj\,A) = (Adj\,A)\,A = |A| I_n$
- $Adj\,(AB) = (Adj\,B).\,(Adj\,A)$

- $A^{-1} = \dfrac{Adj\ A}{|A|}; |A| \neq 0$
- $(A^{-1})^{-1} = A$ and $(A^{-1})^T = (A^T)^{-1}$
- $(AB)^{-1} = B^{-1} A^{-1}$.
- If A is a square matrix of order n then $|Adj\ A| = (\det A)^{n-1} = |A|^{n-1}$ and

  $|Adj\,(Adj\ A)| = |A|^{(n-1)^2}$
- If $|A| \neq 0$ then $|A^{-1}| = \dfrac{1}{|A|}$
- If $A_{n\times n}$ matrix then $|KA| = K^n|A|$
- If A is square matrix then
  - (i) $A + A^T$ is always symmetric
  - (ii) $A - A^T$ always skew-symmetric
- If A and B are symmetric then
  - (i) $A + B$ is also symmetric.
  - (ii) $A - B$ is also symmetric.
  - (iii) $AB + BA$ is symmetric
  - (iv) $AB - BA$ is skew-symmetric
  - (v) A^n and B^n are symmetric
- If A and B are skew-symmetric then,
  - (i) $A + B$ is also skew-symmetric.
  - (ii) $A - B$ is also skew-symmetric.
  - (i) A^n and B^n are symmetric, if 'n' is even
  - (ii) A^n and B^n are skew-symmetric, if 'n' is odd
- The determinant of orthogonal matrix and unitary matrix A has absolute value '1'.
- If $A_{m\times n}$ and $B_{n\times p}$ then product of AB requires
  - (i) mnp multiplications
  - (ii) $m(n-1)p$ additions
  - (iii) for each entry, n multiplications and $(n-1)$ additions.
- $(A^T)^T = A$, $(kA)^T = k(A^T)$
- $(A + B)^T = A^T + B^T$, $(AB)^T = B^TA^T$

- **Rank of Matrix ($r(A)$):** It is the order of its largest non-vanishing (non-zero) minor of the matrix.
- Rank is equal to the number of linearly independent rows or columns in the matrix.
- The system of linear equation $AX = B$ has a solution (consistent) iff rank of A = Rank of $(A|B)$
- The system $AX = B$ has
  (*i*) A unique solution iff Rank (A) = Rank $(A|B)$ = Number of variables
  (*ii*) Infinitely many solutions $\Leftrightarrow$ Rank (A) = Rank $(A|B)$ < number of variables
  (*iii*) No solution if Rank $(A) \neq$ Rank $(A|B)$ i.e. Rank (A) < Rank $(A|B)$
- The system $AX = 0$ has
  (*i*) Unique solution (zero solution or trivial solution) if Rank (A) = number of variables
  (*ii*) Infinitely many number of solutions (non-trivial solutions) if Rank (A) < number of variables
- If Rank $(A) = r$, and number of variables $= n$ then, the number of linearly independent infinite solutions of $AX = 0$ is $(n - r)$
- In the system of homogenous linear equation $AX = 0$
  (*i*) If A is singular then the system possesses non-trivial solution (i.e. infinite solution)
  (*ii*) If A is non-singular then the system possesses trivial (zero) solution (i.e. unique solution)
- Rank of a diagonal matrix = Number of non-zero elements in diagonal.
- If A and B are two matrices
  (*i*) $r(A + B) \leq r(A) + r(B)$
  (*ii*) $r(A - B) \geq r(A) - r(B)$
  (*iii*) $r(AB) \leq \min\{r(A), r(B)\}$
- If a matrix A has rank 'R', then A contains 'R' linearly independent vectors (row/column)
- The system of homogeneous linear equations such that number of unknowns (or variables) exceeds the number of equations necessarily possesses a non-zero solution.

Eigen Value

Let 'A' be a square matrix of order n and λ be a scalar then $|A - \lambda I| = 0$ is the characteristic equation of A. The roots of characteristic equation are called eigen values/lantent roots/Characteristic roots.

- The set of eigen values of matrix is called "spectrum of matrix"
- A matrix of order n will have n latent roots. not necessarily distinct.

Eigen Vector

Corresponding to each eigen value λ, there exists a non-zero solution X such that $(A - \lambda I)X = 0$ then X is eigen vector/latent/vector/characteristic vector of A.

Properties of Eigen Values

- Sum of eigen values of a matrix = sum of elements of principal diagonal (trace).
$$\Sigma\lambda_i = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = \text{Trace of } A$$
- Product of eigen values = Determinant of matrix.
$$\Pi\lambda_i = \lambda_1 \cdot \lambda_2 \cdot \ldots \lambda_n = |A|$$
- If λ is eigen value of A then $\frac{1}{\lambda}$ is eigen value of A^{-1}. (provided $\lambda \neq 0$ i.e. A is non-singular).
- Eigen values of A and A^T are same
- If λ is eigen value of orthogonal matrix then $\frac{1}{\lambda}$ is also its eigen value $[\because\ A^T = A^{-1}]$
- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of matrix 'A', then
  - (*i*) $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ are eigen values of matrix A^m.
  - (*ii*) $\lambda_1 + K, \lambda_2 + K, \ldots, \lambda_n + K$ are eigen values of $A + KI$
  - (*iii*) $(\lambda_1 - K)^2, (\lambda_2 - K)^2, \ldots, (\lambda_n - K)^2$ are eigen values of $(A - KI)^2$
  - (*iv*) $K\lambda_1, K\lambda_2, \ldots, K\lambda_n$ are eigen values of KA.
- The eigen values of symmetric matrix are real.
- The eigen values of skew-symmetric matrix are either purely imaginary or zero.
- The modulus of the eigen values of orthogonal and unitary matrices = 1.
- If a matrix is either lower or upper triangular or diagonal then the principal diagonal elements themselves are the eigen values.
- Zero is eigen value of a matrix iff the matrix is singular.

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Numerical Methods

7

Linear Algebraic Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1}$$

LU Decomposition or LU Factorization or Triangularization (Doolittle's or Crout's triangularisation Method)

- Principle minors of A are non-singular.

$$AX = B \Rightarrow LUX = B \quad [\text{since } A = LU]$$

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ (Doolittle's method)

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$ (Crout's method)

Order of Computation in Doolittle's Method:

In Doolittle's method the equations are set-up row-wise for A's elements.

(a) First row of $A \Rightarrow u_{11} = a_{11}$; $u_{12} = a_{12}$; $u_{13} = a_{13}$

(b) Second row of $A \Rightarrow l_{21} = \dfrac{a_{21}}{a_{11}}$; $u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}} \cdot a_{12}$; $u_{23} = a_{23} - l_{21}\, u_{13}$

(c) Third row of $A \Rightarrow$

$$l_{31} = \frac{a_{31}}{a_{11}};\ l_{32} = \frac{1}{u_{22}}\left[a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{12}\right];\ u_{33} = a_{33} - l_{31} \cdot u_{13} - l_{33}\, u_{23}$$

Similarly, in Crout's method the equations are set-up column-wise for A's elements.

(v) Error is order of h^2 and error term is h^3

(vi) If for the same integration limits, tabulation (step size) is halved then error is reduced by factor '4'.

2. **Simpson's Rule:**

(i) It gives exact result when polynomial of degree ≤ 3.

(ii) If we are evaluating $I = \int_a^b f(x)\,dx$ by Simpson's rule then

$$\boxed{\text{Error} = -\frac{h^5}{90} \cdot f^{iv}(\xi) \times n_i} \qquad (a \le \xi \le b)$$

(iii) Simpson's rule is more accurate as compare to trapezoidal rule since it is a fourth order method as compared to trapezoidal rule which is second order.

(iv) Simpson's 1/3rd Rule:

$$\boxed{\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \ldots + y_{n-2}) + 4(y_1 + y_3 + \ldots + y_{n-1})]}$$

(v) Simpson's 3/8 Rule:

$$\boxed{\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}[(y_0 + y_n) + 2(y_3 + y_6 + \ldots + y_{n-3}) + 3(y_1 + y_2 + \ldots + y_{n-1})]}$$

(vi) If tabulation (step size) is halved, the error in Simpson's rule evaluation of the integral is reduced by a factor of '16'.

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Calculus

8

Limit, Continuity and Differentiability

Limit Existence at a Point

- Left Limit (LHL): $\lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a-h)$

  Right Limit (RHL): $\lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a+h)$

  $\lim\limits_{x \to a} f(x)$ exists, iff both LHL and RHL exists.
- Indeterminate forms: $0/0$, ∞/∞, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, 0^∞
- **L Hospital's Rule:** If $\lim\limits_{x \to a} [f(x)/g(x)]$ is of the form either $0/0$ or ∞/∞ then

  $\lim\limits_{x \to a} [f(x)/g(x)] = \lim\limits_{x \to a} [f'(x)/g'(x)]$

Continuity

A function $f(x)$ is said to be continuous at $x = a$ if:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

- If $f(x)$ and $g(x)$ is continuous function then following are also continuous. (*i*) $f(x) + g(x)$, (*ii*) $f(x) \cdot g(x)$, (*iii*) $f(x) - g(x)$, (*iv*) $f(x)/g(x)$ $[g(x) \neq 0]$
- Polynomial, Exponential, Sin and Cos functions are continuous everywhere.

Differentiability

A function $f(x)$ is said to be differentiable at $x = a$ if it is continuous at $x = a$ and if left derivative $= f'(a) =$ right derivative

$$\text{i.e.} \quad \lim_{h \to 0^-} \frac{f(a) - f(a-h)}{h} = f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

- If $f(x)$ is differentiable in (a, b), then it is always continuous at (a, b), but converse need not be true.

Mean Value Theorems

Rolle's Mean Value Theorem

If (*i*) '$f(x)$' is continuous in $[a, b]$, (*ii*) '$f(x)$' is differentiable in (a, b), (*iii*) $f(a) = f(b)$ then $\exists\, c \in (a, b)$ such that $f'(c) = 0$

Lagrange's Mean Value Theorem

If (*i*) '$f(x)$' is continuous in $[a, b]$, (*ii*) '$f(x)$' is differentiable in (a, b)

then $\exists\, c \in (a, b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Cauchy's Mean Value Theorem

Let $f(x)$ and $g(x)$ are two functions if

(*i*) $f(x)$ and $g(x)$ are continuous in $[a, b]$

(*ii*) $f(x)$ and $g(x)$ are differentiable in (a, b)

(*iii*) $g'(x) \neq 0 \quad \forall x \in (a, b)$

then $\exists\, c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Derivatives

- $\dfrac{d}{dx}(\sin x) = \cos x$
- $\dfrac{d}{dx}(\cos x) = -\sin x$
- $\dfrac{d}{dx}(\tan x) = \sec^2 x$
- $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\dfrac{d}{dx}(\sec x) = \sec x \cdot \tan x$
- $\dfrac{d}{dx}(\operatorname{cosec} x) = \operatorname{cosec} x \cot x$
- $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1+x^2}$
- $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$
- $\dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = \dfrac{-1}{x\sqrt{x^2-1}}$
- $\dfrac{d}{dx}(\sin hx) = \cosh x$
- $\dfrac{d}{dx}(\cosh x) = \sinh x$
- $\dfrac{d}{dx}(e^x) = e^x$
- $\dfrac{d}{dx}(a^x) = a^x \cdot \log a$
- $\dfrac{d}{dx}[f(x)]^n = n \cdot [f(x)]^{n-1}$
- $\dfrac{d}{dx}(ax+b)^n = n \cdot (ax+b)^{n-1} . a$
- $\dfrac{du}{dx} = \dfrac{du}{dy}\dfrac{dy}{dx}$ (chain rule)
- $\dfrac{d}{dx}(uv) = u \cdot \dfrac{dv}{dx} + v \cdot \dfrac{du}{dx}$

- $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v.\dfrac{du}{dx} - u\cdot\dfrac{dv}{dx}}{v^2}$
- $\sinh x = \dfrac{e^x - e^{-x}}{2}$
- $\cosh x = \dfrac{e^x + e^{-x}}{2}$
- $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

Integrals

- $\int x^n \cdot dx = \dfrac{x^{n+1}}{n+1}; \ (n \neq -1)$
- $\int \dfrac{1}{x}\cdot dx = \log_e x$
- $\int e^x \cdot dx = e^x$
- $\int a^x = \dfrac{a^x}{\log a}$
- $\int \dfrac{f'(x)}{f(x)}\cdot dx = \log\,[f(x)]$
- $\int [f(x)]^n \cdot f'(x)\cdot dx = \dfrac{[f(x)]^{n+1}}{n+1}$
- $\int \dfrac{1}{a^2+x^2}dx = \dfrac{1}{a}\cdot\tan^{-1}\dfrac{x}{a}$
- $\int \dfrac{1}{a^2-x^2}dx = \dfrac{1}{2a}\cdot\log\left[\dfrac{a+x}{a-x}\right]$
- $\int \dfrac{1}{x^2-a^2}dx = \dfrac{1}{2a}\cdot\log\left[\dfrac{x-a}{x+a}\right]$
- $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\dfrac{x}{a}$
- $\int \dfrac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\dfrac{x}{a}$
- $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\dfrac{x}{a}$
- $\int \sinh x = \cosh x$
- $\int \cosh x = \sinh x$
- $\int \sin x\cdot dx = -\cos x$
- $\int \cos x\cdot dx = \sin x$
- $\int \tan x\cdot dx = \log \cos x = \log \sec x$
- $\int \cot x = \log \sin x$
- $\int \sec x\cdot dx = \log\,(\sec x + \tan x)$
- $\int \text{cosec}\, x\cdot dx = \log\,(\text{cosec}\, x - \cot x)$
- $\int \sec^2 x\cdot dx = \tan x$
- $\int \text{cosec}^2 x\cdot dx = -\cot x$
- $\int e^{ax}\sin bx\, dx = \dfrac{e^{ax}}{a^2+b^2}\ (a\sin bx - b\cos bx)$
- $\int e^{ax}\cos bx\, dx = \dfrac{e^{ax}}{a^2+b^2}\ (a\cos bx + b\sin bx)$
- $\int \sqrt{a^2-x^2}\cdot dx = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$

- $\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$
- $\int \sqrt{x^2 - a^2} \cdot dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

Definite Integrals

Theorem 1:

If $f(x)$ is continous in $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$ in $[a, b]$ then $\int_a^b f(x)dx = F(b) - F(a)$.

Theorem 2:

If $f(x)$ is continous on $[a, b]$ then $F(x) = \int_a^b f(t) \cdot dt$ is differentiable at every point of x in $[a, b]$ and $\frac{dF}{dx} = \frac{d}{dx}\int_a^b f(t) \cdot dt = f(x)$.

i.e., If $f(x)$ is continous on $[a, b]$ then $\exists$ a function $F(x)$ whose derivative on $[a, b]$ is $f(x)$.

Theorem 3:

If $f(x)$ is continous on $[a, b]$ and $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$ then $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t) \cdot dt = f(v(x))\frac{dv}{dx} - f(u(x)) \cdot \frac{du}{dx}$

Properties of Definite Integrals

1. $\int_a^a f(x) \cdot dx = 0$
2. $\int_a^b f(x) \cdot dx = \int_b^a f(z) \cdot dz$
3. $\int_a^b f(x) \cdot dx = -\int_b^a f(x) \cdot dx$
4. $\int_a^b k \cdot dx = k(b-a)$
5. $\int_a^b f(x) \cdot dx = \int_a^c f(x) \cdot dx + \int_c^b f(x) \cdot dx$
6. $\int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$

7. $\int_{-a}^{a} f(x)\cdot dx = 2\int_{0}^{a} f(x)\cdot dx$; if $f(x)$ is even function $f(-x) = f(x)$

$= 0$; if $f(x)$ is odd function $f(-x) = -f(x)$

8. $\int_{a}^{b} f(x)\cdot dx = \int_{a}^{b} f(a+b-x)\cdot dx$

9. $\int_{0}^{2a} f(x)\cdot dx = \int_{0}^{a} f(x)\cdot dx + \int_{0}^{a} f(2a-x)\cdot dx$

10. $\int_{0}^{2a} f(x)\cdot dx = 2\int_{0}^{a} f(x)\cdot dx$; if $f(2a-x) = f(x)$

$= 0$; If $f(2a-x) = -f(x)$

11. $\int_{0}^{na} f(x)\cdot dx = n\cdot\int_{0}^{a} f(x)\cdot dx$; if $f(x+a) = f(x)$

[$f(x)$ is periodic function with period 'a']

12. $\int_{0}^{a} x\cdot f(x)\cdot dx = \frac{a}{2}\int_{0}^{a} f(x)\cdot dx$; if $f(a-x) = f(x)$

13. **Mean value theorem for definite integrals:** Let f be continuous in (a, b) then $\exists c \in (a, b)$ such that $f(c) = \dfrac{\int_{a}^{b} f(x)dx}{(b-a)}$

$f(c)$ is the average value of the function in the interval (a, b).

14. $\int_{a}^{b} f(x)dx \le \int_{a}^{b} g(x)dx$ If $f(x) \le g(x)$ on (a, b)

15. $\int_{0}^{\pi/2} \sin^n x\cdot dx = \int_{0}^{\pi/2} \cos^n x\cdot dx = \dfrac{(n-1)\cdot(n-3)\cdot(n-5)\ldots}{n\cdot(n-2)(n-4)\ldots}\times K$

Where, $K = 1$; if n is odd and $K = \pi/2$; if n is even

Improper Integrals

1. $\int_a^b f(x)\cdot dx$ is "improper integral";

   (i) If $a = -\infty$ or $b = \infty$ or both

   (ii) If $f(x) = \infty$ for one or more values of x in $[a, b]$.

2. $\int_a^b f(x)\cdot dx$ is "convergent", if the value of integral is "finite".

3. $\int_a^b f(x)\cdot dx$ is "divergent", if the value of integral is "infinite".

4. If $0 \le f(x) \le g(x); \forall x$ and $\int_a^\infty g(x)\cdot dx$ converges then $\int_a^\infty f(x)$ also converges.

5. If $0 \le g(x) \le f(x); \forall x$ and $\int_a^\infty g(x)\cdot dx$ diverges then $\int_a^\infty f(x)$ also diverges.

6. If $f(x)$ and $g(x)$ are two functions, such that $\lim\limits_{n\to\infty} \frac{f(x)}{g(x)} = k$

   where $k =$ finite and $k \neq 0$ then

   $\int_a^\infty f(x)\cdot dx$ and $\int_a^\infty g(x)\cdot dx$ "converge" or "diverge" together.

7. $\int_a^\infty e^{-Px}\cdot dx$ and $\int_{-\infty}^b e^{Px}\cdot dx$ converges for any constant $P > 0$ and diverges for $P \le 0$.

8. $\int_a^b \frac{dx}{(b-x)^P}$ is convergent iff $P < 1$.

9. $\int_a^b \frac{dx}{(x-a)^P}$ is convergent iff $P < 1$.

10. $\int_1^\infty \frac{dx}{x^P}$ converges if $P > 1$ and diverges if $P \le 1$.

Total Derivatives

If $u = f(x, y)$ where x and y are functions of t, then the "total derivative" of u with respect to t is:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dt}\right)$$

By putting $t = x$ in above equation, we get

$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$; where x and y are connected by some relation.

Partial Derivatives

1. If $u = f(x, y)$, then $\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{[f(x+h, y) - f(x, y)]}{h}$ and

   $\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{[f(x, y+k) - f(x, y)]}{k}$.

2. If $u = f(x, y)$, where $x = g(r, s)$ and $y = h(r, s)$ then

   $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

   $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

3. If $u = f(x, y)$ is a homogenous function of degree 'n' then

   $x \cdot u_x + y \cdot u_y = n \cdot u$

   $x^2 u_{xx} + 2xy \cdot u_{xy} + y^2 u_{yy} = n(n-1)u$

4. If $u = f(x, y, z)$ is a homogenous function of degree 'n', then

   $x \cdot u_x + y \cdot u_y + z \cdot u_z = n \cdot u$

5. If $u = f(x, y)$ is not homogenous function, but $F(u)$ is a homogenous function of degree 'n', then

   $x \cdot u_x + y \cdot u_y = n\left[\frac{F(u)}{F'(u)}\right] = G(u)$

   $x^2 \cdot u_{xx} + 2xy \cdot u_{xy} + y^2 \cdot u_{yy} = G(u)\,[G'(u) - 1]$

Maximum and Minima [Extremum]

1. A necessary condition for $f(a)$ to be an extreme value of $f(x)$ is $f'(a) = 0$.
2. The vanishing of $f'(a) = 0$ is only a necessary but not a sufficient condition for $f(a)$ to be an extreme value of $f(x)$.
3. **Stationary Values:** A function $f(x)$ is said to be stationary for $x = C$ and $f(C)$ is a stationary value of $f(x)$ if $f'(C) = 0$.
4. To find points of maximum, points of minimum and saddle points the following steps are required:
   (*i*) Put $f'(x) = 0$ and find stationary points (x_i)
   (*ii*) If $f''(x_i) > 0$ then $f(x)$ is minimum at $x_i \forall x_i$.
   (*iii*) If $f''(x_i) < 0$ then $f(x)$ is maximum at $x_i \forall x_i$.
   (*iv*) $f'(x) = 0$ and $f''(x) = 0$ and so on
      (a) Continue finding derivatives at x_i untill the first non-zero derivative is found.
      (b) If the first non-zero derivative is obtained at odd derivative then x_i is a point of inflection or saddle point. If the first non-zero derivative is obtained at even derivative then x_i is either a point of maximum or a point of minimum depending on whether the derivative is negative or positive respectively.
5. **Sufficient condition for extrema:**
   Theorem: $f(C)$ is an extremum of $f(x)$ iff $f'(x)$ changes sign as x passes through C.
   Case (*i*): If $f'(x) = 0$ and $f'(x)$ changes sign from positive to –ve as 'x' passes through C then $f(C)$ is maximum.
   Case (*ii*): If $f'(x) = 0$ and $f(x)$ changes sign from –ve to +ve as x passes through C then $f(C)$ is minimum.
   Case (*iii*): If $f'(x) = 0$ and $f(x)$ does not changes sign as x passes through C then $f(C)$ is not extremum but rather a point of inflection or saddle point.
6. **For 2-variable functions:**
   (*i*) Put $f_x = 0$ and $f_y = 0$; find stationary points by solving simultaneously.
   (*ii*) If $f_{xx} f_{yy} > f_{xy}^2$ and (a) If $f_{xx} > 0$ then $f(x)$ is "minimum" at a, (b) If $f_{xx} < 0$ then $f(x)$ is "maximum" at a
   (*iii*) If $f_{xx} f_{yy} - f_{xy}^2 < 0$ then $x = a$ is a point of inflection or saddle point.

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