Calculus

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Limit, Continuity and Differentiability

Limit Existence at a Point

- Left Limit (LHL): $\lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a-h)$

  Right Limit (RHL): $\lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a+h)$

  $\lim\limits_{x \to a} f(x)$ exists, iff both LHL and RHL exists.
- Indeterminate forms: $0/0$, ∞/∞, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, 0^∞
- **L Hospital's Rule:** If $\lim\limits_{x \to a} [f(x)/g(x)]$ is of the form either $0/0$ or ∞/∞ then

  $\lim\limits_{x \to a} [f(x)/g(x)] = \lim\limits_{x \to a} [f'(x)/g'(x)]$

Continuity

A function $f(x)$ is said to be continuous at $x = a$ if:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

- If $f(x)$ and $g(x)$ is continuous function then following are also continuous.
  (*i*) $f(x) + g(x)$, (*ii*) $f(x) \cdot g(x)$, (*iii*) $f(x) - g(x)$, (*iv*) $f(x)/g(x)$ $[g(x) \neq 0]$
- Polynomial, Exponential, Sin and Cos functions are continuous everywhere.

Differentiability

A function $f(x)$ is said to be differentiable at $x = a$ if it is continuous at $x = a$ and if left derivative $= f'(a) =$ right derivative

i.e. $$\lim_{h \to 0^-} \frac{f(a) - f(a-h)}{h} = f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

- If $f(x)$ is differentiable in (a, b), then it is always continuous at (a, b), but converse need not be true.

Mean Value Theorems

Rolle's Mean Value Theorem

If (*i*) '$f(x)$' is continuous in $[a, b]$, (*ii*) '$f(x)$' is differentiable in (a, b), (*iii*) $f(a) = f(b)$ then $\exists\, c \in (a, b)$ such that $f'(c) = 0$

Lagrange's Mean Value Theorem

If (i) '$f(x)$' is continuous in $[a, b]$, (ii) '$f(x)$' is differentiable in (a, b) then $\exists\ c \in (a, b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Cauchy's Mean Value Theorem

Let $f(x)$ and $g(x)$ are two functions if

(i) $f(x)$ and $g(x)$ are continuous in $[a, b]$

(ii) $f(x)$ and $g(x)$ are differentiable in (a, b)

(iii) $g'(x) \neq 0 \quad \forall x \in (a, b)$

then $\exists\ c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Derivatives

- $\dfrac{d}{dx}(\sin x) = \cos x$
- $\dfrac{d}{dx}(\cos x) = -\sin x$
- $\dfrac{d}{dx}(\tan x) = \sec^2 x$
- $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\dfrac{d}{dx}(\sec x) = \sec x \cdot \tan x$
- $\dfrac{d}{dx}(\operatorname{cosec} x) = \operatorname{cosec} x \cot x$
- $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1+x^2}$
- $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$
- $\dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = \dfrac{-1}{x\sqrt{x^2-1}}$
- $\dfrac{d}{dx}(\sinh x) = \cosh x$
- $\dfrac{d}{dx}(\cosh x) = \sinh x$
- $\dfrac{d}{dx}(e^x) = e^x$
- $\dfrac{d}{dx}(a^x) = a^x \cdot \log a$
- $\dfrac{d}{dx}[f(x)]^n = n \cdot [f(x)]^{n-1}$
- $\dfrac{d}{dx}(ax+b)^n = n \cdot (ax+b)^{n-1} \cdot a$
- $\dfrac{du}{dx} = \dfrac{du}{dy}\dfrac{dy}{dx}$ (chain rule)
- $\dfrac{d}{dx}(uv) = u \cdot \dfrac{dv}{dx} + v \cdot \dfrac{du}{dx}$

- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2}$
- $\sinh x = \frac{e^x - e^{-x}}{2}$
- $\cosh x = \frac{e^x + e^{-x}}{2}$
- $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

Integrals

- $\int x^n \cdot dx = \frac{x^{n+1}}{n+1};\ (n \neq -1)$
- $\int \frac{1}{x}\cdot dx = \log_e x$
- $\int e^x \cdot dx = e^x$
- $\int a^x = \frac{a^x}{\log a}$
- $\int \frac{f'(x)}{f(x)}\cdot dx = \log\,[f(x)]$
- $\int [f(x)]^n \cdot f'(x)\cdot dx = \frac{[f(x)]^{n+1}}{n+1}$
- $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\cdot\tan^{-1}\frac{x}{a}$
- $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\cdot\log\left[\frac{a+x}{a-x}\right]$
- $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\cdot\log\left[\frac{x-a}{x+a}\right]$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$
- $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}$
- $\int \sinh x = \cosh x$
- $\int \cosh x = \sinh x$
- $\int \sin x\cdot dx = -\cos x$
- $\int \cos x\cdot dx = \sin x$
- $\int \tan x\cdot dx = \log\cos x = \log\sec x$
- $\int \cot x = \log\sin x$
- $\int \sec x\cdot dx = \log\,(\sec x + \tan x)$
- $\int \operatorname{cosec} x\cdot dx = \log\,(\operatorname{cosec} x - \cot x)$
- $\int \sec^2 x\cdot dx = \tan x$
- $\int \operatorname{cosec}^2 x\cdot dx = -\cot x$
- $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\quad(a\sin bx - b\cos bx)$
- $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\quad(a\cos bx + b\sin bx)$
- $\int \sqrt{a^2-x^2}\cdot dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

- $\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$
- $\int \sqrt{x^2 - a^2} \cdot dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

Definite Integrals

Theorem 1:

If $f(x)$ is continous in $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$ in $[a, b]$ then $\int_a^b f(x)dx = F(b) - F(a)$.

Theorem 2:

If $f(x)$ is continous on $[a, b]$ then $F(x) = \int_a^b f(t) \cdot dt$ is differentiable at every point of x in $[a, b]$ and $\frac{dF}{dx} = \frac{d}{dx}\int_a^b f(t) \cdot dt = f(x)$.

i.e., If $f(x)$ is continous on $[a, b]$ then $\exists$ a function $F(x)$ whose derivative on $[a, b]$ is $f(x)$.

Theorem 3:

If $f(x)$ is continous on $[a, b]$ and $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$ then $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t) \cdot dt = f(v(x))\frac{dv}{dx} - f(u(x)) \cdot \frac{du}{dx}$

Properties of Definite Integrals

1. $\int_a^a f(x) \cdot dx = 0$
2. $\int_a^b f(x) \cdot dx = \int_b^a f(z) \cdot dz$
3. $\int_a^b f(x) \cdot dx = -\int_b^a f(x) \cdot dx$
4. $\int_a^b k \cdot dx = k(b-a)$
5. $\int_a^b f(x) \cdot dx = \int_a^c f(x) \cdot dx + \int_c^b f(x) \cdot dx$
6. $\int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$

7. $\int_{-a}^{a} f(x)\cdot dx = 2\int_{0}^{a} f(x)\cdot dx$; if $f(x)$ is even function $f(-x) = f(x)$

$= 0$; if $f(x)$ is odd function $f(-x) = -f(x)$

8. $\int_{a}^{b} f(x)\cdot dx = \int_{a}^{b} f(a+b-x)\cdot dx$

9. $\int_{0}^{2a} f(x)\cdot dx = \int_{0}^{a} f(x)\cdot dx + \int_{0}^{a} f(2a-x)\cdot dx$

10. $\int_{0}^{2a} f(x)\cdot dx = 2\int_{0}^{a} f(x)\cdot dx$; if $f(2a-x) = f(x)$

$= 0$; If $f(2a-x) = -f(x)$

11. $\int_{0}^{na} f(x)\cdot dx = n\cdot\int_{0}^{a} f(x)\cdot dx$; if $f(x+a) = f(x)$

[$f(x)$ is periodic function with period 'a']

12. $\int_{0}^{a} x\cdot f(x)\cdot dx = \frac{a}{2}\int_{0}^{a} f(x)\cdot dx$; if $f(a-x) = f(x)$

13. **Mean value theorem for definite integrals:** Let f be continuous in (a, b) then $\exists c \in (a, b)$ such that $f(c) = \dfrac{\int_{a}^{b} f(x)dx}{(b-a)}$

$f(c)$ is the average value of the function in the interval (a, b).

14. $\int_{a}^{b} f(x)dx \le \int_{a}^{b} g(x)dx$ If $f(x) \le g(x)$ on (a, b)

15. $\int_{0}^{\pi/2} \sin^n x\cdot dx = \int_{0}^{\pi/2} \cos^n x\cdot dx = \dfrac{(n-1)\cdot(n-3)\cdot(n-5)....}{n\cdot(n-2)(n-4)....}\times K$

Where, $K = 1$; if n is odd and $K = \pi/2$; if n is even

Improper Integrals

1. $\int_a^b f(x)\cdot dx$ is "improper integral";

   (i) If $a = -\infty$ or $b = \infty$ or both

   (ii) If $f(x) = \infty$ for one or more values of x in $[a, b]$.

2. $\int_a^b f(x)\cdot dx$ is "convergent", if the value of integral is "finite".

3. $\int_a^b f(x)\cdot dx$ is "divergent", if the value of integral is "infinite".

4. If $0 \le f(x) \le g(x); \forall x$ and $\int_a^\infty g(x)\cdot dx$ converges then $\int_a^\infty f(x)$ also converges.

5. If $0 \le g(x) \le f(x); \forall x$ and $\int_a^\infty g(x)\cdot dx$ diverges then $\int_a^\infty f(x)$ also diverges.

6. If $f(x)$ and $g(x)$ are two functions, such that $\lim\limits_{n\to\infty} \dfrac{f(x)}{g(x)} = k$

   where $k =$ finite and $k \neq 0$ then

   $\int_a^\infty f(x)\cdot dx$ and $\int_a^\infty g(x)\cdot dx$ "converge" or "diverge" together.

7. $\int_a^\infty e^{-Px}\cdot dx$ and $\int_{-\infty}^b e^{Px}\cdot dx$ converges for any constant $P > 0$ and diverges for $P \le 0$.

8. $\int_a^b \dfrac{dx}{(b-x)^P}$ is convergent iff $P < 1$.

9. $\int_a^b \dfrac{dx}{(x-a)^P}$ is convergent iff $P < 1$.

10. $\int_1^\infty \dfrac{dx}{x^P}$ converges if $P > 1$ and diverges if $P \le 1$.

Total Derivatives

If $u = f(x, y)$ where x and y are functions of t, then the "total derivative" of u with respect to t is:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial u}{\partial y}\left(\frac{dy}{dt}\right)$$

By putting $t = x$ in above equation, we get

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$; where x and y are connected by some relation.

Partial Derivatives

1. If $u = f(x, y)$, then $\dfrac{\partial u}{\partial x} = \lim\limits_{h \to 0} \dfrac{[f(x+h, y) - f(x, y)]}{h}$ and

   $$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{[f(x, y+k) - f(x, y)]}{k}.$$

2. If $u = f(x, y)$, where $x = g(r, s)$ and $y = h(r, s)$ then

   $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

   $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

3. If $u = f(x, y)$ is a homogenous function of degree 'n' then

   $x \cdot u_x + y \cdot u_y = n \cdot u$

   $x^2 u_{xx} + 2xy \cdot u_{xy} + y^2 u_{yy} = n(n-1)u$

4. If $u = f(x, y, z)$ is a homogenous function of degree 'n', then

   $x \cdot u_x + y \cdot u_y + z \cdot u_z = n \cdot u$

5. If $u = f(x, y)$ is not homogenous function, but $F(u)$ is a homogenous function of degree 'n', then

   $$x \cdot u_x + y \cdot u_y = n\left[\frac{F(u)}{F'(u)}\right] = G(u)$$

   $$x^2 \cdot u_{xx} + 2xy \cdot u_{xy} + y^2 \cdot u_{yy} = G(u)\,[G'(u) - 1]$$

Maximum and Minima [Extremum]

1. A necessary condition for $f(a)$ to be an extreme value of $f(x)$ is $f'(a) = 0$.
2. The vanishing of $f'(a) = 0$ is only a necessary but not a sufficient condition for $f(a)$ to be an extreme value of $f(x)$.
3. **Stationary Values:** A function $f(x)$ is said to be stationary for $x = C$ and $f(C)$ is a stationary value of $f(x)$ if $f'(C) = 0$.
4. To find points of maximum, points of minimum and saddle points the following steps are required:
   (*i*) Put $f'(x) = 0$ and find stationary points (x_i)
   (*ii*) If $f''(x_i) > 0$ then $f(x)$ is minimum at $x_i\ \forall x_i$.
   (*iii*) If $f''(x_i) < 0$ then $f(x)$ is maximum at $x_i\ \forall x_i$.
   (*iv*) $f'(x) = 0$ and $f''(x) = 0$ and so on
      (a) Continue finding derivatives at x_i untill the first non-zero derivative is found.
      (b) If the first non-zero derivative is obtained at odd derivative then x_i is a point of inflection or saddle point. If the first non-zero derivative is obtained at even derivative then x_i is either a point of maximum or a point of minimum depending on whether the derivative is negative or positive respectively.
5. **Sufficient condition for extrema:**
   Theorem: $f(C)$ is an extremum of $f(x)$ iff $f'(x)$ changes sign as x passes through C.
   Case (*i*): If $f'(x) = 0$ and $f'(x)$ changes sign from positive to –ve as 'x' passes through C then $f(C)$ is maximum.
   Case (*ii*): If $f'(x) = 0$ and $f(x)$ changes sign from –ve to +ve as x passes through C then $f(C)$ is minimum.
   Case (*iii*): If $f'(x) = 0$ and $f(x)$ does not changes sign as x passes through C then $f(C)$ is not extremum but rather a point of inflection or saddle point.
6. **For 2-variable functions:**
   (*i*) Put $f_x = 0$ and $f_y = 0$; find stationary points by solving simultaneously.
   (*ii*) If $f_{xx} f_{yy} > f_{xy}^2$ and (a) If $f_{xx} > 0$ then $f(x)$ is "minimum" at a, (b) If $f_{xx} < 0$ then $f(x)$ is "maximum" at a
   (*iii*) If $f_{xx} f_{yy} - f_{xy}^2 < 0$ then $x = a$ is a point of inflection or saddle point.

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