4 Counting

Sum and Product Rule

Rule of sum:

If there are m choice for one event and n choices for another event, both can not occur at the same time, then their m + n choices for one event.

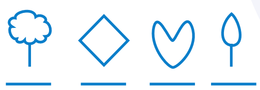
If a sequence of tasks T_1, T_2 ... T_n can be done in W_1, W_2 ..., W_n ways respectively (no tasks can be performed simultaneously), then the number of ways to do one of these tasks is $W_1 + W_2 + ... + W_n$

Note:
In the rule of addition, jobs are performed independently. In this case, we use the word 'or' between various jobs, and the meaning of 'or' is addition.

Solved Examples

1. In how many ways can we draw a face card from a deck?

Solution:
In a deck of 52 cards, there are 4 suits.



$J \rightarrow 4$

$Q \rightarrow 4$

$K \rightarrow 4$

We want to draw a one-face card between all these no of choice = 4 + 4 + 4 = 12

2. A pencil box contains 2 red and 3 black pens. Total number of ways for selecting 1 red pen and 1 black pen or 2 red pens.

Solution:

2 ways for selecting red pens because 2 pens are there.

3 ways for selecting black pens because 3 pens are there.

$\therefore$ Total ways for selecting 1 red and 1 black = 3 × 2 = 6

$\therefore$ Number of ways of selecting 2 red pens = 1

Total ways for selecting 1 red and 2 black or 2 red is 6 + 1, i.e., 7 ways.

Rule of product:

If there are m choices for one event and n choices for another event, then there are m × n choices for both these events to occur.

If a sequence of tasks T_1, T_2 ... T_n can be done in W_1, W_2, ..., W_n ways respectively (every task arrives after the occurrence of the previous task) then the number of ways to perform all the tasks is $W_1 \times W_2 \times ... \times W_n$.

Note:
In the rule of multiplication, different jobs/operations are mutually inclusive, it implies that all jobs are being done in succession. In this case, we use word 'and' to complete all stages of operation, and the meaning of 'and' is multiplication.

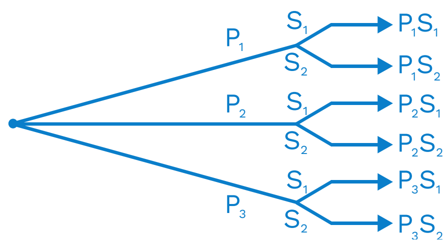
3. Let us consider the following. Rahul has 3 pants and 2 shirts. How many different pairs of pants and a shirt, can he dress up with?

Solution:
There are 3 ways in which a pant can be chosen (because 3, pants available) says P_1, P_2, P_3

Similarly, a shirt can be chosen in 2 ways. (Because 2 shirts available, say S_1, S_2).

For every choice of a pant, there are 2 choices of a shirt.

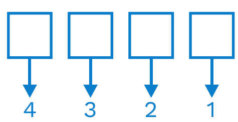
Therefore, there are 3 × 2 = 6 pairs of pants and shirts.



4. Find the number of 4 letter words with or without meaning possible, using the given set of 4 letters, here repetition of letters is not allowed.

Solution:

There are as many words as there are the way of filling in 4 vacant places by the 4 letters (repetition is not allowed).



First place can be filled in 4 different ways by any one of the four letters. Second place can be filled by anyone of the remaining 3 letters in 3 different ways, following third place can be 2 different ways.

Following 4 places can be filled in one way.

Thus, the no of ways in which the 4 places can be filled by the multiplication principle is 4 × 3 × 2 × 1 = 24

Note:

If the repetition of the letters was allowed in the above problem, one could easily understand that each of 4 vacant places can be filled in succession in 4 different ways. Hence the required number of words = 4 × 4 × 4 × 4 = 256

5. If a die is cast and then a coin is tossed, find the number of all possible outcomes

Solution:

A die can fall in 6 different ways, i.e., 1, 2, 3, 4, 5, 6, and a coin can fall in 2 different ways, i.e., head or tail.

Number of possible outcomes from a die and a coin = 6 × 2 = 12 ways.

6. There are 7 trains running between Delhi and Lucknow. In how many ways can a man go from Delhi to Lucknow and return by a different train?

Solution:

A man can go from Delhi to Lucknow in 7 ways (by anyone the 7 trains available).

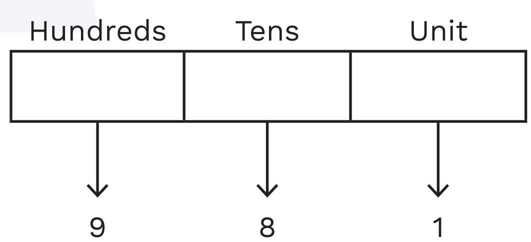
He can return from Lucknow to Delhi in 6 ways by the remaining 6 trains (since, he cannot return by the same train by which he goes from to Lucknow to Delhi).

Required number of ways = 7 × 6

= 42

7. How many 3 digit numbers can be formed, whose unit digit is always zero and repetition of the digit is not allowed?

Solution:



Since, zero is fixed for the unit digit place. So, only 9 digits are available for the hundreds place, and only 8 digits are available for the tens place out of 10 digits, i.e., 0, 1, 2, 3, ..., 9.

Therefore, required possible numbers = 9 × 8 × 1 = 72

Factorial:

Factorial of ‘n’ gives the product of all number the numbers between 1 to n.

$$\boxed{n! = n(n-1)(n-2)\ldots3.2.1}$$

Rack Your Brain

A GATE test paper consists of 65 questions and each question has 5 choices. If each question is necessarily attempted, then find the number of ways of answering the GATE test paper.

Permutation:

- The words permutation comes from the Latin words per + mutare that together mean "by change" or "through change".
- An arrangement of a set of an object is called a permutation.
- In permutation orders matter.
- Factorial plays an important part in determining the total permutation of a given set without repetitions.

$$n! = n(n-1)(n-2)...3.2.1$$

n! represents the number of permutations of n different objects.

- The number of permutation, (arrangements) of n different things taking r at a time is considered as a partial permutation. It is written as $^{n}P_{r}$ ("n permute r")

$$^{n}P_{r} = p(n,r) = \frac{n!}{(n-r)!}$$

- Number of ways of arranging 'n' things in which q things are identical of one type and p things are identical of another type $= \frac{n!}{p!q!}$
- Total permutations of n distinct things round a close curve = (n-1)! (if clock wise and anticlockwise arrangements are considered different)

$= \frac{1}{2}(n-1)!$ (If clock wise and anticlockwise arrangement are considered identical)

- Permutation of n different things taken all at a time = $^{n}p_{n} = n!$
- Permutation of n different thing taken r at a time when a particular thing always occurs $= r^{*(n-1)}P_{(r-1)}$
- Permutation of n different things taken r at a time when a particular thing never occurs $= {}^{(n-1)}P_{r}$

Permutation of n things not all different:

- Number of permutations of n things, taken all at a time, of which p are alike of one kind, q and alike of the second kind, r is alike of the third kind and rest are different $= \frac{n!}{p!q!r!}$
- Number of permutations of n things of which P_1 are alike of one kind, P_2 are alike of second kind, P_3 are alike of third kind ... P_r are alike of r^{th} kind such that

$$P_1 + P_2 + P_3 \ldots + P_r = n \text{ is } \frac{n!}{P_1!P_2!P_3!\ldots P_r!}$$

8. How many 4-digit even numbers are there (repetition is allowed)?

Solution:

$$\underset{9}{\downarrow}\ \underset{10}{\downarrow}\ \underset{10}{\downarrow}\ \underset{5}{\downarrow}$$

First place, one can fill in 9 ways (1 to 9) any value (Because zero cannot come at first position)

Second place, one can fill in 10 ways (0 to 9) any value

Third place, one can fill in 10 ways (0 to 9) any value

Fourth place, one can fill in 5 ways (0, 2, 4, 6, 8) any value

Using product rule = 9 × 10 × 10 × 5 = 4500

Previous Years' Questions

How many 4 digit even numbers have all 4 digits distinct?

(A) 2240 (B) 2296

(C) 2620 (D) 4536

Solution: (B) **[GATE 2001]**

Combination:

- An unordered selection of object is called a combination.
- In combination arrangement of things doesn't matter until the specific thing is to be included.
- The combination of three alphabets {a, b, c} taken two at a time are {a, b}, {a, c}, {b, c}, here {a, b} and {b, a} are not considered separately.
- The word combination and selection are synonymous.

Rack Your Brain

How many ways 65 True/False question can be answered if you can skip any of the question?

(A) ${}^{65}P_3$ (B) ${}^{65}C_3$

(C) 3^{65} (D) 65^3

- The number of combinations of n distinct objects taken r at a time without repetition denoted by ${}^nC_r$ (read as n choose r).

$$ {}^nC_n = \frac{n(n-1)(n-2)...(n-(r-1))}{r(r-1)(r-2)...2.1} $$

- $C(n,r) = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

- ${}^nC_r = \dfrac{{}^nP_r}{r!} = \dfrac{\dfrac{n!}{(n-r)!}}{r!} = \dfrac{n!}{(n-r)!r!}$

- Selecting r objects out of n is the same as selecting (n-r) object out of n {k-selection equal to k rejection}
- Total selections that can be made from 'n' distinct items are given $\sum_{k=0}^{n} {}^nC_k = 2^n$

- ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$

- Total ways of selecting one or more objects from n distinct objects

$$ = 2^n - 1 = ({}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n) $$

- $(x+y)^n = {}^nC_0 x^{n-0} y^0 + {}^nC_1 x^{n-1} y^1 + ... {}^nC_n x^{n-n} y^n$

- $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 ... + {}^nC_n$

  {Put x = 1 and y = 1 in above equation}

- $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... {}^nC_n x^n$

- $n \cdot 2^{n-1} = 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + ... + n \cdot {}^nC_n$

  {By differentiate above and put x = 1}

- ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5$

  $+ \ldots = \dfrac{2^n}{2} = 2^{n-1}$

Combination with repetition:

- The number of combinations of n objects taken r at a time with repetition.
- The number of ways r identical objects can be distributed among n distinct boxes.
- The number of non-negative Integral solution of the equation $x_1 + x_2 + x_3 + ... + x_n = r$ where $x_1 \geq 0$, $x_2 \geq 0$ ... $x_n \geq 0$
- The solution of all the above statement is ${}^{(n-1+r)}C_r$

Properties of ${}^nC_r$ for simplification:

- ${}^nC_0 = 1, {}^nC_n = 1$
- ${}^nC_r = {}^nC_{n-r}$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nC_r = {}^nC_k \quad \Rightarrow \quad r = k \quad \text{or} \quad r + k = n$
- $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$
- $\dfrac{1}{r+1} \cdot {}^nC_r = \dfrac{1}{n+1} \cdot {}^{n+1}C_{r+1}$

9. In how many ways can you distribute 10 similar balls into 3 distinct boxes.

(A) ${}^{12}C_{10}$

(B) ${}^{13}C_{10}$

(C) ${}^{12}C_2$

(D) ${}^{13}C_3$

Solution: (A), (C)

x_1	x_2	x_3

Let x_1 x_2 x_3 are number of balls in box 1, box 2, box 3 respectively.

So, $x_1 + x_2 + x_3 = 10$ [n = 3, r = 10]

Using $^{n-1+r}C_r = {}^{3-1+10}C_{10} = {}^{12}C_{10} = {}^{12}C_2$

Rack Your Brain

In how many ways of placing 20 similar balls into 4 boxes where each box is non empty?

Solved Examples

10. In how many ways you can place atmost 7 similar balls into 4 distinct boxes.

Solution:

Let x_1,x_2,x_3,x_4 are the number of balls in box 1, 2, 3, 4 respectively.

x_1	x_2	x_3	x_4

So, $x_1 + x_2 + x_3 + x_4 \le 7$. $\forall x_i \ge 0$ here r value can be any of between 0 to 7.

So, $= \sum_{r=0}^{7} {}^{n-1+r}C_r$ {n = 4}

$= \sum_{r=0}^{7} {}^{3+r}C_r$

$= {}^3C_0 + {}^4C_1 + {}^5C_2 + {}^6C_3 + {}^7C_4 + {}^8C_5 + {}^9C_6 + {}^{10}C_7$

$= 1 + 4 + 10 + 20 + 35 + 56 + 84 + 120$

$= 330$

Note:

The above problem can also solve using taken n = 5 (one more box as dustbin) and r is 7 in formula $^{n-1+r}C_r$.

Dividing given into groups:

- The number of ways of dividing (p + q) items, into two groups of p and q items, respectively is $\frac{(p+q)!}{p!q!}$
- The number of ways of dividing 2p items into two equal groups of p each is $\frac{(2p)!}{(p!)^2}$ where two groups have distinct identities.
- The number of ways of dividing 2p items into two equal groups of p each is $\frac{(2p)!}{2!(p!)^2}$, where the two groups do not have a distinct identities.
- The number of ways in which (p + q + r) things can be divided into three groups containing p, q, and r things, respectively, is $\frac{(p+q+r)!}{p!q!r!}$

Difference between permutation and combination:

Permutation	Combination
Order is important	Order is not important
Arrangement	Selection
Keywords: password, number, word	Keywords: Sets, subset, team, committee

Derangement:

- Derangement is a permutational arrangement with no fixed points.
- Arrangement of elements in such a way that no is in its correct position. If n distinct items are arranged, the number of ways they can be arranged so that they do not occupy their intended spot is:

$$\boxed{D_n = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \frac{(-1)^n}{n!}\right)}$$

$$\boxed{D_n = \sum_{r=2}^{n} (-1)^r \frac{n!}{r!}}$$

Solved Examples

11. There are 5 tables in a class, each table containing the roll number of the student. How many ways can 5 students be arranged such that no student sits at the table having their own roll number?

Solution:

This is a clear case of derangements of 5 tables and 5 students

$$D_5 = 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)$$

$$= 120\left(1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}\right)$$

$$= 120\times\left(\frac{60-20+5-1}{120}\right)$$

= 44

12. In how many ways can you put 7 letters into their respective envelopes such that exactly 2 go into the right envelope?

Solution:

Number of ways in which the 2 correct envelopes can be selected

$$= {}^7C_2 = {}^7C_5 = \frac{7\times6\times5!}{5!\,2!} = 21$$

Derangement of the remaining 5 envelopes and letters D_5 = 44

Total number of ways of arrangement = 44 × 21 = 924

Previous Years' Questions

In how many ways can we distribute 5 distinct balls, B_1, B_2, ..., B_5 in 5 distinct cells, C_1, C_2, ..., C_5 such that Ball B is not in cell C_i, $\forall_i$ = 1, 2, ... 5, and each cell contains exactly one ball?

(A) 44 (B) 96

(C) 120 (D) 3125

Solution: (A) **[GATE 2004]**

Number of divisors:

If the prime factorisation of n is $P_1^{e_1}\cdot P_2^{e_1}\ldots P_k^{e_k}$, where P_1 are distinct prime numbers then the **Number of divisors is**

d(n) = $(e_1 + 1) \times (e_2 + 1) \ldots \times (e_k + 1)$

Sum of the divisor is

$$\sigma(n) = \frac{P_1^{e_1+1}-1}{P_1-1}\times\frac{P_2^{e_2+1}-1}{P_2-1}\cdots\times\frac{P_k^{e_k+1}-1}{P_k-1}$$

Solved Examples

13. How many divisors of 1500?

Solution:

1500 = $2^2 \times 3^1 \times 5^3$

Number of divisors d(n) = (2 + 1) × (1 + 1) × (3 + 1)

= 3 × 2 × 4

= 24

Binomial theorem:

- $(a + x)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}\, x + {}^nC_2\, a^{n-2}\, x^2 + \ldots + {}^nC_n x^n$ Where ${}^nC_r = \frac{n!}{r!(n-r)!}$.
- $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_n\, x^n$.
- $(1 + x)^n = (x + 1)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2} + \ldots + {}^nC_n$

When expanded in descending powers of x.

- $(a - x)^n = {}^nC_0\, a^n - {}^nC_1\, a^{n-1}\, x + {}^nC_2\, a^{n-2}\, x^2 - \ldots + (-1)^n \cdot {}^nC_n x^n$.

Previous Years' Questions

The number of divisors of 2100 is ______.

(A) 42 (B) 36

(C) 78 (D) 72

Solution: (B) **[GATE 2015]**

Properties of binomial coefficient:

- ${}^nC_r = {}^nC_{n-r}$ (symmetry)
- ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$ (Pascal identity)
- ${}^nC_r \times {}^rC_k = {}^nC_k \times {}^{n-k}C_{r-k}$ (Newton's identity)
- $\sum_{r=0}^{n} {}^nC_r = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots {}^nC_n = 2^n$

  (Row summation)
- $\sum_{r=0}^{n} (-1)^r\, {}^rC_r = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 \cdots = 0$

  (Alternating sign row summation)
- ${}^rC_r + {}^{r+1}C_{r+1} + {}^{r+2}C_{r+2} + \cdots + {}^nC_r = {}^{n+1}C_{r+1}$

  (Column summation)
- $\sum_{K=0}^{r} {}^nC_K \cdot {}^mC_{r-K} = {}^nC_0 \cdot {}^mC_r + {}^nC_1 \cdot {}^mC_r$

  ${}_{-1} + {}^nC_2 \cdot {}^mC_{r-2} + \cdots {}^nC_r \cdot {}^mC_0 = {}^{n+m}C_r$

  (Vandemondes identity)

Solved Examples

14. The coefficient of a^4b^5 in the expansion of $(a + b)^9$ is _______.

Solution:

Coefficient of a^4b^5 is

$$= \frac{9!}{4!5!} = \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 5!} = 126$$

15. The coefficient of a^4b^5 in the expansion of $(2a - 3b)^9$ is_____.

Solution:

Coefficient of a^4b^5 is

$$= \frac{9!}{4!5!}(2)^4 \cdot (-3)^5$$

$= 126 \times 16 \times (-243)$

$= -489{,}888$

Rack Your Brain

The coefficient of $\alpha^3\beta^2\gamma^1\delta^4$ in the expansion of $(\alpha + 3\beta + 2\gamma - 7\delta)^{10}$ is_____.

Multinomial theorem:

The multinomial theorem describes how to expand the power of a sum in terms of the power of the terms in that sum. It is the generalisation of the binomial theorem from binomials to multinomials.

$$(x_1 + x_2 + \cdots x_t)^n = \sum_{\forall n_1 + n_2 + \cdots n_t = n} \binom{n}{n_1, n_2 \cdots n_t} \cdot \prod_{m=1}^{t} x_m^{n_m}$$

where $\binom{n}{n_1, n_2 \cdots n_t} = \frac{n!}{n_1!n_2!\cdots n_t!}$

Note:

The coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \ldots x_t)^n$

where $n = n_1 + n_2 + \ldots n_t$ is $\frac{n!}{n_1!n_2!\cdots n_t!}$.

Catalan numbers:

The n^{th} Catalan number is given directly in terms of binomial coefficients by

$$\frac{{}^{2n}C_n}{n+1} = \frac{1}{n+1}\binom{2n}{n} = \frac{2n!}{(n+1)!n!} = \prod_{K=2}^{n} \frac{n+K}{K} \ \forall\, n \geq 0$$

The first Catalan number for n = 0, 1, 2, 3, ... are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900 ...

Note:

The total number of the path without crossing the diagonal from (0, 0) to (n, n) is:

$$C_n = \binom{2n}{n} - \binom{2n}{n+1}$$

$$\boxed{C_n = \frac{1}{n+1}\, {}^{2n}C_n}$$

Solved Examples

16. In how many ways can we start from (0, 0) and go to (5, 5) without crossing the diagonal? (Assuming that you can move up or right).

Solution:

$$C_n = \frac{1}{n+1}\,{}^{2n}C_n$$

$$= \frac{1}{5+1}\, 2\times 5_{C_5}$$

$$= \frac{1}{6}\, 10_{C_5}$$

$$= \frac{10!}{6\cdot 5!5!}$$

= 42

Application of catalan numbers:

There are number of applications which can be solved using Catalan numbers.

- Balanced parenthesis (form a valid group of parenthesis).
- Polygon triangulation (A convex polygon with n + 2 sides can be triangulated in C_n ways.
- If 2n people are seated around a circular table, the number of ways can all of them be simultaneously shaking hands with another person at the table in such a way that none of the arms crosses each other is C_n.
- Number of full binary trees with n + 1 leaves is C_n.

Generating function:

A generating function is a way of encoding an infinite sequence of numbers (a_n) by treating them as the coefficients of a formal power series. This series is called the generating function of the sequence.

Consider a sequence (a_0,a_1,a_2...an...) of real numbers then a function f(x) is defined as f(x) = $a_0+a_1x+a_2x^2+\ldots a_nx^n+\ldots$ is called generating function of the sequence.

$$\boxed{f(x) = \sum_{r=0}^{\infty} a_r x^r}$$

Solved Examples

17. Find generating function for finite sequence:
a_0, a_1, a_2, a_3, a_4

Solution:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

18. Find generating function for the infinite sequence:
1, 1, 1, 1...

Solution:

Generating function of 1, 1, 1, ...

$= 1 + x + x^2 + x^3 + x^4 + \ldots \infty = \frac{1}{1-x}$ (sum of infinite GP series $\frac{a}{1-x}$)

19. Find the generating function for 1, 2, 3, 4

Solution:

$$f(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

We know that,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \ldots$$

Differentiate both side by $\frac{d}{dx}$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^2 + x^4 + \ldots\right)$$

$$\boxed{\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots}$$

Rack Your Brain

Generating function for $1^2, 2^2, 3^2, 4^2 \ldots$

Identities of generating functions:

1. $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots {}^nC_n x^n = \sum_{r=0}^{n} {}^nC_r x^r$

2. $1 + x + x^2 \ldots + x^{n-1} = \dfrac{1-x^n}{1-x}$

3. $1 + x + x^2 + x^3 \ldots = \sum_{r=0}^{\infty} x^r = \dfrac{1}{1-x}$

4. $\dfrac{1}{(1-x)^n} = \left(\dfrac{1}{1-x}\right)^n = \left(\sum_{r=0}^{\infty} x^r\right)^n = \sum_{r=0}^{\infty} n-1+{}^rC_r x^r$

5. $\dfrac{1}{(1-x)^2} = \sum_{r=0}^{\infty} {}^{(r+1)}C_r x^r = \sum_{r=0}^{\infty} (r+1)x^r$

6. $\dfrac{1}{(1-x)^3} = \sum_{r=0}^{\infty} \dfrac{(r+1)(r+2)}{2} x^r$

7. $\dfrac{1}{1+x} = \sum_{r=0}^{\infty} (-1)^r x^r$

8. $\dfrac{1}{1-ax} = \sum_{r=0}^{\infty} a^r x^r$

9. $\dfrac{x}{(1-x)^2} = \sum_{r=0}^{\infty} {}^{(r+1)}C_r x^{r+1} =$

   $\sum_{r=0}^{\infty} (r+1)x^{r+1} = \sum_{K=1}^{\infty} Kx^K$ (if r+1 = k)

10. $\dfrac{x^2}{(1-x)^2} = \sum_{K=0}^{\infty} K \cdot x^{K+1}$

20. Find coefficient of x^{12} in the expansion of $(x^2 + x^3 + x^4 + \ldots)^2$

Solution:

To simplify the expansion. We extract x^2 coefficient of x^{12} in

$\Rightarrow$ coefficient of x^{12} in $(x^2)^2 (1+x+x^2+\ldots)^2$

coefficient of x^{12} in $x^4 \left(\dfrac{1}{1-x}\right)^2$

$\left\{1 + x + x^2 + x^3 + \ldots = \dfrac{1}{1-x}\right\}$

coefficient of x^8 in the expansion of $\dfrac{1}{(1-x)^2}$

coefficient of x^8 in $\sum_{r=0}^{\infty} (r+1)x^r$ (put r = 8)

so coefficient of x^{12} in the $(x^2 + x^3 + x^4 + \ldots)^2$ is $8 + 1 = 9$

Previous Years' Questions

Let $G(x) = \dfrac{1}{(1-x)^2} = \sum_{i=0}^{\infty} g(i)x^i$, where $|x| < 1$

what is g(i).

(A) i (B) i + 1

(C) 2i (D) 2^i

Solution: (A) **[GATE 2005]**

Rack Your Brain

Find coefficient of x^{12} in the expansion of $(x^3 + x^4 + x^5 + \ldots)^2$ is ________.

The pigeonhole principle:

Principle 1:

If k is a positive integer and k+1 or more objects are placed into k boxes, then there is atleast one box containing two or more of the objects.

Previous Years' Questions

If the ordinary generating function of a sequence $\{a_n\}_{n=0}^{\infty}$ is $\frac{1+z}{(1-z)^3}$, then $a_3 - a_0$ is equal to ________.

Solution: 15 **[GATE 2017]**

Proof:

We use proof by contraposition. Suppose none of the boxes has more than one object. Then the total number of objects would be atmost K. This contradicts statements that we have K+1 objects.

Generalised pigeonhole principle:

If N objects are placed into k boxes. Then, there is atleast one box containing $\left\lceil N/K \right\rceil$ objects.

Theorem:

A function f from a set with K+1 elements to a set with K elements is not one-to-one.

Solved Examples

21. Owner of a house boys '3' distinct colours wall paints. He wants that in his house atleast 10 walls should be the same colour. For his desire to be full filled minimum, how many walls should be there in his house?

Solution:

By using pigeon-hole principle:

$$10 \geq \left\lceil \frac{n-1}{m} \right\rceil + 1 \quad \left\{ \begin{array}{l} n = \text{no. of walls} \\ m = \text{no. of distinct colours} \end{array} \right\}$$

$$10 \geq \left\lceil \frac{n-1}{3} \right\rceil + 1$$

$N \geq 28$

$\therefore$ Atleast 28 walls should be there for the desire of the landlord to be fulfilled.

Previous Years' Questions

Which one of the following is a closed form expression for the generating function of the sequence $\{a_n\}$, where $a_n = 2n + 3$ for all n = 0, 1, 2...?

(A) $\frac{3}{(1-x)^2}$ (B) $\frac{3x}{(1-x)^2}$

(C) $\frac{2-x}{(1-x)^2}$ (D) $\frac{3-x}{(1-x)^2}$

Solution: (D) **[GATE 2018]**

Proofs:

Use the pigeonhole principle

- Create a box for each element Y in the co-domain of f.
- Put in the box for Y all of the elements x the domain such that f(x) = Y.
- Because there are K+1 elements and only K boxes, atleast one box has 2 or more elements.

Hence, f can not be one-to-one.

Application of pigeonhole principle:

- Number theory
- Economics
- Probability
- Finance
- Algorithms
- Computer science
- Mathematical analysis
- Computer programming
- Arithmetic
- Geometry

Rack Your Brain

If 100 cars are painted with 7 colours. Atleast how many cars painted with same colour?

Recurrence relation:

- Many counting problems cannot be solved easily; one such problem can be easily solved by a recurrence relation.
- A recurrence relation is a formula which relates a_n with one or more preceding terms (previous term) a_{n-1}, a_{n-2} ... . Where a_0, a_1, a_2 be a sequence of real numbers.

Examples:

1. For the arithmetic progression {a, a + d, a + 2d, ...}, and d is a common difference.

   The recurrence relation is

   $\boxed{a_n = a_{n-1} + d}$ $n \geq 1$, $a_0 = a$

2. For the geometric progression {a, ar, ar^2, ar^3 ...}. The recurrence relation is

   $\boxed{a_n = (a_{n-1}) \cdot r}$ $n \geq 1$, $a_0 = a$

3. For the Fibonacci series < 1, 1, 2, 3, 5, 8, 13, ...} The recurrence relations is

   $\boxed{a_n = a_{n-1} + a_{n-2}}$ $n \geq 2$, $a_0 = 1$ $a_1 = 1$

Solved Examples

22. The number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in n hours?

Solution:

Let a_n be the number of bacteria at the end of n hours.

and a_{n-1} be the number of bacteria after n – 1 hours.

Since the number of bacteria doubles every hour.

Then the recurrence relation is:

$\boxed{a_n = 2a_{n-1}}$ where a_0 = 5.

Rack Your Brain

Write the recurrence relation of the binary string of length n which does not contain two consecutive zeros.

Types of recurrence relation:

There are two types of recurrence relation:

- Linear homogenous recurrence relation
- Non-linear homogenous recurrence relation

23. What is the solution of the recurrence relation $a_n = n \times a_{n-1}$ where $a_0 = 1$?

Solution:

$a_n = n \times a_{n-1}$

Using substitution method

$a_1 = 1.a_0 = 1(1) = 1!$ Put (n = 1)

$a_2 = 2.a_1 = 2(1!) = 2!$ Put (n = 2)

$a_3 = 3.a_2 = 3(2!) = 3!$ Put (n = 3)

⋮

$\boxed{a_n = n!}$

Linear homogenous recurrence relation:

A linear recurrence relation of degree (order) k with constant coefficients is a recurrence relation of the form.

$$\boxed{C_0 a_n + C_1 a_{n-1} + \ldots + C_k a_{n-k} = f(n)}$$

When f(n) = 0 then it homogenous linear recurrence relation and if f(n) ≠ 0 then non homogenous linear recurrence relation.

Rack Your Brain

What is the solution of the recurrence relation $a_n = a_{n-1} + 3^{n-1}$ where $a_0 = 1$?

Characteristic equation for homogenous recurrence relation:

- If the roots are real and distinct

  $C_1t_1^n + C_2t_2^n + \ldots + C_kt_k^n$

- If the roots are real and equal (two roots are equal)

  $(C_1 + C_2n)\, t_1^n + C_3t_3^n + \ldots + C_kt_k^n$

- If all the rots are real and equal

  $(C_1 + C_2n + C_3n^2 + \ldots + C_kn^k)\, t_1^n$

- If roots are complex $(\alpha \pm i\beta)$
  $\gamma^n(C_1 \cos n\theta + C_2 \sin n\theta)$ where $\gamma = \sqrt{\alpha^2 + \beta^2}$

  and $\theta = \tan^{-1}\left(\dfrac{\beta}{\alpha}\right)$

23. Solve the following linear homogenous recurrence relation by using characteristic root method.
$a_n - 3a_{n-1} + 2a_{n-2} = 0$

Solution:

Characteristic equation

$C(t) = t^2 - 3t^1 + 2t^0 = 0$

$t^2 - 3t + 2 = 0$

$(t - 2)\,(t - 1) = 0$

$t = 2, 1$

Since the roots are real and distinct.

So, $a_n = C_1(2)^n + C_2(1)^n$.

Rack Your Brain

Write the recurrence relation of binary string of length n. Which do not contain two consecutive zero.

Characteristic equation for non-homogenous linear recurrence relation ($f(n)^1$ 0):

- If $f(n) = C(b)^n$ and b is not root of C(t)

  $\boxed{a_n = a_nH + a_nP}$ where a_nH is solution of homogenous linear recurrence relation and a_nP is particular solution (depends on the form of f(n)).

- $f(n) = C(b)^n$ and b is not root of C(t)

  $\boxed{a_nP = D(b)^n}$

- $f(n) = C(b)^n$ and b is root of C(t) of multiplicity m.

  $\boxed{a_nP = Dn^m(b)^n}$

24. Solve the following non-linear homogenous recurrence relation using characteristic method.
$a_n - 3a_{n-1} = 7(2)^n$

Solution:

$a_n = a_nH + a_nP$

$a_n(H) = a_n - 3a_{n-1} = 0$

$C(t) = t - 3 = 0$

$\boxed{t = 3}$

$\boxed{a_n(H) = C_1(3)^n}$

$a_n(P) = f(n) = 7(2)^n$ and 2 is not root of C(t)

$a_n(P) = D(2)^n$

Put this in $a_n - 3a_{n-1} = 7(2)^n$

$D(2)^n - 3D(2)^{n-1} = 7(2)^n$

$2^{n-1}\,[2D - 3D] = 7(2)^n$

$-D = 7 \times 2$

$\boxed{D = -14}$

$a_n(P) = -14(2)^n$

$a_n = a_nH + a_nP$

$\boxed{a_n = C_1(3)^n - 14(2)^n}$

Chapter Summary

	With no Repetition	With Unlimited Repetition	With Limited Repetition
Permutation	n_{p_r}	n^r	$\frac{n!}{n_1!\cdot n_2!\ldots\ldots}$
Combination	^nC_r	$^{n-1+r}C_r$	Generating function

- The number of combinations (selections) of n different things taking r at a time is considered as a combination. It is written as ^nC_r (n choose r)

  $C(n, r) = {}^nC_r = \frac{n!}{r!(n-r)!}$

- Selecting r objects out of n is the same as selecting (n-r) object out of n

  $^nC_r = {}^nC_{c-r}$

  {k-selection equal to k rejection}
- Total selections that can be made from 'n' distinct items is given $\sum_{k=0}^{n} {}^nC_k = 2^n$
- $^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$
- The total number of selections of one or more objects from n different objects

  $= 2^n - 1 = ({}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n)$

- Total ways to select any number of items from n identical items
  = n + 1 (when selection of 0 things is allowed)
  = n (when at least one thing is to be selected)
- The total number of selections from p like things, q like things of another type, and r distinct things
  = $(p + 1)(q + 1)2^r - 1$ (if at least one thing is to be selected)
  = $(p + 1)(q + 1)2^r - 2$ (if none or all cannot be selected)
- The total number of selections of r things from n different things when each thing can be repeated an unlimited number of times $= {}^{n-1+r}C_r$.
- Derangement is an arrangement such that no item should be there in its correct position

  $\boxed{D_n = \sum_{r=2}^{n} (-1)^r \frac{n!}{r!}}$

- The coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \ldots x_t)^n$

  where $n = n_1 + n_2 + \ldots n_t$ is $\frac{n!}{n_1! n_2! \cdots n_t!}$

- The number of shortest paths from point (i, j) to a point (p, q) is $(p+q-i-j)_{C_{p-i}}$

Inclusion-exclusion principle:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
- $n(A \oplus B) = n(A) - n(A \cap B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(B - A) = n(B) - n(A \cap B)$
- $n(\bar{A} \cap \bar{B}) = n(\cup) - n(A \cup B)$
- $n(\bar{A} \cup \bar{B}) = n(\cup) - n(A \cap B)$