3 Differential Equations

Objective

Upon completion of this chapter you will be able to:

- Solve differential equations of first order.
- Determine complimentary function for higher order differential equations.
- Determine particular integral for higher order differential equations.

Introduction

Differential equations form a very important area of engineering applications. Most of the physical problems involve modelling a physical system in terms of differential equations and then determining the solution of differential equations to find system parameters. Like in electrical engineering any circuit involving R, L and C is solved using differential equations. In Mechanical Engineering Heat Equation and even basic equations of motion are an example of differential equations. So, while dealing with a physical system, differential equations are inescapable.

Types of Differential Equations

Ordinary Differential Equations

These equations have all the differential co4 efficients with respect to a single independent variable.

Example:

$$\frac{dy}{dx}=\frac{x}{y};\ \frac{d^2y}{dx^2}+ky=0;$$

$$\left(\frac{d^2y}{dx^2}\right)^4+3\frac{dy}{dx}+5y=x^2$$

Partial Differential Equations

These equations have two or more independent variables and differential coefficients are with respect to any of them.

Example:

$$\frac{dy}{dx}=x\frac{dy}{dz};\frac{dy}{dt}+ky=4\left(\frac{dy}{dx}\right)^2+2\sin 4x$$

Order and Degree of a Differential Equation

Order: It is the order of the highest derivative appearing in a differential equation.

Degree: It is the degree of the highest derivative appearing in a differential equation after the equation has been expressed in a form free from radicals and fractions as far as derivatives are concerned.

Example: $\left(\frac{d^2y}{dx}\right)^4+3\frac{dy}{dx}+5y=x^2$,

this equation has order 2 and degree 4.

Solved Examples

Example: The order and degree of the differential equation

$$\frac{d^3y}{dx^3}+4\sqrt{\left(\frac{dy}{dx}\right)^3+y^2}=0 \text{ are ?}$$

Solution: Removing radicals from the equation

$$\left(\frac{d^3y}{dx^3}\right)^2=16\left[\left(\frac{dy}{dx}\right)^3+y^2\right]$$

The order is 3 as the highest derivative is of 3rd order. The degree is 2 as the power of highest derivative is 2.

Solution of a Differential Equation

The solution of a differential equation refers to the relation between dependent and independent variable in the equation which can satisfy the given differential equation.

The solution involves arbitrary constants and thus the solution of a differential equation represents a family of curves where a different curve is defined for each value of the arbitrary constant.

Any particular solution or curve can be obtained by substituting the value of arbitrary constant.

Example: $y=\frac{x^2}{2}+c$ is a solution of the differential equation $\frac{dy}{dx}=x$.

Here, c represents an arbitrary constant.

A singular solution of a differential equation refers to a solution that cannot be obtained by substituting arbitrary constants in general solution.

Equations of first order and first degree

Variable Separation Method

The general form of a 1^{st} order 1^{st} degree differential equation is given by

$$M+N\frac{dy}{dx}=0 \qquad ...(1)$$

$$Mdx+Ndy=0 \qquad ...(2)$$

Where M, N are functions of variables x and y. Suppose it is possible to express (1) as

$$\frac{dy}{dx}=\frac{f(x)}{g(y)}$$

Then, f (x) dx = g (y) dy

$\int f(x)dx=\int g(y)dy$ gives the solution of differential equation.

Solved Examples

Example: A spherical naphthalene ball exposed to air, losses its volume proportional to its instantaneous surface area due to evaporation. If the initial diameter of the ball is 2 cm, reduces to 1 cm after 3 months , the ball completely evaporates in ___________?

Solution: Volume, $V_1=\frac{4}{3}\pi r^3$

Area, $A = 4\pi r^2$

Assume at time t_1 volume is V_1 and at time t_2 Volume is V_2 Since, it is given

$$\frac{dv}{dt}\propto A \text{ i.e.} \frac{dv}{dt}=-KA$$

Negative sign is used as area decreases when naphthalene evaporates.

$$\text{i.e.} \frac{d}{dt}\left(\frac{4}{3\pi r^3}\right)=-K\left(4\pi r^2\right)$$

$$\frac{4\pi}{3}\times 3r^2\times\frac{dr^2}{dt}=-K4\pi r^2$$

$$\frac{dr}{dt}=-K \text{ .... } (1)$$

i) At t = 0 $\rightarrow$ r = 1 cm

ii) At t = 3 months $\rightarrow r=\frac{1}{2}$cm

iii) At t = n $\rightarrow$ r = 0

From (1) $\frac{dr}{dt}=-K$

dr = –Kdt

Integrating both sides,

r = –Kt + c .... (2)

Substitute (i) in (2) $\Rightarrow$ 1 = c

Substitute (ii) in (2) 3K= $c-\frac{1}{2}$ => $K=\frac{1}{6}$

Substitute (iii) in (2)

$$0=-\frac{1}{6}n+1$$

n = 6 months

Example: A body originally 60°C cools down to 40°C in 15 minutes while kept in air at a temperature of 25°C. Then what will be the temperature of body after 30 minutes?

Solution: To solve the above problem we can apply Newton's law of cooling. According to this law, the temperature of a body changes proportional to the difference between the body temperature and surrounding medium temperature. Assume that θ be the temperature of body and θ_0 the surrounding medium temperature then

$$\frac{d\theta}{dt}\alpha\ \theta-\theta_0$$

$$\frac{d\theta}{dt}=K\left(\theta-\theta_0\right) \qquad ...(1)$$

Where K $\to$ proportionality constant and '-ve' sign indicates body temperature decreases as time increases.

i) at $t = 0,\ \theta = 60°C$ ...(i)

$t = 15,\ \theta = 40°C$ ...(ii)

$t = 30,\ \theta = ?$ ...(iii)

From (1) $\frac{d\theta}{dt} = -K(\theta - \theta_0)$

i.e. $\frac{d\theta}{\theta - \theta_0} = -Kdt$

$$\int \frac{d\theta}{\theta - \theta_0} = -K\int dt$$

$\theta_0 = 25\ \log(\theta - 25) = -Kt + c$ ...(2)

Substitute (i) in (2) $\log 35 = c$ ...(3)

$\log(\theta - 25) = -Kt + \log 35$

$Kt = \log 35 - \log(\theta - 25)$ ...(4)

Substitute (ii) in (4) $15K = \log 35 - \log 15$ ...(5)

(4) $\frac{t}{15} = \frac{\log\left(\frac{35}{\theta - 25}\right)}{\log\left(\frac{35}{15}\right)}$

put $t = 30$

$$\frac{30}{15}\log\frac{35}{15} = \log\frac{35}{\theta - 25}$$

$$\left(\frac{35}{15}\right)^2 = \frac{35}{\theta - 25}$$

i.e. $\theta - 25 = \frac{45}{7} = 6.428$

$\theta = 31.43°C$

Example: If a thermometer is taken outdoor where the temperature is 0°C from a room in which temperature is 21°C and the reading drops to 10°C in 1 minute. Then how long after its removal will the temperature reading be 5°C.

Solution: To solve the above problem we can apply Newton's law of cooling.

$\frac{d\theta}{dt} = K(\theta - \theta_0)$ ...(1)

The temperature at various instants of time is given by,

i) at $t = 0, \theta = 21°C$

ii) at $t = 1\,\text{min}, \theta = 10°C$

iii) at $t = ?, \theta = 5°C$

$$\int \frac{d\theta}{\theta - \theta_0} = -K\int dt$$

$\theta_0 = 0 \Rightarrow \log\theta = -Kt + c$ ...(1)

Substitute (ii) in (1)

$\log 21 = c$

Hence, $Kt = \log 21 - \log\theta = \log\frac{21}{\theta}$ ...(2)

Substitute (ii) in (2)

$K \times 1 = \log 21 - \log 10 = \log\frac{21}{10}$ ...(3)

When temperature is 5°C

$\log\frac{21}{5}$

$t = \log\frac{21}{10} = 1.96$ min

Example: Solve the following differential equation: $x^4\frac{dy}{dx} + x^3y + \operatorname{cosec} xy = 0$

Solution: $x^3\left(x\frac{dy}{dx} + y\right) + \operatorname{cosec} xy = 0$

Let xy=z

$$\frac{dz}{dx} = x\frac{dy}{dx} + y$$

$$x^3\frac{dz}{dx} + \operatorname{cosec} z = 0$$

$$x^3\frac{dz}{dx} = -\operatorname{cosec} z$$

$$\frac{dx}{x^3} = \frac{dz}{-\operatorname{cosec} z} = \sin z\, dz$$

Integrating both sides,

$$\int \sin z\, dz = -\int\frac{dx}{x^3}$$

$$-\cos z = -\left(\frac{-1}{2}x^{-2}\right) + c$$

$$\cos z = \frac{-1}{2}x^2 + c$$

Example: Solve the following differential equation: $\frac{dz}{dx} + 1 - x\tan z = 1$

Solution: $\frac{dz}{dx} = x \tan z$

$$\frac{dz}{\tan z} = xdx$$

$$\int \cot z dx = \int xdx$$

$$\ln(\sin z) = \frac{x^2}{2} + c$$

Homogenous Equations

The differential equations of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$$

Where f (x, y) and $\phi(x, y)$ are homogenous functions of same degree in x and y.

A homogenous function of degree 'n' is represented as, $a\, x^n + a_1 x^{n-1}y + a_2 x^{n-2}y^2$ .... $+ a_n y^n$. Here each term has the degree 'n'.

Example: $\left(x^2 - y^2\right)\tan\frac{y}{x}$ is a homogenous function of degree 2.

To solve such kind of differential equations, we substitute,

$$y = vx \text{ and } \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Solve resultant differential equation by variable separation.

Solved Examples

Example: Solve $(y^2 - x^2)\, dx - 2xydy = 0$

Solution: Given equation can be represented as,

$\frac{dy}{dx} = \frac{y^2 + x^2}{2xy}$ which is homogenous in x and y

Substitute, $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then, the equation becomes

$$v + x\frac{dv}{dx} = \frac{1}{2}\left[\frac{v^2 - 1}{v}\right]$$

$$x\frac{dv}{dx} = -\left[\frac{v^2 + 1}{2v}\right]$$

Separating the variables, $\frac{2v}{v^2 + 1}dv = -\frac{dx}{x}$

Integrate both sides of the equation,

$$\int \frac{2v}{v^2 + 1}dv = -\frac{dx}{\int x}$$

$$\ln\left(v^2 + 1\right) = -\ln x + c = \ln\frac{1}{x} + c = \ln\frac{k}{x}$$

$$v^2 + 1 = \frac{k}{x}$$

Replace v by $\frac{y}{x}$, we get

$$\left(\frac{y}{x}\right)^2 + 1 = \frac{k}{x},$$

$$x^2 + y^2 = kx$$

This represents a family of circles with co-ordinates and radius dependent on k.

Linear Differential Equations

The general form of 1st order linear differential equation in variable y is given by,

$$\frac{dy}{dx} + Py = Q \qquad ...(1)$$

where P and Q are functions of x

Multiply (1) with R where R is some other function of x

$$R\frac{dy}{dx} + RPy = RQ$$

$$\text{Let } RP = \frac{dR}{dx}$$

$$\therefore RP = \frac{dR}{dx}$$

$$\frac{dR}{R} = Pdx$$

$$\therefore R = \exp\left(\int Pdx\right)$$

$$\frac{d[Ry]}{dx} = RQ$$

$$Ry = \int RQdx$$

The solution of above differential equation is then,

$$y = e^{-\int Pdx} \times \int Qe^{\int Pdx}dx + c$$

The factor $R = \exp\left(\int P dx\right)$ is called as integrating factor.

Solved Examples

Example: Determine the solution of $t\frac{dx}{dt} + x = t$ satisfying the condition x(1) = 0.5

Solution: $t\frac{dx}{dt} + x = t$

Divide both sides by 't'

$$\frac{dx}{dt} + \frac{X}{t} = 1$$

$$IF = e^{\int \frac{1}{t}dt} = e^{\log t} = t$$

Solution of the differential equation is then,

$$x \times t = \int 1 \times t\, dt = \frac{t^t}{2} + c$$

Substituting initial conditions,

$$0.5 = \frac{1}{2} + c \Rightarrow c = 0$$

$$xt = \frac{t^2}{2} \Rightarrow x = \frac{t}{2}$$

Example: Determine the solution of $\frac{dy}{dx} + 2y$ tanx = sinx given y = 0 when $x = \frac{\pi}{3}$.

Solution:

$$IF = e^{\int 2\tan x}dx = e^{2\log \sec x} = e^{\log \sec^2 x} = \sec^2 x$$

$y \times \sec^2 x = \int \sin x \sec^2 x\, dx + c = \int \tan x \sec x\, dx + c = \sec x + c$

Substitute Initial Conditions,

0 = 2 + c

$\therefore c = -2$

$y = \cos x - 2\cos^2 x$

Example: Determine the solution of

$$x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$$

Solution: $\frac{dy}{dx} - \frac{y}{x(x-1)} = x(x-1)$

$$IF = e^{\int \frac{-1}{x(x-1)}}dx = e^{\int \frac{(x-1)-x}{x(x-1)}}dx = e^{\log x - \log(x-1)} = \frac{x}{x-1}$$

The solution of differential equation is given by,

$$y\left(\frac{x}{x}\right) = \int x(x-1) \times \frac{x}{x-1}dx$$

$$y\left(\frac{x}{x-1}\right) = \frac{x^3}{3} + c$$

Example: Determine the solution of following differential equation

$(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

Solution: This equation can be rephrased as,

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

$$IF = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

The solution of this differential equation is,

$$x \times e^{\tan-1\, y} = \int \frac{\tan^{-1} y}{1+y^2} \times e^{\tan-1y}dy$$

$t = \tan^{-1} y$

$$dt = \frac{1}{1+y^2}dy$$

$xe^t = \int t \times e^t dt = t \times e^t - \int e^t = e^t\,(t-1)$

Thus, the solution is $x = t - 1 = \tan^{-1} y - 1$

Example: Determine the solution of the equation: $2\frac{dy}{dx} - y\sec x = y^3 \tan x$

Solution: Divide both sides by y^3

$$\frac{2\,dy}{y^3 dx} - \frac{\sec x}{y^2} = \tan x$$

Multiply both sides by -1

$$\frac{-2\,dy}{y^3\,dx} + \frac{\sec x}{y^2} = -\tan x$$

$$\text{Assume}\,\frac{1}{y^2} = z$$

$$\frac{-2\,dy}{y^3\,dx} = \frac{dz}{dx}$$

Thus, this equation can be re-written as,

$$\therefore \frac{dz}{dx} + \sec x \times z = -\tan x$$

$$IF = e^{\int \sec x dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

The solution of above differential equation is then,

z (secx + tanx) = ∫(secxtanx + $\tan^2$ x) dx

Since, $\tan^2 x = \sec^2 x - 1$

z (secx + tanx) = ∫(–secxtanx – $\sec^2$ x + 1) dx

z (secx + tanx) = –secx – tanx + x + c

$$\frac{1}{\tan x\, y^2} \sec x = (-\sec x - \tan x + x + c)$$

Inspection method

Consider the equation Mdx + Ndy = 0

Suppose it is possible to rearrange the terms on LHS of (1) by observing the following formulae we can solve the equation through inspection.

Some useful formulae

$$d\left[\log\left(\frac{x}{y}\right)\right] = \frac{xdy + ydx}{xy}$$

$$d\left[\log\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$$

$$d\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \frac{ydx_2 - xdy_2}{x + y}$$

$$d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{x^2 + y^2}$$

$$d\left[\log\left(x^2 + y^2\right)\right] = 2\left(xdx + ydy\right)$$

$$d\left(\frac{e^x}{y}\right) = \left(\frac{ye^x dx - e^x dy}{y^2}\right)$$

$$d\left(\frac{e^y}{x}\right) = \left(\frac{xe^y dy - e^y dx}{x^2}\right)$$

$$d\left(\frac{x^2}{y}\right) = \left(\frac{2xydx - x^2 dy}{y^2}\right)$$

$$d\left(\frac{y^2}{x}\right) = \left(\frac{2xydy - y^2 dx}{x^2}\right)$$

$$d\left(\frac{x^2}{y^2}\right) = \left(\frac{2xy^2 dx - 2x^2 ydy}{y^4}\right)$$

$$d\left(\frac{y^2}{x^2}\right) = \left(\frac{2x^2 ydy - 2xy^2 dx}{x^4}\right)$$

Solved Examples

Example: Determine the solution of following differential equation

$$\left[y + \cos y + \frac{1}{2\sqrt{x}}\right]dx + \left(x - x\sin y - 1\right)dy = 0$$

Solution: (y dx + x dy) + (cosy dx – xsiny dy)

$$+\frac{1}{2\sqrt{x}}dx - dy = 0$$

This can be expressed as,

$$d(xy) + d(x\cos y) + \frac{1}{2\sqrt{x}}dx - dy = 0$$

Integrating the above equation we get,

$$xy + x\cos y + \sqrt{x} - y = c$$

Example: Determine the solution of

($y + 1 + x^2$) dx + (x^2 siny – x) dy = 0

Solution: The above equation can be rewritten as,

(ydx – xdy) + dx + x^2dx + x^2 siny dy = 0

Divide both sides by x^2

$$\frac{y\,dx - x\,dy}{x^2} + \frac{dx}{x^2} + dx + \sin y\, dy = 0$$

$$-d\left(\frac{y}{x}\right) - d\left(\frac{1}{x}\right) + dx - d\left(\cos y\right) = 0$$

Integrating both sides,

$$-\left(\frac{y}{x}\right) - \frac{1}{x} + x - \cos y = c$$

Example: Determine the solution of differential equation (x^4e^x – $2mxy^2$) dx + $2mx^2y$ dy = 0

Solution: This equation can be rearranged as,

x^4e^xdx – m($2xy^2$) dx – $2x^2y$dy = 0

Divide both sides by x^4

$$e^x dx + m\left(\frac{-2xy^2dx + 2x^2ydy}{x^4}\right) = 0$$

This equation can be expressed in form of differentials as,

$$e^x dx + m \times d\left(\frac{y^2}{x^2}\right) = 0$$

Integrating both sides as,

$$e^x + m\left(\frac{y^2}{x^2}\right) = c$$

Example: Solve the following differential equation,

$$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$$

Solution: The above equation can be expressed in form of differentials as,

$$d\left(\frac{x^2 + y^2}{\left(xdy^2 - ydx\right)}\right) = \sqrt{\frac{a^2 - \left(x^2 + y^2\right)}{x^2 + y^2}}$$

$$\frac{1}{2} - \frac{\sqrt{\left(x^2 + y^2\right)}}{\sqrt{a^2 - \left(x^2 + y^2\right)}} d\left(x_2 + y_2\right) = xdy - ydx$$

Multiply both sides by, $\frac{1}{x^2 + y^2}$

$$\frac{1}{2}\sqrt{x^2 + y^2}\;\frac{1}{\sqrt{a^2 - \left(x^2 + y^2\right)}} d\left(x^2 + y^2\right)$$

$$= \frac{xdy - ydx}{x^2 + y^2}$$

$$\text{Let}\sqrt{x^2 + y^2} = z \Rightarrow dz = \frac{1}{2\sqrt{x^2 + y^2}} \times d\left(x^2 + y^2\right)$$

$$\therefore \frac{dz}{\sqrt{a^2 - \left(x^2 + y^2\right)}} = d\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{dz}{\sqrt{a^2 - z^2}} = d\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{1}{a}\sin^{-1}\left(\frac{z}{a}\right) = \tan^{-1}\left(\frac{y}{x}\right) + c$$

$$\frac{1}{a}\sin^{-1}\left(\frac{\sqrt{x^2 + y^2}}{a}\right) = \tan^{-1}\left(\frac{y}{\cancel{x}}\right) + c$$

Bernoulli's Equation

The equation of the form $\frac{dy}{dx} + Py = Qy^n$

Where, P and Q are functions of x is called as Bernoulli's equation. To solve this type of equation we have to divide both sides by yn The equation now becomes, $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$

Substitute, $z = y^{1-n}$

Thus, $\frac{dz}{dx} = 1\left(-n\ y\right)^{-n}\frac{dy}{dx}$

Thus, this equation reduces to

$$\frac{1\,dz}{1 - n\,dx} + Pz = Q$$

$$\frac{dz}{dx} + P\left(1 - n\right)z = Q\left(1 - n\right)$$

This is a linear equation that can be solved easily.

Exact Differential Equation

Consider the equation Mdx + Ndy = 0 .... (1)

If it is possible to express the above equation as df (x, y) = 0 Then by Integration f (x, y) = c is the solution of (1)

Then we say that the above differential equation is an exact differential equation. The necessary and sufficient condition for an equation to be exact is,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

To get the required solution we can apply any one of the formulae.

∫Mdx + ∫terms of N not containing 'x' dy = c

Treat y as constant

∫Ndy + ∫terms of M not containing 'y' dx = c

Treat x as constant

Solved Examples

Example: Solve the following differential equation:

$$\frac{2x}{y^3}dx + \left(\frac{y^2 - 3x^2}{y^4}\right)dy = 0$$

Solution:

$$\frac{\partial M}{\partial y} = 2x \times \frac{-3}{y^4} = -\frac{6x}{y^4}$$

$$\frac{\partial N}{\partial x} = 0 - \frac{3}{y^4} \times 2x = -\frac{6x}{y^4}$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ this equation is an exact differential equation.

Solution can be given as,

∫Mdx + ∫terms of N not containing 'x' dy = c

$$\frac{2}{y^3} \times \frac{x^2}{2} + \int y^{-2} dy = c$$

$$\frac{x^2}{y^3} \frac{1}{y} = c$$

Note: even if you apply another formula for the solution you will get the same expression.

Example: Solve the following differential equation (2xy + y – tany) dx + (x^2 – xtan^2 y + sec^2 y) dy = 0

Solution: $\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y = 2x - \tan^2 y$

$$\frac{\partial N}{\partial x} = 2x - \tan^2 y$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ this equation is exact

The solution can be computed as,

∫Mdx + ∫terms of N not containing 'x' dy = c

$$y\frac{X^2}{2} + yx - x\tan y + \tan y = c$$

Example: Solve the following differential equation ($y^2e^{xy^2}$ + $4x^3$) dx + ($2xye^{xy^2}$ – $3y^2$) dy = 0

Solution:

$$\frac{\partial M}{\partial y} = y^2 e^{xy^2} \times 2xy + e^{xy^2} \times 2y$$

$$\frac{\partial N}{\partial x} = 2xy \times e^{xy^2}\left(y^2\right) + e^{xy^2} \times 2y = \frac{\partial M}{\partial y}$$

Thus, this equation is exact. Solution is given by,

∫Mdx + ∫terms of N not containing 'x' dy = c

$$\frac{y^2 e^{xy^2}}{y^2} + 4\frac{x^4}{4} + \left(\frac{-3y^3}{3}\right) = c$$

$$e^{xy^2} + x^4 - y^3 = c$$

Example: Solve the following differential equation (xycosxy + sinxy) dx + x^2 cos (xy) dy = 0

Solution: $\frac{\partial M}{\partial y} = -xy\sin xy \times x + x\cos xy + x\cos xy$

$= -x^2y\sin xy + 2x\cos xy$

$$\frac{\partial M}{\partial x} = -x^2 y \sin xy + 2x \cos xy = \frac{\partial M}{\partial y}$$

Solution is given by,

∫Ndy + ∫terms of M not containing 'y' dx = c

$$x^2 \times \left[\frac{\sin xy}{x}\right] = c$$

Thus, x sinxy = c is the solution.

Example: Solve $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

Solution:

$$\frac{\partial M}{\partial y} = e^{x/y} \times \frac{-x}{y^2}$$

$$\frac{\partial N}{\partial X} = e^{x/y} - \left(\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right)e^{x/y} \times \frac{1}{y}$$

$$= -x\frac{e^{x/y}}{y} = \frac{\partial M}{\partial y}$$

Thus, this equation is exact.

Solution is given by,

∫Mdx + ∫terms of N not containing 'x' dy = c

$$x + \frac{e^{x/y}}{1} = c$$

$$x + \frac{y}{ye^{x/y}} = c$$

Equations reducible to exact equation form

Suppose, the differential equation given is ydx – x dy =0 ...(2)

$$\frac{ydx - xdy}{y^2} = 0$$

$$d\left(\frac{x}{y}\right) = 0$$

$$\frac{x}{y} = c$$

In the above equation (2) if the given equation is made an exact equation by multiplying by $\frac{1}{y^2}$, therefore it is called an integrating factor (IF).

We can also observe that $-\frac{1}{x^2}, \frac{1}{xy}, \frac{-1}{xy}, \frac{1}{\left(x^2+y^2\right)}, \frac{-1}{x^2+y^2}$ etc. are suitable IFs of (2). Therefore the IF of a differential equation is not unique.

If the given equation Mdx + Ndy = 0 is not exact equation.

i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

then it can be made exact equation by manipulating with a suitable IF. But to get the appropriate IF of (1) we can follow certain rules.

Rule I

If the given equation Mdx + Ndy = 0 isn't an exact equation but it is a homogenous equation and Mx + Ny ≠ 0, then $\frac{1}{Mx+Ny}$ is an integrating factor of the given equation.

Example: Determine the solution of $(x^2y - 2xy^2)\,dx + (3x^2y - x^3)\,dy = 0$

Solution: If we observe that every term of both M and N of the problem is with same degree, then we can recognize it is a homogenous equation.

$Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y = x^2y^2$

$$IF = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$$

Multiply IF into the given differential equation

$$\frac{x^2y-2xy^2}{x^2y^2}dx + \frac{3x^2y-x^3}{x^2y^2}dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y}\right)dy = 0$$

$$\left(-\frac{2}{x}\right)dx + \left(\frac{3}{y}\right)dy + d\left(\frac{x}{y}\right) = 0$$

Integrating both sides

$$\frac{x}{y} - 2\log x + 3\log y = c$$

Example: Solve the differential equation: $x^2ydx - (x^3 + y^3)\,dy = 0$

Solution: This is a homogenous equation of order 3 but it is not exact $Mx + Ny = x^3y - x^3y - y^4 = -y^4 \neq 0$

$$IF = \frac{1}{Ny} = \frac{1Mx}{y^4}$$

$$\frac{-x^2y}{y^4}dx + \frac{x^3+y^3}{y^4}dy = 0$$

$$\frac{-x^2}{y^3}dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

$$\frac{dy}{y} - d\left(\frac{x^3}{3y^3}\right) = 0$$

Integrating both sides, we get $-\frac{x^3}{3y^3} + \log y = c$

Example: Solve the differential equation: $x^2ydx - (x^3 + y^3)dy = 0$

Solution: This is a homogenous equation of order 3 but it is not exact

$Mx + Ny - x^3y - x^3y - y^4 = -y^4 \neq 0$

$$IF = \frac{1}{Ny} = \frac{1}{y^4}Mx$$

$$\frac{-x^2y^4}{-x^2}dx + \frac{x^3+y^3}{X^3y^4\,\frac{1}{y}}dy = 0$$

$$\frac{-x^2}{y^3}dx + y^4 + dy = 0$$

$$\frac{dy}{y} - d\left(\frac{x^3}{3y^3}\right) = 0$$

Integrate both sides, we get $\frac{x^3}{3y^3} + \log y = c$

Rule II

In the given equation Mdx + Ndy = 0 is not an exact equation but it is in the form yf (xy) dx + xg (xy)dy = 0 and (Mx–Ny) ≠ 0

Then $IF = \frac{1}{Mx - Ny}$

Solved Examples

Example: Solve $(x^2y^2 + xy + 1)\,ydx + (x^2y^2 - xy + 1)\,x\,dy = 0$

Solution: The above differential equation is not exact but it is in the form,

$yf(xy)dx + xg(xy)dy = 0$

$Mx - Ny = x^3y^3 + x^2y^2 + xy - (x^3y^3 - x^2y^2 + xy) = 2x^2y^2$

$$IF = \frac{1}{2x^2y^2}$$

$$\frac{x^2y^2 + xy + 1}{2x^2y^2} ydx + \frac{x^2y^2 - xy + 1}{2x^2y^2} xdy = 0$$

$$\left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx + \left(x - \frac{1}{y} + \frac{1}{xy^2}\right)dy = 0$$

$$\frac{dx}{x} - \frac{dy}{y} - d\left(\frac{1}{xy}\right) + d(xy) = 0$$

Integrate both sides

$$xy + \log x - \frac{1}{xy} - \log y = c$$

Example: Solve

$$[xy\cos(xy) + \sin(xy)]\, ydx$$
$$[xy\cos(xy) + \sin(xy)]\, ydx = 0$$

Solution: The above differential equation is not exact but it is in the form,

$yf(xy)\,dx + xg(xy)dy = 0$

$Mx - Ny = 2xy\sin xy$

$$IF = \frac{1}{2xy\sin xy}$$

$$\frac{xy\cos(xy) + \sin xy}{2xy\sin xy} ydx + \frac{xy\cos(xy)}{2xy\sin xy} xdy = 0$$

$$\frac{1}{2}[y\cot(xy)dx + x\cot(xy)dy]$$

$$+\frac{1}{2x}dx - \frac{1}{2y}dy = 0$$

$$\frac{1}{2}d\log \mathrm{sim}(xy) + \frac{1}{2x}dx - \frac{1}{2}\log\frac{x}{y} = \frac{1}{2}\log c$$

$$\frac{1}{2}\log[y\sin(xy)] = \frac{1}{2}\log c$$

$$x\sin xy = cv$$

Rule III

If the given equation $Mdx + Ndy = 0$ is not exact equation but $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ [only a function of x] then $e^{\int f(x)dx}$ is the integrating factor of the differential equation.

Rule IV

If the given equation $Mdx + Ndy = 0$ is not exact equation but $1\left(\frac{\partial N}{M} - \frac{\partial M}{\partial}\right) = g(y)$ (only a function of y) then $e^{\int g(y)dy}$ is the Integrating Factor of differential equation.

Note: While applying rule III or IV first we consider we must confirm that the equation is not exact.

Solved Examples

Example: Solve $(x^2 + y^2 + x)\,dx + xydy = 0$

Solution: Checking for the exactness of the differential equation

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 2y - y \neq 0$$

$$\text{Rule III}: \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{y}{xy} = \frac{1}{x} = f(x)$$

$$IF = e^{\int 1/x dx} = e^{\log x} = x$$

Multiply IF into the differential equation,

$(x^3 + xy^2 + x^2)\,dx + x^2ydy = 0$

$$x^3dx + x^2dx + d\left(\frac{x^2y^2}{2}\right) = 0$$

Integrate both sides

$$\frac{X^4}{4} + \frac{x^3}{3} + \frac{x^2y^2}{2} = C$$

Example: Solve $(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0$

Solution: Checking for exactness

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x \times 3y^2 + 1 - (4xy^2 + 2) = -(xy^2 + 1)$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial X} - \frac{\partial M}{\partial y}\right) = \frac{xy^2 + 1}{xy^2 + 1} \times \frac{1}{y} = g(y)$$

$$\text{Rule IV: } \therefore\ IF = e^{\int \frac{1}{y}dy} = y$$

Multiply IF on both sides of the equation,

$(xy^4 + y^2)\,dx + 2\,(x^2y^3 + xy + y^5)\,dy = 0$

$(y^2dx + 2xydy) + 2y^5dy + (xy^4dx + 2x^2y^3dy) = 0$

$$d(xy^2) + 2d\left(\frac{y^2}{6}\right) + d\left(\frac{x^2y^4}{2}\right) = 0$$

Integrate both sides,

$$Xy^2 + \frac{y^6}{3} + \frac{x^2y^4}{2} = 0$$

Clairaut's Equation

The differential equations of the form y = px + f(p) where $p = \frac{dy}{dx}$ dy is known as Clairaut's equation.

Differentiate both sides with respect to x

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$[x + f'(p)]\frac{dp}{dx} = 0$$

$$\text{Thus, either } \frac{dp}{dx} = 0 \text{ or } [x + f'(p)] = 0$$

$$\text{IF } \frac{dp}{dx} = 0 => p = c$$

Thus, solution of this differential equation is y = cx + f(c) where c is an arbitrary constant. Solution of a higher order linear differential equation with constant coefficient form.

Linear Differential Equation of n^{th} order

The general form is given by

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \text{.........}k_{n-2}\frac{d^2y}{dx^2} + k_{n-1}\frac{dy}{dx} + k_ny = x$$

Where K_1, K_2.................K_n are constants and X = function of x.

Such equations are most important in the study of electromechanical vibrations and other engineering problems.

If y_1, y_2, y_3 ...., yn are only two solutions of the equations

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \text{.........}k_{n-2}\frac{d^2y}{dx^2} + k_{n-1}\frac{dy}{dx} + k_ny = 0$$

Then, $u = c_1y_1 + c_2y_2 + c_3y_3$ .... + c_ny_n is also its solution. This means,

$$\frac{d^nu}{dx^n} + k_1\frac{d^{n-1}u}{dx^{n-1}} + k_2\frac{d^{n-2}u}{dx^{n-2}} + \text{.........}k_{n-2}\frac{d^2u}{dx^2} + k_{n-1}\frac{du}{dx} + k_nu = 0$$

If y = v be any particular solution of

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^n} + \text{..........}k_ny = x$$

$$\text{Then, } \frac{d^nv}{dx^n} + k_1\frac{d^{n-1}v}{dx^{n-1}} + \text{..........}k_nv = x$$

$$\text{Then, } \frac{d^n(u+v)}{dx^n} + k_1\frac{d^{n-1}(u+v)}{dx^{n-1}} + \text{..........}K(u+v) = x$$

Then, y = u + v is complete solution of the differential equation. The part 'u' is called as Complimentary Function (CF) and v is called as Particular Integral (PI)

$$\text{Let } D = \frac{d}{dx} : \text{Then, } D^n = \frac{d^n}{dx^n}$$

Thus, the differential equation can be represented as,

$(D^n + K_1 D^{n-1} + K_2 D^{n-2} + +$ .... $(K^{n-2}D^n + Kn + D)\ y = X$

i.e. f (D) y = X

Suppose X = 0

We get f(D) y = 0

Which is called homogeneous linear differential equation. Otherwise, if (X ≠ 0), it is called non-homogeneous linear differential equation.

Rules for finding the Complimentary Function

To solve the equation $\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}}{dx^{n-1}} + \text{..........} + k_ny = 0$ where k's are constants.

This equation can be represented as,

$(Dn + K_1 D^{n-1} +$ .... $+ K_n)\ y = 0$

This equation can be represented in symbolic co-efficient form as

$D^n + K_1D^{n-1} +$ .... + Kn = 0

This is called as Auxiliary equation (A.E.). Let m1, m2, m3, ..., mn be its roots.

Case 1: If all roots are real and different. Then, this equation is represented as,

$(D-m_1)(D-m_2)(D-m_3)$ .... $(D-m_n)\ y = 0$

If $(D-m_n)\ y = 0$, then

$$\frac{dy}{dx} - m_ny = 0$$

This is a linear equation, and the integrating factor is e^{-mnx}

The solution is $y = ce^{mnx}$

complimentary function is $y = c_1 e_{m1x} + c_2 e^{m2x} + \dots\dots\dots + c_n e^{mnx}$

Case 2: If two roots are equal i.e. $m_1 = m_2 = m$ and others are distinct Then, the complimentary function is given by

$y = (c_1 + c_2\, x)\, e^{mx} + c_3\, e^{m3x} \dots\dots\dots\dots\dots\dots + c_n\, e^{mnx}$

However, if there are 3 equal roots then the complimentary function is given by.

$y = (c_1 + c_2 x + c_3\, x^2)e^{mx} + c_4 e^{m4x} \dots\dots\dots\dots\dots\dots + c_n e^{mnx}$

Case 3: If one pair of roots are complex

$m_1 = \alpha + i\beta;\ m_2 = \alpha - i\beta$

Then, the solution is given by

$y = e^{\alpha x}\,(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m3x} \dots\dots\dots\dots\dots\dots + c_n e^{mnx}$

Case 4: If two pair of imaginary roots be equal

$m_1 = m_2 = \alpha + i\beta$

$m_3 = m_4 = \alpha - i\beta$

Then, the complimentary function is given by,

$y = e^{ax}\left[(c_1x + c_2)\cos\beta x + (c_3x + c_4)\sin\beta x\right] + c_5 e^{m5x} \dots\dots\dots\dots\dots\dots + c_n e^{mnx}$

Solution of f(D) y = 0 form:

Given Equations	General forms
1. $(D - a)\, y = 0$	1. $y = ce^{ax}$
2. $(D + a)\, b = 0$	2. $y = ce^{-bx}$
3. $Dy = 0$	3. $y = c$
4. $(D - a)(D - b)\, y = 0$	4. $y = c_1e^{ax} + c_2e^{bx}$
5. $(D - a)(D + b)(D - c)\, y = 0$	5. $y = c_1e^{ax} + c_2e^{-bx} + c_3e^{cx}$
6. $(D - a)(D - b)y = 0$ Where $a = \alpha + j\beta; b = \alpha - j\beta$	6. $y = e^{\alpha x}\left[k_1 \cos\beta x + k_2 \sin\beta x\right]$
7. $(D - a)(D - b)y = 0$ Where $a = \alpha + \sqrt{\beta}; b = \alpha - \sqrt{\beta}$	7. $y = e^{\alpha x}\left[k_1 \cos\sqrt{\beta}x + k_2 \sin\sqrt{\beta}x\right]$
8. $(D - a)^2\, y = 0$	8. $y = (c_1 + c_2x)\, e^{ax}$
9. $(D - a)^3\, y = 0$	9. $y = (c_1 + c_2x + c_3x^2)\, e^{ax}$
10. $(D - a)^4\, y = 0$	10. $y = (c_1 + c_2x + c_3x^3 + c_4x^4)\, e^{ax}$
11. $(D - a)^2\,(D + b)^3\, y = 0$	11. $y = (c_1 + c_2x)e^{ax} + (c_3x + c_4x + c_5x^2)\, e^{-bx}$
12. $(D - a)^2(D + b)^2\, y = 0$, $a = \alpha + j\beta$, $b = \alpha - j\beta$	12. $y = e^{\alpha x}\left[(k_1 + k_2x)\cos\beta x + (k_3 + k_4x)\sin\beta x\right]$
13. $(D - a)^2\left(D - b^2\right)y = 0$, $a = \alpha + j\sqrt{\beta}$, $b = \alpha - j\sqrt{\beta}$	13. $y = e^{\alpha x}\dfrac{\left[(k_1 + k_2x)\cosh\sqrt{\beta}x\right]}{\left[+(k_3 + k_4x)\sinh\sqrt{\beta}x\right]}$

14. $(D-a)^2(D-b)(D-c)(D+d)y=0$ $b=\alpha+j\beta$ and $c=\alpha-j\beta$	14. $y=(c_1+c_2x)e^{ax}+e^{\alpha x}(c_3\cos\beta+c_4\sin\beta x)$ $+c_5e^{-dx}$

Note: In writing the general solution for complete solution of the given differential equation it must be observed that the number of arbitrary constants in the solution is equal to the order of the differential equation given.

Solved Examples

Example: Solve the differential equation

$$\frac{d^3y}{dx^3}-\frac{3dy}{dx}+2y=0$$

Solution: Auxiliary equation is $(D^3-3D+2)y=0$

$(D^3-D-2D+2)y=0$

$$\left[D\left(D^2-1\right)-2\left(D-1\right)\right]y=0$$

$$\left(D-1\right)\left[D\left(D+1\right)-2\right]y=0$$

$$\left(D-1\right)\left[\left(D^2+D-2\right)\right]y=0$$

$$\left(D-1\right)\left[\left(D+2\right)\left(D-1\right)\right]y=0$$

$$\left(D-1\right)^2\left(D+2\right)y=0$$

$$D=1,1,-2$$

$$\therefore y=\left(c_1+c_2x\right)e^x+c_2e^{-2x}$$

Example: Solve the differential equation

$$\frac{d^4y}{dx^4}+8\frac{d^2y}{dx^2}+16y=0$$

Solution: The auxiliary equation is $(D^4+8D^2+16)y=0$

$(D^4+4D^2+4D^2+16)y=0$

$$\left[D^2\left(D^2+4\right)+4\left(D^2+4\right)\right]y=0$$

$$\left(D^2+4\right)^2=0$$

$$D^2=-4,-4$$

$$D=\pm2i,\pm2i$$

Thus, complementary function is given by

$$y_1=\left(c_1+c_2x\right)\cos 2x+\left(c_3+c_4x\right)\sin 2x$$

Example: Solve the differential equation

$$\frac{d^4y}{dx^4}-2\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$$

Solution: The auxiliary equation is $(D^4-2D^3-3D^2+4D+4)y=0$

$(D+1)(D^3-3D^2+4)y=0$

$(D+1)(D+1)(D^2-4D+4)=0$

$(D+1)^2(D-2)^2=0$

$D=-1,-1,2,2$

$\therefore y=(c_1+c_2x)e^{-x}+(c_3+c_4x)e^{2x}$

Example: The solution of $y''-2y'+10y=0$ satisfying $y(0)=4$ and $y'(0)=1$

Solution: The auxiliary equation is $(D^2-2D+10)y=0$

$D^2-2D+10=0$

$$D=\frac{2\pm\sqrt{4-40}}{2}=\frac{2\pm6i}{2}=1\pm3i$$

$$y=e^x\left(c_1\cos 3x+c_2\sin 3x\right)$$

$$y\left(0\right)=4$$

$$\therefore c_1=4$$

$$\frac{dy}{dx}=e^x\left(-4\times3\sin 3x+3c_2\cos 3x\right)$$

$$+\left(4\cos 3x+c_2\sin 3x\right)e^x$$

$$y'\left(0\right)=1$$

$$4+3c_2=1 \Rightarrow c_2=-1$$

$$\therefore y=e^x\left(4\cos 3x-\sin 3x\right)$$

Example: The solution of $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+17y=0$ satisfying the condition $y(0)=1, y'\left(\frac{\pi}{4}\right)=0$

Solution: The auxiliary equation is $(D^2+2D+17)y=0$

$D^2+2D+17=0$

$$\text{i.e. } D=\frac{-2+\sqrt{4-68}}{2}=\frac{-2\pm8i}{2}=-1\pm4i$$

$$y=e^{-x}\left(c_1\cos 4x+c_2\sin 4x\right)$$

$$y\left(0\right)=1$$

$$\frac{dy}{dx}=e^{-x}\left(4c_1\sin 4x+4c_2\cos 4x\right)+$$

$-e^{-x}(c_1\cos 4x + c_2\sin 4x)$

$\frac{dy}{dx}\left(\frac{\pi}{4}\right) = 0$

i.e. $c_2 = \frac{1}{4}$

Complimentary Function is $y = e^{-x}\left(\begin{matrix}\cos 4x + \\ \frac{1}{4}\sin 4x\end{matrix}\right)$

Rules for finding the Particular Integral

The differential equation $\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}}{dx^{n-1}} + \ldots\ldots + k_n y = x$ where k's are constants can be represented as, $(D^n + K_1 D^{n-1} + \ldots\ldots + K_n)\, y = X$ or $f(D)y = X$

From this we can observe that $y = \frac{x}{f(D)}$ is also a solution to it which is called particular integral or particular solution and is denoted by y_P.

An example of this is $\frac{X}{D-a} = e^{ax}\int Xe^{-ax}dx$

Assume $\frac{X}{D-a} = y$

Thus, $(D-a)y = x$

This can be expressed in differential form as,

$\frac{dy}{dx} - ay = x$

Integrating factor is e^{-ax}, its solution

is $ye^{-ax} = \int xe^{-ax}dx$

Thus, $y = \frac{x}{D-a} = e^{ax}\int Xe^{-ax}dx$

Case 1: Consider the equation $f(D)\,y = e^{ax+b}$ then

$y_p = \frac{e^{ax+b}}{f(D)} = \frac{e^{ax+b}}{f(a)}$

since, $D\left(e^{ax+b}\right) = a\left(e^{ax+b}\right)$

$D^2\left(e^{ax+b}\right) = a^2\left(e^{ax+b}\right)$

$D^n\left(e^{ax+b}\right) = a^{(n)}\left(e^{ax+b}\right)$

Thus, we can express in general form as

$f(D)e^{ax+b} = f(a)e^{ax+b}$

i.e. $\frac{e^{ax+b}}{f(a)} = \frac{e^{ax+b}}{f(D)}$

If $f(a) = 0$ then above rule fails,

$\frac{e^{ax+b}}{f(D)} = x\frac{e^{ax+b}}{f'(a)}$

Again if $f'(a) = 0$ then $\frac{e^{ax+b}}{f(D)} = x^2\frac{e^{ax+b}}{f''(a)}$

Some examples of this case are,

i) $\frac{e^{2-3x}}{(D+1)(D+2)} = \frac{e^{2-3x}}{(-3+1)(-3+2)} = \frac{e^{2-3x}}{2}$

ii) $\frac{e^{2-3x}}{(D-a)} = e^{ax}\int e^{ax+b}\times e^{-ax}dx = xe^{ax+b}$

iii) $\frac{e^{ax+b}}{(D-a)^2} = \frac{1}{(D-a)}xe^{ax+b} = e^{ax}\int xe^{ax+b}\times e^{-ax}dx$

$= \frac{x^2e^{ax+b}}{2}$

iv) $\frac{e^{2-3x}}{(D-a)^3} = \frac{1}{(D-a)}\times\frac{x^2}{2}e^{ax+b} = e^{ax}\int\frac{x^2e^{ax+b}}{2}$

$\times e^{-ax}dx = \frac{x^3e^{ax+b}}{3!}$

v) $\frac{e^{2-3x}}{(D-a)^k} = \frac{x^ke^{ax+b}}{k!}$

vi) $\frac{e^{ax+b}}{(D-a)^m(D-n)(D+P)^r} = \frac{x^m}{m!}\frac{e^{ax+b}}{(a-n)(a+p)^r}$

Case 2:

$f(D)y = \sin(ax+b)$ or $\cos(ax+b)$

$y_P = \frac{\sin(ax+b)}{s(D)}$

$D(\sin(ax+b)) = a\cos(ax+b)$

$D^2(\sin(ax+b)) = -a^2\sin(ax+b)$

$D^2 = -a^2$

$y_P = \frac{\sin(ax+b)}{f(D^2)} = \frac{\sin(ax+b)}{f(-a^2)}$ Given $f(-a^2) \neq 0$

Similarly

$y_P = \frac{\cos(ax+b)}{f(D^2)} = \frac{\cos(ax+b)}{f(-a^2)}$ $\left[f(-a^2) \neq 0\right]$

Some examples of this case are,

i) $\frac{\cos(3x+4)}{(D^2+4)} = \frac{\cos(3x+4)}{(-9+4)} = \frac{\cos(3x+4)}{-5}$

ii) $\frac{\sin 2-x}{D^3+1}=\frac{\sin(2-x)}{D\times D^2+1}=\frac{\sin(2-x)}{D\times -1+1}$

$=\frac{\sin(2-x)}{1-D}$

$=(1+D)\frac{\sin(2-x)}{1}\frac{\sin(2-x)}{D^3+1}$

$=(1+D)\frac{\sin(2-x)}{1+1}$

$=\frac{1}{2}\left[\sin(2-x)-\cos(2-x)\right]$

iii) $\frac{\cos(ax+b)}{D^2+a^2}=\frac{x}{2a}\sin(ax+b)$

$\text{Since } f(-a^2)=-a^2+a^2=0$

$\cos(ax+b)=\frac{x}{f'(D)}\cos(ax+b)$

$=\left(\frac{x}{2D}\right)\cos(ax+b)=\frac{x}{2a}\sin(ax+b)$

iv) $\frac{\sin(ax+b)}{D^2+a^2}=\frac{-x}{2a}\cos(ax+b)$

$\text{Since, } f(-a^2)=-a^2+a^2=0\frac{\sin(ax+b)}{D^2+a^2}$

$=\frac{x}{f'(D)}\sin(ax+b)$

v) $\frac{\sin(3x+4)}{D^2+9}=\frac{-x}{2\times 3}\cos(3x+4)$

$=\frac{-x}{6}\cos(3x+4)$

vi) $\frac{\cos(2x-3)}{D^2+4}=\frac{x}{2}\sin\frac{(2x-3)}{2}$

$=\frac{x}{4}\sin(2x-3)$

Example: Solve the differential equation: $(D^3 - 5D^2 + 8D - 4)\, y = e^{2x}$

Solution: The homogenous equation for the following differential equation can be written as,

$(D^3 - 5D^2 + 8D - 4) = 0$

$(D - 1)\,(D2 - 4D + 4) = 0$

$(D - 1)\,(D - 2)^2 = 0$

The roots of this equation are,

$D = 1, 2, 2$

$y_c = c_1 e^x + (c_2 + c_3 x)e^{2x}$

$y_P=\frac{x}{f(D)}=\frac{e^{2x}}{(D-1)(D-2)^2}=x^2\frac{e^{2x}}{2!}$

$\text{Since, } f(2)=0 \text{ and } f'(2)=0$

$y_P=\frac{x}{f(D)}=x^2\frac{e^{2x}}{f''(a)}$

$f''(D)=6D-10$

$\text{Thus, } f''(2)=6\times 2-10=2$

Example: Determine the solution of $y''-8y'+16y = 3e^{4x}$ satisfying $y = 0$ at $x = 0$ and $x = 2$

Solution: The given differential equation can be represented as,

$(D^2 - 8D + 16)\, y = 0$

$(D - 4)^2 = 0$

$D = 4, 4$

Complimentary Function is given by, $y_c = (c_1 + c_2 x)e^{4x}$

Particular integral can be calculated as,

$y_P=\frac{3e^{4x}}{(D-4)^2}=\frac{x^2}{2!}3e^{4x}$

$\text{Since, } f(4)=0 \text{ and } f'(4)=0$

$y_P=\frac{3e^{4x}}{(D-4)^2}=\frac{x^2}{f''(4)}3e^{4x}=\frac{x^2}{2}3e^{4x}$

$\text{Complete solution is, } y_c=y_c+y_P$

$=(c_1+c_2x)e^{4x}+\frac{3x^2}{2}e4x$

Satisfying the initial conditions,

$y = 0$ at $x = 0$

$0 = c_1 + 0$

$c_1 = 0$

$y = 0$ at $x = 2$

$0 = 2c_2e^8 + 6e^8$

$c_2 = -3$

$\therefore y=-3xe^{4x}+\frac{3x^2}{2}e^{4x}$

Example: Solve the differential equation

$\frac{d^4y}{dx^4}-y=15\cos 2x$

Solution: The homogenous part of the differential equation can be represented as,

$(D^4 - 1)y = 0$

$(D^2)^2 - 1^2 = 0$

$(D^2 + 1)(D^2 - 1)^2 = 0$

$D = \pm 1, \pm i$

$y_c = c_1e^x + c2e^{-x} + e^0 (c_3\cos x + c_4\sin x)$
$y_c = c_1e^x + c2e^{-x} + c_3\cos x + c_4\sin x$

$$y_P = \frac{15\cos 2x}{(D^2+1)(D^2-1)} = \frac{15\cos 2x}{(-2^2+1)(-2^2-1)}$$

$$= \frac{15\cos 2x}{-3\times -5} = \cos 2x$$

$$y = y_c + y_P = c_1e^x + c_2e^{-x} + c_3 + \cos x + c_4 \sin x + \cos 2x$$

Example: Solve the following differential equation $\frac{d^2y}{dx^2} - 4y = \cosh(2x - 1 + 3)^x$

Solution: Differential form of the given equation is $(D^2 - 4)\, y = \cosh(2x - 1) + 3^x$

The homogenous part of the equation is, $(D^2 - 4) = 0$

$(D + 2)(D - 2) = 0$

$D = \pm 2$

$y_c = \cosh(2x-1) + 3^x$

Particular Integral can be computed as,

$$y_P = \frac{\cosh(2x-1)+3^x}{(D+2)(D-2)}$$

$$y_P = \frac{1}{2}\frac{\left[e^{2x-1}+e^{-2x+1}\right]}{(D+2)(D-2)} - \frac{e^{x\log 3}}{(D+2)(D-2)}$$

$$y_P = \frac{x}{2}\frac{\left[e^{2x-1}+e^{-2x+1}\right]}{f'(D)} + \frac{3^x}{(2+\log 3)(-2+\log 3)}$$

$$= \frac{x}{4}\frac{\left[e^{2x-1}+e^{-2x+1}\right]}{D} + \frac{3^x}{(2+\log 3)(-2+\log 3)}$$

$$y_P = \frac{x}{8}\left[e^{2x-1}+e^{-2x+1}\right] + \frac{3^x}{(2+\log 3)(-2+\log 3)}$$

$$= \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$$

Example: Solve $y'' + y = \sin x \sin 2x$

Solution: The homogenous part of the differential equation is $D^2 + 1 = 0$

$D = \pm i$

Thus, complementary function is $y_c = c_1 \cos x + c_2 \sin x$

$$X = \sin\sin 2x = \frac{1}{2}(2\sin x \sin 2x)$$

$$= \frac{1}{2}[\cos x - \cos 3x]$$

Particular Integral can be computed as,

$$y_P = \frac{\sin_2 \sin 2x}{D^2+1} = \frac{1}{2}\left[\frac{\cos x}{D^2+1} - \frac{\cos 3x}{D^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{x}{f'(D)}\cos x - \frac{\cos 3x}{-3^2+1}\right]$$

$$y_P = \frac{1}{2}\left[\frac{x}{2D}\cos x - \frac{\cos 31}{-3^2+1}\right]$$

$$y_P = \frac{1}{2}\left[\frac{x}{2}\sin x + \frac{\cos 3x}{8}\right]$$

Case 3:

Consider $f(D)y = x^m \,(m \in Z^+)$

$$\text{Then } y_P = \frac{x^m}{f(D)} = f(D)^{-1}x^m$$

$$\text{Then } y_P = \frac{x^m}{f(D)} = f(D)^{-1}x^m$$

By using binomial expansion and then apply on x^m. Some of the common binomial expansions are given as,

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots\ldots$

$(1 + x)^{-1} = 1 - x + x^2 + x^3 + \ldots\ldots\ldots\ldots\ldots$

$(1 \mp x)^{-2} = 1 \pm 2x + 3x^2 \pm 4x^3 + \ldots\ldots\ldots\ldots\ldots$

$(1 \mp x)^{-3} = 1 \pm 3x + 6x^2 \pm 10x^2 + \ldots\ldots\ldots\ldots\ldots$

Solved Examples

Example: Solve the differential equation $y'' - 4y' + 4y = x^3$

Solution: The given differential equation can be represented in differential form as,

$(D^2 - 4D + 4)y = x^3$

To determine the complementary function

$(D - 2)^2 = 0$

$D = 2, 2$

Thus, the complimentary function is given by,

$y_c = (c_1 + c_2\, x)e^{2x}$

To determine particular integral,

$$y_P = \frac{x^3}{(D-2)^2} = \frac{x^3}{\left(3D^2 - \frac{D^2}{2}\right)} = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2}$$

$$y_P = \frac{1}{4}\left[1+2\times\frac{D}{2}+\frac{1}{4}+\frac{4D^3}{8}\right]x^3 = \frac{1}{4}\left[\begin{array}{l}x^2+3x^2+\frac{3}{4}\\ \times 6x+\frac{1}{2}\times 6\end{array}\right]$$

$$= \frac{1}{4}\left[x^3+3x^2+\frac{9}{2}x+3\right]$$

Example: Solve the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^2+2x+4$$

Solution: The homogenous part of differential equation is $(D^2 + D) = 0$

$D(D + 1) = 0$

$D = 0, -1$

$y_c = c_1 + c_2e^{-x}$

To determine the particular integral

$$y_P = \frac{x+2x+4}{D(D+1)} = \frac{1}{D}(D+1)^{-1}\left(x^2+2x+4\right)$$

$$= \frac{1}{D}\left(1-D+D^2-D^3\right)\left(x^2+2x+4\right)$$

$$y_P = \left(\frac{1}{D}-1+D-D^2\right)\left(x^2+2x+4\right)$$

$$= \frac{x^3}{3}+\frac{2x^2}{2}+4x-x^2-2x-4+2x+2-2$$

We are not considering higher powers as higher order derivatives will go to zero,

$$y_P = \frac{x^3}{3}+4x-4$$

Example: Solve the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=x^2+x$$

Solution: The homogenous part of differential equation is $(D^2 - 3D + 2) = 0$

$(D - 1)(D - 2) = 0$

$D = 1, 2$

$$y_c = c_1e^x + c_2^{2x} = x^2 + x$$

$$y_P = \frac{x^2}{\left(D^2-3D+2\right)} = \left(1-\frac{3}{2}D+\frac{D^2}{2}\right)$$

$$= \frac{1}{2}\left[1-\left(\frac{3}{2}-\frac{D^2}{2}\right)\right]^{-1}\left(x^2+x\right)$$

$$y_P = \frac{1}{2}\left[1+\left(\frac{3}{2}D-\frac{D^2}{2}+\frac{9}{4}D^2\right)\right]\left(x^2+x\right)$$

$$= \frac{1}{2}\left[x^2+x+\frac{3}{2}(2x+1)-\frac{1}{2}\times 2+\frac{9}{4}\times 2\right]$$

$$y_P = \frac{1}{2}\left[x^2+4x+5\right]$$

Example: Determine the solution of

$$\frac{d_2y}{dx^2} = 3x-2 \text{ satisfying } y(0)=2 \text{ and } y'(1)=-3$$

Solution:

$$\frac{dy}{dx} = \frac{3x^2}{2}-2x+c_1$$

$$y = \frac{3x^3}{2\times 3}-\frac{2x^2}{2}+c_1x+c$$

$$\text{Since, } y'(1) = -3$$

$$-3 = \frac{3}{2}-2+c_1$$

$$c_1 = \frac{5}{2}$$

$$\text{Since, } y(0) = 2$$

$$\text{i.e. } 2 = \frac{0}{2}+0+c$$

$$c = 2$$

$$\text{Thus, the solution is, } y = \frac{x^3}{2}-x^2-\frac{5}{2}x+2$$

Case 4:

Consider the equations $f(D)\,y = e^{ax}.V$ where V is also a function of x. It may be sin (bx + c) or cos (ax + b) or x^m etc.

$f(D)y = e^{ax}.V$

$$y^P = \frac{e^{ax}.V}{f(d)} = e^{ax}\left[\frac{V}{f(D+a)}\right]$$

$$D\left[e^{ax}.V\right] = e^{ax}.D(V)+aVe^{ax} = e^{ax}\left[D+a\right]V$$

$$D^2\left[e^{ax}.V\right] = e^{ax}\left[D^2V+aDV\right]+(D+a)Ve^{ax}$$

$$= e^{ax}\left[D^2+2aD+a^2\right]V = e^{ax}(D+a)^2V$$

Solved Examples

Example: Solve the following differential equation $(D2 - 5D + 6)y = e^{2x}.x^3$

Solution: The homogenous part of the differential equation is $(D - 2)(D - 3) = 0$

$D = 2, 3$

Thus, the complementary function is

$y_c = c_1e^{2x} + c_2\, e^{3x}$

To determine the particular integral,

$$y_P = \left(D - e^2\right)\left(xD_3 - 3\right) = e^{2x}\left|\begin{matrix}\left(-D + 2 - 2\right)x^3 \\ \left(D + 2 - 3\right)\end{matrix}\right|$$

$$- e^{2x}\left[\frac{D^1\left(1 - D\right)}{x^3}\right]$$

$$y_P = -e^{2x}\left[\frac{1}{D}\left(1 - D\right)^{-1}x^3\right] = -e^{2x}\left[\begin{matrix}\frac{1}{D}1 + D + D^2 \\ +D^3 + D^4\end{matrix}\right]x^3$$

$$y_P = e^{-2x}\left[\left(\frac{1}{2} + 1 + D + D^2 + D^3\right)x^3\right]$$

$$= e^{-2x}\left[\frac{x^4}{4} + x^3 + 3x^2 + 6x + 6\right]$$

Example: Solve the differential equation y" + 4y = $2e^x \sin^2 x$

Solution: The differential equation can be expressed in differential form as, $(D^2 + 4)y = e^x(1 - \cos 2x)$ = ex – ex cos2x

The homogenous part of differential equation is,

$(D^2 + 4)y = 0$

$D = \pm 2i$

Thus, complimentary function is $y_c = c_1 \cos 2x + c^2 \sin 2x$

Particular Integral can be calculated as,

$$y_{P1} = \frac{e^x}{D^2 + 4} = \frac{e^x}{5}$$

$$y_{P2} = \frac{e^x \cos 2x}{D^2 + 4} = e^x \times \frac{\cos 2x}{\left(D + 1\right)^2 + 4}$$

$$= e^x \frac{\cos 2x}{D^2 + 2D + 5} = e^x \frac{\cos 2x}{-4 + 2D + 5}$$

Here, we have replaced D^2 by $-a^2$ as per **Case 5:**

$$y_{P2} = e^x \frac{\left(2D - 1\right)\cos 2x}{4D^2 - 1} = e^x \frac{\left(2D - 1\right)\cos 2x}{-17}$$

$$= \frac{-e^x}{17}\left[-4\sin 2x - \cos 2x\right]$$

$$y_{P2} = \frac{e^x}{17}\left[\cos 2x + 4\sin 2x\right]$$

$$y_P = y_{P1} - y_{P2}\left(X = e^x - e^x \cos 2x\right)$$

Thus, solution is $y = y_c + y_P = c_1 \cos 2x$

$$+c_2 \sin 2x + \frac{e^x}{5} - \frac{e^x}{17}\left[\cos 2x + 4\sin 2x\right]$$

Example: Determine the solution of (D2 – 2D + 4) y = ex cosx

Solution: The homogenous part of the differential equation is, $(D^2 - 2D + 4)y = 0$

$$D = \frac{2 + \sqrt{4 - 4 \times 4}}{2} = 1 \pm i\sqrt{3}$$

Thus, complementary function is,

$$y_c = e^x\left[c_1 \cos\sqrt{3}x + c_2 \cos\sqrt{3}x\right]$$

The Particular Integral can be computed as,

$$y_P = \frac{e^x \cos x}{D^2 - 2D + 4} = e^x \frac{\cos x}{\left(D + 1\right)^2 - 2D + 4 - 2}$$

$$= e^x \frac{\cos x}{D^2 + 2D + 2 - 2D + 4 - 2}$$

$$\left[y_P = \frac{e^{ax}\cos bx}{f(D)} = e^{ax}\frac{\cos bx}{f(D + a)}\right]$$

$$y_P = e^x \frac{\cos x}{D^2 + 4} = e^x \frac{\cos x}{-1 + 4} = \frac{e^x \cos x}{3}$$

Thus, the complete solution is,

$$y = e^x\left[c_1 \cos\sqrt{3}x + c_2 \cos\sqrt{3}x\right] + e^x \frac{\cos x}{3}$$

Example: Determine the solution of y" – 7y' + 6y = $e^{2x}(1 + x)$

Solution: The homogenous part of differential equation is $(D^2 - 7D + 6)y = 0$

$(D - 6)(D - 1) = 0$

D = 6, 1

Thus, complementary function is $y_c = c_1 e_{6x} + c_2 e^x$

The particular integral is,

$$y_P = \frac{e^{2x}\left(1 + x\right)}{D^2 - 7D + 6} = \frac{e^{2x}\left(1 + x\right)}{\left(D - 6\right)\left(D - 1\right)}$$

$$= \frac{e^{2x}1 - x}{\left(D - 4\right)\left(D + 1\right)}\left[y_P = \frac{e^{2x}V(x)}{(D)} = e^{ax}\frac{V(x)}{f(D + a)}\right]$$

$$y_P = e^{2x}\left[\frac{\left(1 + x\right)}{D^2 - 3D - 4}\right] = 4e^{2x}\left[\left(D + 1\right) \times \left(1 - \frac{D}{4}\right)^{-1}\right]$$

$$(1 + x)$$

$$y_P = -4e^{2x}\left[\left(1 - D\right)\left(1 + \frac{D}{4}\right)\right]\left(1 + x\right)$$

$$y_P = -4e^{2x}\left[1 - D + \frac{D}{4} - \frac{D^2}{4}\right]\left(1 + x\right)$$

$$y_P = -4e^{2x}\left[1 + x - \frac{3}{4}\right] = 4e^{2x}\left[x + \frac{1}{4}\right]$$

Method of variation of parameters

Consider the equation $\frac{d^2y}{dx^2}+k_1\frac{dy}{dx}+k_2y=x$

Where K_1, k_2 are constants and x is a function of x. Now we can have the complimentary function $y_c = c_1y_1 + c_2y_2$

Where c_1 and c_2 are arbitrary constants or parameters and y_1, y_2 are functions of x.

By the method of variation of parameters it is possible to write the Particular Integral as in the form of complimentary function given by $y_P = Ay_1 + By_2$ where

$$A=\int\frac{xy_2}{W}dx$$

$$B=\int\frac{xy_1}{W}dx$$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1 & Y_2 \end{vmatrix}\left[\text{called Wronskian of } y_1, y_2\right]$$

Therefore the complete solution of differential equation is given by

$y = y_c + y_P = (c_1 + A)\, y_1 + (c_2 + B)y_2$

Solved Examples

Example: Determine the solution of $(D^2 + a^2)\, y = \tan ax$

Solution: The homogenous part of differential equation is, $(D^2 + a^2)\, y = 0$

$D = \pm ai$

Thus, complimentary function is,

$$y_c = c_1\underbrace{\cos ax}_{y_1}+c_2\underbrace{\sin ax}_{y_2}$$

$$W=\begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}=a\times 1=a$$

$$A=-\int\frac{Xy_2}{W}dx=-\int\frac{\tan ax\times\sin ax}{a}dx$$

$$=\frac{-1}{a}\int\frac{\sin^2 ax}{\cos ax}=\frac{-1}{a}\int\frac{1-\cos^2 ax}{\cos ax}dx$$

$$A=\frac{-1}{a}\left[\left[\frac{\log\left(\sec ax+\tan ax\right)}{a}\right]-\frac{\sin ax}{a}\right]$$

$$=\frac{-1}{a^2}\left[\log\left(\sec ax+\tan ax\right)-\sin ax\right]$$

$$B=\frac{1}{a}\int\tan ax\times\cos ax\,dx=\frac{1}{a}\int\sin ax\,dx$$

$$=\frac{1}{a}\times\frac{-\cos ax}{a}=\frac{-1}{a^2}\cos ax$$

$$y_P=Ay_1+By_2=\frac{-1}{a^2}\left[\log\left(\sec ax+\tan ax\right)-\sin ax\right]$$

$$\cos ax-\frac{1}{a^2}\cos ax\sin ax$$

Example: Solve the differential equation

$$\left(D^2-6D+9\right)y=\frac{e^{3x}}{x^2}$$

Solution: The differential equation is represented as

$$\left(D-3\right)^2 y=\frac{e^{3x}}{x^2}$$

$D = 3, 3$

Complimentary function is $y_c=\left(C_1+c_2x\right)e^{3x}$

$$=c_1\underbrace{e^{3x}}_{y_1}+c_2\underbrace{xe^{3x}}_{y_2}$$

Wronskian is given by,

$$W=\begin{vmatrix} e^{3x} & e^{3x} \\ 3e^{3x} & 3e^{3x}+e^{3x} \end{vmatrix}=e^{6x}$$

$$A=-\int\frac{e^{3x}}{x^2}\times\frac{xe^{3x}}{e^{6x}}dx=-\log x$$

$$B=\int\frac{e^{3x}}{x^2}\times\frac{xe^{3x}}{e^{6x}}dx=-\frac{1}{x}$$

$$y_P=Ay_1+By_2=-e^{3x}\log x-\frac{e^{3x}}{x}$$

Example: Determine the solution of $(D^2 - 1)\, y = e^{2x}\sin(e^{-x})$

Solution: The homogenous part of differential equation is $D^2 - 1 = 0$

$D = \pm 1$

Complimentary Function is given by,

$$y_c=c_1\underbrace{e^{x}}_{y^1}+c_2\underbrace{e^{-x}}_{y^2}$$

$$W=\begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}=-1-1=-2$$

$$A=\int\frac{e^{-2x}\sin\left(e^{-x}\right)e^{-x}}{2}dx$$

$e^{-x}=z$

$dz=-e^{-x}dx$

$$A=-\frac{1}{2}\int z^2\sin z dz=\frac{-1}{2}\left[-z^2\cos z+\int 2z\cos z dx\right]$$

$$A=-\frac{1}{2}\left[-z^2\cos z+2\times\left(z\sin z+\cos z\right)\right]$$

$$=\frac{1}{2}\left[e^{-2x}\cos e^{-x}-2e^{-x}\sin e^{-x}+2\cos e^{-x}\right]$$

$$B = \frac{-1}{2}\int e^{-2x} \sin e^{-x} \times e^{x} dx$$

$e^{-x} = z; \quad dz = -e^{-x}dx$

$$B = \frac{1}{2}\int \sin z dz = \frac{-1}{2}\cos\left(e^{-x}\right)$$

Particular integral is, $y_P = Ay_1 + By_2$

Euler-Cauchy Equation

The differential equation of the form,

$$x^m \frac{d^m y}{dx^m} + k_1 x^{m-1} \frac{d^{m-1}y}{dx^{m-1}} + \dots\dots\dots + k_{m-1} x \frac{dy}{dx}$$

$+k_m y = x$ is called Euler-Cauchy equation.

Substitute $x = e^z \Rightarrow z = \log x D = \frac{d}{dz}$

Then, $x\frac{dy}{dx} = x\frac{dy}{dz}\frac{dz}{dx} = x \times \frac{dy}{dz} \times \frac{1}{x} = Dy$

Similarly, $x^2\frac{d^2y}{dx^2} = D(D-1)y$

Also, $X^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

Then, the equation becomes,

$$D(D-1)\dots.(D-m+1)y + k_1 D(D-1)\dots.(D-m+2)$$

$$y + \dots\dots\dots + k_m y = X$$

This equation can be solved in terms of 'z' by complimentary function and particular integral.

Solved Examples

Example: Solve the equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$

Solution: If we substitute $x = e^z$, the differential equation becomes,

$D(D-1)y + Dy - 4y = 0$

$(D^2 - D + D - 4)y = 0$

$(D + 2)(D - 2)y = 0$

$D = \pm 2$

$$y = c_1 e^{-2z} + c_2 e^{2z} = \frac{c_1}{x^2} + c_2 \times x^2$$

Applying initial conditions, $y(0) = 0$

$$0 = \frac{c_1}{x^2} + c_2 \times 0 \Rightarrow c_1 = 0$$

Since, $y(1) = 1$

$1 = c_2 \times 1$

$\therefore c_2 = 1$

$\therefore y = x^2$

Example: Solve the differential equation $x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$

Solution: If we substitute $x = e^z$, the differential equation becomes,

$D(D-1)y - 2y = e^{2z} + e^{-z}$

$(D^2 - D - 2)y = e^{2z} + e^{-z}$

The homogenous part of differential equation is $(D - 2)(D + 1)y = 0$

$D = 2, -1$

Thus, complimentary function is $y_c = c_1e^{2z} + c_2e^{-z}$

The particular integral is $y_P = \frac{e^{2z} + e^{-z}}{(D-2)(D+1)}$

$$= \frac{z}{3}e^{2z} - \frac{ze^{-z}}{3}$$

Complete solution is given by, $y = y_c + y_P$

$$\left(c_1x^2 + \frac{c_2}{x}\right) + \frac{\log x}{3}\left(x^2 - \frac{1}{x}\right)$$