1 Linear Algebra

Objective

Upon completion of this chapter, you will be able to determine:

- Determinant of a matrix
- Adjoint and inverse of a matrix
- Rank of a matrix
- Solution of homogeneous and non-homogeneous linear equations.
- Eigen values and eigen vectors of a matrix
- Higher powers of matrix using Cayley Hamilton theorem

Introduction

Linear algebra is a branch of mathematics concerned with the study of vectors, with families of vectors called vector spaces or linear spaces and with functions that input one vector and output another, according to certain rules. These functions are called linear maps or linear transformations and are often represented by matrices.

Matrices are rectangular arrays of numbers or symbols, and matrix algebra or linear algebra provides the rules defining the operations that can be formed on such an object. An elementary application of linear algebra is to the solution of a system of linear equations in several unknowns, which often result when linear mathematical models are constructed to represent physical problems.

Matrix

A system of mn numbers arranged in the form of a rectangular array having m rows and n columns is called a matrix of order m×n

If $A = [aij]_{m\times n}$ be any matrix of order m×n then it is written in the form:

$$\begin{bmatrix} a_{11} & a_{12} & \rule{3cm}{0.4pt} & a_{1n} \\ a_{21} & a_{22} & \rule{3cm}{0.4pt} & a_{2n} \end{bmatrix}$$

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} \rule{4cm}{0.4pt} \\ a_{m1} \; a_{m2} \; \rule{2cm}{0.4pt} \; a_{mn} \end{bmatrix}$$

Horizontal lines are called rows and vertical lines are called columns.

Types of matrices:

The square matrix: The matrix in which the number of rows is same as the number of columns is called as square matrix. If it has n rows, then it is called as n-rowed square matrix. The elements $a_{ij/i=j}$ i.e., a_{11}, a_{22} ... are called diagonal elements and the line along which the elements lie is called principal diagonal of matrix. Elements other than a_{11}, a_{22}, etc. are called off – diagonal elements i.e. $a_{ij/i=j}$.

Example: $A = \begin{bmatrix} 1 & 9 & 4 \\ 7 & 2 & 8 \\ 10 & 78 & 45 \end{bmatrix}_{3\times3}$ is a square matrix

The diagonal elements of this matrix are {1, 2, 45}

Note:

A square sub-matrix of a square matrix A is called a "principle sub-matrix" if its diagonal elements are also the diagonal elements the matrix A. So $\begin{bmatrix} 1 & 9 \\ 7 & 2 \end{bmatrix}$ is a principle sub matrix of the matrix A given above, but $\begin{bmatrix} 9 & 4 \\ 2 & 8 \end{bmatrix}$ is not.

Diagonal matrix: A square matrix in which all off-diagonal elements are zero is called a diagonal matrix. The diagonal elements may or may not be zero.

Example: $A = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 89 \end{bmatrix}$ is a diagonal matrix

The above matrix can also be written as $A = \text{diag}\ [11\ ,\ 15\ ,89]$

Properties of diagonal matrix:

diag [x, y, z] + diag [p, q, r] = diag [x + p, y + q, z + r]

diag [x, y, z] × diag [p, q, r] = diag [xp, yq, zr]

$$\left(\text{diag}\left[x,y,z\right]\right)^{-1} = \text{diag}\left[\frac{1}{x},\frac{1}{y},\frac{1}{z}\right]$$

$$\left(\text{diag}\left[x,y,z\right]\right)^{t} = \text{diag}\left[x,y,z\right]$$

$$\left(\text{diag}\left[x,y,z\right]\right)^{n} = \text{diag}\left[x^n,y^n,z^n\right]$$

Eigen values of $\text{diag}\left[x,y,z\right] = x, y$ and z

determinant of $\text{diag}\left[x,y,z\right] = \left[x,y,z\right] = xyz$

Scalar matrix: A scalar matrix is a diagonal matrix with all diagonal elements being equal.

Example : $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix

Unit matrix or identity matrix: A diagonal matrix whose each diagonal element is 1 is called as identity matrix.

Thus a square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$

Example : $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Properties of identity matrix:

1. AI = IA = A
2. $I^n = I$
3. $I^{-1} = I$
4. $|I| = 1$

Null matrix: The m×n matrix whose elements are all zero is called null matrix. Null matrix is Denoted by null matrix need not be square.

Example : $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O_{2\times1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Properties of null matrix:

1. A + O = O + A = A

   So, O is additive identity.

2. A + (–A) = O if

Upper triangular matrix: An upper triangular matrix is a square matrix whose lower off-diagonal elements are zero, i.e. $a_{ij} = 0$ whenever $i > j$

It is denoted by U. The diagonal and upper off-diagonal elements may or may not be zero.

Example : $U = \begin{bmatrix} 79 & 45 & -76 \\ 0 & 54 & 87 \\ 0 & 0 & 0 \end{bmatrix}$

Lower triangular matrix: A lower triangular matrix is a square matrix whose upper off-diagonal triangular elements are zero. i.e. $a_{ij} = 0$ whenever $i < j$. The diagonal and lower off-diagonal elements may or may not be zero. It is denoted by L.

Example : $L \begin{bmatrix} 17 & 0 & 0 \\ 85 & 0 & 0 \\ 42 & 34 & 62 \end{bmatrix}$

Idempotent matrix: A matrix A is called Idempotent if and only if $A^2 = A$.

Example : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

are examples of idempotent matrices.

Involutory matrix: A matrix A is called involutory if and only if $A^2 = I$.

Example : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is Involutory

Also $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ is Involutory since $A^2 = I$.

Nilpotent matrix: A matrix A is said to be nilpotent of class x or index x if $A^x = O$ and $A^{x-1} \neq O$ i.e. x is the smallest index which makes $A^x = O$

Matrix Algebra

Equality of matrices:

Two matrices $A = [a_{ij}]$ and $B = [b^{ij}]$ are said to be equal if.

1. They are of same size.

2. The elements in the two corresponding places of two matrices are the same i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j.

Example : Let $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 42 & 55 \\ & \end{bmatrix}$

$\Rightarrow$ x = 42 , y = 55 , p = 31 and q = 19

Addition of matrices:

Two matrices A and B are compatible for addition only if they both have the same size say $m\times n$. Then their sum is defined to be the matrix of the type $m\times n$ obtained by adding corresponding elements of A and B

i.e. adding the elements that lie in the same position.

Thus if, $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{m\times n}$ then $A + B = [a_{ij} + b_{ij}]_{m\times n}$

Example : $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} B = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 7 & -8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 13 \end{bmatrix}$$

Properties of matrix addition:

1. Matrix addition is commutative A+B = B+A
2. Matrix addition is associative (A+B) + C = A + (B+C)
3. Cancellation laws holds good in case of addition of matrices, which is X = –A

   $A + X = B + X \Rightarrow A = B$

   $X + A = X + B \Rightarrow A = B$
4. The equation A + X = 0 has a unique solution in the set of all $m\times n$ matrices.

Subtraction of two matrices:

If A and B are two $m\times n$ matrices, then we define, A – B = A + (B)

Thus, the difference A – B is obtained by subtracting from each element of A corresponding elements of B.

Note:

Subtraction of matrices is neither commutative nor associative

Multiplication of a matrix by a scalar:

Let A be any $m\times n$ matrix and k be any real number called scalar. The $m\times n$ matrix obtained by multiplying every element of the matrix A by k is called scalar multiple and is denoted by kA.

If $A = [a_{ij}]_{m\times n}$ then $Ak = kA = [kA]_{m\times n}$

If $A = \begin{bmatrix} 5 & 6 & 1 \\ 2 & -5 & 3 \\ 1 & 1 & 6 \end{bmatrix}$ then, $4A = \begin{bmatrix} 20 & 24 & 4 \\ 8 & -24 & 12 \\ 4 & 4 & 24 \end{bmatrix}$

Properties of multiplication of a matrix by a scalar:

1. Scalar multiplication of matrices distributes over the addition of matrices i.e., k (A + B) = kA + kB
2. If p and q are two scalars and A is any $m\times n$ matrix then, (p + q) A = pA + qA
3. If p and q are two matrices and $A = [a_{ij}]_{-m\times n}$ then, p (qA) = (pq) A
4. If $A = [a_{ij}]_{-m\times n}$ be a matrix and k be any scalar then, (–k) A = –(kA) = k (–A)

Multiplication of two matrices:

Let $A = (a_{ij})_{-m\times n}$; $B = [b_{jk}]_{-n\times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B.

Then the matrix $C = [C_{ik}]_{-m\times p}$ such that $Cik = \sum_{j=1}^{n} a_{ij}b_{jk}$ is called the product of matrices A and B.

Properties of matrix multiplication:

1. Multiplication of matrices is not commutative. If AB exits, then it is not guaranteed that BA also exists.

   For example, $A_{2\times 2} \times B_{2\times 7} = C_{2\times 7}$ but $B_{2\times 7} \times A_{2\times 2}$ does not exist since these are not compatible for multiplication.
2. Matrix multiplication is associative. i.e., A (BC) = (AB)C where A, B , C are $m\times n$, $n\times p$, $p\times q$ matrices respectively.
3. Multiplications of matrices are distributive i.e., A (B + C) = AB + AC
4. The equation AB = O does not necessarily imply that at least one of matrices A and B must be a zero matrix

   For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. $AB = AC \Rightarrow B = C$ (iff A is non-singular matrix)

6. $BA = CA \Rightarrow B = C$ (iff A is non – singular matrix)

Trace of a matrix:

The sum of the diagonal elements of a square matrix is termed as trace of a matrix.

Thus if $A = [a_{ij}]_{-m}$ then, $Tr(A) = \sum_{xn_{j=1}}^{n} a_{ji} = a_{11} + a_{22} + \dots a_{nn}$

$$\text{Let } A = \begin{bmatrix} 11 & 2 & 5 \\ -2 & -23 & 1 \\ -1 & 6 & 35 \end{bmatrix}$$

Then, trace (A) = tr (A) = 11 + (–23) + 35 = 23

Properties of trace of a matrix:

Let A and B be two square matrices of order n and k be a scalar. Then,

1. tr (kA) = k tr A
2. tr (A + B) = tr A + trB
3. tr (AB) = tr (BA)

Transpose of a matrix:

Let $A = [aij]_{-m\times n}$. Then the $n\times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A' or A^T.

$$\text{Let } A = \begin{bmatrix} 6 & 8 \\ 3 & 4 \\ 2 & 1 \end{bmatrix} \text{ then, } A^T = A' = \begin{bmatrix} 6 & 3 & 2 \\ 8 & 4 & 1 \end{bmatrix}$$

If $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ then

$$B' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Properties of Transpose of a matrix:

If A^T and B^T be transposes of A and B respectively then,

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$. k being any complex number
4. $(AB)^T = B^T A^T$
5. $(ABC)^T = C^T B^T A^T$

Conjugate of a matrix:

The conjugate of a matrix is obtained by taking the conjugate of each element of the matrix. For real matrices,

Conjugate will yield the same.

$$\text{Let } A = \begin{bmatrix} 6 & 8 \\ 3 & 4 \\ 2 & 1 \end{bmatrix} \text{ then, } A^T = A = \begin{bmatrix} 6 & 3 & 2 \\ 8 & 4 & 1 \end{bmatrix}$$

If $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ then

$$\text{Example : If } A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$$

$$A \begin{bmatrix} 2-3i & 4+7i & 8 \\ i & 6 & 9-i \end{bmatrix}$$

Properties of conjugate of a matrix:

If $\bar{A}$ & $\bar{B}$ be the conjugates of A & B respectively. Then

1. $\overline{(\bar{A})} = A$
2. $\overline{(A+B)} = \bar{A} + \bar{B}$
3. $\overline{(KA)} = \bar{K}\,\bar{A}$, k being any complex number
4. $\overline{(AB)} = \bar{A}\bar{B}$, if A & B can be multiplied
5. $\bar{A} = A$ if and only if A is real matrix
6. $\bar{A} = -A$ if and only if A is purely imaginary matrix

Transposed conjugate of matrix:

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by

A^θ or A^* or $(\bar{A})^T$. It is also called conjugate transpose of A.

$$\textbf{Example : } \text{If } A = \begin{bmatrix} 2+i & 3-i \\ 4 & 1-i \end{bmatrix}$$

To find A^θ , we first find $\underline{A} = \begin{bmatrix} 2-i & 3+i \\ 4 & 1+i \end{bmatrix}$

Then $A^\theta = (\bar{A})^T = \begin{bmatrix} 2-i & 4 \\ 3 & \end{bmatrix}$

Properties:

If A^θ & B^θ be the transposed conjugates of A and B respectively then,

1. $(A^\theta)^\theta = A$
2. $(A + B)^\theta\ A^\theta + B^\theta$

3. $(KA)^\theta = \bar{K}A^\theta$, $K \rightarrow$ complex number

4. $(AB)^\theta = B^\theta A^\theta$

Classification of Real Matrices Symmetric Matrices:

If the transpose of a matrix is identical to the original matrix, then the matrix is called as symmetric matrix.

$A^T = A$

Note: AAT and $\frac{A + A^T}{2}$ are always symmetric matrices.

Skew symmetric matrices:

If the transpose of a matrix is negative of the original matrix, then the matrix is termed as a skew symmetric matrix.

$A^T = -A$

The diagonal elements of a skew symmetric matrix are always zero.

Here, the first term represents a symmetric matrix, and the second term represents skew symmetric matrix.

Orthogonal matrices:

If the product of a matrix and its transpose is equal to the identity matrix or the inverse of a matrix is identical to its transpose, then the matrix is termed as orthogonal matrix.

$A^T = A^{-1}$ or $AA^T = I$

$$|AA^T| = |I| = 1$$

$$\text{Thus, } |A||A^T| = (|A|)^2 = 1$$

$$|A| = \pm 1$$

Classification of complex matrices:

Hermitian matrix:

If the conjugated transpose of a matrix is identical to the original matrix, then the matrix is termed as Hermitian matrix.

$A^\theta = A$

Skew-hermitian matrix:

If the conjugated transpose of a matrix is negative of the original matrix, then the matrix is termed as skew Hermitian matrix.

$A^\theta = -A$

A complex matrix can be resolved into Hermitian and skew hermitian matrix,

$$A = \frac{A + A^\theta}{2} + \frac{A - A^\theta}{2}$$

Here, the first term represents a Hermitian matrix, and the second term represents Skew Hermitian Matrix.

Unitary matrix:

If the product of matrix and its conjugated transpose is the identity matrix. Thus, inverse of matrix is identical to its conjugated transpose of a matrix.

$A^\theta = A^{-1}$ or $AA^\theta = I$

Let a_{11}, a_{12}, a_{21}, a_{22} be any four numbers. The symbol $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ represents the number $a_{11}\, a_{22} - a_{12}\, a_{21}$ and is called determinants of order 2.

Similarly, $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ represent a determinant of order 3.

Minors and cofactors:

Consider the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Leaving the row and column passing through the elements a_{ij} , then the second order determinant thus obtained is called the minor of element aij and we will be denoted by M_{ij}.

As for example, the minor of the element

$$a_{11} \text{ is } M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}\, a_{33} - a_{23}\, a_{32})$$

Similarly, the minor of the element

$$a_{12} \text{ is } M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (a_{21}\, a_{33} - a_{23}\, a_{31})$$

Minor of the element

$$a_{31} \text{ is } M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = (a_{12}\, a_{23} - a_{22}\, a_{13})$$

Cofactors:

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of element a_{ij} . We shall denote the cofactor of an element by corresponding capital letter.

Example: Cofactor of $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$

Cofactor element

$$a_{12} = A_{12} = (-1)^{1+2} M_{12}$$

$$= -\begin{vmatrix} a_{12} & a_{23} \\ a_{13} & a_{33} \end{vmatrix} = -(a_{21}\,a_{33} - a_{23}\,a_{31})$$

Cofactor element

$$a_{31} = A_{31} = (-1)^{3+1} M_{31} = M_{31}$$

$$= \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = (a_{12}\,a_{23} - a_{22}\,a_{13})$$

We define for any matrix, the sum of the products of the elements of any row or column with corresponding cofactors is equal to the determinant of the matrix.

Thus, the determinant of a 3x3 matrix is, $\Delta = a_{11} \times A_{11} + a_{12} \times A_{12} + a_{13} \times A_{13}$

Example : If $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 6 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

$$\mathrm{Cof}(A) = \begin{bmatrix} 12 & 4 & -12 \\ -4 & 2 & 4 \\ 2 & -1 & 8 \end{bmatrix}$$

$|A| = (1 \times 12) + (2 \times 4) + (0 \times -12) = (-1 \times -4) + (6 \times 2) + (1 \times 4) = (2 \times 2) + (0 \times -1) + (2 \times 8) = 20$

Note: The determinant can only be computed for square matrices, and it can be calculated along any row or column.

Properties of determinants:

1. The value of the determinant will not be changed if rows and columns are interchanged.

   i.e. $|A| = |A|$

   $$|A| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 10 - 4 = 6$$

   $$|A|^T = \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = 10 - 4 = 6$$

2. The value of determinant is multiplied by -1 if two rows or two columns are interchanged.
3. The value of determinant can be zero in the following cases.
   a) The elements in two rows or two columns are identical

   $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

   b) The elements in two rows and two columns are proportional to each other.

   $$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 5 & 6 \end{vmatrix} = 0 \qquad \because C_3 = 3C_1$$

   c) All elements in any row or any column are zeros.

   $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

   d) If the elements in the determinant are of consecutive order (valid for 3rd and higher order)

   $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

   e) The first row of each element starts from the 2nd element of previous row such that the elements in that determinant are of consecutive order.

   $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

4. The determinant of upper triangle, lower triangle, diagonal, scalar or identity matrix is the product of its diagonal elements.

   $$\text{Upper triangle} = |A| = \begin{vmatrix} 10 & 7 & 5 \\ 0 & 13 & 1 \\ 0 & 0 & 7 \end{vmatrix}$$

   $$= 10 \times 13 \times 7 = 910$$

Lower triangle $= |A| = \begin{vmatrix} 11 & 0 & 0 \\ 21 & 7 & 0 \\ 54 & 5 & 9 \end{vmatrix}$

$= 11 \times 7 \times 9 = 693$

Diagonal matrix $= \begin{vmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3 \times 6 \times 1 = 18$

Scalar matrix: A diagonal matrix with same diagonal elements.

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2 \times 2 \times 2 = 8$$

- If A is n × n matrix then
  $|KA| = K^n |A|$

  Consider the matrix $|A| = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix}_{2\times 2}$

  $\therefore |3A| = 3^2 \times \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 9 \times 12 = 108$

- If each element of a determinant contains sum of two elements, then that determinant should be expressed as sum of two determinants of the same order.

$$\text{Eg}: |A| = \begin{vmatrix} a & a^2 & a^3+3 \\ b & b^2 & b^3+3 \\ c & c^2 & c^3+3 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 3 \\ b & b^2 & 3 \\ c & c^2 & 3 \end{vmatrix}$$

Solved Examples

Example: If A = $(a)_{ij\ 3\times3}$, B = $(b)_{ij\ 3\times3}$ such that b_{ij} = $2^{i+j} a_{ij}\ \forall_{i,j}$ and $|A| = 2$ then $|B| =$ ________?

Solution: Given, $b_{ij} = 2^{i+j}\ a_{ij} \forall\ ij$

$$\therefore |B| = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix} = 2^2 \times 2^3 \times 2^4$$

$$\begin{vmatrix} a_{11} & 2a_{12} & 2^2 a_{13} \\ a_{21} & 2a_{22} & 2^2 a_{23} \\ a_{31} & 2a_{32} & 2^2 a_{33} \end{vmatrix}$$

$$|B| = 2^2 \times 2^3 \times 2^4 \times 2 \times 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|A| = 2 \Rightarrow |B| = 2^{12} \times 2 = 2^{13}$$

Example: If A = $a_{ij\ m\times n}$ such that a_{ij} = i + j $\forall$i, j then sum of all elements of A is ?

Solution: The determinant of A can be expressed as,

$$|A| = \begin{vmatrix} a_{11} & a_{12} \ldots\ldots\ldots\ldots\ldots & a_m \\ a_{12} & a_{22} \ldots\ldots\ldots\ldots\ldots & a_m \\ : & & \\ : & \ldots\ldots\ldots\ldots\ldots & a_m \\ a_{m1} & a_{m2} \ldots\ldots\ldots\ldots\ldots & a_{mn} \end{vmatrix}$$

Since, $a_{ij} = i + j\ \forall i, j$

$$|A| = \begin{bmatrix} 1+1 & 1+2 \ \ldots\ldots\ldots\ldots & 1+n \\ 2+1 & 2+2 \ \ldots\ldots\ldots\ldots & 2+n \\ : & & \\ : & & \\ m+1 & m+2 \ \ldots\ldots\ldots\ldots & m+n \end{bmatrix}_{m\times n}$$

$$= \begin{vmatrix} 1 & 1 & 1 \ldots\ldots\ldots & 1 \\ 2 & 2 & 2 \ldots\ldots\ldots & 2 \\ : & & & \\ : & & & \\ m & m & \ldots\ldots\ldots & m \end{vmatrix}_{m\times n} + \begin{vmatrix} 1 & 2 & 3 \ldots\ldots\ldots & n \\ 1 & 2 & 3 \ldots\ldots\ldots & n \\ : & & & \\ : & & & \\ 1 & 2 & 3 \ldots\ldots\ldots & n \end{vmatrix}$$

Taking the sum of each column in first matrix. It will be equal to sum of first 'm' natural numbers

$$\frac{m(m+1)}{2}$$

Sum of each column = 2

Sum of 'n' columns = $\frac{mn(m+1)}{2}$

Taking sum of each row in second matrix. Each row sum is equal to sum of 'n' natural numbers.

Sum of each row = $\frac{n(n+1)}{2}$

Sum of 'm' rows = $\frac{mn(n+1)}{2}$

Total Sum = $\frac{mn}{2}[m+n+2]$

Example: Given $A = (a_{ij})$ such that $a_{ij} = i^2 - j^2 \ \forall$ i, j . Find sum of the elements of the matrix.

Solution: The elements of matrix are,

$$A = \begin{bmatrix} 0 & -3 & -8 \text{..........} 1 & -n^2 \\ 3 & 0 & -5 \text{..........} 2^2 & -n^2 \\ 8 & 5 & 0 \text{...............} & -n^2 \\ n^2-1^2 & n^2-2^2 & \text{..........} & 0 \end{bmatrix}$$

$\Rightarrow$ Skew symmetric matrix

$\therefore$ sum of elements = 0

Example: Matrix A has m rows and (m + 5) columns and B has n rows and (11 – n) columns, the orders of A & if AB and BA are defined is?

Solution: Since AB is defined m + 5 = n Since BA is defined, 11 – n = m

Solving above two equations, m = 3 and n = 8

Example: The following represents equation of a straight line. $\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0.$ The line passes through,

(A) 0, 0 (B) 3, 4

(C) 4, 3 (D) 4, 4

Solution: The determinant is given by 8x + 2y – 32 = 0 8x + 2y = 32

Substituting the options we get (3, 4)

Solution:

$$|A| = 5\times(3-0)+2\times(0-2\times3) = 15-12 = 3$$

Note: A 4x4 determinant can be calculated as,

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

By using either row or column or both operations make all elements above and below or to the left and right of the selected element are zeros. And then expand the determinant along the row or column.

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{vmatrix}$$

$$|A| = a_{11}\begin{vmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{vmatrix}$$

b_{ij} are the new elements after row or column transformation.

Example: Determine the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 4 & 5 & 7 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & 4 & 9 \end{bmatrix}$$

Solution: Applying the row transformations,

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - 2R_1$

The resultant matrix is, $A = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 4 & 5 & 7 \\ 0 & -1 & 2 & -2 \\ 0 & -1 & 6 & -1 \end{bmatrix}$

Determinant is given by,

$$|A| = 1\times\begin{bmatrix} 4 & 5 & 7 \\ -1 & 2 & -2 \\ -1 & 6 & -1 \end{bmatrix} = 17$$

Example: Find the value of the determinant

$$A = \begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}$$

Solution: Applying the column transformation, $C_1 \to C_1 + C_2$

$$\begin{vmatrix} 1+2b & b & 1 \\ 1+2b & 1+b & 1 \\ 1+2b & 2b & 1 \end{vmatrix}$$

Since, $C_1 = (1 + 2b)\, C_3$ i.e. both columns are proportional

$\therefore |A| = 0$

Adjoint of a matrix:

Let B be the cofactor element of the matrix A, then adj of A = B^T

Singular matrix:

A matric is said to be singular matrix if $|A| = 0$. It is called as nonsingular if $|A| \neq 0$

The inverse of only non-singular matrix can be computed. Thus, singular matrices are non-invertible.

A matrix A is used to be invertible if we can find some other matrix B such that AB=BA=I, then B is called Inverse of the matrix A.

Inverse of a matrix A is given by, $A^{-1} = \dfrac{\text{adj}A}{|A|}$

Properties of inverse of a matrix:

1. $AA^{-1} = A^{-1}A = I$
2. A and B inverse of each other iff $AB = BA = I$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
5. If A be an $n \times n$ non-singular matrix, then $(AT)^{-1} = (A^{-1})^T$
6. If A be an $n \times n$ non-singular matrix, then $(A^{-1})^{\theta} = (A^{\theta})^{-1}$
7. For a 2×2 matrix there is a short-cut formula for inverse as given below

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

8. Assuming A is matrix of order "n x n"

$|A|\, A^{-1}$ adj A

$|A|\, |A^{-1}| = |\text{adj } A|$

Since, $|KA| = k^n\, |A|$

Thus, $|A|^n\, |A^{-1}| = |\text{adj } A|$

$|A|^n\, |A|^{-1} = |\text{adj } A|$

$|A|^{n-1} = |\text{adj } A|$

Replacing A by adj A

$\Rightarrow |\text{adj (adj } A)| = |\text{adj } A|^{n-1} = |A|^{n-1}$

i.e., $|\text{adj (adj } A)| = |A|^{(n-1)^2}$

Similarly $|\text{adj(adj(adj } A))| = |A|^{(n-1)^3}$

Solved Examples

Example: For the matrix m $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \times & \frac{3}{5} \end{bmatrix}$ and $m^T = m^{-1}$. Find x?

Solution: If m is an orthogonal matrix, its rows and columns must be pairwise orthogonal and periodic orthogonal.

$$\frac{3}{5}x + \frac{4}{5} \times \frac{3}{5} = 0$$

$$\frac{3}{5}x = -\frac{4}{5} \times \frac{3}{5}$$

$$x = -\frac{4}{5}$$

Example: If $A = (a_{ij})^{5\times 5}$ such that $a_{ij} = i - j$. Find A^{-1} in each case.

Solution: $aij = i - j = -(j - i)$

$a_{ij} = -a_{ji}$

Thus, A is a skew symmetric matrix

$A^T = -A$

$|A^T| = |-A|$

$|A| = (-1)^5\, |A| = -|A|$

$|A| + |A| = 0$

$2|A| = 0$

$|A| = 0$

Hence, A is a singular matrix and thus A^{-1} does not exist

Note: Every odd order skew symmetric matrix is a singular matrix $|A|_{n\times n} = 0$ Every even order skew symmetric matrix is a non-singular matrix $|A|_{n\times n} \neq 0$
The determinant of even order skew symmetric matrix is always perfect square.

$$\text{Eg}: \begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} = 9 = 3^2 \neq 0$$

Example: If x and y are two non-zero matrices of the same order, such that xy=0, then

(A) $|x| \neq 0, |y| = 0$

(B) $|x| = 0, |y| \neq 0$

(C) $|x| \neq 0, |y| \neq 0$

(D) $|x| = 0, |y| = 0$

Solution : Let $|x| = 0$ and $|y| \neq 0 \Rightarrow y^{-1}$ exists.

$xy = 0$

$xyy^{-1} = 0y^{-1}$

$x = O$ is false since it is given x and y are non-zero matrices.

$\therefore |y| \neq 0$ is wrong $\Rightarrow |y| = 0$

Similarly, $|x| = 0$

Note: Product of two nonzero matrices is a null matrix if both of them are singular.

Example: Let k be a positive real number and let

$$A = \begin{bmatrix} (2k-1) & 2\sqrt{\cancel{k}} & 2\sqrt{\cancel{k}} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 2k-1 & \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0\sqrt{k} \end{bmatrix}$$

Find A) |adj B| B) |adj A| C) |adj A| $= 10^6$, the value of k is ________?

Solution: (A) Here B is an odd order skew symmetric matrix.

$\therefore |B| = 0 \Rightarrow |\text{adj } B| = |B^{n-1}| = 0$

(B) $|\text{adj } A| = |A|^{n-1} = |A|^2$

$$|A| = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$

$$R_2 \to R_2 + R_3 \Rightarrow \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 0 & 1+2k & -(1+2k) \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$

$$C_2 \to C_2 \to C_3 \Rightarrow \begin{vmatrix} 2k-1 & 4\sqrt{k} & 2\sqrt{k} \\ 0 & 0 & -(1+2k) \\ -2\sqrt{k} & 2k-1 & -1 \end{vmatrix}$$

Expanding the determinant along the second row,

$|A| = (1+2k)\left[(2k-1)+8\right]$

$= (1+2k)\left[4k^2 - 4k + 1 + 8k\right]$

$= (1+2k) = (2k+1)$

$\therefore |\text{adj } A| = |A|^2 \left[(2k+1)^3\right]^2 = (2k+1)^6$

(C) $(2k+1)^6 = 10^6$

ie. $2k + 1 = 10 \Rightarrow 2k = 9 \Rightarrow k = \frac{9}{2}$

Example: Given an orthogonal matrix A

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \text{ The } (A \times A^T)^{-1} \text{ is } \_\_\_\_\_\_?$$

Solution: $(A \times A^T) = I$

$\therefore (A \times A^T)^{-1} = I^{-1} = I$

Example: Find the inverse of matrix.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}?$$

Solution:

Since its rows and columns are pairwise orthogonal. Thus, A is an orthogonal matrix. $A^T = A^{-1}$

$$\therefore A^{-1} = \begin{bmatrix} os\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of a Matrix

Submatrix:

A matrix is obtained after deleting some rows or columns is called a submatrix.

$$\text{Eg} : A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 1 & -1 & 0 & 2 \end{bmatrix}$$

Some of the submatrices of A are

$$B_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}, \qquad B_2 = \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix}_{2\times2}$$

$$B_3 = \begin{bmatrix} 5 & 6 & 7 \\ -1 & 0 & 2 \end{bmatrix}_{2\times3}$$

Thus, minor can also be defined as the determinant of a square sub-matrix.

If the determinant of at-least one highest possible square sub matrix is nonzero then order of the determinant is called rank of a matrix.

Solved Examples

Example: Find the rank of the matrix,

$$A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 4 & 5 \\ 1 & -11 & 14 & 5 \end{bmatrix}_{3\times4} \ ?$$

Solution: Highest possible square matrix is 3×3.

$$|B| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix}$$

$$= (44-14) - 2(42-22) + (12-2) = 0$$

$$|B| = \begin{vmatrix} 3 & -2 & 1 \\ -1 & 4 & 5 \\ -11 & 14 & 5 \end{vmatrix}$$

$$= 3(20-70) + 1(-10-14) - 11(-10-4) \neq 0$$

$$\therefore \text{Rank}(A) = 3$$

Properties of rank of a matrix:

- Rank $(O_{n\times n}) = 0$
- Rank $(I_{n\times n}) = n$
- Rank $(a_{ij}\,(I_{n\times n})) = n$
- Rank $(A + B) \leq$ rank (A) + rank (B)
- Rank $(A - B) \geq$ rank (A) – rank (B)

If A is an $m \times n$ matrix, then rank $(A) \leq \min(m,n)$

- Rank $(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$
- Rank $(AB) \geq$ rank (A) + rank (B) – n. If A and B are $n \times n$ matrices
- Rank (A) = rank (A^T)
- If rank $(A)_{n\times n} = n$ then rank (adj A) = n
- If rank $(A_{n\times n}) = n - 1$ then rank (adj A) = 1

Example: Determine the rank of adjoint of the matrix $A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{vmatrix}$

Solution: Adjoint of A is given by,

$$\text{adj}(A) = \begin{bmatrix} 0 & 0 & -17 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Applying row transformations, $R_1 \to R_1 \to 17R_2$, $R_3 \to R_3 \to 5R_2$ as rank is invariant under row transformation.

$$\text{adj } A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank of (adj A) = number of non-zero rows = 1

If rank $(A_{n\times n}) \leq n - 2$, then rank of (adj A) = 0

Echelon form:

Row Echelon Form: A matrix is said to be in echelon form if it satisfies the following conditions.

- All zero rows should occupy last rows if any.
- The number of zeros before a non-zero entry of each row is less than the number of such zeros before a non- zero entry of

the next row. i.e., zeros increases row by row.

- Rank of a matrix in Echelon form = number of non-zero rows.
- To reduce any matrix to row echelon form we use only row operations.
- The number of non-zero rows in row echelon Form are also called linearly independent vectors or LI rows.

Note: Every upper triangular matrix will be a row Echelon matrix, but converse may or may not be true. Echelon form as it satisfies the conditions.

$$A=\begin{bmatrix}0 & -2 & 5 & 6\\0 & 0 & 5 & 2\\0 & 0 & 0 & 3\\0 & 0 & 0 & 0\end{bmatrix}\rightarrow \text{Echelon form}$$

But not upper triangular matrix as the diagonal elements are all zero.

Example: Determine the rank of matrix A

$$A=\begin{bmatrix}1 & 2 & 2 & -1\\2 & 1 & 2 & 0\\2 & 2 & 1 & 3\end{bmatrix}$$

Solution: Applying row transformation

$R_2 \rightarrow R_2 \rightarrow 2R_1$

$R_3 \rightarrow R_3 \rightarrow 2R_1$

$$\therefore A=\begin{bmatrix}1 & 2 & 2 & -1\\0 & -3 & -2 & 2\\0 & -2 & -3 & 5\end{bmatrix}$$

Now $R_3 \rightarrow 3R_3 - 2R_2$

$$A=\begin{bmatrix}1 & 2 & 2 & -1\\0 & -3 & -2 & 2\\0 & 0 & -5 & 11\end{bmatrix}$$

There are three linearly independent rows and thus rank(A) = 3

Example: Determine the rank of

$$A=\begin{bmatrix}2 & 1 & 1 & 0\\0 & 1 & 1 & 5\\1 & 1 & 2 & 3\end{bmatrix}$$

Solution: Applying row transformation

$R_3 \rightarrow 2R_3 - R_1$

$$A=\begin{bmatrix}2 & 1 & 1 & 0\\0 & 1 & 1 & 5\\0 & 1 & 3 & 6\end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$A=\begin{bmatrix}2 & 1 & 1 & 0\\0 & 1 & 1 & 5\\0 & 0 & 2 & 1\end{bmatrix}$$

Since, there are three linearly independent rows, rank (A) = 3

Note: If rank(A) = n then it has n linearly independent rows and linearly independent columns.

Example: If $A = (a_{ij})$ such that $a_{ij} = i \times j \ \forall\ i, j$ then rank(A) = ?

Solution:

The matrix A when expanded is,

$$A=\begin{bmatrix}1 & 2 & 3 & \ldots\ldots & n\\2 & 4 & 6 & \ldots\ldots & 2n\\: & & & & \\m & 2m & 3m & \ldots\ldots & nm\end{bmatrix}$$

$$= 1\times 2\times 3\times \ldots\ldots\ldots\ldots\ldots \times m\begin{bmatrix}1 & 2 & 3\ldots\ldots n\\1 & 2 & 3\ldots\ldots n\\: & & \\1 & 2 & 3\ldots\ldots n\end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - R_1$

$Rm \rightarrow Rm - R_1$

$$A = m!\begin{bmatrix}1 & 2 & 3\ldots\ldots\ldots n\\0 & 0 & 0\ldots\ldots\ldots 0\\: & & \\0 & 0 & 0\ldots\ldots\ldots 0\end{bmatrix}$$

Number of non-zero rows = 1

$\therefore$ Rank (A) = 1 as there is only 1 linearly independent vector.

Example: If $x = (x_1\ \ x_2\ \ldots\ x_n)^T$ is an n-tuple nonzero vector. Then find rank (xx^T)

Solution: Since x is a n-element column vector. Then, x^T is an n-element row vector.

Thus, both have rank of 1.

$$\rho\left(XX^T\right) \le \min\left\{\rho\left(X\right)\rho\left(X^T\right)\right\}$$
$$P\left(XX^T\right) \le \min\left\{1, 1\right\} = 1.$$
Here, ρ represents rank of matrix.

Example: The rank of 5×6 matrix is 4. Then which of the following statement is true?

(A) Q has 4 linearly independent rows and 4 linearly independent columns.

(B) Q has 4 LI rows and 5 LI columns

(C) $Q \,.\, Q^T$ is invertible

(D) $Q^T \,.\, Q$ is invertible

Solution: Option (A) is correct

If rank of a matrix is 4 then it have 4 LI rows and 4 LI columns. These 4 LI rows can be obtained by reducing the matrix into echelon form and 4 LI columns can be obtained by reducing the matrix into echelon form.

Linearly dependent and independent vectors:

If the elements are written in a horizontal line or in vertical line is called as vector. This means row matrix, or a column matrix can be termed as a vector.

Example: (1 2 3) $\rightarrow$ vector

Two vectors x_1 and x_2 are said to be linearly dependent if it is possible to express one of the vectors as multiple of another vector.

Suppose, one vector is $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$X_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2x_1 \text{ or } x_1 = \frac{1}{2_2}x$$

Two vectors in R^2 (two dimensional) are said to be linearly dependent if they are collinear.

Similarly 3 vectors in R^3 (3-D) are said to be linearly dependent if they are coplanar i.e. lying in the same plane.

If it is not possible to express 1 vector as multiple of other vector the two vectors are said to be linearly independent.

Suppose, $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Hence, $x_1 \neq kx_2$. Thus, x_1 & x_2 are linearly independent Vectors.

Linearly dependent vectors:

A set of r n-vectors (vectors having n-element each), x_1, x_2 ... x_r are said to be linearly dependent if there exists r scalars K_1, K_2 ... k_r such that $K_1x_1 + K_2x_2 + ... + K_rx_r = 0$ where K_1, K_2, ... K_r not all zeros. (At least 1 K should be a non-zero number)

Linearly independent vectors:

'r' vectors x_1, x_2 ... x_r are linearly independent vectors if there exist 'r' scalars. K_1, K_2 ... K_r such that $K_1X_1 + K_2X_2 + ... + K_rX_r = 0$ where K_1, K_2 ... K_r are all zeros.

- If the rank of matrix is less than the dimension of the matrix, then the vectors are said to be linearly dependent. Thus, if the determinant of the matrix is zero the vectors are said to be linearly dependent.
- If the rank of matrix is equal to the dimension of the matrix, then the vectors are said to be linearly independent. Thus, if determinant of the matrix is non-zero the vectors are said to be linearly independent.
- If the number of components of the vectors is more than the number of vectors, then the vectors are said to be linearly independent but if the number of components is less than the number of vectors then the vectors are said to be linearly dependent.

Example: Consider vectors (1 2 3 4)
(2 1 –1 7)
(2 3 4 5)

Number of vectors r = 3 Number of component n = 4

Since r < n, the vectors are linearly independent.

If the vectors are expressed in Matrix form,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 7 \\ 2 & 3 & 4 & 5 \end{bmatrix}_{3\times4}$$

Applying row transformations,

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 2R_1$

$$A\begin{bmatrix}1 & 2 & 3 & 4\\0 & -3 & -7 & -1\\0 & -1 & -2 & -3\end{bmatrix}$$

$R_3 \to 3R_3 - R_2$

$$A\begin{bmatrix}1 & 2 & 3 & 4\\0 & -3 & -7 & -1\\0 & 0 & 1 & 8\end{bmatrix}$$

$\rho(A) = 3$

Thus, the rank of matrix is same as number of vectors. Thus, the vectors are linearly Independent. But if we consider the vectors X_1 = (1 2 3); X_2 = (1 0 3); X_3 = (1 1 – 1); X_4 = (0 1 2) Number of vectors r = 4

Number of components n = 3

Since r > n, so the given vectors are linearly dependent.

Suppose the vectors are linearly dependent, then any 1 of the vectors can be expressed as a linear combination of other vectors.

If the given vectors are linearly independent, then it is impossible to express any of the vectors as a linear combination of other vectors.

Example: Determine whether the given vectors are linearly independent or Dependent X_1 = (1 0 0); X_2 = (1 0 0); X_3 = (0 0 1)

Solution: Expressing the linear combination of vectors and equating them to zero, K_1X_1 + K_2X_2 + K_3X3 = 0

Substituting the vectors, K1(1 0 0) + K_2(0 1 0) + K_3(0 0 1) = (0 0 0)

i.e. (K_1 K_2 K_3) = (0 0 0) K_1 = K_2 = K_3 = 0

Since, all coefficients are zero. Thus, all vectors are linearly independent.

Alternate method:

Expressing the vectors in matrix form,

$$|A| = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = 1 \neq 0$$

$\rho(A) =$ number of non-zero rows = 3

Since, rank is same as number of vectors. The vectors are linearly independent.

The set of vectors having at least one vector is a null vector, the vectors are said to be linearly dependent.

Example: Consider the vectors X_1 = (2 3 4); X_2 = (1 –1 2); X_3 = (0 0 0). Determine whether these vectors are linearly dependent or independent.

Solution: Expressing the vectors in matrix form. The determinant is given by,

$$|A| = \begin{vmatrix}2 & 3 & 4\\1 & -1 & 2\\0 & 0 & 0\end{vmatrix} = 0$$

Since |A| =0, rank(A) < 3, but the determinant of 2×2 square sub-matrix is nonzero

$|B| = \begin{vmatrix}2 & 3\\1 & -1\end{vmatrix} = -5 \neq 0$ Thus, $\rho(A) = 2$

Number of given vectors = 3. Thus, ρ(A) < n and the vectors are linearly dependent.

Orthogonal vectors:

For orthogonal Vectors dot product or inner product of the vectors must be zero. Each term in the row or column vector can be considered as a component of the vector. The dot product is obtained as sum of the product of corresponding components of both vectors.

In matrix form it can be expressed as, $X_1^TX_2 = X_1 X_2^T = 0$

The set of orthogonal vectors in R^n (R vectors with n elements are linearly independent).

Dimension and basis:

- The number of linearly independent vectors in a vector space is called as dimension of vector space.
- For a three-dimensional space, there are only 3 linearly independent vectors and any fourth vectors can be expressed as a linear combination of the other three vectors.
- Number of linearly independent vectors can be determined by expressing the matrix in Row Echelon Form and the number of non-zero rows is equal to the number of linearly independent vectors.

- The set of linearly independent Vectors is called as basis of vector space. Any vector in vector space can be expressed as a linear combination of its basis vectors.

Nullity of a matrix:

Nullity of a matrix is denoted by N(A) and is defined as the difference between order and rank of the matrix.

i.e. $N(A)=n(A)-\rho(A)$

Nullity of a non-singular matrix of any order is always zero as the rank of the matrix is same as the order since the determinant of the matrix is non-zero.

Connection between rank and span:

A set of n vectors $X_1, X_2, X_3 \ldots X_n$ spans R_n iff they are linearly independent which can be checked by constructing a matrix with X1, X_2, $X_3 \ldots X_n$ as its rows (or columns) and checking that the rank of such a matrix is indeed n. If however the rank is less than, n say m, then the vectors span only a subspace of R^n

Solved Examples

Example: Test whether the following vectors are linearly dependent or linearly independent. Also find dimensions and basis?

(1 1 –1 0) (4 4 –3 1) (–6 2 2 2) (9 9 –6 3)

Solution: Expressing the vectors in matrix form,

$$A=\begin{bmatrix}1 & 1 & -1 & 0\\ 4 & 4 & -3 & 1\\ -6 & 2 & 2 & 2\\ 9 & 9 & -6 & 3\end{bmatrix}$$

Applying row transformation, $R_2 \to R_2 - 4R_1$; $R_3 \to R_3 + 6R_1$; $R_4 \to R_4 - 9R_1$

$$A=\begin{vmatrix}1 & 1 & -1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 8 & -4 & 2\\ 0 & 0 & 3 & 3\end{vmatrix}$$

Interchanging row 2 and row 3.

$$R_2 \Rightarrow R_3 \Rightarrow A=\begin{bmatrix}1 & 1 & -1 & 0\\ 0 & 8 & -4 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 3 & 3\end{bmatrix}$$

Applying, $R_4 \to R_4 \to 3R_1$

$$A=\begin{bmatrix}1 & -1 & 0 & 1\\ 0 & 8 & -4 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

In row echelon form, the number of non-zero rows is 3. Thus, rank of the matrix is 3. Thus, the dimension is 3.

Basis is $\{(1\ -1\ 0\ 1), (0\ 8\ -4\ 2), (0\ 0\ 1\ 1)\}$

Example: Determine the value of k if nullity of $A=\begin{bmatrix}k & 1 & 2\\ 1 & -1 & -2\\ 2 & 1 & 1\end{bmatrix}$ is 1?

Solution: The order of matrix A is n(A) = 3 nullity of matrix A is, $N(A)=n(A)-\rho(A)$ $1=3-\rho(A)$

$\therefore\ \rho(A) = 3 - 1 = 2$

Since, Rank is 2. Thus, $|A|=0$

$$|A|=0 \Rightarrow \begin{vmatrix}K & 1 & 2\\ 1 & -1 & -2\\ 2 & 1 & 1\end{vmatrix}=0$$

$$K(-1+2)-1(1+4)+2(1+2)=0$$

$$K=-1$$

Example: The rank of a matrix is 5 and nullity of the matrix is 3. Then what is the order of the matrix.

Solution: $N(A) = n(A) - \rho(A)$

$3 = n(A) - 5$

$n(A) = 8$

Example: Check if the vectors [1 2 –1], [2 3 0], [–1 2 5] span R^3

Solution: Constructing $A=\begin{bmatrix}1 & 2 & -1\\ 2 & 3 & 0\\ -1 & 2 & 5\end{bmatrix}$

Finding determinant to determine the rank,

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 2 & 5 \end{vmatrix} = 1(15-0) - 2(10-0) - 1(4+3)$$

$= 15 - 20 - 7 = -12 \neq 0$

So, rank = 3

$\therefore$ The vectors are linearly independent and hence span R^3

System of Non-Homogenous and Homogeneous Linear Equations

Consider a non-homogenous system of equations, containing two equations and two variables.

$ax + by = e$

$cx + dy = f$

The above system of equations has

- No solution if $\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$. Lines parallel to each other.
- Unique solution if $\frac{a}{c} = \frac{b}{d}$. Lines intersect at 1 particular point only.
- Infinite number of solutions if $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$

  The given system of equations can be brought to the form of number of equations < number of variables. In this case, both the lines are identical and hence overlap or intersect at infinite number of points.

Example: The system of equations $4x + 2y = 7$, $2x + y = 6$ have?

Solution: For the given set of equations,

$$\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f} \text{ as } \frac{4}{2} = \frac{2}{1} \neq \frac{7}{6}$$

$\therefore$ No solution exists for the following system of equations.

Example: The system of equations $x + 3y = 5$, $2x + 5y = -3$ have.

Solution: For the following system of equations,

$$\frac{1}{2} \neq \frac{3}{5} \text{ as } \frac{a}{c} \neq \frac{b}{d}$$

Thus, the following system has a unique solution.

System of Equations with 'n' variables

Consider a system of m equations and n variables.

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{32}x_2 + a_{21}x_3 + \ldots + a_{2n}x_n = b_2$

:

:

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_n$

The above system of equation can be put in matrix form

$$\begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots & a_{2n} \\ : & & \\ a_{21} & a_{m2} \ldots\ldots\ldots & a_{mn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ : \\ X_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{bmatrix}$$

$AX = B$

Where $X \rightarrow$ solution matrix.

If we write the elements of matrix B in the last column of matrix A, the resulting matrix is called augmented matrix and is denoted by (A | B)

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots & a_1 & b_1 \\ a_{22} & a \ldots\ldots\ldots & a_n & b_2 \\ : & & & \\ am_1 & am_2 \ldots\ldots\ldots & a_{mn} & b_m \end{bmatrix}_{(m+1)\times n}$$

Procedure to determine the solution:

1. Reduce (A | B) to row echelon form
2. Find (A| B) and $\rho(A)$
3. If $\rho(A) < \rho(A \mid B)$ or $\rho(A \mid B) \neq \rho(A)$ the given system of equations has no solution, system is inconsistent.
4. If $\rho(A \mid B) = \rho(A)$ = number of unknowns, the given system of equations has unique solution i.e. system is consistent.
5. If $\rho(A \mid B) = \rho(A)$ < number of unknowns, the given system of equations has infinite number of solutions.

6. If the number of equations < number of variables (r < n), the given system of equations will have infinite number of non-zero solutions. This non-zero solution can be found by assigning (n – r) variables or arbitrary constants. These n – r solutions are said to be linearly independent solutions.

r=rank of matrix & n=order of matrix

Inconsistent: The given system of equations is said to be inconsistent if they have no solution.

Consistent: The given system of equations is said to be consistent if they have a solution (unique or infinite).

Note: If the number of equations < number of variables (or) number of variables exceeds the number of equations, the system of equations have ∞ number of solutions.

Homogeneous system of linear equation:

Consider the following homogenous system of linear equations of m equations in n variables.

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0$

:

:

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0$

In matrix form

$$\begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{12} \dots a_{2n} \\ : & \\ a_{m1} & a_{m2} \dots a_{mn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ : \\ X_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ : \\ 0 \end{bmatrix}$$

The following system can be represented as, AX=O.

The solution of this system can have following cases:

Case-1: Inconsistency: This is not possible in a homogeneous system since such a system is always consistent since the trivial solution C $[0, 0, 0.......]^t$ always exists for a homogeneous system.

Case-2: Consistent unique solution: If r = n ; the equation AX = O will have only the trivial unique solution X = [0, 0, 0 ...]

Note: That $r = n \Rightarrow |A| \neq 0$ i.e., A is non-singular.

Case-3: Consistent infinite solution: If r <n we shall have n – r linearly independent non-trivial infinite solutions. Any linear combination of these (n – r) solutions will also be a solution of AX = 0

Thus in this case, the equation AX = O will have infinite solutions.

Note: If $r < n \Rightarrow |A|$ 0 i.e, A is a singular matrix.

Solved Examples

Example: How many solutions does the following system of equations have?

–x + 5y = –1, x – y = 2, x + 3y = 3

Solution: The matrix A and B are given as,

$$A = \begin{bmatrix} -1 & 5 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}; B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

The augmented matrix is given by,

$$A|B = \begin{bmatrix} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

Determinant of augmented matrix is,

$$|A|B| = -1(-3-6) - 5(3-2) - 1(3+1) = 0$$

So rank (A|B) = 2 = rank (A) = number of variables. Thus, this system has unique solution.

Example: Find the value of λ and μ for the system of equation.

x + y + z = 6, x + 4y + 6z = 20, x + 4y λ z = μ to have

(A) No solution

(B) Unique solution

(C) ∞ number of non-zero solution

Solution: The augmented matrix is given by,

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{bmatrix}$$

Applying row transformation,

$R_2 \to R_2 \to R_1$ and $R_3 \to R_3 \to R_1$

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-1 & \mu-6 \end{bmatrix}$$

$R_3 \to R_3 \to R_2$

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda-6 & \mu-20 \end{bmatrix}$$

(A) For No solution

$\rho(A) < \rho(A \mid B)$

i.e. $\lambda - 6 = 0$ so that last row of A is reduced to zero. Then, $\rho(A) = 2$

$\mu - 20 \neq 0$

Thus, last row of [A|B] is non-zero and $\rho(A \mid B) = 3$

$\lambda = 6$ and $\mu \neq 20$

(B) Unique solution

$\rho(A \mid B) = \rho(A)$ = number of unknowns.

$\rho(A \mid B) = \rho(A) = 3$

$\lambda - 6 \neq 0$, $\mu - 20 \neq 0$ or $\mu - 20 = 0$

$\therefore$ $\lambda \neq 6$, μ = any value

Thus, last row will be non-zero for both A and A | B.

(C) Infinite number of solutions

$\rho(A \mid B) = \rho(A) <$ number of unknowns.

$\lambda - 6 = 0$ and $\mu - 20 = 0$

$\therefore$ $\lambda = 6$ and $\mu = 20$

Example: For what value of α and β, the following system of equations

$x + y + z = 5$, $x + 3y + 3z = 9$, $2x + 2y + \alpha z = \beta$ have infinite number of solutions.

Solution: For infinite number of solutions $\rho(A \mid B) = \rho(A) <$ number of unknowns

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 2 & 2 & \alpha & \beta \end{bmatrix}$$

Apply Row Transformations, $R_2 \to R_2 - R_1$; $R_3 \to R_3 - 2R_1$

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \alpha-2 & \beta-10 \end{bmatrix}$$

$\alpha = 2$, $\beta = 10$

This ensures that $\rho(A \mid B) = \rho(A) = 2$

Example: Find the condition on a, b, c for which the following system of equations

$3x + 4y + 5z = a$

$4x + 5y + 6z = b$

$5x + 6y + 7z = c$

do not have a solution?

Solution:

The augmented matrix is given by,

$$[A|B] = \begin{bmatrix} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{bmatrix}$$

Applying row transformations, $R_2 \to 3R_2 - 4R_1$ and $R_3 \to 3R_3 - 5R_1$

$$[A|B] = \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & -2 & -4 & 3c-5a \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$[A|B] = \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & 0 & 0 & 3(a+c-2b) \end{bmatrix}$$

From above matrix $\rho(A) = 2$

If $3(a + c - 2b) \neq 0$, then $\rho(A \mid B) = 3 \neq \rho(A)$. Thus, no solution.

Example: For what values of λ does the system of equations have two linearly independent solutions.

$x + y + z = 0$, $(\lambda + 1)y + (\lambda + 1)z = 0$, $(\lambda^2 - 1)z = 0$

Solution: Since there are two linearly independent solutions $n - r = 2$

$3 - r = 2$

$r = 3 - 2 = 1$

If rank is 1, determinant of the matrix should be zero.

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda+1 & \lambda+1 \\ 0 & 0 & \lambda^2-1 \end{vmatrix}_{3\times3} = 0$$

$$1\left[(\lambda+1)(\lambda^2-1)-0\right]=0$$

$$(\lambda+1)(\lambda+1)(\lambda-1)=0$$

$$\lambda=+1,-1$$

For λ = –1, rank = 1 as the determinant of order 2 square sub-matrix goes to 0. If λ = 1, rank = 2

$\therefore$ λ = –1 is correct.

Example: The rank of $A_{3\times3}$ is 1. The system of equations AX=0 has how many linearly independent Solutions?

Solution: Order of Matrix n=3

Rank of Matrix r =1

Number of linearly independent Solutions = n-r = 2

Example: The system of equations $x + 3y - 2z = 0$, $2x - y + 4z = 0$, $x - 11y + 14z = 0$

trivial solutions?

Solution: The coefficient matrix A is given by,

$$A\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

Determinant of A is

$$|A|-1(-14+44)-3(28-4)-2(-22+1)=0$$

Thus, rank(A) < 3 = number of unknowns

Hence, this system has infinite non-trivial solutions.

Example: Find the value of k for which the system of equations.

$(3k - 8) \times 3y + 3z = 0$

$3x + (3k - 8)\, y + 3z = 0$

$3x + 3y + (3k - 8)\, z = 0$

has nontrivial solutions?

Solution: For non-trivial solutions, rank (A) < 3. Thus |A| = 0

$$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$, $R_2 \to R_2 - R_3$

$$(3k-8+6)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 3k-8-3 & 3-(3k-8) \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

i.e. $(3k-8+6)(3k-8-3)(3k-8-3)=0$

$$k=\frac{2}{3}, k=\frac{11}{3}, \frac{11}{3}$$

Eigen Values and eigen Vectors

Let A be an n $\times$ n matrix and λ is a scalar. The matrix (A – λI) is called a characteristic equation matrix.

|A – λI| is called characteristic determinant or characteristic polynomial. The equation |A – λI| = 0 , is called as characteristic equation.

The roots of the characteristic equation are called characteristic roots of eigen value or proper values or latent roots.

The set of eigen values of a matrix 'A' is called as spectrum of A.

Solved Examples

Example: Determine the eigen values of the matrix $A=\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

Solution: The characteristic matrix is

$$A-\lambda I=\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}=\begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

Characteristic polynomial is

$$|A-\lambda|=\begin{vmatrix} 4-\gamma & 2 \\ 2 & 4-\lambda \end{vmatrix}$$

Characteristic equation is $|A-\lambda|=0$

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}=(4-\lambda)^2-4=0$$

$16 + \lambda^2 - 8\lambda - 4 = 0$

$\lambda^2 - 8\lambda + 12 = 0$

$\lambda = 6,2$

Eigen Vectors: If λ is an eigen value of the matrix A, then there exists a non-zero vector X such that AX = λX. The X is called an eigen vector corresponding to the eigenvalue λ.

From Previous Example, consider the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+2 \\ \end{bmatrix} = \begin{bmatrix} 6 \\ \end{bmatrix} = 6\begin{bmatrix} 1 \\ \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to the eigen value λ = 6.

Similary, $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4+4 \\ 8+2 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an Eigen vector.

Example: Find the eigen values and eigen vectors of matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

Solution: The characteristic equation is,

$$|A - \lambda| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$-(3-\lambda)(3+\lambda) - 16 = 0$$

$$\lambda^2 - 9 - 16 = 0$$

$$\lambda^2 - 25 = 0$$

$$\lambda^2 = 25$$

$$\lambda = \pm 5$$

Case-1: When $\lambda = 5$

$AX = \lambda X$

$AX = \lambda XI$

$AX - \lambda XI = 0$

$(A - \lambda I)\, X = 0$

$$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = 5$

$$\begin{bmatrix} -2 & 4 \\ 4 & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \end{bmatrix}$$

$-2x_1 + 4x_2 = 0 \rightarrow (1)$

$4x_1 - 8x_2 = 0 \rightarrow (2)$

Equation (2) is obtained from Equation (1) by multiplying a factor of '-2'. Thus, unique solution is impossible.

$$\frac{x_1}{x_2} = \frac{2}{1}$$

Therefore the eigen vector corresponding to eigen value λ = 5

$$X_1 = \begin{bmatrix} x \\ x_2^1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case-2: When λ = –5

$AX - \lambda IX = 0$

$$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = -5$

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \end{bmatrix}$$

$8x_1 + 4x_2 = 0$

$4x_1 + 2x_2 = 0$

The first equation in this case can be obtained from second by multiplying a factor of '2'

i.e. $4x_1 = -2x_2$

$$x_1 = \frac{-x_2}{2}$$

$$\therefore \frac{x_1}{x_2} = \frac{-1}{2}$$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$A^T = \begin{bmatrix} 3 & 4 \\ 4 & \end{bmatrix} =$ A. Thus, the matrix A is symmetric.

For the two eigenvectors obtained,

$x_1 x_1^T = 0 \;\&\; {}^T_1 x_2 = 0.$ Thus, both eigen vectors are orthogonal.

Remark:

The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are always orthogonal.

$$\begin{vmatrix} x_1 & x_2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4 - (-1) = 5 \neq 0 \Rightarrow \text{L.I.}$$

The eigen vectors corresponding to distinct eigen values of any square matrix are always linearly independent. The eigen vectors corresponding to repeated eigen values may be linearly independent or linearly dependent vector.

Example: Find the eigen values and corresponding eigen vectors of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: The characteristic matrix is,

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

Characteristic equation is,

$$|A - \lambda| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0$$

$\lambda = 1,1,1$ are the eigenvalues of A.

To determine the eigen-vectors, $(A - \lambda I)\, X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since, $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = x_3 = 0$, x_1 can be any value.

Number of equations < number of variables Infinite number of nonzero solutions, r = 2, n =3. If r < n, put n – r = 3 – 2 = 1

This means one variable can be assigned any arbitrary value.

Note: Zero vectors cannot be eigen vectors.

$$\therefore \text{ Eigen vectors } = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ c can be 1, 2}$$

Corresponding to 3 repeated eigen values, there exists only 1. Linearly independent eigen vector.

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Example: Find the eigen values and corresponding eigen vectors of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: For any upper triangular matrix, diagonal elements represent the eigen values as seen in previous example.

Thus, $\lambda = 1,1,1$

To determine the eigen-vectors, $(A - \lambda I)\, x = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} [x] = 0$$

$$\text{Since, } \lambda = 1; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

In this case, $x_2 = 0$ but x_1 & x_3 can have any values. r < n, put n – r = 3 – 1 = 2

Two variables can be assumed as arbitrary values.

Put $x_1 = c_1$, $x_2 = 0$, $x_3 = c_2$

$$X = \begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The two linearly independent eigen-vectors are,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \& \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Both these vectors cannot be expressed as multiple of one another. Thus, both are linearly independent. Therefore, corresponding to 3 equal eigen values, there exist two linearly independent eigen vectors.

Example: Determine the eigen values and eigen vectors of the matrix

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Solution: The characteristic equation is,

$$|A-\lambda I| = \begin{bmatrix}1-\lambda & 0 & 0\\0 & 1-\lambda & 0\\0 & 0 & 1-\lambda\end{bmatrix} = (1-\lambda)^3$$

λ = 1,1,1

Thus, eigen values of a diagonal matrix are same as its diagonal elements.

To determine eigen vectors, $(A - \lambda I)\, X = 0$

$\lambda=1$

$$\text{i.e.}\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

r = 0, n = 3

r < n, put n – r = 3 – 0 = 3

Three variables can assume arbitrary values.

$x_1 = c_1$, $x_2 = c_2$, $x_3 = c_3$

$$x = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = c_1\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\downarrow\, X_1} + c_2\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\downarrow\, X_2} + c_3\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\downarrow\, X_3}$$

$$\begin{vmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{vmatrix} = 1 \neq 0$$

The vectors are linearly independent.

There exists 3 linearly independent eigen vectors.

$$x_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, x_2 \begin{bmatrix}0\\1\\0\end{bmatrix}, x_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Corresponding to 3 equal eigen values.

Note: If an eigen value λ is repeated m times, then the corresponding number of linearly independent eigen Vectors is given by

P = n–r where $1 \le P \le m$

P = Number of linearly independent eigen vectors. n = order of matrix or number of variables.

$r = \rho\,(A - \lambda I)$ i.e. Rank of characteristic Matrix.

m is the number of times an eigen value is repeated.

If some eigen values are repeated and some are non-repeated, then the corresponding eigen vectors may be linearly independent or linearly dependent.

Example: The number of linearly independent eigen vector of

$$A = \begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}$$

Solution: Since the matrix is upper triangular, diagonal elements represent eigenvalues λ = 2,2

To determine eigen vectors, $(A - \lambda I)\, X = 0$

$$\begin{bmatrix}2-\lambda & 1\\0 & \end{bmatrix} = A - \lambda I$$

Since, $\lambda = 2$

Characteristic Matrix becomes, $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$

P = n – r, where $1 \le P \le 2$

Since, there is only one non-zero row, r =1

P = 2 – 1 = 1 linearly independent eigen vector

Example: The number of LI eigen vector of

$$A = \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$$

Solution: Since, this is a diagonal matrix, the diagonal elements represent eigen values.

λ = 3,3

To determine eigen vectors, $(A - \lambda I)\, X = 0$

$$\begin{bmatrix}3-\lambda & 0\\0 & 3-\kappa\end{bmatrix}$$

Since $\lambda = 3$

Characteristic Matrix becomes, $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

P = n–r = 2 – 0 = 2

Thus, there are 2 linearly independent eigen vectors.

Properties of eigen values and eigen vectors:

Let A be on $n \times n$ matrix. The eigen values of A are $\lambda_1, \lambda_2 \ldots \lambda_n$. Trace (A) = $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n$

Product of the eigen values = Determinant of a matrix.

- The eigen values of a upper triangle or lower triangle or diagonal or scalar or identity matrix is its diagonal elements.
- The eigen values of A and A^T are same.
- The eigen values of A and $P^{-1}AP$ (P is a non-singular matrix) are same.
- The eigen values of orthogonal matrix are unit modular i.e. they have a magnitude of 1.
- If λ is an eigen value $|\lambda| = 1$ of an orthogonal matrix, then $1/\lambda$ is also one of its eigen value.
- The eigen values of real symmetric matric are real.
- The eigen value of skew symmetric matric are purely imaginary or zeros.
- For a real matrix if a+ib is an eigen value then a-ib is also another eigen value of the same matrix.
- The eigen vector of A and A^{-1} are same.
- The eigen vectors of A and A^m are same.
- If λ is an eigen value of a matrix A then $K\lambda$ is an eigen value of kA.
    - $\frac{1}{\lambda}$ is an Eigen value of A^{-1}
    - λ^2 is an eigen value of A^2
    - λ^m is an eigen value of A^m
    - $\frac{|A|}{\lambda}$ is an Eigen value of Adj A
    - $\lambda \pm K$ is an eigen value of $A \pm KI$
    - $(\lambda \pm K)^2$ is an eigen value of $(A \pm KI)^2$
    - $\frac{1}{\lambda \pm K}$ is an Eigen value of $(A \pm KI)^{-1}$

Note:

- A matrix A is said to be singular. i.e. Det A = 0; if one of its eigen value is zero. Its converse is also true.
- If one of the eigen values of a matrix A is zero, then the homogenous system of equations has infinite number of non-zero solutions.

Solved Examples

Example: The eigen value of

$$A = \begin{vmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{vmatrix} \text{ are}$$

(A) 3, 3+5j, –6 –j

(B) $-6 + 5j,\ 3 - j,\ 3+j$

(C) $3 - j,\ 5 + j,\ 3 + j$

(D) $3,\ -1 + 3j,\ -1 - 3j$

Solution: $\text{Tr}[A] = \lambda_1 + \lambda_2 + \lambda_3 = (-1) + (-1) + 3 = 1$

Verifying the options,

Only option (d) satisfies the above equation $3,\ -1 + 3j,\ -1 - 3j$

Example: The eigen values and eigen vectors of a 2×2 matrix are given by eigen value.

$\lambda_1 = 8$ and $\lambda_2 = 4$

Eigen vectors $\gamma_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\gamma_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is

(A) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 8 & 8 \\ 8 & 4 \end{bmatrix}$

Solution: Since Trace of matrix is same as sum of eigen values Trace = 8+4 = 12

Options (A) and (D) have same trace.

Product of eigen values is same as determinant Product = 8 x 4 =32

Only option (A) has the determinant 32.

Example: The vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an Eigen vector of

$A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}$. The eigen value corresponding to the eigen vector is ?

Solution: If X is the eigen-vector of matrix A then $(A - \lambda I)x = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-1 - 2 \times 2 + -\lambda \times -1 = 0$

$-1 - 4 + \lambda = 0$

$\lambda = 5$

Example: For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, the eigen value corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is ________________ ?

Solution: For any eigenvector X,

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} 101 \\ 101 \end{bmatrix} = 0$$

$2 \times 101 + (4 - \lambda)\, 101 = 0$

$101\,(2 + 4 - \lambda) = 0$

$6 - \lambda = 0$

$\lambda = 6$

Example: Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ The eigen values of $3A^3 + 5A^2 - 6A + 2I$ is?

Solution: Since the matrix is upper triangular, diagonal elements represent eigen values.

$\lambda = 1, 3, -2$

If λ is eigen value of A, then eigen value of $3A^3 + 5A^2 - 6A + 2I$ is $3\lambda^3 + 5\lambda^2 - 6\lambda + 2$

$\lambda = 1 \Rightarrow 3(1)^3 + 5(1)^2 - 6\,(1) + 2 = 4$

$\lambda = 3 \Rightarrow 3(3)^3 + 5(3)^2 - 6 \times 3 + 2 = 110$

$\lambda = -2 \Rightarrow 3(-2)^3 + 5(-2)^2 - 6 \times -2 + 2 = 10$

Example: The eigen value of a 3 × 3 matrix are given by 1, 2, 3.

Find

(A) $Tr(A^2 + A^{-1} + adj\ A)$

(B) $Det(A^2 + A^{-1} + adj\ A)$

Solution: eigen Values of A are $\lambda = 1, 2, 3$

$|A| = 1 \times 2 \times 3 = 6$

Eigen Values of $A^{-1} = \frac{1}{\lambda} = 1, \frac{1}{2}, \frac{1}{3}$

Eigen Values of $A^2 = \lambda^2 = 1, 4, 9$

Eigen Values of Adj(A)= $= \frac{|A|}{\lambda} = 6, 3, 2$

(A) EigenValues of $A^2 + A^{-1} + adj\ A$

$= \lambda^2 + \frac{1}{\lambda} + \frac{|A|}{\lambda}$

Eigen Values = 8 + 7.5 +11.33 = 26.83

Thus, $Tr(A^2 + A^{-1} + adj\ A) = 26.83$

(B) Det $\left(A^2 + A^{-1} + adj\ A\right) = 8 \times \frac{15}{2} \times \frac{34}{3} = 680$

Example: The eigen vectors of a 2 × 2 matric $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are given by $\begin{bmatrix} 1 \\ a \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix}$ What is a + b?

Solution: Since the matrix is upper triangular, the diagonal values represent eigen values.

$\lambda = 1, 2$

Since, $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

For $\lambda = 1$

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_2 = 0$ ...(1)

$x_2 = 0$ ...(2)

Thus, there is only one independent variable x_1

Since only one non-zero column, r = 1, n = 2

If r < n, n–r = 2 – 1 =1

Thus, only one variable can 1 have arbitrary value.

$$x_1 = \begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix} \text{or} \begin{bmatrix} 1 \\ b \end{bmatrix} a = 0 \text{ or } b = 0$$

For $\lambda = 2$, $(A - \lambda I)X = 0$

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + 2x_2 = 0$

$$\frac{x_1}{x_2} = \frac{2}{1}$$

If $x_1 = 1$, then $x_2 = \frac{1}{2}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

i.e. $b = \frac{1}{2}$

$$\therefore a + b = 0 + \frac{1}{2} = \frac{1}{2}$$

Method 2:

If X is the eigen vector then, AX = λX

λ = 1, 2

If $\lambda = 1 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 1\begin{bmatrix} 1 \\ \end{bmatrix}$

$$\begin{bmatrix} 1 + 2a \\ 2a \end{bmatrix} = \begin{bmatrix} 1 \\ \end{bmatrix}$$

$1 + 2a = 1$

$a = 0$

If $\lambda = 2$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = 2\begin{bmatrix} 1 \\ \end{bmatrix}$$

$$\begin{bmatrix} 1 + 2b \\ 2b \end{bmatrix} = \begin{bmatrix} 2 \\ 2b \end{bmatrix}$$

$1 + 2b = 2$

$2b = 1$

$b = \frac{1}{2}$

$\therefore a + b = \frac{1}{2}$

Cayley Hamilton Theorem

Every square matrix satisfies its own characteristic equation using Cayley Hamilton theorem, we can find Inverse of a matrix or Higher powers of a matrix

Example: Consider matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ Its characteristic equation is $\lambda^2 - 8\lambda + 12 = 0$

By Cayley Hamilton theorem we have $A^2 - 8A + 12I = 0$

Note: The constant term of any characteristic equation of any order square matrix is equal to the determinant of a matrix.

We can compute positive and negative powers of A as,

Positive Power of A	**Negative Power of A**
Consider A2 – 8A + 12I = 0	$A^2 - 8A + 12I = 0$
$A^2 = 8A - 12I$	$12I = 8A - A^2$
$A^3 = 8A^2 - 12A$	$I = \frac{1}{12}\left[-A^2 + 8A\right]$
$A^4 = 8A^3 - 12A^2$	$A^{-1} = \frac{1}{12}\left[-A + 8I\right]$
....	$A^{-2} = \frac{1}{12}\left[-I + 8A^{-1}\right]$
....	$A^{-3} = \frac{1}{12}\left[-A^{-1} + 8A^{-2}\right]$
....	....
	....

Table 1.1

Consider $A_{2\times2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Characteristic equation is $|A - \lambda| = 0$

i.e. $\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$

$(a - \lambda)(d - \lambda) - bc = 0$

$ad - a\lambda - \lambda d + \lambda^2 - bc = 0$

$\lambda^2 - \lambda(a + d) + (ad - bc) = 0$

$\lambda^2 - \lambda\left[Tr(A)\right] - |A| = 0$ which is the characteristic equation

According to Cayley Hamilton Theorem we have We can replace $\lambda \to A$

$A^2 - A\left(\mathrm{Tr}\left(A\right)\right) + |A| I = 0$

Consider a 3 × 3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Characteristic equation:

$\lambda^3 - \lambda^2\left[\mathrm{Tr}\left(A\right)\right] + \lambda$

$\left\{\begin{vmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\right\} + |A| = 0$

Solved Examples

Example: Find A^8 if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ using Cayley Hamilton theorem

Solution: $|A| = -1 - 4 = -5$

Characteristic equation:

$\lambda^2 - \lambda\left(1 + \left(-1\right)\right) + -5 = 0$

$\lambda^2 - 5 = 0$

By Cayley Hamilton theorem

$A^2 - 5I = 0$

$A^2 = 5I$

$(A^2)^4 = (5I)^4 = 625I = 625\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$

Example: The characteristic equation of a 3 × 3 matrix P is given by $\alpha(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$, where I Identity matrix. The inverse of matrix P will be _______________ ?

Solution: By cayley theorem theorem

$\lambda \to P$

$P^3 + P^2 + 2P + I = 0$

$I = -(P^3 + P^2 + 2P)$

$P^{-1}I = -(P^2 + P + 2I)$

$P^{-1} = -(P^2 + P + 2I)$

Example: Given $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Express A^3 as a linear polynomial?

Solution: Characteristic equation is $\lambda^2 + 5\lambda + 6I = 0$

$A^2 + 5A + 6I = 0$

$A^2 = -(5A + 6I)$ .... (1)

$A^3 = -(5A^2 + 6A) = -5(-5A - 6I) - 6A$ from (1)

$A^3 = 19A + 30I$

Practice Questions

1. If $A = \begin{vmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{vmatrix}$ and

$\mathrm{adj}\left(A\right) = \begin{vmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & k & 7 \end{vmatrix}$ then $k = ?$

(A) -5

(B) 3

(C) -3

(D) 5

2. If $A^T = A^{-1}$,, where A iis a reall matriix ,, then A is

(A) Normall

(B) Symmetriic

(C) Hermiitiian

(D) Orthogonall

3. 3. If a matrix A is m x n and B is n x p, the number of multiplication operations and addition operations needed to calculate the matrix AB, respectively, are:

(A) mn^2p, mpn

(B) mpn (n-1)

(C) mpn, mp (n-1)

(D) mn^2p, (m + p) n

4. The matrices $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ & $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ commute under multiplication.

(A) If $a = b$ or $\theta = n\pi$, n is an integer

(B) Always

(C) Never

(D) If $a\cos\theta \neq b\sin\theta$

5. The determinant of the matrix

$$\begin{bmatrix} 6 & -8 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(A) 11

(B) -48

(C) 0

(D) -24

6. If A and B are two matrices and if AB exists, ten BA exists

(A) Only if A has an many rows as B has columns

(B) Only if both A and B are square matrices

(C) Only if A and B are skew symmetric matrices

(D) Only if both A and B are symmetric

7. The inverse of the matrix $A = \begin{bmatrix} -3 & 5 \\ 2 & 1 \end{bmatrix}$

(A) $\begin{bmatrix} \frac{5}{13} & -\frac{1}{13} \\ \frac{2}{13} & \frac{3}{13} \end{bmatrix}$ (C) $\begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ \frac{2}{13} & \frac{3}{13} \end{bmatrix}$

(B) $\begin{bmatrix} \frac{2}{13} & \frac{5}{13} \\ \frac{-1}{13} & \frac{3}{13} \end{bmatrix}$ (D) $\begin{bmatrix} \frac{1}{13} & -\frac{5}{13} \\ \frac{2}{13} & \frac{3}{13} \end{bmatrix}$

8. If A and B are real symmetric matrices of order n then which of the following is true.

(A) AAT = I

(B) $A = A^{-1}$

(C) AB = BA

(D) $(AB)^T = B^T A^T$

9. The matrix $\begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$ is an inverse of the matrix $\begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}$

(A) True

(B) False

10. Give matrix $L = \begin{vmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 5 \end{vmatrix}$ and $M = \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix}$ then $L \times M$ is

(A) $\begin{vmatrix} 8 & 1 \\ 13 & 2 \\ 22 & 5 \end{vmatrix}$ (B) $\begin{vmatrix} 6 & 5 \\ 9 & 8 \\ 12 & 13 \end{vmatrix}$

(C) $\begin{vmatrix} 2 & 13 \\ 5 & 22 \\ 6 & 2 \end{vmatrix}$ (D) $\begin{vmatrix} 9 & 4 \\ 0 & 5 \end{vmatrix}$

Answer Key

1 – (A)	2 – (D)	3 – (C)	4 – (A)	5 – (B)
6 – (A)	7 – (C)	8 – (D)	9 – True	10 – (B)