A TYPE-FREE ONLINE REUSABLE SERVICE COMPOSITION ALGORITHM WITH INFINITE HORIZON FOR CLOUD MANUFACTURING SYSTEM

ABSTRACT

Dynamic service composition (DSC) plays an important resource allocation role in cloud manufacturing (CMfg) system. We study an online reusable service composition problem for CMfg system under uncertainty. *Reusability* refers to the fact that cloud services are available again after being occupied for a period of time. Different from existing cloud resource allocation practice, it is usually difficult to distinguish typical customer types when allocate reusable services in CMfg environment, as required service types, service amount and serving time vary. Moreover, cloud commerce runs 24 without a break, meaning that CMfg platform has infinite planning horizon. In this regard, we develop an algorithm based on Multiplicative Weight Update scheme and Primal-Dual formulation for reusable cloud services. More importantly, the algorithm accommodates service requests with random units and random service duration for infinite horizon. The algorithm makes effective use of historical records, and balances the trade-off among the reward, dual-cost, and services occupied. Theoretical analysis proves that the algorithm achieves $(1-\epsilon)$ fraction of the optimal expected rewards. The error term diminishes as the horizon extends. Along with analysis, we conduct experiments based on synthetic data, showing the effectiveness of our proposed algorithm.

Keywords Cloud manufacturing · concentration bounds · online algorithm · service composition

A Auxiliary Results

Here, we provide necessary inequalities for analysis purpose

Proposition A.1. ***Azuma-Hoeffding Inequality***

Let N be a positive integer and C be a positive real number. Suppose the random variables $X_1, ..., X_N$ constitute a martingale difference sequence with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^N$, i.e. $\mathbb{E}[X_n|\mathcal{F}_{n-1}] = 0$ almost surely for every $n \in [N]$. In addition, suppose $X_n \in [-C, C]$ almost surely for every $n \in [N]$. For any fixed confidence level $\delta \in (0, 1)$, it holds that[1]

$$\Pr\left[\frac{1}{N}\sum_{n=1}^{N} X_n \geq C\sqrt{\frac{\log(1/\delta)}{2N}}\right] \leq \delta \tag{1}$$

or

$$\Pr\left[\frac{1}{N}\sum_{n=1}^{N} X_n \leq -C\sqrt{\frac{\log(1/\delta)}{2N}}\right] \leq \delta \tag{2}$$

Proposition A.2. ***Multiplicative Chernoff inequality***

We give a simplified extension of multiplicative Chernoff inequality[2]*. Suppose random variables $\{X_n\}_{n=1}^N$ are independent, and that $\Pr\left(X_n \in [0, C] \text{ for all } n \in [N]\right) = 1$ for some $C \in \mathbb{R}_+$. Denote $\mu = \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^N X_n\right]$. The following concentration inequalities hold for any fixed confidence level $\delta \in (0, 1)$*

$$\Pr\left(\frac{1}{N}\sum_{n=1}^{N} X_n - \mu \geq \sqrt{\frac{3C\mu}{N}\log\frac{1}{\delta}}\right) \leq \delta \tag{3}$$

$$\Pr\left(\frac{1}{N}\sum_{n=1}^{N} X_n - \mu \leq -\sqrt{\frac{2C\mu}{N}\log\frac{1}{\delta}}\right) \leq \delta. \tag{4}$$

Proposition A.3. ***Multiplicative Weight Update***

Let $\{l(s)\}_{s=1}^T$ be an arbitrary sequence of vectors, where $l(s) = (l_i(s))_{i\in[I]} \in [-C, C]^I$ for each $s \in [T]$. Consider the sequence of vectors $\vartheta(1), \ldots, \vartheta(T)$, where $\vartheta(s) = (\vartheta_i(s))_{i\in[I]} \in \Delta^I$ is defined as

$$\vartheta_i(s) = \frac{\exp\left[-\eta(s)\sum_{t=1}^{s-1} l_i(t)\right]}{\sum_{j=1}^{I}\exp\left[-\eta(s)\sum_{t=1}^{s-1} l_j(t)\right]}, \text{ where } \eta(s) = \frac{\sqrt{\log I}}{C\sqrt{s}} \tag{5}$$

for each $s \in [T]$ and $i \in [I]$. Then, for any $i \in [I]$, it holds that[3]

$$\frac{1}{T}\sum_{s=1}^{T} l_i(s) \geq \frac{1}{T}\sum_{s=1}^{T}\sum_{j=1}^{I}\vartheta_j(s) l_j(s) - 2C\sqrt{\frac{\log I}{T}} \tag{6}$$

[1]Mohri M., Rostamizadeh A., and Talwalkar A., "Concentration Inequalities," in *Foundations of Machine Learning*, second ed. USA: MIT press, 2018, pp. 441–442.

[2]Mohri M., Rostamizadeh A., and Talwalkar A., "Concentration Inequalities," in *Foundations of Machine Learning*, second ed. USA: MIT press, 2018, pp. 439–440.

[3]Orabona F., "A modern introduction to online learning," in *arXiv*, Version 6, Chapter 7.5. Accessed on: Dec 31 2019, DOI: 1912.13213, [Online].

B Proofs

B.1 Proof of Lemma 1

Proof. Let π be a non-anticipatory feasible policy that achieves the expected optimum $\mathbb{E}\left[\lambda_*^C\right]$ in (IP-C), i.e. $\mathbb{E}\left[\frac{1}{TI}\sum_{t=1}^{T}\sum_{i=1}^{I} W_i(t)A_i(t)D_i(t)X^\pi(t)\right] = \mathbb{E}[\text{opt(IP-C)}]$. Define $\boldsymbol{x} = \{x(t)\}_{t\in[T]}$ as $x(t) = \Pr(X^\pi(t) = 1)$. We claim that $\boldsymbol{x}$ is feasible to (LP-E), and verifying the claims about the feasibility and the objective value proves the claim.

We first verify the feasibility to (LP-E). Since the policy π satisfies the reusable resource constraints, the inequality $\sum_{\tau=1}^{t} \mathbf{1}\left(D_i(\tau) \geq t-\tau+1\right) A_i(\tau)X^\pi(\tau) \leq c_i$ holds for all $i \in [I], t \in [T]$. Taking expectation over $X^\pi(\tau), D_i(\tau)$, and $A_i(\tau)$ for $\tau = 1, 2, \dots, t$ on the L.H.S. gives

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{\tau=1}^{t} \mathbf{1}\left(D_i(\tau) \geq t-\tau+1\right) A_i(\tau)X^\pi(\tau)\right] \\
=&\sum_{\tau=1}^{t} \mathbb{E}\left[\mathbf{1}\left(D_i(\tau) \geq t-\tau+1\right) A_i(\tau)X^\pi(\tau)\right] \\
=&\sum_{\tau=1}^{t} \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left(D_i(\tau) \geq t-\tau+1\right) A_i(\tau) \mid X^\pi(\tau)\right] X^\pi(\tau)\right] \\
=&\sum_{\tau=1}^{t} \mathbb{E}\left[\mathbf{1}\left(D_i \geq t-\tau+1\right) A_i\right] x(\tau) \leq c_i
\end{aligned}
$$

Hence, the claim about the objective value is shown, and the Lemma is proved. ☐

B.2 Proof of Lemma 2

Proof. We prove this lemma by Reductio ad Adsurdum. First, for an arbitrary $i \in [I]$, if $\hat{p}_i v_i > \bar{r}$, then

$$
f(\boldsymbol{p}) = \sum_{t=1}^{T}\sum_{i=1}^{I} c_i p_{it} + \frac{1}{TI}\sum_{t=1}^{T}\left\{\sum_{i=1}^{I} v_i(w_i - TIp_{it})\right\}^+ \geq \sum_{t=1}^{T}\sum_{i=1}^{I} c_i p_{it} + 0
$$

whereas

$$
f(0) = \sum_{i=1}^{I} v_i w_i \leq I\bar{r}
$$

Thus, based on Assumption 1, we have

$$
f(\boldsymbol{p}) \geq \sum_{t=1}^{T}\sum_{i=1}^{I} c_i p_{it} \geq \sum_{t=1}^{T}\sum_{i=1}^{I} v_i p_{it} = \sum_{i=1}^{I} v_i \bar{p}_i > I\bar{r} \geq f(\mathbf{0})
$$

Hence, $\boldsymbol{p}$ cannot be the optimal solution. The lemma is proved. ☐

B.3 Proof of Lemma 3

Proof. The proof relies on a crucial application of Proposition 3, with a judicious choice of $l(1), \dots, l(\tau)$ (where we set $\tau = \tau^{(q-1)}$) that underpins the construction of Algorithm 2. Now, for each $s \in [\tau^{(q-1)}]$, we define

$$
l_i(s) = \begin{cases} W_i(s)V_i(s) - (\hat{\lambda}_*^{(q)} + \epsilon_B^{(q)}), & \forall i \in [I] \\ -\hat{p}_i^{(q)} V_i(s), & \forall i \in [I] \end{cases} \tag{7}
$$

It is evident that $|l_i(s)| \leq \bar{r}, \forall i \in [I], s \in [\tau^{(q-1)/2}]$. In addition, under the specification of $\{l(s)\}_{s=1}^{\tau}$ in (7), it can be directly verified that the MWU weigh vector $\vartheta(s)$ in (5) is equal to $\left\{\boldsymbol{\phi}^{(q)}(s), \boldsymbol{\psi}^{(q)}(s)\right\}$ for each $s \in [\tau^{(q-1)/2}]$. Applying Proposition A.3 gives us the following inequalities (which simultaneously hold with certainty)

$$\left[\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_i(s)V_i(s)\right] - \left(\hat{\lambda}_*^{(q)} + \epsilon_B^{(q)}\right) \geq \Xi^{(q)} - 2\bar{r}\sqrt{\frac{2\log(2I)}{\tau^{(q-1)}}} \quad \forall i \in [I] \tag{8}$$

$$-\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)/2}} \hat{p}_i^{(q)} V_i(s) \geq \Xi^{(q)} - 2\bar{r}\sqrt{\frac{2\log(2I)}{\tau^{(q-1)}}} \quad \forall i \in [I] \tag{9}$$

where

$$\Xi^{(q)} = \frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2}\left\{\sum_{l\in[I]}\phi_l^{(q)}(s)\left[W_i(s)V_i(s) - \left(\hat{\lambda}_*^{(q)} + \epsilon_B^{(q)}\right)\right] + \sum_{l\in[I]}\psi_l^{(q)}(s)\left(-\hat{p}_i^{(q)}V_i(s)\right)\right\} \tag{10}$$

and recall

$$\epsilon_B^{(q)} = 2\bar{r}\sqrt{\frac{\log(1/\delta)}{\tau^{(q-1)}}} \tag{11}$$

Then, to prove this lemma, the following 3 inequalities have to be satisfied:

$$\Pr\left(\Xi^{(q)} \geq -2\bar{r}\sqrt{\frac{\log(1/\delta)}{\tau^{(q-1)}}}\right) \geq 1-\delta \quad \text{or} \quad \Pr\left(\Xi^{(q)} \leq 2\bar{r}\sqrt{\frac{\log(1/\delta)}{\tau^{(q-1)}}}\right) \geq 1-\delta \tag{12}$$

$$\Pr\left(\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_\cdot(s)V_\cdot(s;\phi,\psi) \geq \hat{\lambda}_*^{(q)} + \epsilon_B^{(q)} - \epsilon_A^{(q)}\right) \geq 1-2\delta \tag{13}$$

$$\Pr\left(\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} \hat{p}_\cdot V_\cdot(s;\phi,\psi) \leq \epsilon_A^{(q)}\right) \geq 1-2\delta \tag{14}$$

where

$$\epsilon_A^{(q)} = 2\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}} + 2\bar{r}\sqrt{\frac{2\log(2I)}{\tau^{(q-1)}}} + \bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}} \tag{15}$$

First of all, we prove (12). It is easy to see (10) has another formulation

$$\Xi^{(q)} = \frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2}\left\{\sum_{l\in[I]}\left[\phi_l^{(q)}(s)\left(W_i(s)V_i(s) - \hat{\lambda}_*^{(q)} - \epsilon_B^{(q)}\right) - \psi_l^{(q)}(s)\left(\hat{p}_i^{(q)}V_i(s)\right)\right]\right\} \tag{16}$$

Notice that weight vectors $\{\phi^{(q)}(s), \psi^{(q)}(s)\}^I$ are all within [0, 1], whereas $\left|\hat{p}_i^{(q)}V_i(s)\right| \leq \bar{r}$ and $\left|W_i(s)V_i(s) - \hat{\lambda}_*^{(q)} - \epsilon_B^{(q)}\right| \leq \bar{r}$. By considering $\left[\phi_l^{(q)}(s)\left(W_i(s)V_i(s) - \hat{\lambda}_*^{(q)} - \epsilon_B^{(q)}\right) - \psi_l^{(q)}(s)\left(\hat{p}_i^{(q)}V_i(s)\right)\right] \in [0, \bar{r}]$, we apply Proposition A.1, and give (12).

Secondly, we prove (13). Given (8), (12) and Proposition A.1 for the martingale difference sequence with respect to the filtration $\{\mathcal{F}(s)\}_{s=1}^{\tau^{(q-1)}}$ defined as $\{\mathcal{F}(s)\} = \sigma(\{\hat{\lambda}_*^{(q)}\} \cup \{V_i(s)\})$. The expectation $\mathbb{E}[W_i(s)V_i(s) \mid \mathcal{F}(s-1)]$ is only taken over the randomness in $V_i(s)$, and note that $\{\phi(s), \psi(s)\}^I$ are $\mathcal{F}(s-1)$-measurable. Now we proceed with $\{W_i(s)V_i(s) - W_i(s)V_i(s;\phi,\psi)\}_{s=1}^{\tau^{(q-1)}/2}$ for all $i \in [I]$. By applying Azuma-Hoeffding inequality, we have

$$\Pr\left(\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_i(s)V_i(s) - \frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_i(s)V_i(s;\phi,\psi) \geq \bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}}\right) \leq \frac{\delta}{I}$$

for any arbitrary i. Reformulate and take a union bound for all $i \in [I]$, we have

$$\Pr\left(\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_\cdot(s)V_\cdot(s) \leq \bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}} + \frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_i(s)V_i(s;\phi,\psi)\right) \geq 1-\delta \tag{17}$$

Subtract the term $\left(\hat{\lambda}_*^{(q)} + \epsilon_B^{(q)}\right)$ into both sides of (17), and introduce $\Xi^{(q)}$ in (8) gives

$$\Pr\left(\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_\cdot(s)V_\cdot(s) - \left(\hat{\lambda}_*^{(q)} + \epsilon_B^{(q)}\right) \geq \bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}} + \frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_i(s)V_i(s;\phi,\psi) - \left(\hat{\lambda}_*^{(q)} + \epsilon_B^{(q)}\right)\right) \leq \delta$$

$$\Pr\left(\Xi^{(q)} - 2\bar{r}\sqrt{\frac{2\log(2I)}{\tau^{(q-1)}}} \geq \bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}} + \frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_i(s)V_i(s;\phi,\psi) - \left(\hat{\lambda}_*^{(q)} + \epsilon_B^{(q)}\right)\right) \leq \delta$$

$$\Pr\left(\Xi^{(q)} \geq 2\bar{r}\sqrt{\frac{2\log(2I)}{\tau^{(q-1)}}} + \bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}} + \frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_i(s)V_i(s;\phi,\psi) - \left(\hat{\lambda}_*^{(q)} + \epsilon_B^{(q)}\right)\right) \leq \delta$$

Introduce (12) to eliminate $\Xi^{(q)}$, and recall $\epsilon_A^{(q)}$ (15) gives

$$\Pr\left(\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_i(s)V_i(s;\phi,\psi) \leq \hat{\lambda}_*^{(q)} + \epsilon_B^{(q)} - 2\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}} - 2\bar{r}\sqrt{\frac{2\log(2I)}{\tau^{(q-1)}}} - \bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}}\right) \leq 2\delta$$

$$\Pr\left(\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} W_i(s)V_i(s;\phi,\psi) \geq \hat{\lambda}_*^{(q)} + \epsilon_B^{(q)} - \epsilon_A^{(q)}\right) \geq 1 - 2\delta$$

Therefore, inequality (13) is proved. Likewise, inequality (14) can be proved in a similar way

$$\Pr\left(\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} \hat{p}_i V_i(s;\phi,\psi) \leq 2\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}} + 2\bar{r}\sqrt{\frac{2\log(2I)}{\tau^{(q-1)}}} + \bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}}\right) \geq 1 - 2\delta$$

$$\Pr\left(\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} \hat{p}_i V_i(s;\phi,\psi) \leq \epsilon_A^{(q)}\right) \geq 1 - 2\delta \qquad \forall i \in [I]$$

□

B.4 Proof of Lemma 4

Proof. $\hat{p}_i = \sum_{t=1}^{T} p_{it}$ is the estimated value of the real dual price p_i. Thus, we have to provide the estimation error of p_i. Based on (4) in Proposition A.2, and suppose the maximum dual price of an arbitrary service is $\bar{p}$, we have

$$\Pr\left(\hat{p}_i \leq p_i - \sqrt{\frac{2\bar{p}p_i}{\tau^{(q-1)}/2}\log\frac{I}{\delta}}\right) \leq \frac{\delta}{I} \qquad \forall i \in [I]$$

According to Lemma 2 and the fact that the minimum v_i is at least 1 unit/round (if not 0), we infer $\bar{p} \leq \bar{r}$. Together with the union bound over i, we have

$$\Pr\left(\hat{p}_\cdot \leq p_\cdot - 2\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}}\right) \leq \delta \tag{18}$$

Then, we replace $\hat{p}_i$ in (14) with p_i

$$\Pr\left(\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} \hat{p}_\cdot V_\cdot(s;\phi,\psi) \geq \epsilon_A^{(q)}\right) \leq 2\delta$$

$$\Pr\left[\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} \left(p_\cdot - 2\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}}\right) V_\cdot(s;\phi,\psi) \geq \epsilon_A^{(q)}\right] \leq 2\delta + \delta$$

$$\Pr\left[\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} p_\cdot V_\cdot(s;\phi,\psi) \geq \epsilon_A^{(q)} + 2\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}} \cdot \frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} V_\cdot(s;\phi,\psi)\right] \leq 3\delta$$

$$\Pr\left[\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} p.V.(s;\phi,\psi) \leq \epsilon_A^{(q)} + 2\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}v.(\phi,\psi)}\right] \geq 1-3\delta$$

Since we assumed $v_i \leq c_i$, and it gives

$$\Pr\left[\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} p.V.(s;\phi,\psi) \leq \epsilon_A^{(q)} + 2\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}c.}\right] \geq 1-3\delta \tag{19}$$

For a concise demonstration, we recalculate the error in (19)

$$\epsilon_A^{(q)} + 2\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}c.} \approx 4c_{\max}\bar{r}\sqrt{\frac{\log(2I)/\delta}{\tau^{(q-1)}}} = \epsilon_C^{(q)}$$

Therefore, the lemma

$$\Pr\left[\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2} p.V.(s;\phi,\psi) \leq \epsilon_C^{(q)}\right] \geq 1-3\delta$$

is proved. □

B.5 Proof of Lemma 5

Proof. The Lemma is proved by established three steps. We start by observing that

$$\mathbb{E}[L(t)|\mathcal{H}(\tau^{(q-1)})] \geq 1 - \sum_{i\in[I]}\mathbb{E}\left[\mathbf{1}\left(\sum_{\tau=\max\{t-\bar{d},1\}}^{t-1}\hat{A}_i(\tau)\mathbf{1}(\hat{D}_i(\tau)\geq t-\tau+1) > c_i-\bar{a}\right)\middle|\mathcal{H}(\tau^{(q-1)})\right]$$

we denote the expectation term $\mathbb{E}\left[\mathbf{1}\left(\sum_{\tau=\max\{t-\bar{d},1\}}^{t-1}\hat{A}_i(\tau)\mathbf{1}(\hat{D}_i(\tau)\geq t-\tau+1) > c_i-\bar{a}\right) \mid \mathcal{H}(\tau^{(q-1)})\right]$ as $G_i^{(q)}(t)$. Firstly, we demonstrate that for any $t \in \left\{\tau^{(q-1)}+1+\bar{d},\ldots,\tau^{(q)}\right\}$, any $i\in[I]$ and any fixed $\varepsilon>0$, the inequality

$$G_i^{(q)}(t) \leq \frac{1}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}-1}}\cdot\exp\left[\frac{\varepsilon}{\bar{a}}\cdot\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\cdot\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2}V_i(s;\phi,\psi)\right] \tag{20}$$

holds with at least $1-3\delta$ probability. Secondly, by setting $\varepsilon = \frac{\beta}{1+\epsilon_D^{(q)}}$, we demonstrate the inequality

$$\frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}}\cdot\exp\left[\frac{\varepsilon}{\bar{a}}\cdot\frac{c_{\min}}{\epsilon_C^{(q)}+c_{\min}}\cdot\frac{\epsilon_C^{(q)}+1}{\epsilon_C^{(q)}+1+\beta}\cdot\epsilon_C^{(q)}\right] \leq \frac{\sqrt{\xi}}{I} \tag{21}$$

which holds with certainty since inequality (21) only involves deterministic parameters. To show the Lemma, we just have to prove (20) and (21).

Inequality (20) is shown by the following string of calculations

$$
\begin{aligned}
&G_i^{(q)}(t)\\
&= \mathbb{E}\left[\mathbf{1}\left(\sum_{\tau=\max\{t-\bar{d},1\}}^{t-1} \hat{A}_i(\tau)\mathbf{1}(\hat{D}_i(\tau) \geq t-\tau+1) > c_i - \bar{a}\right)\middle|\mathcal{H}(\tau^{(q-1)})\right]\\
&= \mathbb{E}\left\{\mathbf{1}\left[(1+\varepsilon)^{\sum_{\tau=\max\{t-\bar{d},1\}}^{t-1} \frac{\hat{A}_i(\tau)}{\bar{a}}\mathbf{1}\left(\hat{D}_i(\tau)\geq t-\tau+1\right)} > (1+\varepsilon)^{\frac{c_i}{\bar{a}}-1}\right]\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right\}
\end{aligned}
$$

By Markov inequality, we have

$$
\begin{aligned}
&\leq \frac{1}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}-1}}\mathbb{E}\left[(1+\varepsilon)^{\sum_{\tau=\max\{t-\bar{d},1\}}^{t-1} \frac{\hat{A}_i(\tau)}{\bar{a}}\mathbf{1}\left(\hat{D}_i(\tau)\geq t-\tau+1\right)}\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]\\
&= \frac{1}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}-1}}\prod_{\tau=\max\{t-\bar{d},1\}}^{t-1}\mathbb{E}\left[(1+\varepsilon)^{\frac{\hat{A}_i(\tau)}{\bar{a}}\mathbf{1}\left(\hat{D}_i(\tau)\geq t-\tau+1\right)}\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]
\end{aligned}
$$

By the fact that $(1+\varepsilon)^a \leq 1+\varepsilon\cdot a$ for all $a\in[0,1], \varepsilon>0$

$$
\begin{aligned}
&\leq \frac{1}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}-1}}\prod_{\tau=\max\{t-\bar{d},1\}}^{t-1}\left(1+\varepsilon\cdot\mathbb{E}\left[\frac{\hat{A}_i(\tau)}{\bar{a}}\mathbf{1}\left(\hat{D}_i(\tau)\geq t-\tau+1\right)\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]\right)\\
&\leq \frac{1}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}-1}}\prod_{\tau=\max\{t-\bar{d},1\}}^{t-1}\left(1+\varepsilon\cdot\mathbb{E}[B(\tau)]\mathbb{E}\left[\frac{\tilde{A}_i(\tau)}{\bar{a}}\mathbf{1}\left(\tilde{D}_i(\tau)\geq t-\tau+1\right)\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]\right)
\end{aligned}
$$

By the inequality $1+\varepsilon\leq e^{\epsilon}$ which holds for all $\varepsilon>0$

$$
\begin{aligned}
&\leq \frac{1}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}-1}}\prod_{\tau=\max\{t-\bar{d},1\}}^{t-1}\exp\left(\varepsilon\cdot\mathbb{E}[B(\tau)]\mathbb{E}\left[\frac{\tilde{A}_i(\tau)}{\bar{a}}\mathbf{1}\left(\tilde{D}_i(\tau)\geq t-\tau+1\right)\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]\right)\\
&= \frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}}\exp\left(\frac{\varepsilon}{\bar{a}}\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\sum_{\tau=\max\{t-\bar{d},1\}}^{t-1}\mathbb{E}\left[\tilde{A}_i(\tau)\mathbf{1}\left(\tilde{D}_i(\tau)\geq t-\tau+1\right)\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]\right)\\
&\leq \frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}}\exp\left(\frac{\varepsilon}{\bar{a}}\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\sum_{\tau=t-\bar{d}+1}^{t}\sum_{s=t-\tau+1}^{\bar{d}}\mathbb{E}\left[\tilde{A}_i(\tau)\mathbf{1}\left(\tilde{D}_i(\tau)=s\right)\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]\right)\\
&= \frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}}\exp\left(\frac{\varepsilon}{\bar{a}}\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\sum_{s=1}^{\bar{d}}\sum_{\tau=t-s+1}^{t}\mathbb{E}\left[\tilde{A}_i(\tau)\mathbf{1}\left(\tilde{D}_i(\tau)=s\right)\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]\right)
\end{aligned}
$$

Note that $\{(\tilde{A}_i(\tau),\tilde{D}_i(\tau))\}_{\tau=\max\{t-\bar{d},1\}}^{t}$ are i.i.d. conditioned on $\mathcal{H}\left(\tau^{(q-1)}\right)$, leads to

$$
\begin{aligned}
&= \frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}}\exp\left(\frac{\varepsilon}{\bar{a}}\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\sum_{s=1}^{\bar{d}}\sum_{\tau=t-s+1}^{t}\mathbb{E}\left[\tilde{A}_i(t)\mathbf{1}\left(\tilde{D}_i(t)=s\right)\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]\right)\\
&= \frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}}\exp\left(\frac{\varepsilon}{\bar{a}}\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\sum_{s=1}^{\bar{d}}\mathbb{E}\left[\tilde{A}_i(t)\cdot s\cdot\mathbf{1}\left(\tilde{D}_i(t)=s\right)\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]\right)\\
&= \frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}}\exp\left(\frac{\varepsilon}{\bar{a}}\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\cdot\mathbb{E}\left[\tilde{A}_i(t)\tilde{D}_i(t)\middle|\mathcal{H}\left(\tau^{(q-1)}\right)\right]\right)\\
&= \frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}}\exp\left[\frac{\varepsilon}{\bar{a}}\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\cdot\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2}V_i(s;\phi,\psi)\right]
\end{aligned}
$$

As we assume that $\beta = \sqrt{\xi \log \frac{2I}{\xi}} \in [0,1]$, thus $\varepsilon \in [0,1]$, we then have

$$
\begin{aligned}
&\frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}} \exp\left[\frac{\varepsilon}{\bar{a}} \frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)} \cdot \frac{2}{\tau^{(q-1)}} \sum_{s=1}^{\tau^{(q-1)}/2} V_i(s;\phi,\psi)\right] \\
&\le \frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}} \exp\left[\frac{\varepsilon}{\bar{a}} \cdot \frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)} \cdot \epsilon_C^{(q)}\right] \\
&= \frac{1+\varepsilon}{(1+\varepsilon)^{\frac{c_i}{\bar{a}}}} \exp\left[\frac{\varepsilon}{\bar{a}} \cdot \frac{c_{\min}}{\epsilon_C^{(q)}+c_{\min}} \cdot \frac{\epsilon_C^{(q)}+1}{\epsilon_C^{(q)}+1+\beta} \cdot \epsilon_C^{(q)}\right] \\
&= (1+\varepsilon)\exp\left[\frac{c_i}{\bar{a}} \log\frac{1}{\varepsilon+1}\right] \cdot \exp\left[\frac{\varepsilon}{\bar{a}} \cdot \frac{c_{\min}}{\epsilon_C^{(q)}+c_{\min}} \cdot \frac{1}{\frac{\beta}{\epsilon_C^{(q)}+1}+1} \cdot \epsilon_C^{(q)}\right] \\
&= (1+\varepsilon)\exp\left[\frac{c_i}{\bar{a}} \log\frac{1}{\varepsilon+1}\right] \cdot \exp\left[\frac{\varepsilon}{\bar{a}} \cdot \frac{c_{\min}\cdot\epsilon_C^{(q)}}{\epsilon_C^{(q)}+c_{\min}} \cdot \frac{1}{\varepsilon+1}\right] \\
&= (1+\varepsilon)\left[\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right]^{\frac{c_i}{\bar{a}(1+\varepsilon)}} \\
&\le (1+\varepsilon)\exp\left[-\frac{\varepsilon^2}{(1+\varepsilon)(2+\varepsilon)} \cdot \frac{1}{\xi}\right] \\
&\le 2\exp\left[-\frac{\beta^2}{2\times 3\times(\epsilon_C^{(q)}+1)^2} \cdot \frac{1}{\xi}\right] \\
&\le 2\exp\left[-\frac{\log(2I/\xi)\cdot\tau^{(q-1)}}{6\times 16 c_{\max}^2 \bar{r}^2 \log(2I/\delta)}\right]
\end{aligned}
\tag{22}
$$

Recalling our assumption that $\tau^{(q-1)} \ge 96 c_{\max}^2 \bar{r}^2 \log(2I/\delta)$ and the fact that $\xi \le 1$, (22) can be upper bounded as

$$
G_i^{(q)}(t) \le (22) \le 2\exp\left(-\log\frac{2I}{\xi}\right) = \frac{\xi}{I} \le \frac{\sqrt{\xi}}{I} \qquad w.p. \quad 1-3\delta
$$

Therefore, we have

$$
\mathbb{E}[L(t)|\mathcal{H}(\tau^{(q-1)})] \ge 1 - \sum_{i\in[I]} G_i^{(q)}(t) = 1-\sqrt{\xi} \qquad w.p. \quad 1-3\delta
$$

□

B.6 Proof of Lemma 6

Proof. By the coupling argument that constructs $\hat{J}_i(t)$, for every $t \in \left\{\bar{\tau}^{(q-1)}+\ 1,\ldots,\tau^{(q)}\right\}$ and every $i \in [I]$, with certainty we have

$$
\begin{aligned}
\mathbb{E}\left[\hat{J}_i(t) \mid \mathcal{H}\left(\tau^{(q-1)}\right)\right] &= \mathbb{E}\left[B(t)\tilde{J}_i(t) \mid \mathcal{H}\left(\tau^{(q-1)}\right)\right] \\
&= \mathbb{E}[B(t)] \cdot \mathbb{E}\left[\tilde{J}_i(t) \mid \mathcal{H}\left(\tau^{(q-1)}\right)\right] \\
&= \frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)} \cdot \frac{2}{\tau^{(q-1)}} \sum_{s=1}^{\tau^{(q-1)}/2} W_i(s)V_i(s;\phi,\psi)
\end{aligned}
$$

Applying Lemma 3, we know that the inequality

$$\begin{aligned}
&\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\cdot\frac{2}{\tau^{(q-1)}}\sum_{s=1}^{\tau^{(q-1)}/2}W_i(s)V_i(s;\phi,\psi)\\
\geq&\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\cdot\left(\hat{\lambda}_*^{(q)}+\epsilon_B^{(q)}-\epsilon_A^{(q)}\right)\\
\geq&\frac{c_{\min}(\epsilon_C^{(q)}+1)}{(\epsilon_C^{(q)}+c_{\min})(\epsilon_C^{(q)}+1+\beta)}\cdot\left(\lambda_*-\epsilon_A^{(q)}\right)
\end{aligned}$$

holds for all $i\in[I]$ with probability at least $1-5\delta$. Thus, we obtain the desired lower bound. $\square$

B.7 Proof of Lemma 7

Proof. Indeed, the random variables $\{L(t)\hat{J}_i(t)\}_{t=\bar{\tau}^{(q-1)}+1}^{\tau^{(q)}}$ are correlated even when we condition on $\mathcal{H}\left(\tau^{(q-1)}\right)$. Instead, we apply Lemma 7 and partial proof of Lemma 6 in Zhang and Chi (2022) [4] on suitable subsets of $\{L(t)\hat{J}_i(t)\}_{t=\bar{\tau}^{(q-1)}+1}^{\tau^{(q)}}$ that partition $\{L(t)\hat{J}_i(t)\}_{t=\bar{\tau}^{(q-1)}+1}^{\tau^{(q)}}$. To this end, we define $N=\left\lceil\frac{\tau^{(q)}-\tau^{(q-1)}}{\bar{d}+1}\right\rceil$. For $l\in[\bar{d}+1]$ and $n\in[N]$, we define a new time index

$$k(n,l)=\bar{\tau}^{(q-1)}+l+(n-1)\cdot\left(\bar{d}+1\right)$$

Clearly, we have $\{k(n,l)\}_{n\in[N],l\in[\bar{d}+1]}\supseteq\left\{\bar{\tau}^{(q-1)}+1,\ldots,\tau^{(q)}\right\}$. Crucially, we observe that for any $l\in[\bar{d}]$, the random variables in the collection

$$\Gamma(l)=\left\{L(k(n,l))\cdot\hat{J}_i(k(n,l))\right\}_{n=1}^{N}$$

are independent and identically distributed conditioned on $\mathcal{H}\left(\tau^{(q-1)}\right)$. We first show the conditional independence. For every t, the random variable $L(t)\hat{J}_i(t)$ is $\sigma\left(\left\{(\hat{J}(\tau),\hat{A}(\tau),\hat{D}(\tau))\right\}_{\tau=t-\bar{d}}^{t}\right)$-measurable. More precisely, by the definition of $L(t),\hat{J}_i(t)$, we know that there is a deterministic function g_i such that $L(t)\hat{J}_i(t)=g_i(\{(\hat{J}(\tau),\hat{A}(\tau),\hat{D}(\tau))\}_{\tau=t-\bar{d}}^{t})$, where g_i does not vary with t and only depends on i. Since the time indexes in $\Gamma(l)$ are at least $\bar{d}+1$ time steps apart, we know that for any two distinct $n,n'\in[N]$, the time indexes sets $\left\{k(n,l)-\bar{d},\ldots,k(n,l)\right\}$ and $\left\{k(n',l)-\bar{d},\ldots,k(n',l)\right\}$ are disjoint. By observing that $\{(\hat{J}(\tau),\hat{A}(\tau),\hat{D}(\tau))\}_{\tau=\tau^{(q-1)}-\bar{d}}^{\tau^{(q)}}$ are independent conditioned on $\mathcal{H}\left(\tau^{(q-1)}\right)$, we know that the random variables in $\Gamma(l)$ are independent conditioned on $\mathcal{H}\left(\tau^{(q-1)}\right)$.

The identically distributed part follows from the fact that, for any $t\in\left\{\bar{\tau}^{(q-1)}+1,\ldots,\tau^{(q)}\right\}$, we know that $\left\{t-\bar{d},\ldots,t\right\}\subset\left\{\tau^{(q-1)}+1,\ldots,\tau^{(q)}\right\}$. In addition, by the coupling argument on the construction of $\hat{J},\hat{A},\hat{D}$, we know that $\{(\hat{J}(t),\hat{A}(t),\hat{D}(t))\}_{t=\tau^{(q-1)}+1}^{(q)}$ are identically distributed conditioned on $\mathcal{H}\left(\tau^{(q-1)}\right)$, since all the time indexes in $\left\{\tau^{(q-1)}+1,\ldots,\tau^{(q)}\right\}$ belong to phase q. Since $L(t)\hat{J}_i(t)=g_i(\{(\hat{J}(\tau),\hat{A}(\tau),\hat{D}(\tau))\}_{\tau=t-\bar{d}}^{t})$ and g_i does not vary with t, we know that $\{L(t)\hat{J}_i(t)\}_{t=\tau^{(q-1)}+1}^{\tau^{(q)}}$ are identically distributed conditioned on $\mathcal{H}\left(\tau^{(q-1)}\right)$, which in particular implies that the random variables in $\Gamma(l)$ are identically distributed conditioned on $\mathcal{H}\left(\tau^{(q-1)}\right)$.

After establishing the claim that the random variables in $\Gamma(l)$ are conditionally i.i.d. for any l, we apply the conditional Chernoff inequality (Lemma 7 in Zhang and Chi (2022)) on the random variables in $\Gamma(l)$, along with $\mathcal{F}=\mathcal{H}\left(\tau^{(q-1)}\right)$ and $C=\bar{r}$. Summing (20) over $t\in\{k(n,l)\}_{n=1}^{N}$ gives us that, with probability at least $1-5\delta$, it holds that

$$\sum_{n=1}^{N}\mathbb{E}\left[\hat{J}_i(k(n,l))L(k(n,l))\mid\mathcal{H}\left(\tau^{(q-1)}\right)\right]=\mu_-(l)\geq\frac{\left(1-\sqrt{\xi}\right)\left(\lambda_*-\epsilon_A^{(q)}\right)}{\epsilon_C^{(q)}+2}$$

for all $l\in[\bar{d}+1]$. Observe that $\mu_-(l)\leq N\frac{\bar{r}}{2}$ for all $l\in[\bar{d}+1]$ a.s. The conditional Chernoff inequality shows us that with probability $\geq 1-6\delta$, we have

[4] Xilin, Zhang and Cheungwang Chi. Online Resource Allocation for Reusable Resources. 2022, [Online]. Available: `https://arxiv.org/abs/2212.02855`

$$\sum_{n=1}^{N} \hat{J}_i(k(n,l))L(t(n;l)) \geq \mu_-(l) - \sqrt{2C\mu_- \log\left(\frac{\bar{d}+1}{\delta}\right)} \geq \mu_-(l) - \sqrt{N\bar{r}^2 \log\left(\frac{\bar{d}+1}{\delta}\right)} \tag{23}$$

for all $l \in [\bar{d}+1]$. Summing (23) over l gives

$$\begin{aligned}
&\sum_{t=\tau^{(q-1)}+1}^{\tau^{(q)}} \hat{J}_i(t)L(t) \\
\geq & \frac{\left(1-\sqrt{\xi}\right)\left(\lambda_* - \epsilon_A^{(q)}\right)}{\epsilon_C^{(q)}+2}\left(\tau^{(q)} - \bar{\tau}^{(q-1)}\right) - \left(\bar{d}+1\right)\bar{r}\sqrt{N\log\left(\frac{\bar{d}+1}{\delta}\right)} \\
\geq & \frac{\left(1-\sqrt{\xi}\right)\left(\lambda_* - \epsilon_A^{(q)}\right)}{\epsilon_C^{(q)}+2}\left(\tau^{(q)} - \tau^{(q-1)}\right) - \left(\bar{d}+1\right)\bar{r}\sqrt{\frac{\tau^{(q-1)}}{\bar{d}+1}\log\left(\frac{\bar{d}+1}{\delta}\right)} \\
\geq & \frac{1-\sqrt{\xi}}{\epsilon_C^{(q)}+2}\lambda_*\left(\tau^{(q)} - \tau^{(q-1)}\right) - \frac{1-\sqrt{\xi}}{\epsilon_C^{(q)}+2}\epsilon_A^{(q)}\left(\tau^{(q)} - \tau^{(q-1)}\right) - \left(\bar{d}+1\right)\bar{r}\sqrt{\frac{\tau^{(q)}-\tau^{(q-1)}}{\bar{d}+1}\log\left(\frac{\bar{d}+1}{\delta}\right)}
\end{aligned}$$

with the fact that $\epsilon_A^{(q)} \leq 6\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}}$, we have

$$\begin{aligned}
\geq & \frac{1-\sqrt{\xi}}{\epsilon_C^{(q)}+2}\lambda_*\left(\tau^{(q)} - \tau^{(q-1)}\right) - \frac{1-\sqrt{\xi}}{\epsilon_C^{(q)}+2}\cdot 6\bar{r}\sqrt{\frac{\log(I/\delta)}{\tau^{(q-1)}}}\left(\tau^{(q)} - \tau^{(q-1)}\right) \\
& \qquad - \bar{r}\sqrt{(\bar{d}+1)(\tau^{(q)}-\tau^{(q-1)})\log\left(\frac{\bar{d}+1}{\delta}\right)} \\
= & \frac{1-\sqrt{\xi}}{\epsilon_C^{(q)}+2}\lambda_*\left(\tau^{(q)} - \tau^{(q-1)}\right) \\
& \qquad - \frac{6\bar{r}(1-\sqrt{\xi})\left(\tau^{(q)}-\tau^{(q-1)}\right)}{\epsilon_C^{(q)}+2}\left[\sqrt{\log\left(\frac{I}{\delta}\right)} + \bar{r}\sqrt{(\bar{d}+1)\log\left(\frac{\bar{d}+1}{\delta}\right)}\right]
\end{aligned}$$

If phase q goes to infinity, $\tau^{(q)} - \tau^{(q-1)}$ and $\epsilon_C^{(q)}$ expand as well. Recall $\tau^{(q-1)} \geq 96c_{\max}^2\bar{r}^2\log(\frac{2I}{\delta})$ and (22)

$$\geq \frac{1-\sqrt{\xi}}{\frac{1}{\sqrt{6}}+2}\left(\tau^{(q)} - \tau^{(q-1)}\right)\lambda_* - O\left(\sqrt{\tau^{(q)} - \tau^{(q-1)}}\right)$$

$\square$