Exam 10

(1.) $p = 0.6$

a) This has a geometric distrib. because it describes an event that stops on 1^{st} success.

$$\text{pdf } X = \begin{pmatrix} k \\ p \cdot q^k \end{pmatrix} = \begin{pmatrix} k \\ 0.6 \cdot 0.4^k \end{pmatrix}, k \in \mathbb{N}^*$$

(4 fails)

$$P_4 = 0.6 \cdot 0.4^4$$

$$= 0.6 \cdot 0.0256$$

$$= \boxed{0.1536}$$

$$\begin{array}{r} 16 \\ 16 \\ \hline 96 \\ 16 \\ \hline 256 \end{array}$$

b) Pascal Model - stop on n^{th} success

NB ↑

$$P_k = C^k_{n+k-1} \, q^k \cdot p^n$$

Stop on $n = 7^{th}$ success

$k - 0$ fails

$$\Rightarrow P_0 = C_8^0 \cdot 0.4^0 \cdot 0.6^7 = 1 \cdot 1 \cdot 0.6^7 = \boxed{0.6^7}$$

2.) Let $X \in \chi^2(2, 1/2)$. Find pdf of $Y = \sqrt{X}$.

$n=2$?

$$g(x) = \sqrt{x}$$

$$\boxed{f_Y y = \frac{f_X\left(g^{-1}(y)\right)}{\left|g'\left(g^{-1}(y)\right)\right|}}$$

$$g(x) = \sqrt{x} = y \Rightarrow x = y^2$$

$$\boxed{g^{-1}(y) = y^2.} \qquad g'(x) = \frac{1}{2\sqrt{x}} \quad \Bigg| \quad g'(g^{-1}(y)) = \boxed{\frac{1}{2y}}$$

$$f_X(x) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, \quad x>0$$

$$f_Y(y) = \frac{f_X(y^2)}{\left|\frac{1}{2y}\right|} = \frac{y^{n-2} \cdot e^{-\frac{y^2}{2}}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \cdot 2y$$

$$f_Y(y) = \frac{y e^{-\frac{y^2}{2}}}{\Gamma(1)} = \boxed{y e^{-\frac{y^2}{2}}} \quad \underline{\text{pdf}}$$

$$\Gamma(1) = \int_0^\infty x^0 \cdot e^{-x}\, dx = \int e^{-x}\, dx = -\int (e^{-x})' =$$

$$= -e^{-x}\Big|_0^{\infty} = -\left(\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right) = \boxed{1.}$$

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3) $X_1, X_2 \dots X_n$ pdf: $f(x,\theta) = \theta x^{\theta-1}$

$x \in (0,1)$, $\theta > 0$ unknown

a) Method of moments estimator $\hat{\theta}$ for θ

$$\overline{X_n} = \frac{1}{n}\sum X_i = E(X)$$

$$f(x,\theta) = \theta x^{\theta-1}$$

$$E(X) = \int_{\mathbb{R}} x \cdot f(x,\theta) = \int_0^1 x \cdot \theta x^{\theta-1} = \theta \int_0^1 x^{\theta} =$$

$$E(x) = \theta \cdot \frac{x^{\theta+1}}{\theta+1}\Big|_0^1 = \theta\left(\frac{1}{\theta+1}\right) = \boxed{\frac{\theta}{\theta+1}}$$

$$\overline{X_n} = E(x) = \frac{\theta}{\theta+1}$$

$$\Leftrightarrow (\theta+1)\overline{X_n} = \theta$$

$$\theta\overline{X_n} - \theta = -\overline{X_n}$$

$$\theta(\overline{X_n} - 1) = -\overline{X_n}$$

$\hat{\theta} = -\overline{X_n}$ ... $\overline{X}$ ... Method of

$$\hat{\theta} = \frac{\overline{X_n}}{\overline{X_n} - 1} = \boxed{\frac{\overline{X_n}}{1 - \overline{X_n}}}$$ moments Estimator.

b) Max Likelihood Estimator $\overline{\theta}$ of θ.

$$\frac{\partial \ln L(x_1, \dots, x_n; \theta)}{\partial \theta} = 0.$$

$$L(X_1, X_2 \dots X_n, \theta) = \prod_{i=1}^{n} f(X_i, \theta) = (*)$$

$$(*) = \boxed{\theta^n \cdot (X_1 \cdot X_2 \cdot \dots \cdot X_n)^{\theta - 1}}$$

$$\ln L(X_1, X_2 \dots X_n; \theta) = n \ln \theta + (\theta - 1) \ln\left(\prod_{i=1}^{n} x_i\right), \theta > 0$$

$$\frac{\partial \ln L(x_1, x_2 \dots x_n; \theta)}{\partial \theta} = \frac{n}{\theta} + \ln\left(\prod_{i=1}^{n} x_i\right) = 0$$

$$\Leftrightarrow \ln\left(\prod x_i\right) = \frac{-n}{\theta} \Leftrightarrow \boxed{\overline{\theta} = \frac{-n}{\ln\left(\prod_{i=1}^{n} x_i\right)}}$$

Exam 8.

2. $X \in N(0,1)$, pdf of $Y = X^2$. What type of distrib is it? | χ^2 distrib with $k=1$ degree of freedom.

$g(x) = x^2$ $\qquad$ $x^2 = y \Rightarrow \boxed{x = \sqrt{y}}$

$\boxed{g^{-1}(y) = \sqrt{y}}$ $\qquad$ $\boxed{g'(x) = 2x}$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|} = \frac{f_X(\sqrt{y})}{|g'(\sqrt{y})|} =$$

$$= \frac{\frac{1}{1 \cdot \sqrt{2\pi}} e^{\frac{y}{2}}}{2\sqrt{y}} = \boxed{\frac{e^{\frac{y}{2}}}{2\sqrt{2\pi y}}}.$$

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3. $X_1, X_2, \dots X_n$ – random sample

$f(x, \theta) = \frac{1}{\theta}$ $\qquad$ $x \in (0, \theta)$, $\theta > 0$ unknown.

a) Method of moments estimator

$$\overline{X_n} = \frac{1}{n}\sum X_i = E(X)$$

$$E(X) = \int_{\mathbb{R}} x \cdot f(x) = \int_0^\theta x \cdot \frac{1}{\theta} = \frac{1}{\theta} \cdot \frac{x^2}{2}\Big|_0^\theta$$

$$= \frac{1}{\theta} \cdot \frac{\theta^2}{2} = \boxed{\frac{\theta}{2}}$$

$$\overline{X_n} = \frac{\theta}{2} \Rightarrow \boxed{\hat{\theta} = 2\overline{X_n}}$$!

2) Is $\hat{\theta}$ an absolutely correct estimator?

! Absolutely correct $\Leftrightarrow$ $E(\bar{\theta}) = \theta$ (unbiased)

AND $V(\bar{\theta}) \to 0$ as $n \to \infty$.

$$E(\bar{\theta}) = E\left(2\overline{X_n}\right) = E\left(2 \cdot \frac{1}{n}\sum X_i\right) = \frac{2}{n} E\left(\sum X_i\right)$$

$$= \frac{2}{n}\sum E(X_i) = \frac{\cancel{2}}{\cancel{n}} \cdot \cancel{n} \cdot \frac{\theta}{\cancel{2}} = \theta.$$

$\Rightarrow \hat{\theta}$ unbiased.

$$V(\bar{\theta}) = V\left(2 \cdot \overline{X_n}\right) = V\left(\frac{2}{n}\sum X_i\right) = \frac{4}{n^2} V\left(\sum X_i\right)$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \boxed{\frac{\theta^2}{12}}$$

$$E(X^2) = \int_{\mathbb{R}} x^2 f(x,\theta) = \int_0^\theta x^2 \cdot \frac{1}{\theta} = \frac{1}{\theta} \cdot \frac{x^3}{3}\Big|_0^\theta =$$

$$= \boxed{\frac{\theta^2}{3}}$$

$V(\sum X_i) = \sum V(X_i)$ if X_i indep ✓

$$V(\sum X_i) = n \cdot V(X) = \boxed{\frac{n\theta^2}{12}}$$

$$V(\bar{\theta}) = \frac{\cancel{4}}{\cancel{n^2}\, n} \cdot \frac{\cancel{n} \cdot \theta^2}{\cancel{12}\, 3} = \frac{\theta^2}{3n}$$

$$\lim_{n\to\infty} V(\bar{\theta}) = \lim_{n\to\infty} \frac{\theta^2}{3n} = 0.$$

$\Rightarrow$ Yes, $\hat{\theta}$ is absolutely correct.