$$P\left(X \in (\mu - 3\sigma, \mu + 3\sigma)\right) =$$

$$P(\mu - 3\sigma < X < \mu + 3\sigma) = \big| - \mu$$

$$P(-3\sigma < X - \mu < 3\sigma) =$$

$$P(|X - \mu| < 3\sigma) = \quad \text{chose } \boxed{\varepsilon = 3\sigma}$$

$$1 - P(|X - \mu| \geq 3\sigma) \geq 1 - \frac{V(X)}{9\sigma^2}$$

$$1 - P(|X - \mu| \geq 3\sigma) \geq 1 - \frac{1}{9}$$

$$P\left(|X - \mu| < 3\sigma\right) \geq \frac{8}{9} = 0.888 > 88\%$$

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② $P(450 < X < 550) \geq 90\%$

1) Find the distribution, the name and parameter.

2) Look at the table with $E(X)$ and $V(X)$.

Sol:

$n = 1000$

1) Binomial $\Rightarrow$ $Bino(n, p)$ $p = 0.5$

2) $E(X) = n \cdot p$ $\qquad V(X) = n \cdot p(1-p)$

$E(X) = 500$ $\qquad V(X) = 1000 \cdot 0.5 \cdot 0.5$

$= \boxed{250}$

$P(450 \leq X < 550) =$

$P(-50 < X - 500 < 50) = P(|X - 500| < 50)$

$1 - P(|X - 500| \geq 50) \geq 1 - \frac{V(X)}{\varepsilon^2}$

$\boxed{\varepsilon = 50}$

$1 - P(|X - 500| \geq 50) \geq 1 - \frac{250}{2500}$

$P(|X - 500| < 50) \geq \frac{2500 - 250}{2500}$

$P(|X - 500| < 50) \geq \frac{2250}{2500} = 0.9 = \underline{\underline{90\%}}$

Law of Large Numbers: $X_1 \dots X_n$ independent, identically distributed with X (iid)

$$\overline{X_n} = \frac{1}{n}(X_1 + X_2 + \dots X_n)$$

LLN. $\overline{X_n} \to E(X) = \mu$

CLT $\overline{X_n} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ $n \to \infty$

On Average everything is normal.

(3.) X_i = time to download file, $i = \overline{1,82}$

$E(X_i) = 15$ sec

$V(X_i) = 16$

$$P\left(\sum X_i < 1200\right) = ?$$

$V(S_{82}) = \sqrt{V(\sum X_i)} = \sqrt{\sum V(X_i)} = \sqrt{82 \cdot 16} = 36$

$\sum X_i = S_n$, $E(S_{82}) = 82 \cdot 15 = 1230 = \sum E(X_i)$

$$P(S_n < 1200) = P\left(\frac{S_n - E(S_n)}{\sigma(S_n)} < \frac{1200 - E(S_n)}{\sigma(S_n)}\right)$$

$$= P\left(\frac{S_n - 1230}{36} < \frac{1200 - 1230}{36}\right)$$

$$= P\left(\underbrace{\frac{S_n - 1230}{36}} < \underbrace{-\frac{30}{36}}\right)$$

$$\simeq \text{normcdf}(-0.83) \simeq \boxed{20\%}$$

(Z above; -0.83 above)

(4.) $X_1 = 0.4$, $X_2 = 0.7$, $X_3 = 0.9$ iid with X.

p.d.f. of X: $f(x,\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$

$\theta > 0$, Estimate θ

Method of moments:

$$\overline{X}_n = \frac{1}{n}(X_1 + X_2 + \dots + X_n) = E(X)$$

$$\overline{X}_3 = \frac{1}{3}(X_1 + X_2 + X_3) = \int_0^1 x \cdot f(x,\theta) = \boxed{\frac{\theta}{\theta+1}}$$

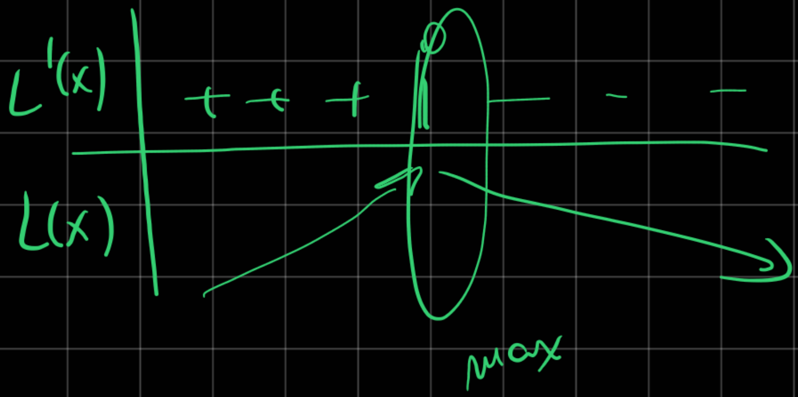
$$\boxed{\frac{2}{3} = \frac{\theta}{\theta+1}} \Rightarrow \boxed{\theta = 2}$$

Method of Maximum Likelihood

Sol $L(X_1, \dots X_n; \theta) = \prod_{i=1}^{n} f(x_i, \theta)$

$$L(0.4, 0.7, 0.3; \theta) = \theta^3 (0.4 \cdot 0.7 \cdot 0.3)^{\theta-1}$$

max of L, use $L'(\,)$ → make table



S2| $\ln L(\dots;\theta) = 3\ln\theta + (\theta-1)\ln(0.252)$, $\theta>0$

S3| $\dfrac{dL}{d\theta}(\dots\theta) = \dfrac{3}{\theta} + \ln(0.252) = 0$

$\Leftrightarrow \quad \theta = -\dfrac{3}{\ln(0.252)} \simeq \boxed{2.17}$

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5. $X_1, \dots X_n$ — iid

$$f(x,\theta) = \frac{1}{2\theta} e^{-\frac{x}{2\theta}}, \quad x>0 \quad (\theta>0)$$

$\mu = E(X) = 2\theta \qquad \sigma^2 = V(X) = 4\theta^2$

a) $\bar\theta$ for θ. (method of moments estimator)

$$\overline{X}_n = \underbrace{E(X)}_{2\theta} \Leftrightarrow \boxed{\bar\theta = \frac{1}{2}\overline{X}_n}$$

b) $e(\bar\theta)$ → verify if $\bar\theta$ unbiased $\Leftrightarrow E(\bar\theta) = \theta$

b) $E(\theta) \Rightarrow$ verify if θ unbiased $\Rightarrow E(\theta) = \theta$

$$\Leftrightarrow E\left(\frac{1}{2}\bar{X}_n\right) = E\left(\frac{1}{2}\cdot\frac{1}{n}\left(\sum X_i\right)\right) =$$

$$\frac{1}{2n} E\left(\sum X_i\right) = \frac{1}{2n}\sum E(X_i) = \frac{1}{\cancel{2n}}\cdot \cancel{n \cdot 2}\theta = \theta$$

$\Rightarrow \theta$ is unbiased.

$$e(\bar{\theta}) = \frac{1}{I_n(\theta)\cdot V(\bar{\theta})}$$

$$I_n(\theta) = -E\left(\frac{d^2 \ln L(x_1 \ldots x_n;\theta)}{d\theta^2}\right)$$

$I_n(\theta) = n \cdot I_1(\theta)$ if THE RANGE OF X does NOT depend on θ.

$\boxed{\text{Range of X = set of input vals. where pdf} \neq 0.}$

range of $X = (0, \infty)$

$$I_1 = -E\left(\frac{d^2 \ln L}{d\theta^2}(X_1;\theta)\right)$$

$$L(X_1, \theta) = f(X_1, \theta) = \frac{1}{2\theta} e^{-\frac{X_1}{2\theta}}$$

$$\ln L(x_1, \theta) = -\ln(2\theta) - \frac{x_1}{2\theta}$$

$$\frac{\partial \ln L(x_1, \theta)}{\partial \theta} = -\frac{2}{2\theta} + \frac{x_1}{2\theta^2}$$

$$\frac{\partial^2 \ln L(x_1, \theta)}{\partial^2 \theta} = +\frac{1}{\theta^2} - \frac{x_1}{\theta^3}$$

$$I_1 = -n \cdot E\left(\frac{1}{\theta^2} - \frac{x_1}{\theta^3}\right) = -\frac{n}{\theta^2} + \frac{n}{\theta^3} \cdot E(x_1)$$

$$= \boxed{\frac{n}{\theta^2}}$$

$$V(\bar{\theta}) = V\left(\frac{1}{2n}(\Sigma x_i)\right) = \frac{1}{4n^2} \Sigma V(x_i) = \frac{\theta^2}{n}$$

$$E = \frac{1}{\cancel{\frac{n}{\theta^2}} \cdot \cancel{\frac{\theta^2}{n}}} = \boxed{1.}$$

Chebyshev's Ineq.: $P(|X - E(X)| \geq \varepsilon) < \frac{V(X)}{\varepsilon^2}$

CLT: $\frac{S_n - E(S_n)}{} = \frac{S_n - n\mu}{} \sim Z \in$

$$\frac{\quad}{\sigma(S_n)} = \frac{\quad}{\sigma\sqrt{n}} \sim N(0,1)$$

$\Rightarrow$ normcdf (val)

Law of Large Nr: $\overline{S_n} = E(X) = \mu$

Method of Moments: $\overline{X_n} = E(X)$

Method of Max. Likelihood $\boxed{\dfrac{d \ln L(x_1 \dots x_n)}{d\theta} = 0.}$

$$L(x_1 \dots x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta).$$

$E(\bar{\theta}) = \theta \iff \bar{\theta}$ unbiased.

$$e(\bar{\theta}) = \frac{1}{I_n(\theta) \cdot V(\bar{\theta})}, \quad I_n(\theta) = n \cdot I_1(\theta) \text{ if}$$

the range of X doesn't depend on θ.

$$I_n(\theta) = -E\left[\frac{\partial^2 \ln L(x_1, x_2 \dots x_n; \theta)}{\partial \theta^2}\right]$$

$$\bar{\theta} = \frac{1}{n} S_n = \frac{1}{n} \sum x_i$$