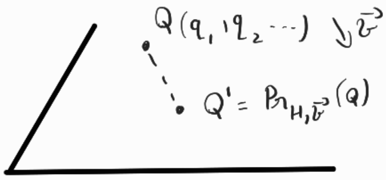
(Parallel) Projections (& reflections) with respect to a hyperplane

Hyperplane: In an n-dim euclidian space, is a set of the form:

$$H=\{(x_1,x_2,\dots,x_n)\in E^n \mid a_1x_1+a_2x_2+\dots a_nx_n+a_{n+1}=0\}$$

If $n=2 \Rightarrow H=$ line

If $n=3 \Rightarrow H=$ plane



$\vec{v}(v_1,v_2,\dots v_n)\in V^n$

$Q'=\ell\cap H$ where $\ell=Q+\langle\vec{v}\rangle: \begin{cases} x_1=q_1+tv_1\\ x_2=q_2+tv_2\\ \dots\\ x_n=q_n+tv_n\end{cases}$ $\quad Q'=\begin{cases} x_1=q_1+tv_1\\ \dots\\ x_n=q_n+tv_n\\ a_1x_1+\dots+a_nx_n+a_{n+1}=0\end{cases}$

$$a_1(q_1+tv_1)+a_2(q_2+tv_2)+\dots+a_n(q_n+tv_n)+a_{n+1}=0$$

$$\Rightarrow (a_1q_1+a_2q_2+\dots+a_nq_n)+t(a_1v_1+\dots+a_nv_n)+a_{n+1}=0$$

$$t=-\frac{a_1q_1+a_2q_2+\dots+a_nq_n+a_{n+1}}{a_1v_1+a_2v_2+\dots+a_nv_n}$$

coordinates of Q':

$$\begin{cases} x_1=q_1-\dfrac{a_1q_1+\dots}{a_1v_1+\dots}\cdot v_1\\ \dots\\ x_n=q_n-\dfrac{a_1q_1+\dots+a_{n-1}}{a_1v_1+\dots}\cdot v_n\end{cases}$$

Let $p(x_1,\dots,x_n)$; $p'=pr_{H,\vec{v}}(P)=(x_1'\dots x_n')$

$$\Rightarrow \begin{pmatrix}x_1'\\x_2'\\ \vdots\\ x_n'\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\ \vdots\\ x_n\end{pmatrix}-\frac{a_1x_1+\dots a_nx_n+a_{n+1}}{a_nv_1+a_nv_n}\begin{pmatrix}v_1\\v_2\\ \vdots\\ v_n\end{pmatrix}$$

The tensor Product

$\vec{v}\,(v_1, v_2, \dots, v_n)$, $\vec{w}\,(\vec{w}_1, \dots, \vec{w}_n) \in V^n$

$$\vec{v} \cdot \vec{w} = (v_1, v_2, \dots v_n) \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \vec{v} \cdot \vec{w}$$

$$\vec{v} \otimes \vec{w} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \cdot (w_1 \dots w_n) = \begin{pmatrix} v_1 w_1 & v_1 w_2 & \dots & v_1 w_n \\ \vdots & \vdots & & \vdots \\ v_n w_1 & v_n w_2 & \dots & v_n w_n \end{pmatrix}$$

tensor prod.

$$A, B \in M_n(\mathbb{R}) \; ; \; A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{11} \cdot B & a_{12} B & \dots & a_{1n} B \\ \vdots & \vdots & & \vdots \\ a_{n1} B & a_{n2} B & \dots & a_{nn} B \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 0 & 1 & 0 & 2 \\ -1 & -1 & -2 & -2 \\ 0 & 3 & 0 & 4 \\ -3 & -3 & -4 & -4 \end{pmatrix}$$

Prop: The matrix form of the parallel projection is: $f = \overbrace{a_1 x_1 + \dots + a_n x_n + a_{n+1}}^{\lim f}$

$$[Pr_{H, \vec{v}}(P)]_R = \frac{1}{(\lim f)(\vec{v})} \left(\vec{a}\,\vec{v}\, Id_n - \vec{v} \otimes \vec{a} \right) \cdot [P]_R - \frac{a_{n+1}}{(\lim f)\vec{v}} [\vec{v}]_R$$

$\vec{a} = \vec{n}_H = (a_1, a_2 \dots a_n)$

$\vec{v} = (v_1, \dots, v_n)$

Changing coordinates systems

$$\left.\begin{array}{l} R = (O, \overbrace{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}}^{b}) \\ \\ R' = (O', \underbrace{\vec{e'_1}, \vec{e'_2}, \dots, \vec{e'_n}}_{b'}) \end{array}\right\} \text{reference systems}$$

$O, O' \in E^n$

$\forall i,\ \vec{e_i}, \vec{e'_i} \in V^n$; $\forall P \in E^n : [P]_R = [\vec{OP}]_b$

The base-change matrix from R to R' is the base change matrix from b to b'

$$M_{R',R} = M_{b',b} = [id]_{b,b'} = T_{b',b}$$

$\forall P \in E^n$

$$[P]_{R'} = [\vec{O'P}]_{R'} = [\vec{O'P}]_{b'} = [id]_{bb'} [O'P]_b$$

$$= M_{R',R}[\underbrace{\vec{O'P}}_{\vec{OP}-\vec{OO'}}]_b = M_{R',R}[O']_R = M_{R',R}\cdot[P]_R + \underbrace{M_{R',R}[\vec{O'O}]_b}_{[\vec{O'O}]_{b'}} = M_{R',R}\cdot[P]_R + [O]_{R'}$$

Eigen values/vectors:

Let $A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$P_A(x)=\det(A-xI_2)=\begin{vmatrix}1-x&0\\0&-1-x\end{vmatrix}=(1-x)(-1-x)$ => The eigen values are $\lambda_1=1$; $\lambda_2=-1$

egalezi cu 0 => eigen value

$V(\lambda_1)=\{(x,y)\in\mathbb{R}^2 \mid A\begin{pmatrix}x\\y\end{pmatrix}=\underbrace{\lambda_1\begin{pmatrix}x\\y\end{pmatrix}}_{\lambda_1 I_2\begin{pmatrix}x\\y\end{pmatrix}}\}$

the eigen vector for the eigen val.

$=\{(x,y)\in\mathbb{R}^2 \mid (A-\lambda_1 I_2)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\}=\{(x,y)\in\mathbb{R}^2 \mid \begin{pmatrix}0&0\\0&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\}$

$=\{(x,y)\in\mathbb{R}^2 \mid -2y=0\}=\{(x,0)\mid x\in\mathbb{R}\}$

$=\langle(1,0)\rangle$

$ma(\lambda_1)=1$

algebric multiplicity (de câte ori λ_1 e root, ex: pt $(x-1)^2$ => $x=1$ $ma(x)=2$)

$mg(\lambda_1)=\dim V(\lambda_1)=1$

M diagonalisable if $ma(\lambda)=mg(\lambda)$, $\forall\ \lambda\in\mathbb{R}$

Axis and angle of rotation

$O(n)=\{A\in M_n(\mathbb{R}) \mid A\cdot A^t=I_n\}$

$SO(n)=\{A\in O(n) \mid \det(A)=1\}$

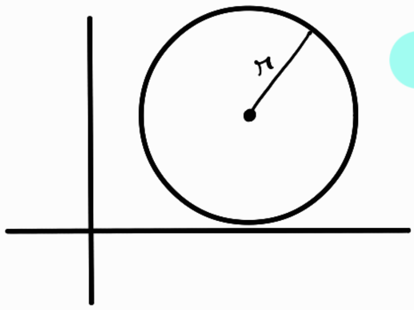
If $A\in SO(n)$ => $Tr(A)=1+2\cos\theta$; θ - angle of rotation

$Fix(A)=\{(x,y,z) \mid A\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}\}$ - axis of rotation

Example:

$Fix(A)=\{(-\frac{a}{2},0,a)\mid a\in\mathbb{R}\}=\langle(-\frac{1}{2},0,1)\rangle$

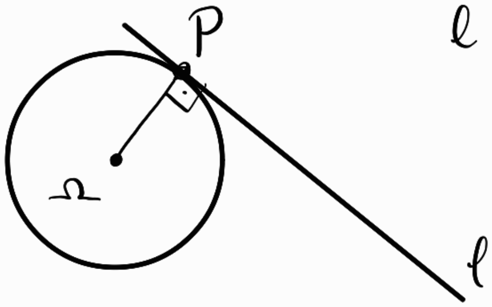
Circle and ellipses



locus - set of points that have a certain property

$C(\Omega, r) = (x-x_0)^2 + (y-y_0)^2 = r^2$ → implicit eq.

$C(\Omega, r): \begin{cases} x = x_0 + r\cos t \\ y = y_0 + r\sin t \end{cases}, \ t \in [0, 2\pi)$



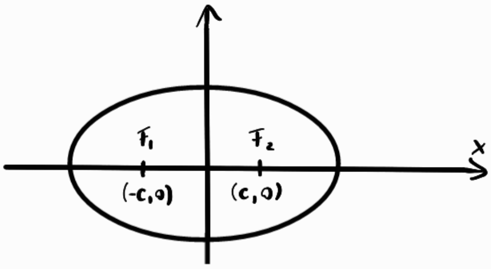
ℓ tangent to $C(\Omega, r)$
$\Rightarrow \Omega P \perp \ell$

* A point $P(x_P, y_P)$ is on the line $\ell: x - y - c = 0$ if $x_P - y_P - c = 0$

$d(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$; $A(x_A, y_A)$ $B(x_B, y_B)$

$d(A, \ell) = \dfrac{|a \cdot x_0 + b \cdot y_0 + c|}{\sqrt{a^2 + b^2}}$; $A(x_0, y_0)$ $\ell: ax + by + c$

Ellipses:



Fix F_1, F_2 two distinct points and $a \in \mathbb{R}$

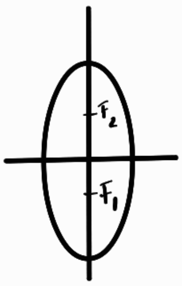
$\mathcal{E}$ = the locus of points P in the plane for which

$PF_1 + PF_2 = 2a$

$$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$c = \sqrt{a^2 - b^2}$ $(a > b)$ $\quad e = \frac{c}{a}$ → eccentricity

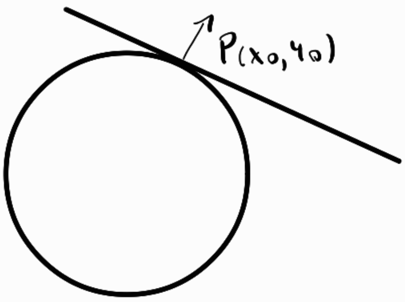
F_1, F_2 foci has to be on the longer axis



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$c = \sqrt{b^2 - a^2}$$

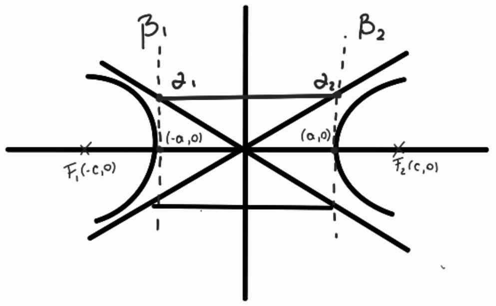
$$e = \frac{c}{b}$$



$$T_{\mathcal{E}}(x_0,y_0) : \frac{x \cdot x_0}{a^2} + \frac{y \cdot y_0}{b^2} = 1 \iff \frac{\partial F}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial F}{\partial y}(x_0,y_0)(y-y_0) = 0$$

$$F(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

Hyperbolas and parabolas



Fix p_1, p_2

H- locus of points M in the plane for which $(MF_1 - MF_2) = 2a$

F_1, F_2 foci of the elipse

$$H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \;;\; y = \pm\frac{b}{a}\sqrt{x^2-a^2} \;;\; c = \sqrt{a^2-b^2} \;;\; a,b>0$$

H has 2 asimptotes at $\pm\infty$:
- $\alpha_1: y = -\frac{b}{a}x$
- $\alpha_2: y = \frac{b}{a}x$

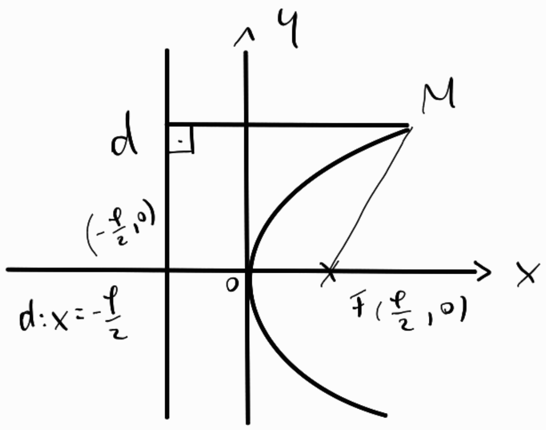
If we trace the lines α_1, α_2 along with $\beta_1 = x=-a \;; \beta_2 = x=a$

we obtain a rectangle with sides $2a$ and $2b$

$$T_H(x_0,y_0): \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1 \iff \frac{\partial F}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial F}{\partial y}(x_0,y_0)(y-y_0) = 0$$

$$F(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Parabolas



F point (the focus)

d line (the directrix)

P = locus of points in the plane equidistant to F and d

$MF = \text{dist}(M,d)$

$$P: y^2 = 2px \iff x = \frac{y^2}{2p} \iff y = \pm\sqrt{2px}$$

$$T_P(x_0,y_0): yy_0 = p(x+x_0) \iff \frac{\partial F}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial F}{\partial y}(x_0,y_0)(y-y_0) = 0$$

$$F(x,y) = y^2 - 2px$$

Canonical form:

$$Q: a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + a_{10}x + a_{01}y + a_{00} = 0$$

$$M(Q) = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

→ Find the eigen values (root of $P_{M(Q)}(x) = \begin{vmatrix} a_{11}-x & a_{12} \\ a_{12} & a_{22}-x \end{vmatrix}$)

→ Find the eigen vectors and from them select a right oriented orthonormal basis (B)

Step I:

$B = (\vec{v_1}, \vec{v_2})$; $E = (\vec{e_1}, \vec{e_2})$ initial basis $\qquad (AB)^t = B^t A^t$

$$M_{E,B} = [id]_{B,E} = ([\vec{v_1}]_E, [\vec{v_2}]_E)$$

We have $\begin{pmatrix} x \\ y \end{pmatrix} = M_{E,B} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

coord in the old basis E $\qquad$ coord in the new basis B

$$Q: (x \ \ y) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (a_{10} \ \ a_{01}) \begin{pmatrix} x \\ y \end{pmatrix} + a_{00} = 0$$

$$(x \ \ y) = \begin{pmatrix} x \\ y \end{pmatrix}^t = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}^t \cdot M_{E,B}^t = (x_1 \ \ y_1) M_{E,B}^t$$

$$Q: (x_1 \ \ y_1) M_{E,B}^t M_{(Q)} M_{E,B} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (a_{10} \ \ a_{01}) M_{E,B} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad M_{E,B}^t M_{(Q)} M_{E,B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

λ_1, λ_2 - eigenvalues of $M_{(Q)}$

$$Q: \lambda_1 x_1^2 + \lambda_2 y_1^2 + \alpha x_1 + \beta y_1 + a_{00} = 0$$

Step II force squares $(\lambda_1, \lambda_2 \neq 0)$

ELLIPSES
HYPERBOLA

$$Q: \lambda_1\left(x_1^2 + 2\frac{\alpha}{2\lambda_1}x_1 + \frac{\alpha^2}{4\cdot\lambda_1^2}\right) + \lambda_2\left(y_1^2 + 2\frac{\beta}{2\lambda_2} + \frac{\beta^2}{4\lambda_2^2}\right) + \underbrace{a_{00} - \frac{\alpha^2}{4\lambda_1} - \frac{\beta^2}{4\lambda_2}}_{K}$$

$$Q: \lambda_1\left(x_1 + \frac{\alpha}{2\lambda_1}\right)^2 + \lambda_2\left(y_1 + \frac{\beta}{2\lambda_2}\right)^2 + K = 0$$

$$\begin{cases} x_2 = x_1 + \frac{\alpha}{2\lambda_1} \\ y_2 = y_1 + \frac{\beta}{2\lambda_2} \end{cases} \Rightarrow Q: \lambda_1 x_2^2 + \lambda_2 y_2^2 + k = 0$$