Seminar 7.

① $K = (O, i, j)$, $K' = (O', i', j')$

$[O']_K = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$ $[i']_K = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $[j']_K = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$M_{KK'} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$M_{K'K} = M_{KK'}^{-1} = \boxed{\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \cdot \frac{1}{-5}}$$

$[A]_K = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

$$[A_{K'}] = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \cdot \frac{1}{-5} \cdot \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = -\frac{1}{5} \cdot \begin{pmatrix} -12 & -3 \\ 6 & -6 \end{pmatrix} =$$

$$= -\frac{1}{5} \begin{bmatrix} -15 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$[B_{K'}] = -\frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \cdot \Big( \underbrace{\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix}}_{\begin{bmatrix} -5 \\ 5 \end{bmatrix}} \Big) =$$

$$= -\frac{1}{5} \begin{bmatrix} -10 & -5 \\ 5 & -10 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -15 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$[C]_{K'} = -\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix}\right) =$$

$$\begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -8 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$\overrightarrow{AB}$: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} x = 3 \\ y = t. \end{cases}$$

Cart. eq: $x = 3$

(4.) $K = (O, i, j, k)$, $K' = (O', i', j', k')$

$$M_{KK'} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = A$$

$$M_{K'K} = M_{KK'}^{t} =$$

$$A^{+} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

$a_{11} = 2$

$a_{12} = -(-4) = 4$

$a_{13} = -2$

$a_{21} = -(-4) = 4$

$a_{22} = -2$

$a_{23} = -(-1) = 1$

$a_{31} = 0$

$a_{32} = -(0) = 0$

$a_{33} = -1 - 4 = -5$

$\det = -2 - 8 = -10$

$$M_{K'K} = -\frac{1}{10}\begin{bmatrix} 2 & 4 & -2 \\ 4 & -2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

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Projections and reflections in/on hyperplanes.

6\. $v(2,1,1) \in \mathbb{V}^3$ and $Q(2,2,2) \in \mathbb{E}^3$

a) $\pi : z = 0, \quad Q + \langle v \rangle$

Matrix form for the parallel projection on plane $\pi$ along the line $v$

$$[Pr_{H,v}(P)] = [I_3 - v \otimes a][P] = (*)$$

ec. of plane $\pi$

$$v \otimes a = \begin{bmatrix} 2 \\ 1 \\ \end{bmatrix} [0\ 0\ 1] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ \end{bmatrix}$$

$$\begin{matrix} 0 & 0 & 1 \end{matrix}$$

$$(A) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

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(OR) $\quad l_p: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_p + 2t \\ y_p + t \\ z_p + t \end{bmatrix}$

$$Pr_{H,v}(P) = l_p \cap H \Leftrightarrow z_p + t = 0$$

$$t = -z_p$$

$$\Rightarrow \begin{bmatrix} x_p - 2z_p \\ y_p - z_p \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

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b) PARALLEL REFLECTION

$$\left[ Pr_{H,v}(P) \right] = \frac{[P] + \left[ Ref_{H,v}(P) \right]}{2}$$

$$Ref_{H,v}(P) = \left( I_n - 2\frac{v \otimes a}{v \cdot a} \right)[P] - 2 \cdot \frac{a_{n+1}}{v \cdot a} v$$

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(7.) $A(2, 11, -5) \qquad H: x + 4y - 3z + 7 = 0$

$\vec{n}$

$$\pi_H \quad (1, 4, -3) = a$$

Perp reflection:

$$[P'] = \left( I_n - \frac{v \otimes a}{v \cdot a} \right) [P] - \frac{a_{n+1}}{a^t v} [v]$$

?

$$[A'] = \left( I_n - \frac{(1,4,-3) \otimes (1,4,-3)}{(1,4,-3) \cdot (1,4,-3)} \right) \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} - \dots$$

$$a \otimes v = \begin{pmatrix} 1 & 4 & -3 \\ 4 & 16 & -12 \\ -3 & -12 & 9 \end{pmatrix}$$

$$v \cdot a = 1 + 16 + 9 = 26$$

$$[A'] = \frac{1}{26} \begin{pmatrix} 25 & 4 & -3 \\ 4 & 10 & -12 \\ -3 & -12 & 17 \end{pmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} - \frac{1}{26} \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

$\downarrow$

coord lui A

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8. $P(6, -5, 5)$, $2x - 3y + z - 4 = 0$

$$n_H = a = (2, -3, 1)$$

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9. $\ell: \begin{cases} 2x - y - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$ $\qquad \pi: x + 2y - z = 0$

$y = 2x - 1$

$x + 2x - 1 - z + 1 = 0$

$\boxed{3x - z = 0.}$

$P_{H,v}[P] = [I_3 - v \otimes a][P]$

$= [I_3 - (1,0,3) \otimes (1,2,-1)][P]$

$\begin{cases} x = x \\ y = 0 \\ z = 3x \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 + t \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

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10. $\ell \ni M(1,0,7)$, $\ell \parallel \Pi$, $\ell \cap \ell_1$

$\ell_1 : \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}$

$\Pi : 3x - y + 2z - 15 = 0$

$(n_\Pi : (3, -1, 2))$ $\quad v_\ell \perp n_H$

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$v_\ell \cdot n_\Pi = 0$ $\quad$ Let $v_\ell = (1, 1, -1)$

$\iff \boxed{3v_1 - v_2 + 2v_3 = 0} \; (v_\ell)$

$v_\ell \cap \ell$

$\ell_1 : \begin{cases} x = 4k + 1 \\ y = 2k + 3 \end{cases} \iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} k + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$z = k$ $\quad [z] \quad [1] \quad [0]$

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$$l: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$l \cap l_1 \iff l: \begin{cases} x = 1 + t = 4k + 1 \\ y = t = 2k + 3 \\ z = 7 - t = k \end{cases}$$

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SOL.

$\pi: \quad 3x - y + 2z - 15 = 0$

$n_\pi: (3, -1, 2)$

$\pi_\alpha: \quad 3x - y + 2z + \alpha = 0$

$\pi_M: \quad 3 + 14 + \alpha = 0$

$\alpha = -17.$

$\pi_M \ni M, \ni l \quad : \quad \boxed{3x - y + 2z - 17 = 0.}$

$l_1: \quad \dfrac{x-1}{4} = \dfrac{y-3}{2} = \dfrac{z}{1} \quad \iff$

$$\begin{cases} x = 4k + 1 \\ y = 2k + 3 \\ z = k \end{cases}, \quad k \in \mathbb{R}$$

$$3(4k+1) - \cancel{2k} + 3 + \cancel{2k} - 17 = 0$$

$$12k + 3 - 14 = 0$$

$$12k - 11 = 0$$

$$\boxed{k = \frac{11}{12}}$$

$$\Rightarrow \begin{cases} x = 4 \cdot \frac{11}{12} + 1 \\ y = 2 \cdot \frac{11}{12} + 3 \\ z = \frac{11}{12} \end{cases} \Leftrightarrow \begin{cases} x = \frac{14}{3} \\ y = \frac{11}{6} + 3 = \frac{29}{6} \\ z = \frac{11}{12} \end{cases}$$

$$\pi_M \cap l_1 = P\left(\frac{14}{3}, \frac{29}{6}, \frac{11}{12}\right)$$

$$\Rightarrow l : (MP) : \quad \overrightarrow{MP}\left(\frac{14}{3} - 1, \frac{29}{6}, \frac{11}{12} - 7\right) =$$

$$\boxed{\overrightarrow{MP}\left(\frac{11}{3}, \frac{29}{6}, -\frac{73}{12}\right)}$$

$$l: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} + t \cdot \begin{bmatrix} \frac{11}{3} \\ \frac{29}{6} \\ -\frac{73}{12} \end{bmatrix}$$

$$\frac{x-1}{\frac{11}{3}} = \frac{y}{\frac{29}{6}} = \frac{z-7}{-\frac{73}{12}}$$

$$\Leftrightarrow \frac{6(x-1)}{29} = \frac{3y}{11} \Leftrightarrow \begin{cases} \frac{6}{29}(x-1) - \frac{3y}{11} = 0 \\ - \quad 4 \quad = 1 \end{cases}$$

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(12) $\pi_1 : 3x - 4z = -1$ $\qquad a(3, 0, -4)$, $v$

$a = n_\pi$ ?

$$\mathrm{Ref}_{\pi}^{\dagger}(P) = \left( I_3 - 2 \frac{n_\pi \otimes a}{n_\pi \cdot a} \right) [P] - 2 \cdot \frac{(1)}{n_\pi \cdot a} [n_\pi]$$

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(6.) $v(2,1,1)$, $Q(2,2,2)$, $\underset{a}{\pi} : z = 0$

$$\mathrm{Pr}_{\pi}^{v}(P) = \left[ I_3 - \frac{v \otimes a}{v \cdot a} \right] [P] - \frac{a_{n+1}}{v \cdot a} [v]$$

$$= \left[ I_3 - \frac{(2,1,1) \otimes (0,0,1)}{(2,1,1)(0,0,1)} \right] [P] - \frac{0}{---} [v]$$

$$= \left( I_3 - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{1} \right) P = \boxed{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} [P]}$$

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$$[\mathrm{Ref}_{\pi, v}(P)] = [2 \cdot \mathrm{Pr}_{\pi, v}(P)] - [P] =$$

Or

$$\left[Ref_{H,u}(P)\right] = \left[I_n - 2\cdot\frac{v \otimes a}{v\cdot a}\right][P] - 2\cdot\frac{a_{n+1}}{v\cdot a}[v]$$

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(14.) $\Pi: ax+by+c=0$ $\quad \vec{a}(a,b)$

$\Pi \nparallel \ell$

$$\ell: \frac{x-x_0}{v_1} = \frac{y-y_0}{v_2}$$

$$\begin{cases} x = v_1 k + x_0 \\ y = v_2 k + y_0 \end{cases} \iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + k\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$Pr_{\Pi,\ell}(P) = \left(I_2 - \frac{(v_1,v_2)\otimes(a,b)}{v_1 a + v_2 b}\right)[P] - \frac{c}{v_1 a + v_2 b}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

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(15.) H - hyperplane,

$$Pr_{\Pi,u}(P) = \left(I_2 - \frac{(v_1,v_2)\otimes(a,b)}{v_1 a + v_2 b}\right)[P] - \frac{c}{v_1 a + v_2 b}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$R\ (R\ (P)) =$ this lies on the hyperplane

$$\left(I_2 - \frac{n \otimes a}{n \cdot a}\right)\left[\left(I_2 - \frac{n \otimes a}{n \cdot a}\right)[P] - \frac{a_{n+1}}{n \cdot a}[n]\right] - \frac{a_{n+1}}{n \cdot a}[n]$$

$$\left[I_2 - 2\frac{n \otimes a}{n \cdot a} + \left(\frac{n \otimes a}{n \cdot a}\right)^2\right][P] - \frac{(n \otimes a)\, a_{n+1}}{(n \cdot a)^2}[n] - \frac{a_{n+1}}{n \cdot a}[n]$$

## Compact matrix form (ChatGPT)

$$Pr_{H,u}(P) \Rightarrow P' = \left[I - \frac{a \otimes u}{a \cdot u}\right][P] \quad ?$$

$u(2,1,1)$, $\pi : z = 0$

$$Pr_{H,u}(P) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}[P]$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p_i' = p_i - \frac{a_1 p_1 n_i + \dots + a_n n_i p_n}{a_1 n_1 + \dots + a_n n_n} - \frac{a_{n+i} n_i}{a^+_n}$$

Sem 8

(2.) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  $\det(A - \lambda I_2) =$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda_{1,2} = \pm i$$

PS S8 ?

Alg. mult – the nr of roots
geom mult – nr of lin indep eigenvect.

Rotations.

(7.) $A(1,1)$, $B(4,1)$, $C(2,3)$

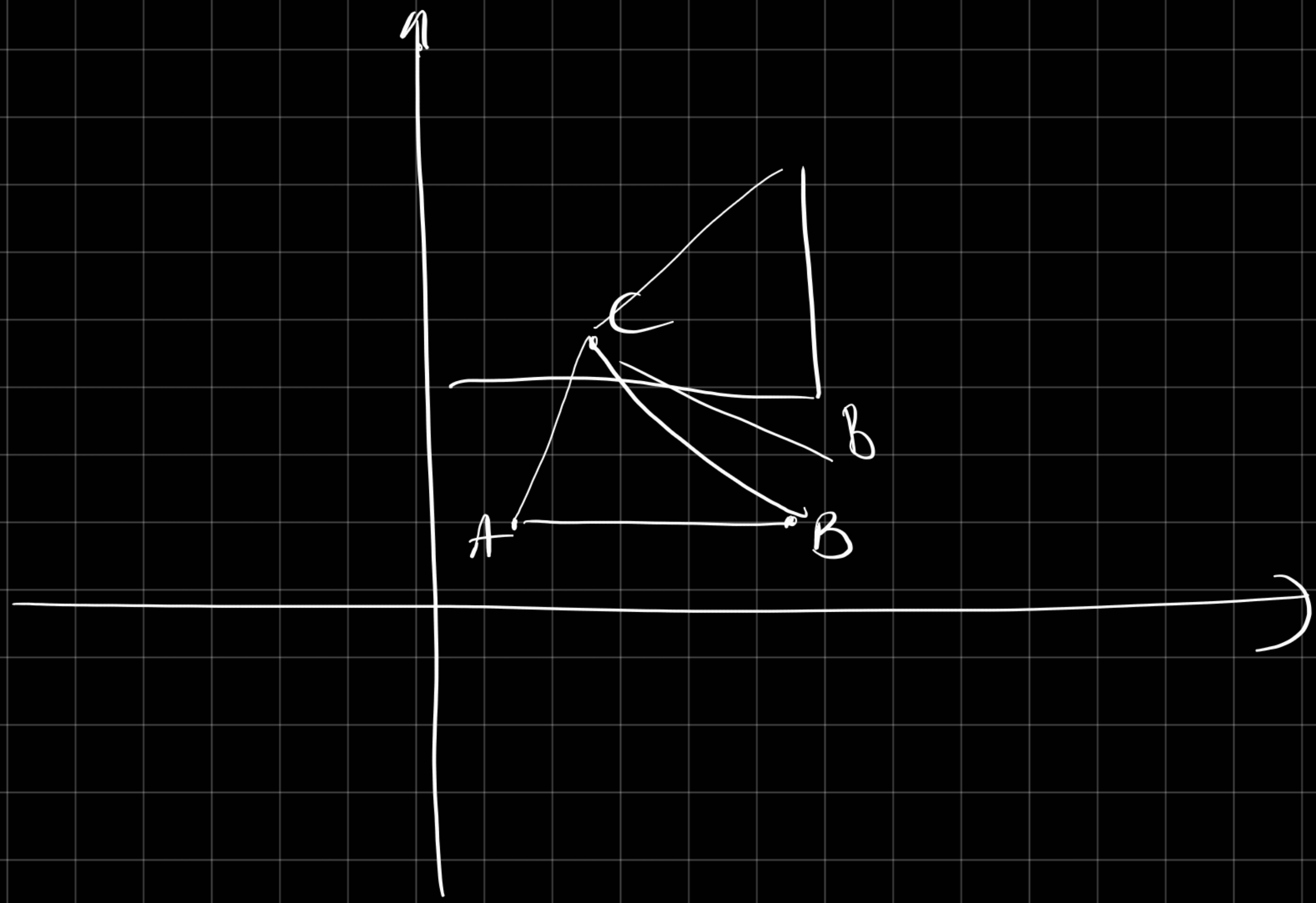
$$\mathrm{Rot}_{C,90°}(P) = \underbrace{T_{\overrightarrow{OC}}}_{\text{shift}} \cdot \mathrm{Rot}_{C,90°} \cdot \underbrace{T_{-\overrightarrow{OC}}}_{\text{unshift}} [P]$$

$$T_{-\overrightarrow{OC}}[P] = \begin{bmatrix} x-2 \\ \dots \end{bmatrix}$$

$$\begin{bmatrix} \\ y-3 \end{bmatrix}$$

$$Rot_{C,90^\circ} \cdot T_{-\vec{OC}}[P] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x-2 \\ y-3 \end{bmatrix} = \begin{bmatrix} 3-y \\ x-2 \end{bmatrix}$$

$$T_{\vec{OC}} \cdot (\qquad) = \begin{bmatrix} 3-y+2 \\ x-2+3 \end{bmatrix} = \begin{bmatrix} -y+5 \\ x+1 \end{bmatrix}$$




$Ref_{\perp AB}(P) : \Big[$

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$$\Big[ \cos\left(-\frac{\pi}{\ }\right) \quad -\sin\left(-\frac{\pi}{\ }\right) \Big]$$

(9.) $T \longrightarrow Rot_{-\frac{\pi}{3}} = \begin{bmatrix} \cos\left(-\frac{\pi}{3}\right) & -\sin\left(-\frac{\pi}{3}\right) \\ \sin\left(-\frac{\pi}{3}\right) & \cos\left(-\frac{\pi}{3}\right) \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$T^{-1} = T_{-(-2,5)} \left[ \quad Rot \quad \right] = \begin{bmatrix} x-2 \\ y+5 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

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(12.) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \leftarrow$ Rot vector

If $f$ is a rot $\Leftrightarrow$ Rot Axis is given by

$$\{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = (x,y,z)\}$$

<u>Rot-Axis</u>: $\left| \; A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right.$

$\Leftrightarrow$ <u>Rot angle</u>.

$Tr(A) = 1 + 2\cos\theta = \frac{1}{3}(-1-2+2)$

$= -\frac{1}{3}$

$\cos\theta = \dfrac{-\frac{1}{3}-1}{2} = -\frac{4}{2\cdot 3} = -\frac{2}{3}$

$\theta = \arccos -\frac{2}{3}$

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10. Center of rot:

$f(x) = x \Leftrightarrow$

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13. Isometry is bijective:

$d(A,B) = d(\phi(A), \phi(B))$

$\phi(A) = \phi(B) \Rightarrow d(\quad) = 0$

$d(A,B) = 0 \Leftrightarrow A = B$

inj.

Sug:

dim mabe

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(15.) $Pr_{H,v}(Pr_{H,v}(P)) = Pr_{H,v}(P)$

$$\left[I_n - \frac{v\otimes a}{\mu\cdot a}\right]\left(\left[I_n - \frac{v\otimes a}{\mu\cdot a}\right][P] - \frac{a_{n+1}}{\mu\cdot a}[v]\right) - \frac{a_{n+1}}{\mu\cdot a}[v]$$

$$= Pr$$

$$Pr - \frac{v\otimes a}{\mu\cdot a}\left(\left[I_n - \frac{v\otimes a}{\mu\cdot a}\right][P] - a_{n+1} \ldots\right) \ldots Pr$$

$$\Leftrightarrow \frac{v\otimes a}{\cancel{\mu\cdot a}}\left(\left[I_n - \frac{v\otimes a}{\mu\cdot a}\right][P] - \frac{a_{n+1}}{\mu\cdot a}[v]\right) = -\frac{a_{n+1}}{\mu\cdot a}[v] \quad \cdot \mu\cdot a$$

$$v\otimes a\left([P] - \frac{v\otimes a}{\mu\cdot a}[P] - \frac{a_{n+1}[v]}{\mu\cdot a}\right) = -a_{n+1}[v]$$

$$(a\cdot[P])\cdot v - \frac{v\otimes a}{\mu\cdot a}\cdot(a\cdot[P])v - \frac{v\otimes a[v]\,a_{n+1}}{\mu\cdot a}$$

$$= \cancel{(a\cdot[P])v} - \frac{[\cancel{a}\cdot(a\cdot[P])\cancel{v}]\cdot\cancel{v}}{\cancel{\mu\cdot a}} - \frac{(\cancel{a\cdot v})\cdot[v]\,a_{n+1}}{\cancel{\mu\cdot a}}$$

$$= -a_{n+1}[v] = -a_{n+1}[v]$$

Circles. Sem 9

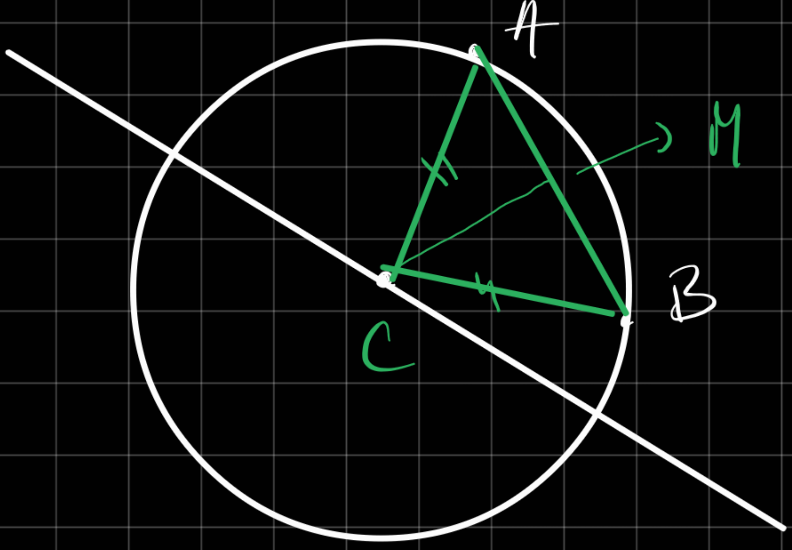
$O = \left(-1, \frac{1}{2}\right)$ $\quad R = \frac{5}{2}$

$\Leftrightarrow \boxed{(x+1)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{5}{2}\right)^2}$

$(x - x_0)^2 + (y - y_0)^2 = R^2$

$C(x_0, y_0),\ R = R$

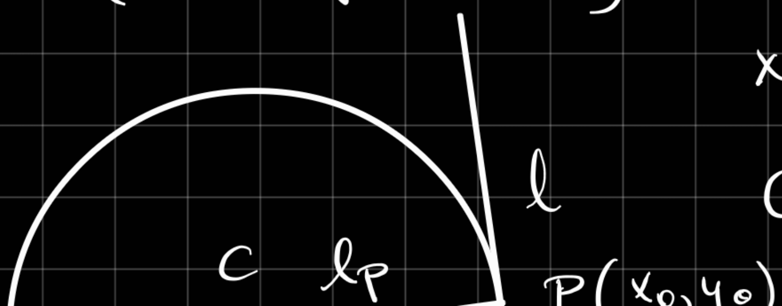
$CM \perp AB$

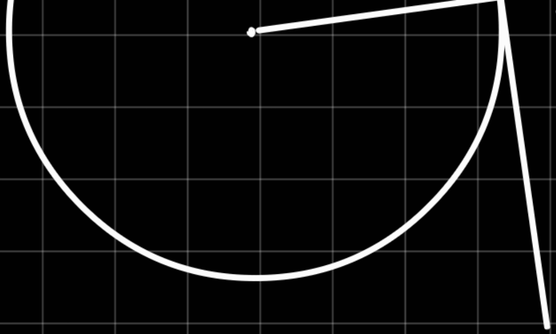



(2.) $x \to \left(x, \pm\sqrt{R^2 - x^2}\right)$

$x^2 + y^2 = R^2$

$C(0,0)$





$$\overrightarrow{CP}\left(x_0, y_0\right) \Rightarrow (CP): \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$(CP): \frac{x}{x_0} = \frac{y}{y_0}$$

$$\Leftrightarrow \boxed{x y_0 - y x_0 = 0.} \qquad \boxed{\overrightarrow{n_{CP}}\left(y_0, -x_0\right)}$$

$$\Rightarrow l: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} y_0 \\ -x_0 \end{bmatrix}$$

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$$x y_0 - \pm\sqrt{R^2 - x^2}\, x_0 = 0$$

$$x y_0 = \pm\sqrt{R^2 - x^2}\, x_0 \quad \| (\,)^2$$

$$x^2 y_0^2 = \left(R^2 - x^2\right) x_0^2$$

$$\Leftrightarrow x^2 y_0^2 + x^2 x_0^2 = R^2 x_0^2 \quad \Big| : x_0^2$$

$$\boxed{x^2\left(\frac{y_0^2}{x_0^2} + 1\right) = R^2} \qquad ?$$

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9.4.25

Ellipses.

(3.) $9x^2 + 25y^2 - 225 = 0 \mid :225$

$$\boxed{\frac{x^2}{25} + \frac{y^2}{9} = 1.}$$

$a^2 = 25$
$b^2 = 9$

$b^2 = a^2 - c^2 \iff c^2 = a^2 - b^2$

$c = \sqrt{25 - 9} = \boxed{4.}$

$\Rightarrow F_1(-4,0),\ F_2(4,0).$

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(6.) $l' = ?$ , $l : 2x - 2y - 13 = 0$ , $\mathcal{E} : x^2 + 4y^2 - 20 = 0$

$l' : n_l(2,-2),\ P(x_0, y_0)$

$(l') : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\frac{x - x_0}{2} = \frac{y - y_0}{-2} \iff -2x + 2x_0 = 2y - 2y_0$

$$\begin{cases} \boxed{2x + 2y - 2y_0 - 2x_0 = 0.} \mid :2 \\ x^2 + 4y^2 - 20 = 0 \end{cases}$$

$x + y - (x_0 + y_0) = 0. \iff x = -y + (x_0 + y_0)$

$$\Rightarrow (-y+x_0+y_0)^2+4y^2-20=0$$

$$(x_0+y_0)^2-2(x_0+y_0)y+y^2+4y^2-20=0$$

$$5y^2-2y(x_0+y_0)+(x_0+y_0)^2-20=0.$$

$$\Delta = 4(x_0+y_0)^2-4\cdot 5\cdot[(x_0+y_0)^2-20]$$

$$\Delta = 4[(x_0+y_0)^2-5(x_0+y_0)^2+100]$$

$$\Delta = 4(-4(x_0+y_0)^2+100)$$

$$\Delta = -16(x_0+y_0)^2+400 = 0 \text{ to be tangent.}$$

$$\Leftrightarrow (x_0+y_0)^2 = \frac{-400}{-16} = \frac{100}{4} = 25$$

$$x_0+y_0 = \pm 5$$

Let $P(1,4)$

$$\Rightarrow \ell': \boxed{x+y-5=0.}$$

$$x+y+5=0$$

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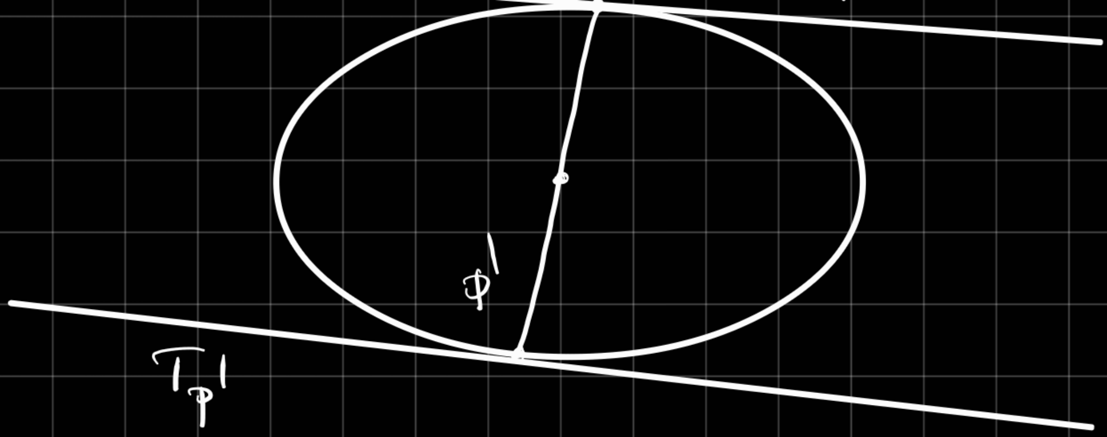
7.

$P$ $T_p$




$\mathcal{E}: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ Tangent at P:

$$\frac{x \cdot x_P}{a^2} + \frac{y \cdot y_P}{b^2} = 1.$$

$$T_P \; \boxed{\frac{x_P}{a^2} \cdot x + \frac{y_P}{b^2} \cdot y - 1 = 0.}$$

$$\boxed{\frac{x_{P'}}{a^2} \cdot x + \frac{y_{P'}}{b^2} \cdot y - 1 = 0}$$

$$C(0,0) \Rightarrow \frac{x_P + x_{P'}}{2} = \frac{y_P + y_{P'}}{2} = 0$$

$$\Leftrightarrow x_{P'} = -x_P, \quad y_{P'} = -y_P$$

$$\Rightarrow -\frac{x_P}{a^2} \cdot x - \frac{y_P}{b^2} \cdot y - 1 = 0 \;\Big|\; \cdot(-1)$$

$$T_{P'} \; \boxed{\frac{x_P}{a^2} x + \frac{y_P}{b^2} y + 1 = 0}$$

$\Rightarrow T_P \parallel T_{P'}$ since $m_{T_P} = m_{T_{P'}}$

---

(8.) $\mathcal{E}_a: \dfrac{x^2}{a^2} + \dfrac{y^2}{16} = 1.$

major axis minor axis

$l: x - y + 5 = 0$

$x = y - 5$

$\dfrac{y^2 - 10y + 25}{a^2} + \dfrac{y^2}{16} = 1 \quad \Big| \cdot 16a^2$

$16y^2 - 160y + 16 \cdot 25 + y^2a^2 = 16a^2$

$y^2(16 + a^2) - 160y + 16(25 - a^2) = 0$

$\Delta = 160^2 - 4(a^2 + 16) \cdot 16(25 - a^2) = 0$

$\vdots$

$a(a^2 - 25a + 16) = 0$

$a = 0$, but $a \neq 0$

$\Delta = 561$

$a_{1,2} = \dfrac{25 \pm \sqrt{561}}{2}$

(10.) Det. the common tangents to the ellipses:

$$T_{\mathcal{E}_1}: \frac{x x_0}{45} + \frac{y y_0}{9} = 1 \iff n_{T_{\mathcal{E}_1}}\left(\frac{x_0}{45}, \frac{y_0}{9}\right)$$

$$\text{—— " ——} \Rightarrow n_{T_{\mathcal{E}_2}}\left(\frac{x_1}{9}, \frac{y_1}{18}\right)$$

$$\iff \frac{x_0}{45} = \frac{x_1}{9} \iff 9x_0 = 45x_1 \mid :9$$

$$\boxed{x_0 = 5x_1}$$

And $18y_0 = 9y_1$

$$\boxed{2y_0 = y_1}$$

$(x_0, y_0) \in \mathcal{E}_1$

$\Rightarrow \left(\frac{x_0}{5}, 2y_0\right) \in \mathcal{E}_2$

$$\frac{\left(\frac{x_0}{5}\right)^2}{9} + \frac{(2y_0)^2}{18} = 1 \quad \Big| \cdot 18$$

$$\frac{2x_0^2}{25} + 4y_0^2 = 18$$

$x^2 + 5y^2 = 45$

$x^2 = 45 - 5y^2$

$$\frac{2(45 - 5y_0^2)}{25} + 4y_0^2 = 18$$

$$\frac{2 \cdot \cancel{5}(9 - y_0^2)}{2\cancel{5}} + 4y_0^2 = \overset{4}{18} \quad \Big| \cdot 5$$

$$18 - 2y_0^2 + 20y_0^2 = 90 \quad \Big| -18$$

$$18y_0^2 = 72$$

$$y_0^2 = \frac{72}{18} = 4$$

$$\boxed{y_0 = \pm 2} \Rightarrow x_0^2 = 45 - 5 \cdot 4 = 25$$

$$\boxed{x_0 = \pm 5}$$

$\Rightarrow$ The 4 tangent intersection points with $\mathcal{E}_1$:

$(-2,-5), (-2,5), (2,-5), (2,5)$

common tangents.

$$\Rightarrow -\frac{2}{45}x - \frac{5}{9}y = 1.$$

-4- with $\mathcal{E}_2$

$$\left(-\frac{2}{5}, -10\right) \Rightarrow -\frac{2}{5} \cdot \frac{1}{9}x - \frac{10 \cdot}{18}y = 1$$

$$-\frac{2}{45}x - \frac{5}{9}y = 1.$$

---

Formula for tangent with Slope (Elipse)

$$\boxed{y = mx \pm \sqrt{a^2m^2 + b^2}}$$

Solve 2-

$$T\mathcal{E}_1: \; y = mx \pm \sqrt{45m^2+9}$$

$$T\mathcal{E}_2: \; y = mx \pm \sqrt{9m^2+18}$$

Same $\Leftrightarrow$

$$45m^2+9 = 9m^2+18$$

$$36m^2 = 9$$

$$m^2 = \frac{9}{36} = \boxed{\frac{1}{4}}$$

$$m = \pm\frac{1}{2}$$

$$y = \frac{1}{2}x \pm \sqrt{\frac{9}{4}+18}$$

$$\frac{9+4\cdot\overset{3}{18}}{4} = \frac{9+72}{4} = \frac{81}{4}$$

$$\boxed{y = \pm\frac{1}{2}x \pm \frac{9}{2}} \;\Big|\cdot 2$$

$$2y = x+9$$

$$x = 2y-9.$$

$$x\cdot\frac{2}{45} + y\cdot\frac{5}{9} = 1$$

$$M_1\left(\pm\frac{2}{45}, \pm\frac{5}{9}\right)$$

$$\pm 2y \pm x - 9 = 0$$

$$M_2\left(\pm 2, \pm 1\right)$$

$y \mp \frac{1}{2}x \mp \frac{9}{2} = 0$

$\frac{x \cdot \pm 2}{45} + \frac{y \cdot \pm 5}{9} = 1 \quad \Big| \cdot \frac{9}{2}$

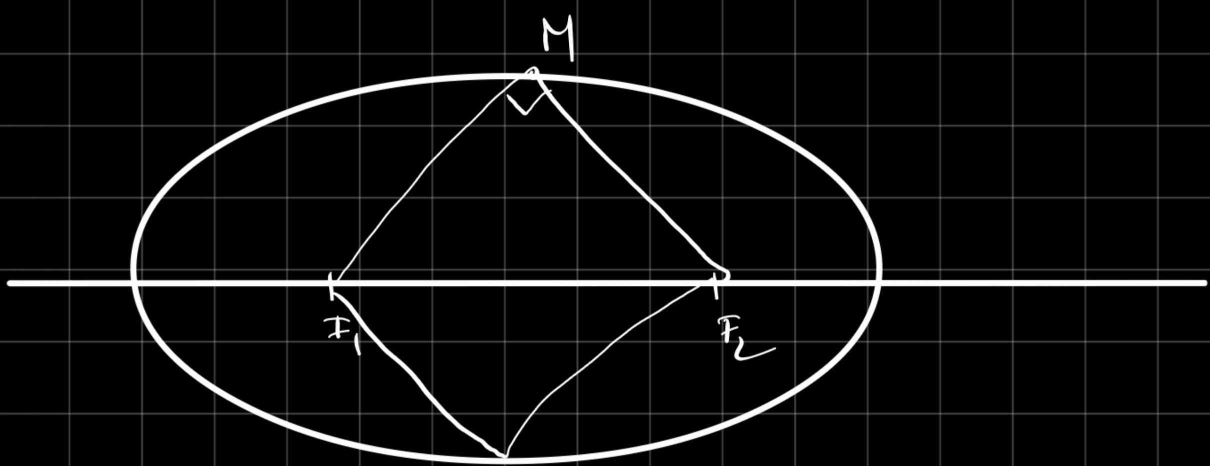
$x \pm \frac{1}{5} \pm y \pm \frac{5}{2} = \frac{9}{2}$

---

(11.) $\mathcal{E}: \frac{x^2}{4} + y^2 - 1 = 0 \quad \Big| \quad \frac{x^2}{4} + y^2 = 1 \qquad a^2 = 4$
$b^2 = 1$

$c^2 = a^2 - b^2 = 3 \qquad c = \pm\sqrt{3}$

$F_1(-\sqrt{3}, 0)$, $F_2(\sqrt{3}, 0)$.




$MF_1 \perp MF_2$ ?

$\overrightarrow{MF_1}(-\sqrt{3} - x_M, -y_M) \qquad \overrightarrow{MF_1} \cdot \overrightarrow{MF_2} =$

$\overrightarrow{MF_2}(\sqrt{3}-x_M, -y_M)$ $\quad -3+\cancel{\sqrt{3}x_M}-\cancel{\sqrt{3}x_M}+x_M^2$

$+ y_M^2 = 0$

$\Leftrightarrow \boxed{x_M^2+y_M^2=3.}$

$y_M = \pm\sqrt{3-x_M^2}$ $\qquad M\left(x, \pm\sqrt{3-x^2}\right)$

---

$\overrightarrow{MF_1}\cdot\overrightarrow{MF_2} = |MF_1|\cdot|MF_2|\cdot\cos(\phi)$

$\Leftrightarrow x\cdot(2a-x)\cdot\cos(\phi)$

$(2ax-x^2)\cdot\cos\theta$

---

Max value $\Rightarrow \cos(\theta)=1$

$\Leftrightarrow |MF_1|\cdot|MF_2| = \sqrt{(-\sqrt{3}-x_M)^2+(-y_M)^2}\cdot$

$\sqrt{(\sqrt{3}-x_M)^2+(-y_M)^2} =$

---

$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

$\cos M = |MF_1|^2+|MF_2|^2-(2c)^2$

$$\frac{\ldots}{2|MF_1|\cdot|MF_2|} =$$

$+y_M^2$

$$= \frac{x_M^2 + \cancel{2\sqrt{3}x_M} + 3 + y_M^2 + 3 - \cancel{2\sqrt{3}x_M} + x_M^2 - 4c^2}{2\sqrt{\left(x_M^2 + 2\sqrt{3}x_M + 3 + y_M^2\right)\left(x_M^2 + 3 + y_M^2 - 2\sqrt{3}x_M\right)}}$$

$$= \frac{2x_M^2 + 2y_M^2 - 6}{2\sqrt{\quad - \text{"} - \quad}} = 1$$

$$\Leftrightarrow \quad x_M^2 + y_M^2 - 3 = \sqrt{\quad\quad} \quad \Big| \,(\,)^2$$

$\vdots$

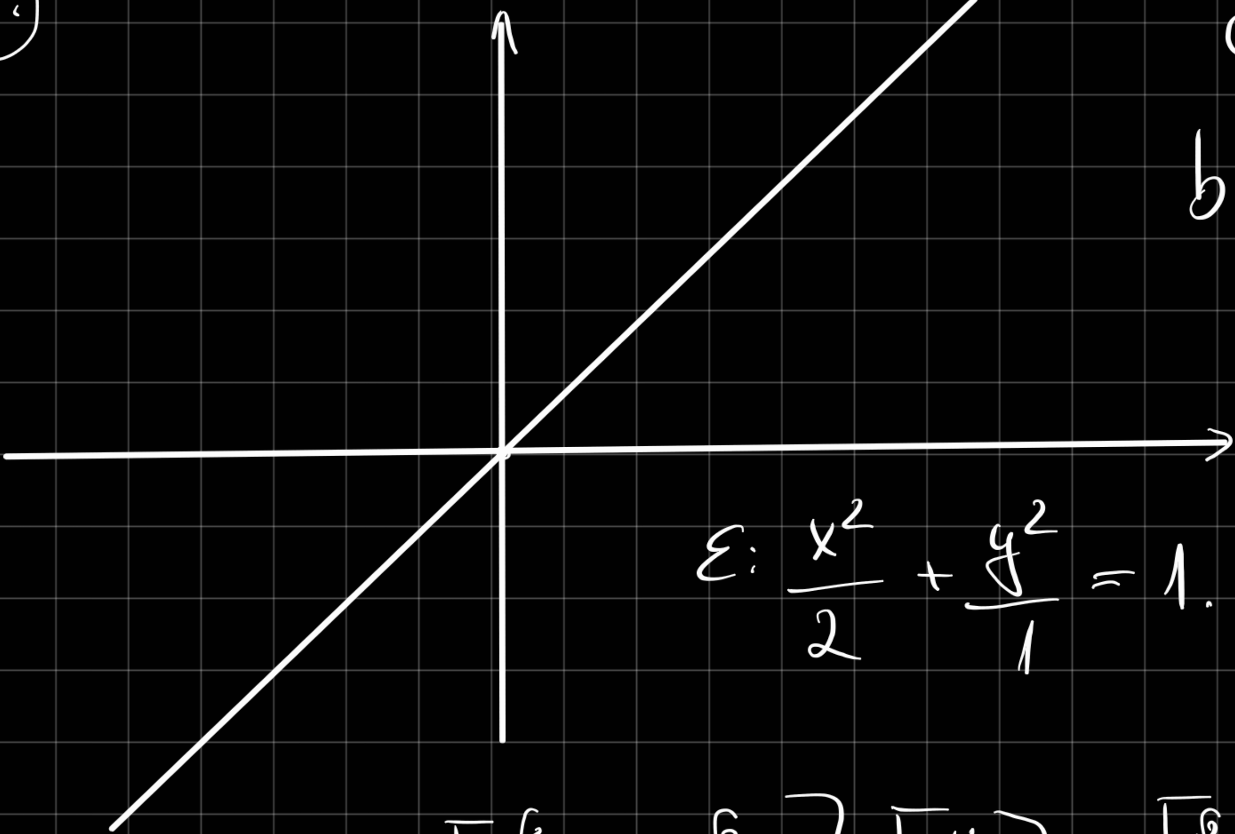
---

(12.)

$a = 2$

$c = \sqrt{3}$

$b = 1$



$$\mathcal{E}: \frac{x^2}{2} + \frac{y^2}{1} = 1. \quad \Big| \text{ în } Oxy$$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \ldots & \ldots \end{bmatrix} \begin{bmatrix} x \\ \ldots \end{bmatrix} \quad \begin{bmatrix} \frac{\sqrt{2}}{2}(x-y) \\ \ldots \end{bmatrix}$$

$\mathrm{Rot}\ \frac{\pi}{4} \Rightarrow \begin{bmatrix} \dots & \dots \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \dots \\ y \end{bmatrix} = \begin{bmatrix} \dots \\ \frac{\sqrt{2}}{2}(x+y) \end{bmatrix}$

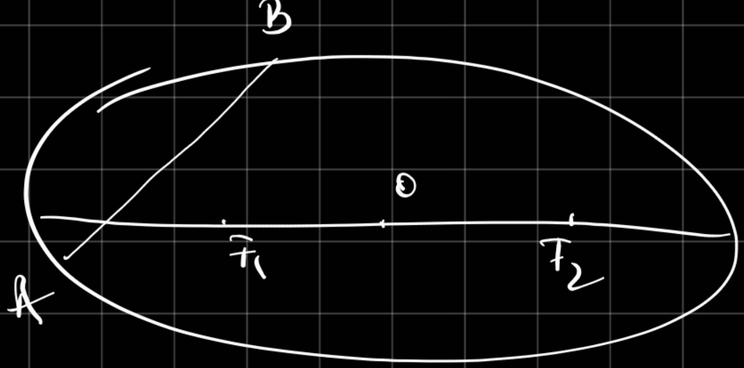
$$\mathcal{E}_{\mathrm{Rot}_{\frac{\pi}{4}}} : \frac{\left[\frac{\sqrt{2}}{2}(x-y)\right]^2}{2} + \left[\left(\frac{\sqrt{2}}{2}\right)(x+y)\right]^2 = 1$$

$$\frac{1}{2}(x-y)^2 \cdot \frac{1}{2} + \frac{1}{2}(x+y)^2 = 1 \quad \Big| \cdot 4$$

$$x^2 - 2xy + y^2 + 2x^2 + 4xy + 2y^2 = 4$$

$$\boxed{3x^2 + 2xy + 3y^2 = 4}$$

13. $\mathcal{E} : x^2 + 4y^2 = 25$




$$\frac{x^2}{25} + \frac{y^2}{\frac{25}{4}} = 1$$

$a = 5$
$b = \frac{5}{2}$

$c^2 = a^2 - b^2$

$c^2 = 25 - \frac{25}{4} = \frac{3 \cdot 25}{4}$

$$\boxed{c = \frac{5}{2}\sqrt{3}}$$

$A(x_A, y_A) : \quad x_A^2 + 4y_A^2 = 25.$

$B(x_B, y_B) : \quad \frac{x_A + x_B}{2} = \frac{7}{2}$

$$\frac{y_A + y_B}{2} = \frac{7}{4}$$

$F_1\left(-\frac{5}{2}\sqrt{3}, 0\right)$

$F_2\left(\frac{5}{2}\sqrt{3}, 0\right)$

$\Leftrightarrow \left( x \quad 7 \quad x \dots \right.$

$x_B = 7 - x_A$

$y_B = \frac{7}{2} - y_A$

$B \in \mathcal{E}: \quad (7 - x_A)^2 + 4\left(\frac{7}{2} - y_A\right)^2 = 25$

$$49 - 14x_A + x_A^2 + 4\left(\frac{49}{4} - 7y_A + y_A^2\right) = 25$$

$$49 - 14x_A + \underline{x_A^2} + 49 - 28y_a + \underline{4y_A^2} = 25$$

$$\underbrace{(x_A + 4y_A^2)}_{\cancel{25}} + 98 - 14x_A - 28y_a = \cancel{25}$$

$$14x_A + 28y_a = 98 \mid :14$$

$$\boxed{x_A + 2y_A = 7}$$

---

14. $\mathcal{E}: \frac{x^2}{25} + \frac{y^2}{9} = 1$ $\qquad \ell: x + 2y = 1$

$\ell_c: x + 2y = c \qquad \boxed{y = -\frac{1}{2}x + c}$

$$\frac{x^2}{25} + \frac{\left(-\frac{1}{2}x + c\right)^2}{9} = 1$$

$$36/ \quad \frac{x^2}{\phantom{0}} + \frac{c^2 - \frac{1}{2}xc + \frac{1}{4}x^2}{\phantom{0}} = 1$$

$$\frac{\quad}{25} \qquad \frac{\quad}{9}$$

$$\frac{61}{36\cdot 25}x^2 - x\cdot\frac{c}{9} + \left(\frac{c^2}{9} - 1\right) = 0$$

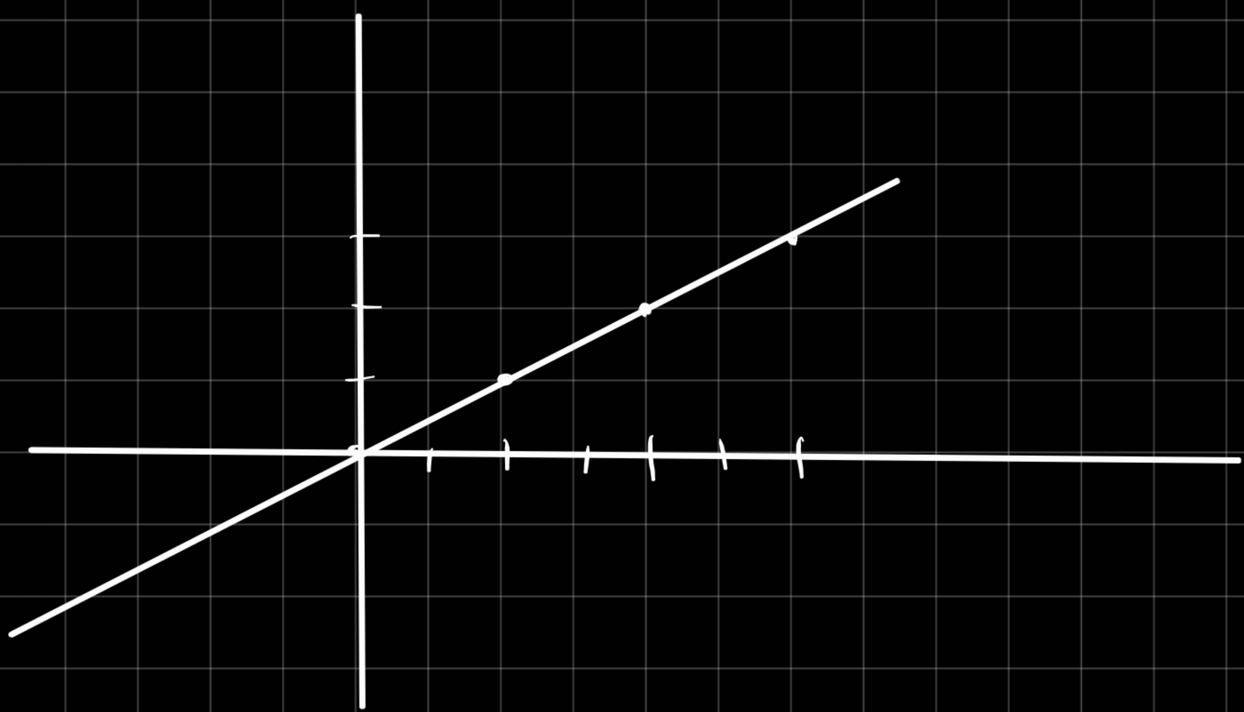
By Viète's rel: $\dfrac{x_A + x_B}{2} = \dfrac{\frac{c}{9}}{\frac{61}{36\cdot 25}} = \boxed{\dfrac{100c}{61}} = \dfrac{50c}{61}$

Same for $y_A$

$$\Rightarrow M\left(\frac{50}{61}c, \frac{36}{61}c\right) = c\left(\frac{50}{61}, \frac{36}{61}\right)$$

Ex

$c\cdot(2,1) \Rightarrow (0,0), (2,1), (4,2), (6,3)$



Seminar 10

(2) Tangents with slope: (Hyperbola)

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$y = mx \pm \sqrt{16m^2 - 8}$$

---

$$-2y = 4x - 5 \quad \Big| \cdot \frac{1}{-2}$$

$$\boxed{y = -2x + \frac{5}{2}} \Rightarrow m = -2.$$

$$\Rightarrow \boxed{y = -2x \pm \sqrt{64 - 8}}$$

(4.) $(x_0, y_0) \in H.$

$$\frac{x x_0}{4} - \frac{y y_0}{9} = 1$$

$$\boxed{x \cdot \frac{x_0}{4} - y\,\frac{y_0}{9} = 1.}$$

$$x_p \cdot \frac{x_0}{4} - y_p \cdot \frac{y_0}{9} \neq 1.$$

$(x_p, y_p)$, where $y_p \neq x \cdot \frac{x_0}{4} -$

5. Eq of the asymptotes of a hyperbola

$$\boxed{y = -\frac{b}{a}x.}$$

$(a_1): y = -\frac{3}{2}x.$ $\qquad (a_2): y = \frac{3}{2}x$

$l_1 \cap a_1 \Rightarrow A_1(4,-6)$

$l_2 \cap a_2 \Rightarrow A_2(2,3)$

$O(0,0)$

$$A_{OA_1A_2} = \frac{1}{2}\left\| \begin{vmatrix} 0 & 0 & 1 \\ 4 & -6 & 1 \\ 2 & 3 & 1 \end{vmatrix} \right\| = \boxed{12.}$$

6. a) $T_H: \frac{x x_0}{20} - \frac{y y_0}{5} - 1 = 0 \qquad (x_0, y_0) \in H$

$l: 4x + 3y - 7 = 0$

$n_{T_H}: \left(\frac{x_0}{20}, \frac{y_0}{5}\right) \qquad n_l: (4,3)$

$n_{T_H} \perp n_\ell \iff \frac{x_0}{5} + \frac{3y_0}{5} = 0 \mid \cdot 5$

$$x_0 = -3y_0$$

$$\frac{9y_0^2}{4} - \frac{y_0^2}{9} = 1 \mid \cdot 36$$

$$81y_0^2 - 4y_0^2 = 36$$

$$x_0 = -\frac{18}{\sqrt{77}}$$

$$77y_0^2 = 36$$

$$y_0^2 = \frac{36}{77} \qquad y_0 = \frac{6}{\sqrt{77}}$$

---

b) $T_P: \; yy_0 - 4(x+x_0) = 0$

$$yy_0 - 4x - 4x_0 = 0 \qquad n_{T_P}(-4, y_0)$$

$$\ell: \; 2x + 2y - 3 = 0 \qquad n_\ell(2,2)$$

$$\boxed{y_0 = -4.}$$

$$16 - 8x_0 = 0$$

$$\boxed{x_0 = 2.} \Rightarrow T_P: \; -4y - 4x - 8 = 0$$

$$\boxed{4x+4y+8=0}$$

---

(7.) $\mathcal{H}: \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0 \qquad P(1,-5)$

$T_{\mathcal{H}}: \frac{x x_0}{3} - \frac{y y_0}{5} = 1$

$T_{\mathcal{H}} \ni P \iff \frac{x_0}{3} + y_0 = 1 \quad \Big| \cdot 3 \qquad \boxed{x_0 + 3y_0 = 3}$

$$\boxed{x_0 = 3 - 3y_0}$$

$$\frac{9(1 - 2y_0 + y_0^2)}{3} - \frac{y_0^2}{5} = 1 \quad \Big| \cdot 15$$

$$45 - 90y_0 + 45y_0^2 - 3y_0^2 = 15$$

$$45 - 90y_0 + 42y_0^2 = 15$$

$$30 - 90y_0 + 42y_0^2 = 0$$

$$42y_0^2 - 90y_0 + 30 = 0 \quad \Big| : 3$$

$$14y_0^2 - 30y_0 + 10 = 0 \quad \Big| : 2$$

$$\begin{array}{r|l} 85 & 5 \\ 17 & 17 \\ 1 & \end{array}$$

$$7y_0^2 - 15y_0 + 5 = 0$$

---

$$\Delta = 15^2 - 4 \cdot 7 \cdot 5 = 225 - 140 = \boxed{85}$$

$$y_0 = \frac{15 \pm \sqrt{85}}{14}$$

---

(8.) $\mathcal{H}: x^2 - \frac{y^2}{4} - 1 = 0$

Hyperbola

$a^2 = 1, \quad b^2 = 4$

$\boxed{b^2 = c^2 - a^2}$

$c^2 = a^2 + b^2 = 5$

$F_1(-\sqrt{5}, 0), \quad F_2(\sqrt{5}, 0), \quad M(x, y)$

$\boxed{|MF_1| - |MF_2| = 2a}$ $\qquad |MF_1| = 2a + |MF_2|$

$$\cos M = \frac{|F_2P|^2 + |F_1P|^2 - |F_1F_2|^2}{2|F_2P| \cdot |F_1P|}$$

$$\cos M = \frac{|F_2M|^2 + (2a + |MF_2|)^2 - 2c}{2|F_2M| \cdot (2a + F_2M)}$$

Let $|F_2M| = x$

$$\cos M = \frac{x^2 + 4 + 4 \cdot x + x^2 - 2\sqrt{5}}{2x(2+x)}$$

$$= \frac{2x^2 + 4x + (4 - 2\sqrt{5})}{2x^2 + 4x}$$

$\hat{M} = 90^\circ \iff \cos M = 0$

$\iff 2x^2 + 4x + 4 - 2\sqrt{5} = 0$

$\Delta = 16 - 4 \cdot 2 \cdot (4 - 2\sqrt{5})$

$\Delta = 16 - 32 + 16\sqrt{5}$

$\Delta = 16(\sqrt{5} - 1)$

$$x_{1,2} = \frac{-4 \pm 4\sqrt{\sqrt{5}-1}}{4} = \boxed{-1 \pm \sqrt{(\sqrt{5}-1)}}$$

$$x = \left|\overrightarrow{MF_2}\left(\sqrt{5} - x_M, -y_M\right)\right| = \sqrt{(\sqrt{5} - x_M)^2 + y_M^2}$$

$= -1 \pm \sqrt{(\sqrt{5}-1)}$ . . . .

---

Seminar 10 ex 8 Să înțeleg

9. $P: y^2 - 10x = 0$

$P: y^2 = 2px$

$y^2 = 2\cdot 5x \quad F_1\left(\frac{5}{2}, 0\right)$

$\Rightarrow p = 5$

directrix: $d: x = -\frac{p}{2}$

---

$P: yy_0 - 5(x + x_0) = 0 \quad , \quad (x_0, y_0) \in P$

$12y_0 - 5(x_0 - 3) = 0$

$12y_0 - 5x_0 + 15 = 0$

$$\boxed{x_0 = \frac{12}{5}y_0 + 3}$$

$y_0^2 - \cancel{10}^{2}\left(\frac{12}{\cancel{5}}y_0 + 3\right) = 0$

$y_0^2 - 24y_0 - 30 = 0$

$\Delta = 24^2 + 4\cdot 30 = 576 + 120 = \boxed{696}$

$y_0 = \frac{24 \pm 2\sqrt{174}}{2} = \boxed{12 \pm \sqrt{174}}$

$\begin{array}{r} 24 \\ 24 \\ \hline 96 \\ 48 \\ \hline 576 \end{array} \qquad \begin{array}{r|l} 696 & 2 \\ 348 & 2 \\ 174 & 2 \\ 87 & 3 \\ 29 & \end{array}$

$d(P, \ell) = \frac{|ax_p + by_p + c|}{\sqrt{a^2 + b^2}}$

(10) $\mathcal{H}: x^2 - 2y^2 = 1$. $\quad \ell: 2x - y = 0$

$$\begin{cases} x^2 - 2y^2 = 1 & \Leftrightarrow \\ 2x - y + c = 0 \Leftrightarrow & x = \frac{y}{2} + c \end{cases}$$

$$\left(\frac{y}{2} + c\right)^2 - 2y^2 = 1 \quad \Big| \cdot 2$$

$$\frac{y^2}{4} + yc + c^2 - 2y^2 = 1 \quad \Big| \cdot 4$$

$$y^2 + 4yc + 4c^2 - 8y^2 = 4$$

$$-7y^2 + 4yc + 4c^2 = 4$$

$$\Delta = (4c)^2 - 4(-7) \cdot 4c^2$$

$$\Delta = 16c^2 + 16c^2 \cdot 7$$

$$\Delta = 16c^2 \cdot 8$$

$$y_{1,2} = \frac{-4c \pm 4c \cdot 2\sqrt{2}}{2} = \boxed{-2c \pm 4c\sqrt{2}}$$

$x = \frac{y}{2} + c$

$$x_{1,2} = \left(\cancel{-c} - 2c\sqrt{2} + \cancel{c}, \; \cancel{-c} + 2c\sqrt{2} + \cancel{c}\right) = \left(-2\sqrt{2}c, 2\sqrt{2}c\right)$$

$$y = \left(-2c - 4c\sqrt{2}, \; -2c + 4c\sqrt{2}\right)$$

$y_M = (0, -2c) = \boxed{c(0,-2)}$

---

(OR) $\mathcal{H}: x^2 - 2y^2 = 1 \qquad a^2=1,\ b^2=\frac{1}{2}$

Slope formula for Parabola

$$\boxed{y = mx + \frac{p}{2m}}$$

11. $k=?$ $\qquad y = kx+2 \qquad P: y^2=4x$

$y^2 = 2px \Rightarrow p=2.$

$T_p: y = mx + \frac{p}{2m}$

$m_e = k$

$\boxed{y = mx + \frac{1}{m}} \Rightarrow y = kx + \frac{1}{k}$

$y = kx+2$

$\Leftrightarrow \frac{1}{k} = 2 \Rightarrow \boxed{k = \frac{1}{2}}$

---

12. $P: y^2 = 16x. \qquad p=8$

$T_p: y = mx + \frac{8}{2m} = \boxed{mx + \frac{4}{m}}$

$T_p \parallel \ell$: $\boxed{m_\ell = \frac{3}{2}}$

$y = \frac{3}{2}x + 15$

$$y = \frac{3}{2}x + \frac{4 \cdot 2}{3} = \boxed{\frac{3}{2}x + \frac{8}{3}}$$

---

b) $m_\ell = -\frac{4}{2} = -2$

$m_\ell^{\perp} = -\frac{1}{m_\ell} = -\frac{1}{-2} = \boxed{\frac{1}{2}}$

---

(13.) $P: y^2 = 16x$ $\quad$ $p = 8$.

$T_p: y = mx + \frac{8}{2m} = mx + \frac{4}{m}$

$2 = -2m + \frac{4}{m} \;\Big|\; \cdot m$

---

Seminar 11

(2.) $A = \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$

Quadratic. eq with associated matrix $A$:

$$\boxed{6x^2 + 4xy + 9y^2 = 0.}$$

$$\det(A - \lambda I_n) = 0$$

$$\Leftrightarrow \begin{bmatrix} 6-\lambda & 2 \\ 2 & 9-\lambda \end{bmatrix} = (6-\lambda)(9-\lambda) - 4 = 0$$

$$54 - 15\lambda + \lambda^2 - 4 = 0$$

$$\boxed{\lambda^2 - 15\lambda + 50 = 0}$$

$$\Delta = 15^2 - 4 \cdot 50 = 225 - 200 = 25$$

$$\lambda_{1,2} = \frac{15 \pm 5}{2} \quad \lambda_1 = \frac{20}{2} = 10, \quad \lambda_2 = \frac{10}{2} = 5$$

---

$$A \cdot v = \lambda_1 v$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 10v_1 \\ 10v_2 \end{bmatrix} = \begin{bmatrix} 6v_1 + 2v_2 \\ 2v_1 + 9v_2 \end{bmatrix} = \begin{bmatrix} 5v_1 \\ 5v_2 \end{bmatrix}$$

$$\begin{cases} 2v_2 - 4v_1 = 0 \\ 2v_1 - v_2 = 0 \end{cases} \qquad v_2 = 2v_1$$

$$\Rightarrow v = (v_1, 2v_1) = \boxed{\langle 1, 2 \rangle}$$

$$\begin{cases} v_1 + 2v_2 = 0 \iff v_1 = -2v_2 \\ \end{cases}$$

$$\Rightarrow v = (-2v_2, v_2) = \boxed{\langle -2, 1 \rangle}$$

$B = (\langle 1, 2 \rangle, \langle -2, 1 \rangle)$

Choose $v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$M = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \in O(2)$$

$$\Rightarrow \quad M^t \cdot A \cdot M = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \frac{1}{\sqrt{5}}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 10 & -10 \\ 20 & 5 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 50 & -10+10 \\ -20+20 & 20+5 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$$

$$[x' \ y'] \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \boxed{10x'^2 + 5y'^2}$$

---

(4.) a) $-x^2 + xy - y^2 = 0$

$A = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$ $\quad \det(A - \lambda I_2) = \dots$

$\lambda \in \left\{ -\frac{1}{2}, -\frac{3}{2} \right\}$

$\Rightarrow \boxed{-\frac{x'}{2} - \frac{3y'}{2} = 0.}$ Canonical form.

OR

$-x^2 + xy - y^2 = 0.$

$-\left(x^2 - xy + \frac{y^2}{4}\right) + \frac{y^2}{4} - y^2 = 0$

$-\underbrace{\left(x - \frac{y}{2}\right)}^2 - \frac{3}{4}y^2 = 0$

$x' = x - \frac{y}{2}$

$\boxed{-x'^2 - \frac{3}{4}y^2 = 0.}$ $(-1)$

$x'^2 + \frac{3}{4}y^2 = 0.$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ \dots \end{bmatrix}$

$$\boxed{x''^2 + y''^2 = 0.}$$

b) $6xy + x - y = 0$

$A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ $\det(A - \lambda I_n) = \begin{pmatrix} -\lambda & 3 \\ 3 & -\lambda \end{pmatrix} = 0$

$\lambda^2 - 9 = 0$

$\lambda = \pm 3.$ ?

$x = x' + y'$ $y = y'$

$6x'y' + 6y'^2 + x' + \cancel{y'} - \cancel{y'} = 0$

$6\left(y'^2 + x'y' + \frac{x'^2}{4}\right) - \frac{3}{2}x'^2 + x = 0$

$$\boxed{6y'' - \frac{3}{2}x''^2 + x'' = 0.}$$

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2. b) $A = \begin{bmatrix} 5 & -13 \\ -15 & 5 \end{bmatrix}$

$\det(A - \lambda I_n) = \begin{bmatrix} 5-\lambda & -13 \\ \end{bmatrix} = 0$

$$\det(A-\lambda I_2) = \begin{vmatrix} \dots & \dots \\ -15 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)^2 - 15\cdot 15 = 0$$

$$(5-\lambda)^2 = 15\cdot 13$$

$$5-\lambda = \pm\sqrt{13\cdot 15}$$

$$\boxed{\lambda = 5 \pm \sqrt{13\cdot 15}}$$

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c) $\begin{bmatrix} 7 & -2 \\ -2 & \frac{5}{3} \end{bmatrix} = A$ $\qquad \boxed{7x^2 - 4xy + \frac{5}{3}y^2 = 0}$

$$\det(A-\lambda I_2) = \begin{vmatrix} 7-\lambda & -2 \\ -2 & \frac{5}{3}-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)\left(\frac{5}{3}-\lambda\right) - 4 = 0$$

$$\frac{35}{3} - \frac{5}{3}\lambda - 7\lambda + \lambda^2 - 4 = 0 \quad \Big| \cdot 3$$

$$35 - 5\lambda - 21\lambda + 3\lambda^2 - 12 = 0$$

$$3\lambda^2 - 26\lambda + 23 = 0$$

$$\Delta = 26^2 - 4\cdot 3\cdot 23 = 676 - 276 = \boxed{400}$$

$26\cdot 26$: $156 + 52 = 676$ $\qquad$ $12\cdot 23$: $36 + 24 = 276$

$$\lambda_{1,2} = \frac{26 \pm 20}{\dots} \qquad \lambda_1 = \frac{46}{\dots} = \frac{23}{\dots}$$

$x_{1,2} = \frac{\quad}{6}$ ... $3$

$x_2 = 1.$

$\Rightarrow$ - - -

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(5) $x^2 + 4xy + y^2 = a$

$x^2 + 4xy + 4y^2 - 3y^2 = a$

$(x+2y)^2 - 3y^2 = a$

$x' = x + 2y \qquad y' = y$

$$\boxed{x'^2 - 3y'^2 = a.}$$

Hyperbola.

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(6.) $R_{90^\circ}, T_v \to (1,0)$

$$T_v \circ R_{90^\circ} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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$$R_{90}, T_v \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

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(7.) Find the canonical eq. for each case:

a) $5x^2 + 4xy + 8y^2 - 32x - 56y + 80 = 0$

$$5x^2 + 4xy + \frac{4}{5}y^2 - \frac{4}{5}y^2 + 8y^2 - 3 - u - = 0$$

$$\left(x\sqrt{5} + \frac{2}{\sqrt{5}}y\right)^2 + \frac{36}{5}y^2 - 32x - 56y + 80 = 0$$

Let $x' = x\sqrt{5} + \frac{2}{\sqrt{5}}y$

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c) $x^2 - 4xy + y^2 - 6x + 2y + 1 = 0$

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \qquad \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$b = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$\Leftrightarrow (1-\lambda)^2 - 4 = 0$$

$$(1-\lambda)^2 = 4$$

$$1 - \lambda = \pm 2 \qquad \lambda_1 = -1, \quad \lambda_2 = 3$$

$$A \cdot u = \lambda_1 v \Leftrightarrow \begin{bmatrix} v_1 - 2v_2 \\ -2v_1 + v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix} = \begin{bmatrix} 3v_1 \\ 3v_2 \end{bmatrix}$$

$$2v_1 - 2v_2 = 0 \Leftrightarrow \boxed{v_1 = v_2} \qquad \langle 1, 1 \rangle$$

$$-2v_1 - 2v_2 = 0 \Leftrightarrow v_1 = -v_2 \qquad \langle -1, 1 \rangle$$

$M_{t,e} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} -6 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -6-2 \\ -6+2 \end{bmatrix}$

$\Rightarrow \quad -x^2 + 3y^2 - 8x - 4y + 1 = 0$

$-\underbrace{(x^2 + 8x + 16)} + 16 + 3y^2 - 4y + 1 = 0$

$-x'^2 + 3\left(y^2 - \frac{4}{3}y + \frac{4}{9}\right) - \frac{4}{3} + 17 = 0$

$-x' + 3y'^2 + \frac{47}{3} = 0$

$3y'^2 - x' + \frac{47}{3} = 0$

$\Rightarrow$ Hyperbola

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(10.) c) $x^2 + xy + yz + zx = 1$

$\underbrace{x^2 + xy + \frac{y^2}{4}} - \frac{y^2}{4} + yz + zx = 1$

$x' = x + \frac{y}{2} \quad y' = y \quad \Rightarrow x = x' - \frac{y'}{2}$

$x'^2 - \frac{y'^2}{4} + yz + z\left(x' - \frac{y'}{2}\right) = 1$

$x^2 \quad y^2$

$$x_1^2 - \frac{y_1^2}{4} + y_1 z_1 + z_1 x_1 - \frac{z_1 y_1}{2} = 1$$

$$x_1^2 + z_1 x_1 + \frac{z_1^2}{4} - \frac{z_1^2}{4} - \frac{y_1^2}{4} + \frac{y_1 z_1}{2} = 1$$

$$x_2 = x_1 + \frac{z_1}{2} \qquad y_2 = y_1 \qquad z_2 = z_1$$

$$x_2^2 - \frac{z_1^2}{4} - \frac{y_1^2}{4} + \frac{y_2 z_2}{2} = 1.$$

$$x_2^2 - \left(\left(\frac{y_1}{2}\right)^2 - \frac{y_2 z_2}{2} + \left(\frac{z_1}{2}\right)^2\right) = 1$$

Adică $\frac{y_1}{2} + \frac{z_1}{2} = y_3$

Cylinder on a

$$\boxed{x_2^2 - y_3^2 = 1}$$

$\Rightarrow$ Hyperbola

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9. $x^2 + \lambda xy + y^2 - 6x - 16 = 0$ ?

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