GEOMETRY

T. PITAGORA:
$c_1^2 + c_2^2 = ip^2$

T. COS:
$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos A$

$A_\Delta = \frac{l^2\sqrt{3}}{4}$ (equilateral triangle)

$A_\Delta = \frac{AB \cdot AC \cdot \sin A}{2}$ (any triangle)

$S = \sqrt{p(p-a)(p-b)(p-c)}$ $p = \frac{a+b+c}{2}$

$A_\Delta = \frac{c_1 \cdot c_2}{2}$

T. SIN:

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$; R (radius of the circumscribed circle)

$A(x_1, y_1)$, $B(x_2, y_2)$
$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

M middle of AB => $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$\vec{PM} = \frac{1}{2}(\vec{PA} + \vec{PB})$

$A(x, y)$
$\vec{OA} = x\vec{i} + y\vec{j}$

$A(x_1, y_1)$, $B(x_2, y_2)$
$\vec{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$

eq. of line AB: $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$ or $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

$A(x_0, y_0)$ m - slope
line : $y - y_0 = m(x - x_0)$

A, B, C - collinear
$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

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• SCALAR / DOT PRODUCT

$a, b \in V^n$ $\qquad a^2 = \|a\|^2$

$a \cdot b = \begin{cases} 0, \text{ if one is } 0 \\ \|a\| \cdot \|b\| \cdot \cos\measuredangle(a,b) \end{cases}$ $\qquad$ a-unit vector: $\frac{1}{\|a\|} a = \frac{1}{\sqrt{a \cdot a}}$

a unit vector $\Leftrightarrow \|a\| = 1$

$a \in V^n \quad \|a\| = \sqrt{a_1^2 + a_2^2} \quad \|a\|^2 = a_1^2 + a_2^2 + \dots + a_n^2$

$a, b \in V^n \qquad b \perp a \Leftrightarrow b \cdot a = 0$

$v \in V^n$, $v^\perp$ - set of all vectors which are orthogonal to v

$Pr_v^\perp : V^n \to \langle v \rangle$, $Pr_v^\perp(w) = w_0 v$ is the orth. proj. on $\langle v \rangle$

$pr_v^\perp : V^n \to \mathbb{R}$, $pr_v^\perp(w) = w_0$ is the orth. proj. on the dir of v.

$Pr_v^\perp(w) = pr_v^\perp(w) v \qquad Pr_v^\perp(\vec{w}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|^2} \cdot \vec{v}$

$a \cdot b = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\cos\measuredangle(a,b) = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}} = \frac{a \cdot b}{\|a\| \cdot \|b\|}$

$a \perp b \Leftrightarrow a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

$A(A_1, A_2, A_3), B(B_1, B_2, B_3)$

$|AB| = \sqrt{(B_1 - A_1)^2 + (B_2 - A_2)^2 + (B_3 - A_3)^2} = \|\vec{AB}\|$

• Gram-Schmidt

$\vec{v}_1' = v_1$

$\vec{v}_2' = \vec{v}_2 - Pr_{\vec{v}_1'}^\perp(\vec{v}_2)$

$\vec{v}_3' = \vec{v}_3 - Pr_{\vec{v}_1'}^\perp(\vec{v}_3) - Pr_{\vec{v}_2'}^\perp(\vec{v}_3)$

• VECTOR / CROSS PRODUCT

I. if a and b are collinear:

$$a \times b = 0$$

II. if a and b are not collinear:

$$\|a \times b\| = \|a\| \cdot \|b\| \sin \sphericalangle(a,b)$$

$a \times b \perp a$ and $a \times b \perp b$

$(a, b, a \times b)$ is right oriented basis of V^3

$\|a \times b\|$ - area of the parallelogram spanned by the vectors.

~~$a \times b = 0 \Rightarrow$~~ a, b parallel $\Leftrightarrow a \times b = 0$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$\|a \times b\| = \sqrt{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2}$$

ΔABC $\quad \vec{AB} = a$, $\vec{AC} = b$

$$Area_\Delta(a, b) = \frac{1}{2} \|a \times b\|$$

$A(x_A, y_A, 0)$, $B(x_B, y_B, 0)$, $C(x_C, y_C, 0)$

$$a \times b = \begin{vmatrix} i & j & k \\ x_B - x_A & y_B - y_A & 0 \\ x_C - x_A & y_C - y_A & 0 \end{vmatrix} = \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix} \cancel{k} = \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} k$$

$$\|a \times b\| = \pm \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} \qquad Area_\Delta(a, b) = \pm \frac{1}{2} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}$$

• BOX PRODUCT

$[a,b,c] = (a\times b)\cdot c$

$\vec{OA}=a,\ \vec{OB}=b,\ \vec{OC}=c \in V^3$

P be the parallelipiped spanned by a,b,c

$[a,b,c] = $ Vol of P if (a,b,c) is right-oriented
$\qquad\qquad$ $-$Vol of P if (a,b,c) is left oriented

OABC Tetrahedron

$\text{Vol}_{OABC} = \left|\frac{1}{6}[\vec{OA},\vec{OB},\vec{OC}]\right|$

(a,b,c) is right oriented $\Rightarrow [a,b,c]>0$
$\qquad$ left oriented $\Leftrightarrow [a,b,c]<0$

an orthonormal basis (a,b,c) is r.o. if $[a,b,c]=1$
$\qquad$ l.o. if $[a,b,c]=-1$

a,b,c coplanar $\Leftrightarrow [a,b,c]=0$

$$[a,b,c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Lagrange identity:

$(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d)-(b\cdot c)(a\cdot d) = \begin{vmatrix} a\cdot c & a\cdot d \\ b\cdot c & b\cdot d \end{vmatrix}$

Triple vector product:

$(a\times b)\times(c\times d) = b\cdot[a,c,d]-a\cdot[b,c,d] = c\cdot[a,b,d]-d\cdot[a,b,c]$

GEOMETRY - 2

• LINES in E^2

Parametric equation:

$$\begin{cases} x = x_A + t v_x \\ y = y_A + t v_y \end{cases} \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

$A = A(x_A, y_A)$, $v = v(\cancel{x_A}, v_y)$

Cartesian equation:

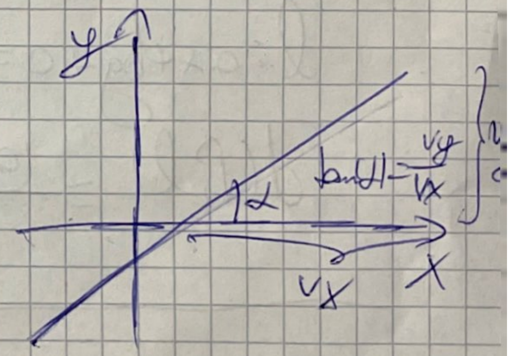
$$\frac{x - x_A}{v_x} = \frac{y - y_A}{v_y} \qquad v_x, v_y \neq 0$$

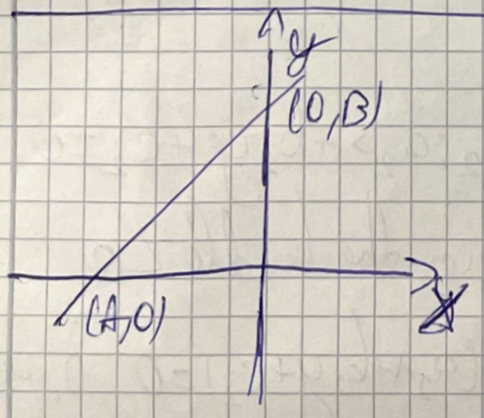
$v_x = 0 \Rightarrow x = x_A \mid v_y = 0 \Rightarrow y = y_A$

Rearrange:

$y = kx + m$, $k = \frac{v_y}{v_x}$ $m = -\frac{v_y}{v_x} x_A + y_A$

$\alpha = \sphericalangle(i, v) \Rightarrow k = \tan(\alpha) \Rightarrow$
$\Rightarrow y - y_A = \tan(\alpha)(x - x_A)$

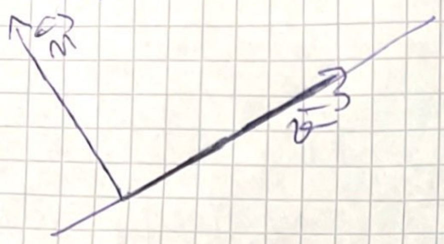




$\frac{x}{A} = \frac{y}{B} = 1$, $A = -\frac{c}{a}$ $B = -\frac{c}{b}$

line: $ax + by + c = 0$

• NORMAL VECTORS



$$\vec{l}^{\perp} = \{\vec{w} \in V^2 \mid l \perp \vec{w}\} = \langle (a,b)^{\perp} \rangle$$
$$= \langle (b,-a) \rangle$$

ex: $\vec{l} = \langle (3,-1) \rangle$
$\vec{l}^{\perp} = \langle (-1,-3) \rangle$

$l_1 : a_1x + b_1y + c_1 = 0 \quad l_2 : a_2x + b_2y + c_2 = 0$

(a_1, b_1) normal vector for l_1

(a_2, b_2) normal vector for l_2

$l : ax + by + c = 0 \quad P(x_P, y_P)$ - point in E^2

$$d(P, l) = \frac{|ax_P + by_P + c|}{\sqrt{a^2 + b^2}}$$

• BUNDLES OF LINES

$l_1 : a_1x + b_1y + c_1 = 0 \quad l_2 : a_2x + b_2y + c_2 = 0$

$l_1, l_2 \to 2$ distinct lines in the bundle $\mathcal{L}_Q$

$l_{\lambda,\mu} : \lambda(a_1x + b_1y + c_1) + \mu(a_2x + b_2y + c_2) = 0 \quad \lambda, \mu \in \mathbb{R} \neq 0$

$Q = Q(x_0, y_0)$ (intersection point) $\quad l_1 : x = x_0 \quad l_2 : y = y_0$

$\mathcal{L}_Q = \{ l_{\lambda,\mu} : \lambda(x - x_0) + \mu(y - y_0) = 0 : \lambda, \mu \in \mathbb{R}$ not both zero $\}$

If $\lambda \neq 0 \Rightarrow l_{1,t} : (a_1x + b_1y + c_1) + t(a_2x + b_2y + c_2) = 0$

$$t = \frac{\mu}{\lambda} \in \mathbb{R}$$

$$l_{1,t} = l_{\lambda,\mu}$$

• LINES AND PLANES

Parametric equation:

$$\begin{cases} x = x_A + s v_x + t w_x \\ y = y_A + s v_y + t w_y \\ z = z_A + s v_z + t w_z \end{cases} \text{ or } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} + s \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + t \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

$A = A(x_A, y_A, z_A)$ - point

$\left.\begin{array}{l} v = v(v_x, v_y, v_z) \\ w = w(w_x, w_y, w_z) \end{array}\right\}$ dir. vectors.

Cartesian equation

$$\left(\frac{v_x}{w_x} - \frac{v_z}{w_z}\right)\left(\frac{x - x_A}{w_x} - \frac{y - y_A}{w_y}\right) = \left(\frac{v_x}{w_x} - \frac{v_y}{w_y}\right)\left(\frac{x - x_A}{w_x} - \frac{z - z_A}{w_z}\right)$$

$\vec{AB}$ vector lin. ind. on $\vec{v}, \vec{w}$

$B(x, y, z) \in$ plane $S \Leftrightarrow$

$$\Leftrightarrow \begin{vmatrix} x - x_A & y - y_A & z - z_A \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0 \Leftrightarrow [\vec{AB}, v, w] = 0$$

Every plane in E^3 can be described with a linear equation in three variables.

$$ax + by + cz + d = 0 \quad (1)$$

$$(1) \Leftrightarrow \frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1$$

$$A = -\frac{d}{a},\ B = -\frac{d}{b},\ C = -\frac{d}{c}$$

NORMAL VECTORS:

$\vec{OA} = (x_A, y_A, z_A) = A \in S$

$n = n(n_x, n_y, n_z)$

$\Rightarrow n_x(x - x_A) + n_y(y - y_A) + n_z(z - z_A) = 0$

$\pi_1 : a_1 x + b_1 y + c_1 z + d_1 = 0$

$\pi_2 : a_2 x + b_2 y + c_2 z + d_2 = 0$ } planes

In order to determine if they intersect we have to discuss their system.

① They intersect in a line if the coordinates of the points on the line will be solutions.
This happens when the rank of A or $\bar{A} = 2$

② They don't intersect if the system does not have solutions. $\Rightarrow$ parallel planes
This happens if the rank A is strictly less than the rank of $\bar{A}$

③ $\pi_1 = \pi_2$ if the solution of the system depends on 2 parameters
This happens if the rank A = rank $\bar{A} = 1$

(a_1, b_1, c_1) - n. vector for π_1

(a_2, b_2, c_2) - n. vector for π_2

$$d(P, \pi) = \frac{|a x_P + b y_P + c z_P + d|}{\sqrt{a^2 + b^2 + c^2}}$$

BUNDLES:

$\pi_{\lambda,\mu} : \lambda(a_1 x + b_1 y + c_1 z + d_1) + \mu(a_2 x + b_2 y + c_2 z + d_2) = 0$

GEOMETRY - 3

• LINES IN E^3

Parametric equation:

$$\begin{cases} x = x_A + t v_x \\ y = y_A + t v_y \\ z = z_A + t v_z \end{cases} \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$A = A(x_A, y_A, z_A)$, $v = (v_x, v_y, v_z)$

Cartesian equation:

$$\frac{x - x_A}{v_x} = \frac{y - y_A}{v_y} = \frac{z - z_A}{v_z}$$

Every line in E^3 can be described with 2 linear equations in 3 variables.

$$\begin{cases} a_1 x + b_1 y + c_1 z + d_1 = 0 \\ a_2 x + b_2 y + c_2 z + d_2 = 0 \end{cases}$$

$$l_1 : \begin{cases} a_1 x + b_1 y + c_1 z + d_1 = 0 \\ a_2 x + b_2 y + c_2 z + d_2 = 0 \end{cases}$$

$$l_2 : \begin{cases} a_3 x + b_3 y + c_3 z + d_3 = 0 \\ a_4 x + b_4 y + c_4 z + d_4 = 0 \end{cases}$$

One way to determine if they intersect:

$$\begin{cases} l_1 = 0 \\ l_2 = 0 \end{cases}$$

It's easier to discuss the relative positions of lines in E^3 via their parametric equations:

$$l_1 : \begin{cases} x = x_1 + tv_x \\ y = y_1 + tv_y \\ z = z_1 + tv_z \end{cases} \qquad l_2 : \begin{cases} x = x_2 + tu_x \\ y = y_2 + tu_y \\ z = z_2 + tu_z \end{cases}$$

We have the cases:

1. If the dir. vectors v and u are proportional then $l_1 \parallel l_2$
2. If ~~they~~ $l_1 \parallel l_2$ and have a point in common, then $l_1 = l_2$
3. $l_1 \nparallel l_2$ and coplanar if:

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0$$

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l_1, l_2 lines

v_1 dir vector for l_1

v_2 dir vector for l_2

$$\cos \measuredangle(v_1, v_2) = \frac{v_1 \cdot v_2}{\|v_1\| \cdot \|v_2\|}$$

$$l_1 : \begin{cases} x = x_A + tv_x \\ y = y_A + tv_y \\ z = z_A + tv_z \end{cases} \qquad P(x_P, y_P, z_P) \in E^3$$

$$d(P, l) = \frac{\|\overrightarrow{PA} \times v\|}{\|v\|}$$

l_1, l_2, lines

- if $l_1 \cap l_2 \Rightarrow d(l_1, l_2) = 0$
- $l_1 \parallel l_2$

$$d(l_1, l_2) = d(P_1, l_2) = d(l_1, P_2)$$

$$P_1 \in l_1, \ P_2 \in l_2$$

- $l_1 \subset \pi_1 \ ; \ \pi_1 \parallel l_2$

$l_2 \subset \pi_2 \ , \ \pi_2 \parallel l_1$

$$d(l_1, l_2) = d(\pi_1, P_2) = d(P_1, \pi_2)$$

$$P_1 \in l_1 \quad P_2 \in l_2$$

$$l_1 : \begin{cases} x = x_1 + t v_x \\ y = y_1 + t v_y \\ z = z_1 + t v_z \end{cases} \qquad l_2 : \begin{cases} x = x_2 + t a_x \\ y = y_2 + t a_y \\ z = z_2 + t a_z \end{cases}$$

$$\pi_1 : \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0$$

$$d(l_1, l_2) = d(\pi_1, P_2) = \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix}}{\sqrt{a_x^2 + a_y^2 + a_z^2}} \right|$$

a - normal vector of π_1

$$a_x = \begin{vmatrix} v_y & v_z \\ u_y & u_z \end{vmatrix} \quad a_y = \begin{vmatrix} v_z & v_x \\ u_z & u_x \end{vmatrix} \quad a_z = \begin{vmatrix} v_x & v_y \\ u_x & u_y \end{vmatrix}$$

$\pi: ax+by+cz+d=0$

$$l: \begin{cases} x = x_0 + tv_x \\ y = y_0 + tv_y \\ z = z_0 + tv_z \end{cases}$$

In order to see if they intersect, we check which points in l satisfy:

$a(x_0+tv_x)+b(y_0+tv_y)+c(z_0+tv_z)+d=0 \Leftrightarrow$

$\Leftrightarrow (av_x+bv_y+cv_z)t+ax_0+by_0+cz_0+d=0$

$n(a,b,c)$ normal vector for π

- $n \cdot v = 0 \Rightarrow l \parallel \pi$
- if $l \parallel \pi$ and the eq. above is not satisfied, then l is outside of π

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GEOMETRY – 4

- Changing coordinate systems

$K = O\,i\,j \quad K' = O'\,i'\,j'$

$[O']_K = \begin{bmatrix} 7 \\ -1 \end{bmatrix}_K \quad i' = -2i + j = \begin{bmatrix} -2 \\ 1 \end{bmatrix}_K \quad j' = i + 2j = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_K$

<u>First step</u>: Change origin

$[\overrightarrow{O'A}]_{K'} = [\overrightarrow{O'A}]_K = [\overrightarrow{OA}]_K - [\overrightarrow{OO'}]_K$

<u>Second step</u>: change the directions: $O\,i\,j \to O\,i'\,j'$

$[\overrightarrow{OA}]_K = M_{K,K'} [\overrightarrow{OA}]_{K'}$ and $[\overrightarrow{OA}]_{K'} = M_{K',K} [\overrightarrow{OA}]_K$

$$! \quad \boxed{[A]_{K'} = M_{K',K} \left([A]_K - [O']_K\right)} = M_{K',K} [A]_K + [O]_{K'}$$

A in $K = (1, 2)$

A in $K' \Rightarrow [A]_{K'} = M_{K',K} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_K + [O]_{K'}$

$M_{K',K} = M_{K,K'}^{-1}$

$$[A]_{K'} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \cdot \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}_K - \begin{bmatrix} 7 \\ -1 \end{bmatrix}_K \right) = \frac{1}{-5} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}_K + \begin{bmatrix} -7 \\ 1 \end{bmatrix}_K \right)$$

$$= \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Same for E^n

• Projections and reflections in a hyperplane

A hyperplane in E^n is a subset of points H

$$H: a_1x_1 + \dots + a_nx_n + a_{n+1} = 0$$

with respect to some coordinate system

H in E^2 are line

E^3 are planes

$l \subseteq E^n$ - line

$l = \{Q + tv, t \in \mathbb{R}\}$

$Q(q_1, \dots, q_n)$ - point

$v(v_1, \dots, v_n)$ - dir. vector

$H \subseteq E^n$ hyperplane

$$H: a_1x_1 + \dots + a_nx_n + a_{n+1} = 0$$

$l \cap H \Leftrightarrow Q + tv \in l \cap H \Leftrightarrow a_1(q_1 + tv_1) + \dots + a_n(q_n + tv_n) + a_{n+1}$

The intersection point (if exists) is:

$$Q' = Q - \frac{a_1q_1 + \dots + a_nq_n + a_{n+1}}{v_1q_1 + \dots + v_nq_n} v$$

- Tensor products (Outer Product)

$v(v_1, \dots, v_n)$ and $w(w_1, \dots, w_n)$ - vectors

$v \otimes w$ - tensor product $= n \times n$ matrix defined by $(v \times w)_{ij} = v_i w_j$

$$v \otimes w = v \cdot w^t = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot [w_1, \dots, w_n] = \begin{bmatrix} v_1 w_1 & \dots & v_1 w_n \\ \vdots & & \\ v_n w_1 & \dots & v_n w_n \end{bmatrix}$$

$$(v \otimes w)^t = w \otimes v$$

Inner Product:

$$\langle v, w \rangle = v^t \cdot w = [v_1, \dots, v_n] \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + \dots + v_n w_n$$

$$u, v, w \in V^n \Rightarrow (u \otimes v) w = (v \cdot w) \cdot u$$

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GEOMETRY - 5

• Isometries

An isometry is a map $\phi : E^n \to E^n$ which preserves distances

$$d(\phi(P), \phi(Q)) = d(P, Q), \ \forall P, Q \in E^n$$

Let $\phi \in AGL(E^n)$ be an affine transformation

$\phi(x) = Ax + b$ with respect to some orthonormal c. sys

$$\Rightarrow \begin{cases} 1.\ \phi \text{ is an isometry} \\ A^{-1} = A^t \end{cases}$$

$A \in Mat_{n \times n}(\mathbb{R})$ s.t. $A^t A = I_n \Rightarrow A$ orthogonal

$\det(A) \in \{\pm 1\}$ $\quad A \in O(n)$

If $\det(A) = +1 \Rightarrow A \in SO(n)$ (special orthogonal)

$\Rightarrow \phi$ displacement (direct isometry)

DIM 2

$$A \text{ in } SO(2) \Leftrightarrow A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \theta \in \mathbb{R}$$

A direct isometry ϕ of E^3 that fixes a point is either the identity or a rotation around an axis that passes through that point. θ angle of the rotation

$$\cos(\theta) = \frac{tr(lin(\phi))}{2}$$

GEOMETRY - 5

- Isometries

An isometry is a map $\phi: E^n \to E^n$ which preserves distances

$$d(\phi(P), \phi(Q)) = d(P, Q), \forall P, Q \in E^n$$

Let $\phi \in AGL(E^n)$ be an affine transformation

$\phi(x) = Ax + b$ with respect to some orthonormal c. sys.

$$\Rightarrow \begin{cases} 1.\ \phi \text{ is an isometry} \\ A^{-1} = A^{t} \end{cases}$$

$A \in Mat_{n \times n}(\mathbb{R})$ s.t. $A^{t}A = I_n \Rightarrow A$ orthogonal

$\det(A) \in \{\pm 1\}$ $\quad A \in O(n)$

If $\det(A) = +1 \Rightarrow A \in SO(n)$ (special orthogonal)

$\Rightarrow \phi$ displacement (direct isometry)

$$A \text{ in } SO(2) \Leftrightarrow A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \theta \in \mathbb{R}$$

DIM 2

A direct isometry ϕ of E^3 that fixes a point is either the identity or a rotation around an axis that passes through that point. θ angle of the rotation

$$\cos(\theta) = \frac{tr(\text{lin}(\phi)) - 1}{2}$$

• DIM 3 rotations

$$\cos(\theta) = \frac{tr(lin(\phi)) - 1}{2}$$

• Classification of isometries

• In E^2

direct isometry – identity
– translation
– rotation

indirect isometry – reflection
– glidereflection

• In E^3

direct isometry – identity
– translation
– rotation around axis
– ~~glidereflection~~ gliderotation

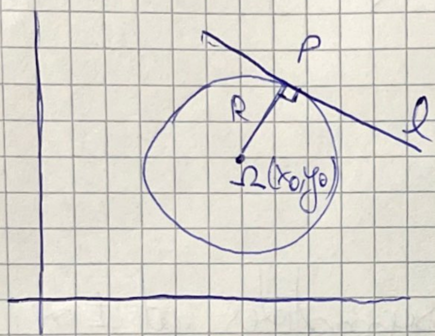
indirect isometry – reflection
– glidereflection
– rotation – reflection

GEOMETRY - 6

• Circle

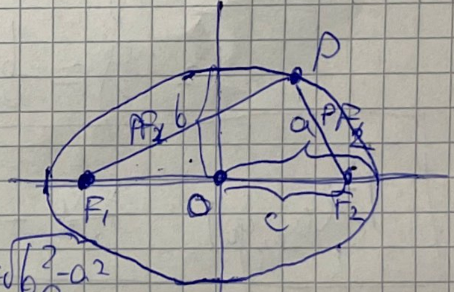
$\mathcal{C}(\Omega, R): (x-x_0)^2+(y-y_0)^2=R^2$

$\mathcal{C}(\Omega, R): \begin{cases} x = x_0 + R\cos t \\ y = y_0 + R\sin t \end{cases} \quad t\in[0,2\pi)$



l - tangent to $\mathcal{C}(\Omega, R) \Rightarrow$
$\Rightarrow \Omega P \perp l$

• Ellipse

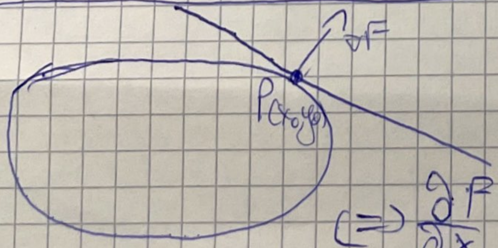


$\mathcal{E}_{a,b}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$a>b,\ c=\sqrt{a^2-b^2}\ \mid a<b\ \ c=\sqrt{b^2-a^2}$
$e=\frac{c}{b}$

eccentricity: $e=\frac{c}{a}=\sqrt{1-\frac{b^2}{a^2}}\in[0,1)$

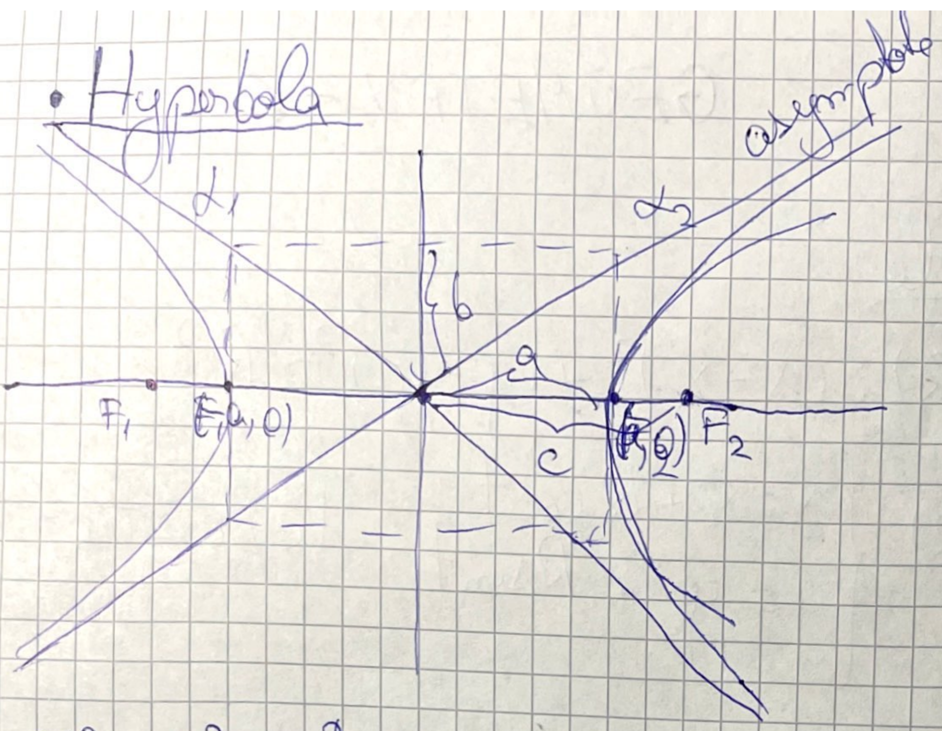
$PF_1+PF_2=2a$



$T_{\mathcal{E}}(x_0,y_0): \frac{x\cdot x_0}{a^2}+\frac{y\cdot y_0}{b^2}=1 \Leftrightarrow$

$\Leftrightarrow \underbrace{\frac{\partial F}{\partial x}(x_0,y_0)}_{\frac{2x_0}{a^2}}\cdot(x-x_0)+\underbrace{\frac{\partial F}{\partial y}(x_0,y_0)}_{\frac{2y_0}{b^2}}\cdot(y-y_0)=0$

• Hyperbola



$$\mathcal{H}: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$c = \sqrt{a^2 + b^2}$$

The hyperbola has 2 asymptotes at $\pm\infty$

$$d_1: y = -\frac{b}{a}x$$

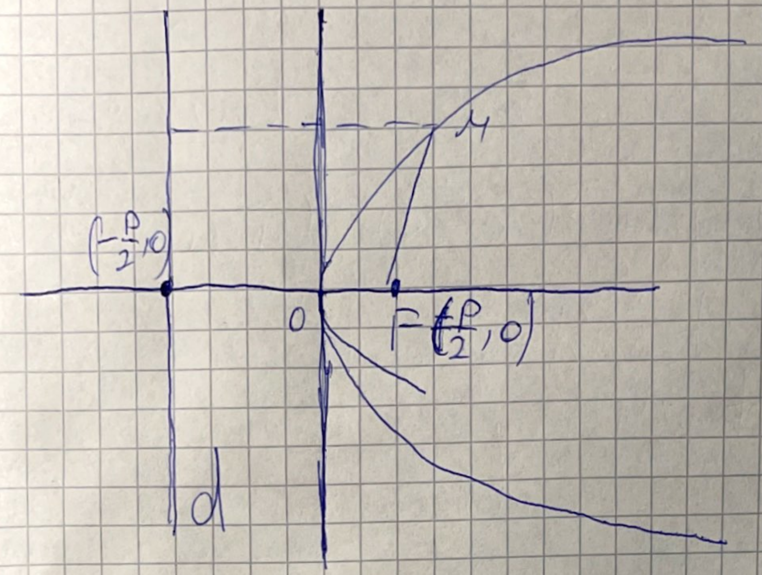
$$d_2: y = \frac{b}{a}x$$

$$\boxed{T_{\mathcal{H}(x_0,y_0)}: \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1} \iff \frac{\partial F}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial F}{\partial y}(x_0,y_0)(y-y_0) = 0$$

$$F(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$$

$$e = \frac{c}{a} = \sqrt{1 + \frac{b^2}{a^2}} \in (1, \infty)$$

Parabolas



$P_p : y^2 = 2px \iff x = \frac{y^2}{2p} \iff y = \pm\sqrt{2px}$

$d : x = -\frac{p}{2}$ d – directix

$T_{P(x_0,y_0)} : yy_0 = p(x+x_0)$

$\iff \frac{\partial F}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial F}{\partial y}(x_0,y_0)(y-y_0) = 0$

$F := y^2 - 2px$