$\xrightarrow{\text{param}} \boxed{\qquad\qquad} \xrightarrow{\text{rand nr}}$

$X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in \underline{I}}$, $I \subseteq \mathbb{N}$ — distribution

ex: $X = \begin{cases} 1, & \text{"heads"} \\ 0, & \text{"tails"} \end{cases}$ → Bernoulli

p = prob of "heads"

• X = the result of rolling a die with n faces

$X \begin{pmatrix} 1 & 2 & \dots & n \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{pmatrix}$ → Discrete, Uniform

$pdf(x, \text{param}) = P(X = x)$

$cdf(x, \text{param}) = P(X \leq x)$

1) 2 files: $\overset{F_1}{0.4}$, $\overset{F_2}{0.3}$ independently

X = nr of corrupted files out of 2 files.

$P(X=0) = P(\overline{F_1}) \cdot P(\overline{F_2}) = 0.6 \cdot 0.7 = \boxed{0.42}$

F_i: "file i is corrupted", $i = 1, 2$

$P(X=2) = P(F_1) \cdot P(F_2) = 0.3 \cdot 0.4 = \boxed{0.12}$

$P(X=1) = 1 - P(X=0) - P(X=2) = 1 - 0.54$

$= \boxed{0.46}$

$$\Rightarrow X \begin{pmatrix} 0 & 1 & 2 \\ 0.42 & 0.46 & 0.12 \end{pmatrix}$$

$P(X=1) = P(F_1) \cdot P(\overline{F_2}) + P(F_2) \cdot P(\overline{F_1}) =$

$= 0.4 \cdot 0.7 + 0.6 \cdot 0.3 = 0.28 + 0.18$

$= \boxed{0.46}$

---

(2) X — nr of heads in 3 flips

a) Binomial with $p = \frac{1}{2}$, $n = 3$

$$X \begin{pmatrix} 0 & 1 & 2 & 3 \\ C_3^0 \left(\frac{1}{2}\right)^3 & C_3^1 \left(\frac{1}{2}\right)^3 & C_3^2 \left(\frac{1}{2}\right)^3 & C_3^3 \left(\frac{1}{2}\right)^3 \end{pmatrix}$$

b) $P(X \leq 2)$, $P(X < 2) = ?$

$P(X \leq 2) = P(X = 0 \cup X = 1 \cup X = 2) = P(X=0) + P(X=1) +$

$P(X=2) = \underbrace{C_3^0 \left(\frac{1}{2}\right)^3 + C_3^1 \left(\frac{1}{2}\right)^3} + C_3^2 \left(\frac{1}{2}\right)^3.$

$P(X < 2) = P(X=0 \cup X=1) =$ — " —

(3.) avg rate 10 accounts / day.

a) $P(X_1 > 8)$?

b) $P(X_2 \leq 16)$

"rare events" + "avg/frequency"

$\Rightarrow$ Poisson Distribution

a) Poisson distribution with $\lambda_1 = 10$

$X_1 \Big(\; k \quad \frac{10^k}{\dots} e^{-10} \dots$

$\left(\frac{10^k}{k!} e^{-10}\right)_{k=0,1,\dots}$

$$P(X_1 > 8) \overset{\lambda_1=10}{=} \boxed{\sum_{k=9}^{10} \frac{10^k}{k!} e^{-10}}$$

b) $P(X_2 \le 16) \overset{\lambda_2=20}{=} \sum_{i=1}^{16} \frac{20^k}{k!} e^{-20}$

* within the next 2 days $\to$ average of 20 new acc. $\Rightarrow \boxed{\lambda = 20}$

(4.) $p = 0.7$; X = nr of attempts needed to gain access to the computer.

Theory Review

Bernoulli Distribution with parameter $p \in (0,1)$ pdf: $X \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$

Binomial Distribution with parameters $n \in \mathbb{N}, p \in (0,1)$ pdf: $X \begin{pmatrix} k \\ C_n^k p^k q^{n-k} \end{pmatrix}_{k=\overline{0,n}}$

Discrete Uniform Distribution with parameter $m \in \mathbb{N}$ pdf: $X \begin{pmatrix} k \\ \frac{1}{m} \end{pmatrix}_{k=\overline{1,m}}$

Hypergeometric Distribution with parameters $N, n_1, n \in \mathbb{N}$ $(n_1 \le N)$ pdf: $X \begin{pmatrix} k \\ \frac{C_{n_1}^k C_{N-n_1}^{n-k}}{C_N^n} \end{pmatrix}_{k=\overline{0,n}}$

Poisson Distribution with parameter $\lambda > 0$ pdf: $X \begin{pmatrix} k \\ \frac{\lambda^k}{k!} e^{-\lambda} \end{pmatrix}_{k=0,1,\dots}$

X represents the number of "rare events" that occur in a fixed period of time; λ represents the frequency, the average number of events during that time.

(Negative Binomial) Pascal Distribution with parameters $n \in \mathbb{N}, p \in (0,1)$ pdf:

$X \begin{pmatrix} k \\ C_{n+k-1}^k p^n q^k \end{pmatrix}_{k=0,1,\dots}$

Geometric Distribution with parameter $p \in (0,1)$ pdf: $X \begin{pmatrix} k \\ pq^k \end{pmatrix}_{k=0,1,\dots}$

Cumulative Distribution Function (cdf) $F_X : \mathbb{R} \to \mathbb{R}$, $F_X(x) = P(X \le x) = \sum_{x_i \le x} p_i$

$(X,Y) : S \to \mathbb{R}^2$ **discrete random vector**:
– **(joint) pdf** $p_{ij} = P(X = x_i, Y = y_j)$, $(i,j) \in I \times J$,
– **(joint) cdf** $F = F_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$, $F(x,y) = P(X \le x, Y \le y) = \sum_{x_i \le x} \sum_{y_j \le y} p_{ij}$, $\forall (x,y) \in \mathbb{R}^2$,
– **marginal densities** $p_i = P(X = x_i) = \sum_{j \in J} p_{ij}$, $\forall i \in I$, $q_j = P(Y = y_j) = \sum_{i \in I} p_{ij}$, $\forall j \in J$.

For $X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I}$, $Y \begin{pmatrix} y_j \\ q_j \end{pmatrix}_{j \in J}$,
X and Y are **independent** $<=>$ $p_{ij} = P(X = x_i, Y = y_j) = P(X = x_i) P(Y = y_j) = p_i q_j$.

$X+Y \begin{pmatrix} x_i + y_j \\ p_{ij} \end{pmatrix}_{(i,j) \in I \times J}$, $\alpha X \begin{pmatrix} \alpha x_i \\ p_i \end{pmatrix}_{i \in I}$, $XY \begin{pmatrix} x_i y_j \\ p_{ij} \end{pmatrix}_{(i,j) \in I \times J}$, $X/Y \begin{pmatrix} x_i / y_j \\ p_{ij} \end{pmatrix}_{(i,j) \in I \times J}$ $(y_j \ne 0)$

a) $X \begin{pmatrix} k \\ 0.7 \cdot 0.3^{k-1} \end{pmatrix}$ $\Rightarrow$ Geometric Distribution

$$\begin{pmatrix} & 0.7 & \cdot & 0.3 \end{pmatrix}_{k=0,1,\ldots}$$

b) $P(X \le 4) = \sum_{k=1}^{4} \frac{7}{10} \cdot \left(\frac{3}{10}\right)^{k-1}$

c) $P(X \ge 3) = 1 - P(X<3) = 1 - \sum_{k=1}^{2} \frac{7}{10}\left(\frac{3}{10}\right)^{k-1}$

(5.) a) X – discrete uniform, $n=5$; $X\begin{pmatrix} k \\ \frac{1}{5} \end{pmatrix}_{k=\overline{1,5}}$

a) $X\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$, $Y\begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$.

$\frac{1}{5} \downarrow$ $x=1$; $\frac{3}{5} \downarrow$ prime ; $\frac{1}{5} \downarrow$ otherwise

b) $pdf(X+Y) = \begin{pmatrix} x_i + y_j \\ p_{ij} \end{pmatrix}$

$X=1 \Rightarrow X+Y = 1+1 = 2$

$X=2 \Rightarrow X+Y = 4$ For $x=2 \Rightarrow y=2$ (prime nr)

$X=3 \Rightarrow X+Y = 5$ For $x=3 \Rightarrow y=2$ (prime nr)

$X=4 \Rightarrow X+Y = 7$ For $x=4 \Rightarrow y=3$ (otherwise)

$X=5 \Rightarrow X+Y = 7$ For $x=5 \Rightarrow y=2$ (prime nr)

$$X+Y\begin{pmatrix} 2 & 4 & 5 & 7 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

$X=1 \Rightarrow XY = 1$

$X=2 \Rightarrow XY = 4$

$X=3 \Rightarrow XY = 6$

$$\Rightarrow XY\begin{pmatrix} 1 & 4 & 6 & 10 & 12 \\ \frac{1}{} & \frac{1}{} & \frac{1}{} & \frac{1}{} & \frac{1}{} \end{pmatrix}$$

$X=4 \Rightarrow XY = 4\cdot 3 = 12$

$X=5 \Rightarrow XY = 10$

Seminar 5

parameters $\longrightarrow \boxed{} \xrightarrow{\text{rand nr}}$

X random variable continuous with pdf $f:\mathbb{R}\to\mathbb{R}$,

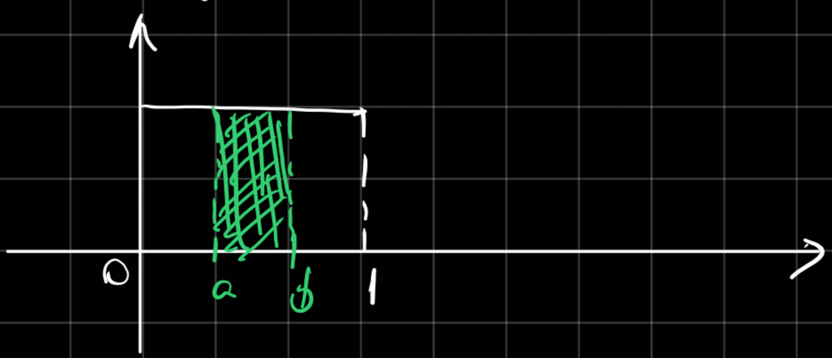
$$\boxed{P\left(X\in(a,b)\right) = \int_a^b f(x)\,dx}$$

ex1| Uniform continuous distribution on $[c,d]$

$$P\left(X\in(a,b)\right) = \frac{len(a,b)}{len(c,d)} = \int_a^b f(x)\,dx.$$

$$f(x) = \begin{cases} \frac{1}{d-c}, & x\in(c,d) \\ 0, & \text{otherwise} \end{cases}$$

ex2| Uniform $[0,1]$



$P(X=x) = 0 \quad (\forall)\, x \in \mathbb{R}$

$P(X = x_0) = 0, \ (\forall) x_0 \in \mathbb{R}$

$$P(X = x_0) = P(X \in [x_0, x_0]) = \int_{x_0}^{x_0} f(x)\, dt = 0.$$

Theory Review

$X : S \to \mathbb{R}$ continuous random variable with pdf $f : \mathbb{R} \to \mathbb{R}$ and cdf $F : \mathbb{R} \to \mathbb{R}$. Properties:

1. $F(x) = P(X \leq x) = \int\limits_{-\infty}^{x} f(t)dt$
2. $\boxed{f(x) \geq 0, \forall x \in \mathbb{R}, \int\limits_{\mathbb{R}} f(x) = 1}$
3. $P(X = x) = 0, \forall x \in \mathbb{R}, P(a < X < b) = P(a \leq X \leq b) = \int\limits_{a}^{b} f(t)dt$
4. $\boxed{F(-\infty) = 0, F(\infty) = 1}$

$(X, Y) : S \to \mathbb{R}^2$ continuous random vector with pdf $f = f_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$ and

cdf $F = F_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}, \ F(x, y) = P(X \leq x, Y \leq y) = \int\limits_{-\infty}^{x} \int\limits_{-\infty}^{y} f(u, v)\, dv\, du, \ \forall (x, y) \in \mathbb{R}^2$. Properties:

1. $P(a_1 < X \leq b_1, a_2 < Y \leq b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$
2. $F(\infty, \infty) = 1, \ F(-\infty, y) = F(x, -\infty) = 0, \ \forall x, y \in \mathbb{R}$
3. $F_X(x) = F(x, \infty), \ F_Y(y) = F(\infty, y), \ \forall x, y \in \mathbb{R}$ (marginal cdf's)
4. $P((X, Y) \in D) = \iint\limits_{D} f(x, y)\, dy\, dx$
5. $f_X(x) = \int\limits_{\mathbb{R}} f(x, y)dy, \ \forall x \in \mathbb{R}, \ f_Y(y) = \int\limits_{\mathbb{R}} f(x, y)dx, \ \forall y \in \mathbb{R}$ (marginal densities)
6. X and Y are independent $<=> f_{(X,Y)}(x, y) = f_X(x) f_Y(y), \ \forall (x, y) \in \mathbb{R}^2$.

Function $Y = g(X)$: X r.v., $g : \mathbb{R} \to \mathbb{R}$ differentiable with $g' \neq 0$, strictly monotone

$f_Y(y) = \dfrac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}, \ y \in g(\mathbb{R})$

Uniform distribution $U(a, b), \ -\infty < a < b < \infty$: pdf $f(x) = \dfrac{1}{b - a}, x \in [a, b]$.

Normal distribution $N(\mu, \sigma), \mu \in \mathbb{R}, \sigma > 0$: pdf $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}$.

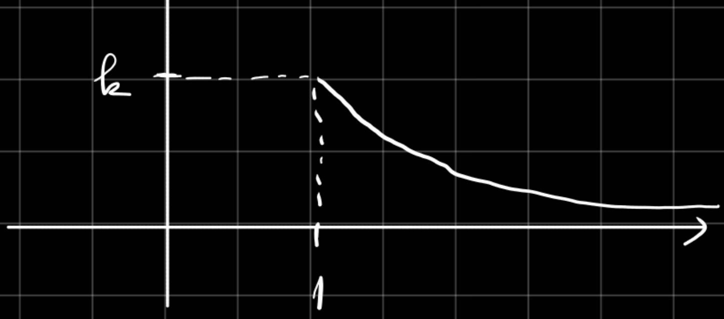
Gamma distribution $Gamma(a, b), \ a, b > 0$: pdf $f(x) = \dfrac{1}{\Gamma(a) b^a} x^{a-1} e^{-\frac{x}{b}}, \ x > 0$.

Exponential distribution $Exp(\lambda) = Gamma(1, 1/\lambda), \ \lambda > 0$: pdf $f(x) = \lambda e^{-\lambda x}, x > 0$.
- Exponential distribution models *time*: waiting time, interarrival time, failure time, time between rare events, etc; the parameter λ represents the frequency of rare events, measured in time^{-1}.
- Gamma distribution models the *total* time of a multistage scheme.
- For $\alpha \in \mathbb{N}$, a $Gamma(\alpha, 1/\lambda)$ variable is the sum of α independent $Exp(\lambda)$ variables.

① The lifetime in years, of some electric comp. is a random var. with density:

$\Big\{ \dfrac{k}{\ } , \ x > 1$

$$f(x) = \begin{cases} \frac{k}{x^4}, & \\ 0, & x < 1 \end{cases}$$



a) Find constant k.

$$\boxed{\int_1^\infty \frac{k}{x^4} = 1} \Rightarrow \int_1^\infty \frac{k}{x^4} = k \cdot \frac{-3}{x^3}\Big|_1^\infty = k\left(\frac{-3}{\infty} - \frac{(-3)}{1}\right)$$

$$= 3k = 1 \Rightarrow \boxed{k = \frac{1}{3}}$$

b) cdf $F = ?$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x < 1 \\ \underbrace{\int_{-\infty}^{1} 0\,dt + \int_1^x \frac{k}{t^4}}, & x \ge 1. \end{cases}$$

c) $$P(X > 2) = 1 - P(X \le 2) = 1 - \int_1^2 \frac{\frac{1}{3}}{t^4}\,dt =$$

$$= 1 - \frac{1}{3}\left(\frac{-3}{x^3}\right)\Big|_1^2 = 1 - \frac{1}{3}\left(-\frac{3}{8} - \frac{-3}{1}\right) = 1 - \frac{1}{3}\left(3 - \frac{3}{8}\right)$$

$$= 1 - 1 + \frac{1}{8} = \boxed{\frac{1}{8}}$$

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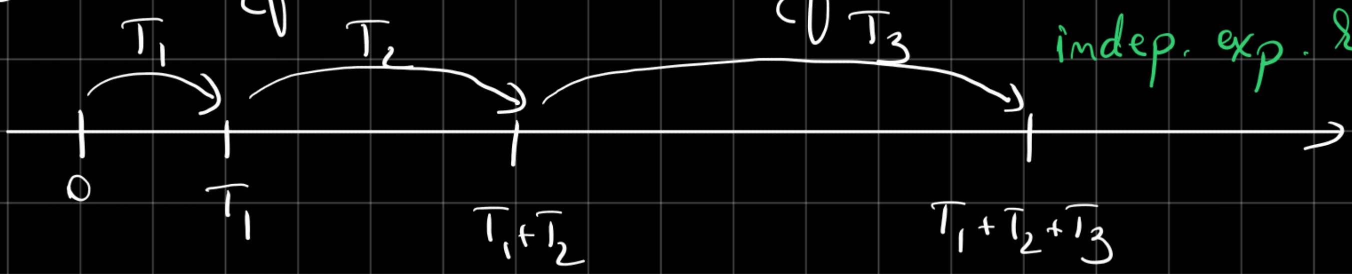
(3) on avg: breakdown every 5 months.

indep. exp. rand. vars



a) $P(\ T\ \ <\ 2)\ -\ 2$

a) $P(T \le 9) = ?$

T = time in months ; avg rate $= \lambda = \boxed{\frac{1}{5}}$

$\Rightarrow \text{Gamma}\left(1, \frac{1}{\lambda}\right) = \text{Exp}(\lambda)$

$\Rightarrow \text{Gamma}\left(\alpha, \frac{1}{\lambda}\right) \xrightarrow[\qquad]{\lambda = \frac{1}{5},\ \alpha = 3} \text{Gamma}(3, 5)$

α = nr of indep. Exp variables

a) $P(T \le 9) = \text{gammacdf}(9, 3, 5)$

b) $P(T > 16 \mid T > 12) = \dfrac{P((T>16) \cap P(T>12))}{P(T>12)} =$

$= \dfrac{P(T>16)}{P(T>12)} = \dfrac{1 - P(T \le 16)}{1 - P(T \le 12)}$

4\.

$$f_{x,y}(x,y) = \begin{cases} \frac{1}{16} x^3 y^3, & x, y \in [0,2] \\ 0, & \text{otherwise} \end{cases}$$

a) $f_X(x) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dy = \begin{cases} \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy, & x \in [0,2] \\ 0, & \text{otherwise} \end{cases}$

$= \frac{x^3}{16} \int_{-\infty}^{\infty} y^3\,dy = \frac{x^3}{16} \cdot \frac{y^4}{4} \Big|_{-\infty}^{\infty} = \frac{x^3}{16}\left(0 + \frac{16}{4} - \frac{0}{4}\right)$

$$= \frac{x^3}{16} \cdot 4 = \boxed{\frac{x^3}{4}}$$

Analog $f_Y(y) = \begin{cases} \frac{y^3}{4}, & y \in [0,2] \\ 0, & \text{otherwise.} \end{cases}$

b) X, Y indep $\Leftrightarrow f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$f_X(x) \cdot f_Y(y) = \begin{cases} \frac{x^3 y^3}{16}, & x,y \in [0,2] \\ 0, & \text{otherwise} \end{cases} \Rightarrow f_{X,Y}(x,y) \Rightarrow X, Y$ <u>indep.</u>

c) $P(X \leq 1) = P(-\infty < X \leq 1) = \int_{-\infty}^{1} f_X(x) =$

$$= \int_0^1 \frac{x^3}{4} = \frac{1}{4} \cdot \frac{x^4}{4} \Big|_0^1 = \boxed{\frac{1}{16}}.$$

Seminar 6

Numerical Characteristics of Random Variables

X: discrete → pdf; continuous →

$\begin{pmatrix} x_i \\ \end{pmatrix}$

$f: \mathbb{R} \to [0, \infty)$

$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx.$

(p_i)

ex| $X = \begin{pmatrix} 1 & \cdots & 6 \\ \frac{1}{6} & \cdots & \frac{1}{6} \end{pmatrix}$

$$E(X) = 1 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = \boxed{3.5}$$

Proof: $X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I}$

↳ call it N times ($N \simeq 1$ bil)

Arithmetic mean of the results:

$$\frac{x_1 \cdot (p_1 \cdot \cancel{N}) + \ldots + x_i p_i \cancel{N}}{\cancel{N}} = E(x).$$

ex| X = result of "rand"

$$f(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x\, dx = \boxed{\frac{1}{2}}$$

$$\boxed{V(x) = E(x^2) - E(x)^2 \geq 0}$$

Variance

Norm (m, σ)

$7-3\sigma \quad 7 \quad 7+3\sigma \qquad 7-3\sigma \quad 7 \quad 7+3\sigma$

$$STD(X) = \sqrt{V(X)} = \sigma(X)$$

$$E(aX) = a\,E(X).$$

$$V(aX) = a^2\,V(X)$$

$$E(X+Y) = E(X) + E(Y)$$

$$X, Y \text{ indep} \iff \begin{cases} V(X+Y) = V(X) + V(Y) \\ E(XY) = E(X) \cdot E(Y) \end{cases}$$

$$E(h(X)) = \sum_{i \in I} h(x_i) \cdot p_i = \int_{-\infty}^{\infty} h(x) \cdot f(x)$$

ex| $X\begin{pmatrix} -1 & 1 \\ 0.5 & 0.5 \end{pmatrix}$, $E(X^2) = 1$

$$X^2\begin{pmatrix} (-1)^2 & 1^2 \\ 0.5 & 0.5 \end{pmatrix} = X^2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

---

1) $X\begin{pmatrix} 0 & 1 & 2 \\ 0.7 & 0.2 & 0.1 \end{pmatrix}$ $\qquad Y$ = cost per day

a) $E(Y) = ?$

$$Y = \begin{pmatrix} 0\,\$ & 500\,\$ & 1000\,\$ \\ 0.7 & 0.2 & 0.1 \end{pmatrix} \qquad \boxed{Y = 500X.}$$

$$E(Y) = 0 \cdot 0.7 + 500 \cdot 0.2 + 1000 \cdot 0.1 = 100 + 100 = \boxed{200}$$

$$Y^2 \begin{pmatrix} 0 & 250000 & 1000000 \\ 0.7 & 0.2 & 0.1 \end{pmatrix}$$

$$E(Y^2) = 500^2 \cdot 0.2 + 1000^2 \cdot 0.1 = \ldots$$

b) Standard deviation ?

$$V(x) = E(x^2) - E(x)^2$$

$$x^2 = \begin{pmatrix} 0 & 1 & 4 \\ 0.7 & 0.2 & 0.1 \end{pmatrix} \qquad E(x^2) = 0.2 + 0.4 = 0.6$$

$$E(x)^2 = (0.2 + 0.2)^2 = \boxed{0.16}$$

$$\Rightarrow V(x) = 0.6 - 0.16 = \boxed{0.44}$$

$$\Rightarrow Std(Y) = \sqrt{V(Y)} = \sqrt{500^2\, V(x)} = \boxed{500 \cdot \sqrt{0.44}}$$

$$V(Y) = 500 \cdot V(x)$$

---

③ $$f(x) = \begin{cases} \frac{3}{x^4}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

How many years, on average, can we expect the electronic equipment to last?

$$E(x) = \int^{\infty} x f(x)\, dx = \int^{1} x f(x)\, dx \quad \int^{\infty} \frac{3}{} \cdot x\, dx =$$

a) $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \underbrace{\int_{-\infty}^{1} x f(x)\, dx}_{0} + \int_{1}^{\infty} \frac{3}{x^4}$

$$= \int_1^\infty \frac{3}{x^3}\, dx = 3 \cdot \frac{x^{-2}}{-2} \Big|_1^\infty = \boxed{\frac{3}{2}}$$

b) $Std(X) = ?$

Let $h(x) = x^2$. $E(x^2) = E(h(x)) = \int_{-\infty}^{\infty} h(x) \cdot f(x)\, dx =$

$$\int_1^\infty x^2 f(x)\, dx = \int x^2 \cdot \frac{3}{x^4} = \int \frac{3}{x^2} = 3\,\frac{x^{-1}}{-1}\Big|_1^\infty = 3$$

$$Std(X) = \sqrt{V(X)} = \sqrt{E(x^2) - E(x)^2} = \sqrt{3 - \frac{9}{4}} =$$

$$= \sqrt{\frac{12-9}{4}} = \sqrt{\frac{3}{4}} = \boxed{\frac{\sqrt{3}}{2}}.$$

4. 600 \$

A ---- 20\$ / share ---- return $\begin{pmatrix} -1 & 0 & 2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$

B ------ 30\$ / share ---- return $\begin{pmatrix} -1 & 0 & 3 \\ 0.3 & 0.1 & 0.6 \end{pmatrix}$

E(total return)

risk = V(total return)

a) $X = 30A$ (has 600\$ to invest 600/20 = 30 shares)

$E(A) = 1$, $E(A^2) = 2.6$

$V(A) = 2.6 - 1 = 1.6$

$\boxed{E(X) = 30.}$ $V(X) = 30^2 \cdot 1.6 = 90\cancel{0} \cdot \frac{16}{1\cancel{0}} = 1440$

b) $Y = 20B$

$E(B) = -1 \cdot 0.3 + 3 \cdot 0.6 = -0.3 + 1.8 = \boxed{1.5}$

$E(B^2) = 5.7$

$V(B) = 5.7 - 1.5^2 = \boxed{3.45}$

$E(Y) = 20 \cdot E(B) = 20 \cdot 1.5 = \boxed{30.}$

$V(Y) = V(20B) = 20^2 \, V(B) = 20^2 \cdot 3.45 = \boxed{1380}$

c) $Z = 15A + 10B$

$E(Z) = E(15A + 10B) = 15E(A) + 10E(B) = 30.$

$V(Z) = 15^2 V(A) + 10^2 V(B) = 705.$

5) Let X - random variable

mean $E(X)$, standard deviation $\sigma(X) = \sqrt{V(X)}$

Find mean + variance of $Y = \frac{X - E(X)}{\quad}$

$$E(Y) = E\left(\frac{X - E(X)}{\sigma(X)}\right) = \frac{1}{\sigma(X)} E(X - E(X)) = \frac{1}{\sigma}(E(X) - E(X))$$

$$\Rightarrow \boxed{E(Y) = 0.}$$

$$V(Y) = V\left(\frac{X - E(X)}{\sigma(X)}\right) = \frac{1}{\sigma^2(X)} V(X - E(X)) = \frac{V(X)}{\sigma^2(X)} - \frac{V(E(X))}{\sigma^2(X)}$$

$$V(E(X)) = 0 \Rightarrow \boxed{\frac{V(X)}{\sigma^2(X)}} = V(Y)$$

Theory Review

Expectation:

- if $X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I}$ is discrete, then $E(X) = \sum_{i \in I} x_i p_i$.

- if X is continuous with pdf f, then $E(X) = \int_{\mathbb{R}} x f(x) dx$.

Variance: $V(X) = E\left((X - E(X))^2\right) = E\left(X^2\right) - (E(X))^2$.

Standard Deviation: $\sigma(X) = \mathrm{Std}(X) = \sqrt{V(X)}$.

Moments:
- **moment of order k**: $\nu_k = E\left(X^k\right)$.
- **absolute moment of order k**: $\underline{\nu_k} = E\left(|X|^k\right)$.
- **central moment of order k**: $\mu_k = E\left((X - E(X))^k\right)$.

Properties:
1. $E(aX + b) = aE(X) + b$, $V(aX + b) = a^2 V(X)$
2. $E(X + Y) = E(X) + E(Y)$
3. if X and Y are independent, then $E(XY) = E(X)E(Y)$ and $V(X + Y) = V(X) + V(Y)$
4. if $h : \mathbb{R} \to \mathbb{R}$ is a measurable function, X a random variable;
- if X is discrete, then $E(h(X)) = \sum_{i \in I} h(x_i) p_i$
- if X is continuous, then $E(h(X)) = \int_{\mathbb{R}} h(x) f(x) dx$

Covariance: $\mathrm{cov}(X, Y) = E((X - E(X))(Y - E(Y)))$

Correlation Coefficient: $\rho(X, Y) = \dfrac{\mathrm{cov}(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}}$

Properties:
1. $\mathrm{cov}(X, Y) = E(XY) - E(X)E(Y)$
2. $V\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 V(X_i) + 2 \sum_{1 \le i < j \le n} a_i a_j \mathrm{cov}(X_i, X_j)$
3. X, Y independent $=>$ $\mathrm{cov}(X, Y) = \rho(X, Y) = 0$ (X and Y are *uncorrelated*)

4. $-1 \leq \rho(X,Y) \leq 1$; $\rho(X,Y) = \pm 1 \iff \exists\, a, b \in \mathbb{R},\ a \neq 0$ s.t. $Y = aX + b$

Let (X,Y) be a continuous random vector with pdf $f(x,y)$, let $h : \mathbb{R}^2 \to \mathbb{R}^2$ a measurable function, then

$$E(h(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f(x,y)\,dx\,dy$$

7.

a) marginal pdf $P(X = x_i) = p_i$

$$p_i = \sum_{j=1}^{4} P(X = x_i, y_j) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0.5 & 0.26 & 0.1 & 0.14 \end{pmatrix}$$

$$p_i = \sum_{j=1}^{4} p_{ij}$$

$$p_j = P(Y = y_i) = \sum_{i=1}^{4} p_{ij} = \begin{pmatrix} 1 & 2 & 3 \\ 0.22 & 0.28 & 0.5 \end{pmatrix}$$

b) $P(2,1) = \boxed{0.06}$

c) $P(X,Y \mid X \geq 3,\ Y \leq 2) =$

$P(3,1) + P(3,2) + P(4,1) + \dots =$

$$\sum_{x \geq 3} P(x,1) + \sum_{x \geq 3} P(x,2) =$$

$$1-\sum_{1}^{2} P(x,1) + 1-\sum_{1}^{2} P(x,2) =$$

$$2 - P(1,1) - P(2,1) - P(1,2) - P(2,2) =$$

d) Z – total charge for a customer

$$P(X * Y) = ?$$

<u>Bonus Problems</u>

8.) X Y – indep $\Rightarrow p_{ij} = p_i \cdot q_j$

$$X = \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad Y \begin{pmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$U = X+Y, \quad V = X-Y$$

$$U = \begin{pmatrix} 1 & 2 & 3 \\ \frac{2}{6} & \frac{3}{6} & \frac{1}{6} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{6} & \frac{3}{6} & \frac{2}{6} \end{pmatrix}$$

$$P_{ij} = P(U = u_i, V = v_j) =$$

U \ V	0	1	2

1	$\frac{2}{36}$	$\frac{6}{36}$	$\frac{4}{36}$
2	$\frac{3}{36}$	$\frac{9}{36}$	$\frac{6}{36}$
3	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{2}{36}$

b) $P(V=u_i, V=v_j) = \sum_{j=1}^{n} p_{ij} = \begin{pmatrix} 1 & 2 & 3 \\ \frac{2}{6} & \frac{3}{6} & \frac{1}{6} \end{pmatrix}$

c) V indep because $p_{ij} = p_i \cdot q_j$

9. $\underset{6\ \ \ 6}{(X, Y)} \Rightarrow P = 6^2 = 36$

$x^2 + y^2 = 10 \Leftrightarrow 1^2 + 3^2 = 10 \quad (1,3), (3,1)$

$P = \frac{2}{36} = \boxed{\frac{1}{18}}$

Theory Review (Sem 5)

$X: S \to \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$ (pdf), $F: \mathbb{R} \to \mathbb{R}$ (cdf).

1. $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\, dt$.

2. $f(x) \ge 0$, $(\forall)\ x \in \mathbb{R}$, $\boxed{\int_{\mathbb{R}} f(x) = 1.}$

3. $P(X = x) = 0$

$$P(a < X < b) = \int_a^b f(t)\, dt = P(a \le X \le b).$$

4. $F(-\infty) = 0$, $F(\infty) = 1$.

$$F_{(X,Y)}: \mathbb{R}^2 \to \mathbb{R},\quad F(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\, du\, dv$$

Properties:

1. $P(a_1 < X \le b_1,\ a_2 < Y \le b_2) =$

$$F(a_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$$

2. $F(\infty, \infty) = 1$, $F(-\infty, y) = F(x, -\infty) = 0$.

3. Marginal CDF's

$$F_X(x) = F(x, \infty)$$

$$F_Y(y) = F(\infty, y).$$

4. $P((x,y) \in D) = \iint_D f(x,y)\, dy\, dx$

5. Marginal Densities

$f_X(x) = \int_{\mathbb{R}} f(x,y)\, dy \quad ; \quad f_Y(y) = \int_{\mathbb{R}} f(x,y)\, dx.$

6. X and Y independent $\Leftrightarrow \boxed{f_{(X,Y)}(x,y) = f_X(x) \cdot f_Y(y).}$

Distributions

- Exponential distrib. $Exp(\lambda)$ models TIME:

λ = freq. of rare events measured in time^{-1}.

- Gamma distrib $Gamma(a,b)$ - TOTAL TIME of a multistage scheme.

Sem 5

$$f(x) = \begin{cases} \frac{k}{x^4}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

$$\int f(x) = 1 \Leftrightarrow \int^{\infty} \frac{k}{x^4} = k \int^{\infty} x^{-4} = k \cdot x^{-3} \Big|^{\infty}$$

$\mathbb{R}$ $\quad 1 \quad x^4 \quad 1 \quad -3\Big|_1$

$$= k\left(\frac{1}{-3\,\infty^3} - \frac{1}{-3}\right) = \frac{k}{3} = 1$$

$$\iff \boxed{k = 3.}$$

b) $\text{cdf} = F(x) = P(X \leq x) = \begin{cases} \int_1^x \frac{3}{t^4}\,dt, & x \geq 1. \\ 0, & x < 1 \end{cases}$

$$= 3\int_1^x t^{-4} = 3 \cdot \frac{t^{-3}}{-3}\Big|_1^x = -t^{-3}\Big|_1^x = \boxed{-\frac{1}{x^3} - 1}$$

$$F(x) = \begin{cases} -1 - \frac{1}{x^3}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

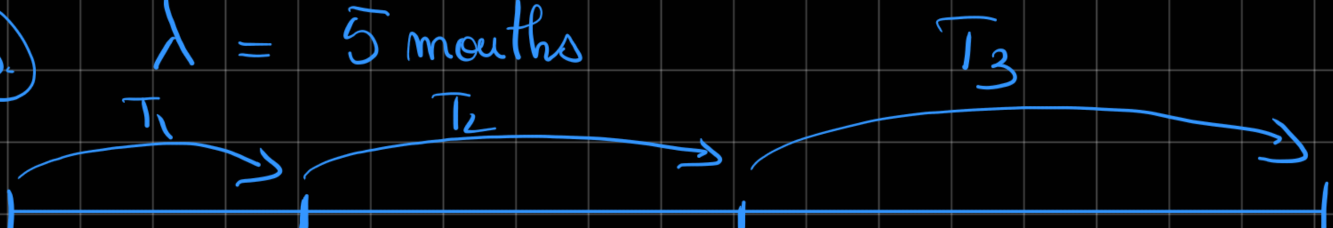
---

c) $P(X > 2) = 1 - P(X \leq 2) = 1 - F(2) = 1 + 1 + \frac{1}{8} =$

$$= 2 + \frac{1}{8} = \boxed{\frac{17}{8}}$$

---

(3) $\lambda = 5$ months



T_1 $\quad$ T_1+T_2 $\quad$ $T_1+T_2+T_3$

a) $P(T \le 9) = ?$

T - time in months, avg rate in time^{-1} $\Rightarrow \boxed{\lambda = \frac{1}{5}}$

$$Exp(\lambda) = Gamma\left(1, \frac{1}{\lambda}\right)$$

$$Gamma\left(\alpha, \frac{1}{\lambda}\right)$$

nr of indep. Exponential variables.

a) $P(T \le 9) = Gamma(3, 5) = \underline{gammacdf(9,3,5)}$

b) $P(T>16 \mid T>12) = \dfrac{P(T>12 \cap T>16)}{P(T>12)} =$

$$= \frac{P(T>16)}{P(T>12)} = \frac{1-P(T\le 16)}{1-P(T\le 12)} = \frac{1-gamacdf(16,3,5)}{1-gamacdf(12,3,5)}$$

4. joint density: $f_{(X,Y)}(x,y) = \frac{1}{16}x^3y^3$, $x,y \in [0,2]$.

a) $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \frac{x^3}{16}\int_{\mathbb{R}} y^3 = \frac{x^3}{16} \cdot \frac{y^4}{4}\Big|_{-\infty}^{\infty} = (*)$

For $x \in [0,2]$ $\rightarrow$

For $y \in [0,2] \Rightarrow$

$$(*) = \frac{x^3}{16}\int_0^2 y^3 = \frac{x^3}{16}\cdot\frac{y^4}{4}\Big|_0^2 = \frac{x^3}{16}\cdot 4 = \boxed{\frac{4x^3}{16}}$$

$$f_X(x) = \begin{cases} \frac{x^3}{4}, & x\in[0,2] \\ 0, & \text{otherwise.} \end{cases}$$

<u>Analog</u> $f_Y(y) = \begin{cases} \frac{y^3}{4}, & y\in[0,2] \\ 0, & \text{otherwise} \end{cases}$

b) X, Y indep $\Leftrightarrow f_{(X,Y)}(x,y) = f_X(x)\cdot f_Y(y) \Leftrightarrow \frac{x^3y^3}{16} = \frac{x^3}{4}\frac{y^3}{4}$

$\Rightarrow$ indep.

c) $P(X\leq 1) = \int_{-\infty}^{1} f_X(x) = \int_{-\infty}^{0} 0 + \int_0^1 \frac{x^3}{4} =$

$$= \frac{1}{4}\cdot\frac{x^4}{4}\Big|_0^1 = \boxed{\frac{1}{16}}$$

---

(2) $X\in U(a,b) \Rightarrow f(x) = \frac{1}{b-a}, \; x\in[a,b]$

$P(s < X < s+h) = P(t < X < t+h)$

<u>length of interval.</u>

a) Uniform distrib.

b) $P(4:50 < X < 4:55) = P(5 < X < 5:05)$

because the length of both intervals is 5 min.

5. $f_X(x) = \frac{1}{4} x e^{-\frac{x}{2}}$, $x \geq 0$; $Y = \frac{1}{2}X + 2$.

$f_Y = ?$

Use this ✓

If $Y = g(X)$ differentiable with $\boxed{g' \neq 0.}$, strictly monotone. $\Rightarrow$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}$$

$g(x) = \frac{1}{2}x + 2$ $\quad g'(x) = \frac{1}{2} > 0$, differentiable

strictly increasing

$\frac{1}{2}x + 2 = y \iff x = 2(y-2)$

$\Rightarrow \boxed{g^{-1}(x) = 2(x-2)}$

$$f_Y(x) = \frac{f_X\left(2(x-2)\right)}{\frac{1}{2}} = \frac{1}{4}\, 2(x-2)\, e^{2-x} =$$

$$= \boxed{\frac{x-2}{2}\, e^{2-x}}$$

(6.) $X \in N(0,1) \iff Y = |X|$

$Y = |X| \iff Y \in \{-X, X\} \Rightarrow$

$$\boxed{f_Y(y) = 2 \cdot f_X(x)}$$

Twice as likely.

Seminar 6

$$E(X) = \begin{cases} \sum x_i p_i & , X \text{ discrete} \\ \int x f(x) & , X \text{ continuous} \end{cases}$$

$$V(X) = E(X^2) - \left(E(X)\right)^2$$

$$\sigma(X) = \text{Std}(X) = \sqrt{V(X)}.$$

---

moment of order k: $\boxed{\nu_{\ } = E(X^K)}$

moment of order k: $\boxed{m_k = E(X^k)}$

absolute moment of order k: $\underline{\nu_k} = E(|X^k|)$

central moment of order k: $\mu_k = E\left((X - E(X))^k\right)$.

X, Y indep $\Leftrightarrow$ $E(XY) = E(X) \cdot E(Y)$

$$V(X+Y) = V(X) + V(Y).$$

$$E(h(X)) = \begin{cases} \sum h(x_i)\, p_i \\ \int_{\mathbb{R}} h(x)\, f(x)\, dx \end{cases}$$

$$\operatorname{cov}(X,Y) = E\left((X - E(X)) \cdot (Y - E(Y))\right)$$

<u>Correlation Coefficient:</u>

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{V(X) \cdot V(Y)}}$$

<u>Properties:</u> 1. $\boxed{\operatorname{cov}(X,Y) = E(XY) - E(X) \cdot E(Y)}$.

2. $V\left(\sum a_i X_i\right) = \sum_{i=1}^{n} a_i^2 V(X_i) + 2 \sum a_i a_j \operatorname{cov}(X_i, X_j)$,

3. X, Y ... (X and Y

3. X, Y indep $\Leftrightarrow$ $cov(X,Y) = \rho(X,Y) = 0$. (uncorrelated)

4. $\rho(X,Y) \in [-1, 1]$

$\rho(X,Y) = \pm 1 \Leftrightarrow Y = aX + b$. !

Problems

1. $X \begin{pmatrix} 0 & 1 & 2 \\ 0.7 & 0.2 & 0.1 \end{pmatrix}$

a) $E(X) = \sum x_i p_i = 0.2 + 0.2 = 0.4$

$E(Y) = E(X) \cdot 500 = \frac{500 \cdot 4}{10} = \boxed{200}$

b) $Std(X) = \sqrt{V(X)}$

$V(X) = E(X^2) - (E(X))^2 = 0.6 - 0.16 = \boxed{0.44}$

$X^2 \begin{pmatrix} 0 & 1 & 4 \\ 0.7 & 0.2 & 0.1 \end{pmatrix} \Rightarrow E(X^2) = 0.2 + 0.4 = 0.6$

$Std(X) = \sqrt{V(X)} = \sqrt{\frac{4 \cdot 11}{100}} = \frac{2}{10}\sqrt{11} = 0.2\sqrt{11}$

$\Rightarrow \boxed{Std(Y) = 500 \cdot \sqrt{0.44}}$

3. $E(X) = \int_{\mathbb{R}} x f(x) = \int_{-\infty}^{1} x \cdot 0 + \int_{1}^{\infty} x \cdot \frac{3}{x^4} =$

$= \int_{1}^{\infty} 3 \cdot x^{-3} = 3 \cdot \frac{x^{-2}}{-2} \Big|_{1}^{\infty} = -\frac{3}{2}\left(\left(\frac{1}{\infty}\right)^{0} - 1\right) =$

$= \boxed{\frac{3}{2}}$

b) $Std(X) = ?$

$V(X) = E(X^2) - (E(X))^2$

Let $h(x) = x^2 \Rightarrow E(X^2) = E(h(x)) = \int_{1}^{\infty} h(x) \cdot f(x)\,dx$

$= \int_{1}^{\infty} x^2 \cdot \frac{3}{x^4} = 3 \cdot \frac{x^{-1}}{-1} \Big|_{1}^{\infty} = -3(0 - 1) = \boxed{3.}$

$Std = \sqrt{3 - \frac{9}{4}} = \sqrt{\frac{3}{4}} = \boxed{\frac{\sqrt{3}}{2}}$

4. 600 \$

A – 20 \$ / share

B – 30 \$ / share

$X_A = \begin{pmatrix} -1 & 0 & 2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}, X_B = \begin{pmatrix} -1 & 0 & 3 \\ 0.3 & 0.1 & 0.6 \end{pmatrix}$

a) $600/20 = 30$ shares

$E(X_A) = \sum x_i p_i = -0.2 + 1.2 = \boxed{1}$

$\boxed{Y = 30X_A}$

$30 \cdot E(X_A) = 30.$

$E(X_A^2) - (E(X_A))^2 = (0.2 + 2.4) - 1 = \boxed{1.6}$

$V(X_A) = \longrightarrow$

$Std(X_A) = \sqrt{1.6} = \sqrt{\frac{16}{10}} = \frac{16\sqrt{10}}{10} = \boxed{1.6\sqrt{10}}.$

$V(Y) = 30^2 V(X_A) = \boxed{1440.}$

b) $E(X_B) = 1.8 - 0.3 = 1.5$ $\quad 600/30 = 20$ shares

$20 \cdot E(X_B) = 20 \cdot 1.5 = 30.$ $\quad Y = 20X_B \quad E(Y) = 20E(X_B)$

$V(X_B) = E(X_B^2) - E^2(X_B) = (0.3 + 5.4) - 2.25 =$

$= 5.70 - 2.25 = \boxed{3.45}$

$Std(X_B) = \sqrt{3.45}.$ $\quad V(Y) = 20^2 V(X_B) = \boxed{1380}$

c) $300/20 = 15$ shares A $\quad 300/30 = 10$ shares B

$\frac{20}{100}$ $\quad Y = (15X_A + 10X_B)$

$15 \cdot E(X_A) + 10 \cdot E(X_B) = 15 + 15 = 30.$

$V(15X_A + 10X_B) = 15^2 V(X_A) + 10^2 V(X_B) =$

$$V(15X_A + 10X_B) = 15^2\, V(X_A) + 10^2\, V(X_B) =$$

$$= 225 \cdot 1.6 + 100 \cdot 3.45 = \boxed{705}$$

$\Rightarrow$ So we should invest in option C.

---

(3) mean $E(X)$, $\sigma(X) = \sqrt{V(X)}$

$$E(Y) = E\left(\frac{X - E(X)}{\sigma(X)}\right) = \frac{1}{\sigma(X)} E(X - E(X))$$

$$= \frac{1}{\sigma(X)}\Big(E(X) - E(E(X))\Big) = \frac{1}{\sigma(X)}\big(E(X) - E(X)\big)_{=0}$$

$$V(Y) = V\left(\frac{X - E(X)}{\sigma(X)}\right) = \frac{1}{\sigma^2(X)}\Big(V(X) - \underbrace{V(E(X))}_{0}\Big)$$

$$= \frac{V(X)}{\sigma^2(X)}$$

---

(2) This is a Geometric distrib.

$$pdf = \begin{pmatrix} k \\ pq^k \end{pmatrix} = \begin{pmatrix} k \\ 0.1 \cdot 0.9^k \end{pmatrix}$$

$$E(X) = \sum_{1} k \cdot 0.1 \cdot 0.9^k = 0.09 + \; !!!$$

(6.) joint dens. funct $f(x,y) = x+y \quad (x,y) \in [0,1] \times [0,1$

$$f_X(x) = \int_{\mathbb{R}} f(x+y)\,dy = \int_0^1 (x+y)\,dy = x \cdot y \Big|_0^1 + \frac{y^2}{2}\Big|_0^1 =$$

$$= \boxed{x + \frac{1}{2}}$$

$$f_Y(y) = \int_{\mathbb{R}} f(x+y)\,dx = \boxed{y + \frac{1}{2}}$$

a) $$E(X) = \int_{\mathbb{R}} x f(x) = \int_0^1 x\left(x+\frac{1}{2}\right) = \int_0^1 x^2 + \frac{x}{2} =$$

$$= \frac{x^3}{3}\Big|_0^1 + \frac{1}{2} \cdot \frac{x^2}{2}\Big|_0^1 = {}^{4/}\frac{1}{3} + {}^{3/}\frac{1}{4} = \boxed{\frac{7}{12}}$$

$$E(Y) = \frac{7}{12}$$

$$V(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_0^1 x^2\left(x+\frac{1}{2}\right) = \frac{x^4}{4} + \frac{x^3}{6}\Big|_0^1 = {}^{3/}\frac{1}{4} + {}^{2/}\frac{1}{6} = \boxed{\frac{5}{12}}$$

$$V(X) = {}^{12/}\frac{5}{\ldots} - \frac{49}{\ldots} = \boxed{\frac{11}{\ldots}}$$

12   144 $\boxed{144}$

b) $$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{V(X)\cdot V(Y)}}$$

$$cov(X,Y) = E\big((X-E(X))(Y-E(X))\big) =$$

$$E(XY) - E(X)E(Y)$$

---

7. $$\begin{pmatrix} -1 & 0 & 1 \\ \sin^2 a & \cos 2a & \sin^2 a \end{pmatrix} \qquad a \in \left(0, \frac{\pi}{4}\right)$$

$$Y_K = X^{2k-1}, \quad Z_K = X^{2K}$$

$$\rho(Y_K, Z_K) = \frac{cov(Y_K, Z_K)}{\sqrt{V(Y_K)\, V(Z_K)}}$$

$$cov(Y_K, Z_K) = E(Y_K Z_K) - E(Y_K)E(Z_K)$$

$$= E\left(X^{4k^2-2k}\right) - E\left(X^{2k-1}\right)\cdot E\left(X^{2k}\right)$$

$$E\left(X^{2k}\right) = (-1)^{2k}\sin^2 a + \sin^2 a = 2\sin^2 a$$

$$E\left(X^{2k-1}\right) = (-1)^{2k-1}\sin^2 a + \sin^2 a = 0.$$

$$E\left(X^{4k^2-2k}\right) = (-1)^{\overbrace{2k(2k-1)}^{\text{par}}} \quad \sin^2 a + \sin^2 a$$

$$= \boxed{2\sin^2 a}$$

$$\boxed{\operatorname{cov}(Y_K, Z_K) = 2\sin^2 a}$$

$$V(Y_K) = E(Y_K^2) - (E(Y_K))^2$$

$$= E\left(X^{4k}\right) - \left(E\left(X^{2k}\right)\right)^2 = \sin^2 a - \sin^4 a$$

$$= \sin^2 a\,(1 - \sin^2 a) = \sin^2 a \cos^2 a.$$

$$V(Z_K) = V\left(X^{2k-1}\right) = E\left(X^{(2k-1)2}\right) - \left(E\left(X^{2k-1}\right)\right)^2$$

$$= \sin^2 a - \left((-1)\sin^2 a\right)^2 = \sin^2 a - \sin^4 a =$$

$$= \sin^2 a \cos^2 a$$

$$\rho = \frac{2\sin^2 a}{\sqrt{(\sin^2 a \cos^2 a)^2}} = \frac{2\cancel{\sin^2 a}}{\cancel{\sin^2 a}\cos^2 a} = \boxed{\frac{2}{\cos^2 a}}$$