June 14, 2023

1.) $x_{k+1} = x_k + \lambda x_k (2 - x_k)$, $k \in \mathbb{N}$, $\lambda \in (0,1)$

a) Find constant solutions (fixed points) + stability

b) Fix $\lambda = 0.5$. If there is an attractor, estimate the basin of attraction using the stair-step (cobweb) diagram.

a) Fixed $\Rightarrow$ $x_{k+1} = x_k = \eta$

$$\cancel{\eta} = \cancel{\eta} + \lambda \eta (2 - \eta)$$

$$0 = \lambda \eta (2 - \eta) \Leftrightarrow \eta \in \{0, 2\}$$

$$f(x) = x + \lambda x (2 - x) = x + 2\lambda x - \lambda x^2$$

$$f'(x) = 1 + 2\lambda - 2x\lambda$$

$\Rightarrow$ For $\eta = 0 \Rightarrow f'(0) = 1 + 2\lambda \in (1, 3)$

For $\eta = 1 \Rightarrow f'(2) = 1 - 2\lambda \in (-1, 1)$

$\Rightarrow$ 0 $\rightarrow$ unstable, 2 $\rightarrow$ attractor

b) $\boxed{\lambda = 0.5}$

$$f(x) = -0.5x^2 + x + x = \boxed{-0.5x^2 + 2x} = 2x - \frac{x^2}{2}$$

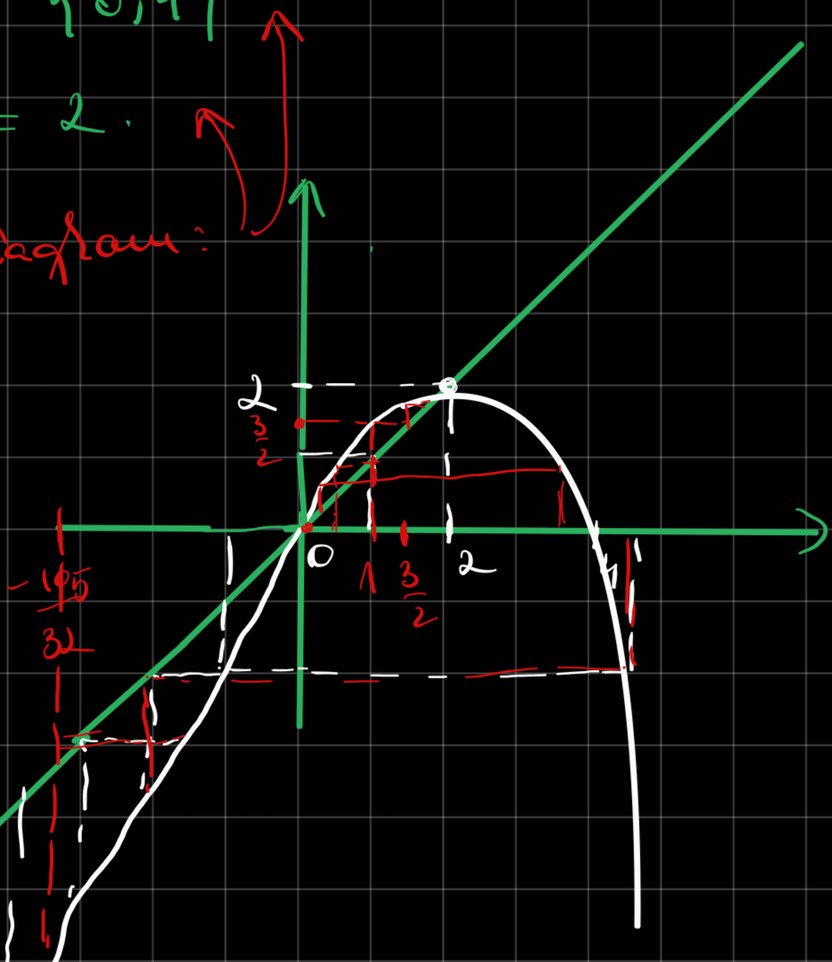
$$f'(x) = -x + 2$$

$$f(x) = 0 \iff x = \{0, 4\}$$

$$f'(x) = 0 \iff x = 2.$$

stair-step (cobweb) diagram:

x		0		2		4
$f'(x)$	+ + +			0	− − −	
$f(x)$		0		2		0



$$f(1) = 2 - \frac{1}{2} = \frac{3}{2}$$

$$f^2(1) = 3 - \frac{9}{8} = \frac{24-9}{8} = \frac{15}{8}$$

$$f^3(1) = \frac{15}{4} - \frac{9\cdot 25}{16\cdot 2} = \frac{15}{4} - \frac{225}{32} = \frac{120-225}{32} = \frac{-105}{32} \simeq -4$$

$\Rightarrow$ Basin of attraction is $(0, \infty)$

② $\begin{cases} x' = -2x \\ y' = x - \sqrt{5}\, y \end{cases}$ $\qquad\qquad$ $\begin{cases} x' = -2x \\ y' = x + 3x^2 - \sqrt{5}(y + y^3) \end{cases}$

a) Flow of linear system

$$X' = \underbrace{\begin{pmatrix} -2 & 0 \\ 1 & -\sqrt{5} \end{pmatrix}}_{A} X$$

$$\det(A - \lambda I_2) = \begin{pmatrix} -2-\lambda & 0 \\ 1 & -\sqrt{5}-\lambda \end{pmatrix} = (-2-\lambda)(-\sqrt{5}-\lambda) = 0$$

$$\Leftrightarrow \lambda \in \{-2, -\sqrt{5}\}$$

$$\Rightarrow A\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} -2x = -2x \\ x - \sqrt{5}y = -2y \end{cases}$$

$$\Leftrightarrow \begin{cases} x = (-2+\sqrt{5})y \\ y = y \end{cases} \Leftrightarrow u_1 = \begin{pmatrix} \sqrt{5}-2 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow A\begin{pmatrix} x \\ y \end{pmatrix} = -\sqrt{5}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} -2x = -\sqrt{5}x \\ x - \sqrt{5}y = -\sqrt{5}y \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y \in \mathbb{R} \end{cases} \Leftrightarrow u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow X = c_1 e^{-2t}\begin{pmatrix} \sqrt{5}-2 \\ 1 \end{pmatrix} + c_2 e^{-\sqrt{5}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x = c_1 e^{-2t}(\sqrt{5}-2) \\ y = c_1 e^{-2t} + c_2 e^{-\sqrt{5}t} \end{cases}$$

$$X(0,0) = (\eta_1, \eta_2)$$

$$\begin{cases} \eta_1 = (\sqrt{5}-2)c_1 \end{cases} \Leftrightarrow \boxed{c_1 = \frac{\eta_1}{\ldots}}$$

$$\begin{cases} \ldots \\ \eta_2 = c_1 + c_2 \end{cases} \qquad \boxed{c_2 = \eta_2 - \frac{\eta_1}{\sqrt{5}-2}}$$

$$\Rightarrow \varphi(t, \eta_1, \eta_2) = \left(\eta_1 e^{-2t}, \frac{\eta_1 e^{-2t}}{\sqrt{5}-2} + \frac{\eta_2(\sqrt{5}-2) - \eta_1}{\sqrt{5}-2} \cdot e^{-\sqrt{5}t} \right)$$

b) $\lim_{t \to \infty} \varphi(t, \eta) = (0, 0)$

c) $\begin{cases} -2x = 0 \\ x + 3x^2 - \sqrt{5}\, y(1+y^2) = 0 \end{cases} \iff x = 0$

$\iff -\sqrt{5}\, y(1+y^2) = 0 \quad \begin{cases} y = 0 \\ y = \pm i \cancel{\in} \mathbb{R} \end{cases}$

$\Rightarrow S = (0,0), \cancel{(0, i)}, \cancel{(0, -i)}$ eq. points.

$$\text{jac } J = \begin{pmatrix} -2 & 0 \\ 1 + 6x & -\sqrt{5} - 3\sqrt{5} y^2 \end{pmatrix}$$

For $(0,0) \Rightarrow J_0 = \begin{pmatrix} -2 & 0 \\ 1 & -\sqrt{5} \end{pmatrix}$

$$\det(J_0 - \lambda I_2) = \begin{pmatrix} -2-\lambda & 0 \\ 1 & -\sqrt{5}-\lambda \end{pmatrix} = 0$$

$$(-2-\lambda)(-\sqrt{5}-\lambda) = 0 \qquad \lambda \in \{-2, -\sqrt{5}\}$$

$\Rightarrow (0,0)$ stable, Node, attractor

For $(0,i) \Rightarrow$
$$J_{(0,i)} = \begin{pmatrix} -2 & 0 \\ 1 & -\sqrt{5}+3\sqrt{5} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & 2\sqrt{5} \end{pmatrix}$$

$$\det(J_{0,i} - \lambda I_2) = \begin{vmatrix} -2-\lambda & 0 \\ 1 & 2\sqrt{5}-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)(2\sqrt{5}-\lambda) = 0$$

$$\boxed{\lambda \in \{-2, 2\sqrt{5}\}}$$ $\Rightarrow$ global repellor, NODE, unstable.

ONLY REAL SOLUTIONS ACCEPTED FOR EQ POINTS

For $(i,0) \Rightarrow$
$$J_{(i,0)} = \begin{pmatrix} -2 & 0 \\ 1+6i & -\sqrt{5} \end{pmatrix}$$

$$\det(J_{(i,0)} - \lambda I_2) = (-2-\lambda)(-\sqrt{5}-\lambda) = 0$$

$$\lambda \in \{-\sqrt{5}, -2\}$$

$\Rightarrow$ stable, attractor, NODE.

---

d) $\psi(t, \eta)$ – flow

We can deduce from c) that for

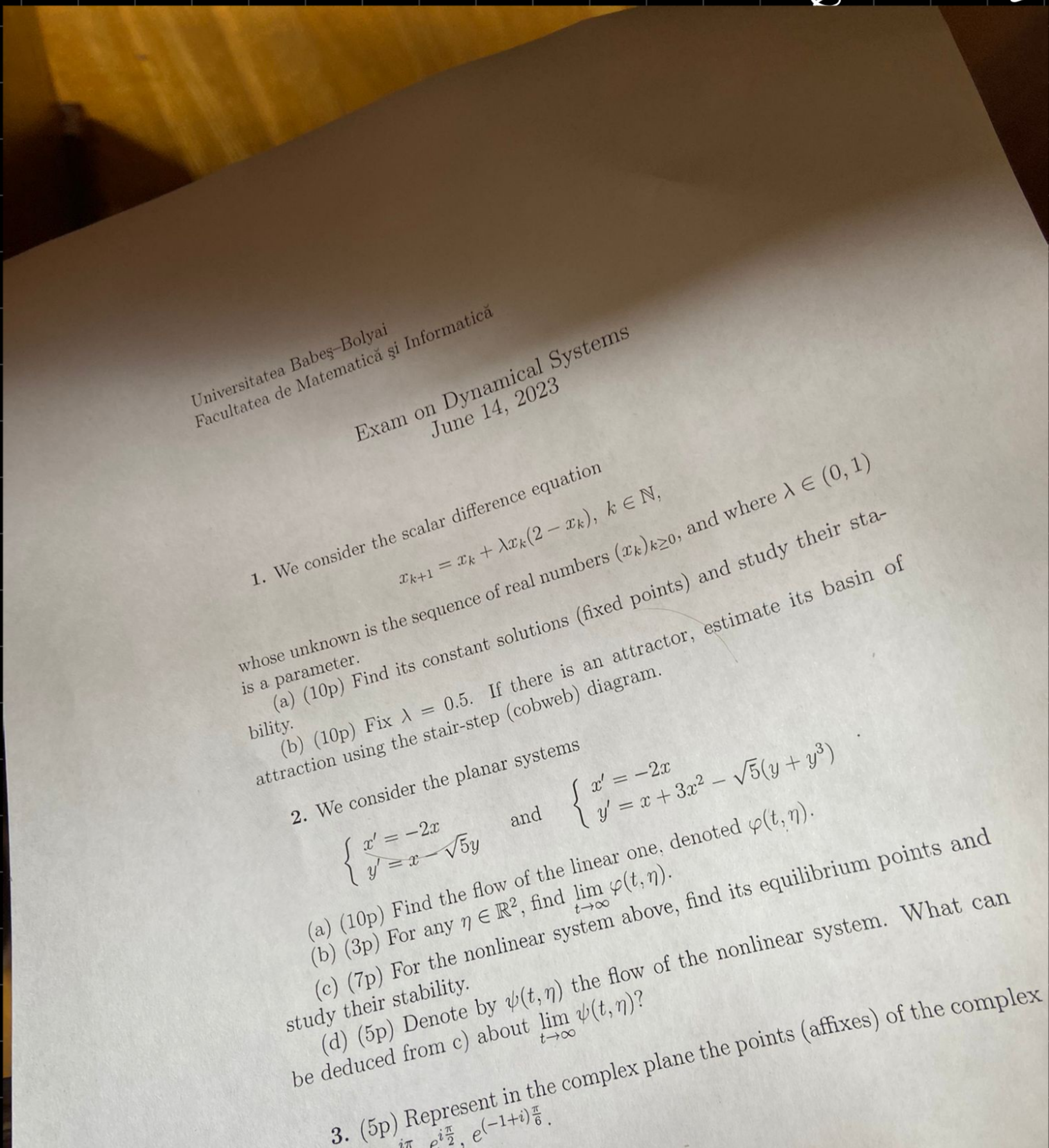
(ii) $\forall\, t, \eta$, because $(0,0)$ a attractor $\Rightarrow$

$$\lim_{t\to\infty} \psi(t,\eta) = (0,0).$$

3. $e^{i\pi} = e^{0}\cos\pi + i e^{0}\sin\pi = -1$

$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$e^{(-1+i)\frac{\pi}{6}} = e^{-\frac{\pi}{6}}\cos\frac{\pi}{6} + i e^{-\frac{\pi}{6}}\sin\frac{\pi}{6} = e^{-\frac{\pi}{6}}\cdot\frac{\sqrt{3}}{2} + i e^{-\frac{\pi}{6}}\cdot\frac{1}{2}$



Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 14, 2023

1. We consider the scalar difference equation

$$x_{k+1} = x_k + \lambda x_k(2 - x_k), \quad k \in \mathbb{N},$$

whose unknown is the sequence of real numbers $(x_k)_{k\geq 0}$, and where $\lambda \in (0,1)$ is a parameter.

(a) (10p) Find its constant solutions (fixed points) and study their stability.

(b) (10p) Fix $\lambda = 0.5$. If there is an attractor, estimate its basin of attraction using the stair-step (cobweb) diagram.

2. We consider the planar systems

$$\begin{cases} x' = -2x \\ y' = x - \sqrt{5}y \end{cases} \quad \text{and} \quad \begin{cases} x' = -2x \\ y' = x + 3x^2 - \sqrt{5}(y + y^3) \end{cases}$$

(a) (10p) Find the flow of the linear one, denoted $\varphi(t,\eta)$.

(b) (3p) For any $\eta \in \mathbb{R}^2$, find $\lim\limits_{t\to\infty} \varphi(t,\eta)$.

(c) (7p) For the nonlinear system above, find its equilibrium points and study their stability.

(d) (5p) Denote by $\psi(t,\eta)$ the flow of the nonlinear system. What can be deduced from c) about $\lim\limits_{t\to\infty} \psi(t,\eta)$?

3. (5p) Represent in the complex plane the points (affixes) of the complex $e^{i\pi}$, $e^{i\frac{\pi}{2}}$, $e^{(-1+i)\frac{\pi}{6}}$.

numbers $e^{\cdots}$,

1

Exam June 10, 2020

(1.) Find LHDE with c.c of min. order that has as solution the sequence with:

(a) $x_k = 7 \cdot \mathrm{Re}(i^k) + 3\,\mathrm{Im}(i^k)$, $k \geq 0$

(b) First terms of x_k are: $1, 0, 0, 1, -2, 5$

(a) $r = \pm i \Rightarrow (r-i)(r+i) = r^2+1 = 0$

$\Rightarrow$ LHDE $\boxed{x_{k+2} + x_k = 0.}$

(b) • Note that this is not a geometric progression, thus it is not the sol. of a first order differential eq.

$x_0 = 1,\ x_1 = 0,\ x_2 = 0,\ x_3 = 1,\ x_4 = -2,\ x_5 = 5$

• Try 2^{nd} order: $x_{k+2} + a x_{k+1} + b x_k = 0$

$k=0 \Rightarrow \quad x_2 + a x_1 + b x_0 = 0 + 0 + b = 0$

$\Leftrightarrow \boxed{b = 0}$

$k = 1 \Rightarrow \quad x_3 + a x_2 + b x_1 = \boxed{1 = 0}$ contradiction

- ~~Try~~ 3^{rd} order: $x_{k+3}+ax_{k+2}+bx_{k+1}+cx_k=0$

$k=0 \Rightarrow \quad x_3+ax_2+bx_1+cx_0=0$

$$1+c=0 \Rightarrow \boxed{c=-1}$$

$k=1 \Rightarrow \quad x_4+ax_3+bx_2+cx_1=0$

$$-2+a=0 \Rightarrow \boxed{a=2}$$

$k=2 \Rightarrow \quad x_5+2x_4+bx_3-x_2=0$

$$5-4+b=0 \quad \boxed{b=-1.}$$

$\Rightarrow$ Solution: $\boxed{x_{k+3}+2x_{k+2}-x_{k+1}-x_k=0}$

$k \geq 0$

2. $\begin{cases} y' = x^2+y^2 \\ y(0)=1 \end{cases}$, $\quad y(x)=\sum_{n=0}^{\infty} a_n x^n$

Find $a_0, a_1, a_2 \in \mathbb{R}$

$y(x)=a_0+a_1x+a_2x^2+\dots+a_nx^n \qquad \boxed{a_0=1.}$

$y'(x)= \quad a_1+2a_2x+3a_3x^2+\dots+na_nx^{n-1}$

$$\begin{cases} a_1+2a_2x+3a_3x^2+\dots = \underbrace{x^2+(a_0+a_1x+\dots+a_nx^n)^2} \\ y(0)=1 \end{cases}$$

$x^2+(a_0+a_1x+a_2x^2+\dots+a_nx^n)^2 = x^2+a_0^2+2a_0a_1x+a_1^2x^2+$

$2a_0a_2x^2+\dots$

$\Rightarrow a_1 = a_0^2 = 1 \Leftrightarrow a_0 a_1 = a_2 \Rightarrow \boxed{a_2 = 1.}$

$$\Rightarrow \boxed{a_0 = a_1 = a_2 = 1.}$$

(4.) Type + stability $\begin{cases} \dot{x} = -6x \\ \dot{y} = 3y \end{cases}$

$\Rightarrow (0,0)$ eq. point

$$A = \begin{pmatrix} -6 & 0 \\ 0 & 3 \end{pmatrix} \Rightarrow \text{SADDLE point, unstable}$$

$\lambda_1 = -6,\ \lambda_2 = 3$

$$\frac{dx}{dy} = \frac{-6x}{3y} \Leftrightarrow -\frac{1}{6x}dx = \frac{1}{3y}dy \quad \Big| \int$$

$$-\frac{1}{6}\ln|x| = \frac{1}{3}\ln|y| + c \quad \Big| \cdot(-6)$$

$$\ln|x| = -2\ln|y| + \ln|c|$$

$$\ln|x| - \ln|y|^{-2} = \ln|c|$$

$$\ln\left|\frac{x}{y^{-2}}\right| = \ln|k|$$

$$\boxed{xy^2 = c} \Rightarrow H: xy^2 \text{ first integral}$$

Check if global first integral:

$$y^2\cdot(-6x) + 2xy\cdot 3y = -6xy^2 + 6xy^2 = 0$$

$\Rightarrow$ GLOBAL FIRST integral.

Or using def. find flow

$$\varphi(t,\eta_1,\eta_2) = \left(\eta_1 e^{-6t}, \eta_2 e^{3t}\right)$$

Since $\left(\eta_1 e^{-6t}\right)\left(\eta_2 e^{3t}\right)^2 = \eta_1 \eta_2^2$

$\Rightarrow \boxed{H: \quad xy^2.}$ $\rightarrow$ check

5. a) Represent in the complex plane the curve:

$$\left\{ \frac{1+i}{\sqrt{2}} e^{-3t-it} : t \geq 0 \right\}$$

$$e^{-3t-i\cdot t} = e^{-3t}\cdot\cos(-t) + e^{-3t}\sin(-t)$$

$$= e^{-3t}\cos t - e^{-3t}\sin t$$

$$= e^{-3t}(\cos t - \sin t)$$

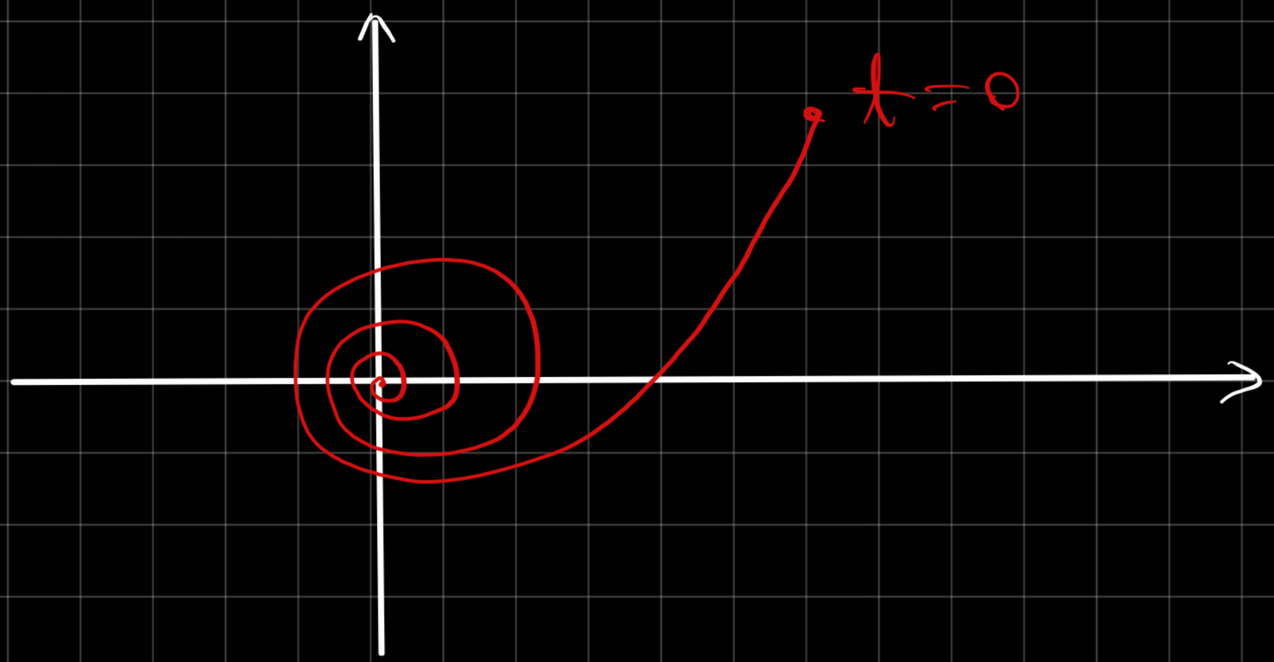
$$\frac{1+i}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$z(t) =$

$$\Rightarrow \frac{1+i}{\sqrt{2}} e^{-3t-ti} = e^{-3t}\left[\cos\left(\frac{\pi}{4}-t\right) + i\sin\left(\frac{\pi}{4}-t\right)\right]$$

$$\Rightarrow |z(t)| = e^{-3t} \quad (\forall)\ t \geq 0 \quad \text{and } \arg(z(t)) = \frac{\pi}{4} - t,\ t \geq 0$$

- As t increases towards ∞, $z(t)$ decreases exponentially towards 0 and $\arg(z(t))$ is strictly decreasing



- $\left(\frac{\pi}{4} - t\right) \rightarrow$ counterclockwise rotation

5) b) $\begin{cases} \dot{x} = -3x + y \\ \dot{y} = -x - 3y \\ x(0) = y(0) = 1 \end{cases}$

$A = \begin{pmatrix} -3 & 1 \\ -1 & -3 \end{pmatrix}$ $\quad \det(A - \lambda I_2) = (-3-\lambda)^2 + 1 =$

$$= 10 + 6\lambda + \lambda^2 = 0 \qquad \lambda^2 + 6\lambda + 10 = 0$$

$$\Delta = 36 - 40 = -4 < 0$$

$$\lambda_{1,2} = \frac{-6 \pm 2i}{2} = -3 \pm i \quad \begin{cases} -3+i \\ -3-i \end{cases}$$

$$A u_1 = (-3+i) u_2 \Leftrightarrow \begin{cases} \cancel{-3x} + y = \cancel{-3x} + ix \\ -x \cancel{-3y} = \cancel{-3y} + iy \end{cases}$$

$$\Leftrightarrow \begin{cases} y = ix \\ -x = iy \end{cases} \Leftrightarrow u_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$A u_2 = (-3-i) u_2 \Leftrightarrow \begin{cases} \cancel{-3x} + y = \cancel{-3x} - ix \\ -x \cancel{-3y} = \cancel{-3y} - iy \end{cases}$$

$$u_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\Rightarrow W(u_1, u_2) = i + i = 2i \neq 0 \quad \underline{\text{lin indep}}$$

$$\Rightarrow X = c_1 e^{-3t} \cos t \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{-3t} \sin(-t) \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\begin{cases} x = e^{-3t} \cos t - e^{-3t} \sin t \\ y = e^{-3t} \cos t \cdot i + e^{-3t} \sin t \cdot i \end{cases}$$

$\lambda_{1,2} \to$ focus point, global attractor

$$\boxed{\text{Global attractor} \Rightarrow \text{NO GLOBAL FIRST INTEGRAL}}$$

5) d) Let $\eta = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Find $e^{2A}\eta$, where A is the matrix of the system from (c)

$\boxed{e^{tA}\eta}$ is the unique solution of the IVP

$$\begin{cases} \dot{x} = Ax \\ x(0) = \eta \end{cases} \qquad x = \begin{pmatrix} e^{-3t}\cos t - e^{-3t}\sin t \\ e^{-3t}\cos t + e^{-3t}\sin t \end{pmatrix}$$

$$e^{2A}\eta = \begin{pmatrix} e^{-6}\cos 2 - e^{-6}\sin 2 \\ e^{-6}\cos 2 + e^{-3t}\sin 2 \end{pmatrix}$$

June 6 2022

1. $f : (0, \infty) \to \mathbb{R}$, $f(x) = \dfrac{x^2+5}{2x}$

a) $\dfrac{x^2+5}{2x} = x \iff \dfrac{x^2+5-2x^2}{2x} = 0$

$2x \qquad\qquad 2x$

$$\frac{5-x^2}{2x}=0 \iff 5-x^2=0 \iff x=\pm\sqrt{5}$$

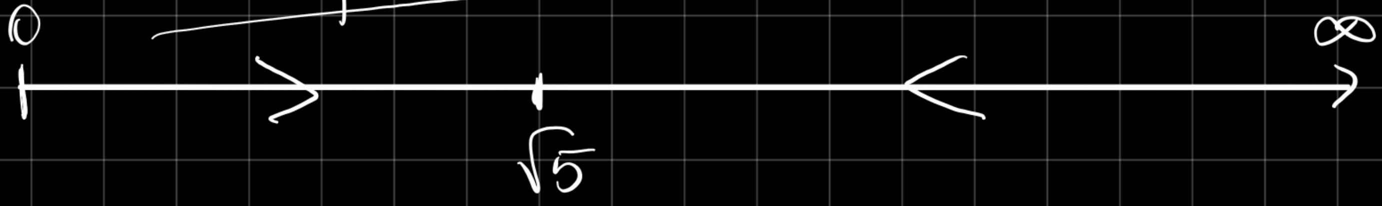
But $x\in(0,\infty) \Rightarrow \boxed{x=\sqrt{5}}$ unique fixed point

b) $$f'(x)=\frac{2x(2x)-(x^2+5)\cdot 2}{4x^2}=\frac{4x^2-2x^2-10}{4x^2}$$

$$f'(x)=\frac{2x^2-10}{4x^2} \iff f'(\sqrt{5})=\frac{10-10}{20}=0$$

$f'(x)$

Phase portrait



$$f'(1)=-2<0, \qquad f'(5)=\frac{2}{5}>0$$

To establish attractivity, we need $\boxed{|f'(x^*)|<1}$ which is right in our case

$\Rightarrow$ attractor.

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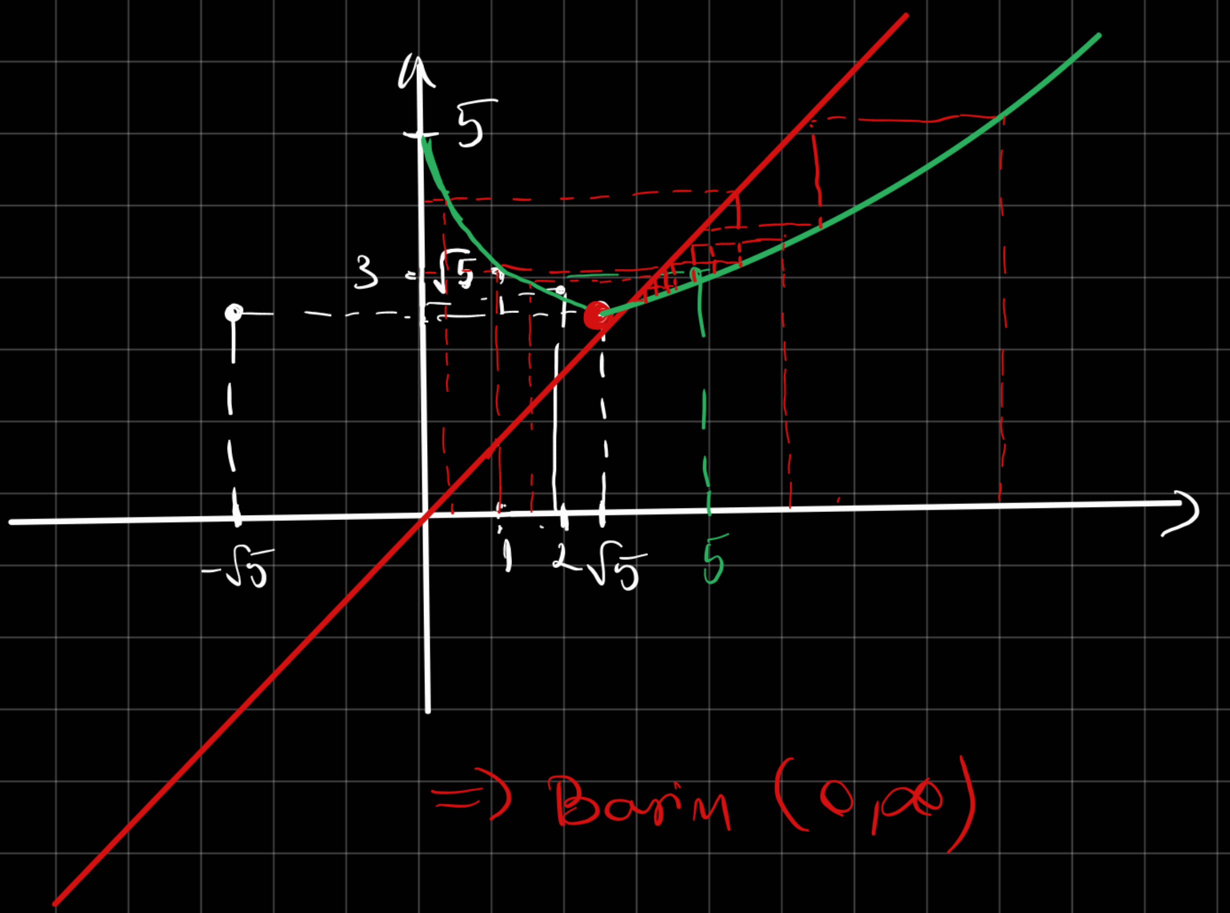
c) $$f(x)=\frac{x^2+5}{2x} \qquad \boxed{f'(x)=\frac{x^2-5}{2x^2}}$$

x	0	1	2	$\sqrt{5}$	5	∞

$f'(x)$	$-$	$-$	-2	$-$	$-$	0	$+$	$+$	$+$	$+$
$f(x)$	5	$+$	$+3$	$+\frac{9}{4}$	$+\sqrt{5}$		$+3$	$+$	$+$	∞



$\Rightarrow$ Barim $(0,\infty)$

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② $\begin{cases} \dot{x} = -y(x^2+y^2) = -yx^2 - y^3 \\ \dot{y} = x(x^2+y^2) = x^3 + xy^2 \end{cases}$

a) Eq. points: $\begin{cases} -y(x^2+y^2) = 0 \\ x(x^2+y^2) = 0 \end{cases}$ Since $x^2+y^2 \geq 0$ $\Rightarrow \boxed{x = y = 0.}$ eq pt.

b) Lin. method $\quad \text{jac}A = \begin{pmatrix} -2yx & -x^2-3y^2 \\ 3x^2+y^2 & 2yx \end{pmatrix}$

$\text{jac}A(0,0) = \begin{pmatrix} 0 & 0 \end{pmatrix}$

$$J_{\varphi}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_2 \Rightarrow \lambda_{1,2} = 0$$

! If $Re(\lambda_{1,2}) > 0 \Rightarrow$ hyperbolic

Else $\Rightarrow$ non-hyperbolic

$\Rightarrow$ non-hyperbolic

---

e) $\varphi(t,1,0) = (\overset{x}{\cos t}, \overset{y}{\sin t})$

$$\Leftrightarrow \begin{cases} -\sin t = -\sin t(\cos^2 t + \sin^2 t) \\ \cos t = \sin t\,(\cos^2 t + \sin^2 t) \end{cases} \Rightarrow \text{YES}$$

$$\boxed{\varphi(0,0) = (1,0)}$$

$\varphi(t,2,0) = (2\overset{x}{\cos 4t}, 2\overset{y}{\sin 4t})$

$$\begin{cases} -2\sin 4t \cdot 4 = -2\sin 4t \cdot 4 \quad \checkmark \\ 8\cos 4t = 2\cos 4t \cdot 4 \quad \checkmark \end{cases} \quad \text{YES}$$

$\varphi(t,3,0) = (3\overset{x}{\cos 9t}, 3\overset{y}{\sin 9t})$

$$\begin{cases} -27\sin 9t = -3\sin 9t \cdot 9 \cdot 1 \\ 27\cos 9t = 3\cos 9t \cdot 9 \end{cases} \quad \text{YES}$$

d) First integral:

$$\frac{dx}{dy} = \frac{-y\,\cancel{(x^2+y^2)}}{x\,\cancel{(x^2+y^2)}} = -\frac{y}{x}$$

$$x\,dx = -y\,dy \quad \Big|\int$$

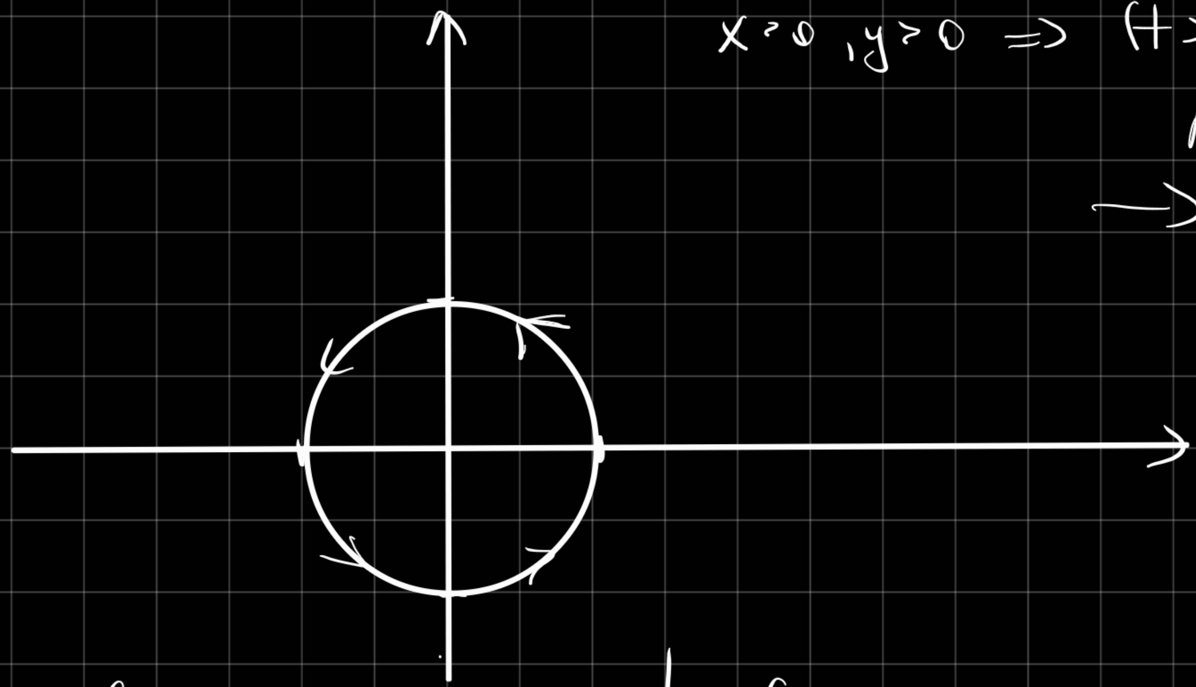
$$\frac{x^2}{2} + c_1 = -\frac{y^2}{2} + c_2$$

$$\frac{x^2+y^2}{2} + c = 0$$

$$\boxed{x^2+y^2 = c} \qquad H: x^2+y^2$$

e)

$x>0,\ y>0 \Rightarrow H>0$



f) $\begin{cases} \dot{x} = -y(x^2+y^2) \end{cases} \quad \Big| \quad \begin{cases} x = \rho\cos\theta \end{cases}$

$$\begin{cases} \dot{y} = x(x^2+y^2) \end{cases} \quad \begin{cases} y = \rho \sin\theta \end{cases}$$

$$\begin{cases} \rho \cdot \dot{\rho} = x \cdot \dot{x} + y \cdot \dot{y} \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{\dot{y}x - \dot{x}y}{x^2} \end{cases} \Leftrightarrow \begin{cases} \rho \cdot \dot{\rho} = -xy(x^2+y^2) + xy(x^2+y^2) \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{x^2(x^2+y^2)+y^2(x^2+y^2)}{x^2} \end{cases}$$

$$\begin{cases} \rho \cdot \dot{\rho} = 0 \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{(x^2+y^2)^2}{x^2} = \dfrac{\cancel{\rho}^4}{\cancel{\rho^2}\cos^2\theta} = \dfrac{\rho^2}{\cos^2\theta} \end{cases}$$

$$\Rightarrow \dot{\rho} = 0 \Leftrightarrow \boxed{\rho = c.}$$

$$\rho^2 = \dot\theta \Rightarrow \rho^2 > 0 \Rightarrow \theta \text{ increases over time}$$

3. $\begin{cases} \dot{x} = ax - 5y \\ \dot{y} = x - 2y \end{cases}$ (0,0) eq point.

$$A = \begin{pmatrix} a & -5 \\ 1 & -2 \end{pmatrix} \qquad \det(A - \lambda I_2) = (a-\lambda)(-2-\lambda) + 5 = 0$$

$$-2a + 2\lambda - \lambda a + \lambda^2 + 5 = 0$$

$$\lambda^2 + \lambda(2-a) + (5-2a) = 0$$

$$\Delta = (2-a)^2 - 4(5-2a) =$$

$$\Delta = 4 - 4a + a^2 - 20 + 8a$$

$$\Delta = a^2 + 4a - 16$$

$$\Delta_1 = 16 + 4 \cdot 16 = 16 \cdot 5$$

$$a_{1,2} = \frac{-4 \pm 4\sqrt{5}}{2} = \boxed{-2 \pm 2\sqrt{5}}$$

CENTER $\Rightarrow \lambda_{1,2} \in \mathbb{C}$, $\boxed{\alpha = 0}$

$\Rightarrow \Delta < 0$

$a \in (-2-2\sqrt{5}, -2+2\sqrt{5}) \Leftrightarrow \Delta < 0$

$$\lambda_{1,2} = \frac{-(2-a) \pm i\sqrt{a^2+4a-16}}{2}$$

$$\Leftrightarrow -\frac{(2-a)}{2} = 0 \Leftrightarrow \boxed{a = 2} \in (\quad)$$

b) For $a = 0 \Rightarrow \begin{cases} \dot{x} = -5y \\ \dot{y} = x - 2y \end{cases}$

$$A = \begin{pmatrix} 0 & -5 \\ 1 & -2 \end{pmatrix}$$

$$-\lambda(-2-\lambda) + 5 = 0$$

$$2\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\Delta = 4 - 4 \cdot 5 = 4 - 20 = -16$$

$$\lambda_{1,2} = \frac{-2 \pm 4i}{2} = \boxed{-1 \pm 2i}$$

$\Rightarrow$ FOCUS, stable.

*** 2 f)**

- The planar system is a simplified version of the Lotka-Volterra equations
- Circular trajectories around origin, radius determined by initial conditions. Fixed eq point $(0,0)$.

The Lotka-Volterra equations for a predator - prey system with 2 populations, x, y:

$$\boxed{\begin{cases} \dot{x} = \alpha x - \beta x y \\ \dot{y} = \delta x y - \gamma y \end{cases}}$$

2021 (open book)

1) $\begin{cases} \dot{x} = 2x+y \\ \dot{y} = -5x+4y \end{cases}$ $\quad A = \begin{pmatrix} 2 & 1 \\ -5 & 4 \end{pmatrix}$

$$\det(A - \lambda I_2) = (2-\lambda)(4-\lambda) + 5 = 0$$

$$8 - 4\lambda - 2\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 - 6\lambda + 13 = 0$$

$$\Delta = 36 - 4 \cdot 13 = 36 - 52$$

$$\Delta = -16$$

$$\lambda_{1,2} = \frac{6 \pm i \cdot 4}{2} = 3 \pm 2i$$

$$A \cdot u = \lambda u \iff \begin{cases} 2x+y = (3+2i)x \\ -5x+4y = (3+2i)y \end{cases}$$

$$\iff \begin{cases} y = (1+2i)x \\ -5x = (-1+2i)y \end{cases} \iff \boxed{u_1 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}}$$

$$-5x + 4(1+2i)x = (3+2i)(1+2i)x$$

$$-5x + 4x + (8x)i = (3 + 2i + 6i - 4)x$$

$$-x + 8xi = (-1 + 8i)x \quad \checkmark$$

$$A \cdot u = \lambda_2 u \iff \begin{cases} 2x + y = (3-2i)x \\ -5x + 4y = (3-2i)y \end{cases}$$

$$\iff \boxed{u_2 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}}$$

$$\boxed{X = c_1 e^{3t} \cos 2t \cdot \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} + c_2 e^{3t} \cos 2t \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}}$$

$$X = \begin{pmatrix} c_1 e^{3t} \cos 2t + c_2 e^{3t} \cos 2t \\ c_1 e^{3t} \cos 2t (1+2i) + c_2 e^{3t} \cos 2t (1-2i) \end{pmatrix}$$

(2.) $x' - 2x = e^{at} \mid \cdot e^{2t}$

$$x' e^{2t} - 2x e^{2t} = e^{(a+2)t}$$

$$(x \cdot e^{2t})' = e^{(a+2)t} \mid \int$$

$$x \cdot e^{2t} = \frac{1}{(a+2)} \int (a+2)\, e^{(a+2)t}$$

$$x e^{2t} = \frac{e^{(a+2)t}}{a+2} \qquad \text{For } a \neq -2$$

$$x = \frac{e^{at} \cdot \cancel{e^{2t}}}{(a+2)\cancel{e^{2t}}} = \boxed{\frac{e^{at}}{a+2}}$$

$$r - 2 = 0 \iff r = 2$$

$$\Rightarrow x_h = C_1 e^{2t} \qquad x = \varphi(t) \cdot e^{2t}$$

$$x' - 2x = e^{at}$$

$$\varphi'(t) e^{2t} + \cancel{2\varphi(t) e^{2t}} - \cancel{2\varphi(t) e^{2t}} = e^{at}$$

$$\varphi'(t) = e^{(a-2)t} \quad \int$$

$$\varphi(t) = \frac{e^{(a-2)t}}{(a-2)} \Rightarrow$$

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3. $f \to$ injective, C^1 function, $f(0) = 1$, $f(1) = -2$

a) $\dot{x} = f(x)$ has a global attractor eq.

point?

C^1 means it is 1 time differentiable

$f(x) = 0$ has a solution since f is an injective, C^1 function and $x \in (0,1)$ because $f(x)$ is continuous and decreasing since $\underline{\underline{\dot{0} > \dot{1}}}$ $\Rightarrow$ Using the Darboux property $\Rightarrow (\exists)\ c \in (0,1)$ such that $f(c) = 0$, which is unique

And that