GROUND STATE OF A RAREFIED FERMI GAS OF RIGID SPHERES

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The properties of the ground state of a Fermi system are discussed. The ground state energy of a Fermi gas of rigid spheres of low density is computed accurate to $(p_F a)^3$, where p_F is the Fermi momentum and a the diameter of the sphere.

IN this paper, we consider correlations in the ground states of Fermi systems with pair interaction. A method is used which allows us to take successively into account the correlations of higher and higher orders (two-particle, three-particle, etc.), and which involves the writing down of a chain of equations for the n-particle correlators, which is equivalent to the exact Schrödinger equation. The ground-state energy is expressed only in terms of a pair correlator.

The method is applied to a Fermi gas of rigid spheres, which is a convenient model of systems with strong repulsive interaction. At low density (which is characterized by a low value of the parameter $\gamma = p_F a$, where p_F is the Fermi momentum and a is the diameter of the sphere) this problem has been investigated by many authors.[1] The ground state energy per particle, with accuracy to γ^2, was first determined by Lee and Yang.[1] The contribution to the energy, which is proportional to γ, is determined by two-particle correlations in the s-state; the presence of Fermi filling affects the s-scattering and leads to a contribution which is proportional to γ^2. In the calculation of the succeeding terms of the expansion, it is necessary to take into account correlations of higher order than the two-particle correlations. A comparative estimate of their contributions is of interest.

The ground state energy per particle is computed here accurate to γ^3. The contribution proportional to γ^3 is equal to

$$\delta\varepsilon = p_F^2 \gamma^3 [(v - 1)$$
$$\times (1/15\pi + 0{,}064 + 0{.}012 + (v - 3)0{.}059)$$
$$+ (v + 1)/5\pi], \qquad (1)$$

where v is the number of spin and isospin degrees of freedom (more exact figures are given below). The first two terms are determined as before by pair correlations in the s-state and by

the effect of Fermi filling on them. The last term is determined by pair correlations in the p-state. The effect of other correlated pairs on a given pair leads to the third term in (1). Finally, the term with the factor 0.059 is due to triple correlations.

For $v = 4$ (a system with neutron-proton degrees of freedom), Eq. (1) is identical with the unpublished results of Martin (see [2]).

The determination of the correlators is given in Sec. 1. Their equations are discussed in Sec. 2, while rigid spheres are considered in Sects. 3 and 4. Details of the derivation of the equations are given in the Appendix. A system of units is employed in which $\hbar = 2m = 1$ (m = fermion mass).

1. CORRELATION FUNCTIONS

Let us expand the wave function of the ground state of a system of fermions $\Psi(1, \dots, N)$ in some, for the time being arbitrary, complete set of (basis) functions of the independent motion of the particles:

$$\Psi = \Psi_0 + \sum_{n=1}^{N} \Psi_n. \qquad (2)$$

Here Ψ_0 is the function with minimum energy (normalized to unity) in the set:

$$\Psi_0 = (N!)^{-1/2} A \prod_{p=1}^{N} \varphi_p(p) \qquad (3)$$

(A is the coordinate antisymmetrization operator: $\langle \varphi_p \mid \varphi_{p'} \rangle = \delta_{pp'}$). All the functions of the set can be generated by a transfer of a certain number of particles from the states φ_p to the states φ_m which are not occupied in Ψ_0.

The function Ψ_n in (2) represents that part of Ψ which is expanded in functions of the set with n transferred particles. The Ψ_n obey the conditions

$$\langle \Psi_n | \Psi_{n'} \rangle = 0 \qquad (n \neq n'), \tag{4a}$$

$$\langle \Psi_0 | \Psi_n \rangle = 0, \tag{4b}$$

which follow from the method of their construction and from the orthogonality of the functions of the complete set. It follows from (4b) that Ψ is normalized by the condition

$$\langle \Psi_0 | \Psi \rangle = 1. \tag{4c}$$

Let us consider the structure of Ψ_n. We begin with Ψ_1. It is expanded in a sum of functions of $\Psi_1{}^{p_1}$ ($p_1 = 1, \ldots, N$), corresponding to the possibility of transfer from each state φ_{p_1}. It is easy to see that $\Psi_1{}^{p_1}$ is obtained from (3) by replacing φ_{p_1} with a function orthogonal to any of the functions φ_p. We call it ψ_p and introduce the transfer operator M_1, such that $\psi_p = M_1 \varphi_{p_1}$. Then

$$\Psi_1{}^{p_1} = (N!)^{-1/2} A \left[\prod_{p \neq p_1} \varphi_p(p) M_1 \varphi_{p_1}(p_1) \right]. \tag{5}$$

The function Ψ_1 is completely determined by the N functions $M_1 \varphi_{p_1}$ or by the matrix elements M_1, with

$$\langle \varphi_{p_2} | M_1 | \varphi_{p_1} \rangle = 0 \quad (p_1, p_2 = 1, \ldots, N). \tag{6}$$

Similarly, Ψ_n is determined by the operator F_n ($F_1 = M_1$), which transfers n particles from the states φ_p to the states φ_m. It is convenient to separate from it the parts corresponding to all possible independent transfers of n particles. For example, for $n = 2$,

$$\Psi_2{}^{p_1 p_2} = (N!)^{-1/2} A \left[\prod_{p \neq p_1, p_2} \varphi_p(p) F_2 \varphi_{p_1}(p_1) \varphi_{p_2}(p_2) \right], \tag{7a}$$

$$F_2 \varphi_{p_1}(1) \varphi_{p_2}(2) = (M_1{}^1 M_1{}^2 + M_2) \varphi_{p_1}(1) \varphi_{p_2}(2) \tag{7b}$$

(here and below, the superscript on the operator denotes the variables on which it acts).

The operator M_n satisfies by definition orthogonality conditions of the type (6) with n p-functions on the right and at least one p-function on the left. As will be seen below, M_n ($n \geq 2$) describes the correlation between the n particles (more exactly, the quasi-particles). Therefore, we call M_n the n-correlator, and the result of its action on φ_p the n-correlation function.

2. EQUATION FOR CORRELATION FUNCTIONS. GROUND STATE ENERGY

We get the equations for M_n by the variational method, varying the mean value of the Hamiltonian for each Ψ_n^*. The value of $\delta(\Psi_n)$ is determined as the part of $\delta \Psi$ which is expanded in the functions of the set with n transferred particles: $\delta(\Psi_n) = (\delta \Psi)_n$. In other words, the variation of Ψ_n reduces to the variation of F_n. Account of the

orthogonality conditions of the type (6) for F_n is taken, as usual, by the method of undetermined Lagrangian multipliers.

The equations for M_n are obtained in the Appendix. Here, we carry out the selection of the basis by means of the equation with $n = 1$. For this purpose, we note that the intuitive representation of the presence of independent motions in the system corresponds to isolation from Ψ in (2) of Ψ_0—the wave function of several quasi-particles. The field in which they move is naturally determined from the condition $M_1 = 0$.

Substituting $M_1 = 0$ in Eq. (A.5) in the Appendix for $n = 1$, we find the equation for the basis functions:

$$T |p\rangle + \sum_{p_1}{}' \langle p_1 | T_{p_1} | A M_2 p p_1 \rangle + \sum_{p_1}{}' \langle p_1 | V | A (1 + M_2) p p_1 \rangle$$
$$+ \sum_{p_1 < p_2}{}' \langle p_1 p_2 | V_{p_1 p_2} | (A' M_2 + A M_3) p p_1 p_2 \rangle = E_p |p\rangle \tag{8}$$

(the prime on the summation sign indicates that the values of p_1 and p_2 equal to p are omitted). The other notation is defined in the Appendix. For $M_1 = 0$, we have $F_2 = M_2$ and $F_3 = M_3$.

Multiplying (8) on the left by $\langle p |$ and using the orthogonality conditions for M_2 and M_3, we get the energy of the quasiparticle in the state φ_p:

$$E_p = \langle p | T | p \rangle + \sum_p{}' \langle p p_1 | V | A (1 + M_2) p p_1 \rangle. \tag{9}$$

We find the total energy by multiplying the Schrödinger equation for the system on the left by $\langle \Psi_0 |$, using the condition (4) and the orthonormality of the functions φ_p:

$$E = \langle \Psi_0 | T | \Psi_0 \rangle + \langle \Psi_0 | V | \Psi_0 + \Psi_2 \rangle = \sum_{p_1} \langle p_1 | T | p_1 \rangle$$
$$+ \sum_{p_1 < p_2} \langle p_1 p_2 | V | A (1 + M_2) p_1 p_2 \rangle. \tag{10}$$

We henceforth consider only a homogeneous set of Fermi particles. Before we specify the interaction, let us discuss the equation for the pair correlation function as applied to this case. Using the explicit form of the quasi-particle functions [$\varphi_p(r, \sigma) = \Omega^{1/2} e^{ip \cdot r} \chi(\sigma)$] and Eqs. (9) and (10), we get from Eq. (A.5) of the Appendix, for $n = 2$,

$$(T_1 + T_2 - E_p - E_{p'}) | A M_2 p p' \rangle + V | A (1 + M_2) p p' \rangle$$
$$+ \sum_{p_1} \langle p_1 | V_{1p_1} + V_{2p_1} | (A' M_2 + A M_3) p p' p_1 \rangle$$
$$+ \sum_{p_1 < p_2} \langle p_1 p_2 | V_{p_1 p_2} | (A' M_3 + A'' M_2 M_2 + A M_4) p p' p_1 p_2 \rangle$$
$$+ \sum_{p_1 < p_2} | A M_2 p_1 p_2 \rangle \langle p_1 p_2 | V | A (1 + M_2) p p' \rangle$$
$$+ \sum_\beta \lambda_{2\beta}^{pp'} | 2_\beta \rangle = 0. \tag{11}$$

The double prime on the operator A means that only those permutations are admitted in which the matrix element containing this symbol does not break up into a product of two factors. For example, the term $M_2{}^{12} M_2{}^{34} \mid pp' p_1 p_2 \rangle$ is absent, since in this case the matrix element is equal to $\mid M_2 pp' \rangle \langle p_1 p_2 \mid V \mid M_2 p_1 p_2 \rangle$.

The first two terms in (11) correspond to the scattering of the quasiparticles by one another; the third and fourth terms describe the effect of three and four particle correlations on the pair correlations, the fifth term—the mutual effect of pair correlations and, finally, the last term, the effect of the quasiparticle filled Fermi sphere (the Pauli principle).

It is easy to eliminate the Lagrangian multipliers in (11). We operate on the left of (11) with the operator $(1 - P_1)(1 - P_2)$, where P_i is the projection operator in the Fermi sphere in the variable i. Taking into account the equality

$$(1 - P_i)(1 - P_2)|2_\beta\rangle = 0, \quad (1 - P_1)(1 - P_2)M_n{}^{12\cdots} = M_n{}^{12\cdots}$$

$$(1 - P_i)T_i = T_i(1 - P_i),$$

we get

$$(T_1 + T_2 - E_p - E_{p'}) \mid A M_2 pp' \rangle$$
$$+ (1 - P_1)(1 - P_2) V \mid A (1 + M_2) pp' \rangle$$
$$+ \sum_{p_i} \langle p_1 \mid (1 - P_1) V_{1 p_i}$$
$$+ (1 - P_2) V_{2 p_i} \mid (A' M_2 + A M_3) pp' p_1 \rangle$$
$$+ \sum_{p_1 < p_2} \langle p_1 p_2 \mid V_{p_1 p_2} \mid (A' M_3 + A'' M_2 M_2 + A M_4) pp' p_1 p_2 \rangle$$
$$+ \sum_{p_1 < p_2} \mid A M_3 p_1 p_2 \rangle \langle p_1 p_2 \mid V \mid A (1 + M_2) pp' \rangle = 0. \quad (12)$$

The relations (12) and (10) are basic for the following sections.

3. A GAS OF RIGID SPHERES. PAIR CORRELATIONS

Equation (12), taken separately, does not determine the pair correlator, since it contains correlators of higher orders. However, sometimes their role is unimportant, and the corresponding terms in (12) can be treated as a perturbation. Such is the situation in the case of a rarefied Fermi gas with strong repulsive interaction. We shall consider this case in Secs. 3 and 4.

We denote the repulsion radius by a. Then the degree of rarefaction is characterized by the nondimensional parameter $p_F a \equiv \gamma$. It is clear that only pair correlations are significant at low density ($\gamma \to 0$). The contribution to the energy under these conditions is principally determined by correlations in states with small orbital momenta, inasmuch as it is determined, in accord

with Eq. (10), by a quantity of the type of the scattering amplitude.

The three-particle correlator, considered as a perturbation, should generally be found first from the equation for M_3. We shall show in Sec. 4 that, in first approximation, the correction in (12) due to the three-particle correlations can be expressed in terms of the pair correlation function and the two-particle amplitude of s-scattering.

We shall work with the interaction

$$V (r) = \begin{cases} \infty, & r < a \\ 0, & r > a, \end{cases}$$

which does not depend on the spin variables. It follows from (12) that, for this interaction $M_2 \mid pp' \rangle = - \mid pp' \rangle$ when $r \le a$; the product $V (1 + M_2) \mid pp' \rangle = 0$ everywhere except for r = a.

Starting out from the presented physical considerations we begin the determination of M_2 by analyzing Eq. (12) for the s-wave. This equation is obtained in the following way. Denoting by **g** the total momentum, **u** the relative momentum of the pair, $\mid M_2 \mid pp' \rangle = \Omega^{-1} e^{i\mathbf{q} \cdot \mathbf{R}} M_2 (\mathbf{g}) \mid \mathbf{u} \rangle$ (R is the coordinate of the center of mass) and separating the spin variables, we get from (12)

$$(T - u^2) M_2(\mathbf{g}) \mid \mathbf{u} \rangle =$$
$$-\tfrac{1}{2}(1 - \theta_g) V \mid (1 + M_2(\mathbf{g})) \mathbf{u} \rangle - \tfrac{1}{2} I(\mathbf{g}) \mid \mathbf{u} \rangle,$$
$$1 - \theta_g = (1 - P_{g/2})(1 - P_{-g/2}). \quad (13)$$

Here $I(\mathbf{g}) \mid \mathbf{u} \rangle$ is the part of (12) which is not explicitly written out in (13); $P_{g/2}$ is the projection operator in the momentum sphere of radius p_F with center g/2. Equation (13) is written for the coordinate wave function of different quasiparticles (i.e., those which have different spin directions). For identical quasiparticles, it is necessary to antisymmetrize the function. By writing down Eq. (13) in integral form and expanding all functions in partial waves:

$$\langle \mathbf{r} \mid M_2(\mathbf{g}) \mid \mathbf{u} \rangle = \sum_{lm} R_{lm}^{gu} (r) Y_{lm} (\mathbf{n}_r),$$

$$e^{i\mathbf{kr}} = \sum_{lm} g_l (kr) Y_{lm}{}^* (\mathbf{n}_k) Y_{lm} (\mathbf{n}_r);$$

$$\langle \mathbf{k} \mid \tfrac{1}{2} V \mid (1 + M_2(\mathbf{g})) \mathbf{u} \rangle = \sum_{lm} c_{lm}^{gu} g_l (ka) Y_{lm} (\mathbf{n}_k),$$

$$g_l (kr) = 4\pi i^l j_l (kr), \quad (14)$$

we find the desired equation

$$R_{00}^{gu} (r) = - \frac{1}{(2\pi)^3 4\pi} c_{00}^{gu} \int \frac{1 - \theta_g (\mathbf{k})}{k^2 - u^2} g_0 (kr) g_0 (ka) d\mathbf{k}$$

$$- \sqrt{4\pi} \, I^{gu} (r) - W (r), \quad (15)$$

$$I^{gu} (r) = \frac{1}{8\pi (2\pi)^3} \int \frac{1 - \theta_g (\mathbf{k})}{k^2 - u^2} g_0 (kr) \langle \mathbf{k} \mid I (\mathbf{g}) \mid \mathbf{u} \rangle d\mathbf{k}. \quad (16)$$

$W(r)$ describes the contribution of the other partial waves which arise from the non-symmetry of $\theta_g(\mathbf{k})$.

We note that one can express the ground-state energy per particle by c_{lm}^{gu}. In accord with (10) and (14),

$$\varepsilon = \frac{E}{N} = \frac{3}{5}p_F^2 + \frac{1}{(2\pi)^3 \, 4\pi p_F^3}$$

$$\times \int dp\,dp' \sum_{lm} c_{lm}^{gu} \, g_l(ua) \, Y_{lm}(\mathbf{n}_u) \, [\nu - (-1)^l]. \quad (17)$$

Now, by means of (15), we shall successively determine the terms of the expansion of $c^{gu}(\gamma)$ in powers of γ. Equation (17) then gives the contribution to the energy from correlations in the s-state, also in the form of a power series in γ. In the present work, we limit ourselves to terms of the expansion up to γ^3.

Using the boundary condition $R_{00}^{gu}(a) = -g_0(ua)Y_{00}^*(\mathbf{n}_u)$, for $r = a$, we get from (15)

$$c_{00}^{gu} = [g_0(ua)Y_{00}^*(\mathbf{n}_u) - \sqrt{4\pi}\,\bar{I}^{gu}(a) - W(a)] / \frac{1}{4\pi(2\pi)^3}$$

$$\times \int \frac{1-\theta_g(\mathbf{k})}{k^2-u^2} g_0^2(ka)\,d\mathbf{k}. \quad (18)$$

It is clear beforehand that $c_{00} \sim a$ as $\gamma \to 0$. Actually, in this limit, Eq. (13) describes the pair scattering in vacuum. Since, as is well known, the scattering amplitude of the l-th partial wave is $t_l \sim a(ua)^{2l}$ $(ua \to 0)$, and, in accord with (14), $t_l \sim c_l(ua)^l$, we have

$$c_{lm} \sim a(ua)^l \sim a\gamma^l. \quad (19)$$

In the numerator of (18), the first term is of the order of unity, the second, as shall be shown in Sec. 4, is of the order of γ^2, and the third of the order of γ^5. In fact, owing to the parity of $\theta_g(\mathbf{k})$, only l of the same parity are coupled. Consequently, the first term in $W(a)$ in (18) corresponds to the d-wave and is equal to $(\delta \to +0)$

$$\frac{c_{2m}}{(2\pi)^3} \int \frac{1-\theta_g(\mathbf{k})}{k^2-u^2} g_0(ka)\,g_2(ka)\,Y_{00}^*(\mathbf{n}_k)\,Y_{2m}(\mathbf{n}_k)\,d\mathbf{k}$$

$$= -\frac{c_{2m}}{(2\pi)^3} \int \frac{\theta_g(\mathbf{k})}{k^2-u^2-i\delta} g_0(ka)\,g_2(ka)\,Y_{00}^*(\mathbf{n}_k)$$

$$\times Y_{2m}(\mathbf{n}_k)\,d\mathbf{k} \sim (a\gamma^2)(p_F\gamma^2) = \gamma^5.$$

Therefore, the first two terms of the expansion of c_{00} are determined by the denominator of (18). In order to obtain ε accurate to γ^3, it is necessary to keep terms up to γ^2 in (18):

$$c_{00}^{gu} \approx \sqrt{4\pi}\,a\,[1 + \gamma B_1 + \gamma^2(B_1^2 + u^2/2p_F^2) - \bar{I}^{gu}(a)]. \quad (20)$$

The quantity $B_1(g, u)$ is determined by the expression

$$B_1(g, u) = \frac{2}{\pi}\left[\frac{1}{2} + \frac{g}{4p_F} + \frac{u}{2p_F}\ln\frac{p_F + g/2 - u}{p_F + g/2 + u}\right.$$

$$\left. + \frac{p_F}{2g}\left(1 - \frac{g^2}{4p_F^2} - \frac{u^2}{p_F^2}\right)\ln\frac{(p_F + g/2)^2 - u^2}{p_F^2 - g^2/4 - u^2}\right].$$

The contribution of $u^2/2p_F^2$ in (20) is due to the "vacuum" pair correlations. The function B_1 arises from the integral with $\theta_g(\mathbf{k})$ and describes the effect of the Pauli principle on the pair correlations.

Substituting (20) in (17), we get the correction to the energy due to the s-wave:

$$\delta\varepsilon^{(s)} = \frac{3}{(2\pi)^3 p_F^4}\gamma(\nu - 1)\int dp\,dp'$$

$$\times \left[1 + \gamma B_1 + \gamma^2\left(B_1^2 + \frac{u^2}{3p_F^2}\right) - \bar{I}^{gu}(a)\right]. \quad (21)$$

Upon integration, the first two terms lead to a well-known result:[1]

$$\delta\varepsilon_1^{(s)} + \delta\varepsilon_2^{(s)} = p_F^2\gamma(\nu - 1)\left(\frac{2}{3\pi} + \gamma\frac{12}{105\pi^2}(11 - 2\ln 2)\right).$$

Integration of the term with $\gamma^2 u^2$ gives

$$\delta\varepsilon_3^{(s)} = p_F^2\gamma^3(\nu - 1) / 15\pi. \quad (22)$$

The integral of B_1^2 can not be expressed analytically. Numerical integration leads to the result

$$\delta\varepsilon_4^{(s)} = p_F^2\gamma^3(\nu - 1)0.0638 \quad (\pm 0.0005). \quad (23)$$

The contribution from $\bar{I}^{gu}(a)$ will be computed in the next section.

Let us now consider the contribution of the p-wave in (17). It is necessary here to retain in c_{1m} terms up to γ. According to (19), the expansion of c_{1m} begins with terms of the order of γ. Therefore, account of any contributions to the vacuum scattering in the p-state is superfluous in this approximation. Directing the z axis along $\mathbf{u}$, we get, as in (18),

$$c_{1,-1} = 0, \quad c_{1,1} = 0;$$

$$c_{10}^{gu} = g_1(ua)Y_{10}^*(\mathbf{n}_z) / \frac{1}{4\pi(2\pi)^3}\int \frac{g_1^2(ka)}{k^2 - u^2 - i\delta}\,d\mathbf{k}$$

$$\approx -i\sqrt{12\pi}\,a\gamma\frac{u}{p_F}. \quad (24)$$

We substitute this in (17):

$$\delta\varepsilon^{(p)} = \frac{1}{(2\pi)^3 p_F^4}\gamma(\nu + 1)\left(\frac{3}{4\pi}\right)^{1/2}i\int dp\,dp'\,c_{10}^{gu}\,u$$

$$= p_F^2\gamma^3(\nu + 1)\frac{1}{5\pi}. \quad (25)$$

The contribution to the energy from the subsequent partial waves is proportional to powers of γ higher than the third, and we shall not consider them.

4. CORRECTIONS TO s-SCATTERING

In the correction $I(g) | u \rangle$, we first consider the mutual effect of pair correlations:

$$I_1(g) | u \rangle = \frac{1}{\Omega} \sum_{u_1} | M_2(g) u_1 \rangle \langle u_1 | V | (1 + M_2(g)) u \rangle \bar{\theta}_g(u_1), \quad (26)$$

where the summation is carried out over the relative momentum of the pairs $p_1 p_2$ having the same total momentum g as the pair pp'. This is marked by the introduction of the corresponding step function $\bar{\theta}_g(u_1)$. Summation over the spin states has already been carried out. Substituting in (16) and expanding all quantities in γ, we find the first term of the expansion of $\bar{I}_1^{gu}(a)$:

$$\bar{I}_1^{gu}(a) = -\frac{(4\pi)^2}{(2\pi)^6} a^2 \int \frac{1 - \theta_g(k)}{k^2 - u^2} dk \int \frac{\bar{\theta}_g(u_1)}{k^2 - u_1^2} du_1. \quad (27)$$

The corresponding correction $\delta\epsilon_5^{(s)}$ is obtained from Eq. (21). Calculation gives

$$\delta\epsilon_5^{(s)} = p_F^2 \gamma^3 (\nu - 1) \cdot 0.0117 (\pm 0.0015). \quad (28)$$

The calculation of corrections of the three-particle correlations is more involved. It is convenient to compute it without transforming in the equation to relative coordinates for the different quasiparticles. After summation over the spin states, we have $I_2 | pp' \rangle$ equal to

$$I_2 | pp' \rangle = \sum_{p_1} \langle p_1 | V_{23}(M_2^{12} + M_2^{13} + M_3)$$

$$+ V_{13}(M_2^{12} + M_2^{23} + M_3) | \nu pp'p_1 - pp_1p' - p_1p'p \rangle$$

$$\equiv \sum_{p_1} \langle p_1 | J | \nu pp'p_1 - pp_1p' - p_1p'p \rangle. \quad (29)$$

We now show that in first approximation in γ, (29) is expressed in terms of the two-particle scattering amplitude and the pair correlation function in the following way:

$$I_2 | pp' \rangle = \sum_{p_1} \langle p_1 | \bar{t}^{23}(M_2^{12} + M_2^{13})$$

$$+ \bar{t}^{13}(M_2^{12} + M_2^{23}) | \nu pp'p_1 - pp_1p' - p_1p'p \rangle, \quad (30a)$$

$$\langle q_1 q_2 | \bar{t} | q_1'q_2' \rangle = \Omega^{-1} 8\pi a \delta_{q_1+q_2,q_1'+q_2'}. \quad (30b)$$

Using (A.5) for $n = 3$, we write out the equation which determines M_3 at low density, i.e., leaving only the direct interaction of three quasiparticles:

$$(T_1 + T_2 + T_3 - p^2 - p'^2 - p''^2) | M_3 pp'p'' \rangle$$

$$= -(1 - P_1)(1 - P_2)(1 - P_3)$$

$$\times [\Gamma + (V_{12} + V_{23} + V_{13})M_3] | pp'p'' \rangle, \quad (31a)$$

$$\Gamma = V_{12}(M_2^{23} + M_2^{13}) + V_{13}(M_2^{12} + M_2^{23})$$

$$+ V_{23}(M_2^{12} + M_2^{13}). \quad (31b)$$

Equation (31) describes the scattering of the quasiparticle by the correlated pair. The corre-

lated quasiparticles are located in the mean at a distance $r_{av} \sim 1/p_F$ from one another, which is γ^{-1} times greater than the scattering amplitude $t_0 \sim a$. Consequently, as $\gamma \to 0$, it should be possible to express Eq. (29) in terms of the characteristics of two-particle scattering.

In order to obtain (30), we substitute the formal solution of Eq. (31):

$$M_3 = [1 - G_3(V_{12} + V_{13} + V_{23})]^{-1} G_3 \Gamma,$$

$$G_3 = -(1 - P_1)(1 - P_2)(1 - P_3)$$

$$\times (T_1 + T_2 + T_3 - p^2 - p'^2 - p''^2)^{-1}$$

in (29). For symmetry, we supplement the term J with the term $V_{12}(M_2^{13} + M_2^{23} + M_3)$. The matrix element (29) of this operator vanishes because of the orthogonality condition $\langle p_1 | M_2 = \langle p_1 | M_3 = 0$, so that the addition has no effect on the result. Then,

$$J = \{1 + (V_{12} + V_{13} + V_{23})$$

$$\times [1 - G_3(V_{12} + V_{13} + V_{23})]^{-1} G_3\} \Gamma. \quad (32)$$

The value of J can be written down in terms of the correlators M_2^{12}, M_2^{13}, M_2^{23}, which satisfy by definition the equations

$$\bar{M}_2^{ik} = G_3 V_{ik}(1 + \bar{M}_2^{ik}), \quad (33)$$

and the corresponding scattering amplitudes $\bar{t}^{ik} = V_{ik}(1 + \bar{M}_2^{ik})$

$$\bar{t}^{ik} = V_{ik} + V_{ik}G_3\bar{t}^{ik} \quad (34)$$

[we note that Eq. (33) differs from Eq. (13) as $\gamma \to 0$

$$M_2 = G_2 V(1 + M_2),$$

$$G_2 = -(1 - P_1)(1 - P_2)(T_1 + T_2 - p_1^2 - p_2^2)^{-1} \quad (35)$$

only in the replacement of G_2 by G_3].

Actually, using (31b), (32)–(34), and the equality

$$[1 - G_3(V_{12} + V_{13} + V_{23})]^{-1} = 1 + \bar{M}_2^{12} + \bar{M}_2^{13} + \bar{M}_2^{23}$$

$$+ \bar{M}_2^{12}\bar{M}_2^{13} + \bar{M}_2^{12}\bar{M}_2^{23} + \dots,$$

on the right side of which are contained all possible products $\bar{M}_2^{ik}$ with the exception of cases in which identical $\bar{M}_2^{ik}$ appear successively in the product, we find

$$J = \bar{t}^{12}(M_2^{23} + M_2^{13}) + \bar{t}^{13}(M_2^{12} + M_2^{23})$$

$$+ \bar{t}^{23}(M_2^{13} + M_2^{12}) + \dots \quad (36)$$

the terms not written down in (36) contain products of more than two operators.

It remains to estimate the values of $\bar{t}$ and $\bar{M}$. The matrices $\bar{t}^{12}$ and $\bar{M}_2^{12}$ are diagonal in the momentum representation over the third variable. The diagonal matrix element $\langle m_3 | \bar{M}_2^{13} | m_3 \rangle \equiv \bar{M}m_3^{12}$ satisfies an equation that follows from (33):

$$\bar{M}_{m_3}{}^{12} = (1 - P_1)(1 - P_2)[p^2 + p'^2 + p''^2 - m_3^2 - T_1$$
$$- T_2]^{-1}V_{12}(1 + \bar{M}_{m_3}{}^{12}),$$

and differs from (35) only by a number in the denominator of the Green's function. As $a \to 0$, according to (14) and (20), $\langle q_1 q_2 | \bar{t}_{m_3}{}^{12} | q_1' q_2' \rangle$ does not depend on this number, and is equal to $\Omega^{-1} 8\pi a \delta_{q_1+q_2, q_1'+q_2'}$. The correlators M_2 and $\bar{M}_2{}^{ik}$ are proportional under these conditions to a. Therefore, it suffices to limit ourselves in J to the terms written down in (36); this leads to Eq. (30).[1]

Before calculating $\delta\epsilon$, we separate from (30a) the terms with $M_2 | pp' \rangle$. They have the structure $(U_1 + U_2) | M_2 pp' \rangle$, where U is the single-particle operator. We have

$$\sum_{p_1}[\langle p_1 | (\bar{t}^{23} + \bar{t}^{13})M_2{}^{12} | v\, pp'p_1 \rangle$$
$$- \langle p_1 | \bar{t}^{23} M_2{}^{13} | p p_1 p' \rangle - \langle p_1 | \bar{t}^{13} M_2{}^{23} | p_1 p' p \rangle]$$
$$= \frac{8}{3\pi} p_F^2 \gamma (v - 1) | M_2 pp' \rangle. \tag{37}$$

This part of $I_2 | pp' \rangle$ is compensated by the correction $I_3 | pp' \rangle$, which is connected with the change in the dispersion law of the quasi-particles. Actually, according to (9), (14), and (20), the energy of the quasiparticles in the first approximation in γ is equal to

$$E_p = p^2 + \delta E_p = p^2 + \frac{4}{3\pi} p_F^2 \gamma (v - 1), \quad I_3 | pp' \rangle$$
$$= -(\delta E_p + \delta E_{p'}) | M_2 pp' \rangle,$$

which cancels Eq. (37).

We now compute $\delta\epsilon$ in the usual way. From (16) we find the first term of the expansion of $\bar{I}_2{}^{gu}(a)$:

$$\bar{I}_2{}^{gu}(a) = -(v - 3) 4a^2 \frac{(4\pi)^2}{(2\pi)^6}$$
$$\times \int dm\, dm'\, dm_1\, dp_1 \frac{\delta(\mathbf{m} + \mathbf{m'} - \mathbf{p} - \mathbf{p'})}{m^2 + m'^2 - p^2 - p'^2}$$
$$\times \left[\frac{\delta(\mathbf{m} + \mathbf{m_1} - \mathbf{p} - \mathbf{p_1})}{m^2 + m_1^2 - p^2 - p_1^2} + \frac{\delta(\mathbf{m'} + \mathbf{m_1} - \mathbf{p'} - \mathbf{p_1})}{m'^2 + m_1^2 - p'^2 - p_1^2} \right].$$
$$\tag{38}$$

Substituting this in (21), we obtain

$$\delta\varepsilon_6{}^{(s)} = \frac{24(4\pi)^2}{(2\pi)^9 p_F^5} \gamma^3 (v-1)(v-3) \int dp\, dm$$
$$\times \left(\int \frac{\delta(\mathbf{m}+\mathbf{m'}-\mathbf{p}-\mathbf{p'})}{m^2 + m'^2 - p^2 - p'^2} dm' dp' \right)^2. \tag{39}$$

Calculation of the integral leads to the result

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[1] Integrals over the momentum which diverge at the upper limit can be cut off in the estimate at values of the order of $1/a$.

$$\delta\varepsilon_6{}^{(s)} = p_F^2 \gamma^3 (v - 1)(v - 3) 0.059 \qquad (\pm 0.003). \tag{40}$$

The origin of the factor $(\nu - 3)$ is evident from (30a). We write down the wave function on the right there in the form

$$(v - 2) | pp' p_1 \rangle + (A_{pp_1} + A_{p'p_1}) | pp'p_1 \rangle.$$

Antisymmetrization in the two latter components corresponds to cases in which the spin state of p_1 is the same as for p or p'. After separation of $M_2 | pp' \rangle$, the two direct terms in the matrix element of (30a), which contains these components, correspond to three exchange terms. Since all the terms contribute the same amount to $\delta\epsilon$, (30a) is proportional to $2(\nu - 2) + 2(2 - 3) = 2(\nu - 3)$.

It is obvious that the contribution of the four-particle correlations to the energy begins with terms of higher order in γ than the three-particle correlations, and can be omitted in the given approximation. This confirms the estimate of the corresponding term in Eq. (22).

Collecting all the $\delta\epsilon$ proportional to γ^3, we get Eq. (1).

In conclusion, we emphasize the following properties of the method used.

1. The fundamental objects of investigation, the correlation functions, have the intuitive meaning of wave functions of correlated groups.

2. In the language of field theory, the correlators represent blocks of diagrams. In the presence of a small parameter (for example, γ), the selection of terms in the equations for the correlation functions automatically guarantees the isolation of a sequence of principal diagrams. For the problem under consideration, the proposed method is, in our opinion, the most natural and economical.

3. The method is equally applicable to finite and infinite systems.

4. Inasmuch as a variational principle is used, any physical information on the structure of the correlation functions (both in infinite and finite systems) can be taken into account in the best fashion.

APPENDIX

The matrix elements of normalization $D_{nn'} = \langle \Psi_n | \Psi_{n'} \rangle$ and of the kinetic and potential energies $T_{nn'} = \langle \Psi_n | T | \Psi_{n'} \rangle$ and $V_{nn'} = \langle \Psi_n | V | \Psi_{n'} \rangle$ enter into the expression for the

mean value of the Hamiltonian. By virtue of (4a) and (4b), the matrix $D_{nn'}$ is diagonal in nn', $T_{nn'}$ $\neq 0$ only for $n - 1 \leq n' \leq n + 1$; $V_{nn'} \neq 0$ for $n - 2 \leq n' \leq n + 2$.

Before the calculation, we introduce a notation that simplifies the writing down of the formulas. We denote by $| n_\alpha \rangle$ the product of n p-functions, chosen from the N in any sequence α. The symbol $F_l | n_\alpha \rangle$ ($l \leq n$) denotes the effect of F_l on all possible l p-functions of the $| n_\alpha \rangle$. For example,

$$F_2 | 3_\alpha \rangle = (F_2{}^{12} + F_2{}^{13} + F_2{}^{23}) | 3_\alpha \rangle.$$

Then Ψ_n is written in the form

$$| \Psi_n \rangle = (N!)^{-\frac{1}{2}} | AF_n N \rangle.$$

We calculate the matrix elements, taking into account the condition of orthonormality of the p functions and the conditions (4)

$$D_{nn'} = (N!)^{-1} \langle AF_n N | AF_n N \rangle = \langle F_n N | AF_n N \rangle$$

$$= \sum_\alpha \langle F_n n_\alpha | AF_n n_\alpha \rangle, \tag{A.1}$$

$$\sum_{i=-1}^{1} T_{n,n+i} = \sum_\alpha \langle F_n n_\alpha | \sum_{s=1}^{n} T_s | (A'F_{n-1} + AF_n) n_\alpha \rangle$$

$$+ \sum_{\substack{\alpha \\ p_1 \neq (\alpha)}} \langle (F_n n_\alpha) p_1 | T_{p_1} | (A'F_n + AF_{n+1}) n_\alpha p_1 \rangle. \tag{A.2}$$

In the second sum of (A.2), after selection of the n p-functions by the sequence α, summation is carried out in the remaining N – n functions. This is denoted by the symbol $p_1 \neq (\alpha)$. The operator T_p acts on the variable on which depends the function φ_p in the left side of the matrix element. As a consequence of the conditions of the orthogonality of F_n and the p-functions, some permutations drop out; this is denoted in (A.2) by the prime on A. For example, in the first term of (A.2), the only possible permutations are when T_s acts on a single, uncorrelated p-function in the right side.

Similar designations are applicable to the matrix elements of the potential energy:

$$\sum_{i=-2}^{2} V_{n,n+i} = \sum_\alpha \langle F_n n_\alpha | \sum_{s<s'}^{n} V_{ss'} | (A'F_{n-2} + A'F_{n-1} + AF_n) n_\alpha \rangle$$

$$+ \sum_{\substack{\alpha \\ p_1 \neq (\alpha)}} \langle (F_n n_\alpha) p_1 | \sum_{s=1}^{n} V_{sp_1} |$$

$$\times (A'F_{n-1} + A'F_n + AF_{n+1}) n_\alpha p_1 \rangle$$

$$+ \sum_{\substack{\alpha \\ p_1 < p_2; \ p_1, \ p_2 \neq (\alpha)}} \langle (F_n n_\alpha) p_1 p_2 | V_{p_1 p_2} |$$

$$\times (A'F_n + A'F_{n+1} + AF_{n+2}) n_\alpha p_1 p_2 \rangle. \tag{A.3}$$

The orthogonality conditions on $| F_n n_\alpha \rangle$ and on the functions φ_p are taken into account by adding to $\langle \Psi | H | \Psi \rangle / \langle \psi | \psi \rangle$ the component

$$\sum_{n,\alpha,\beta} \langle F_n n_\alpha | n_\beta \rangle \bar\lambda_{n_\alpha n_\beta}, \tag{A.4}$$

where $\bar\lambda_{n_\alpha n_\beta}$ are Lagrangian multipliers and $| n_\beta \rangle$ is the product of n basis functions, chosen in some fashion β, among which is at least one p function.

By carrying out variations with respect to $\langle F_n n_\alpha |$, we get the equation for $| M_n n_\alpha \rangle$:

$$\sum_{s=1}^{n} T_s | (A'F_{n-1} + AF_n) n_\alpha \rangle$$

$$+ \sum_{p_1 \neq (\alpha)} \langle p_1 | T_{p_1} | (A'F_n + AF_{n+1}) n_\alpha p_1 \rangle$$

$$+ \sum_{s<s'}^{n} V_{ss'} | (A'F_{n-2} + A'F_{n-1} + AF_n) n_\alpha \rangle$$

$$+ \sum_{p_1 \neq (\alpha)} \langle p_1 | \sum_{s=1}^{n} V_{sp_1} | (A'F_{n-1} + A'F_n + AF_{n+1}) n_\alpha p_1 \rangle$$

$$+ \sum_{\substack{p_1 < p_2; \ p_1, \ p_2 \neq (\alpha)}} \langle p_1 p_2 | V_{p_1 p_2} |$$

$$\times (A'F_n + A'F_{n+1} + AF_{n+2}) n_\alpha p_1 p_2 \rangle$$

$$+ \sum_\beta \lambda_{n_\alpha n_\beta} | n_\beta \rangle - E | AF_n n_\alpha \rangle = 0,$$

$$\lambda_{n_\alpha n_\beta} = \bar\lambda_{n_\alpha n_\beta} \langle \Psi | \Psi \rangle, \tag{A.5}$$

where E is the energy of the system.

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[2] Gomes, Walecka and Weisskopf, Ann of Phys. 3, 241 (1958).