Optical Waves in Crystals

Propagation and Control of Laser Radiation

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Polarization of Light Waves

The field quantities **E** and **H** that describe light waves are vectors. In the previous chapter, when we discussed Gaussian-beam propagation, we used the scalar-wave approximation and were not concerned with the direction of oscillation of the electric field vector, except to note that the electric vector lies in a plane perpendicular to the direction of propagation. Many cases involving the propagation of light waves depend crucially on the direction of oscillation of the electric field. In fact, throughout most of the book, we deal almost exclusively with the propagation and control of polarized light. In this chapter we describe many aspects of polarized light and discuss many of the techniques used to study its propagation.

3.1. THE CONCEPT OF POLARIZATION

Light waves are electromagnetic fields and require the four basic field vectors **E**, **H**, **D**, and **B** for their complete description. The electric field vector **E** is chosen to define the state of polarization of the light waves. This choice is convenient because, in most optical media, physical interactions with the wave involve the electric field. The main reason for studying the polarization of light waves is that in many substances (anisotropic media) the index of refraction depends on the direction of oscillation of the electric field vector **E**. This phenomenon can be explained in terms of the motion of electrons which are driven by the electric field of the light waves. To illustrate this point, assume that the anisotropic material consists of non-spherical, needle-like molecules, and suppose that these molecules are all aligned with their long axes parallel to one another. Consider an electromagnetic wave passing through this substance. Because of the structure of the molecule, the electrons in the substance are pushed further from their equilibrium positions by electric fields that are parallel to the axes of the molecules than by those at right angles to the molecular axes. We thus expect a larger induced electronic polarization in the first case than in the second.

There are many other physical phenomena which are only associated with polarized light waves. Before we study these optical phenomena, it is important to understand in detail the characteristics of polarized waves. We start by reviewing the polarization states of monochromatic plane waves.

3.2 POLARIZATION OF MONOCHROMATIC PLANE WAVES

The polarization of light waves is specified by the electric field vector $\mathbf{E}(\mathbf{r}, t)$ at a fixed point in space, $\mathbf{r}$, at time t. The time variation of the electric field vector $\mathbf{E}$ of a monochromatic wave is exactly sinusoidal, that is, the electric field must oscillate at a definite frequency. If we assume that the light is propagating in the z direction, the electric field vector will lie on the xy plane. Since the x component and the y component of the field vector can oscillate independently at a definite frequency, one must first consider the effect produced by the vector addition of these two oscillating orthogonal components. The problem of superposing two independent oscillations at right angles to each other and with the same frequency is well known and is completely analogous to the classical motion of a two-dimensional harmonic oscillator. The general motion of the oscillator is an ellipse, which corresponds to oscillations in which the x and y components are not in phase. There are, of course, many special cases which are of great importance in optics. We start with a discussion of the general properties of elliptic polarization and follow with a consideration of some special cases.

In the complex-function representation, the electric field vector of a monochromatic plane wave propagating in the z direction is given by

$$\mathbf{E}(z, t) = \mathrm{Re}[\mathbf{A}e^{i(\omega t - kz)}], \tag{3.2-1}$$

where $\mathbf{A}$ is a complex vector which lies in the xy plane. We now consider the nature of the curve which the end point of the electric field vector $\mathbf{E}$ describes at a typical point in space. This curve is the time-evolution locus of the points whose coordinates (E_x, E_y) are

$$\begin{aligned} E_x &= A_x\cos(\omega t - kz + \delta_x), \\ E_y &= A_y\cos(\omega t - kz + \delta_y), \end{aligned} \tag{3.2-2}$$

where we have defined the complex vector $\mathbf{A}$ as

$$\mathbf{A} = \hat{\mathbf{x}}A_x e^{i\delta_x} + \hat{\mathbf{y}}A_y e^{i\delta_y}, \tag{3.2-3}$$

where A_x and A_y are positive numbers, $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors. The curve described by the end point of the electric vector as time evolves can be obtained by eliminating $\omega t - kz$ between the equations (3.2-2). After several steps of elementary algebra, we obtain

$$\left(\frac{E_x}{A_x}\right)^2 + \left(\frac{E_y}{A_y}\right)^2 - 2\frac{\cos\delta}{A_x A_y}E_x E_y = \sin^2\delta, \tag{3.2-4}$$

where

$$\delta = \delta_y - \delta_x. \tag{3.2-5}$$

All the phase angles are defined in the range $-\pi < \delta \leqslant \pi$.

Equation (3.2-4) is the equation of a conic. From (3.2-2) it is obvious that this conic is confined in a rectangular region with sides parallel to the coordinate axes and whose lengths are $2A_x$ and $2A_y$. Therefore, the curve must be an ellipse. The wave (3.2-1) is then said to be elliptically polarized. A complete description of an elliptical polarization includes the orientation of the ellipse with respect to the coordinate axes, the shape, and the sense of revolution of $\mathbf{E}$. In general, the principal axes of the ellipse are not in the x and y directions. By using a transformation (rotation) of the coordinate system, we are able to diagonalize Eq. (3.2-4). Let x' and y' be a new set of axes along the principal axes of the ellipse. Then the equation of the ellipse in this new coordinate system becomes

$$\left(\frac{E_{x'}}{a}\right)^2 + \left(\frac{E_{y'}}{b}\right)^2 = 1, \tag{3.2-6}$$

where a and b are the principal axes of the ellipse, and $E_{x'}$ and $E_{y'}$ are the components of the electric field vector in this principal coordinate system.

Let $\phi(0 \leqslant \phi < \pi)$ be the angle between the direction of the major axis x' and the x axis (see Fig. 3.1) Then the lengths of the principal axes are given by

$$\begin{aligned} a^2 &= A_x^2\cos^2\phi + A_y^2\sin^2\phi + 2A_xA_y\cos\delta\cos\phi\sin\phi \\ b^2 &= A_x^2\sin^2\phi + A_y^2\cos^2\phi - 2A_xA_y\cos\delta\cos\phi\sin\phi. \end{aligned} \tag{3.2-7}$$

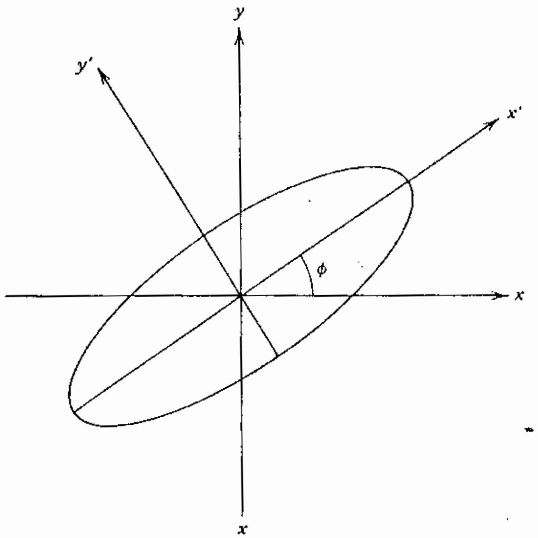


Figure 3.1. A typical polarization ellipse.

The angle ϕ can be expressed in terms of A_x, A_y, and $\cos\delta$ as

$$\tan 2\phi = \frac{2A_xA_y}{A_x^2 - A_y^2}\cos\delta. \tag{3.2-8}$$

The sense of revolution of an elliptical polarization is determined by the sign of $\sin\delta$. The end point of the electric vector will revolve in a clockwise direction if $\sin\delta > 0$ and in a counterclockwise direction if $\sin\delta < 0$. Figure 3.2 illustrates how the polarization ellipse changes with varying phase difference δ.

Before discussing some special cases of polarization, it is important to familiarize ourselves with the terminology. Light is linearly polarized when the tip of the electric field vector **E** moves along a straight line. When it describes an ellipse, the light is elliptically polarized. When it describes a circle, the light is circularly polarized. If the end point of the electric field vector is seen to move in a counterclockwise direction by an observer facing the approaching wave, the field is said to possess right-handed polarization. Figure 3.2 also illustrates the sense of revolution of the ellipse. Our convention for labeling right-hand and left-hand polarization is consistent with the terminology of modern physics in which a photon with a right-hand circular polarization has a positive angular momentum along the direction

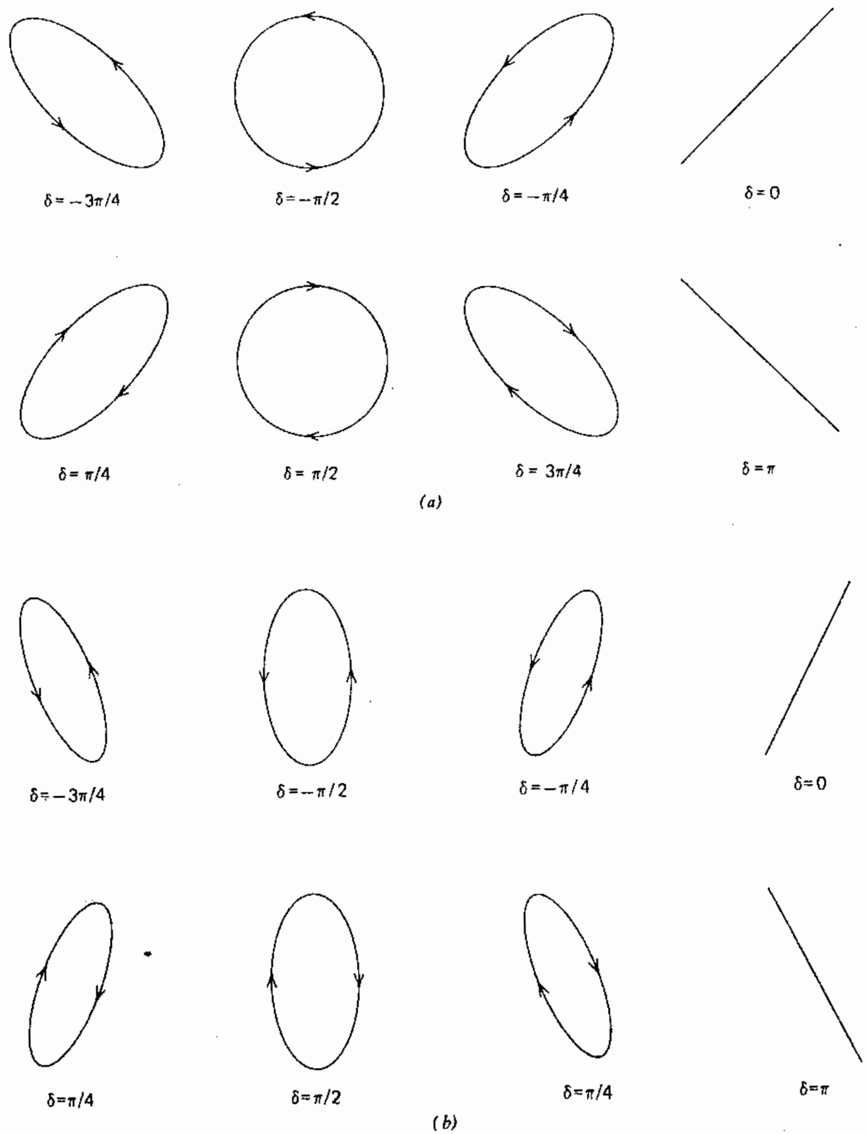


Figure 3.2. Polarization ellipses at various phase angles δ, where: (*a*) $E_x = \cos(\omega t - kz)$, $E_y = \cos(\omega t - kz + \delta)$; (*b*) $E_x = \frac{1}{2}\cos(\omega t - kz)$, $E_y = \cos(\omega t - kz + \delta)$.

of propagation (see Table 3.1 and Problem 3.4). However, some optics books adopt the opposite convention.

3.2.1. Linear and Circular Polarizations

Two special cases are of significant importance, namely, when the polarization ellipse degenerates into a straight line or a circle. According to Eq. (3.2-4) the ellipse will reduce to a straight line when

$$\delta = \delta_y - \delta_x = m\pi \qquad (m = 0, 1). \tag{3.2-9}$$

Here we recall that all the phase angles are defined to be in the range $-\pi < \delta \leqslant \pi$. In this case, the ratio of the components of the electric field vector is a constant

$$\frac{E_y}{E_x} = (-1)^m \frac{A_y}{A_x} \tag{3.2-10}$$

and the light is linearly polarized.

The other special case of importance is that of a circularly polarized wave. According to Eqs. (3.2-4) and (3.2-7) the ellipse will reduce to a circle when

$$\delta = \delta_y - \delta_x = \pm\tfrac{1}{2}\pi \tag{3.2-11}$$

and

$$A_y = A_x. \tag{3.2-12}$$

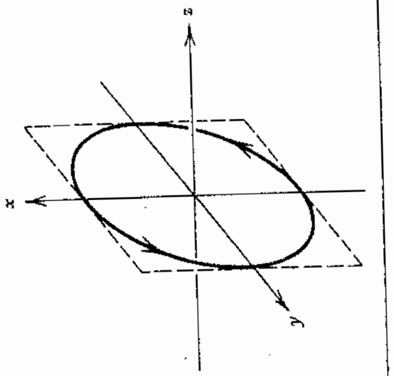
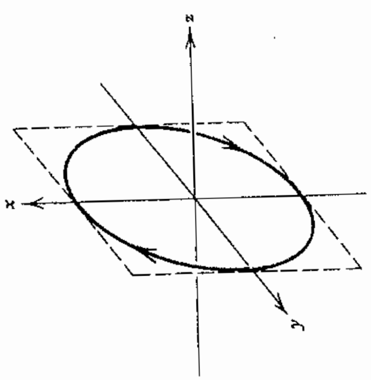
According to our convention, the light is right-hand circularly polarized when $\delta = -\frac{1}{2}\pi$, which corresponds to a counter-clockwise rotation of the electric field vector, and left-hand circularly polarized when $\delta = \frac{1}{2}\pi$, which corresponds to a clockwise rotation of the electric field vector (see Table 3.1).

The ellipticity of a polarization ellipse is defined as

$$e = \pm\frac{b}{a}, \tag{3.2-13}$$

where a and b are the lengths of the principal axes. The ellipticity is taken as positive when the rotation of the electric field vector is right-handed and negative otherwise.

Table 3.1. Relations between Angular Momentum and Handedness of Circularly Polarized Light

Field of a single photon ($\hbar\omega$)		L_z		Helicity	Handedness
$E_x = A\cos(\omega t - kz)$ $E_y = A\cos(\omega t - kz - \frac{1}{2}\pi)$		$\hbar$	$k > 0$	$\hbar$	Right-handed
			$k < 0$	$-\hbar$	Left-handed
$E_x = A\cos(\omega t - kz)$ $E_y = A\cos(\omega t - kz + \frac{1}{2}\pi)$		$-\hbar$	$k > 0$	$-\hbar$	Left-handed
			$k < 0$	$\hbar$	Right-handed

3.3. COMPLEX-NUMBER REPRESENTATION

From the discussion in the previous section we found how the polarization state of a light wave can be described in terms of the amplitudes and the phase angles of the x and y components of the electric field vector. In fact, all the information about the polarization of a wave is contained in the complex amplitude **A** of the plane wave (3.2-1). Therefore, a complex number χ defined as

$$\chi = e^{i\delta} \tan \psi = \frac{A_y}{A_x} e^{i(\delta_y - \delta_x)} \tag{3.3-1}$$

is sufficient to describe the polarization states. The angle ψ is defined to be between 0 and $\pi/2$. A complete description of the ellipse of polarization, which includes the orientation, sense of revolution, and ellipticity [see Eq. (3.2-13)], can be expressed in terms of δ and ψ. Figure 3.3 illustrates various different polarization states in the complex plane. It can be seen from the figure that all the right-handed elliptical polarization states are in the lower half of the plane, whereas the left-handed elliptical polarization states are in the upper half of the plane. The origin corresponds to a linear polarization state with direction of oscillation parallel to the x axis. Thus, each point on

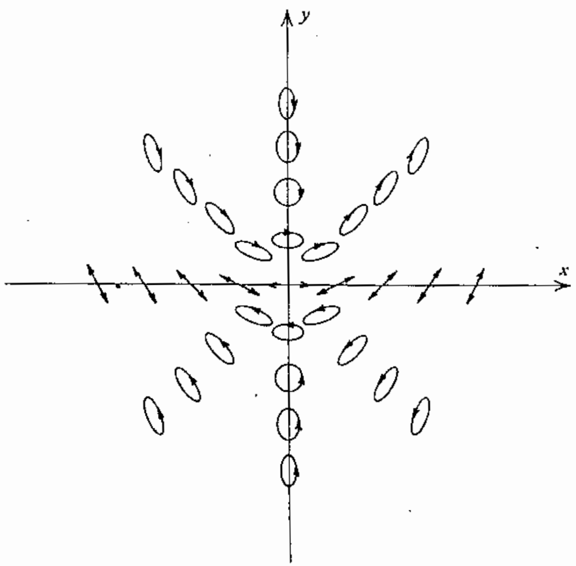


Figure 3.3. Each point of the complex plane is associated with a polarization state.

the complex plane represents a unique polarization state. Each point on the x axis represents a linearly polarized state with different azimuth angles of oscillation. Only two points $(0, \pm 1)$ correspond to circular polarization. Each point of the rest of the complex plane corresponds to a unique elliptical polarization state.

The inclination angle ϕ and the ellipticity angle θ ($\theta \equiv \tan^{-1} e$) of the polarization ellipse correspond to a given complex number χ and are given by

$$\tan 2\phi = \frac{2\,\mathrm{Re}[\chi]}{1 - |\chi|^2} \tag{3.3-2}$$

and

$$\sin 2\theta = -\frac{2\,\mathrm{Im}[\chi]}{1 + |\chi|^2}. \tag{3.3-3}$$

3.4. JONES-VECTOR REPRESENTATION

The Jones vector, introduced in 1941 by R. C. Jones [1], describes efficiently the polarization state of a plane wave. In this representation, the plane wave (3.2-1) is expressed in terms of its complex amplitudes as a column vector

$$\mathbf{J} = \begin{pmatrix} A_x e^{i\delta_x} \\ A_y e^{i\delta_y} \end{pmatrix}. \tag{3.4-1}$$

Notice that the Jones vector is a complex vector, that is, its elements are complex numbers. **J** is not a vector in the real physical space; rather it is a vector in an abstract mathematical space. To obtain, as an example, the real x component of the electric field, we must perform the operation $E_x(t) = \mathrm{Re}[J_x e^{i\omega t}] = \mathrm{Re}[A_x e^{i(\omega t + \delta_x)}]$.

The Jones vector contains complete information about the amplitudes and the phases of the electric-field-vector components. It thus specifies the wave uniquely. If we are only interested in the polarization state of the wave, it is convenient to use the normalized Jones vector which satisfies the condition that

$$\mathbf{J}^* \cdot \mathbf{J} = 1 \tag{3.4-2}$$

where the asterisk (*) denotes complex conjugation. Thus, a linearly polarized light wave with the electric field vector oscillating along a given direction can be represented by the Jones vector

$$\begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}, \tag{3.4-3}$$

where ψ is the azimuth angle of the oscillation direction with respect to the x axis. The state of polarization which is orthogonal to the state represented by Eq. (3.4-3) can be obtained by the substitution of ψ by $\psi + \frac{1}{2}\pi$, leading to a Jones vector

$$\begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix}. \tag{3.4-4}$$

For special case when $\psi = 0$ represents linearly polarized waves whose electric field vector oscillates along the coordinate axes, the Jones vectors are given by

$$\hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.4-5}$$

Jones vectors for the right- and left-hand circularly polarized light waves are given by

$$\hat{\mathbf{R}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{3.4-6}$$

$$\hat{\mathbf{L}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}. \tag{3.4-7}$$

These two circular polarizations are mutually orthogonal in the sense that

$$\hat{\mathbf{R}}^* \cdot \hat{\mathbf{L}} = 0. \tag{3.4-8}$$

Since the Jones vector is a column matrix of rank 2, any pair of orthogonal Jones vectors can be used as a basis of the mathematical space spanned by all the Jones vectors. Any polarization can be represented as a superposition of two mutually orthogonal polarizations $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, or $\hat{\mathbf{R}}$ and $\hat{\mathbf{L}}$. In particular, we can resolve the basic linear polarization $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ into two circular

polarizations $\hat{\mathbf{R}}$ and $\hat{\mathbf{L}}$ and vice versa. These relations are given by

$$\hat{\mathbf{R}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}), \tag{3.4-9}$$

$$\hat{\mathbf{L}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}), \tag{3.4-10}$$

$$\hat{\mathbf{x}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{R}} + \hat{\mathbf{L}}), \tag{3.4-11}$$

$$\hat{\mathbf{y}} = \frac{i}{\sqrt{2}}(\hat{\mathbf{R}} - \hat{\mathbf{L}}). \tag{3.4-12}$$

Circular polarizations are seen to consist of linear oscillations along the x and y directions with equal amplitude $1/\sqrt{2}$, but with a phase difference of $\frac{1}{2}\pi$. Similarly, a linear polarization can be viewed as a superposition of two oppositely sensed circular polarizations.

We have so far discussed only the Jones vectors of some simple special cases of polarization. It is easy to show that a general elliptic polarization can be represented by the following Jones vector:

$$\mathbf{J}(\psi, \delta) = \begin{pmatrix} \cos\psi \\ e^{i\delta}\sin\psi \end{pmatrix}. \tag{3.4-13}$$

This Jones vector represents the same polarization state as the one represented by the complex number $\chi = e^{i\delta}\tan\psi$. Table 3.2 shows the Jones vectors of some typical polarization states.

The most important application of Jones vectors is in conjunction with the Jones calculus. This is a powerful technique used for studying the propagation of plane waves with arbitrary states of polarizations through an arbitrary sequence of birefringent elements and polarizers. This topic will be considered in some detail in Chapter 5.

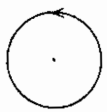
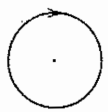
PROBLEMS

3.1. Derive Eq. (3.2-4).

3.2. Derive Eqs. (3.2-7) and (3.2-8).

3.3. Show that the end point of the electric vector of an elliptically polarized light will revolve in a clockwise direction if $\sin\delta > 0$ and in a counterclockwise direction if $\sin\delta < 0$.

Table 3.2. **Jones Vectors of Some Typical Polarization States**

Polarization Ellipse	Jones Vector
	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$
	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$
	$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}$
	$\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -i \end{pmatrix}$

3.4. (a) A right-hand circularly polarized wave ($\sin\delta < 0$) propagating in the z direction has a finite extent in the x and y directions. Assuming that the amplitude modulation is slowly varying (the wave is many wavelengths broad), show that the electric and magnetic fields are given approximately by

$$\mathbf{E}(x, y, z, t) \simeq E_0(x, y)(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) + \frac{-i}{k}\left(\frac{\partial E_0}{\partial x} - i\frac{\partial E_0}{\partial y}\right)\hat{\mathbf{z}}e^{i(\omega t - kz)},$$

$$\mathbf{H}(x, y, z, t) \simeq i\frac{k}{\omega\mu}\mathbf{E}(x, y, z, t).$$

(b) Calculate the time-averaged component of angular momentum along the direction of propagation ($+z$). Show that this component of the angular momentum is $\hbar$ provided the energy of the wave is normalized to $\hbar\omega$. This shows that a right-hand circularly polarized photon carries a positive angular momentum $\hbar$ along the direction of its momentum vector (helicity).

(c) Show that the transverse components of the angular momentum vanish.

3.5. *Orthogonal polarization states.*

(a) Find a polarization state which is orthogonal to the polarization state

$$\mathbf{J}(\psi, \delta) = \begin{pmatrix} \cos\psi \\ e^{i\delta}\sin\psi \end{pmatrix}.$$

Answer:

$$\begin{pmatrix} \sin\psi \\ e^{i(\pi+\delta)}\cos\psi \end{pmatrix}.$$

(b) Show that the major axes of the ellipses of two mutually orthogonal polarization states are perpendicular to each other and the senses of revolution are opposite.

3.6. (a) An elliptically polarized beam propagating in the z direction has a finite extent in the x and y directions:

$$\mathbf{E}(x, y, z, t) \simeq E_0(x, y)(\alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}})e^{i(\omega t - kz)}$$

where $\alpha = \cos\psi$, $\beta = \sin\psi\, e^{i\delta}$. Show that the electric field must have a component in the z direction [see Problem 3.4(a)], and derive the expressions for the electric field and the magnetic field.

(b) Calculate the z component of the angular momentum, assuming that the total energy of the wave is $\hbar\omega$. Answer: $L_z = -\hbar \sin 2\psi \sin\delta$.

(c) Decompose elliptically polarized light into a linear superposition of right-hand and left-hand polarized states $\hat{\mathbf{R}}$ and $\hat{\mathbf{L}}$, i.e., if $\mathbf{J}$ is the Jones vector of the polarization state in (a), find r and l such that

$$\mathbf{J} = r\hat{\mathbf{R}} + l\hat{\mathbf{L}}.$$

(d) If r and l are the probability amplitudes that the photon is right-hand and left-hand circularly polarized, respectively, show that the angular momentum can be obtained by evaluating $|r|^2 - |l|^2$:

$$L_z = \hbar\left(|r|^2 - |l|^2\right).$$

(e) Express the angular momentum in terms of the ellipticity angle θ.

3.7. Derive Eqs. (3.3-2) and (3.3-3).

3.8. *Orthogonal polarization states.* Consider two monochromatic plane waves of the form

$$\mathbf{E}_a(z,t) = \mathrm{Re}\left[\mathbf{A}e^{i(\omega t - kz)}\right],$$

$$\mathbf{E}_b(z,t) = \mathrm{Re}\left[\mathbf{B}e^{i(\omega t - kz)}\right].$$

The polarization states of these two waves are orthogonal, that is, $\mathbf{A}^* \cdot \mathbf{B} = 0$.

(a) Let δ_a, δ_b be the phase angles defined in Eq. (3.2-2). Show that

$$\delta_a - \delta_b = \pm\pi.$$

(b) Since δ_a, δ_b are all in the range $-\pi < \delta \leqslant \pi$, show that

$$\delta_a\delta_b \leqslant 0.$$

(c) Let χ_a, χ_b be the complex numbers representing the polarization states of these two waves. Show that

$$\chi_a\chi_b = -1.$$

(d) Show that the major axes of the polarization ellipses are mutually orthogonal and the ellipiticities are of the same magnitude with opposite signs.

REFERENCES AND SUGGESTED READING

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3. R. M. A. Azzam and N. M. Bashara, *Ellipsometry and Polarized Light*, North-Holland, Amsterdam, 1977.
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5

Jones Calculus and its Application to Birefringent Optical Systems

Many sophisticated birefringent optical systems, such as wide-angle electro-optic modulators [1], Lyot filters [2–5] and Šolc filters [6, 7] involve the passage of light through a train of polarizers and retardation plates. The effect of each individual element, either polarizer or retardation plate, on the polarization state of the transmitted light can be easily pictured without the aid of any matrix algebra. However, when an optical system consists of many such elements, each oriented at a different azimuth angle, the calculation of the overall transmission becomes complicated and is greatly facilitated by a systematic approach. The Jones calculus, invented in 1940 by R. C. Jones [8], is a powerful 2 × 2-matrix method in which the state of polarization is represented by a two-component vector (see Section 3.4) while each optical element is represented by a 2 × 2 matrix. The overall matrix for the whole system is obtained by multiplying all the matrices, and the polarization state of the transmitted light is computed by multiplying the vector representing the input beam by the overall matrix. We will first derive the mathematical formulation of the Jones matrix method, and then apply it to some birefringent filters.

5.1. JONES MATRIX FORMULATION

We have shown in the previous chapter that light propagation in a birefringent crystal consists of a linear superposition of two eigenwaves. These eigenwaves have well-defined phase velocities and directions of polarization. The birefringent crystals may be either uniaxial or biaxial. However, most commonly used materials such as calcite and quartz are uniaxial. In a uniaxial crystal, these eigenwaves are the ordinary and the extraordinary wave. The directions of polarization for these eigenwaves are mutually

orthogonal and are called the "slow" and "fast" axes of the crystal for that direction of propagation. Retardation plates are usually cut in such a way that the c axis lies in the plane of the plate surfaces. Thus the propagation direction of normally incident light is perpendicular to the c axis.

Retardation plates (also called wave plates) and phase shifters are polarization-state converters, or transformers. The polarization state of a light beam can be converted to any other polarization state by means of a suitable retardation plate. In formulating the Jones matrix method, we assume that there is no reflection of light from either surface of the plate and the light is totally transmitted through the plate surfaces. In practice, there is reflection, though most retardation plates are coated so as to reduce the surface reflection loss. The Fresnel reflections at the plate surfaces not only decrease the transmitted intensity, but also affect the fine structure of the spectral transmittance because of multiple-reflection interference (see Section 5.5). Referring to Fig. 5.1, we consider an incident light beam with polarization state described by the Jones vector

$$\mathbf{V} = \begin{pmatrix} V_x \\ V_y \end{pmatrix}, \tag{5.1-1}$$

where V_x and V_y are two complex numbers. The x and y axes are fixed laboratory axes. To determine how the light propagates in the retardation

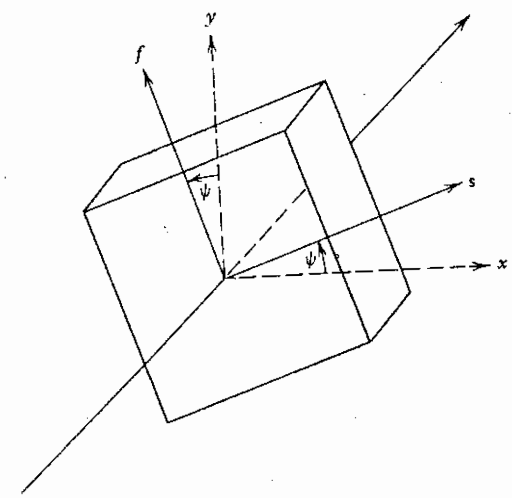


Figure 5.1. A retardation plate with azimuth angle ψ.

plate, we need to decompose the light into a linear combination of the "fast" and "slow" eigenwaves of the crystal. This is done by the coordinate transformation

$$\begin{pmatrix} V_s \\ V_f \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} \equiv R(\psi) \begin{pmatrix} V_x \\ V_y \end{pmatrix}. \qquad (5.1\text{-}2)$$

V_s is the slow component of the polarization vector V, whereas V_f is the fast component. The "slow" and "fast" axes are fixed in the crystal. These two components are eigenwaves of the retardation plate and will propagate with their own phase velocities and polarizations. Because of the difference in phase velocity, one component is retarded relative to the other. This retardation changes the polarization state of the emerging beam.

Let n_s and n_f be the refractive indices for the "slow" and "fast" components, respectively. The polarization state of the emerging beam in the crystal sf coordinate system is given by

$$\begin{pmatrix} V'_s \\ V'_f \end{pmatrix} = \begin{pmatrix} \exp\left(-in_s \frac{\omega}{c} l\right) & 0 \\ 0 & \exp\left(-in_f \frac{\omega}{c} l\right) \end{pmatrix} \begin{pmatrix} V_s \\ V_f \end{pmatrix} \qquad (5.1\text{-}3)$$

where l is the thickness of the plate and ω is the frequency of the light beam. The phase retardation is given by the difference of the exponents in (5.1-3) and is equal to

$$\Gamma = (n_s - n_f)\frac{\omega l}{c}. \qquad (5.1\text{-}4)$$

Notice that the phase retardation Γ is a measure of the relative change in phase, not the absolute change. The birefringence of a typical crystal retardation plate is small, that is, $|n_s - n_f| \ll n_s, n_f$. Consequently, the absolute change in phase caused by the plate may be hundreds of times greater than the phase retardation. Let ϕ be the mean absolute phase change,

$$\phi = \tfrac{1}{2}(n_s + n_f)\frac{\omega l}{c}. \qquad (5.1\text{-}5)$$

Then Eq. (5.1-3) can be written in terms of ϕ and Γ as

$$\begin{pmatrix} V'_s \\ V'_f \end{pmatrix} = e^{-i\phi} \begin{pmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{i\Gamma/2} \end{pmatrix} \begin{pmatrix} V_s \\ V_f \end{pmatrix}. \qquad (5.1\text{-}6)$$

The Jones vector of the polarization state of the emerging beam in the xy coordinate is given by transforming back from the crystal sf coordinate system:

$$\begin{pmatrix} V'_x \\ V'_y \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} V'_s \\ V'_f \end{pmatrix}. \qquad (5.1\text{-}7)$$

By combining Eqs. (5.1-2), (5.1-6), and (5.1-7), we can write the transformation due to the retardation plate as

$$\begin{pmatrix} V'_x \\ V'_y \end{pmatrix} = R(-\psi) W_0 R(\psi) \begin{pmatrix} V_x \\ V_y \end{pmatrix}, \qquad (5.1\text{-}8)$$

where $R(\psi)$ is the rotation matrix and W_0 is the Jones matrix for the retardation plate. These are given, respectively, by

$$R(\psi) = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \qquad (5.1\text{-}9a)$$

and

$$W_0 = e^{-i\phi} \begin{pmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{i\Gamma/2} \end{pmatrix}. \qquad (5.1\text{-}9b)$$

The phase factor $e^{-i\phi}$ can be neglected if interference effects are not important, or not observable. A retardation plate is characterized by its phase retardation Γ and its azimuth angle ψ, and is represented by the product of three matrices (5.1-8):

$$W = R(-\psi) W_0 R(\psi). \qquad (5.1\text{-}10)$$

Note that the Jones matrix of a wave plate is a unitary matrix, that is,

$$W^\dagger W = 1,$$

where the dagger † means Hermitian conjugate. The passage of a polarized light beam through a wave plate is mathematically described as a unitary transformation. Many physical properties are invariant under unitary transformations; these include the orthogonal relation between the Jones vectors and the magnitude of the Jones vectors. Thus if the polarization states of two beams are mutually orthogonal, they will remain orthogonal after passing through an arbitrary wave plate.

The Jones matrix of an ideal homogeneous linear sheet polarizer oriented with its transmission axis parallel to the laboratory x axis is

$$P_0 = e^{-i\phi}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.1-11}$$

where ϕ is the absolute phase accumulated due to the finite optical thickness of the polarizer. The Jones matrix of a polarizer rotated by an angle ψ about z is given by

$$P = R(-\psi)P_0R(\psi). \tag{5.1-12}$$

Thus, if we neglect the absolute phase ϕ, the Jones matrix representations of the polarizers transmitting light with electric field vectors parallel to the x and y axes, respectively, are given by

$$P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

To find the effect of a train of retardation plates and polarizers on the polarization state of a polarized light, we write down the Jones vector of the incident beam, and then write down the Jones matrices of the various elements. The Jones vector of the emerging beam is obtained by carrying out the matrix multiplication in sequence.

5.1.1. Example: A Half-Wave Retardation Plate

A half-wave plate has a phase retardation of $\Gamma = \pi$. According, to Eq. (5.1-4) an x-cut* (or y-cut) uniaxial crystal will act as a half-wave plate provided the thickness is $t = \lambda/2(n_e - n_o)$. We will determine the effect of a half-wave plate on the polarization state of a transmitted light beam. The azimuth angle of the wave plate is taken as 45°, and the incident beam as vertically polarized. The Jones vector for the incident beam can be written as

$$\mathbf{V} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5.1-13}$$

and the Jones matrix for the half-wave plate is obtained by using Eqs.

*A crystal plate is called x-cut if its faces are perpendicular to the principal x axis.

(5.1-9) and (5.1-10):

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{5.1-14}$$

The Jones vector for the emerging beam is obtained by multiplying Eqs. (5.1-14) and (5.1-13); the result is

$$\mathbf{V}' = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.1-15}$$

This is horizontally polarized light. The effect of the half-wave plate is to rotate the polarization by 90°. It can be shown that for a general azimuth angle ψ, the half-wave plate will rotate the polarization by an angle 2ψ (see Problem 5.1). In other words, linearly polarized light remains linearly polarized, except that the plane of polarization is rotated by an angle of 2ψ.

When the incident light is circularly polarized, a half-wave plate will convert right-hand circularly polarized light into left-hand circularly polarized light and vice versa, regardless of the azimuth angle. The proof is left as an exercise (see Problem 5.1). Figure 5.2 illustrates the effect of a half-wave plate.

5.1.2. Example: A Quarter-Wave Plate

A quarter-wave plate has a phase retardation of $\Gamma = \frac{1}{2}\pi$. If the plate is made of an x-cut (or y-cut) uniaxially anisotropic crystal, the thickness is $t = \lambda/4(n_e - n_o)$ (or odd multiples thereof). Suppose again that the azimuth angle of the plate is $\psi = 45°$ and the incident beam is vertically polarized. The Jones vector for the incident beam is given again by Eq. (5.1-13). The Jones matrix for this quarter-wave plate, according to Eq. (5.1-10), is

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{5.1-16}$$

The Jones vector for the emerging beam is obtained by multiplying Eqs. (5.1-16) and (5.1-13) and is given by

$$\mathbf{V}' = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{-i}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}. \tag{5.1-17}$$

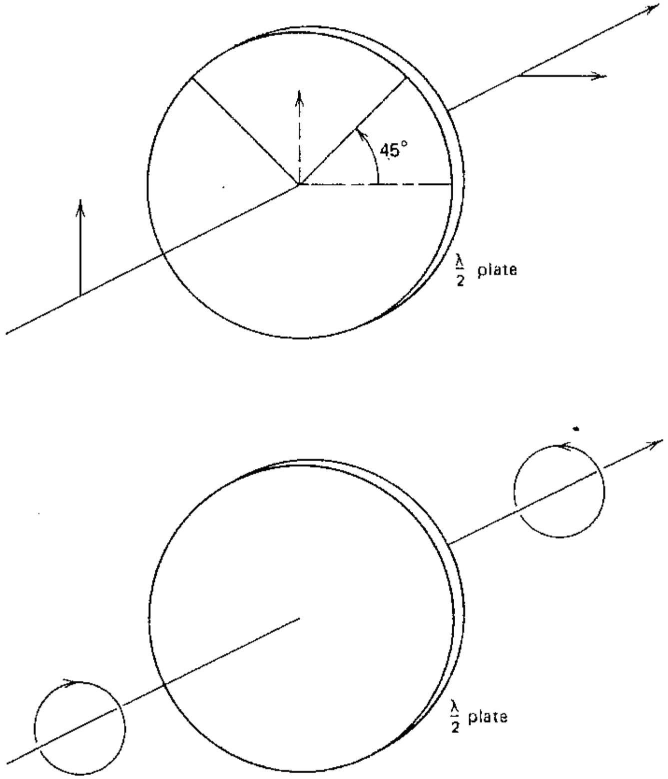


Figure 5.2. The effect of a half-wave plate on the polarization state of a beam.

This is left-hand circularly polarized light. The effect of a 45°-oriented quarter-wave plate is to convert vertically polarized light into left-hand circularly polarized light. If the incident beam is horizontally polarized, the emerging beam will be right-hand circularly polarized. The effect of this quarter-wave plate is illustrated in Fig. 5.3.

5.2. INTENSITY TRANSMISSION

So far our development of the Jones calculus was concerned with the polarization state of the light beam. In many cases, we need to determine the transmitted intensity. A narrowband filter, for example, transmits radiation only in a small spectral regime and rejects (or absorbs) radiation at other wavelengths. To change the intensity of the transmitted beam, an

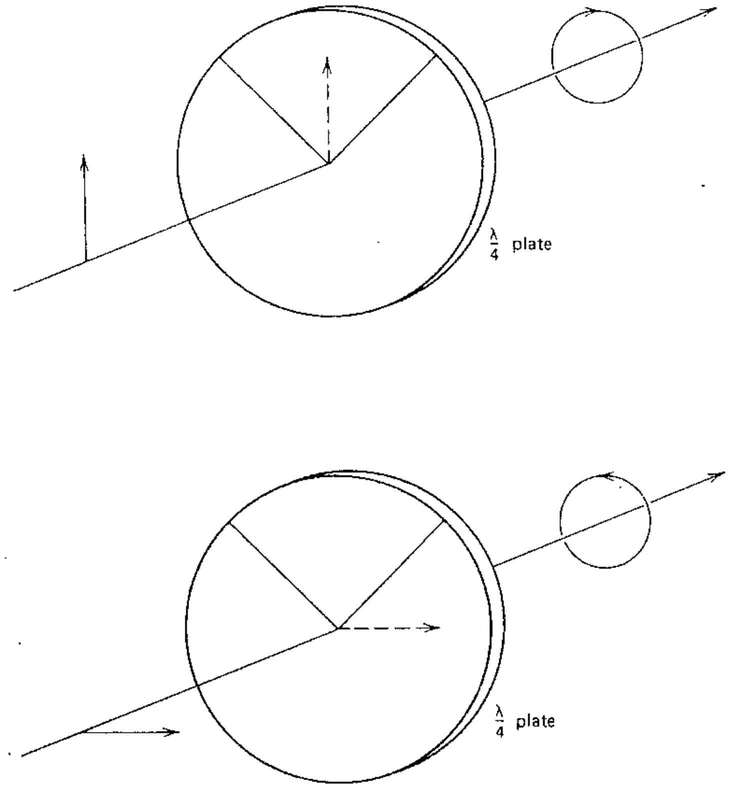


Figure 5.3. The effect of a quarter-wave plate on the polarization state of a linearly polarized beam.

analyzer is usually required. An analyzer is basically a polarizer. It is called an analyzer simply because of its location in the optical system. In most birefringent optical systems, a polarizer is placed in front of the system in order to "prepare" a polarized light. A second polarizer (analyzer) is placed at the output to analyze the polarization state of the emerging beam. Because the phase retardation of each wave-plate is wavelength-dependent, the polarization state of the emerging beam depends on the wavelength of the light. A polarizer at the rear will cause the overall transmitted intensity to be wavelength-dependent.

The Jones vector representation of a light beam contains information about not only the polarization state but also the intensity of light. Let us now consider the light beam after it passes through the polarizer. Its electric vector can be written as a Jones vector

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{5.2-1}$$

The intensity is calculated as follows:

$$I = \mathbf{E}^{\dagger} \cdot \mathbf{E} = |E_x|^2 + |E_y|^2. \tag{5.2-2}$$

where the dagger indicates the Hermitian conjugate. If the Jones vector of the emerging beam after it passes through the analyzer is written as

$$\mathbf{E}' = \begin{pmatrix} E'_x \\ E'_y \end{pmatrix}, \tag{5.2-3}$$

the transmissivity of the birefringent optical system is calculated as

$$T = \frac{|E'_x|^2 + |E'_y|^2}{|E_x|^2 + |E_y|^2}. \tag{5.2-4}$$

Intercala de

5.2.1. Example: A Birefringent Plate Sandwiched between Parallel Polarizers

Referring to Fig. 5.4, we consider a birefringent plate sandwiched between a pair of parallel polarizers. The plate is oriented so that the "slow" and "fast" axes are at 45° with respect to the polarizer. Let the birefringence be $n_e - n_o$ and the plate thickness be d. The phase retardation is then given by

$$\Gamma = 2\pi(n_e - n_o)\frac{d}{\lambda}, \tag{5.2-5}$$

and the corresponding Jones matrix is, according to Eq. (5.1-10),

$$W = \begin{pmatrix} \cos\frac{1}{2}\Gamma & -i\sin\frac{1}{2}\Gamma \\ -i\sin\frac{1}{2}\Gamma & \cos\frac{1}{2}\Gamma \end{pmatrix}. \tag{5.2-6}$$

Let the incident beam be unpolarized, so that after it passes through the front polarizer, the electric field vector can be represented by the following

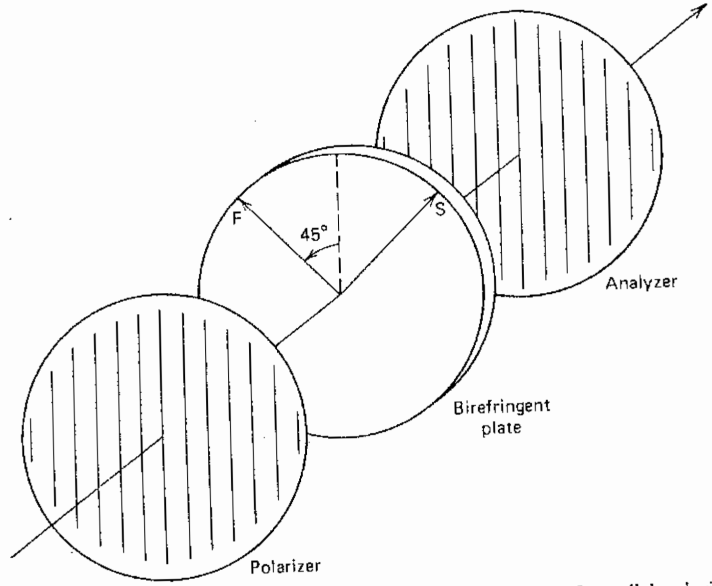


Figure 5.4. A birefringent plate sandwiched between a pair of parallel polarizers.

Jones vector:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5.2-7}$$

where we assume that the intensity of the incident beam is unity and only half of the intensity passes through the polarizer. The Jones vector representation of the electric field vector of the transmitted beam is obtained as follows:

$$\mathbf{E}' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\frac{1}{2}\Gamma & -i\sin\frac{1}{2}\Gamma \\ -i\sin\frac{1}{2}\Gamma & \cos\frac{1}{2}\Gamma \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \cos\frac{1}{2}\Gamma \end{pmatrix}. \tag{5.2-8}$$

The transmitted beam is vertically (y) polarized with an intensity given by

$$I = \tfrac{1}{2}\cos^2\tfrac{1}{2}\Gamma = \tfrac{1}{2}\cos^2\left[\frac{\pi(n_e - n_o)d}{\lambda}\right]. \tag{5.2-9}$$

It can be seen from Eq. (5.2-9) that the transmitted intensity is a sinusoidal function of the wave number and peaks at $\lambda = (n_e - n_o)d$, $(n_e - n_o)d/2$, $(n_e - n_o)d/3, \ldots$. The wave-number separation between transmission maxima increases with decreasing plate thickness.

5.2.2. Example: A Birefringent Plate Sandwiched between a Pair of Crossed Polarizers

If we rotate the analyzer shown in Fig. 5.4 by 90°, then the input and output polarizers are crossed. The transmitted beam for this case is obtained as follows:

$$E' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{1}{2}\Gamma & -i\sin\frac{1}{2}\Gamma \\ -i\sin\frac{1}{2}\Gamma & \cos\frac{1}{2}\Gamma \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{-i}{\sqrt{2}} \begin{pmatrix} \sin\frac{1}{2}\Gamma \\ 0 \end{pmatrix}. \tag{5.2-10}$$

The transmitted beam is horizontally (x) polarized with an intensity given by

$$I = \tfrac{1}{2}\sin^2\tfrac{1}{2}\Gamma = \tfrac{1}{2}\sin^2\left[\frac{\pi(n_e - n_o)d}{\lambda}\right]. \tag{5.2-11}$$

This is again a sinusoidal function of the wave number. The transmission spectrum consists of a series of maxima at $\lambda = 2(n_e - n_o)d$, $2(n_e - n_o)d/3, \ldots$. These wavelengths correspond to phase retardations of π, 3π, $5\pi, \ldots$, that is, when the wave plate becomes a half-wave plate or odd integral multiple of a half-wave plate.

5.3. POLARIZATION INTERFERENCE FILTERS

Spectral filters can be based on the interference of polarized light. These filters play an important role in many optical systems where filters of extremely narrow bandwidth with wide angular fields or tuning capability are required. In solar physics, for example, the distribution of hydrogen may be measured by photographing the solar corona in the light of the H_α ($\lambda = 6563$ Å) line. In view of the large amount of light present at neighboring wavelengths, a filter of extremely narrow bandwidth (~ 1 Å) is required if reasonable discrimination is to be attained. Such filters consist of birefringent crystal plates (wave plates) and polarizers. The two basic versions of these birefringent filters are Lyot–Öhman filters [2–5, 12] and Šolc filters [6, 7]. They are based on the interference of polarized light, which requires a phase retardation between the components of the light polarized parallel to the fast and slow axes of the crystal when radiation passes through it. Since the phase retardation introduced by a waveplate is proportional to the birefringence of the crystal, it is desirable to have crystals with large birefringence ($n_e - n_o$) for filter construction. Currently, the most commonly used materials are quartz, calcite, and $(NH_4)H_2PO_4$ (ADP).

The basic principle of the Lyot–Öhman filter is very simple and is outlined in Problem 5.5.

The Jones calculus developed in the previous two sections will now be applied to study the transmission characteristics of the Šolc filters. Šolc filters, named after their inventor [6, 7], are composed of a pile of identical birefringent plates each oriented at a prescribed azimuth angle. The azimuth angle of each plate is measured from the transmission axis of the front polarizer. The whole stack of birefringent plates is placed between a pair of polarizers.

Table 5.1. The Folded Šolc Filter

Element	Azimuth
Front polarizer	0°
Plate 1	ρ
Plate 2	$-\rho$
Plate 3	ρ
⋮	⋮
Plate N	$(-1)^{N-1}\rho$
Rear polarizer	90°

5.3.1. Folded Šolc Filters

There are two basic types of Šolc filters: folded and fan filters. The folded Šolc filter works between crossed polarizers. The azimuth angles of the individual plates are prescribed in Table 5.1. The geometrical arrangement of a six-plate Šolc filter is sketched in Fig. 5-5. As described in Table 5.1, the front polarizer has its transmission axis parallel to the x axis; and the rear polarizer parallel to the y-axis. The overall Jones matrix for these N plates is

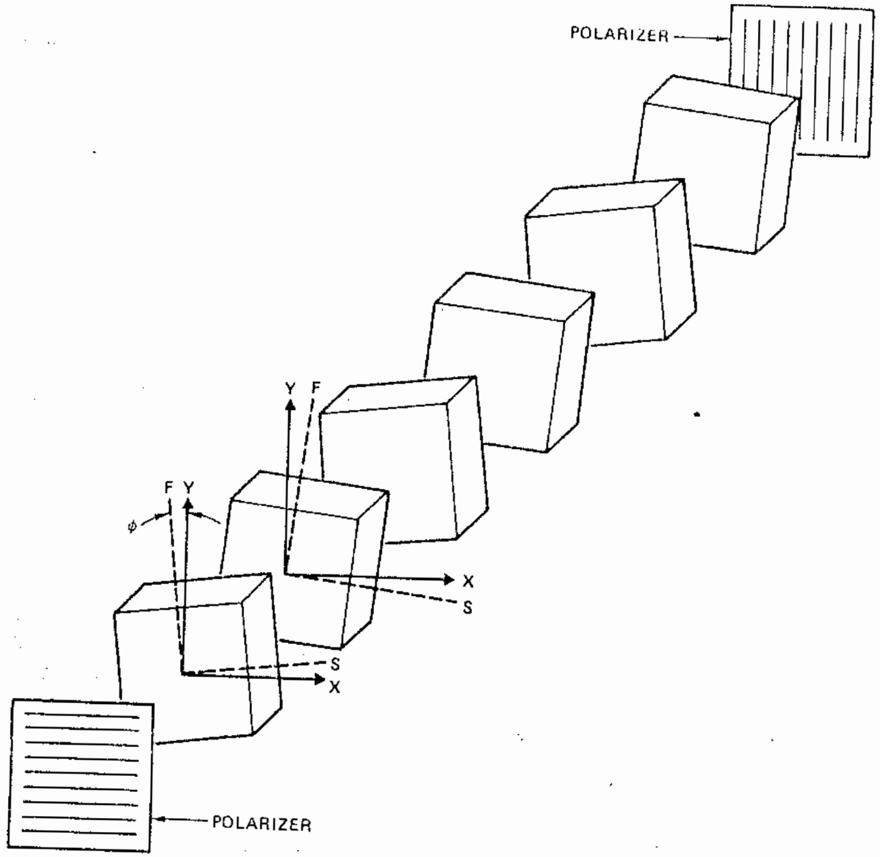


Figure 5.5. A six-stage folded Šolc filter.

given by

$$M = \left[R(\rho)W_0R(-\rho)R(-\rho)W_0R(\rho)\right]^m, \tag{5.3-1}$$

where we assume that the number of plates is an even number, $N = 2m$. Substituting Eq. (5.1-9) into Eq. (5.3-1) and carrying out the matrix multiplication results in

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^m, \tag{5.3-2}$$

with

$$\begin{aligned} A &= \left(\cos\tfrac{1}{2}\Gamma - i\cos 2\rho \sin\tfrac{1}{2}\Gamma\right)^2 + \sin^2 2\rho \sin^2\tfrac{1}{2}\Gamma, \\ B &= \sin 4\rho \sin^2\tfrac{1}{2}\Gamma, \\ C &= -B, \\ D &= \left(\cos\tfrac{1}{2}\Gamma + i\cos 2\rho \sin\tfrac{1}{2}\Gamma\right)^2 + \sin^2 2\rho \sin^2\tfrac{1}{2}\Gamma, \end{aligned} \tag{5.3-3}$$

where Γ is the phase retardation of each plate. Note that this matrix is unimodular (i.e., $AD - BC = 1$), since all the matrices in Eq. (5.3-1) are unimodular. Equation (5.3-2) can be simplified by using Chebyshev's identity [9] to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^m = \begin{pmatrix} \dfrac{A \sin mK\Lambda - \sin(m-1)K\Lambda}{\sin K\Lambda} & B\dfrac{\sin mK\Lambda}{\sin k\Lambda} \\ C\dfrac{\sin mK\Lambda}{\sin K\Lambda} & \dfrac{D \sin mK\Lambda - \sin(m-1)K\Lambda}{\sin K\Lambda} \end{pmatrix} \tag{5.3-4}$$

with

$$K\Lambda = \cos^{-1}\left[\tfrac{1}{2}(A + D)\right]. \tag{5.3-5}$$

We use the notation $K\Lambda$ for the purpose of comparing this result with that obtained from coupled-mode theory (see Section 6.4).

The incident wave and the emerging wave are related by

$$\begin{pmatrix} E_x' \\ E_y' \end{pmatrix} = P_y M P_x \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{5.3-6}$$

The emerging beam is polarized in the y direction, with a field amplitude given by

$$E_y' = M_{21} E_x \tag{5.3-7}$$

If the incident light is linearly polarized in the x direction, the transmissivity of this filter is

$$T = |M_{21}|^2. \tag{5.3-8}$$

From Eqs. (5.3-2) and (5.3-3), we obtain

$$T = \left| \sin 4\rho \sin^2 \tfrac{1}{2}\Gamma \frac{\sin mK\Lambda}{\sin K\Lambda} \right|^2 \tag{5.3-9}$$

with

$$\cos K\Lambda = 1 - 2\cos^2 2\rho \sin^2 \tfrac{1}{2}\Gamma. \tag{5.3-10}$$

The transmissivity T is often expressed in terms of a new variable χ, defined by

$$K\Lambda = \pi - 2\chi. \tag{5.3-11}$$

In terms of this new variable χ, the transmissivity is

$$T = \left| \tan 2\rho \cos \chi \frac{\sin N\chi}{\sin \chi} \right|^2 \tag{5.3-12}$$

with

$$\cos \chi = \cos 2\rho \sin \tfrac{1}{2}\Gamma. \tag{5.3-13}$$

According to Eqs. (5.3-12) and (5.3-13), transmissivity becomes $T = \sin^2 2N\rho$ when the phase retardation of each plate is $\Gamma = \pi, 3\pi, 5\pi, \ldots$, that is, when each plate becomes a half-wave plate. This transmissivity is 100% if the azimuth angle ρ is such that

$$\rho = \frac{\pi}{4N}. \tag{5.3-14}$$

The transmission under these conditions can be easily understood if we examine the polarization state after passing through each plate within the Šolc filter. We recall that in passing through a half-wave ($\Gamma = \pi, 3\pi, \ldots$) plate, the azimuthal angle between the polarization vector and the fast (or slow) axis of the crystal changes sign. Past the front polarizer, the light is linearly polarized in the x direction (azimuth $\psi = 0$). Since the first plate is at the azimuth angle ρ, the emerging beam after passage through the first plate is linearly polarized at $\psi = 2\rho$. The second plate is oriented at the

azimuth angle $-\rho$, making an angle of 3ρ with respect to the polarization direction of light incident on it. The polarization direction at its output face will be rotated by 6ρ and oriented at the azimuth angle -4ρ (see Fig. 5.6). The plates are oriented successively at $+\rho, -\rho, +\rho, -\rho, \ldots$, while the polarization direction at the exit of the plates assumes the values $2\rho, -4\rho, 6\rho, -8\rho, \ldots$. The final azimuthal angle after N plates is thus $2N\rho$. If this final azimuth angle is 90° (i.e., $2N\rho = \frac{1}{2}\pi$), then the light passes through the rear polarizer without any loss of intensity. Light at other wavelengths, where the plates are not half-wave plates, does not experience a 90° rotation of polarization and suffers loss at the rear polarizer.

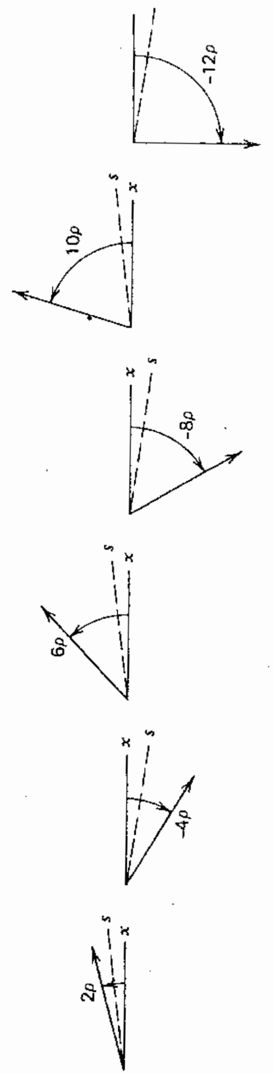
The Šolc filter can also be viewed as a periodic medium as far as wave propagation is concerned. The alternating azimuth angles of the crystal axes constitute a periodic perturbation to the propagation of both eigenwaves. This perturbation couples the fast and slow eigenwaves. Because these waves propagate at different phase velocities, complete exchange of electromagnetic energy is possible only when the perturbation is periodic so as to maintain the relations necessary to cause continuous power transfer from the fast to the slow wave and vice versa. This is the first manifestation of the principle of phase matching by a periodic perturbation, to which we will return in subsequent chapters. The basic physical explanation is as follows: If power is to be transferred gradually with distance from mode A to mode B by a static perturbation, then it is necessary that both waves travel with the same phase velocity. If the phase velocities are not equal, the incident wave A gets progressively out of phase with the wave B into which it couples. This limits the total fraction of power that can be exchanged. This situation can be avoided if the sign of the perturbation is reversed whenever the phase mismatch (between the coupled field and the field into which it couples) is equal to π. This reverses the sign of the coupled power, and thus maintains the proper phase for continuous power transfer. A coupled mode theory of the folded Šolc filter is given in Section 6.5.

The transmission characteristics around the peak and its sidelobes in a Šolc filter are interesting and deserve some investigation. Assume that each plate is characterized by refractive indices n_e and n_o and thickness d. Let λ_ν denote the wavelength at which the phase retardation is $(2\nu + 1)\pi$. The phase retardation at a general wavelength is

$$\Gamma = \frac{2\pi}{\lambda}(n_e - n_o)d. \tag{5.3-15}$$

If λ is slightly away from λ_ν [i.e., $(\lambda - \lambda_\nu) \ll \lambda_\nu$], Γ can be approximately given by

$$\Gamma = (2\nu + 1)\pi + \Delta\Gamma$$



Birefringent plate	1	2	3	4	5	6
Azimuth angle	7.5°	−7.5°	7.5°	−7.5°	7.5°	−7.5°
Polarization azimuth	15°	−30°	45°	−60°	75°	−90°

Figure 5.6. Polarization state within a folded six-stage Šolc filter. The dashed lines indicate the directions of the slow axis. The arrows indicate the direction of polarization.

where

$$\Delta\Gamma = -\frac{(2\nu+1)\pi}{\lambda_\nu}(\lambda - \lambda_\nu). \tag{5.3-16}$$

We assume further that the azimuth angle of the plate obeys the condition (5.3-14), and N is much larger than 1. Under these conditions, the trigonometric functions in Eq. (5.3-13) can be expanded to yield

$$\chi \simeq \frac{\pi}{2N}\left[1 + \left(\frac{N\Delta\Gamma}{\pi}\right)^2\right]^{1/2}. \tag{5.3-17}$$

Substituting χ into Eq. (5.3-12), we obtain

$$T = \left(\frac{\sin\left(\frac{1}{2}\pi\sqrt{1 + (N\Delta\Gamma/\pi)^2}\right)}{\sqrt{1 + (N\Delta\Gamma/\pi)^2}}\right)^2. \tag{5.3-18}$$

This approximate expression for the transmissivity is valid provided $N \gg 1$ and $(\lambda - \lambda_\nu) \ll \lambda_\nu$. From (5.3-18), the full bandwidth at half maximum (FWHM) of the main transmission peak is given approximately by $\Delta\Gamma_{1/2} = 1.60\pi/N$, which in terms of the wavelength is

$$\Delta\lambda_{1/2} \simeq 1.60\left[\frac{\lambda_\nu}{(2\nu+1)N}\right]. \tag{5.3-19}$$

Thus, to build a narrowband Šolc filter with a bandwidth of 1 Å to observe the H_α line ($\lambda_0 = 6563$ Å) requires the number of half-wave ($\nu = 0$) plates be approximately 10^4. The transmission spectrum consists of a main peak at λ_0 and a series of sidelobes around it. According to Eq. (5.3-18), these secondary maxima occur approximately at

$$\sqrt{1 + \left(\frac{N\Delta\Gamma}{\pi}\right)^2} \simeq 2l + 1, \qquad l = 1, 2, 3, \ldots, \tag{5.3-20}$$

with transmissivity given by

$$T \simeq \frac{1}{(2l+1)^2}. \tag{5.3-21}$$

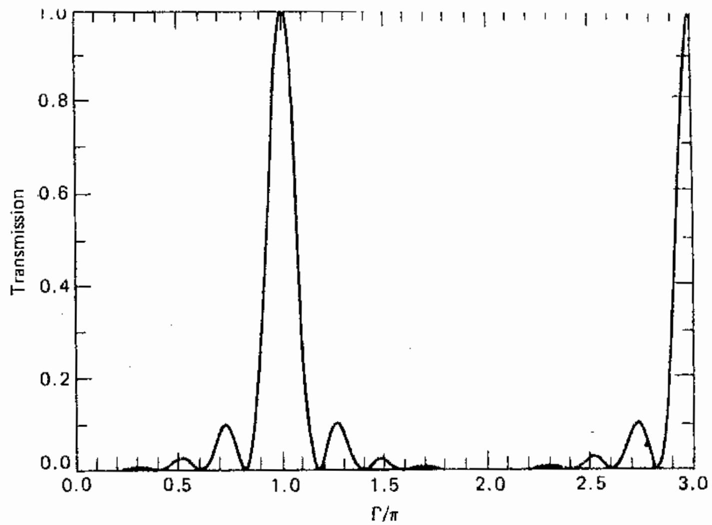


Figure 5.7. A calculated transmission spectrum of a folded Šolc filter.

A calculated transmission spectrum is shown in Fig. 5.7. Notice that the bandwidth is inversely proportional to the total number of plates.

5.3.2. Fan Šolc Filters

A fan Šolc filter also consists of a stack of identical birefringent plates, each oriented at a prescribed azimuth. A brief description of the basic type of fan Šolc filter is given in Table 5.2. The geometrical arrangement is sketched in

Table 5.2. The Fan Šolc Filter

Element	Azimuth
Front polarizer	0°
Plate 1	ρ
Plate 2	3ρ
Plate 3	5ρ
⋮	⋮
Plate N	$(2N-1)\rho = \frac{1}{2}\pi - \rho$
Rear polarizer	0°

Fig. 5.8. According to the Jones matrix method formulated in the previous section, the overall matrix for these N plates is given by

$$M = R(-\tfrac{1}{2}\pi + \rho)W_0 R(\tfrac{1}{2}\pi - \rho)\cdots$$
$$\times R(-5\rho)W_0 R(5\rho)R(-3\rho)W_0 R(3\rho)R(-\rho)W_0 R(\rho)$$
$$= R(-\tfrac{1}{2}\pi + \rho)[W_0 R(2\rho)]^N R(-\rho), \qquad (5.3\text{-}22)$$

where we have used the following identity for the rotation matrix:

$$R(\rho_1)R(\rho_2) = R(\rho_1 + \rho_2). \qquad (5.3\text{-}23)$$

Notice that the last plate always appears first in the product (5.3-22).

By using the Chebyshev identity (5.3-4) and carrying out the matrix multiplication in Eq. (5.3-22), we obtain

$$M_{11} = \sin 2\rho \cos\tfrac{1}{2}\Gamma \frac{\sin N\chi}{\sin\chi}$$
$$M_{12} = -\cos N\chi - i\sin\tfrac{1}{2}\Gamma \frac{\sin N\chi}{\sin\chi}$$
$$M_{21} = \cos N\chi - i\sin\tfrac{1}{2}\Gamma \frac{\sin N\chi}{\sin\chi} \qquad (5.3\text{-}24)$$
$$M_{22} = M_{11}$$

with

$$\cos\chi = \cos 2\rho \cos\tfrac{1}{2}\Gamma. \qquad (5.3\text{-}25)$$

These are the elements of the overall Jones matrix not including the polarizers.

The incident wave **E** and the emerging wave **E**′ are thus related by

$$\begin{pmatrix} E_x' \\ E_y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix}. \qquad (5.3\text{-}26)$$

The emerging beam is horizontally polarized (x) with amplitude given by

$$E_x' = M_{11}E_x. \qquad (5.3\text{-}27)$$

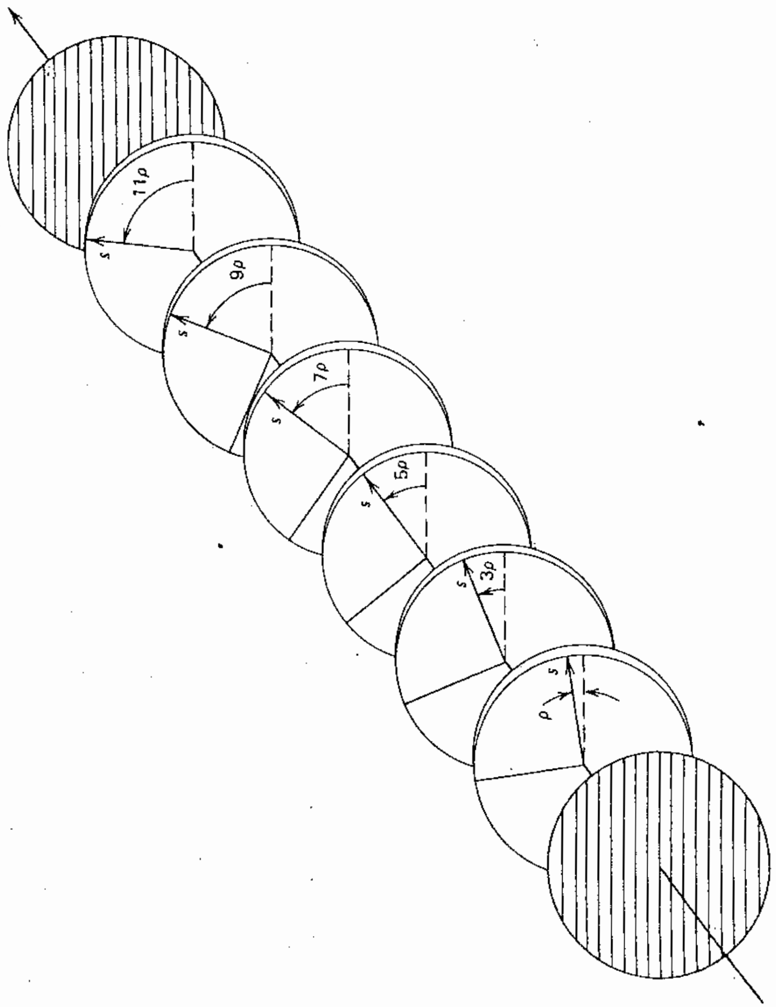


Figure 5.8. A six-stage fan Šolc filter.

If the incident wave is linearly polarized in the x direction, the transmissivity is given by

$$T = |M_{11}|^2. \tag{5.3-28}$$

From Eq. (5.3-24), we have the following expression for the transmissivity:

$$T = \left|\tan 2\rho \cos\chi \frac{\sin N\chi}{\sin\chi}\right|^2 \tag{5.3-29}$$

with

$$\cos\chi = \cos 2\rho \cos\tfrac{1}{2}\Gamma. \tag{5.3-30}$$

Notice that the transmission formula (5.3-29) is formally identical to Eq. (5.3-12). Maximum transmissivity ($T = 1$) occurs when $\Gamma = 0, 2\pi, 4\pi, \ldots,$ and $\rho = \pi/4N$. This unity transmission results simply from the fact that at these wavelengths the plates are full-wave plates. The light will remain linearly polarized in the x direction after going through each plate and will suffer no loss at the rear polarizer. Light at other wavelengths, where the plates are not full-wave plates, does not remain linearly polarized in the x direction and suffers loss at the rear polarizer. Let λ_ν be the wavelength at which the phase retardation $\Gamma = 2\nu\pi$. If λ differs slightly from λ_ν (i.e., $\lambda - \lambda_\nu \ll \lambda_\nu$), Γ is given approximately by

$$\Gamma = 2\nu\pi + \Delta\Gamma = 2\nu\pi - \frac{2\nu\pi}{\lambda_\nu}(\lambda - \lambda_\nu), \tag{5.3-31}$$

where $\nu = 1, 2, 3, \ldots$. The case $\nu = 0$ occurs only when the birefringence vanishes at some particular wavelength λ_0. This special case has some important applications in wide-angle narrowband filters and deserves special attention (see Problem 5.8). If we now further assume that N is much larger than 1 and follow the same procedure as in Eq. (5.3-17), we obtain the following approximate expression for the transmissivity:

$$T = \left(\frac{\sin\frac{1}{2}\pi\sqrt{1 + (N\Delta\Gamma/\pi)^2}}{\sqrt{1 + (N\Delta\Gamma/\pi)^2}}\right)^2, \tag{5.3-32}$$

which is identical to Eq. (5.3-18). The FWHM $\Delta\lambda_{1/2}$ of the transmission maxima is again given approximately by

$$\Delta\lambda_{1/2} \simeq 1.60\frac{\lambda_\nu}{2\nu N}, \qquad \nu = 1, 2, 3, \ldots. \tag{5.3-33}$$

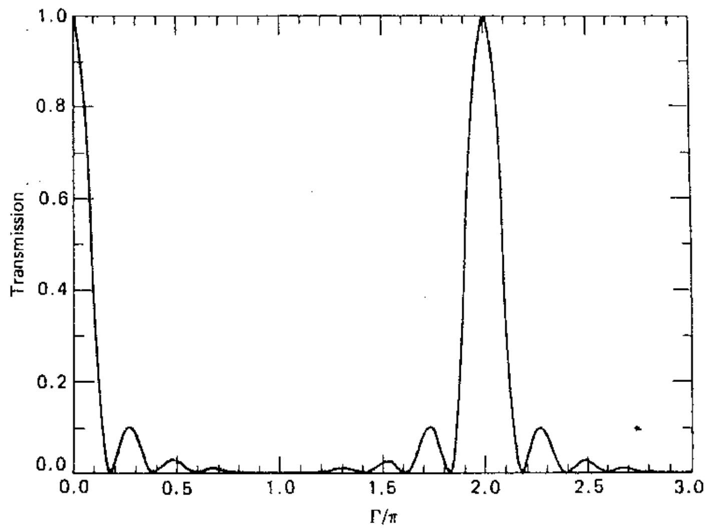


Figure 5.9. A calculated transmission spectrum of a fan Šolc filter.

The transmission spectrum of the fan Šolc filter is identical to that of the folded Šolc filter except that the curves are shifted by $\Gamma = \pi$. In other words, the transmissivity of the fan Šolc filter at a phase retardation Γ is identical to that of the folded Šolc filter at phase retardation $\Gamma + \pi$. This can also be seen from the expression for the transmissivity in Eqs. (5.3-12) and (5.3-29). A calculated transmission spectrum of a fan Šolc filter is shown in Fig. 5.9.

Šolc filters play an important role in many modern optical devices such as electro-optic tunable filters [10, 11] and wide-field-of-view narrowband filters. More information on Šolc filters can be found in [12].

5.4. LIGHT PROPAGATION IN TWISTED ANISOTROPIC MEDIA

In this section the propagation of electromagnetic radiation through a slowly twisting anisotropic medium is described by the Jones calculus. The transmission of light through a twisted nematic liquid crystal is a typical example. This situation is similar to a fan-type Šolc filter structure with the number of plates, N, tending to infinity and the plate thickness tending to zero as $1/N$. In fact, we are going to subdivide the twisted anisotropic medium into N plates and assume that each plate is a wave plate with a

phase retardation and an azimuth angle. The overall Jones matrix can then be obtained by multiplying together all the matrices associated with these plates.

We will limit ourselves to the case when the twisting is linear and the azimuth angle of the axes is

$$\psi(z) = \alpha z, \tag{5.4-1}$$

where z is the distance in the direction of propagation and α is a constant.

Let Γ be the phase retardation of the plate when it is untwisted. In particular, for the case of nematic liquid crystal with c axis parallel to the plate surfaces, Γ is given by

$$\Gamma = \frac{2\pi}{\lambda}(n_e - n_o)l, \tag{5.4-2}$$

where l is the thickness of the plate. The total twist angle is

$$\phi \equiv \psi(l) = \alpha l. \tag{5.4-3}$$

To derive the Jones matrix for such a structure, we need to divide this plate into N equally thick plates. Each plate has a phase retardation of Γ/N. The plates are oriented at azimuth angles $\rho, 2\rho, 3\rho, \ldots, (N-1)\rho, N\rho$ with $\rho = \phi/N$. The overall Jones matrix for these N plates is given by

$$M = \prod_{m=1}^{N} R(m\rho) W_0 R(-m\rho). \tag{5.4-4}$$

It is important to remember that in the above matrix product, $m = 1$ appears at the right-hand end. Following the procedure (5.3-23), this matrix can be written

$$M = R(\phi)\left[W_0 R\left(-\frac{\phi}{N}\right)\right]^N, \tag{5.4-5}$$

where

$$W_0 = \begin{pmatrix} e^{-i\Gamma/2N} & 0 \\ 0 & e^{i\Gamma/2N} \end{pmatrix}. \tag{5.4-6}$$

Using Eqs. (5.1-9a) and (5.4-6), we obtain

$$M = R(\phi)\begin{pmatrix} \cos\frac{\phi}{N} e^{-i\Gamma/2N} & -\sin\frac{\phi}{N} e^{-i\Gamma/2N} \\ \sin\frac{\phi}{N} e^{i\Gamma/2N} & \cos\frac{\phi}{N} e^{i\Gamma/2N} \end{pmatrix}^N. \tag{5.4-7}$$

Equation (5.4-7) can be further simplified by using Chebyshev's identity (5.3-4). In the limit when N tends to infinite ($N \to \infty$), the result is given by (see Problem 5.10)

$$M = R(\phi)\begin{pmatrix} \cos X - i\dfrac{\Gamma}{2}\dfrac{\sin X}{X} & -\phi\dfrac{\sin X}{X} \\ \phi\dfrac{\sin X}{X} & \cos X + i\dfrac{\Gamma}{2}\dfrac{\sin X}{X} \end{pmatrix} \tag{5.4-8}$$

where

$$X = \sqrt{\phi^2 + \left(\frac{\Gamma}{2}\right)^2}. \tag{5.4-9}$$

Here we have an exact expression for the Jones matrix of a linearly twisted anisotropic plate.

Let $\mathbf{V}$ be the initial polarization state, the polarization state $\mathbf{V}'$ after exiting the plate can be written

$$\mathbf{V}' = M\mathbf{V}. \tag{5.4-10}$$

5.4.1. Adiabatic Following

If often happens, especially in twisted nematic liquid crystals, that the phase retardation Γ is much larger than the twist angle ϕ. For example, consider a liquid-crystal layer 25 μm thick with a twist angle of $\frac{1}{2}\pi$. The birefringence of liquid crystal is typically $n_e - n_o = 0.1$. For wavelength $\lambda = 0.5\ \mu$m, we have $\Gamma/\phi = 20$. This number can be even bigger if the layer is thicker. If we assume $\Gamma \gg \phi$, the overall Jones matrix (5.4-8) becomes

$$M = R(\phi)\begin{pmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{i\Gamma/2} \end{pmatrix}. \tag{5.4-11}$$

If the incident light is linearly polarized along either the slow or the fast axis at the entrance plane, then according to Eq. (5.4-11), the light will remain linearly polarized along the "local" slow or fast axis. In a sense, the polarization vector follows the rotation of the local axes, provided the polarization vector is along one of the axes. The operation of the Jones matrix on any polarization vector can be divided into two parts. First, a phase-retardation matrix operates on the Jones vector of the incident wave. For light linearly polarized along one of the principal axes, this matrix only

gives a phase shift to the light beam and leaves the polarization state unchanged. Second, the operation of $R(\phi)$ is to rotate the Jones vector by an angle ϕ. For linearly polarized light, this rotation makes the polarization vector parallel to the principal axes at the exit face of the plate. Thus, if the incident light is polarized along the direction of the normal modes at the input plane ($z = 0$), the polarization vector of the light wave will follow the rotation of principal axes and remain parallel to the local slow (or fast) axis provided the twist rate is small. This is called adiabatic following. This phenomenon has an important application in liquid-crystal light valves. The principle of operation of these light valves is discussed below.

We consider the case of twisted nematic liquid crystal with a quarter turn ($\phi = \frac{1}{2}\pi$). If this layer is placed between a pair of parallel polarizers with their transmission axes (x) parallel to the c axis of the liquid crystal at the entrance plane ($z = 0$), the Jones vector representation of the wave immediately after passage through the first polarizer can be written

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.4-12}$$

The rotation matrix $R(\phi)$ in Eq. (5.4-8) can be written

$$R(\tfrac{1}{2}\pi) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.4-13}$$

After passing through the liquid crystal, the polarization state of the beam, according to Eqs. (5.4-8), (5.4-12), and (5.4-13), becomes

$$\mathbf{V}' = \begin{pmatrix} \phi\dfrac{\sin X}{X} \\ \cos X - i\dfrac{\Gamma}{2}\dfrac{\sin X}{X} \end{pmatrix}. \tag{5.4-14}$$

The y component will be blocked by the second polarizer. The transmissivity of the whole structure is thus given by

$$T = \frac{\sin^2\left(\frac{1}{2}\pi\sqrt{1 + (\Gamma/\pi)^2}\right)}{1 + (\Gamma/\pi)^2}, \tag{5.4-15}$$

where we have used Eq. (5.4-9) for X and $\phi = \frac{1}{2}\pi$.

For a crystal layer with thickness large enough, the phase retardation is much larger than π (i.e., $\Gamma \gg \pi$). The transmissivity is virtually zero, according to Eq. (5.4-15). This is a result of the adiabatic following. The polarization vector follows the rotation of the axes and therefore is rotated by an angle $\phi = \frac{1}{2}\pi$ equal to the twist angle. Since this direction is orthogonal to the transmission axis of the analyzer, the transmission is zero.

In many twisted liquid crystals, the c axis can be forced to align along a given direction by the application of an electric field (or stress) (see Section 7.7). The application of an electric field along the z direction will destroy this twisted structure (see Section 7.7). This leads to $\Gamma = 0$ and to total transmission of light, according to Eq. (5.4-15). When the electric field is removed, the liquid crystal recovers its twisted structure and the light is extinguished. This is the basic principal of liquid-crystal light valves.

5.5. THE PROBLEM OF FRESNEL REFLECTION AND PHASE SHIFT

In the Jones-calculus formulation, the reflections of light from the surface of the wave plate are neglected. These reflections will normally decrease the throughput of electromagnetic energy. However, if the plate surfaces are optically flat, the interference effect can result in a decrease or increase in the transmissivity, depending on the optical path lengths.

The exact approaches derived from the electromagnetic theory involve the use of a 4×4-matrix method [13–15]. The light wave of a beam is represented mathematically by a column vector that consists of the complex amplitudes of the incident wave and the reflected wave. Each wave has a slow and a fast component. (The details of this method are beyond the scope of this book; interested readers are referred to the cited references.)

This exact approach gives the exact transmission spectrum of the birefringent filter, because the interference effects due to the reflected waves are accounted for. The practical calculation of the transmission requires a computer. The calculated transmission spectrum of a Šolc Filter (see Fig. 5.10) shows fine structure due to interference. This fine structure should be experimentally observable if the birefringent plates are thin and optically flat. This approach also predicts the existence of a superfine structure that comes from the Fabry–Perot interference fringes of the whole stack of crystals rather than those of the individual plates; however, this superfine structure is beyond the resolution limit of the plotter at the scale shown in Fig. 5.10. The figure also compares the Jones-calculus method, which neglects reflected waves, and the exact 4×4-matrix method.

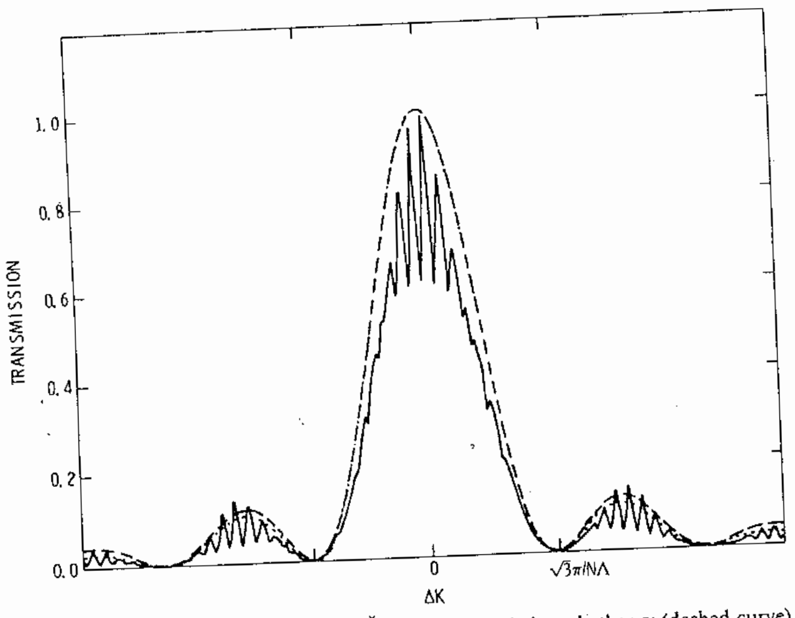


Figure 5.10. Transmission spectrum of a Šolc filter: coupled-mode theory (dashed curve) and Jones calculus (dotted curve) with the exact 4×4-matrix method (solid curve).

PROBLEMS

5.1. *Half-wave plate.* A half-wave plate has a phase retardation of $\Gamma = \pi$. Assume that the plate is oriented so that the azimuth angle (i.e., the angle between the x axis and the slow axis of the plate) is ψ.

(a) Find the polarization state of the transmitted beam, assuming that the incident beam is linearly polarized in the y direction.

(b) Show that a half-wave plate will convert right-hand circularly polarized light into left-hand circularly polarized light, and vice versa, regardless of the azimuth angle of the plate.

(c) Lithium tantalate ($LiTaO_3$) is a uniaxial crystal with $n_o = 2.1391$ and $n_e = 2.1432$ at $\lambda = 1\ \mu m$. Find the half-wave-plate thickness at this wavelength, assuming the plate is cut in such a way that the surfaces are perpendicular to the x axis of the principal coordinate (i.e., x-cut).

5.2. *Quarter-wave plate.* A quarter-wave plate has a phase retardation of $\Gamma = \frac{1}{2}\pi$. Assume that the plate is oriented in a direction with azimuth angle ψ.

(a) Find the polarization state of the transmitted beam, assuming that the incident beam is polarized in the y direction.

(b) If the polarization state resulting from (a) is represented by a complex number on the complex plane, show that the locus of these points as ψ varies from 0 to $\frac{1}{2}\pi$ is a branch of a hyperbola. Obtain the equation of the hyperbola.

(c) Quartz (α-SiO_2) is a uniaxial crystal with $n_o = 1.53283$ and $n_e = 1.54152$ at $\lambda = 1.1592\ \mu$m. Find the thickness of an x-cut quartz quarter-wave plate at this wavelength.

5.3. *Polarization transformation by a wave plate.* A wave plate is characterized by its phase retardation Γ and azimuth angle ψ.

(a) Find the polarization state of the emerging beam, assuming that the incident beam is polarized in the x direction.

(b) Use a complex number to represent the resulting polarization state obtained in (a).

(c) The polarization state of the incident x-polarized beam is represented by a point at the origin of the complex plane. Show that the transformed polarization state can be anywhere on the complex plane, provided Γ can be varied from 0 to 2π and ψ can be varied from 0 to $\frac{1}{2}\pi$. Physically, this means that any polarization state can be produced from linearly polarized light, provided a proper wave plate is available.

(d) Show that the locus of these points in the complex plane obtained by rotating a wave plate from $\psi = 0$ to $\psi = \frac{1}{2}\pi$ is a hyperbola. Derive the equation of this hyperbola.

(e) Show that the Jones matrix W of a wave plate is unitary, that is,

$$W^{\dagger}W = 1,$$

where the dagger indicates Hermitian conjugation.

(f) Let $\mathbf{V}_1'$ and $\mathbf{V}_2'$ be the transformed Jones vectors from $\mathbf{V}_1$ and $\mathbf{V}_2$, respectively. Show that if $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal, so are $\mathbf{V}_1'$ and $\mathbf{V}_2'$.

5.4. *Polarizers and projection operators.* An ideal polarizer can be considered as a projection operator which acts on the incident polarization state and projects the polarization vector along the transmission axis of the polarizer.

(a) If we neglect the absolute phase factor in Eq. (5.1-11), show that

$$P_0^2 = P_0 \quad \text{and} \quad P^2 = P.$$

Operators satisfying these conditions are called projection operators in linear algebra.

(b) Show that if $\mathbf{E}_1$ is the amplitude of the relative field of the electric field, the amplitude of the beam after it passes through the polarizer is given by

$$\mathbf{p}(\mathbf{p} \cdot \mathbf{E}_1),$$

where $\mathbf{p}$ is the unit vector along the transmission axis of the polarizer.

(c) If the incident beam is vertically polarized (i.e., $\mathbf{E}_1 = \hat{\mathbf{y}}E_0$), the polarizer transmission axis is in the x direction (i.e., $\mathbf{p} = \hat{\mathbf{x}}$). The transmitted beam has zero amplitude, since $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$. However, if a second polarizer is placed in front of the first polarizer and is oriented at 45° with respect to it, the transmitted amplitude is not zero. Find this amplitude.

(d) Consider a series of N polarizers with the first one oriented at $\psi_1 = \pi/2N$, the second one at $\psi_2 = 2(\pi/2N)$, the third one at $\psi_3 = 3(\pi/2N), \ldots,$ and the nth one at $N \cdot (\pi/2N)$. Let the incident beam be horizontally polarized. Show that the transmitted beam is vertically polarized with an amplitude of

$$\left[\cos\left(\frac{\pi}{2N}\right)\right]^N.$$

Evaluate the amplitude for $N = 1, 2, 3, \ldots, 10$. Show that in the limit of $N \to \infty$, the amplitude becomes one. In other words, a series of polarizers oriented like a fan can rotate the polarization of the light without attenuation.

5.5. *Lyot–Öhman filter.* In solar physics, the distribution of hydrogen in the solar corona is measured by photographing at the wavelength of H_α line ($\lambda = 6563$ Å). To enhance the signal-to-noise ratio, a filter of extremely narrow bandwidth (~ 1 Å) is required. The polarization filter devised by Lyot and Öhman consists of a set of birefringent plates separated by parallel polarizers. The plate thicknesses are in geometric progression, that is, $d, 2d, 4d, 8d, \ldots$. All the plates are oriented at an azimuth angle of 45°.

(a) Show that if n_o and n_e are the refractive indices of the plates, the transmissivity of the whole stack of N plates is given by

$$T = \tfrac{1}{2}\cos^2 x \cos^2 2x \cos 4x \cdots \cos^2 2^{N-1}x$$

with

$$x = \frac{\pi d(n_e - n_o)}{\lambda} = \frac{\pi d(n_e - n_o)\nu}{c}.$$

(b) Show that the transmission can be written

$$T = \frac{1}{2}\left(\frac{\sin 2^N x}{2^N \sin x}\right)^2.$$

(c) Show that the transmission bandwidth (FWHM) of the whole system is governed by that of the bands of the thickest plate, that is,

$$\Delta\nu_{1/2} \sim \frac{c}{2^N d(n_e - n_o)}$$

and the free spectral range $\Delta\nu$ is governed by that of the bands of the thinnest plate, that is,

$$\Delta\nu \sim \frac{c}{d(n_e - n_o)}.$$

The finesse F of the system, defined as $\Delta\nu/\Delta\nu_{1/2}$, is then

$$F \sim 2^N.$$

(d) Design a filter with a bandwidth of 1 Å at the H_α line, using quartz as the birefringent material. Assume that $n_o = 1.5416$ and $n_e = 1.5506$ at $\lambda = 6563$ Å. Find the required thickness of the thickest plate.

(e) Show that according to (b) the bandwidth (FWHM) is given by

$$\Delta\nu_{1/2} = 0.886\frac{c}{2^N d(n_e - n_o)}.$$

5.6. *Off-axis effect.* A wave plate made of uniaxial crystal with its c axis parallel to the plate surfaces has a phase retardation of $2\pi(n_e - n_o)d/\lambda$ for a normal incident beam. For an off-axis beam, the light will "see" different birefringence because the refractive index for the extraordinary wave is dependent on the direction of the beam. In addition, the path length in the crystal plate is no longer d (see Fig. 5.11).

(a) Let θ_o, θ_e be the refractive angles for the ordinary and extraordinary waves, respectively. Show that the phase retardation is

$$\Gamma = \frac{2\pi}{\lambda}(n_e\cos\theta_e - n_o\cos\theta_o)d.$$

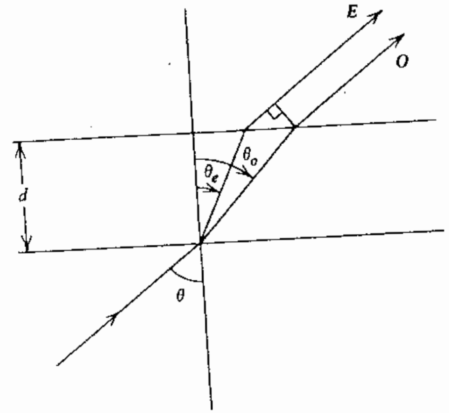


Figure 5.11. Off-axis rays.

(b) Let θ be the angle of incidence, and ϕ be the angle between the c axis and the tangential component of the wave vector. Show that the phase retardation can be expressed in terms of θ and ϕ as

$$\Gamma = \frac{2\pi}{\lambda}d\left[n_e\sqrt{1 - \frac{\sin^2\theta\sin^2\phi}{n_e^2} - \frac{\sin^2\theta\cos^2\phi}{n_o^2}} - n_o\sqrt{1 - \frac{\sin^2\theta}{n_o^2}}\right].$$

(c) Show that in the case when $\sin^2\theta$ is much smaller than n_o^2 and n_e^2, the phase retardation can be written

$$\Gamma = \frac{2\pi}{\lambda}(n_e - n_o)d\left\{1 + \sin^2\theta\left[\frac{\sin^2\phi}{2n_o n_e} - \frac{\cos^2\phi}{2n_o^2}\right]\right\}.$$

(d) According to the result in (c), a Lyot–Öhman filter with a passband of λ_0 at normal incidence will have a passband of $\lambda_0 \pm \Delta\lambda$ for off-axis light. Show that with $n_{eff}^2 = n_o^2 \simeq n_e^2$, $\Delta\lambda$ is given by

$$\Delta\lambda = \frac{\lambda\sin^2\theta}{2n_{eff}^2}.$$

(e) Show that a narrowband Lyot–Öhman filter has a limited aperture of

$$\theta = \pm n_o \left(\frac{\Delta\lambda_{1/2}}{\lambda} \right)^{1/2},$$

where we assume $n_e - n_o \ll n_o$.

5.7. *Šolc filters.* The transmissivity near a transmission peak can be written as

$$T = \frac{\sin^2 \frac{1}{2}\pi\sqrt{1 + x^2}}{1 + x^2}$$

with

$$x = N\Delta\Gamma/\pi.$$

(a) Show that $T = 0.5$ at $x = 0.8$, and derive the expression (5.3-19) for the FWHM.

(b) Find the wavelengths for which the transmissivity vanishes.

(c) Find the peak transmissivity of the sidelobes.

(d) Estimate the integrated transmissivity over all the sidelobes and compare it with the area under the main peak.

5.8. *Iso-index Lyot–Öhman filters.* Consider a Lyot–Öhman filter (Problem 5.5) made of iso-index crystals (i.e., crystals with $n_e = n_o$ at certain wavelength λ_c).

(a) Derive an expression for the bandwidth (FWHM) as a function of the plate thickness and the rate of change α of the birefringence.

(b) Derive an expression for the free spectral range as a function of α and the plate thickness d.

(c) Design an iso-index Lyot–Öhman filter using CdS at 5245 Å with a bandwidth of 0.1 Å and a free spectral range of at least 10 Å.

(d) Explain why the total thickness required for a given bandwidth $\Delta\lambda_{1/2}$ is usually smaller than in the conventional Lyot–Öhman filter made of quartz or calcite. Show that this is true only when α satisfies

$$\alpha > \left| \frac{n_e - n_o}{\lambda} \right|,$$

where n_e, n_o are the refractive indices of calcite or quartz.

5.9. *Field of view of iso-index filters.* Use the result obtained in Problem 5.6(c) to study the field of view of the iso-index Lyot–Öhman filter.

(a) Show that the passband is independent of the angle of incidence.

(b) Derive an expression for the bandwidth for off-axis light as a function of θ and ϕ.

(c) Show that for the most extreme angle of incidence, the bandwidth is increased or decreased by a factor of $(1 \pm 1/2n_o^2)$.

5.10. *Twisted nematic liquid crystals.*

(a) Use Chebyshev's identity (5.3-4) to simplify Eq. (5.4-7).

(b) Derive Eq. (5.4-8).

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