

# Index Notation for Vector Calculus

by

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Index notation, also commonly known as subscript notation or tensor notation, is an extremely useful tool for performing vector algebra. Consider the coordinate system illustrated in Figure 1. Instead of using the typical axis labels  $x$ ,  $y$ , and  $z$ , we use  $x_1$ ,  $x_2$ , and  $x_3$ , or

$$x_i \quad i = 1, 2, 3$$

The corresponding unit basis vectors are then  $\hat{e}_1$ ,  $\hat{e}_2$ , and  $\hat{e}_3$ , or

$$\hat{e}_i \quad i = 1, 2, 3$$

The basis vectors  $\hat{e}_1$ ,  $\hat{e}_2$ , and  $\hat{e}_3$  have the following properties:

$$\hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1 \quad (1)$$

$$\hat{e}_1 \cdot \hat{e}_2 = \hat{e}_1 \cdot \hat{e}_3 = \hat{e}_2 \cdot \hat{e}_3 = 0 \quad (2)$$

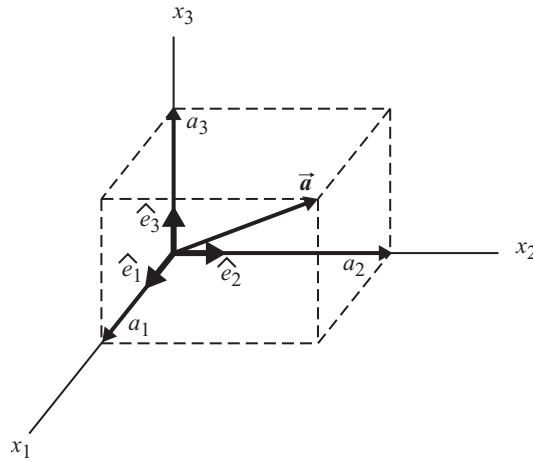


Figure 1: Reference coordinate system.

We now introduce the **Kronecker delta** symbol  $\delta_{ij}$ .  $\delta_{ij}$  has the following properties:

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad i, j = 1, 2, 3 \quad (3)$$

Using Eqn 3, Eqns 1 and 2 may be written in index notation as follows:

$$\boxed{\hat{e}_i \cdot \hat{e}_j = \delta_{ij}} \quad i, j = 1, 2, 3 \quad (4)$$

In standard vector notation, a vector  $\vec{A}$  may be written in component form as

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad (5)$$

Using index notation, we can express the vector  $\vec{A}$  as

$$\begin{aligned} \vec{A} &= A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \\ &= \sum_{i=1}^3 A_i \hat{e}_i \end{aligned} \quad (6)$$

Notice that in the expression within the summation, the index  $i$  is *repeated*. Repeated indices are always contained within summations, or phrased differently a repeated index *implies* a summation. Therefore, the summation symbol is typically dropped, so that  $\vec{A}$  can be expressed as

$$\vec{A} = A_i \hat{e}_i \equiv \sum_{i=1}^3 A_i \hat{e}_i \quad (7)$$

This repeated index notation is known as Einstein's convention. Any repeated index is called a *dummy index*. Since a repeated index implies a summation over all possible values of the index, one can always relabel a dummy index, *i.e.*

$$\begin{aligned} \vec{A} &= A_i \hat{e}_i = A_j \hat{e}_j = A_k \hat{e}_k \quad etc. \\ &\equiv A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \end{aligned} \quad (8)$$

## The Scalar Product in Index Notation

We now show how to express *scalar products* (also known as inner products or dot products) using index notation. Consider the vectors  $\vec{a}$  and  $\vec{b}$ , which can be expressed using index notation as

$$\begin{aligned}\vec{a} &= a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3 = a_i\hat{e}_i \\ \vec{b} &= b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3 = b_j\hat{e}_j\end{aligned}\tag{9}$$

Note that we use different indices ( $i$  and  $j$ ) for the two vectors to indicate that the index for  $\vec{b}$  is completely independent of that used for  $\vec{a}$ . We will first write out the scalar product  $\vec{a} \cdot \vec{b}$  in long-hand form, and then express it more compactly using some of the properties of index notation.

$$\begin{aligned}\vec{a} \cdot \vec{b} &= \left( \sum_{i=1}^3 a_i \hat{e}_i \right) \cdot \left( \sum_{j=1}^3 b_j \hat{e}_j \right) \\ &= \sum_{i=1}^3 \sum_{j=1}^3 [(a_i \hat{e}_i) \cdot (b_j \hat{e}_j)] \\ &= \sum_{i=1}^3 \sum_{j=1}^3 [a_i b_j (\hat{e}_i \cdot \hat{e}_j)] \quad (\text{commutative property}) \\ &= \sum_{i=1}^3 \sum_{j=1}^3 (a_i b_j \delta_{ij}) \quad (\text{from Eqn 3})\end{aligned}$$

Summing over all values of  $i$  and  $j$ , we get

$$\begin{aligned}\vec{a} \cdot \vec{b} &= a_1 b_1 \delta_{11} + a_1 b_2 \delta_{12} + a_1 b_3 \delta_{13} \\ &\quad + a_2 b_1 \delta_{21} + a_2 b_2 \delta_{22} + a_2 b_3 \delta_{23} \\ &\quad + a_3 b_1 \delta_{31} + a_3 b_2 \delta_{32} + a_3 b_3 \delta_{33} \\ &= a_1 b_1 \delta_{11} + a_2 b_2 \delta_{22} + a_3 b_3 \delta_{33} \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= \sum_{i=1}^3 a_i b_i \\ &= a_i b_i = a_j b_j = a_k b_k\end{aligned}$$

Doing this in a more compact notation gives us

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (a_i \hat{e}_i) \cdot (b_j \hat{e}_j) \\ &= a_i b_j \hat{e}_i \cdot \hat{e}_j = a_i b_j \delta_{ij} \\ &= a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3\end{aligned}$$

Notice that when we have an expression containing  $\delta_{ij}$ , we simply get rid of the  $\delta_{ij}$  and set  $i = j$  everywhere in the expression.

### Example 1: Kronecker delta reduction

Reduce  $\delta_{ij}\delta_{jk}\delta_{ki}$ :

$$\begin{aligned}\delta_{ij}\delta_{jk}\delta_{ki} &= \delta_{ik}\delta_{ki} \quad (\text{remove } \delta_{ij}, \text{ set } j = i \text{ everywhere}) \\ &= \delta_{ii} \quad (\text{remove } \delta_{ik}, \text{ set } k = i \text{ everywhere}) \\ &= \sum_{i=1}^3 \delta_{ii} = \sum_{i=1}^3 1 = 1 + 1 + 1 = 3\end{aligned}$$

Here we can see that

$$\boxed{\delta_{ii} = 3} \quad (\text{Einstein convention implied}) \quad (10)$$

Note also that

$$\boxed{\delta_{ij}\delta_{jk} = \delta_{ik}} \quad (11)$$

### Example 2: $\vec{r}$ and $\hat{r}$ in index notation

(a) Express  $\vec{r}$  using index notation.

$$\vec{r} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3 = \boxed{x_i \hat{e}_i}$$

(b) Express  $\hat{r}$  using index notation.

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{(\vec{r} \cdot \vec{r})^{1/2}} = \boxed{\frac{x_i \hat{e}_i}{(x_j x_j)^{1/2}}}$$

(c) Express  $\vec{a} \cdot \hat{r}$  using index notation.

$$\vec{a} \cdot \hat{r} = \frac{\vec{a} \cdot \vec{r}}{|\vec{r}|} = \boxed{\frac{a_i x_i}{(x_j x_j)^{1/2}}}$$

## The Cross Product in Index Notation

Consider again the coordinate system in Figure 1. Using the conventional right-hand rule for cross products, we have

$$\begin{aligned} \hat{e}_1 \times \hat{e}_1 &= \hat{e}_2 \times \hat{e}_2 = \hat{e}_3 \times \hat{e}_3 = 0 \\ \hat{e}_1 \times \hat{e}_2 &= \hat{e}_3 & \hat{e}_2 \times \hat{e}_1 &= -\hat{e}_3 \\ \hat{e}_2 \times \hat{e}_3 &= \hat{e}_1 & \hat{e}_3 \times \hat{e}_2 &= -\hat{e}_1 \\ \hat{e}_3 \times \hat{e}_1 &= \hat{e}_2 & \hat{e}_1 \times \hat{e}_3 &= -\hat{e}_2 \end{aligned} \tag{12}$$

To write the expressions in Eqn 12 using index notation, we must introduce the symbol  $\epsilon_{ijk}$ , which is commonly known as the Levi-Civita tensor, the alternating unit tensor, or the permutation symbol (in this text it will be referred to as the permutation symbol).  $\epsilon_{ijk}$  has the following properties:

$$\begin{aligned} \epsilon_{ijk} &= 1 && \text{if } (ijk) \text{ is an even (cyclic) permutation} \\ &&& \text{of } (123), \text{ i.e. } \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{ijk} &= -1 && \text{if } (ijk) \text{ is an odd (noncyclic) permutation} \\ &&& \text{of } (123), \text{ i.e. } \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1 \\ \epsilon_{ijk} &= 0 && \text{if two or more subscripts are the same,} \\ &&& \text{i.e. } \epsilon_{111} = \epsilon_{112} = \epsilon_{313} = 0 \text{ etc.} \end{aligned}$$

Hence, we may rewrite the expressions in Eqn 12 as follows:

$$\begin{aligned}
 \hat{e}_1 \times \hat{e}_2 &= \epsilon_{123} \hat{e}_3 & \hat{e}_2 \times \hat{e}_1 &= \epsilon_{213} \hat{e}_3 \\
 \hat{e}_2 \times \hat{e}_3 &= \epsilon_{231} \hat{e}_1 & \hat{e}_3 \times \hat{e}_2 &= \epsilon_{321} \hat{e}_1 \\
 \hat{e}_3 \times \hat{e}_1 &= \epsilon_{312} \hat{e}_2 & \hat{e}_1 \times \hat{e}_3 &= \epsilon_{132} \hat{e}_2
 \end{aligned} \tag{13}$$

Now, we may write a single generalized expression for all the terms in Eqn 13:

$$\boxed{\hat{e}_i \times \hat{e}_j = \epsilon_{ijk} \hat{e}_k} \tag{14}$$

Here  $\epsilon_{ijk} \hat{e}_k \equiv \sum_{k=1}^3 \epsilon_{ijk} \hat{e}_k$  ( $k$  is a dummy index). That is, this works because

$$\begin{aligned}
 \hat{e}_1 \times \hat{e}_2 &= \epsilon_{12k} \hat{e}_k = \sum_{k=1}^3 \epsilon_{12k} \hat{e}_k \\
 &= \epsilon_{121} \hat{e}_1 + \epsilon_{122} \hat{e}_2 + \epsilon_{123} \hat{e}_3 = \hat{e}_3
 \end{aligned}$$

The same is true for all of the other expressions in Eqn 13. Note that  $\hat{e}_i \times \hat{e}_i = \epsilon_{iik} \hat{e}_k = 0$ , since  $\epsilon_{iik} = 0$  for all values of  $k$ .  $\epsilon_{ijk}$  is also given by the following formula.

$$\boxed{\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)} \quad i, j, k = 1, 2, 3 \tag{15}$$

This is a remarkable formula that works for  $\epsilon_{ijk}$  if you do not want to calculate the parity of the permutation  $(ijk)$ . Also note the following property of  $\epsilon_{ijk}$ :

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kji}$$

*i.e.* switching any two subscripts reverses the sign of the permutation symbol (or in other words  $\epsilon_{ijk}$  is *anti-symmetric*). Also,

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$

*i.e.* cyclic permutations of the subscripts do not change the sign of  $\epsilon_{ijk}$ . These

properties also follow from the formula in Eqn 15.

Now, let's consider the cross product of two vectors  $\vec{a}$  and  $\vec{b}$ , where

$$\begin{aligned}\vec{a} &= a_i \hat{e}_i \\ \vec{b} &= b_j \hat{e}_j\end{aligned}$$

Then

$$\vec{a} \times \vec{b} = (a_i \hat{e}_i) \times (b_j \hat{e}_j) = a_i b_j \hat{e}_i \times \hat{e}_j = a_i b_j \epsilon_{ijk} \hat{e}_k$$

Thus we write for the cross product:

$$\boxed{\vec{a} \times \vec{b} = \epsilon_{ijk} a_i b_j \hat{e}_k} \quad (16)$$

All indices in Eqn 16 are dummy indices (and are therefore summed over) since they are repeated. We can always relabel dummy indices, so Eqn 16 may be written equivalently as

$$\vec{a} \times \vec{b} = \epsilon_{pqr} a_p b_q \hat{e}_r$$

Returning to Eqn 16, the  $k$ th component of  $\vec{a} \times \vec{b}$  is

$$\left( \vec{a} \times \vec{b} \right)_k = \epsilon_{ijk} a_i b_j$$

where now only  $i$  and  $j$  are dummy indices. Note that the cross product may also be written in determinant form as follows:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (17)$$

The following is a very important identity involving the product of two permutation symbols.

$$\epsilon_{ijk} \epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \quad (18)$$

The proof of this identity is as follows:

- If any two of the indices  $i, j, k$  or  $l, m, n$  are the same, then clearly the left-hand side of Eqn 18 must be zero. This condition would also result in two of the rows or two of the columns in the determinant being the same, so therefore the right-hand side must also equal zero.
- If  $(i, j, k)$  and  $(l, m, n)$  both equal  $(1, 2, 3)$ , then both sides of Eqn 18 are equal to one. The left-hand side will be  $1 \times 1$ , and the right-hand side will be the determinant of the identity matrix.
- If any two of the indices  $i, j, k$  or  $l, m, n$  are interchanged, the corresponding permutation symbol on the left-hand side will change signs, thus reversing the sign of the left-hand side. On the right-hand side, an interchange of two indices results in an interchange of two rows or two columns in the determinant, thus reversing its sign.

Therefore, all possible combinations of indices result in the two sides of Eqn 18 being equal. Now consider the special case of Eqn 18 where  $n = k$ . In this case, the repeated index  $k$  implies a summation over all values of  $k$ . The product of the two permutation symbols is now

$$\begin{aligned}
 \epsilon_{ijk}\epsilon_{lmk} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{ik} \\ \delta_{jl} & \delta_{jm} & \delta_{jk} \\ \delta_{kl} & \delta_{km} & \delta_{kk} \end{vmatrix} && \text{(note } \delta_{kk} = 3\text{)} \\
 &= 3\delta_{il}\delta_{jm} - 3\delta_{im}\delta_{jl} + \delta_{im}\delta_{jk}\delta_{kl} \\
 &\quad - \delta_{ik}\delta_{jm}\delta_{kl} + \delta_{ik}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jk}\delta_{km} \\
 &= 3\delta_{il}\delta_{jm} - 3\delta_{im}\delta_{jl} + \delta_{im}\delta_{jl} \\
 &\quad - \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} && \text{(from Eqn 11)}
 \end{aligned} \tag{19}$$

Or finally

$$\boxed{\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} \tag{20}$$



Eqn 20 is an *extremely* useful property in vector algebra and vector calculus applications. It can also be expressed compactly in determinant form as

$$\epsilon_{ijk}\epsilon_{lmk} = \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix} \quad (21)$$

The cyclic property of the permutation symbol allows us to write also

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

To recap:

$$\begin{aligned} \hat{e}_i \cdot \hat{e}_j &= \delta_{ij} & \text{and} & & \vec{a} \cdot \vec{b} &= a_i b_i \\ \hat{e}_i \times \hat{e}_j &= \epsilon_{ijk} \hat{e}_k & \text{and} & & \vec{a} \times \vec{b} &= \epsilon_{ijk} a_i b_j \hat{e}_k \end{aligned}$$

These relationships, along with Eqn 20, allow us to prove any vector identity.

### Example 3: Vector identity proof

Show for the double cross product:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

Start with the left-hand side (LHS):

$$\begin{aligned} \vec{a} \times (\vec{b} \times \vec{c}) &= (a_i \hat{e}_i) \times (\epsilon_{jkl} b_j c_k \hat{e}_l) \\ &= \epsilon_{jkl} a_i b_j c_k (\hat{e}_i \times \hat{e}_l) \\ &= \epsilon_{jkl} a_i b_j c_k \epsilon_{ilh} \hat{e}_h \\ &= \epsilon_{jkl} \epsilon_{hil} a_i b_j c_k \hat{e}_h \\ &= (\delta_{jh} \delta_{ki} - \delta_{ji} \delta_{kh}) a_i b_j c_k \hat{e}_h \\ &= \delta_{jh} \delta_{ki} a_i b_j c_k \hat{e}_h - \delta_{ji} \delta_{kh} a_i b_j c_k \hat{e}_h \\ &= a_i b_j c_i \hat{e}_j - a_i b_i c_k \hat{e}_k \\ &= (a_i c_i) (b_j \hat{e}_j) - (a_i b_i) (c_k \hat{e}_k) \\ &= \boxed{(\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}} \end{aligned}$$

**Example 4: The scalar triple product**

Show that  $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

$$\begin{aligned}
 \vec{a} \cdot (\vec{b} \times \vec{c}) &= (a_i \hat{e}_i) \cdot (\epsilon_{jkm} b_j c_k \hat{e}_m) \\
 &= \epsilon_{jkm} a_i b_j c_k (\hat{e}_i \cdot \hat{e}_m) \\
 &= \epsilon_{jkm} a_i b_j c_k \delta_{im} \\
 &= \epsilon_{jki} a_i b_j c_k
 \end{aligned}$$

or

$$\boxed{\vec{a} \cdot (\vec{b} \times \vec{c}) = \epsilon_{ijk} a_i b_j c_k}$$

From our permutation rules, it follows that

$$\begin{aligned}
 \vec{a} \cdot (\vec{b} \times \vec{c}) &= \epsilon_{ijk} a_i b_j c_k \\
 &= \epsilon_{kij} c_k a_i b_j = \boxed{\vec{c} \cdot (\vec{a} \times \vec{b})} \\
 &= \epsilon_{jki} b_j c_k a_i = \boxed{\vec{b} \cdot (\vec{c} \times \vec{a})}
 \end{aligned}$$