Project 1 in FYS-3150

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September 25, 2014

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1 The Jacobi Method

1.1 Theory behind the Jacobi method

The Jacobi method is an iterative method to make an approximated diagonal matrix in an eigenvalue problem by multiplying it several times with a rotational matrix, $\mathbf{S}$, that is chosen so it sets some off diagonal elements to 0. When multiplying it with the rotational matrix some of the already 0 elements may get a nonzero value, so by always choosing the largest off-diagonal element hopefully it will produce a near diagonal end matrix. By also doing the rotation transformation on the right and side the equation will be equal on both sides throughout and we can extract the eiganvalues easily at the end.

$$\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}} \tag{1}$$

Doing a transformation with the rotation matrix $\mathbf{S} = \begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & & & \\ & & \cos\theta & \cdots & -\sin\theta \\ & 0 & \vdots & 1 & \vdots \\ & & \sin\theta & \cdots & \cos\theta \end{pmatrix}$, in which we choose θ so the wanted elements in $\mathbf{A}$ becomes 0

$$\mathbf{S}\mathbf{A}\vec{\mathbf{x}} = \lambda\mathbf{S}\vec{\mathbf{x}} \tag{2}$$

$$\mathbf{S}\mathbf{A}(\mathbf{S}^{-\mathbf{1}}\mathbf{S})\vec{\mathbf{x}} = \lambda\mathbf{S}\vec{\mathbf{x}} \tag{3}$$

Then we introduce a new vector $\vec{\mathbf{y}} = \mathbf{S}\vec{\mathbf{x}}$ and a new matrix $\mathbf{B} = \mathbf{S}\mathbf{A}\mathbf{S}^{-\mathbf{1}}$, where $\mathbf{B}$ is more diagonal than $\mathbf{A}$

$$\mathbf{B}\vec{\mathbf{y}} = \lambda\vec{\mathbf{y}} \tag{4}$$

We do this again untill desired level of diagonality of the matrix is achieved

1.2 The Jacobi method computationally

Since the matrix inversion `<http://en.wikibooks.org/wiki/LaTeX/Hyperlinks#.5Chyperref>` and multiplication $O(n^3)$ (`<http://www.ee.ucla.edu/ee236b/lectures/num-lin-alg.pdf>`) (Put in proper references later bibtex) used in the Jacobi method is unecessarily heavy for the sparse rotational matrix $\mathbf{S}$ we used a quicker method to do it.

For a simplified 3×3 symmetric system the transformation $\mathbf{B} = \mathbf{S}\mathbf{A}\mathbf{S}^{-\mathbf{1}}$ becomes:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \tag{5}$$

where $c = \cos\theta, s = \sin\theta$

$$\tag{6}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ ca_{12} - sa_{13} & ca_{22} - sa_{23} & ca_{23} - sa_{33} \\ sa_{12} + ca_{13} & sa_{22} + ca_{23} & sa_{23} + ca_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} a_{11} & c(a_{12}) - s(a_{13}) & s(a_{12}) + c(a_{13}) \\ ca_{12} - sa_{13} & c(ca_{22} - sa_{23}) - s(ca_{23} - sa_{33}) & s(ca_{22} - sa_{23}) + c(ca_{23} - sa_{33}) \\ sa_{12} + ca_{13} & c(sa_{22} + ca_{23}) - s(sa_{23} + ca_{33}) & s(sa_{22} + ca_{23}) + c(sa_{23} + ca_{33}) \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} a_{11} & ca_{12} - sa_{13} & sa_{12} + ca_{13} \\ ca_{12} - sa_{13} & c^2a_{22} + s^2a_33 - 2sca_{23} & a_{23}(c^2 - s^2) + sc(a_{22} - a_{33}) \\ sa_{12} + ca_{13} & a_{23}(c^2 - s^2) + sc(a_{22} - a_{33}) & s^2a_{22} + c^2a_{33} + 2sca_{23} \end{pmatrix} \tag{9}$$

Then we choose θ so that $B_{23} = 0$

$$\mathbf{B} = \begin{pmatrix} a_{11} & ca_{12} - sa_{13} & sa_{12} + ca_{13} \\ ca_{12} - sa_{13} & c^2a_{22} + s^2a_33 - 2sca_{23} & 0 \\ sa_{12} + ca_{13} & 0 & s^2a_{22} + c^2a_{33} + 2sca_{23} \end{pmatrix} \tag{10}$$

All the components of $\mathbf{B}$ is now known so they can be calculated straightforward for θ. If we precalculate $\cos\theta$ and $\sin\theta$ the flops needed to calculate $\mathbf{B}$ will be:

- First row: 4 multiplications and 2 additions
- Second row: 10 multiplications and 3 additions
- Third row: 10 multiplications and 3 additions