Chapter 1

Verification

1.1 Electron plasma oscillations in unmagnetized plasma

1.1.1 Physical mechanism

Here we will consider an electron fluid under local thermal equilibrium, LTE, experiencing small perturbations from an equilibrium state of an homogeneous electron density, n_0, and pressure, p_0, and vanishing flow, $\mathbf{u}_0 = 0$, and electric field, $\mathbf{E}_0 = 0$. See Goldston and Rutherford (1995) for a more thorough overview.

The small perturbation of the electron density will cause an electric field electric field working to restore the equilibrium. When the electrons reach the equilibrium position they have a kinetic energy and will overshoot causing a new perturbation away from the equilibrium.

The fluid is governed by the following equations:

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0 \tag{1.1a}$$

$$m_e n_e \left(\frac{\partial}{\partial t} + \mathbf{u}_e \cdot \nabla \right) \mathbf{u}_e = e n_e \nabla \phi - \nabla p_e \tag{1.1b}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_e \cdot \nabla \right) p_e + \frac{5}{3} p_e \nabla \cdot \mathbf{u}_e = 0 \tag{1.1c}$$

$$\epsilon_0 \nabla^2 \phi = e \left(n_e - n_0 \right) \tag{1.1d}$$

Assuming the small perturbation to the equilibrium.

$$\text{Perturbation} \rightarrow \begin{cases} n_e = n_0 + \tilde{n}_e \\ p_e = p_0 + \tilde{p}_e \\ \mathbf{u}_e = \tilde{\mathbf{u}}_e \\ \phi = \tilde{\phi} \end{cases}$$

Inserting the perturbation and linearizing the equations we get:

$$\frac{\partial \tilde{n}_e}{\partial t} + \nabla \cdot (n_0 \tilde{\mathbf{u}}_e) = 0 \tag{1.2a}$$

$$m_e \frac{\partial \tilde{\mathbf{u}}_e}{\partial t} = e\nabla\tilde{\phi} - \frac{\nabla \tilde{p}_e}{n_0} \tag{1.2b}$$

$$\frac{\partial \tilde{p}}{\partial t} + \frac{5}{3} p_0 \nabla \cdot \tilde{\mathbf{u}}_e = 0 \tag{1.2c}$$

$$\epsilon_0 \nabla^2 \tilde{\phi} = e\tilde{n}_e \tag{1.2d}$$

Then we combine the continuity and energy equations, eq. (1.2a) and eq. (1.2c).

$$\frac{\partial}{\partial t}\left(\frac{\tilde{p}_e}{p_0} + \frac{5}{3}\frac{\tilde{n}_e}{n_0}\right) = 0 \tag{1.3}$$

The perturbed pressure and density is proportional, $\nabla \tilde{p}_e = (5p_0/3n_0)\,\nabla \tilde{n}_e$. Assuming plane wave solutions along the x-axis, so the differential operators become $\nabla \to ik$ and $\frac{\partial}{\partial t} \to -i\omega$, we can solve for the dispersion relation.

$$\epsilon(\omega, k) = 1 + \frac{5}{3}\lambda_{se}^2 k^2 - \frac{\omega^2}{\omega_{pe}^2} \tag{1.4}$$

Here we have subsituted in the electron debye length $\lambda_{se} = (\epsilon_0 p_0)/(e^2 n_0^2)$, and the electron plasma frequency $\omega_{pe} = e^2 n_0$

1.1.2 Test Case

Since the physics behind the electron plasma wave is quite simple, we want to verify that our code works by letting it reproduce the waves.

- Homogenous plasma
- No fields
- Equilibrium? (Lattice structure?) First test this
- No thermal velocity / (Not necessary?)
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To ensure the correctness of the solver we have used a few different methods to improve our confidence in it. The first layer is a collection of unittests built to verify the small modular parts of the model. were it is possible, for example restricting a predefined grid and checking that the coarser grid is correct, and these tests are described in chapter 2

The multigrid method has several different steps in the algorithm, as a developmental help and to ensure that the program works correctly during as many different conditions as possible we want to test the whole code, as well as the constituent parts where possible. The method is quite modular and several parts of it can be tested alone. The GS-RB, used for smoothing, can be independently tested, since on it's own it converges to a solution, just at a higher computational cost than the multigrid method. To test it we will use an initial density field with a length between the grid steps that results in an exact answer. The restriction and prolongation operators can also tested to a degree by checking that they preserve a constant grid through several grid level changes.

NB! Rewrite (see if something is usable) below

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1.1.3 The Multigrid method

We use both of the aforemented tests to check that the whole multigrid method works as intended. Since a constant source term will give a trivial solution of the potential, $\phi(x, y, z) = \mathbf{0}$, we use that as a test. In addition we also test that it converges on a sinusoidal source term as we did the smoother.

1.1.4 Prolongation and Restriction

The prolongation and restriction operators with the earlier proposed stencils will average out the grid points when applied. So the idea here is to set up a system with a constant charge density, $\rho(\mathbf{r}) = C$, and then apply a restriction. After performing the restriction we can check that the grid points values are preserved. Then we can do the same with the prolongation. While this does not completely verify that the operators work as wanted, it gives an indication that we have not lost any grid points and the total mass of the charge density should be conserved.

1.1.5 Smoothers

The iterative method GS-RB used for the pre- and postsmoothing of the grid in our implementation of the multigrid method is also a direct solver. So we can test it, or most other smoothers, by testing them on a small system where the problem has an analytical solution. Then we can let them run for a while and ensure that they are converging towards the solution. If we let the source term be sinusoidal in one direction, and constant in the other directions it has an easy solution given below

$$\nabla^2 \phi(x, y, z) = \sin(x) \tag{1.5}$$

This has a solution when $\phi(x, y, z) = -\sin(x)$ and we can test that the solver converges to the solution. If we let the source term be constant in the x direction and instead vary in the other directions we can get verify that the solver works in all three directions independently.

Chapter 2

Unit Tests