0.1 General idea

When an iterative solver solves a problem, it starts with an initial guess then for each cycle it improves the guess to come closer to the wanted solution. The difference between the guess and the correct solution, the residual, does not necessarily converge equally fast for different frequencies. A solver can be very efficient on reducing the local error, while it takes many cycles to reduce the errors due to distant influence. A multigrid solver attacks this problem by applying iterative methods on different discretizations of the problem, by solving on a very coarse grids the error due to distant influence will be reduced faster, while solving on a fine grid reduces the local error fast. So by solving on both fine and coarse grids the needed cycles will be reduced. To implement a multigrid algorithm we then need algorithms to solver the problem on a grid **??**, restriction **??** and prolongation **??** operators to transfer the problem between grids, as well as a method to compute the residual.

0.1.1 V-cycle

The simplest multigrid cycle is called a V-cycle, which starts at the finest grid, goes down to the coarsest grid and then goes back up to the finest grid. First the problem is smoothed on the finest level, then we compute the residual, or the rest after inserting the guess solution in the equation. The residual is then used as the source term for the next level, and we restrict it down as the source term for the next coarser level and repeat untill we reach the coarsest level. When we reach the coarsest level the problem is solved there and we obtain a correction term. The correction term is prolongated to the next finer level and added to the solution there, improving the solution, following by a new smoothing to obtain a new correction. This continue untill we reach the finest level again and a multigrid cycle is completed, see fig. 1 for a 3 level schematic.

In the following description of the steps in the MG method, we will use ϕ, ρ, d and ω to signify the solution, source, defect and correction respectively. A subscript means the grid level, where 0 si the finest level, while the superscript 0 implies an initial guess is used. Hats and tildes are also used to signify the stage the solution is in, with a hat meaning the solution is smoothed and a tilde meaning the correction from the grid below is added.

A level in the multigrid V cycle will then look like the following:

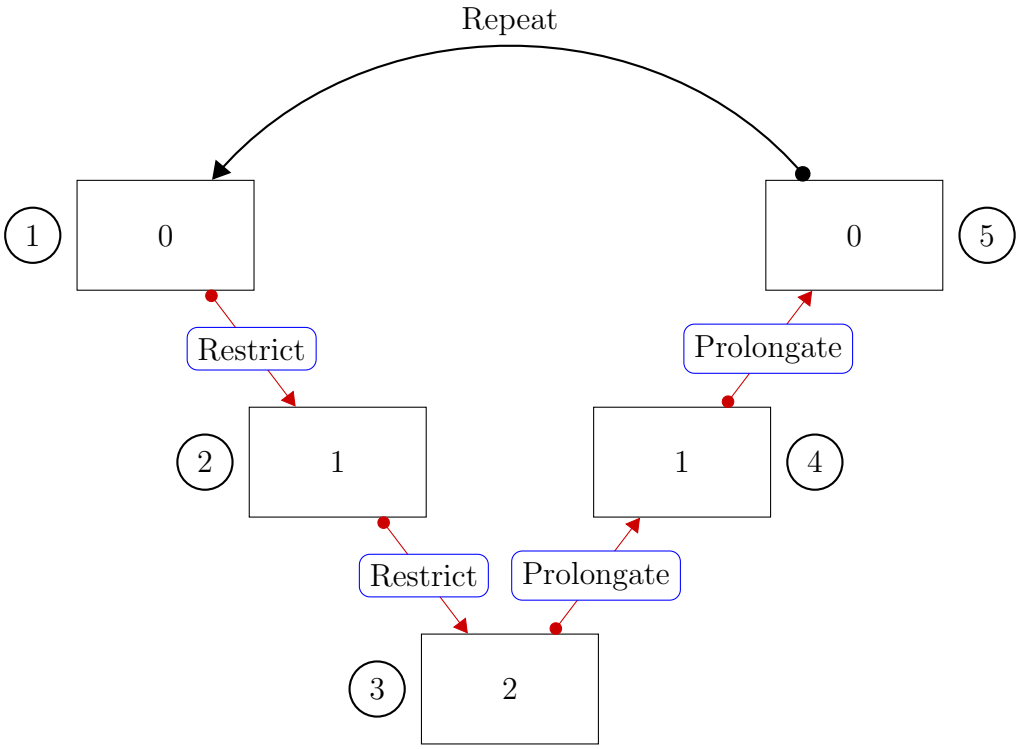

Figure 1: Schematic overview of the PIC method. In a three level MG implementation, there is 5 main steps in a cycle that needs to be consider.

1: Smooth	$\widehat{\phi}_l = \mathcal{S}(\phi_l, \rho_l)$
2: Residual	$d_l = \nabla^2 \widehat{\phi}_l - \rho_l$
3: Restrict	$\rho_{l+1} = \mathcal{R} d_l$
4: Go down, recieve correction	$\omega_l = \mathcal{I} \phi_{l+1}$
5: Add correction	$\widetilde{\phi}_l = \widehat{\phi}_l + \omega_l$
6: Smooth	$\phi_l = \mathcal{S}(\widetilde{\phi}_l, \rho_l)$
7: Interpolate correction	$\omega_{l-1} = \mathcal{I} \phi_l$

0.2 Ex: 3 level V cycle

① Compute defect on grid 0, the finest grid:

- $\widehat{\phi}_0 = \mathcal{S}(\phi_0, \rho_0)$
- $d_0 = \nabla^2 \widehat{\phi}_0 - \rho_0$
- Restrict defect: $\rho_1 = \mathcal{R} d_0$

② Compute defect on grid 1:

- $\widehat{\phi}_1 = \mathcal{S}(\phi_1^0, \rho_1)$

- $d_1 = \nabla^2 \widehat{\phi}_1 - \rho_1$
- Restrict defect: $\rho_2 = \mathcal{R} d_1$

③ Solve Coarse Grid for correction ω

- $\phi_2 = \mathcal{S}(\phi_2^0, \rho_2)$
- Interpolate as correction:$\omega_1 = \mathcal{I}\phi_2$

④ Add correction on level 1:

- $\widetilde{\phi}_1 = \widehat{\phi}_1 + \omega_1$
- $\phi_1 = \mathcal{S}(\widetilde{\phi}_1, \rho_1)$
- Interpolate correction:$\omega_0 = \mathcal{I}\phi_1$

⑤ Compute solution.

- $\widetilde{\phi}_0 = \widehat{\phi}_0 + \omega_0$
- $\phi_0 = \mathcal{S}(\widetilde{\phi}_0, \rho_0)$