First Order ODEs

- A **first-order** differential equations contain only y' and may contain y and given functions of x; hence we can write them

$$F(x, y, y') = 0$$

or in **standard(or normal)** form

$$\frac{dy}{dx} = f(x, y) \;\text{ or }\; y' = f(x, y)$$

Example. $y' = y,\ \dfrac{dy}{dx} = \cos x,\ y' = y^2x^3.$

- Many, but not all, first–order DEs can be written in standard form by algebraically solving for y' and then setting $f(x, y)$ equal to the right side of the resulting equation.
- In the standard form the right side can always be written as a quotient of two other functions $M(x, y)$ and $-N(x, y)$. i.e.

$$\frac{dy}{dx} = \frac{M(x, y)}{-N(x, y)}$$

which is equivalent to the **differential form**

$$M(x, y)dx + N(x, y)dy = 0$$

- There is no general method of analytical solution of equations of this form, but there are a number of special types of first–order equation for which **standard methods of solution** are available.
- The most important of these are:
  - Equations with the variables separable
  - Homogeneous equations
  - Exact equations
  - Linear equations
- We will discuss each of these equations in the following sections.

Separable Differential Equations

- The easiest kind of DE to solve is a separable DE, which is one that can be written in the form

$$P(x) + Q(y)\frac{dy}{dx} = 0$$

or in the more symmetric differential notation,

$$\boxed{P(x)dx + Q(y)dy = 0}$$

- Notice that the x's and y's are separated from one another, hence the name "separable differential equation".

- To solve $P(x)dx+Q(y)dy=0$, we integrate both sides with respect to x. This yields

$$\int P(x)dx+\int Q(y)dy=C$$

which is also equivalent to

$$\int Q(y)dy=-\int P(x)dx+C$$

where C is a constant.
- This is the **general solution** of the DE.

Notice:

- The integrals obtained above may be, for all practical purposes, impossible to evaluate. In such cases, **numerical techniques** are used to obtain an approximate solution.
- Even if the indicated integration can be performed, it may not be algebraically possible to solve for y **explicitly** in terms of x. In that case, the solution is left in implicit form.

Example 1. Solve $xdx - y^2dy = 0$.

Solution:

Integrating both sides with respect to x, we have

$$\int x dx + \int (-y^2) dy = C$$

$$\frac{x^2}{2} - \frac{y^3}{3} = C$$

Solving for y explicitly, we obtain the solution as

$$y = \left(\frac{3}{2}x^2 + C^*\right)^{\frac{1}{3}}; \quad C^* = -3C. \ \square$$

Note: In the above line, we have introduced a new arbitrary constant C^* instead of retaining C in order to put the result in a better form. Such changes are quite freely made.

Example 2. Solve $y' = y^2x^3$.

Solution: We first rewrite this equation in the separable (differential) form

$$x^3dx - \left(\frac{1}{y^2}\right)dy = 0$$

Then integrating both sides, we have

$$\int x^3 dx - \int \left(\frac{1}{y^2}\right) dy = C$$

Performing the indicated integrations, we have

$$\frac{x^4}{4} + \frac{1}{y} = C$$

Solving explicitly for y , we get

$$y = \frac{-4}{x^4 + C}. \quad \square$$

Exercise

1. Solve $\dfrac{dy}{dx} = e^{3x-2y} + x^2e^{-2y}$.

   Ans. $\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + C$ or

   $$3e^{2y} = 2(e^{3x} + x^3) + C.$$

2. Solve $\dfrac{dy}{dx} = \dfrac{x^2+2}{y}$. **Ans.** $\frac{1}{3}x^3 + 2x - \frac{1}{2}y^2 = C$

3. Solve $y' = \dfrac{x+1}{y^4+1}$. **Ans.** $\frac{x^2}{2} + x - \frac{y^5}{5} - y = C$

Solutions to the IVP

- The solution to the initial–value problem

$$A(x)dx + B(y)dy = 0; \quad y(x_0) = y_0$$

can be obtained, as usual, by first using

$$\int A(x)dx + \int B(y)dy = C$$

to solve the DE and then applying the initial condition directly to evaluate C.

Example. Solve $e^x dx - y\,dy = 0; \quad y(0) = 1.$

Solution: The solution to the DE is

$$\int e^x dx + \int (-y)\,dy = C$$

which implies $y^2 = 2e^x + C^*, \; C^* = -2C$

Applying the initial condition, we obtain

$$(1)^2 = 2e^0 + C^* \implies C^* = -1.$$

Thus, the solution to the IVP is

$$y^2 = 2e^x - 1 \;\text{ or }\; y = \sqrt{2e^x - 1}. \quad \square$$

(Note that we can not choose the negative square root, since then $y(0) = -1$, which violates the initial condition.)

Exercise: Find the solution of the DE

$$(x+1)dy + (y-1)dx = 0$$

subject to the condition that $y = 3$ when $x = 0$.

Answer: Separating the variables, we obtain

$$\frac{dx}{x+1} = \frac{-dy}{y-1}$$

Integrating both sides, we get

$$\ln(x+1) = -\ln(y-1) + \ln C$$

$$\text{or} \quad (x+1)(y-1) = C$$

Substituting $x = 0$, $y = 3$ in this equation, we obtain $C = 2$. Hence, the solution of the given IVP is $(x+1)(y-1) = 2$. ❑

Homogeneous Equations

- A first–order DE in standard form ($y' = f(x, y)$) is **homogeneous** if $f(tx, ty) = f(x, y)$ for every real number t.
- In the general frame work of DEs, the word "homogeneous" has an entirely different meaning. Only in the context of first order DEs does homogeneous have the meaning defined above.
- A first–order DE in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **homogeneous** if both coefficient functions M and N are homogeneous functions of the same degree. In other words, if

$$M(tx, ty) = t^n M(x, y) \text{ and } N(tx, ty) = t^n N(x, y)$$

Note: If a function f possesses the property

$$f(tx, ty) = t^n f(x, y)$$

for some real number n, then f is said to be a **homogeneous function of degree n**.

- For example, $f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3, since $f(tx, ty) = (tx)^3 + (ty^3) = t^3(x^3 + y^3) = t^3 f(x, y)$.
- We can also define homogeneous as follows:

- **Homogeneous equations** are equations which can be expressed in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad (*)$$

- A homogeneous equation can always be converted to an equation with **separable** variables by making the substitution $\frac{y}{x} = v$ or $y = xv$, and eliminating y in favor of the new variable v.

$$y = xv \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}, \text{ and } f\left(\frac{y}{x}\right) = f(v)$$

- Now, equation $(*)$ becomes

$$v + x\frac{dv}{dx} = f(v), \text{ or } \frac{1}{x} + \frac{1}{v - f(v)}\frac{dv}{dx} = 0,$$

$$\text{or } \frac{1}{x}dx + \frac{1}{v - f(v)}dv = 0$$

which is separable.

Example 1. Solve $y' = \dfrac{y+x}{x}$.

Solution: Here $f(x, y) = \frac{y+x}{x}$. Now since

$$f(tx, ty) = \frac{ty + tx}{tx} = \frac{t(y+x)}{tx} = \frac{y+x}{x} =$$

$f(x, y)$, it is homogeneous.

let $y = xv$ so that $\dfrac{dy}{dx} = y' = v + x\dfrac{dv}{dx}$.

Now the given equation $y' = \dfrac{y+x}{x}$ becomes

$$v + x\,\frac{dv}{dx} = \frac{xv + x}{x} = v + 1$$

$$\Rightarrow x\,\frac{dv}{dx} = 1 \quad \text{or} \quad dv = \frac{1}{x}\,dx$$

This last equation is separable; its solution is

$$\int dv - \int \frac{1}{x}\,dx = C$$

$$v = \ln|x| + C \quad \text{or} \quad v = \ln|x| + \ln|C^*|$$

$$\Rightarrow v = \ln|xC^*|, \text{ we have set } C = \ln|C^*|$$

Finally, substituting $v = \frac{y}{x}$ back, we obtain the solution

$$y = x \ln|xC^*|. \quad \square$$

Example 2. Solve the DE $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$.

Solution.

First, check it is homogeneous:

$$f(tx, ty) = \frac{(ty)^2 - (tx)^2}{2(tx)(ty)} = \frac{t^2(y^2 - x^2)}{2t^2xy}$$

$$= \frac{y^2 - x^2}{2xy} = f(x, y)\ .$$

Putting $\frac{y}{x} = v$ or $y = vx$, we obtain by differentiation $\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$, and the given DE becomes

$$v + x\,\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v}$$

Or, after rearrangement $\quad x\,\dfrac{dv}{dx} + \dfrac{v^2+1}{2v} = 0.$

Separating variables and integrating now gives

$$\int \frac{2v}{v^2+1}\,dv + \int \frac{dx}{x} = C$$

$$\ln(v^2+1) + \ln x = \ln C^*, \;\; \text{here } C = \ln C^*$$

$$\text{or} \quad x(v^2+1) = C^*$$

To complete the solution now substitute back $\frac{y}{x}$ for v to obtain, after a little rearrangement,

$$y^2 + x^2 - xC^* = 0. \;\square$$

Exercise:

1. Solve $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$.

   Ans. $\ln x = \tan^{-1} v + C = \tan^{-1}(y/x) + \ln C^*$

   or $x = C^* e^{\tan^{-1}(y/x)}$.

2. Solve the IVP $y' = \dfrac{2xy}{x^2 - y^2}$; $y(1) = 1$.

   Ans. $x^2 + y^2 = 2y$.