

Exact Differential Equations

- A first-order DE of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is called **exact** if its left side is the total or exact differential of some function $u(x, y)$. i.e. $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

- Recall that if a function $u(x, y)$ has continuous partial derivatives, its **total or exact differential** is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

- Thus, if $u(x, y) = c = \text{constant}$, then $du = 0$.

Note: The necessary and sufficient condition for the DE $M(x, y)dx + N(x, y)dy = 0$ to be **exact** is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

- Let us derive this. Comparing the equations

$$M(x, y)dx + N(x, y)dy = 0 \quad (1)$$

$$\text{and } du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \quad (2)$$

we see that (1) is **exact** if there is some function

$u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N \quad (3)$$

- Suppose that M and N are defined and have continuous first partial derivatives in a region in the xy -plane whose boundary is a closed curve having no self-intersections. Then from (3),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

- By the assumption of **continuity** the two second derivatives are equal. Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- This condition is not only necessary but also sufficient for $Mdx + Ndy$ to be an exact differential.
- If (1) is exact, the function $\mathbf{u}(x, y)$ can be found by guessing or in the following systematic way:
 - From the first equation of (3), we have by integration with respect to x

$$u = \int_y^y M dx + h(y) \quad (4)$$

- In this integration, y is to be regarded as a constant, and $h(y)$ plays the role of a constant of

integration.

- To determine $h(y)$, we derive $\frac{\partial u}{\partial y}$ from (4), use (3) to get $\frac{dh}{dy}$, and integrate $\frac{dh}{dy}$ to get h .
- **Note** that formula (4) was obtained from the first equation of (3). Instead, we may equally well use the second equation of (3).

Example 1. Solve $2xy \, dx + (1 + x^2) \, dy = 0$.

Solution: Here $M(x, y) = 2xy$, $N(x, y) = 1+x^2$.

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x$, the DE is exact. We now determine a function $u(x, y)$ that satisfies

$$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial u}{\partial y} = N.$$

Thus $\frac{\partial u}{\partial x} = 2xy$. Integrating both sides of this equation with respect to x , we find that

$$\begin{aligned} \int \frac{\partial u}{\partial x} dx &= \int 2xy dx \\ \Rightarrow u(x, y) &= x^2y + h(y) \end{aligned} \quad (*)$$

We now determine $h(y)$. Differentiating equation $(*)$ with respect to y , we obtain

$$\frac{\partial u}{\partial y} = x^2 + h'(y) = N(x, y) = 1 + x^2$$

$\Rightarrow h'(y) = 1$. Integrating this last equation with respect to y , we obtain

$$h(y) = y + C_1.$$

Substituting this expression into $(*)$ yields

$$u(x, y) = x^2y + y + C_1$$

The solution to the DE, which is given implicitly as $u(x, y) = C$ is

$$x^2y + y = C_2, \quad (C_2 = C - C_1)$$

Solving for y explicitly, we obtain the solution as

$$y = \frac{C_2}{(x^2 + 1)}. \quad \square$$

Example 2.

Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$.

Solution: First test for exactness. Here

$M = x^3 + 3xy^2$ and $N = 3x^2y + y^3$. Thus
 $\frac{\partial M}{\partial y} = 6xy$, and $\frac{\partial N}{\partial x} = 6xy$. It is exact.

From (4) we have $u = \int_y M dx + h(y) =$

$$\int (x^3 + 3xy^2)dx + h(y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + h(y)$$

To find $h(y)$, we differentiate this equation with

respect to y and use (3), obtaining

$$\frac{\partial u}{\partial y} = 3x^2y + \frac{dh}{dy} = N = 3x^2y + y^3.$$

Hence $\frac{dh}{dy} = y^3$, so by integration $h = \frac{y^4}{4} + c^*$.

Thus $u(x, y) = \frac{1}{4}(x^4 + 6x^2y^2 + y^4) = c^{**}$,

where $(c^{**} = c - 4c^*)$. \square

- Sometimes, we **check our result**. For this we can differentiate $u(x, y) = c$ implicitly and see whether this leads to $\frac{dy}{dx} = -\frac{M}{N}$.

Note: The solution of the exact DE

$Mdx + Ndy = 0$ can easily be given by

$$\int_{y \text{ const.}} M dx + \int (\text{terms of } N \text{ not containing } x) dy = C$$

- Let's solve Example 2. using this method:

$$\int_{y \text{ const.}} (x^3 + 3xy^2) dx + \int (y^3) dy = C$$

$$\frac{x^4}{4} + \frac{3}{2}x^2y^2 + \frac{y^4}{4} = C$$

$$\frac{1}{4}(x^4 + 6x^2y^2 + y^4) = 4C$$

which is similar to the above result. \square

Exercise

1. Solve $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0.$

Ans. $e^{xy^2} + x^4 - y^3 = C.$

2. Solve $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$

Ans. $y \sin x + (\sin y + y)x = C.$

3. Solve the IVP

$$(\sin x \cosh y)dx - (\cos x \sinh y)dy = 0; \quad y(0) = 0.$$

Ans. $\cos x \cosh y = 1.$

Integrating Factors

(Reducing to the Exact Equations)

- We sometimes have an equation $M(x, y)dx + N(x, y)dy = 0$, that is not exact, but if we multiply it by a suitable function $I(x, y)$, the new equation

$$I(x, y)M(x, y)dx + I(x, y)N(x, y)dy = 0$$

is exact, so that it can be solved by the previous method. The function $I(x, y)$ is then called an *integrating factor*.

- How to find integrating factors?

- In simpler cases, integrating factors may be found by inspection or perhaps after some trials. In the general case, the idea is this:
- The equation $IMdx + INdy = 0$ is exact, and hence the exactness criterion $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ becomes

$$\frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial x}(IN), \text{ i.e.}$$

$$I_yM + IM_y = I_xN + IN_x.$$

- In the general case, this would be complicated and useless. Hence we look for an integrating factor depending only on **one variable**;

fortunately, in many practical cases, there are such factors, as we shall see.

- Thus, let $I = I(x)$. Then $I_y = 0$ and $I_x = I' = \frac{dI}{dx}$, so that $\frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial x}(IN)$ becomes

$$IM_y = I'N + IN_x$$

Dividing by IN and reshuffling terms, we have

$$\frac{1}{I} \frac{dI}{dx} = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \quad (*)$$

- Similarly, if $I = I(y)$, we get

$$\frac{1}{I} \frac{dI}{dy} = \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \quad (**)$$

- If $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \equiv g(x)$, a function of x alone,
then $I(x, y) = e^{\int g(x) dx}$. (by integrating $(*)$)
and taking exponents on both sides). Similarly,
- If $\frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \equiv h(y)$, a function of y alone,
then $I(x, y) = e^{-\int h(y) dy}$.
- If a DE does not have one of the forms given above, then a search for an integrating factor likely will not be successful, and other methods of solution are recommended.

Example 1.

Solve $2 \sin(y^2)dx + xy \cos(y^2)dy = 0$.

Solution: Here $M = 2 \sin(y^2)$, $N = xy \cos(y^2)$.

Now

$$\begin{aligned}\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) &= \frac{1}{xy \cos(y^2)} (4y \cos(y^2) - y \cos(y^2)) \\ &= \frac{3}{x} = g(x) \text{ and thus } I(x) = e^{\int \frac{3}{x} dx} = x^3.\end{aligned}$$

Finally, multiplying the given DE by this integrating factor, it becomes exact and its solution is $x^4 \sin(y^2) = C$. \square

Example 2.

Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$.

Solution: In this case $M = xy^3 + y$, and

$$N = 2(x^2y^2 + x + y^4). \text{ Hence } \frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) =$$

$$\frac{1}{y(xy^2+1)} (3xy^2 + 1 - (4xy^2 + 2)) = -\frac{1}{y},$$

which is a function of y alone. Therefore

$$I = e^{-\int -\frac{1}{y} dy} = e^{\int \frac{1}{y} dy} = e^{\ln y} = y.$$

Multiplying throughout by y , the DE becomes

$$(xy^4 + y^2)dx + (2x^2y^3 + 2xy + 2y^5)dy = 0,$$

which is an exact equation. The solution is

$$\frac{1}{2}x^2y^4 + xy^2 + \frac{1}{3}y^6 = C. \quad \square$$

Exercise

1. Convert $y' = 2xy - x$ into an exact DE.

Ans. $(-2xye^{-x^2} + xe^{-x^2})dx + e^{-x^2}dy = 0$

2. Solve $(2xy^2 + ye^x)dx = e^x dy$.

Ans. $\frac{e^x}{y} + x^2 = C$ (Here $I.F = \frac{1}{y^2}$)

3. Solve the IVP

$$2xy dx + (4y + 3x^2)dy = 0; y(0.2) = -1.5.$$

Ans. $y^4 + x^2y^3 = 4.9275$.