Differential Equations I

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Introduction

Why We Study Differential Equations?

- Differential equations are a fundamental branch of mathematics because they model how quantities change with respect to one another.
- Here are some reasons why studying them is so important:

- **Modeling Natural Phenomena:** Many physical systems, from the motion of planets to the behavior of electrical circuits, can be described by differential equations.

For example, Newtons laws of motion involve differential equations.

- **Engineering Applications:** In engineering, differential equations are used to model systems such as mechanical vibrations, heat transfer, fluid dynamics, and control systems. This helps in designing and analyzing various technologies.
- **Economics and Biology:** In economics, differential equations can model growth rates, market dynamics, and financial systems. In biology, they can describe population growth, the spread

of diseases, and other dynamic processes.

- **Mathematical Theory:** Studying differential equations also deepens our understanding of mathematical theory, including analysis and numerical methods. Its a rich field that connects various areas of mathematics.
- **Problem-Solving Skills:** Learning to solve differential equations develops critical thinking and problem-solving skills. It teaches us to approach complex problems systematically and to apply mathematical techniques in diverse contexts.

What is Differential Equation?

Definition: An equation that involves an unknown function y (dependent variable), various derivatives of y, and possibly other known functions is called a ***Differential Equation(DE)***.

- Differential equations are of fundamental importance in applications, not only in mathematics but in virtually all sciences and engineering.
- Many physical laws and relations appear mathematically in the form of DEs.

- For an applied mathematician, the study of a DE consists of three phase:
  i. formulation of the DE from the given physical situation, called **modeling**.
  ii. solutions of this DE, evaluating the arbitrary constants from the given conditions.
  iii. physical interpretation of the solution.

Examples

1. $\dfrac{dy}{dt} = -9.8t$, the motion of a falling body.
2. $\dfrac{dy}{dt} + 2y = 100$, the change in the size of a population.
3. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 9y = \cos t$, the flow of current in an electric circuit.
4. $\dfrac{d^2y}{dx^2} + \sin y = 0$, the motion of a pendulum.

Classification of Differential Equations

- The subject of DEs is vast, and the first task is to break it down into more manageable parts.
- The major division is into:
  1. **Ordinary Differential Equation (ODE)**, which involve only ordinary derivatives of a function of a single independent variable.
  2. A differential equation involving partial derivatives of a function of more than one independent variable is called **Partial Differential Equation (PDE)**.

Examples:

1. All the above equations are ODEs.
2. $e^x\,dx + e^y\,dy = 0$, $\dfrac{d^2x}{dt^2} + n^2x = 0$,
   $y' = \cos x$, and $y'' + 4y = x$ are ODEs.
3. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$, $\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} - z = 0$, and
   $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$ are examples of PDEs.

- The other main classification is by **order**.

  Definition: The **order** of a DE is the order of the highest derivative that appears in the equation.

- The **degree** of a DE is the exponent to which the highest derivative is raised after fractions and radicals involving y or its derivatives have been removed from the equation.

  Example. $\frac{dy}{dx} + 2y = 100$ is of the 1st order and 1st degree, $y'' - 2xy' + y = e^x$ is 2nd order 1st degree, and $y = x\frac{dy}{dx} + \frac{5}{\frac{dy}{dx}}$ is of order 1 degree 2.

Exercise: Find the order and degree of the following differential equations.

1. $(x + y^2)\, dx + (x^2 - y)\, dy = 0.$
   Ans. order 1, degree 1
2. $(y''')^4 + 5(y'')^5 - 2y' = x^2.$
   Ans. order 3, degree 4
3. $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}.$ **Ans.** order 2, degree 1
4. $(y')^{\frac{3}{2}} = y'' + 1.$ **Ans.** order 2, degree 2

Note:

- We can express an $\mathbf{n^{th}}$**–order** ODE in one dependent variable by the **general form**
$$F(x, y, y', \cdots, y^{(n)}) = 0$$
where F is a real–valued function of $n+2$ variables.

- The differential equation
$$\frac{d^n y}{dx^n} = f(x, y, y', \cdots, y^{(n-1)}),$$
where f is a real-valued continuous function, is referred to as the **normal form**.

- Thus when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx} = f(x, y) \quad \text{and} \quad \frac{d^2y}{dx^2} = f(x, y, y')$$

  to represent **general first-** and **second-order** ordinary differential equations.

- For **example**, the normal form of the first–order equation $4xy' + y = x$ is $y' = \frac{(x-y)}{4x}$, and the normal form of the second–order equation

$$y'' - y' + 6y = 0 \text{ is } y'' = y' - 6y.$$

Classification by Linearity

- A differential equation is said to be **linear** if it is linear (first degree) in the dependent variable and its derivatives, and those coefficients are a function of the independent variable.
- That is, a differential equation is linear if the dependent variable and its derivatives are not multiplied together, not raised to powers, do not occur as the arguments of functions.
- A **nonlinear** differential equation is simply one that is **not** linear.

- Or an n^{th} order differential equation is **linear** if it can be written in the form
$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1y'+a_0(x)y = g(x)$,
the coefficients $a_i(x)$ are function of x alone.
- Two important special cases are **linear first-order** $(n = 1)$ and **linear second-order** $(n = 2)$ equations:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \text{ and}$$

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

- A **solution** of a DE in the unknown function y and the independent variable x on the interval I, is a function $y(x)$ that satisfies the DE identically for all x in I.
- For **example**, $y = x^2$ is a solution of the DE $xy' = 2y$. How?

Homework: Show that e^{2x} and e^{3x} are solutions of $y'' - 5y' + 6y = 0$.

- The **general solution** includes all possible solutions and typically includes arbitrary constants (in the case of an ODE) or arbitrary functions (in the case of a PDE.)
- A solution without arbitrary constants/functions is called a **particular solution**. A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants or it is any one solution.
- Often we find a particular solution to a differential equation by giving extra conditions in the form of initial or boundary conditions.

Example. $y = c_1x + c_2$ is a general solution of the differential equation $\frac{d^2y}{dx^2} = 0$ and $y = -3x+5$ is its particular solution.

- A DE may sometimes have an additional solution that **cannot** be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution**.
- If a solution of a differential equation is given explicitly as $y = f(x)$ we call it an **explicit solution**, otherwise if it is of the form $h(x, y) = 0$ it is called an **implicit solution**.

Initial-Value and Boundary-Value Problems

- In application one may be interested to find a solution to a differential equation satisfying certain defined conditions.
- If all the conditions are given at one point(value) of the independent variable the conditions are called **initial conditions**, and if the conditions are given at more than one point of the independent variable the conditions are called **boundary conditions**.
- A differential equation together with initial

conditions is called an **initial-value problem (IVP)**, and a differential equation along with boundary conditions constitutes a **boundary-value problem (BVP)**.

Example:

1. The problem $y'' + 2y' = e^x; y(\pi) = 1, y'(\pi) = 2$ is an IVP because the two subsidiary conditions are both given at $x = \pi$.
2. The DE $y'' + y = 0$ subject to the conditions $y(0) = 0, y'(\pi/2) = 1$ is a BVP, because the two subsidiary conditions are given at the different values $x = 0$ and $x = \frac{\pi}{2}$.

Note: A solution to an initial–value problem or boundary–value problem is a function $y(x)$ that both solves the DE and satisfies all given subsidiary conditions.

Exercise: Let $y = Ce^{-2t} + 50$. Find the value of C for which y is the particular solution of

$$\frac{dy}{dt} + 2y = 100; \; y(0) = 10.$$

Answer: $C = -40$