Applications of First-Order DEs

- It is often desirable to describe the behavior of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms.
- The mathematical description of a system of phenomenon is called a **mathematical model** and is constructed with certain goals in mind.
- For example, we may wish to understand the mechanisms of a certain ecosystem by studying the growth of animal populations in that system.

1. Population Dynamics

- One of the earliest attempts to model human **population growth** by means of mathematics was by the English economist *Thomas Malthus* in 1798.
- Basically, the idea behind the Malthusian model is the assumption that the rate at which the population of a country grows at a certain time is **proportional** to the total population of the country at that time.
- In other words, the more people there are at time t, the more there are going to be in the future.

- In mathematical terms, if $P(t)$ denotes the total population at time t, then this assumption can be expressed as

$$\frac{dP}{dt} \propto P \quad \text{or} \quad \frac{dP}{dt} = kP,$$

where k is a constant of proportionality.

- Populations that grow at a rate described by this model are rare; nevertheless, it is still used to model growth of small populations over short intervals of time (bacteria growing in a petri dish, for example).

Note:

- If two quantities a and b are **proportional**, we write $a \propto b$. This means that one quantity is a constant multiple of the other: $a = kb$.
- The initial-value problem

$$\frac{dx}{dt} = kx, \;\; x(t_0) = x_0, \qquad\qquad (*)$$

  where k is a constant of proportionality, serves as a model for diverse phenomena involving either **growth or decay**.
- As we have seen above, in biological applications the rate of growth of certain populations

(bacteria, small animals) over short periods of time is proportional to the population present at time t.

- Knowing the population at some arbitrary initial time t_0, we can then use the solution of $(*)$ to predict the population in the future–that is, at times $t > t_0$.
- The constant of proportionality k in $(*)$ can be determined from the solution of the initial-value problem, using a subsequent measurement of x at a time $t_1 > t_0$.

Example. A population of a small town grows proportion to its current population. The initial population is 5000 and grows 4% per year. This can be modeled by

$$\frac{dP}{dt} = 0.04P, \; P(0) = 5,000$$

a) Find an equation to model the population.

b) Determine the population after 3 years.

c) How long will it take the population to double?

Solution:

a) From $\frac{dP}{dt} = 0.04P$, we have $dP = 0.04Pdt$ or

$\frac{1}{P}dP = 0.04dt$, which is **separable** DE.

Now integrating both sides, we obtain

$$\int \frac{1}{P}\, dP = \int 0.04\, dt$$

$$\ln P = 0.04t + C_1 \quad (P \text{ is positive})$$

$$P(t) = e^{0.04t + C_1} = e^{0.04t} \cdot e^{C_1}$$

$$= C_2 e^{0.04t} \quad (\text{here } C_2 = e^{C_1})$$

Finally, using the initial condition we have $P(0) = C_2 e^0 = 5,000$ which implies $C_2 = 5,000$. Hence, the required equation is

$$\boldsymbol{P(t) = 5000\, e^{0.04t}}$$

b) The population after 3 years is

$$P(3) = 5000\, e^{0.04(3)} \approx \mathbf{5637} \text{ people.}$$

c) Here $10,000 = 5,000\, e^{0.04t}$ or $2 = e^{0.04t}$ so that

$$t = \frac{\ln 2}{0.04} \approx \mathbf{17.3} \text{ years.} \ \square$$

Exercise: Bacterial Growth

A culture initially has P_0 number of bacteria. At $t = 1$hr the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

Answer: We first solve the DE $\frac{dP}{dt} = kP$.
With $t_0 = 0$ the initial condition is $P(0) = P_0$.
We then use the empirical observation that $P(1) = \frac{3}{2}P_0$ to determine the constant of proportionality k.

Thus, solving the DE we have $P(t) = P_0 e^{kt}$.
At $t = 1$ we have $\frac{3}{2}P_0 = P_0 e^k$ or $e^k = \frac{3}{2}$
from which we get $k = \ln\frac{3}{2} = 0.4055$. Hence

$$P(t) = P_0\, e^{0.4055t}$$

To find the time at which the number of bacteria has tripled, we solve $3P_0 = P_0\, e^{0.4055t}$ for t.
It follows that

$$0.4055t = \ln 3, \quad \text{or} \quad t = \frac{\ln 3}{0.4055} \approx \mathbf{2.71}\,\text{hr.} \quad ❑$$

Notice:

- In exercise above the actual number P_0 of bacteria present at time $t = 0$ played no part in determining the time required for the number in the culture to triple.
- The time necessary for an initial population of, say, 100 or $1,000,000$ bacteria to triple is still approximately 2.71 hours.
- Problems describing **growth** (whether of populations, bacteria, or even capital) are characterized by a positive value of $\boldsymbol{k}$, whereas problems involving **decay**, as in radioactive disintegration,

yield a **negative *k*** value.

- Accordingly, we say that k is either a **growth constant** ($k > 0$) or a **decay constant** ($k < 0$).

2. Newton's Law of Cooling

- According to Newton's empirical law of cooling/warming, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, the so-called ambient temperature.
- If $T(t)$ represents the temperature of a body at time t, T_s the temperature of the surrounding medium, and dT/dt the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the

mathematical statement

$$\frac{dT}{dt} \propto T - T_s \quad \text{or} \quad \frac{dT}{dt} = k(T - T_s),$$

where k is a constant of proportionality.

- In either case, cooling or warming, if T_s is a constant, it stands to reason that $k < 0$.

Example. A cup of fast-food coffee is $180^\circ F$ when freshly poured. After 2 minutes in a room at $70^\circ F$, the coffee has cooled to $165^\circ F$.

a) Find the function that models the cooling of the coffee.

b) What will the temperature be after 10 minutes?

c) Find the time that it will take for the coffee to cool to $120°F$.

Solution:

a) Here $T_s = 70$, $T_0 = T(0) = 180$, and $T(2) = 165$. Then we should solve the IVP

$$\frac{dT}{dt} = k(T - T_s) = k(T - 70), \; T(0) = 180$$

It is both linear and separable. Separating variables, we have

$$\frac{dT}{T-70} = kdt,$$

which gives on integration

$$\ln|T-70| = kt + C_1, \text{ or}$$

$$T(t) = 70 + C_2e^{kt}$$

Now using $T(0) = 180$, we have $180 = 70 + C_2e^0$ which gives $C_2 = 110$. Therefore,

$$T(t) = 70 + 110e^{kt}$$

To find k, we use $T(2) = 165$:

$$T(2) = 70 + 110e^{2k} = 165$$

$$\frac{95}{110} = e^{2k} \quad \text{or} \quad \ln\left(\frac{95}{110}\right) = 2k$$

$$k = \frac{\ln\left(\frac{95}{110}\right)}{2} \approx -0.073$$

Therefore $\boldsymbol{T(t) = 70 + 110e^{-0.073t}}$.

b) After 10 minutes, the temperature will be

$$T(10) = 70 + 110e^{-0.073(10)} = \mathbf{123}.$$

c) When $T = 120$, we have

$$120 = 70 + 110e^{-0.073\,t}$$

$$\frac{50}{110} = e^{-0.073\,t} \quad \text{or} \quad \ln\left(\frac{5}{11}\right) = -0.073\,t$$

which implies $\boldsymbol{t \approx 10.8}$ minutes. ❑

Exercise

1. A bowl of soup at $190^\circ F$ is left in a room of $70^\circ F$. At time $t = 0$, the soup is cooling at $15^\circ F$ per minute.
   i) Find the function that models the cooling of the soup.
   ii) How long will it take for the temperature to reach $143^\circ F$?

Answer: i) $T(t) = 70 + 120e^{-0.125t}$
ii) $t \approx 3.98$

2. A glass of hot water has an initial temperature $80°C$, placed in a room where the temperature is $30°C$. After one minute the water temperature drops to $70°C$. What will be the temperature after 3 minutes? At what time the water cools down to $40°C$?

Answer: Here $T(t) = 30 + 50e^{\ln(0.8)t}$ so that

$$T(3) = 30 + 50e^{3\ln(0.8)} = \mathbf{55.6}°C$$

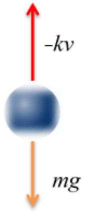
To find the time when $T = 40°C$, we use

$$40 = 30 + 50e^{\ln(0.8)t}$$

which gives $\boldsymbol{t \approx 7.2}$ minutes.

3. Falling Objects/Bodies

- Consider a falling object influenced by gravity and an air resistance proportional to the velocity of the object.
- There are two forces acting on the object, the force due to gravity given by the weight $w = mg$ and the force due to air resistance which is $-kv$.



The **net force** is $F = mg - kv$.

- Now using **Newton's** second law of motion we perform a substitution.

$$F = ma = m\frac{dv}{dt} = mg - kv$$

Hence $m\dfrac{dv}{dt} + kv = mg$ or $\boxed{\boldsymbol{\dfrac{dv}{dt} + \dfrac{k}{m}v = g}}$

- Note here that air resistance must be proportional to velocity.
- The last equation is linear first order differential equation.

Example 1. A steel ball weighing 1 pound (lb) is dropped from $2500\, ft$ with no velocity. As it fall, air resistance is equal to $v/8$ in pounds where v is the velocity of the ball in feet per second. Find the limiting velocity and the time it takes for the ball to hit the ground.

Solution: Here $g = 32\,\text{ft/sec}^2$, $k = \frac{1}{8}$, and $m = \frac{w}{g} = \frac{1}{32}$. Now using $\frac{dv}{dt} + \frac{k}{m}v = g$ we have

$$\frac{dv}{dt} + \frac{1/8}{1/32}v = 32 \quad \text{or} \quad \frac{dv}{dt} + 4v = 32.$$

Solving this linear first-order DE, we get

$$v(t) = 8 + Ce^{-4t}$$

To find C, use the initial condition $v(0) = 0$ from which $C = -8$. Hence, the velocity function is

$$\boldsymbol{v(t) = 8 - 8e^{-4t}}.$$

Limiting velocity $= \lim_{t\to\infty} v(t) = \lim_{t\to\infty}(8 - 8e^{-4t}) =$ **8** ft/sec.

To find the time it takes to hit the ground, we need to find the position function $s(t)$:

$$s(t) = \int v(t)dt = \int (8 - 8e^{-4t})dt$$

$$= 8t + 2e^{-4t} + C$$

Letting $s(0) = 0$, we get $C = -2$. Thus, the position function is $\boldsymbol{s(t) = 8t + 2e^{-4t} - 2}$.

Now solving

$$2500 = 8t + 2e^{-4t} - 2$$

we get $\boldsymbol{t \approx 312.75}$ seconds. ❑

Example 2. A body is dropped from a height of 300 ft with an initial velocity of 30 ft/sec. Assuming no air resistance (**free fall**), find the time required for the body to hit the ground.

Solution: In this case the equation becomes

$$\frac{dv}{dt} = g \quad \text{or} \quad \frac{dv}{dt} = 32$$

which is separable DE. Integrating both sides we have $v(t) = 32t + C_1$.

Using $v(0) = 30$, we get $C_1 = 30$. Therefore,

$$\boldsymbol{v(t) = 32t + 30}$$

Integrating both sides of this equation, we obtain

$$s(t) = \int (32t + 30)dt = 16t^2 + 30t + C_2$$

Assuming $s(0) = 0$, we get $C_2 = 0$. Hence

$$\boldsymbol{s(t) = 16t^2 + 30t}$$

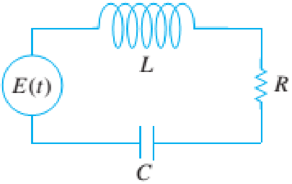
Now $300 = 16t^2 + 30t$ implies $16t^2 + 30t - 300 = 0$ which gives $\boldsymbol{t \approx 3.49}$. ❑

Exercise: In Example 2 above, what is an expression for the velocity of the body at any time t? Also find an expression for the position of the body at any time t.

4. Electrical Circuits

- Consider the single-loop series circuit shown in figure below, containing an inductor, resistor, and capacitor. The current in a circuit after a switch is closed is denoted by $I(t)$; the charge on a capacitor at time t is denoted by $q(t)$.
- The letters L, R, and C are known as inductance, resistance, and capacitance, respectively, and are generally constants.
- Now according to **Kirchhoff's second law**, the impressed voltage $E(t)$ on a closed loop must equal the sum of the voltage drops in the loop.

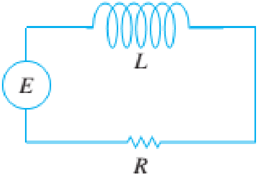
- The current $I(t)$ is related to charge $q(t)$ on the capacitor by $I = \dfrac{dq}{dt}$.



LRC-series circuit

RL-series circuit:

- For a series circuit containing only a resistor and an inductor, Kirchhoff's second law states that the sum of the voltage drop across the inductor ($L\frac{dI}{dt}$) and the voltage drop across the resistor (IR) is the same as the impressed voltage ($E(t)$) on the circuit.



LR series circuit

- Thus we obtain the linear differential equation for the current $I(t)$,

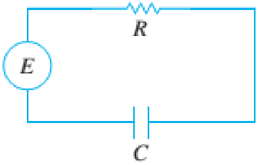
$$\boxed{L\frac{dI}{dt}+RI = E(t)}$$

where L and R are constants known as the inductance and the resistance, respectively. The current $I(t)$ is also called the **response** of the system.

RC-series circuit:

- The voltage drop across a capacitor with capacitance C is given by $\frac{q(t)}{C}$, where q is the charge on the capacitor. Hence, for the series circuit shown in figure below, Kirchhoff's second law gives

$$RI + \frac{1}{C}q = E(t)$$



RC series circuit

- But current I and charge q are related by $I = dq/dt$, so the equation becomes the linear differential equation

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Example 1. A 12−volt battery is connected to an RL series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current $i(t)$ if the initial current is zero.

Solution: From $L\dfrac{di}{dt} + Ri = E(t)$ we see that we must solve

$$\frac{1}{2}\frac{di}{dt} + 10i = 12,$$

subject to $i(0) = 0$. First, we multiply the differential equation by 2 and read off the integrating factor e^{20t}. We then obtain

$$\frac{d}{dt}(ie^{20t}) = 24e^{20t}$$

Integrating each side of the last equation and solving for i gives

$$i(t) = \frac{6}{5} + Ce^{-20t}$$

Now $i(0) = 0$ implies that $0 = \frac{6}{5} + C$ or $C = -\frac{6}{5}$. Therefore the current is

$$\boldsymbol{i(t)} = \frac{6}{5} - \frac{6}{5}e^{-20t}. \quad \square$$

Example 2. A 100 volt electromotive force is applied to an RC series circuit in which the resistance is $200\,\Omega$ and the capacitance is 10^{-4} farad. Find the charge q(t) on the capacitor if $q(0) = 0$. Also, find the current $I(t)$.

Solution: Using $\displaystyle R\frac{dq}{dt} + \frac{q}{C} = E(t)$ we have

$$200\frac{dq}{dt} + \frac{1}{10^{-4}}q = 100 \quad \text{or} \quad \frac{dq}{dt} + 50q = \frac{1}{2}$$

Solving this last linear DE, we obtain

$$q(t) = Ce^{-50t} + \frac{1}{100}$$

From the initial condition, $q(0) = 0$, we obtain $C = -\frac{1}{100}$. Thus,

$$\boldsymbol{q(t)} = -\frac{1}{100}e^{-50t} + \frac{1}{100}, \quad \text{and}$$

$$\boldsymbol{I(t)} = \frac{dq}{dt} = \frac{1}{2}e^{-50t}. \quad ❑$$

Exercise: An RC circuit has an emf of 100 volts, a resistance of 5 ohms, a capacitance of 0.02 farad, and an initial charge on the capacitor of 5 coulombs. Find an expression for the charge on the capacitor at any time t.

Answer: $q(t) = 3e^{-10t} + 2$

Next!

Second-Order DEs