Linear Differential Equations

- A first-order DE is said to be **linear** if it can be written in the (standard) form
$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (*)$$
- The characteristic feature of this equation is that it is linear in y and y', whereas P and Q on the right may be any given functions of x.
- If the right side $Q(x)$ is **zero** for all x in the interval in which we consider the equation, the differential equation is said to be **homogeneous**; otherwise it is said to be **non-homogeneous**.

- Let us find a formula for the **general solution** of $(*)$ in some interval I, assuming that P and Q are continuous in I.
- For the **homogeneous** equation

$$y' + P(x)y = 0$$

this is vey simple. Indeed, by separating variables we have

$$\frac{dy}{y} = -P(x)dx, \text{ thus}$$

$$\ln|y| = -\int P(x)dx + C^*$$

and by taking exponents on both sides

$$y(x) = Ce^{-\int P(x)dx} \quad (C = \pm e^{C^*})$$

Here we may take $C = 0$ and obtain the **trivial solution** $y = 0$.

- The **non-homogeneous** equation $(*)$ will now be solved. It turns out that it has the pleasant property of possessing an integrating factor depending only on x. That is,

$$\boxed{I.F = e^{\int P dx}}$$

- Now we multiply both sides of $(*)$ by this I.F so that we get

$$\frac{dy}{dx}\, e^{\int P dx} + y\, e^{\int P dx} P(x) = Q(x)\, e^{\int P dx}$$
$$\Rightarrow \frac{d}{dx}\left(y e^{\int P dx}\right) = Q(x)\, e^{\int P dx}$$

- Finally integrate both sides of this last equation with respect to x, and then solve the resulting equation for y. i.e.

$$y\, e^{\int P dx} = \int Q(x)\, e^{\int P dx} dx + C$$

Example 1. Solve $y' - 3y = 6$.

Solution:

It is linear with $P(x) = -3$ and $Q(x) = 6$.

Here $\displaystyle\int P(x)dx = \int -3\,dx = -3x.$

Thus $I(x) = e^{\int P(x)dx} = \boldsymbol{e^{-3x}}$.

Multiplying the DE by the I.F, we obtain

$$e^{-3x}y' - 3\,e^{-3x}y = 6\,e^{-3x}$$

$$\text{or} \quad \frac{d}{dx}\left(ye^{-3x}\right) = 6\,e^{-3x}$$

Integrating both sides with respect to x, we have

$$\int \frac{d}{dx}\left(ye^{-3x}\right)dx = \int 6\,e^{-3x}dx$$

$$ye^{-3x} = -2\,e^{-3x} + C$$

$$y = C\,e^{3x} - 2. \ \square$$

Note: No need to include the constant of integration while calculating I.F.

Example 2. Find an integrating factor and then solve the DE $\dfrac{dy}{dx} - 2xy = x$.

Solution: The DE is linear with $P(x) = -2x$

and $Q(x) = x$. Hence

$$I(x) = e^{\int P(x)dx} = e^{\int(-2x)dx} = \boldsymbol{e^{-x^2}}.$$

Now multiplying the DE by this $I(x)$, we get

$$e^{-x^2}y' - 2xe^{-x^2}y = x\,e^{-x^2}$$

$$\frac{d}{dx}\left(ye^{-x^2}\right) = x\,e^{-x^2}$$

Integrating both sides, we have

$$\int \frac{d}{dx}\left(ye^{-x2}\right)dx = \int x\,e^{-x^2}dx$$

$$ye^{-x^2} = -\tfrac{1}{2}e^{-x^2} + C \quad \text{or} \quad \boldsymbol{y = Ce^{x^2} - \tfrac{1}{2}}. \;\square$$

Exercises

1. Solve $y' + y = \sin x$.

   Ans. $y = Ce^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$

2. Solve $\dfrac{dz}{dx} - xz = -x$.

   Ans. $z(x) = Ce^{\frac{x^2}{2}} + 1$.

3. Solve the IVP $\; y' + y\tan x = \sin 2x, \; y(0) = 1$.

   Ans. $y(x) = C\cos x - 2\cos^2 x$,

   and using the initial condition, we get $C = 3$.

   Therefore, $\; y = 3\cos x - 2\cos^2 x$.

Bernoulli's Equation

(Reduction to Linear form)

- It is named after the Swiss mathematician *Jakob Bernoulli* who is known for his basic work in probability and elasticity theory.
- Certain nonlinear DEs can be reduced to linear form. The practically most famous of these is the Bernoulli equation. It has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad (**)$$

where n is a real number.

If $n = 0$ or $n = 1$, the equation is **linear**.

- To solve $(**)$, divide both sides by y^n, so that

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \qquad (***)$$

Put $y^{1-n} = z$ so that $(1-n)y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

Thus $(***)$ becomes

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x) \quad \text{or}$$

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

which is **linear** in $\boldsymbol{z}$.

Example 1. Solve $y' + xy = xy^2$.

Solution: This equation is not linear. It is, however, a Bernoulli DE with $P(x) = Q(x) = x$, and $n = 2$. We now make the substitution $z = y^{1-n} = y^{-1}$, from which follow

$$y = \frac{1}{z} \text{ and } y' = -\frac{1}{z^2} z'$$

Substituting these equations into the DE, we get

$$-\frac{z'}{z^2} + \frac{x}{z} = \frac{x}{z^2}$$

$$\Rightarrow \; z' - xz = -x, \text{ which is linear.}$$

Its solution is $z = Ce^{\frac{x^2}{2}} + 1$. (Show!)

The solution of the original DE is then

$$y = \frac{1}{z} = \frac{1}{Ce^{\frac{x^2}{2}} + 1}. \quad \square$$

Example 2. Solve $y' - \frac{3}{x}y = x^4 y^{\frac{1}{3}}$.

Solution: This is a Bernoulli DE with $P(x) = -\frac{3}{x}$, $Q(x) = x^4$, and $n = 1/3$. Using the substitution $z = y^{1-n} = y^{1-\frac{1}{3}} = y^{\frac{2}{3}}$, we have

$$y = z^{\frac{3}{2}} \text{ and } y' = \frac{3}{2} z^{\frac{1}{2}} z'.$$

Substituting these values into the DE, we get

$$\frac{3}{2}z^{\frac{1}{2}}z' - \frac{3}{x}z^{\frac{3}{2}} = x^4 z^{\frac{1}{2}}$$

$$\text{or} \quad z' - \frac{2}{x}z = \frac{2}{3}x^4, \ \text{ which is linear.}$$

Its solution is $z = Cx^2 + \frac{2}{9}x^5$.

Since $z = y^{\frac{2}{3}}$, the solution of the original problem is given implicitly by

$$y^{\frac{2}{3}} = Cx^2 + \frac{2}{9}x^5. \ \square$$

Exercise

1. Solve $x\dfrac{dy}{dx} + y = x^3y^6$.

Ans. $1 = (\frac{5}{2} + Cx^2)x^3y^5$.

2. Solve the DE $(x + 3y + 2)\dfrac{dy}{dx} = 1$.

Ans. It is not linear in y and $\frac{dy}{dx}$.
But writing it in the form $\frac{dx}{dy} = x + 3y + 2$
or $\frac{dx}{dy} - x = 3y + 2$, it is linear in x and $\frac{dx}{dy}$.
Its solution is $\boldsymbol{x = -3y - 5 + Ce^{y}}$.