Mathematical Background for Cryptography

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Part I - Modular Arithmetic and Congruence

Preliminary

- The division relationship $a = q \times n + r$ has two inputs (a and n) and two outputs (q and r).
- In modular arithmetic, we are interested in only one of the outputs, the remainder r.
- We use modular arithmetic in our daily life; for example, we use a clock to measure time. Our clock system uses modulo 12 arithmetic. However, instead of a 0 we use the number 12.

Modulo Operator

- The modulo operator is shown as **mod**. The second input (n) is called the **modulus**. The output r is called the **residue**.

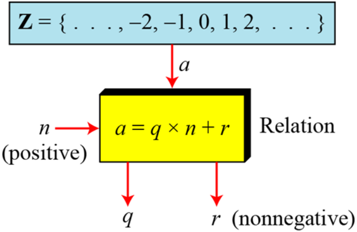


Figure: Division algorithm and modulo operator

Modulo Operator(cont.)

Find the result of the following operations :

- 27 mod 5
  Dividing 27 by 5 results in r = 2
- 36 mod 12
  Dividing 36 by 12 results in r = 0
- -18 mod 14
  Dividing -18 by 14 results in r = -4. After adding the modulus r = 10
- -7 mod 10
  Dividing -7 by 10 results in r = -7. After adding the modulus to -7, r = 3

Set of Residues

The modulo operation creates a set, which in modular arithmetic is referred to as the **set of least residues modulo n**, or Z_n.

$$\boxed{\mathbf{Z}_n = \{\, 0, 1, 2, 3, \ \ldots\ , (n-1) \,\}}$$

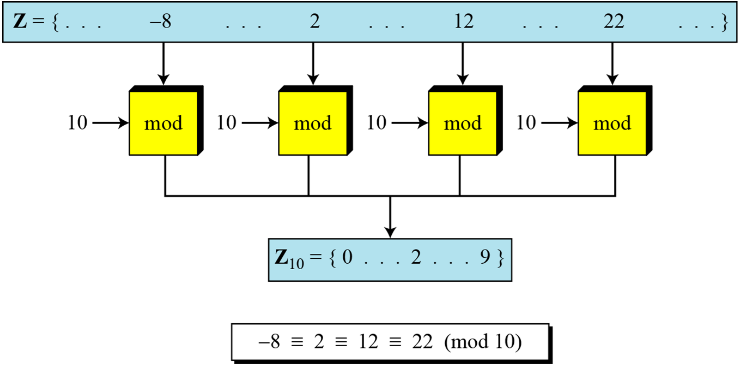
$$\boxed{\mathbf{Z}_2 = \{\, 0, 1 \,\}} \quad \boxed{\mathbf{Z}_6 = \{\, 0, 1, 2, 3, 4, 5 \,\}} \quad \boxed{\mathbf{Z}_{11} = \{\, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \,\}}$$

Figure: Some Z_n sets

Congruence

To show that two integers are congruent, we use the congruence operator $\equiv$. For example, we write:

$$2 \equiv 12 \pmod{10} \qquad 13 \equiv 23 \pmod{10}$$
$$3 \equiv 8 \pmod{5} \qquad 8 \equiv 13 \pmod{5}$$



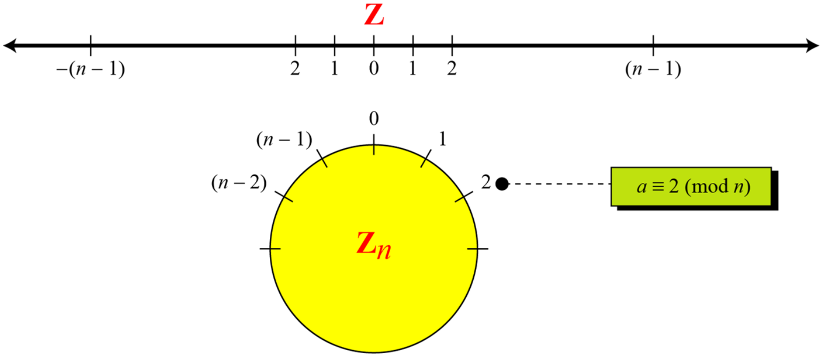
Congruence Relationship

Residue Classes

- A residue class $[a]$ or $[a]_n$ is the set of integers congruent modulo n.
- It is the set of all integers such that $x \equiv a(mod)n$
- E.g. for $n = 5$, we have five sets as shown below:

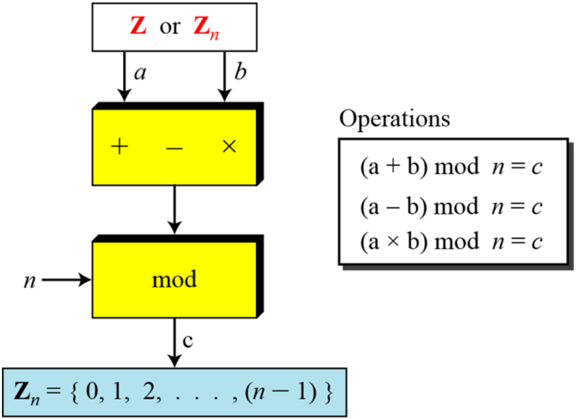
$$
\begin{aligned}
[0] &= \{\ldots, -15, \ -10, \ -5, 0, \ \ 5, 10, 15, \ldots\} \\
[1] &= \{\ldots, -14, \ \ -9, \ -4, 1, \ \ 6, 11, 16, \ldots\} \\
[2] &= \{\ldots, -13, \ \ -8, \ -3, 2, \ \ 7, 12, 17, \ldots\} \\
[3] &= \{\ldots, -12, \ \ -7, \ -5, 3, \ \ 8, 13, 18, \ldots\} \\
[4] &= \{\ldots, -11, \ \ -6, \ -1, 4, \ \ 9, 14, 19, \ldots\}
\end{aligned}
$$

Comparison of Z and Z_n using graphs



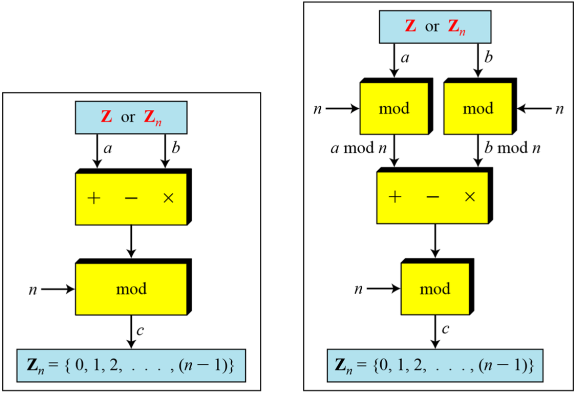
Operation in Z_n

The three binary operations that we discussed for the set Z can also be defined for the set Z_n. The result may need to be mapped to Z_n using the mod operator.



Operation in Z_n

First Property:	$(a + b) \bmod n = [(a \bmod n) + (b \bmod n)] \bmod n$
Second Property:	$(a - b) \bmod n = [(a \bmod n) - (b \bmod n)] \bmod n$
Third Property:	$(a \times b) \bmod n = [(a \bmod n) \times (b \bmod n)] \bmod n$



a. Original process   b. Applying properties

Operation in Z_n

In arithmetic, we often need to find the remainder of powers of 10 when divided by an integer.

$$10^n \bmod x = (10 \bmod x)^n \qquad \text{Applying the third property } n \text{ times.}$$

$$\begin{aligned}
10 \bmod 3 = 1 \quad &\rightarrow \quad 10^n \bmod 3 = (10 \bmod 3)^n = 1 \\
10 \bmod 9 = 1 \quad &\rightarrow \quad 10^n \bmod 9 = (10 \bmod 9)^n = 1 \\
10 \bmod 7 = 3 \quad &\rightarrow \quad 10^n \bmod 7 = (10 \bmod 7)^n = 3^n \bmod 7
\end{aligned}$$

Exercise: We have been told in arithmetic that the remainder of an integer divided by 3 is the same as the remainder of the sum of its decimal digits. In other words, the remainder of dividing 6371 by 3 is same as dividing 17 by 3. Prove this claim using the properties of the mod operator.

Inverses

- When we are working in modular arithmetic, we often need to find the inverse of a number relative to an operation.
- We are normally looking for an additive inverse (relative to an addition operation) or a multiplicative inverse (relative to a multiplication operation).
- In Z_n, two numbers a and b are additive inverses of each other if,
  $a + b \equiv 0 mod n$

In modular arithmetic, each integer has an additive inverse. The sum of an integer and its additive inverse is congruent to 0 modulo n.

- Find all additive inverse pairs in Z_{10}.
  Solution : The six pairs of additive inverses are (0, 0), (1, 9), (2, 8), (3, 7), (4, 6), and (5, 5).

Multiplicative Inverses

In Z_n, two numbers a and b are the multiplicative inverse of each other if,
$a \times b \equiv 1 mod n$
In modular arithmetic, an integer may or may not have a multiplicative inverse. When it does, the product of the integer and its multiplicative inverse is congruent to 1 modulo n.

Multiplicative Inverses(cont.)

- Find the multiplicative inverse of 8 in Z_{10}.
  Solution : There is no multiplicative inverse because $\gcd(10, 8) = 2 \neq 1$. In other words, we cannot find any number between 0 and 9 such that when multiplied by 8, the result is congruent to 1.
- Find all multiplicative inverses in Z_{10}.
  Solution : There are only three pairs: (1, 1), (3, 7) and (9, 9). The numbers 0, 2, 4, 5, 6, and 8 do not have a multiplicative inverse.
- Find all multiplicative inverses in Z_{11}.
  Solution : We have seven pairs: (1, 1), (2, 6), (3, 4), (5, 9), (7, 8), (9, 9), and (10, 10).

Euclidean algorithm

- The extended Euclidean algorithm finds the multiplicative inverses of b in Z_n when n and b are given and $gcd(n, b) = 1$.
- The multiplicative inverse of b is the value of t after being mapped to Z_n.

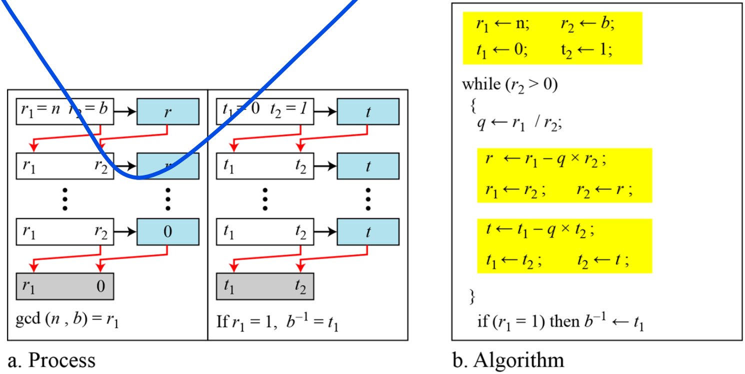


Figure: Using extended Euclidean algorithm to find multiplicative inverse

Euclidean algorithm (cont.)

Find the multiplicative inverse of 11 in Z_{26}.

q	r_1	r_2	r	t_1	t_2	t
2	26	11	4	0	1	−2
2	11	4	3	1	−2	5
1	4	3	1	−2	5	−7
3	3	1	0	5	−7	26
	1	0		−7	26	

The gcd (26, 11) is 1; the inverse of 11 is -7 or 19.

Euclidean algorithm (cont.)

Find the multiplicative inverse of 12 in Z_{26}.

q	r_1	r_2	r	t_1	t_2	t
2	26	12	2	0	1	−2
6	12	2	0	1	−2	13
	2	0		−2	13	

The gcd (26, 12) is 2; the inverse does not exist.

Euclidean algorithm (cont.)

Exercise: Find the multiplicative inverse of 23 in Z_{100}.

Euclidean algorithm (cont.)

Solution: Find the multiplicative inverse of 23 in Z_{100}.

q	r_1	r_2	r	t_1	t_2	t
4	100	23	8	0	1	−4
2	23	8	7	1	−4	19
1	8	7	1	−4	9	−13
7	7	1	0	9	−13	100
	1	0		−13	100	

The gcd (100, 23) is 1; the inverse of 23 is -13 or 87.

Addition and Multiplication Tables

	0	1	2	3	4	5	6	7	8	9
0	**0**	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	**0**
2	2	3	4	5	6	7	8	9	**0**	1
3	3	4	5	6	7	8	9	**0**	1	2
4	4	5	6	7	8	9	**0**	1	2	3
5	5	6	7	8	9	**0**	1	2	3	4
6	6	7	8	9	**0**	1	2	3	4	5
7	7	8	9	**0**	1	2	3	4	5	6
8	8	9	**0**	1	2	3	4	5	6	7
9	9	**0**	1	2	3	4	5	6	7	8

Addition Table in $\mathbf{Z}_{10}$

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	**1**	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	**1**	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	**1**	8	0	2	9	6	3
8	0	8	6	4	2	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	**1**

Multiplication Table in $\mathbf{Z}_{10}$

Some important sets

$$\mathbf{Z}_6 = \{0, 1, 2, 3, 4, 5\}$$

$$\mathbf{Z}_6{}^* = \{1, 5\}$$

$$\mathbf{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$$

$$\mathbf{Z}_7{}^* = \{1, 2, 3, 4, 5, 6\}$$

$$\mathbf{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\mathbf{Z}_{10}{}^* = \{1, 3, 7, 9\}$$

We need to use Z_n when additive inverses are needed; we need to use Z_n^* when multiplicative inverses are needed.
Cryptography often uses two more sets: Z_p and Z_p^* . The modulus in these two sets is a prime number.

$$Z_{13} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$
$$Z_{13}{}^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

References

1. Forouzan, Behrouz A. "Cryptography & Network Security. 2011."

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