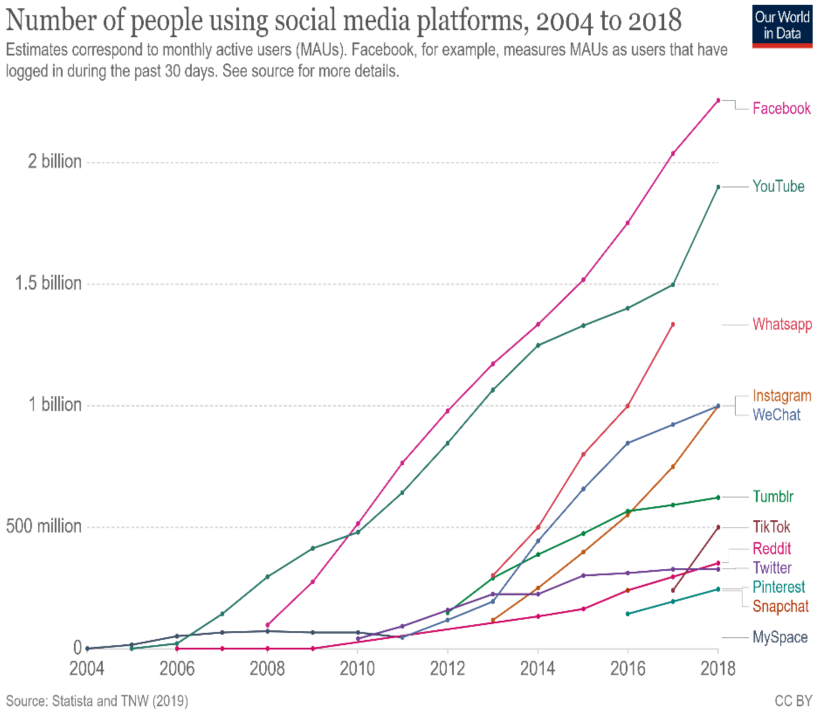
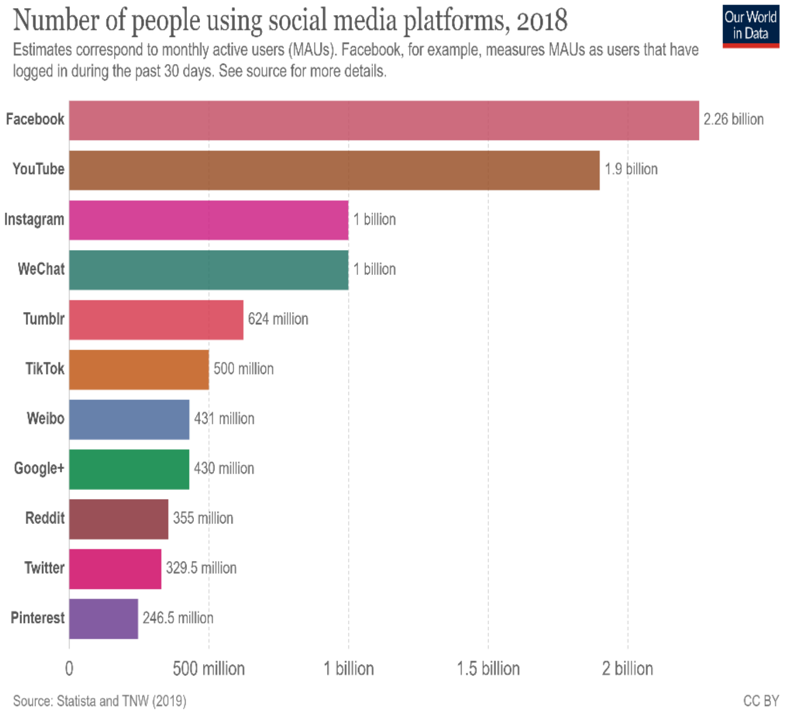
Network Growth

Rise of Online Social Networks



- Online social networks growing rapidly year-by-year
- Their sizes are huge
- In 2014, Facebook possessed 1.39 billion active users and 400 billion friendship links
- March 2015, Twitter has 288 million active users and 60 billion followers



Such huge volume of these online networks restrict the active research with real social networks

https://ourworldindata.org/rise-of-social-media

Synthetic Networks

- Generated using theoretical network models
- Often possesses strong underlying mathematical foundation
- Often can simulate important real-world network characteristics
- Help getting insights of the real-life networks
- Allow experimentation through simulation when real networks are unavailable
- can establish network insights on concrete theoretical foundations

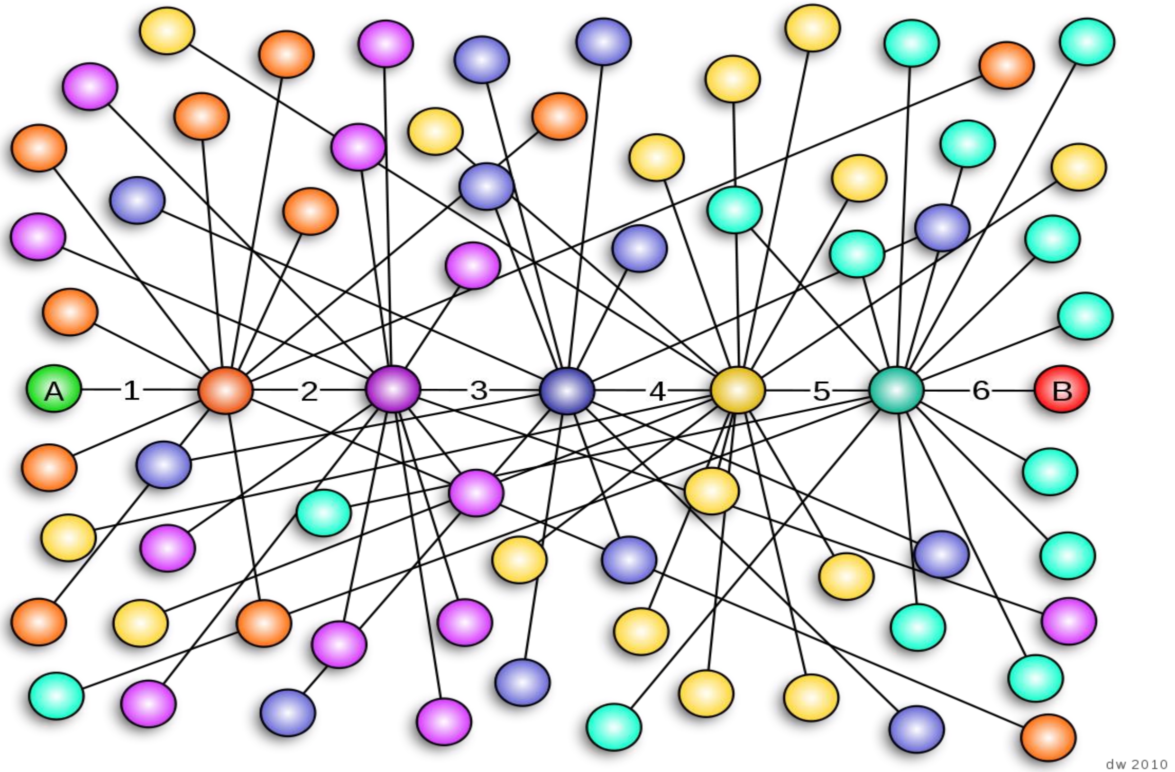
Properties of Real-world Networks

- High average local clustering coefficient
- Small-world property
- Scale-free property

High Average Local Clustering Coefficient

- Neighbors of a node tend to be highly connected with each other at individual level
- In Facebook social graph, for users with 100 friends have average local clustering coefficient of 0.14
- For a graph of 150 million nodes, the number is unexpectedly high

Small-world Property



The "six degrees of separation" model

https://en.wikipedia.org/wiki/Small-world_experiment

- An outcome of Stanley Milgram's small-world experiment (1967) to measure the probability of two random persons being known to each other

Six Degrees of Separation: Milgram's Small-World Experiment

A possible path of a letter

https://en.wikipedia.org/wiki/Small-world_experiment

- Source cities: Omaha, Nebraska, and Wichita, Kansas
- Destination city: Boston, Massachusetts
- Information packets sent initially to random persons in Omaha/Wichita
- Recipient was to forward the letter
  - directly to target, if she knew the target personally
  - Else to some friend/relative who more likely to know the target
- 64 out of 296 letters eventually reach the target contact
- Number of intermediates among the traversal path chains:

Maximum: 11 Minimum: 1 Median: 5.2 (≈6)

Small-world of Social Media!!!

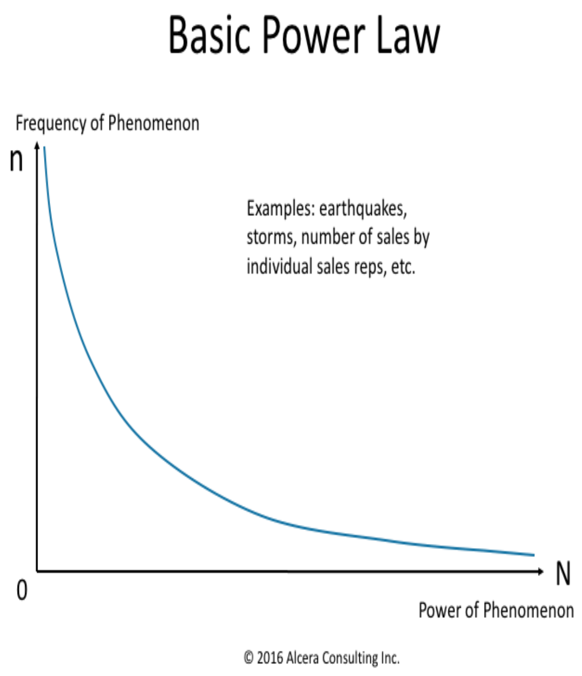
- Average chain of contacts in Microsoft Messenger was 6.6 people [Leskovec and Horvitz 2007]
- Average distance between two random Facebook users in 2011 was 4.74 with 3.74 intermediaries [Four Degree of Separation by Backstrom et al. (2012)]
- Average distance between two random Facebook users in 2016 was 4.57 with 3.57 intermediaries [Repeat Experiment by Facebook in 2016]

Small-world Property for Social Networks

- Can be viewed in light of the average path length between two randomly chosen nodes in a network
- A network G is said to follow the small-world property if the average path length of the network is logarithmically proportional to the network size.

$$\textit{Average Path Length} \propto \log(\textit{Network Size})$$

Scale-free Property: Power Law



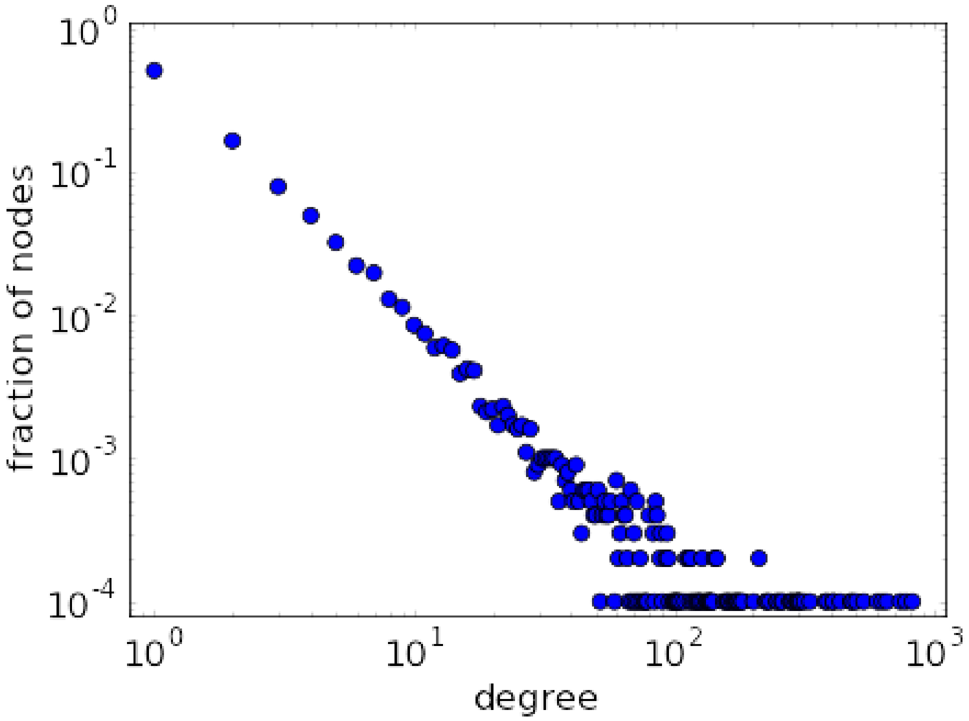
https://exploitingchange.com/2016/09/14/another-powerful-idea-the-power-law/

- Power Law: a relative change in the value of one variable leads to the proportional change in the value of other variable
- Independent of the initial values of both the variable
- Mathematically,

$$y \propto x^{-b}, b \in \mathbb{R}$$

- Functions that follows power-law are scale-invariant
- Pareto Principle (or the 80/20 rule) in Economics: 80% of the outcomes are results of 20% of the causes
- Power law principle is also coined as Pareto Distribution

Scale-free Property: Real Networks

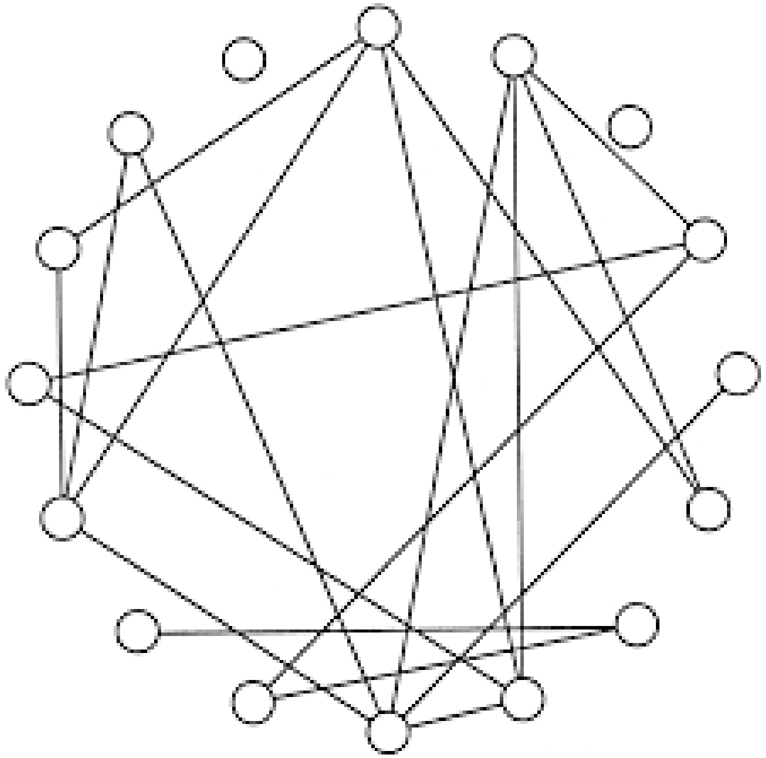


Degree distribution of a scale-free network
https://mathinsight.org/scale_free_network

- Networks whose degree distribution follows power-law are known as scale-free networks

- Most real-world networks are found to be scale-free

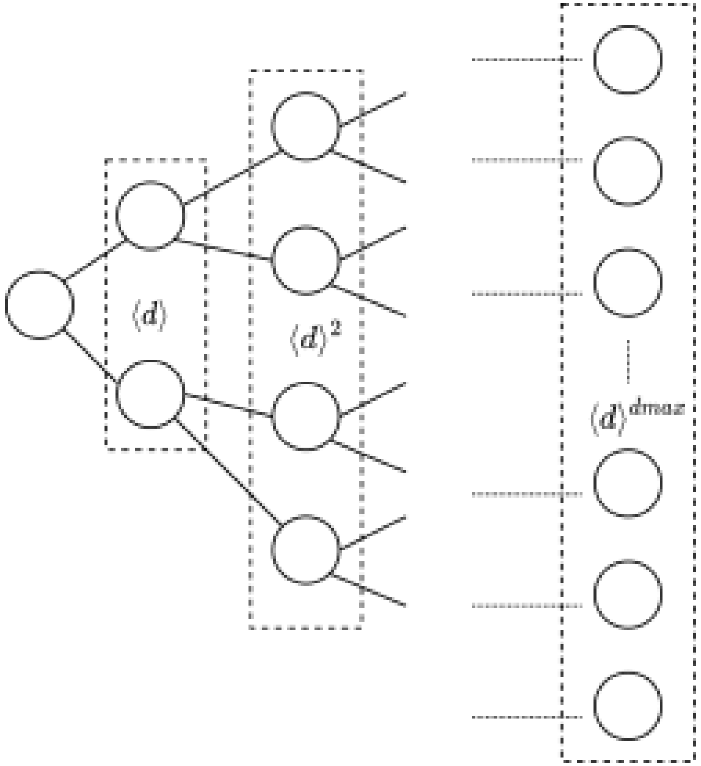
Synthetic Networks: Random Network Model



An instance of $G(16, \frac{1}{7})$ network

- Also popularly known as Erdős and Rényi model (or ER model)
- A number of variants of the model
- Popular variants:
  - $G(N, K)$ model [Erdős and Rényi 1959]: From the set of all networks of N nodes and K edges. a network is chosen uniformly at random
  - $G(N, p)$ model [Gilbert 1959]: Network has N nodes, and any random pair of nodes has a probability p of being adjacent independently with any other pair of nodes in the network
- Both the variants behave identically in the limiting case
- $G(N, p)$ model considered as the standard random network model

Erdős-Rényi Network: Average Path Length



A depiction of random network in tree format

- When l_{max} represent the maximum path length of $G(N,p)$,

$$1 + \langle d \rangle + \langle d \rangle^2 + \langle d \rangle^3 + \cdots\cdots\cdots + \langle d \rangle^{l_{max}} = N$$

- When $\langle d \rangle \gg 1$, the above yields,

$$l_{max} \approx \frac{logN}{log\langle d \rangle}$$

- Further approximation yields,

$$\langle l \rangle \propto logN$$

Theorem: Erdős-Rényi Networks follow small-world property.

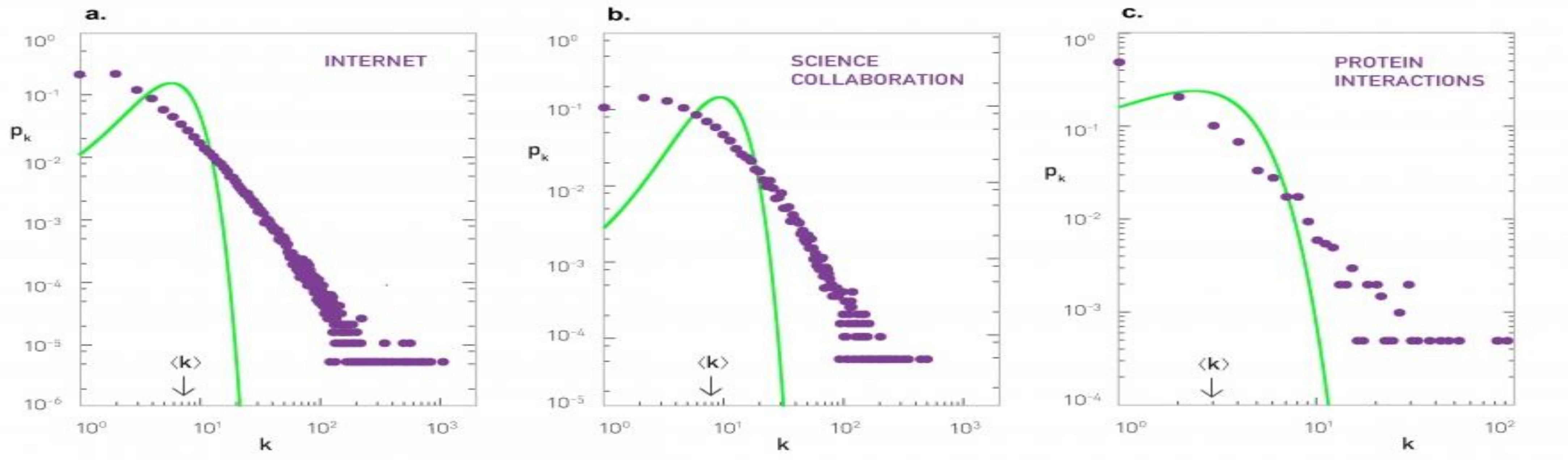
Erdős-Rényi Network: Clustering Coefficient

- In $G(N, p)$
  - Number of possible edges between neighbours of a node: $\binom{\langle d \rangle}{2}$
  - Expected number of edges between these nodes: $p \times \binom{\langle d \rangle}{2}$
- The above yields, the local clustering coefficient for a node $v_i \in G$

$$C_i = p \approx \frac{\langle d \rangle}{N}$$

Theorem: The local clustering coefficient for any node in an Erdős-Rényi Network is inversely proportional to the size of the network

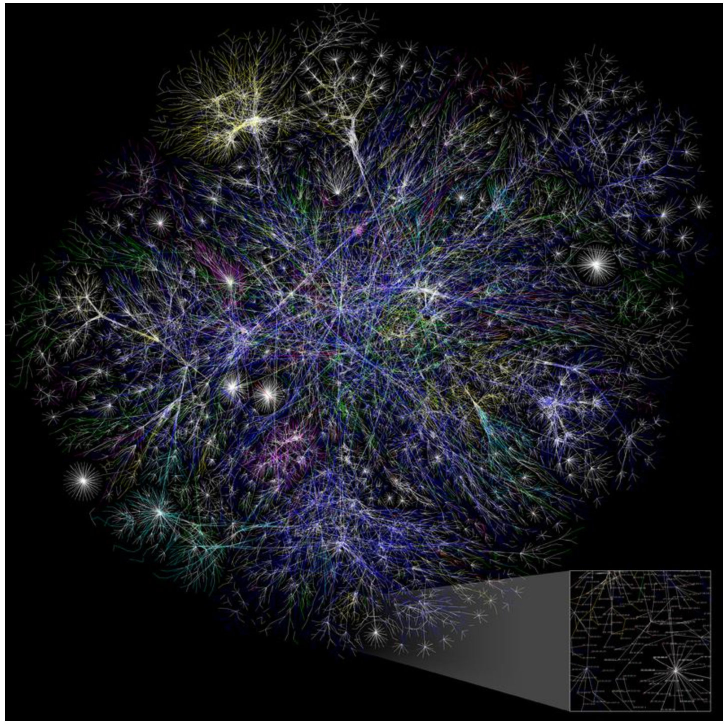
Erdős-Rényi Networks vs. Real-life Networks: Degree Distribution



Real-life networks are often scale-free; however, Erdős-Rényi Networks are not

http://networksciencebook.com/chapter/3#not-poisson

Erdős-Rényi Networks vs. Real-life Networks: Presence of Outliers



Partial map of the Internet based on the January 15, 2005. Hubs are highlighted

https://en.wikipedia.org/wiki/Hub_(network_science)

- In an Erdős-Rényi Network, probability of having a node with a high degree is extremely low
- probability of a node with 2000 neighbours is 10^{-27}!
- In real-world networks, such nodes exist (Hubs)
- Celebrities in social networks

Erdős-Rényi Networks vs. Real-life Networks: Small-world Property

- Erdős-Rényi Networks follow small-world property as the maximum path length in Erdős-Rényi Networks $\approx \frac{logN}{log\langle d\rangle}$

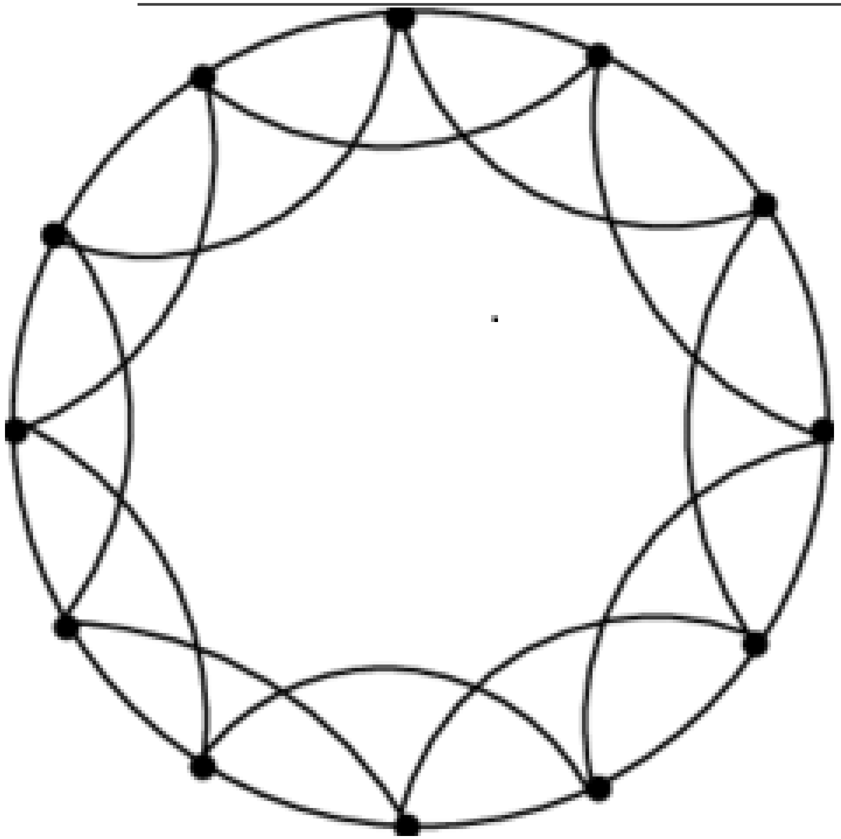
Erdős-Rényi Networks vs. Real-life Networks: Clustering Coefficient

- local clustering coefficients in Erdős-Rényi Networks decreases with the increase in network size

- Nodes having high local clustering coefficient exist in enormously large real-life networks

- Echo chambers in large social networks like Facebook

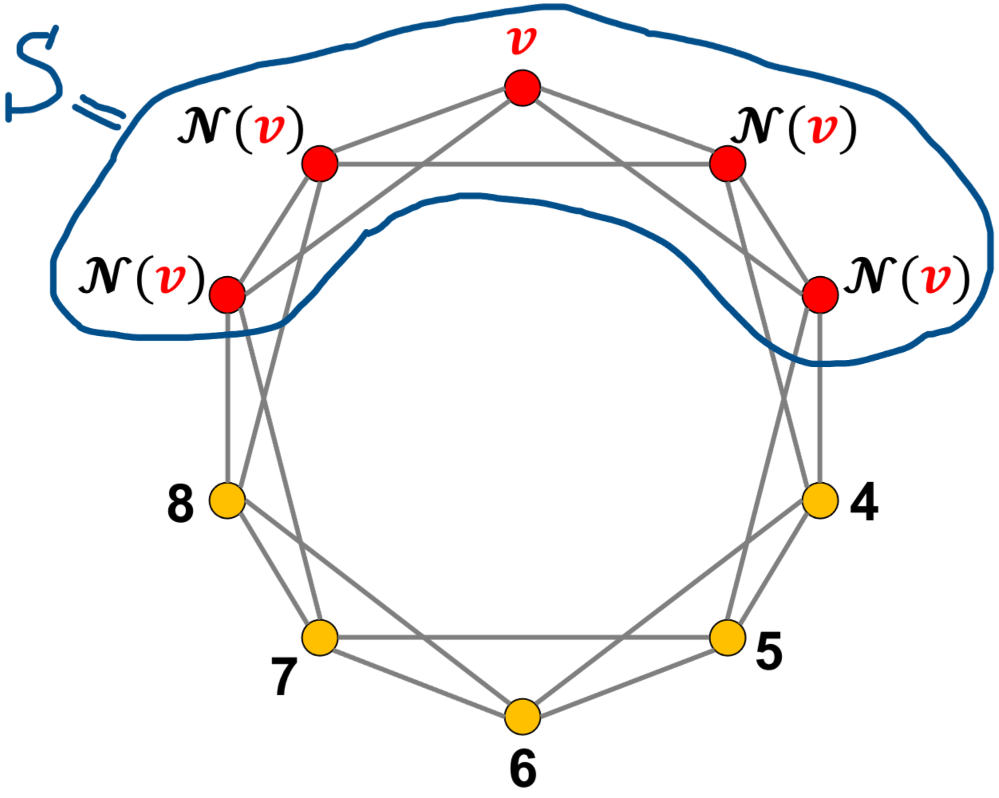
Regular Ring Lattice Network Model



4-Regular Ring Lattice Network

- ❑ A ring lattice network consists of N nodes labeled $0, 1, 2, \cdots, N-1$ arranged in circular order
- ❑ Every node in the network is connected to exactly k other nodes, immediate $\frac{k}{2}$ rightmost nodes and $\frac{k}{2}$ leftmost nodes relative to the position of the node in the network

Regular Ring Lattice Network Model: Local Clustering Coefficient

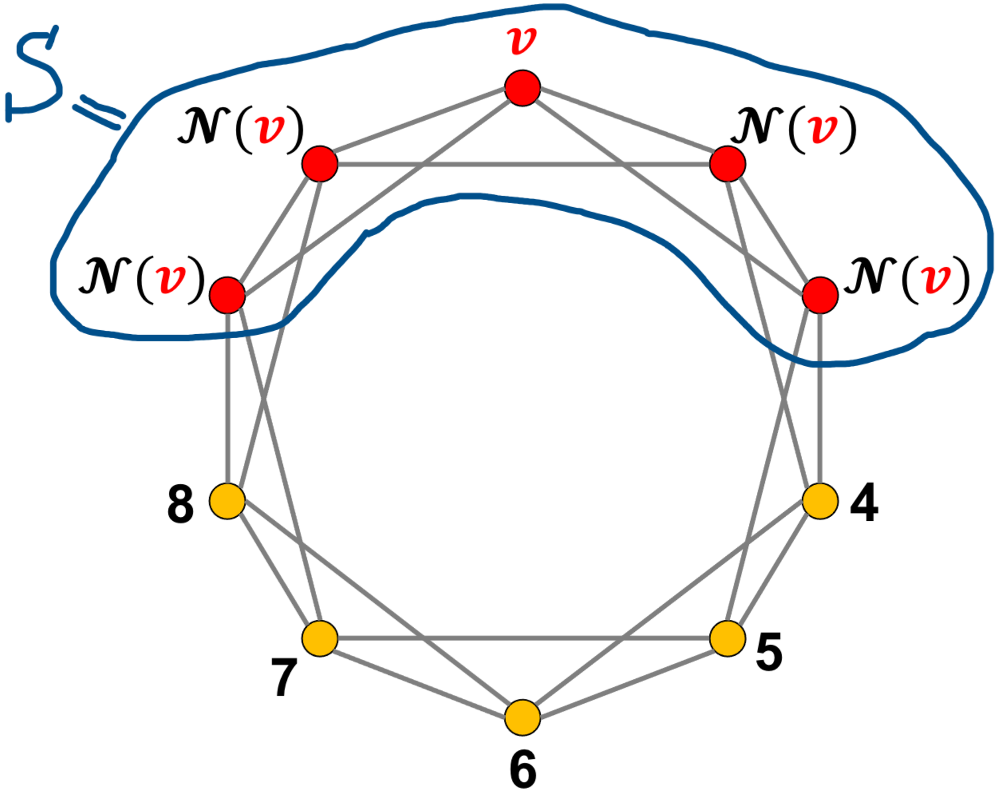


4-Regular Ring Lattice Network

Theorem: Local clustering coefficient of any node v in a $k - Regular\ Ring\ Lattice\ Network$ is $\frac{3(k-2)}{4(k-1)}$

- When k is large, then the above expression boils down to $\frac{3}{4}$
- Regular Ring Lattice networks has high local clustering coefficients

Regular Ring Lattice Network Model: Small-world Property

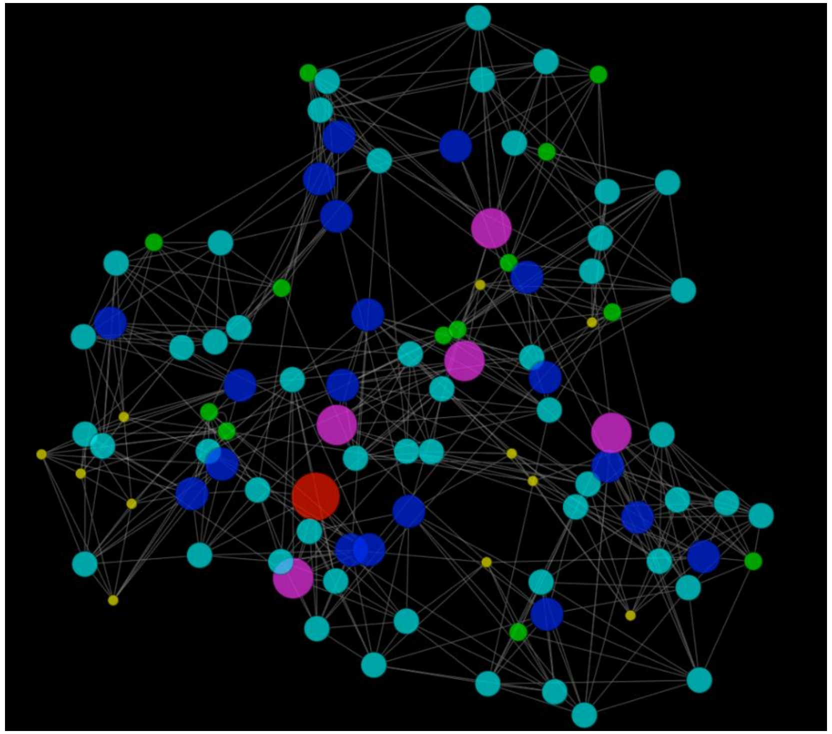


4-Regular Ring Lattice Network

- ❑ All the connection in ring lattice are to nearby nodes
- ❑ Two nodes that are in the opposite sides of the ring will be connected by long chains

Theorem: Regular Ring Lattice networks does not follow small-world property

Watts-Strogatz Network Model

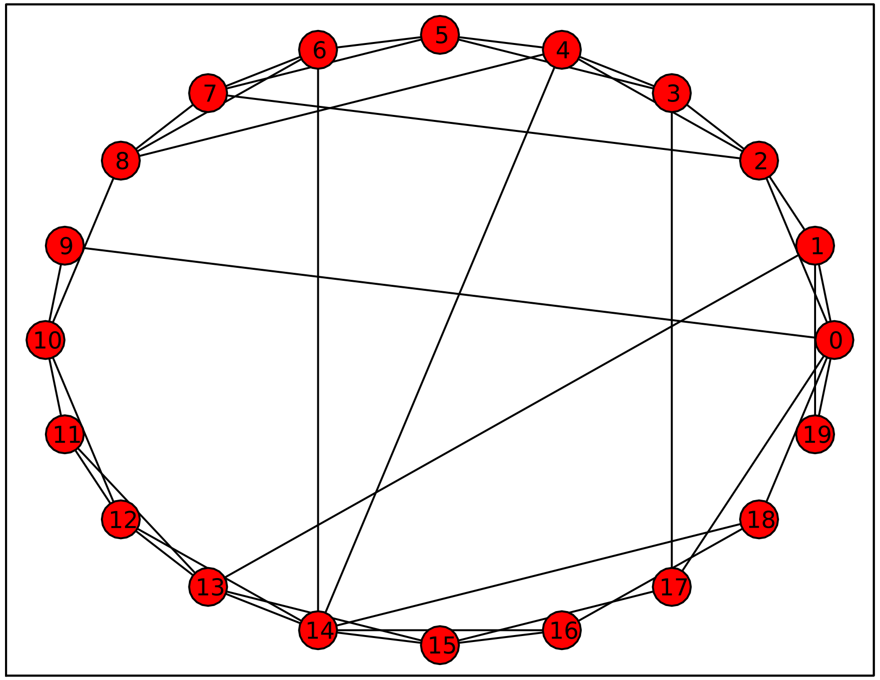


Watts–Strogatz network with 100 nodes formed by igraph and visualized by Cytoscape 2.5
https://en.wikipedia.org/wiki/Watts%E2%80%93Strogatz_model

- Erdős-Rényi Networks
  - possesses the small-world property,
  - have small local clustering coefficients for all the nodes when the network size is large
- Regular Ring Lattice networks
  - possess high local clustering coefficients for all the node
  - does not follow small-world property
- Watts-Strogatz Model form networks that has a high local clustering coefficient and possesses the small-world property
- Proposed by Watts and Strogatz in 1998

Watts-Strogatz Network Model: Network Formation

Watts-Strogatz model N=20, K=4, β=0.2



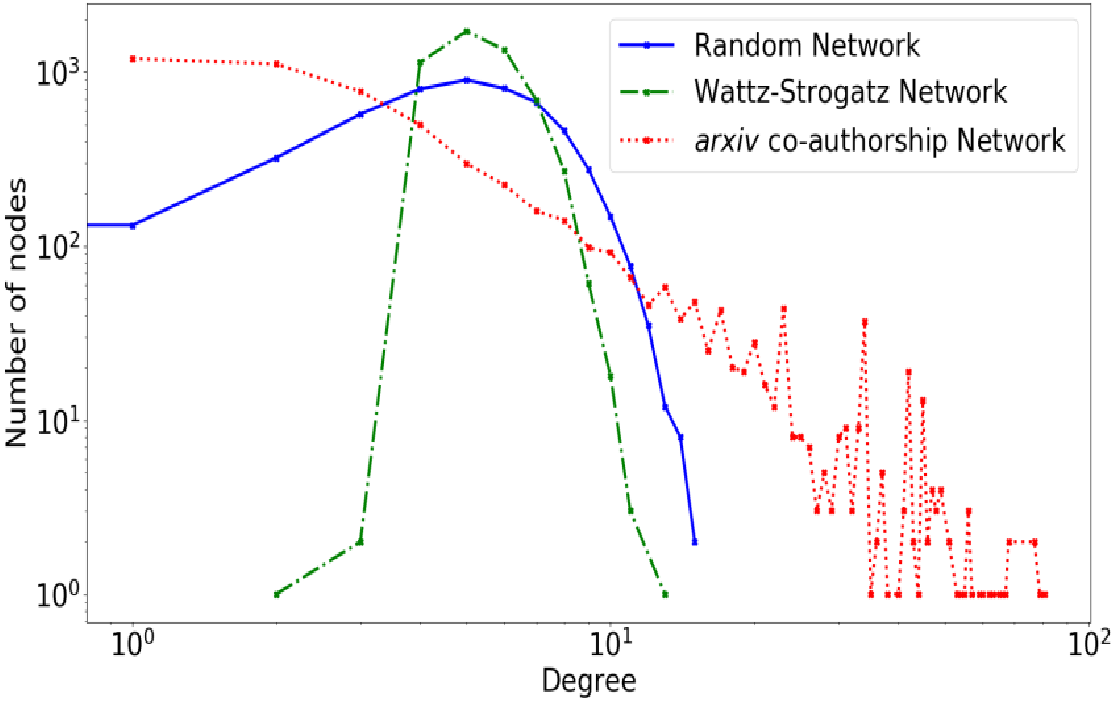
https://www.hindawi.com/journals/mpe/2014/693743/fig2/

1) Start with a k-regular lattice network of size N
2) List the nodes of the lattice as $1, 2, \cdots, N$
3) Choose i^{th} node from the list, $i = 1, 2, \cdots, N$
4) Select edges that that link i^{th} node to some j^{th} node ($j > i$)
5) With a fixed rewiring probability β, rewire the other end of these edges
6) Avoid formation self-loops and link-duplication
7) Repeat steps 3 through 6 until all the nodes are scanned

Watts-Strogatz Network Model: Properties

- What if $\beta \rightarrow 0$?
  - What is the effect on small world property and clustering coefficient ?
- What if $\beta \rightarrow 1$?
  - What is the effect on small world property and clustering coefficient ?
- Can we have outliers in Watts-Strogatz Network ? What about real networks ?

Watts-Strogatz Networks vs. Real-world Networks



Comparisons of Degree Distribution

- ❑Watts-Strogatz networks have only few outliers, whereas, real-life networks have significantly high number of outliers
- ❑Watts-Strogatz networks follow small-world property if $\beta \rightarrow 1$, however, real-world networks always follow small-world
- ❑Degree distribution in Watts-Strogatz does not follow power law, whereas, real networks often does that
- ❑Both Watts-Strogatz networks and real-world networks have high clustering coefficient

Preferential Attachment Model

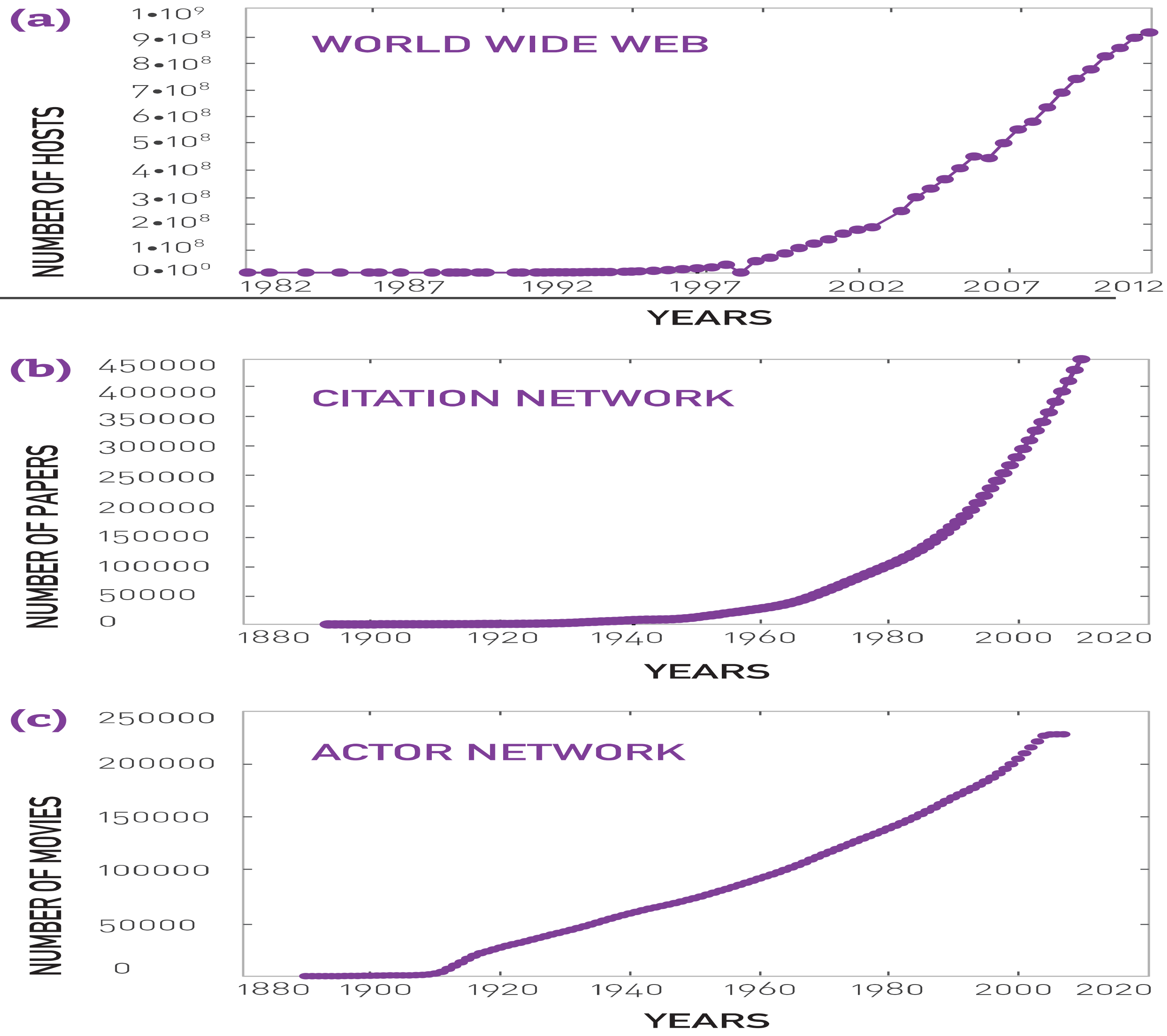
What is the common property of the following networks ?

- Citation Network
- WWW Network
- Social Network
- Collaboration Network

They are dynamic, evolve over time, expanding by adding new nodes and edges…

Growth:

Erdos-Renyi ?
Watts-Strogatz ?



Preferential Attachment

New nodes prefer to connect to the more connected nodes

- Do we observe this property in real networks ? Can you give example ?
- Higher degree nodes are preferred over less degree nodes !!!
  - **Rich gets richer**

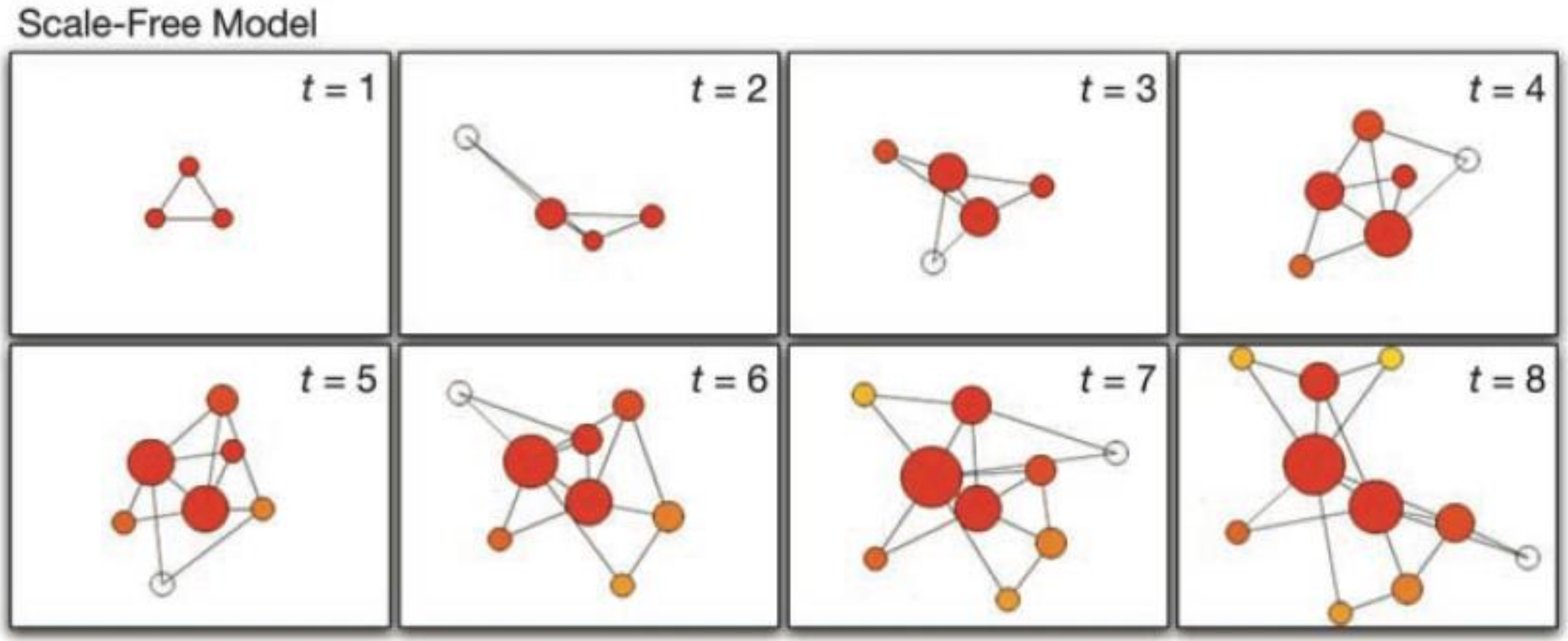
Barabasi-Albert Model

- **Dynamic Growth:** start at t=0 with n_0 nodes and $m_0 \geq n_0$ edges
  - 1) **Growth**
    - At each time step add a new node with m edges ($m \leq n_0$), connecting to m nodes already in network $k_i(i) = m$
  - 2) Preferential Attachment
    - The probability of linking to existing node i is proportional to the node degree k_i

$$\Pi(k_i) = \frac{k_i}{\sum_i k_i}$$

    - after t timesteps: t + n_0 nodes, mt + m_0 edges

Barabasi-Albert Model



Barabasi-Albert Model

Let see if the degree of a node $k_i(t)$ at time t i.e. $k_i(t)$ is dependent on initial time.

$$k_i(t+\delta t) = k_i(t) + m\Pi(k_i)\delta t$$

$$\frac{dk_i(t)}{dt} = m\Pi(k_i) = m\frac{k_i}{\sum_i k_i} = \frac{mk_i}{2mt} = \frac{k_i(t)}{2t}$$

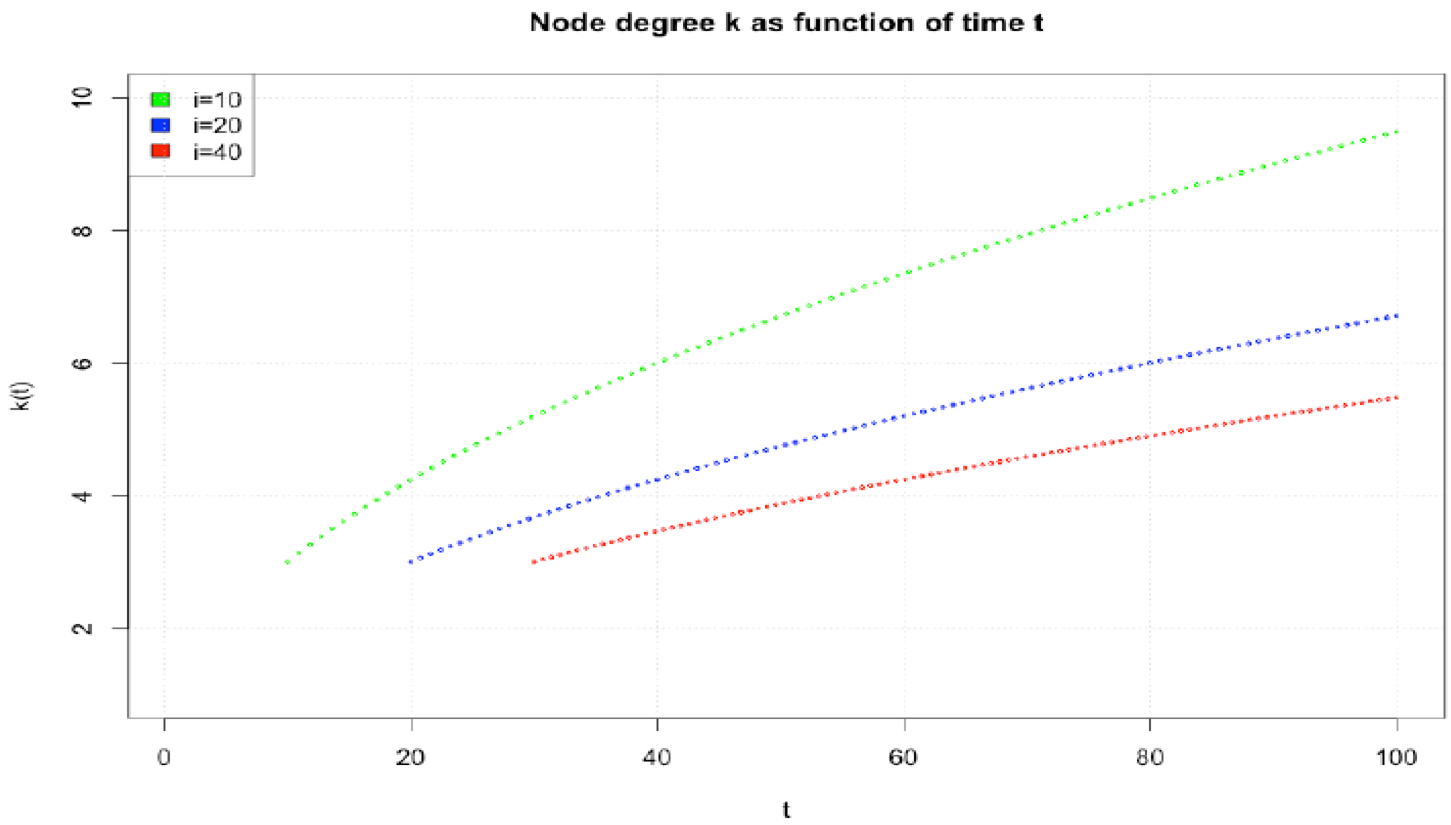
node *i* is added at time t_i: $k_i(t_i) = m$

$$\int_m^{k_i(t)} \frac{dk_i}{k_i} = \int_{t_i}^{t} \frac{dt}{2t}$$

Solution:

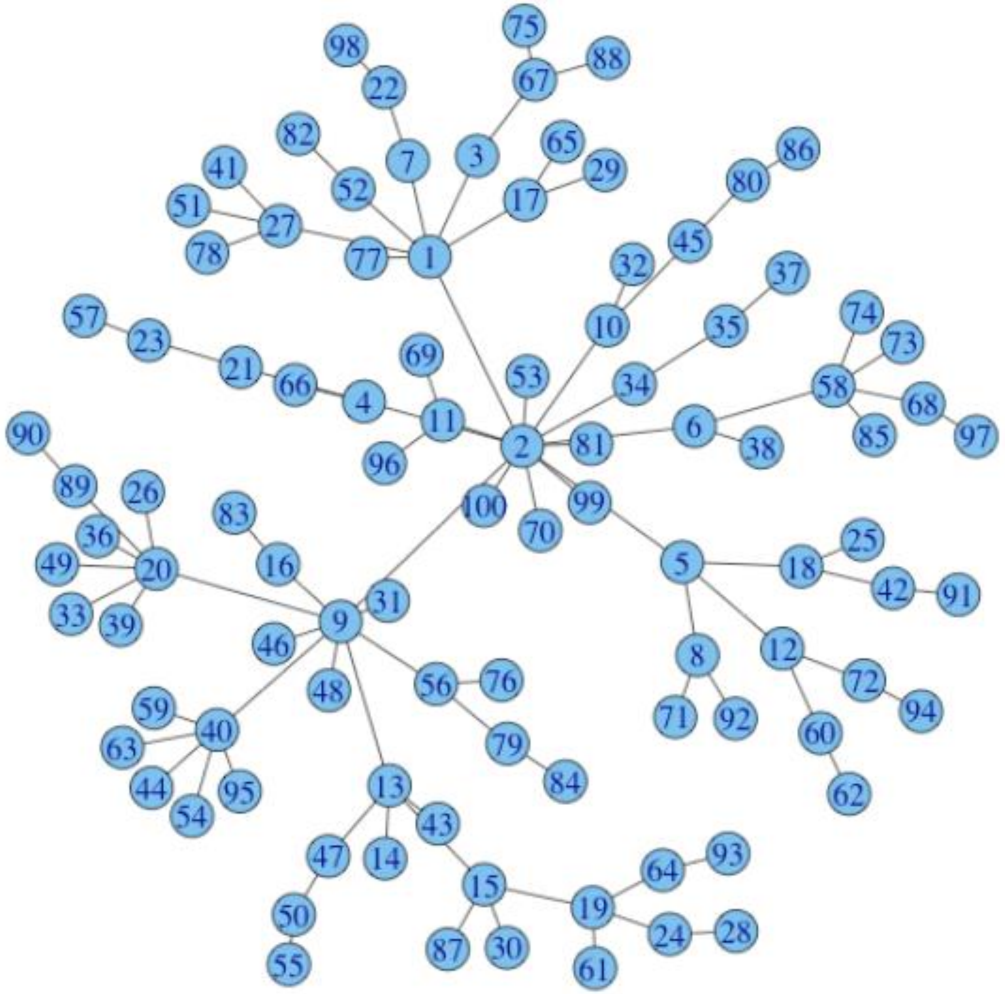
$$k_i(t) = m\left(\frac{t}{t_i}\right)^{1/2}$$

Barabasi-Albert Model



$$k_i(t) = m\left(\frac{t}{t_i}\right)^{1/2}; \qquad \frac{dk_i(t)}{dt} = \frac{m}{2}\frac{1}{\sqrt{tt_i}}$$

Barabasi-Albert Model

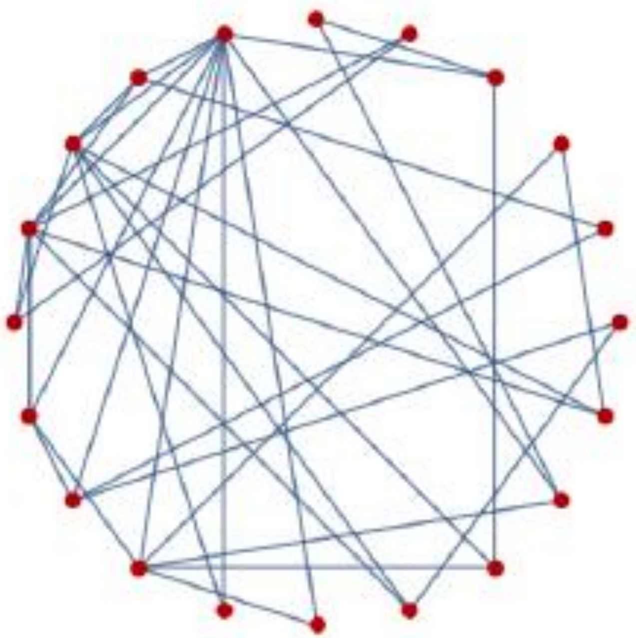


Barabasi-Albert Network Model

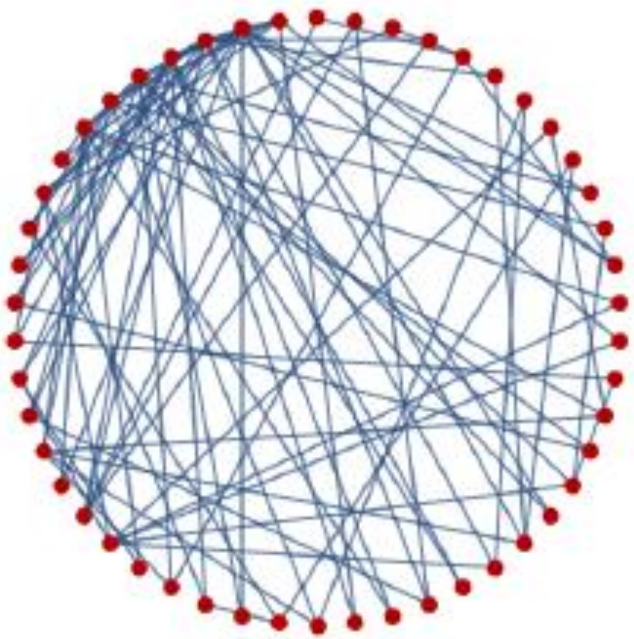
- Generates scale-free networks as per the following degree distribution (the derivation is skipped)

$$P(k) = \frac{2m^2}{k^3}$$

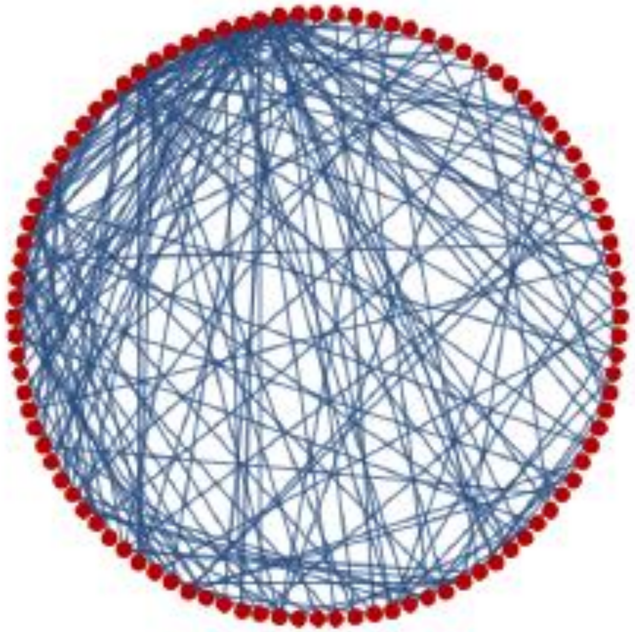
Barabasi-Albert Model: Network Growth



$|G| = 20$
Average Path Length = 2.16



$|G| = 50$
Average Path Length = 2.69



$|G| = 100$
Average Path Length = 3.02

https://www.researchgate.net/publication/259742981_Information_Theory_Kolmogorov_Complexity_and_Algorithmic_Probability_in_Network_Biology