S^3 : $x^2+y^2+z^2+t^2=1$

parametrization $\Phi(\theta,\varphi,w) = (x=\sin w \sin\theta \cos\varphi,$
$y=\sin w \sin\theta \sin\varphi,$
$z=\sin w \cos\theta,$
$t=\cos w \;)$

Check $x^2+y^2+z^2+t^2=1.$

$$x^2+y^2+z^2+t^2 = \underline{\sin^2 w \sin^2\theta \cos^2\varphi + \sin^2 w \sin^2\theta \sin^2\varphi} + \sin^2 w \cos^2\theta + \cos^2 w$$
$$= \sin^2 w \sin^2\theta + \sin^2 w \cos^2\theta + \cos^2 w$$
$$= 1.$$

Derive velocity vectors.

$$\partial_\theta = (\sin w \cos\theta \cos\varphi,\ \sin w \cos\theta \sin\varphi,\ -\sin w \sin\theta,\ 0)$$
$$\partial_\varphi = (-\sin w \sin\theta \sin\varphi,\ \sin w \sin\theta \cos\varphi,\ 0,\ 0)$$
$$\partial_w = (\cos w \sin\theta \cos\varphi,\ \cos w \sin\theta \sin\varphi,\ \cos w \cos\theta,\ -\sin w)$$

Calculate inner products (in $\mathbb{R}^4$)

$$\partial_\theta \cdot \partial_\theta = \underline{\sin^2 w \cos^2\theta \cos^2\varphi + \sin^2 w \cos^2\theta \sin^2\varphi} + \sin^2 w \sin^2\theta$$
$$= \sin^2 w \cos^2\theta + \sin^2 w \sin^2\theta$$
$$= \sin^2 w$$

$$\partial_\theta \cdot \partial_\varphi = -\sin^2 w \sin\theta\cos\theta \sin\varphi\cos\varphi + \sin^2 w \sin\theta\cos\theta \sin\varphi\cos\varphi$$
$$= 0.$$

$$\partial_\theta \cdot \partial_w = \sin w\cos w \sin\theta\cos\theta\cos^2\varphi + \sin w\cos w\sin\theta\cos\theta\sin^2\varphi - \sin w\cos w\sin\theta\cos\theta$$
$$= \sin w\cos w\sin\theta\cos\theta - \sin w\cos w\sin\theta\cos\theta$$
$$= 0$$

$$\partial_\varphi \cdot \partial_\varphi = \sin^2 w\sin^2\theta\sin^2\varphi + \sin^2 w\sin^2\theta\cos^2\varphi$$
$$= \sin^2 w\sin^2\theta$$

$$\partial_\varphi \cdot \partial_w = -\sin w\cos w\sin^2\theta\sin\varphi\cos\varphi + \sin w\cos w\sin^2\theta\sin\varphi\cos\varphi$$
$$= 0$$

$$\partial_w \cdot \partial_w = \cos^2 w\sin^2\theta\cos^2\varphi + \cos^2 w\sin^2\theta\sin^2\varphi + \cos^2 w\cos^2\theta + \sin^2 w$$
$$= \cos^2 w\sin^2\theta + \cos^2 w\cos^2\theta + \sin^2 w$$
$$= 1$$

$\therefore$ metric $$g_{ij} = \begin{pmatrix} \sin^2 w & 0 & 0 \\ 0 & \sin^2 w\sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

inverse of g, $$g^{ij} = \begin{pmatrix} \frac{1}{\sin^2 w} & 0 & 0 \\ 0 & \frac{1}{\sin^2 w\sin^2\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Derive Christoffel symbol (of Levi-Civita connection).

Using $\Gamma^i{}_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk})$,

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk})$$

$$= \frac{1}{2} g^{ii} (\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk})$$

using $g^{ij} = g_{ij} = 0$ for $i \neq j$.

$$= \frac{1}{2} g^{ii} (\partial_j g_{ii} \delta_{ik} + \partial_k g_{ii} \delta_{ij} - \partial_i g_{jj} \delta_{jk})$$

$$\Gamma^\theta{}_{\theta\theta} = \frac{1}{2} \frac{1}{\sin^2 w} (\partial_\theta \sin^2 w + \partial_\theta \sin^2 w - \partial_\theta \sin^2 w)$$
$$= 0$$

$$\Gamma^\theta{}_{\theta\varphi} = \frac{1}{2} \frac{1}{\sin^2 w} (0 + \partial_\varphi \sin^2 w - 0)$$
$$= 0.$$

$$\Gamma^\theta{}_{\theta w} = \frac{1}{2} \frac{1}{\sin^2 w} (0 + \partial_w \sin^2 w - 0)$$

$$= \frac{1}{2} \frac{1}{\sin^2 w} \times 2 \sin w \cos w$$

$$= \frac{\cos w}{\sin w}.$$

$$\Gamma^\theta{}_{\varphi\varphi} = \frac{1}{2} \frac{1}{\sin^2 w} (0 + 0 - \partial_\theta \sin^2 w \sin^2 \theta)$$

$$= \frac{1}{2} \frac{1}{\sin^2 w} \times - 2 \sin^2 w \sin\theta \cos\theta$$

$$= - \sin\theta \cos\theta.$$

$$\Gamma^{\theta}{}_{\varphi\omega} = \frac{1}{2}\frac{1}{\sin^2\omega}\left(0+0-0\right)$$

$$= 0.$$

$$\Gamma^{\theta}{}_{\omega\omega} = \frac{1}{2}\frac{1}{\sin^2\omega}\left(0+0-\partial_\theta 1\right)$$

$$= 0.$$

$$\Gamma^{\varphi}{}_{\theta\theta} = \frac{1}{2}\frac{1}{\sin^2\omega\sin^2\theta}\left(0+0-\partial_\varphi \sin^2\omega\right)$$

$$= 0.$$

$$\Gamma^{\varphi}{}_{\theta\varphi} = \frac{1}{2}\frac{1}{\sin^2\omega\sin^2\theta}\left(\partial_\theta \sin^2\omega\sin^2\theta + 0 - 0\right)$$

$$= \frac{1}{2}\frac{1}{\sin^2\omega\sin^2\theta} \times 2\sin^2\omega\sin\theta\cos\theta$$

$$= \frac{\cos\theta}{\sin\theta}.$$

$$\Gamma^{\varphi}{}_{\theta\omega} = \frac{1}{2}\frac{1}{\sin^2\omega\sin^2\theta}\left(0+0-0\right)$$

$$= 0.$$

$$\Gamma^{\varphi}{}_{\varphi\varphi} = \frac{1}{2}\frac{1}{\sin^2\omega\sin^2\theta}\left(\partial_\varphi \sin^2\omega\sin^2\theta + \partial_\varphi \sin^2\omega\sin^2\theta - \partial_\varphi \sin^2\omega\sin^2\theta\right)$$

$$= 0.$$

$$\Gamma^{\varphi}{}_{\varphi w} = \frac{1}{2}\frac{1}{\sin^2 w \sin^2\theta}\left(0 + \partial_w \sin^2 w \sin^2\theta - 0\right)$$

$$= \frac{1}{2}\frac{1}{\sin^2 w \sin^2\theta} \times 2\sin w \cos w \sin^2\theta$$

$$= \frac{\cos w}{\sin w}$$

$$\Gamma^{\varphi}{}_{ww} = \frac{1}{2}\frac{1}{\sin^2 w \sin^2\theta}\left(0 + 0 - \partial_\varphi 1\right)$$

$$= 0.$$

$$\Gamma^{w}{}_{\theta\theta} = \frac{1}{2}\left(0 + 0 - \partial_w \sin^2 w\right)$$

$$= \frac{1}{2} \times -2\sin w \cos w$$

$$= -\sin w \cos w.$$

$$\Gamma^{w}{}_{\theta\varphi} = \frac{1}{2}\left(0 + 0 - 0\right)$$

$$= 0.$$

$$\Gamma^{w}{}_{\theta w} = \frac{1}{2}\left(\partial_\theta 1 + 0 - 0\right)$$

$$= 0$$

$$\Gamma^{w}{}_{\varphi\varphi} = \frac{1}{2}\left(0+0-\partial_w \sin^2 w \sin^2\theta\right)$$
$$= \frac{1}{2} \times -2\sin w \cos w \sin^2\theta$$
$$= -\sin w \cos w \sin^2\theta .$$

$$\Gamma^{w}{}_{\varphi w} = \frac{1}{2}\left(\partial_\varphi 1 + 0 - 0\right)$$
$$= 0.$$

$$\Gamma^{w}{}_{ww} = \frac{1}{2}\left(\partial_w 1 + \partial_w 1 - \partial_w 1\right)$$
$$= 0.$$

$$\boxed{\begin{aligned}
&\therefore\ \Gamma^{\theta}{}_{\theta\theta} = \Gamma^{\theta}{}_{\theta\varphi} = \Gamma^{\theta}{}_{\varphi w} = \Gamma^{\theta}{}_{ww} = 0 \\
&\Gamma^{\theta}{}_{\theta w} = \cot w \ , \ \Gamma^{\theta}{}_{\varphi\varphi} = -\sin\theta\cos\theta \\
&\Gamma^{\varphi}{}_{\theta\theta} = \Gamma^{\varphi}{}_{\theta w} = \Gamma^{\varphi}{}_{\varphi\varphi} = \Gamma^{\varphi}{}_{ww} = 0 \\
&\Gamma^{\varphi}{}_{\theta\varphi} = \cot\theta \ , \ \Gamma^{\varphi}{}_{\varphi w} = \cot w \\
&\Gamma^{w}{}_{\theta\varphi} = \Gamma_{\theta w} = \Gamma^{w}{}_{\varphi w} = \Gamma^{w}{}_{ww} = 0. \\
&\Gamma^{w}{}_{\theta\theta} = -\sin w \cos w \ , \ \Gamma^{w}{}_{\varphi\varphi} = -\sin w \cos w \sin^2\theta .
\end{aligned}}$$

Derive Riemann curvature tensor

using $R^i{}_{jkl} = \partial_k \Gamma^i{}_{lj} - \partial_l \Gamma^i{}_{kj} + (\Gamma^i{}_{kp}\Gamma^p{}_{lj} - \Gamma^i{}_{lp}\Gamma^p{}_{kj})$

$$R^\theta{}_{\varphi\theta\varphi} = \partial_\theta \Gamma^\theta{}_{\varphi\varphi} - \partial_\varphi \Gamma^\theta{}_{\theta\varphi} + (\Gamma^\theta{}_{\theta p}\Gamma^p{}_{\varphi\varphi} - \Gamma^\theta{}_{\varphi p}\Gamma^p{}_{\theta\varphi})$$

$$= \partial_\theta(-\sin\theta\cos\theta) - \partial_\varphi(0) + \Big(\cot w \times -\sin w\cos w\sin^2\theta$$

$$- \ -\sin\theta\cos\theta \times \cot\theta\Big)$$

$$= -\cos^2\theta + \sin^2\theta - \cos^2 w\sin^2\theta + \cos^2\theta$$

$$= \sin^2\theta - \cos^2 w\sin^2\theta$$

$$= \sin^2 w\sin^2\theta .$$

$$R^\theta{}_{\varphi\theta w} = \partial_\theta \Gamma^\theta{}_{w\varphi} - \partial_w \Gamma^\theta{}_{\theta\varphi} + (\Gamma^\theta{}_{\theta p}\Gamma^p{}_{w\varphi} - \Gamma^\theta{}_{wp}\Gamma^p{}_{\theta\varphi})$$

$$= \partial_\theta(0) - \partial_w(0) + (0 - 0)$$

$$= 0 .$$

$$R^\theta{}_{\varphi\varphi w} = \partial_\varphi \Gamma^\theta{}_{w\varphi} - \partial_w \Gamma^\theta{}_{\varphi\varphi} + (\Gamma^\theta{}_{\varphi p}\Gamma^p{}_{w\varphi} - \Gamma^\theta{}_{wp}\Gamma^p{}_{\varphi\varphi})$$

$$= \partial_\varphi(0) - \partial_w(-\sin\theta\cos\theta) + \big(-\sin\theta\cos\theta \times \cot w - \cot w \times -\sin\theta\cos\theta\big)$$

$$= 0 .$$

$$R^{\theta}{}_{\omega\theta\omega} = \partial_\theta \Gamma^{\theta}{}_{\omega\omega} - \partial_\omega \Gamma^{\theta}{}_{\theta\omega} + \left(\Gamma^{\theta}{}_{\theta p}\Gamma^{p}{}_{\omega\omega} - \Gamma^{\theta}{}_{\omega p}\Gamma^{p}{}_{\theta\omega}\right)$$

$$= \partial_\theta (0) - \partial_\omega (\cot \omega) + (0 - \cot \omega \times \cot \omega)$$

$$= \csc^2 \omega - \frac{\cos^2 \omega}{\sin^2 \omega}$$

$$= 1.$$

$$R^{\theta}{}_{\omega\varphi\omega} = \partial_\varphi \Gamma^{\theta}{}_{\omega\omega} - \partial_\omega \Gamma^{\theta}{}_{\varphi\omega} + \left(\Gamma^{\theta}{}_{\varphi p}\Gamma^{p}{}_{\omega\omega} - \Gamma^{\theta}{}_{\omega p}\Gamma^{p}{}_{\varphi\omega}\right)$$

$$= \partial_\varphi (0) - \partial_\omega (0) + (0 - 0)$$

$$= 0.$$

$$R^{\varphi}{}_{\omega\varphi\omega} = \partial_\varphi \Gamma^{\varphi}{}_{\omega\omega} - \partial_\omega \Gamma^{\varphi}{}_{\varphi\omega} + \left(\Gamma^{\varphi}{}_{\varphi p}\Gamma^{p}{}_{\omega\omega} - \Gamma^{\varphi}{}_{\omega p}\Gamma^{p}{}_{\varphi\omega}\right)$$

$$= \partial_\varphi (0) - \partial_\omega (\cot \omega) + (0 - \cot \omega \times \cot \omega)$$

$$= \csc^2 \omega - \frac{\cos^2 \omega}{\sin^2 \omega}$$

$$= 1$$

$\therefore$ $R_{\theta\varphi\theta\varphi} = \sin^4 \omega \sin^2 \theta$, $R_{\theta\omega\theta\omega} = \sin^2 \omega$, $R_{\varphi\omega\varphi\omega} = \sin^2 \omega \sin^2 \theta$.

$R_{\theta\varphi\theta\omega} = R_{\theta\varphi\varphi\omega} = R_{\theta\omega\varphi\omega} = 0$.

Caculate sectional curvature

Let $X = X^i \partial_i$, $Y = Y^i \partial_i$.

$$\Rightarrow \langle X,X\rangle = (X^\theta)^2 \sin^2 w + (X^\varphi)^2 \sin^2 w \sin^2\theta + (X^w)^2$$

$$\langle Y,Y\rangle = (Y^\theta)^2 \sin^2 w + (Y^\varphi)^2 \sin^2 w \sin^2\theta + (Y^w)^2$$

$$\langle X,Y\rangle = X^\theta Y^\theta \sin^2 w + X^\varphi Y^\varphi \sin^2 w \sin^2\theta + X^w Y^w .$$

$$\therefore \langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2$$

$$\begin{aligned}
= \; & \cancel{(X^\theta)^2 (Y^\theta)^2 \sin^4 w} + (X^\theta)^2 (Y^\varphi)^2 \sin^4 w \sin^2\theta + (X^\theta)^2 (Y^w)^2 \sin^2 w \\
& + (X^\varphi)^2 (Y^\theta)^2 \sin^4 w \sin^2\theta + \cancel{(X^\varphi)^2 (Y^\varphi)^2 \sin^4 w \sin^4\theta} + (X^\varphi)^2 (Y^w)^2 \sin^2 w \sin^2\theta \\
& + (X^w)^2 (Y^\theta)^2 \sin^2 w + (X^w)^2 (Y^\varphi)^2 \sin^2 w \sin^2\theta + \cancel{(X^w)^2 (Y^w)^2} \\
& - \cancel{(X^\theta)^2 (Y^\theta)^2 \sin^4 w} - \cancel{(X^\varphi)^2 (Y^\varphi)^2 \sin^4 w \sin^4\theta} - \cancel{(X^w)^2 (Y^w)^2} \\
& - 2 X^\theta X^\varphi Y^\theta Y^\varphi \sin^4 w \sin^2\theta - 2 X^\theta X^w Y^\theta Y^w \sin^2 w - 2 X^\varphi X^w Y^\varphi Y^w \sin^2 w \sin^2\theta
\end{aligned}$$

$$\begin{aligned}
= \; & (X^\theta Y^w - X^w Y^\theta)^2 \sin^2 w + (X^\theta Y^\varphi - X^\varphi Y^\theta)^2 \sin^4 w \sin^2\theta \\
& + (X^\varphi Y^w - X^w Y^\varphi)^2 \sin^2 w \sin^2\theta
\end{aligned}$$

$$\langle R(X,Y)Y, X \rangle$$

$$= \cancel{R_{ijkl} X^i Y^j Y^k X^l}$$

$$\begin{aligned}
= \; & R_{\theta\varphi\theta\varphi} X^\theta Y^\varphi Y^\theta X^\varphi + R_{\theta\varphi\varphi\theta} X^\theta Y^\varphi Y^\varphi X^\theta \\
& + R_{\theta\omega\theta\omega} X^\theta Y^\omega Y^\theta X^\omega + R_{\theta\omega\omega\theta} X^\theta Y^\omega Y^\omega X^\theta \\
& + R_{\varphi\theta\varphi\theta} X^\varphi Y^\theta Y^\varphi X^\theta + R_{\varphi\theta\theta\varphi} X^\varphi Y^\theta Y^\theta X^\varphi \\
& + R_{\varphi\omega\varphi\omega} X^\varphi Y^\omega Y^\varphi X^\omega + R_{\varphi\omega\omega\varphi} X^\varphi Y^\omega Y^\omega X^\varphi \\
& + R_{\omega\theta\omega\theta} X^\omega Y^\theta Y^\omega X^\theta + R_{\omega\theta\theta\omega} X^\omega Y^\theta Y^\theta X^\omega \\
& + R_{\omega\varphi\omega\varphi} X^\omega Y^\varphi Y^\omega X^\varphi + R_{\omega\varphi\varphi\omega} X^\omega Y^\varphi Y^\varphi X^\omega
\end{aligned}$$

$$\begin{aligned}
= \; & \sin^4\omega \sin^2\theta \, X^\theta X^\varphi Y^\theta Y^\varphi - \sin^4\omega \sin^2\theta \, (X^\theta)^2 (Y^\varphi)^2 \\
& + \sin^2\omega \, X^\theta X^\omega Y^\theta Y^\omega - \sin^2\omega \, (X^\theta)^2 (Y^\omega)^2 \\
& + \sin^4\omega \sin^2\theta \, X^\theta X^\varphi Y^\theta Y^\varphi - \sin^4\omega \sin^2\theta \, (X^\varphi)^2 (Y^\theta)^2 \\
& + \sin^2\omega \sin^2\theta \, X^\varphi X^\omega Y^\varphi Y^\omega - \sin^2\omega \sin^2\theta \, (X^\varphi)^2 (Y^\omega)^2 \\
& + \sin^2\omega \, X^\theta X^\omega Y^\theta Y^\omega - \sin^2\omega \, (X^\omega)^2 (Y^\theta)^2 \\
& + \sin^2\omega \sin^2\theta \, X^\varphi X^\omega Y^\varphi Y^\omega - \sin^2\omega \sin^2\theta \, (X^\omega)^2 (Y^\varphi)^2
\end{aligned}$$

$$= -(X^\theta Y^\varphi - X^\varphi Y^\theta)^2 \sin^4 w \sin^2\theta$$

$$- (X^\theta Y^w - X^w Y^\theta) \sin^2 w$$

$$-(X^\varphi Y^w - X^w Y^\varphi) \sin^2 w \sin^2\theta$$

$$= -\left(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2\right)$$

$$\therefore K = -1 \quad ??$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

$$\nabla_X \nabla_Y Z = \nabla_{X^i\partial_i} \nabla_{Y^j\partial_j} Z^k \partial_k$$

$$= X^i \nabla_{\partial_i} \left(Y^j \nabla_{\partial_j} Z^k \partial_k\right)$$

$$= X^i \nabla_{\partial_i} \left(Y^j \left((\partial_j Z^k)\partial_k + Z^k \Gamma^l_{jk} \partial_l\right)\right)$$

$$= X^i \partial_i\left(Y^j(\partial_j Z^k)\right) \partial_k + X^i Y^j (\partial_j Z^k) \Gamma^l_{ik} \partial_l$$

$$+ X^i \partial_i\left(Y^j Z^k \Gamma^l_{jk}\right) \partial_l + X^i Y^j Z^k \Gamma^l_{jk} \Gamma^m_{il} \partial_m$$

$$= \cancel{X^i(\partial_i Y^j)(\partial_j Z^l)} + \cancel{X^i Y^j(\partial_i \partial_j Z^l)} + \cancel{X^i Y^j(\partial_j Z^k)\Gamma^l_{ik}}$$

$$+ \cancel{X^i(\partial_i Y^j) Z^k} \Gamma^l_{jk} + \cancel{X^i Y^j(\partial_i Z^k)\Gamma^l_{jk}} + X^i Y^j Z^k(\partial_i \Gamma^l_{jk})$$

$$+ X^i Y^j Z^k \Gamma^m_{jk} \Gamma^l_{im}$$

$$\nabla_Y \nabla_X Z = \cancel{Y^i(\partial_i X^j)(\partial_j Z^l)} + \cancel{Y^i X^j(\partial_i \partial_j Z^l)} + \cancel{Y^i X^j(\partial_j Z^k)\Gamma^l_{ik}}$$

$$+ \cancel{Y^i(\partial_i X^j) Z^k} \Gamma^l_{jk} + \cancel{Y^i X^j(\partial_i Z^k)\Gamma^l_{jk}} + Y^i X^j Z^k(\partial_i \Gamma^l_{jk})$$

$$+ Y^i X^j Z^k \Gamma^m_{jk} \Gamma^l_{im}$$

$$\nabla_{[X,Y]} Z = \nabla_{(X^i(\partial_i Y^j) - Y^i(\partial_i X^j))\partial_j} Z^k \partial_k$$

$$= \left(X^i(\partial_i Y^j) - Y^i(\partial_i X^j)\right) \nabla_{\partial_j} Z^k \partial_k$$

$$= \left(X^i(\partial_i Y^j) - Y^i(\partial_i X^j)\right)\left((\partial_j Z^l) + Z^k \Gamma^l_{jk}\right)$$

$$= \cancel{X^i(\partial_i Y^j)(\partial_j Z^l)} + \cancel{X^i(\partial_i Y^j) Z^k}\Gamma^l_{jk}$$

$$- \cancel{Y^i(\partial_i X^j)}(\partial_j Z^l) - \cancel{Y_i(\partial_i X^j) Z^k}\Gamma^l_{jk}.$$

$$\therefore\ R(X,Y)Z = X^i Y^j Z^k \left(\underline{\partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}} \right)$$

$$= X^i Y^j Z^k R^l{}_{kij}\, \partial_l$$

$$\langle R(X,Y)Y, X\rangle$$
$$= R_{ijkl} X^k Y^l Y^j X^i$$
$$= -R_{ijlk} X^i Y^j Y^k X^l$$
$$= \langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2$$

$$\therefore K = 1.$$