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SOME ISOPERIMETRIC INEQUALITIES FOR HARMONIC FUNCTIONS*

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1. Introduction. In this paper we present a new isoperimetric lower bound for the first nonzero eigenvalue p_2 in the Stekloff problem [13] for a bounded convex domain D in R_2. Such an inequality is of interest in itself and is also useful for determining a priori error bounds in the Neumann problem for second order elliptic equations (see [3]).

The first isoperimetric inequality for p_2 to appear in the literature was that due to Weinstock [15], i.e.,

$$p_2 \leqq 2\pi/L, \tag{1.1}$$

where L is the length of ∂D. Lower bounds of various types (generally somewhat complicated) have been computed by Bramble and Payne [2], Kuttler and Sigillito [9], [10], [11], and Bandle [1]. Other results are due to Troesch [14] and to Hersch and Payne [5]. In this note we show that for convex domains,

$$p_2 \geqq K_{\min}, \tag{1.2}$$

where $K_{\min}$ denotes the minimum curvature of ∂D. We obtain, in fact, the two-sided bound

$$K_{\max} \geqq p_2 \geqq K_{\min}. \tag{1.3}$$

In § 3 we consider the eigenvalue problem characterized by

$$\nu_1 = \inf_{\Delta h = 0 \text{ in } D} \frac{\oint_{\partial D} h^2\, ds}{\int_D h^2\, dx} \tag{1.4}$$

for a convex domain D with Lipschitz boundary ∂D. We establish the inequality

$$\nu_1 \geqq 2K_{\min}. \tag{1.5}$$

We shall henceforth refer to ν_1 as the first Dirichlet eigenvalue.

2. The Stekloff problem. The first nonzero Stekloff eigenvalue p_2 is characterized as follows:

$$p_2 = \inf_{\oint_{\partial D} \phi\, ds = 0} \frac{\int_D |\operatorname{grad} \phi|^2\, dx}{\oint_{\partial D} \phi^2\, ds}, \qquad \phi \in H_1(D). \tag{2.1}$$

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It can be shown that if ∂D is a Lipschitz boundary then the minimizing function H exists and satisfies

$$\Delta H = 0 \quad \text{in } D,$$
$$\frac{\partial H}{\partial n} - p_2 H = 0 \quad \text{on } \partial D, \tag{2.2}$$
$$\oint_{\partial D} H\, ds = 0.$$

Here $\partial/\partial n$ denotes the outward normal derivative on ∂D.

Let us assume for the moment that $\partial D \in C^\infty$. Then the differential equation will be satisfied on the boundary. We now set

$$v = |\text{grad } H|^2. \tag{2.3}$$

Clearly v is subharmonic in D and hence v takes its maximum value at a point P of ∂D. But by Hopf's second principle [8] either

$$v \equiv \text{const.} \quad \text{in } D \tag{2.4}$$

or

$$\frac{\partial v}{\partial n}(P) > 0. \tag{2.5}$$

Since the tangential derivative of v must vanish at P we thus have the following two conditions at P (assuming $v \not\equiv$ const. in D):

$$2\left[\frac{\partial H}{\partial n}\frac{\partial^2 H}{\partial n^2} + \frac{\partial H}{\partial s}\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial n}\right) - K\left(\frac{\partial H}{\partial s}\right)^2\right] > 0, \tag{2.6a}$$

$$2\left[\frac{\partial H}{\partial n}\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial n}\right) + \frac{\partial H}{\partial s}\frac{\partial^2 H}{\partial s^2}\right] = 0. \tag{2.6b}$$

Here (2.6a) is obtained by rewriting (2.5) and K denotes the curvature at P. The differential equation in normal coordinates is given by

$$\frac{\partial^2 H}{\partial n^2} + K\frac{\partial H}{\partial n} + \frac{\partial^2 H}{\partial s^2} = 0 \quad \text{at } P. \tag{2.7}$$

Inserting the expression for $\partial^2 H/\partial n^2$ from (2.7) into (2.6) we obtain

$$\frac{\partial H}{\partial s}\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial n}\right) - \frac{\partial H}{\partial n}\frac{\partial^2 H}{\partial s^2} - K|\text{grad } H|^2 > 0. \tag{2.8}$$

Let us assume for the moment that $\partial H/\partial s \neq 0$ at P. Then (2.6b) implies

$$\frac{\partial^2 H}{\partial s^2}(P) = -p_2\frac{\partial H}{\partial n}(P). \tag{2.9}$$

Insertion of (2.9) into (2.8) and use of the boundary condition in (2.2) lead to

$$(p_2 - K)|\text{grad } H|^2 > 0 \tag{2.10}$$

which implies

$$p_2 > K(P) \geqq K_{\min}. \tag{2.11}$$

Thus if $\partial D \in C^\infty$, $v \not\equiv$ const. in D, and $\partial H(P)/\partial s \neq 0$, the desired result has been obtained. If $\partial H(P)/\partial s = 0$, we use the fact that at P

$$\partial^2 v/\partial s^2 \leqq 0 \tag{2.12}$$

to obtain

$$p_2^2 H \frac{\partial^2 H}{\partial s^2} + \left(\frac{\partial^2 H}{\partial s^2}\right)^2 \leqq 0 \tag{2.13}$$

at P. Also in this case (2.8) becomes

$$H \frac{\partial^2 H}{\partial s^2} + K p_2 H^2 < 0. \tag{2.14}$$

Multiplying (2.14) by p_2^2 and adding to (2.13) we have

$$\left(\frac{\partial^2 H}{\partial s^2} + p_2^2 H\right)^2 + p_2^3(K - p_2)H^2 < 0 \tag{2.15}$$

which again implies (2.11). We must now investigate the consequence of $v \equiv$ const.

If H is harmonic and the square of its gradient is constant, then H must be a linear function of x and y, i.e.,

$$H = ax + by + c, \qquad a, b, c, \text{ arbitrary constants}. \tag{2.16}$$

The boundary conditions give

$$\begin{aligned} &\frac{\partial H}{\partial n} - p_2 H \equiv a n_x + b n_y - p_2(ax + by + c) = 0 \quad \text{on } \partial D. \\ &\oint_{\partial D} (ax + by + c)\, ds = 0. \end{aligned} \tag{2.17}$$

If we now set

$$\xi = ax + by, \tag{2.18}$$

then (2.17) becomes

$$\begin{aligned} &n_\xi - p_2(\xi + c) = 0 \quad \text{in } \partial D, \\ &\oint_{\partial D} (\xi + c)\, ds = 0, \end{aligned} \tag{2.19}$$

which is clearly satisfied only for the circle (let $y = f(x)$ describe an arc of ∂D and use (2.19)) in which case $c = 0$ if the origin is taken at the center of the circle. But for the circle we know that

$$p_2 = K = K_{\min}. \tag{2.20}$$

Thus we have proved that if $\partial D \in C^\infty$, $p_2 \geqq K_{\min}$. We remark now that if ∂D is not in C^∞ we may approximate it by C^∞ curves, take the limit, and observe that the result remains valid for any Lipschitz boundary. (Note if ∂D is convex, then at a corner K tends to $+\infty$ so that $K_{\min}$ would not occur there. On the other hand, if the boundary has any straight-line segments, the inequality just states the obvious fact that $p_2 > 0$. Also the result would be of no interest though true if the domain were nonconvex.)

We now show that

$$p_2 \leqq K_{\max}. \tag{2.21}$$

To do this we observe that from Weinstock's inequality (1.1) and the Gauss–Bonnet formula it follows that

$$p_2 L \leqq 2\pi = \int_{\partial D} K\, ds \leqq K_{\max} L. \tag{2.22}$$

We have thus established the following theorem.

THEOREM 1. *If D is a simply connected bounded domain in R_2 with Lipschitz boundary ∂D, then the first nonzero Stekloff eigenvalue p_2 satisfies* (1.3) *with the equality signs holding if and only if the domain is in the interior of a circle.*

The lower bound is of course of no interest unless ∂D is convex.

3. The Dirichlet eigenvalue problem. In this section we wish to establish (1.5). It can be shown (see, e.g., Fichera [4]) that if the boundary ∂D is sufficiently smooth, the eigenvalue ν_1 satisfies the following system:

$$\begin{aligned} &\Delta^2 B = 0 \quad \text{in } D,\\ &B = 0 \quad \text{on } \partial D,\\ &\Delta B - \nu_1 \frac{\partial B}{\partial n} = 0 \quad \text{on } \partial D, \end{aligned} \tag{3.1}$$

and that the minimizing function h_1 of (1.4) and B are related through the identity

$$h_1 = \Delta B. \tag{3.2}$$

(See also Bramble and Payne [2], Hersch and Payne [6].)

Let us again assume for the moment that $\partial D \in C^\infty$. Then clearly (3.1) holds. But Miranda [12] has shown that the quantity

$$W = |\text{grad } B|^2 - B\Delta B \tag{3.3}$$

assumes its maximum value on the boundary. This follows from the fact that

$$\begin{aligned} \Delta W &= 2 \sum_{i,j=1}^{2} \left(\frac{\partial^2 B}{\partial x_i \partial x_j}\right)^2 - (\Delta B)^2 \\ &= \left(\frac{\partial^2 B}{\partial x^2} - \frac{\partial^2 B}{\partial y^2}\right)^2 + 4\left(\frac{\partial^2 B}{\partial x \partial y}\right)^2 \geqq 0. \end{aligned} \tag{3.4}$$

Thus at the point P_1 on ∂D where W assumes its maximum value we have (if $W \not\equiv$ const. in D)

$$\partial W/\partial n > 0. \tag{3.5}$$

Since B vanishes on ∂D this expression may be written

$$2\frac{\partial B}{\partial n}\frac{\partial^2 B}{\partial n^2} - \frac{\partial B}{\partial n}\Delta B > 0 \quad \text{at } P_1. \tag{3.6}$$

However, at P_1

$$\begin{aligned} \Delta B &= \nu_1 \frac{\partial B}{\partial n}, \\ \frac{\partial^2 B}{\partial n^2} &= \Delta B - K\frac{\partial B}{\partial n} = (\nu_1 - K)\frac{\partial B}{\partial n}. \end{aligned} \tag{3.7}$$

Inserting these expressions into (3.6), we obtain

$$(\nu_1 - 2K)\left(\frac{\partial B}{\partial n}\right)^2 > 0 \quad \text{at } P_1. \tag{3.8}$$

This clearly implies

$$\nu_1 > 2K(P) \geqq 2K_{\min}. \tag{3.9}$$

If $W \equiv$ const. in D, it follows that

$$W = \left(\frac{\partial B}{\partial n}\right)^2 = \text{const.} \quad \text{on } \partial D. \tag{3.10}$$

This in turn implies by (3.1) that ΔB is constant on ∂D and hence that B is proportional to the torsion function in D. Thus it follows that if $W \equiv$ const. in D,

$$\begin{aligned} &\Delta B = k \quad \text{in } D, \\ &B = 0 \quad \text{on } \partial D, \\ &\frac{\partial B}{\partial n} = \frac{kA}{L} \quad \text{on } \partial D, \end{aligned} \tag{3.11}$$

where A denotes the area of D and L the length of its perimeter.

Now if $W =$ const., then at every point of ∂B it follows from the arguments of (3.5) through (3.8) with the inequality replaced by an equality sign that either

$$\partial B/\partial n = 0 \tag{3.12}$$

or

$$\nu_1 = 2K = \text{const.} \tag{3.13}$$

Clearly the condition (3.12) implies $B \equiv 0$ and hence we would be led to the inadmissible trivial solution. Since the only closed domain for which $K =$ const. is the circle, it then follows that the only smooth domain for which the equality sign in (1.5) can hold is the interior of a circle.

Again if $\partial D \notin C^\infty$, we may approximate ∂D by C^∞ curves and take the limit. We thus obtain the following theorem.

THEOREM 2. *If D is a bounded two-dimensional domain with convex Lipschitz boundary ∂D, then the first Dirichlet eigenvalue ν_1 satisfies* (1.5) *with equality if and only if D is the interior of a circle.*

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