Solve the geodesic equations.

$$\frac{d^2 \gamma^i}{dt^2} + \Gamma^i_{jk} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0.$$

$$\Rightarrow \begin{cases} \dfrac{d^2\theta}{dt^2} + 2\cot\omega \dfrac{d\theta}{dt}\dfrac{d\omega}{dt} - \sin\theta\cos\theta\left(\dfrac{d\varphi}{dt}\right)^2 = 0. \\ \dfrac{d^2\varphi}{dt^2} + 2\cot\theta \dfrac{d\theta}{dt}\dfrac{d\varphi}{dt} + 2\cot\omega \dfrac{d\varphi}{dt}\dfrac{d\omega}{dt} = 0. \\ \dfrac{d^2\omega}{dt^2} - \sin\omega\cos\omega\left(\dfrac{d\theta}{dt}\right)^2 - \sin\omega\cos\omega\sin^2\theta\left(\dfrac{d\varphi}{dt}\right)^2 = 0. \end{cases}$$

???

Embed S^3 in $\mathbb{R}^4$.

For any given curve $\gamma : [-\varepsilon, \varepsilon] \to S^3 \subset \mathbb{R}^4$

$$\gamma(s) \cdot \gamma(s) = 1 \quad \text{in } \mathbb{R}^4.$$

Differentiate both sides. $\Rightarrow \quad 2\gamma \cdot \dot{\gamma} = 0 \quad \text{in } \mathbb{R}^4.$

$\therefore \gamma \perp \dot{\gamma}$ in $\mathbb{R}^4$.

Consider $\ddot{r}$ in $\mathbb{R}^4$.

$$\ddot{r} = \ddot{r}_{tan} + \ddot{r}_{nor}$$

where $\ddot{r}_{tan} \in d\Phi^{-1}(T_\gamma S^3)$, $\ddot{r}_{nor} \in (d\Phi^{-1}(T_\gamma S^3))^\perp$

Claim : $\ddot{r}_{tan} = d\Phi^{-1}\left(\frac{D\dot{r}}{ds}\right)$

equivalently, $\ddot{r} - d\Phi^{-1}\left(\frac{D\dot{r}}{ds}\right) = k \cdot r$ for some $k \in \mathbb{R}$ in $\mathbb{R}^4$.

$$\frac{D\dot{r}}{ds} = (\ddot{\theta} + 2\cot w\,\dot{\theta}\dot{w} - \sin\theta\cos\theta\,\dot{\varphi}^2)\,\partial_\theta$$
$$+ (\ddot{\varphi} + 2\cot\theta\,\dot{\theta}\dot{\varphi} + 2\cot w\,\dot{\varphi}\dot{w})\,\partial_\varphi$$
$$+ (\ddot{w} - \sin w\cos w\,\dot{\theta}^2 - \sin w\cos w\sin^2\theta\,\dot{\varphi}^2)\,\partial_w.$$

$$d\Phi^{-1}(\partial_\theta) = (\sin w\cos\theta\cos\varphi,\ \sin w\cos\theta\sin\varphi,\ -\sin w\sin\theta,\ 0)$$
$$d\Phi^{-1}(\partial_\varphi) = (-\sin w\sin\theta\sin\varphi,\ \sin w\sin\theta\cos\varphi,\ 0,\ 0)$$
$$d\Phi^{-1}(\partial_w) = (\cos w\sin\theta\cos\varphi,\ \cos w\sin\theta\sin\varphi,\ \cos w\cos\theta,\ -\sin w)$$

$$\therefore\ d\Phi^{-1}\left(\frac{D\dot r}{ds}\right)_x =$$

$$\cancel{\ddot\theta \sin w \cos\theta \cos\varphi} + \cancel{2\dot\theta \dot w \cos w \cos\theta \cos\varphi} - \dot\varphi^2 \sin w \sin\theta \cos^2\theta \cos\varphi$$

$$- \cancel{\ddot\varphi \sin w \sin\theta \sin\varphi} - \cancel{2\dot\theta\dot\varphi \sin w \cos\theta \sin\varphi} - \cancel{2\dot\varphi \dot w \cos w \sin\theta \sin\varphi}$$

$$+ \cancel{\ddot w \cos w \sin\theta \cos\varphi} - \dot\theta^2 \sin w \cos^2 w \sin\theta \cos\varphi - \dot\varphi^2 \sin w \cos^2 w \sin^3\theta \cos\varphi$$

$$x = \sin w \sin\theta \cos\varphi$$

$$\dot x = \sin w \cos\theta \cos\varphi\, \dot\theta - \sin w \sin\theta \sin\varphi\, \dot\varphi + \cos w \sin\theta \cos\varphi\, \dot w$$

$$\ddot x = \cancel{\sin w \cos\theta \cos\varphi\, \ddot\theta} - \sin w \sin\theta \cos\varphi\, \dot\theta^2 - \cancel{\sin w \cos\theta \sin\varphi\, \dot\theta\dot\varphi} + \cancel{\cos w \cos\theta \cos\varphi\, \dot\theta \dot w}$$

$$- \cancel{\sin w \sin\theta \sin\varphi\, \ddot\varphi} - \cancel{\sin w \cos\theta \sin\varphi\, \dot\theta\dot\varphi} - \sin w \sin\theta \cos\varphi\, \dot\varphi^2 - \cancel{\cos w \sin\theta \sin\varphi\, \dot\varphi \dot w}$$

$$+ \cancel{\cos w \sin\theta \cos\varphi\, \ddot w} + \cancel{\cos w \cos\theta \cos\varphi\, \dot\theta \dot w} - \cancel{\cos w \sin\theta \sin\varphi\, \dot\varphi \dot w} - \sin w \sin\theta \cos\varphi\, \dot w^2.$$

$$\ddot x - d\Phi^{-1}\left(\frac{D\dot r}{ds}\right)_x = -\dot\theta^2 \sin w \sin\theta \cos\varphi \left(1 - \cos^2 w\right)$$

$$- \dot\varphi^2 \sin w \sin\theta \cos\varphi \left(1 - \cos^2\theta - \cos^2 w \sin^2\theta\right)$$

$$- \dot w^2 \sin w \sin\theta \cos\varphi$$

$$= \sin w \sin\theta \cos\varphi \left(-\dot\theta^2(1-\cos^2 w) - \dot\varphi^2(1-\cos^2\theta - \cos^2 w \sin^2\theta) - \dot w^2\right)$$

$$= x\left(-\dot\theta^2(1-\cos^2 w) - \dot\varphi^2(1-\cos^2\theta - \cos^2 w\sin^2\theta) - \dot w^2\right)$$

$$
\begin{aligned}
d\Phi^{-1}\left(\frac{D\dot r}{ds}\right)_y = {} & \cancel{\ddot\theta \sin w \cos\theta \sin\varphi} + \cancel{2\dot\theta\dot w \cos w \cos\theta \sin\varphi} - \dot\varphi^2 \sin w \sin\theta \cos^2\theta \sin\varphi \\
& + \cancel{\ddot\varphi \sin w \sin\theta \cos\varphi} + \cancel{2\dot\theta\dot\varphi \sin w \cos\theta \cos\varphi} + \cancel{2\dot\varphi\dot w \cos w \sin\theta \cos\varphi} \\
& + \cancel{\ddot w \cos w \sin\theta \sin\varphi} - \dot\theta^2 \sin w \cos^2 w \sin\theta \sin\varphi - \dot\varphi^2 \sin w \cos^2 w \sin^3\theta \sin\varphi
\end{aligned}
$$

$y = \sin w \sin\theta \sin\varphi.$

$\dot y = \sin w \cos\theta \sin\varphi\, \dot\theta + \sin w \sin\theta \cos\varphi\, \dot\varphi + \cos w \sin\theta \sin\varphi\, \dot w$

$$
\begin{aligned}
\ddot y = {} & \cancel{\sin w \cos\theta \sin\varphi\, \ddot\theta} - \sin w \sin\theta \sin\varphi\, \dot\theta^2 + \cancel{\sin w \cos\theta \cos\varphi\, \dot\theta\dot\varphi} + \cancel{\cos w \cos\theta \sin\varphi\, \dot\theta\dot w} \\
& + \cancel{\sin w \sin\theta \cos\varphi\, \ddot\varphi} + \cancel{\sin w \cos\theta \cos\varphi\, \dot\theta\dot\varphi} - \sin w \sin\theta \sin\varphi\, \dot\varphi^2 + \cancel{\cos w \sin\theta \cos\varphi\, \dot\varphi\dot w} \\
& + \cancel{\cos w \sin\theta \sin\varphi\, \ddot w} + \cancel{\cos w \cos\theta \sin\varphi\, \dot\theta\dot w} + \cancel{\cos w \sin\theta \cos\varphi\, \dot\varphi\dot w} - \sin w \sin\theta \sin\varphi\, \dot w^2
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \ddot y - d\Phi^{-1}\left(\frac{D\dot r}{ds}\right)_y = {} & -\dot\theta^2 \sin w \sin\theta \sin\varphi\,(1-\cos^2 w) \\
& - \dot\varphi^2 \sin w \sin\theta \sin\varphi\,(1-\cos^2\theta - \cos^2 w \sin^2\theta) \\
& - \dot w^2 \sin w \sin\theta \sin\varphi \\
= {} & \sin w \sin\theta \sin\varphi \left(-\dot\theta^2(1-\cos^2 w) - \dot\varphi^2(1-\cos^2\theta-\cos^2 w\sin^2\theta) - \dot w^2\right) \\
= {} & y\left(-\dot\theta^2(1-\cos^2 w) - \dot\varphi^2(1-\cos^2\theta-\cos^2 w \sin^2\theta) - \dot w^2\right)
\end{aligned}
$$

$$d\Phi^{-1}\left(\frac{Dr}{ds}\right)_z = -\cancel{\ddot{\theta}\sin w\sin\theta} - \cancel{2\dot{\theta}\dot{w}\cos w\sin\theta} + \dot{\varphi}^2\sin w\sin^2\theta\cos\theta$$

$$+\cancel{\ddot{w}\cos w\cos\theta} - \underline{\dot{\theta}^2\sin w\cos^2 w\cos\theta} - \dot{\varphi}^2\sin w\cos^2 w\sin^2\theta\cos\theta$$

$$z = \sin w\cos\theta$$

$$\dot{z} = -\sin w\sin\theta\,\dot{\theta} + \cos w\cos\theta\,\dot{w}$$

$$\ddot{z} = -\cancel{\sin w\sin\theta\,\ddot{\theta}} - \underline{\sin w\cos\theta\,\dot{\theta}^2} - \cancel{\cos w\sin\theta\,\dot{\theta}\dot{w}}$$

$$+\cancel{\cos w\cos\theta\,\ddot{w}} - \cancel{\cos w\sin\theta\,\dot{\theta}\dot{w}} - \sin w\cos\theta\,\dot{w}^2$$

$$\therefore \ddot{z} - d\Phi^{-1}\left(\frac{Dr}{ds}\right)_z = -\dot{\theta}^2\sin w\cos\theta\left(1-\cos^2 w\right)$$

$$-\dot{\varphi}^2\sin w\cos\theta\left(\sin^2\theta - \sin^2\theta\cos^2 w\right)$$

$$-\dot{w}^2\sin w\cos\theta$$

$$= \sin w\cos\theta\left(-\dot{\theta}^2(1-\cos^2 w) - \dot{\varphi}^2\sin^2\theta(1-\cos^2 w) - \dot{w}^2\right)$$

$$= z\left(-\dot{\theta}^2(1-\cos^2 w) - \dot{\varphi}^2\sin^2\theta(1-\cos^2 w) - \dot{w}^2\right)$$

$$d\Phi^{-1}\left(\frac{Dr}{ds}\right)_t = -\cancel{\ddot{w}\sin w} + \dot{\theta}^2\sin^2 w\cos w + \dot{\varphi}^2\sin^2 w\cos w\sin^2\theta$$

$$t = \cos w$$

$$\dot{t} = -\sin w\,\dot{w}$$

$$\ddot{t} = \cancel{-\sin w\, \ddot{w}} - \cos w\, \dot{w}^2$$

$$\therefore\ \ddot{t} - d\Phi^{-1}\left(\frac{D\dot{r}}{ds}\right)_t = -\dot{\theta}^2 \sin^2 w \cos w - \dot{\varphi}^2 \sin^2 w \cos w \sin^2\theta - \dot{w}^2 \cos w$$

$$= \cos w \left(-\dot{\theta}^2 \sin^2 w - \dot{\varphi}^2 \sin^2 w \sin^2\theta - \dot{w}^2\right)$$

$$= t \left(-\dot{\theta}^2 \sin^2 w - \dot{\varphi}^2 \sin^2 w \sin^2\theta - \dot{w}^2\right)$$

$$\therefore\ \ddot{r} - d\Phi^{-1}\left(\frac{D\dot{r}}{ds}\right) = \Big(x\left(-\dot{\theta}^2(1-\cos^2 w) - \dot{\varphi}^2(1-\cos^2\theta - \cos^2 w \sin^2\theta) - \dot{w}^2\right),$$

$$y\left(-\dot{\theta}^2(1-\cos^2 w) - \dot{\varphi}^2(1-\cos^2\theta - \cos^2 w \sin^2\theta) - \dot{w}^2\right),$$

$$z\left(-\dot{\theta}^2(1-\cos^2 w) - \dot{\varphi}^2 \sin^2\theta(1-\cos^2 w) - \dot{w}^2\right),$$

$$t\left(-\dot{\theta}^2 \sin^2 w - \dot{\varphi}^2 \sin^2 w \sin^2\theta - \dot{w}^2\right) \Big)$$

$$= \Big(x\left(-\dot{\theta} \sin^2 w - \dot{\varphi}^2 \sin^2 w \sin^2\theta - \dot{w}^2\right),$$

$$y\left(-\dot{\theta} \sin^2 w - \dot{\varphi}^2 \sin^2 w \sin^2\theta - \dot{w}^2\right),$$

$$z\left(-\dot{\theta} \sin^2 w - \dot{\varphi}^2 \sin^2 w \sin^2\theta - \dot{w}^2\right),$$

$$t\left(-\dot{\theta} \sin^2 w - \dot{\varphi}^2 \sin^2 w \sin^2\theta - \dot{w}^2\right) \Big)$$

$$= \left(-\dot{\theta} \sin^2 w - \dot{\varphi}^2 \sin^2 w \sin^2\theta - \dot{w}^2\right) \cdot (x, y, z, t)$$

$$= -(\dot{\theta}\sin^2 w + \dot{\varphi}^2\sin^2 w\sin^2\theta + \dot{w})\cdot\gamma$$

Note that $\dot{\theta}\sin^2 w + \dot{\varphi}^2\sin^2 w\sin^2\theta + \dot{w} = g_{ij}\dot{\gamma}^i\dot{\gamma}^j$

$$= \|\dot{\gamma}\|^2$$

where $\|\cdot\|$ is natural metric in $\mathbb{R}^4$.

$$\therefore \ddot{\gamma} = d\Phi^{-1}\left(\frac{D\dot{\gamma}}{ds}\right) - \|\dot{\gamma}\|^2\cdot\gamma$$

Because $\gamma \perp \dot{\gamma}$ in $\mathbb{R}^4$, $\ddot{\gamma}_{tan} = d\Phi^{-1}\left(\frac{D\dot{\gamma}}{ds}\right)$, $\ddot{\gamma}_{nor} = -\|\dot{\gamma}\|\cdot\gamma$ $\boxtimes$

If γ is a geodesic on S^3, $\frac{D\dot{\gamma}}{ds} = 0$.

$$\therefore \ddot{\gamma} = -\|\dot{\gamma}\|^2\cdot\gamma.$$

WOLG, set $\|\dot{\gamma}\| = \text{constant} = \|\dot{\gamma}(0)\|$ for each $s\in[-\varepsilon,\varepsilon]$

since the reparametrization $\tilde{\gamma}(s) = \gamma(as)$ gives $\dot{\tilde{\gamma}} = a\dot{\gamma}$.

The solution of $\ddot{\gamma} = -\|\dot{\gamma}(0)\|\cdot\gamma$ with initial condition $\gamma(0)$, $\dot{\gamma}(0)$

is $\quad r(s) = r(0)\cos(\|\dot r(0)\| s) + \dfrac{\dot r(0)}{\|\dot r(0)\|}\sin(\|\dot r(0)\| s)$

which is the circle on plane spanned by $r(0), \dot r(0)$ centered at $(0,0,0,0)$.

Question.

Claim : $\ddot r_{tan} = d\Phi^{-1}\left(\frac{D\dot r}{ds}\right)$

equivalently, $\ddot r - d\Phi^{-1}\left(\frac{D\dot r}{ds}\right) = k\cdot r$ for some $k \in \mathbb{R}$ in $\mathbb{R}^4$.

generalization.

Claim : (M,g) : Riemannian manifold with Levi-Civita connection ∇.

(N,h) : $N \subset M$, submanifold. h = induced metric.

$\tilde\nabla$: Levi-Civita connection of (N,h).

$p \in N$, $pr_{T_pN} : T_pM \to T_pN$ as $T_pN \subset T_pM$, g-orthogonal projection

$Z \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$: satisfying $Y = Z$ around p.

$\Rightarrow pr_{T_pN}(\nabla_X Z)_p = (\tilde\nabla_X Y)_p$? How to prove?