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Equivalence of invariant metrics via Bergman kernel on complete noncompact Kähler manifolds

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Abstract

We study equivalence of invariant metrics on noncompact Kähler manifolds with a complete Bergman metric of bounded curvature. Especially only the boundedness of the ratio between Bergman kernel and the n-times wedge product of Bergman metric in any fundamental domain of such a Kähler manifold is required to obtain the equivalence of the Bergman metric and the complete Kähler–Einstein metric. To demonstrate the effectiveness of this method, we consider a two-parameter family of 3-dimensional bounded pseudoconvex domains

$$E_{p,\lambda} = \{(x, y, z) \in \mathbb{C}^3; (|x|^{2p} + |y|^2)^{1/\lambda} + |z|^2 < 1\}, \qquad p, \lambda > 0.$$

For this family, boundary limits of the holomorphic sectional curvature of the Bergman metric are not well-defined, and hence previously known methods for comparison of invariant metrics do not work. Lastly, we provide an estimate of lower bound of the integrated Carathéodory–Reiffen metric on complete noncompact simply-connected Kähler manifolds with negative sectional curvature.

1 Introduction

As the Bergman metric, the complete Kähler–Einstein metric of negative scalar curvature, the Kobayashi–Royden metric, and the Carathéodory–Reiffen metric are generalizations of the Poincaré–Bergman metric on the complex hyperbolic space,

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equivalence of these four invariant metrics on negatively curved complex manifolds has been studied in complex geometry. In addition, since these four metrics have the property that any automorphism becomes an isometry [31, 35], it makes sense to study them from the viewpoint of differential geometry. Hermitian metrics and Finsler metrics with this property are called *invariant metrics*. Some well-known classes having equivalence of these metrics are complex manifolds with uniform squeezing property, smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$, and weakly pseudoconvex domains of finite type in $\mathbb{C}^2$ [4, 36]. In complex dimension 3, the equivalence of these metrics breaks down for some weakly pseudoconvex domains with analytic boundary [16].

In this context, Wu and Yau proved the following remarkable theorems based on the quasi-bounded geometry and Shi's estimate [30] with Kähler–Ricci flow.

Theorem 1 ([33], Corollary 7) *Let (M, ω) be a complete simply-connected noncompact Kähler manifold whose Riemannian sectional curvature is negatively pinched. Then the base Kähler metric is uniformly equivalent to the Kobayashi–Royden metric, the Bergman metric and the complete Kähler–Einstein metric of negative scalar curvature.*

Theorem 2 ([33], Theorems 2, 3) *Let (M, ω) be a complete Kähler manifold whose holomorphic sectional curvature is negatively pinched. Then the base Kähler metric is uniformly equivalent to the Kobayashi–Royden metric and the complete Kähler–Einstein metric of negative scalar curvature.*

As an interesting application of equivalence of invariant metrics, it is recently showed by the first-named author that the non-equivalence of invariant metrics can be used to show the non-existence of complete Kähler metric whose holomorphic sectional curvature is negatively pinched on pseudoconex domains in $\mathbb{C}^n$ under some conditions (see [12]).

Based on Theorem 2, one possible method to show the equivalence of the invariant metrics on a complete Kähler manifold (M, ω) is to prove that the holomorphic sectional curvature of ω has a negative range. As explicit formulas are recently obtained for the Bergman kernels on certain weakly pseudoconvex domains (e.g., see [2, 3, 14, 28] and references therein), one could attempt to compute the holomorphic sectional curvature of the Bergman metric to establish the equivalence of the invariant metrics (for example, see [13]). However, in general, it seems to be a daunting task to compute the holomorphic sectional curvature for nontrivial pseudoconvex domains even with explicit formulas of the Bergman kernels.

Indeed, for the bounded pseudoconvex domains, even for the class of convex domains or strictly pseudoconvex domains, the curvature information of Bergman metric is known only near the boundary and not in the interior. The holomorphic sectional curvature of the Bergman metric has values between $-\infty$ and $+2$ [17, 24], but there is an example [21] of a semi-finite type pseudoconvex domain in which the holomorphic sectional curvature of Bergman metric blows up to $-\infty$.

Our main result in this paper is that, neither requiring the negative range of curvature as Wu–Yau theorems do, nor specifying the type of pseudoconvex domains, we provide a concrete approach to compare invariant metrics. Our method is based on knowledge of the Bergman kernel and can be applied to general bounded pseudoconvex domains Ω in $\mathbb{C}^n$ when an explicit description of the Bergman kernel near the boundary of Ω is available.

To state the main result (Theorem 1) below, we define the fundamental domain $\widetilde{M}$ of a complex manifold M to be the subset of M which contains exactly one point from each of the orbits of the group action by the automorphism group of M. An automorphism f of M means f and its inverse are holomorphic.

Theorem A *Let (M, ω_B) be an n-dimensional noncompact Kähler manifold with a complete Bergman metric ω_B of bounded curvature, where B denotes the Bergman kernel on M (as the (n, n)-form). Then the following statements hold:*

1. *Assume that $\frac{B}{\omega_B^n}$ is a bounded function for some fundamental domain $\widetilde{M}$. Here $\omega_B^n := \omega_B \wedge \cdots \wedge \omega_B$ (n-times). Then there exist a complete Kähler–Einstein metric ω_{KE} of negative scalar curvature and a constant $C_1 > 0$ such that ω_{KE} is uniformly equivalent to ω_B by C_1, i.e.,*

$$\frac{1}{C_1}\omega_{KE}(v, v) \leq \omega_B(v, v) \leq C_1\omega_{KE}(v, v) \qquad \text{for all } v \in T'M.$$

2. *Assume that there exists a compact subset K in M such that the holomorphic sectional curvature of ω_B is negative outside of K, and that M is biholomorphically and properly embedded into B_N, $N \geq n$, where B_N is the unit ball in $\mathbb{C}^N$. Then the Carathéodory–Reiffen metric γ_M is not essentially zero, and the Bergman metric is uniformly equivalent to the Kobayashi–Royden metric, i.e., there exists $C_2 > 0$ such that*

$$\frac{1}{C_2}\chi_M(p; v) \leq \sqrt{\omega_B(v, v)} \leq C_2\chi_M(p; v) \qquad \text{for all } v \in T'_pM,\ p \in M,$$

where χ_M is the Kobayashi–Royden metric on M. Moreover, if $N = n$, the Bergman metric is uniformly equivalent to the complete Kähler–Einstein metric of negative scalar curvature.

Remark 3 Under the same assumptions of Theorem A, but without additional assumptions of the first and second statements, we obtain the following from [32]: there exists $C_0 > 0$, which only depends on n and the curvature range of ω_B, such that

$$\chi_M(p; v) \leq C_0\sqrt{\omega_B(v, v)} \qquad \text{for all } v \in T'_pM,\ p \in M.$$

(See Remark 11 for the details.)

The second statement of Theorem A differs from the Wu–Yau theorems (Theorems 1 and 2) in that the Bergman metric's holomorphic sectional curvature is not required to be everywhere negative, but it still ensures the equivalence of invariant metrics. For the other assumption, we note that every bounded strictly pseudoconvex domain in $\mathbb{C}^n$ admits a proper holomorphic embedding into a ball (for example, see [18, p.11]).

To demonstrate the effectiveness of our method, we consider invariant metrics on a two-parameter family of 3-dimensional bounded domains defined by

$$E_{p,\lambda} = \{(x, y, z) \in \mathbb{C}^3; (|x|^{2p} + |y|^2)^{1/\lambda} + |z|^2 < 1\}, \qquad p, \lambda > 0. \tag{1.1}$$

When $p = \lambda = 1$, the domain $E_{p,\lambda}$ is the unit ball in $\mathbb{C}^3$. When $\lambda = 1$ and $p \geq 1/2$, this reduces to the well-known convex egg (Thullen) domains whose invariant metrics are uniformly equivalent ([13, 23]). With other pairs of (p, λ) for (1.1), the boundary limits of the holomorphic sectional curvature of the Bergman metric are not well-defined, so neither squeezing functions nor the Wu–Yau theorems can be applied. However, we show that Theorem A can be applied. For this purpose, we use a concrete formula for the Bergman kernel of $E_{p,\lambda}$, which is obtained in [2]. We also verify the Cheng's conjecture on $E_{p,\lambda}$ in the process of calculation. Namely, we show that the Bergman metric and the complete Kähler–Einstein metric is the same on $E_{p,\lambda}$ if and only if $p = \lambda = 1$ (Proposition 25).

In the last section, we obtain a result on the Carathéodory–Reiffen metric which is missing in the Wu–Yau theorems. Classical invariant metrics include the Carathéodory–Reiffen metric whose definition is based on the existence of non-constant bounded holomorphic functions on noncompact complex manifolds. However, showing the existence of such functions still remains as a big challenge in hyperbolic complex geometry.

The upper bounds of the Carathéodory–Reiffen metric have been studied extensively. As for comparison between Carathéodory–Reiffen metric and the Bergman metric on the bounded domains, the first result is obtained by Qi-Keng Lu [26] and then on manifolds by Hahn [19, 20]. Further developments are made by Ahn, Gaussier and Kim [1]. Very recently, a comparison of Carathéodory distance and Kähler–Einstein distance of Ricci curvature -1 for certain weakly pseudoconvex domains is established by the first-named author [11].

Our result in the last section is a lower bound of the integrated Carathéodory–Reiffen metric (Theorem 7). The positive lower bound of the Carathéodory–Reiffen metric is important in that it is the smallest invariant metric among invariant metrics [11, 22], and it provides quantitative information about non-constant bounded holomorphic functions (also, see [5]).

The article is organized as follows: In Sect. 2, we review the definitions of the invariant metrics. In the next section, we recall the quasi-bounded geometry and a result on comparison with the Kobayashi–Royden metric. In Sect. 4, we apply Shi's estimate on Kähler–Ricci flow outside of a compact subset on noncompact Kähler manifold. In Sect. 5, we prove Theorem A by generating a complete Kähler metric with negatively pinched holomorphic sectional curvature and applying the Wu–Yau theorems. In Sect. 6, we perform explicit calculation on $E_{p,\lambda}$ for any (p, λ) to verify the bounded curvature of the complete Bergman metric, and the hypothesis of Theorem A-3. In the last section, we prove Theorem 7 to obtain an integrated lower bound of the Carathéodory–Reiffen metric in the setting of Theorem 1.

2 Preliminaries

Let M be an n-dimensional complex manifold equipped with a complex structure J and a Hermitian metric g. The complex structure $J : T_{\mathbb{R}}M \to T_{\mathbb{R}}M$ is a real linear endomorphism that satisfies for every $x \in M$, and $X, Y \in T_{\mathbb{R},x}M$, $g_x(J_xX, Y) = -g_x(X, J_xY)$, and for every $x \in M$, $J_x^2 = -\mathbf{Id}_{T_xM}$. We decompose the complexified tangent bundle $T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C} = T'M \oplus \overline{T'M}$, where $T'M$ is the eigenspace of J with respect to the eigenvalue $\sqrt{-1}$ and $\overline{T'M}$ is the eigenspace of J with respect to the eigenvalue $-\sqrt{-1}$. We can regard v, w as real tangent vectors, and η, ξ as corresponding holomorphic $(1, 0)$ tangent vectors under the $\mathbb{R}$-linear isomorphism $T_{\mathbb{R}}M \to T'M$, i.e. $\eta = \frac{1}{\sqrt{2}}(v - \sqrt{-1}Jv)$, $\xi = \frac{1}{\sqrt{2}}(w - \sqrt{-1}Jw)$.

A Hermitian metric on M is a positive definite Hermitian inner product

$$g_p : T_p'M \otimes \overline{T_p'M} \to \mathbb{C}$$

which varies smoothly for each $p \in M$. The metric g can be decomposed into the real part denoted by $\mathrm{Re}(g)$, and the imaginary part denoted by $\mathrm{Im}(g)$. The real part $\mathrm{Re}(g)$ induces an inner product called the induced Riemannian metric of g, an alternating $\mathbb{R}$-differential 2-form. Define the $(1, 1)$-form $\omega := -\frac{1}{2}\,\mathrm{Im}(g)$, which is called the fundamental $(1, 1)$-form of g or the Kähler metric. In local coordinates this form can written as

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{i\overline{j}} dz_i \wedge d\overline{z_j}.$$

The components of the curvature 4-tensor of the Chern connection associated with the Hermitian metric g are given by

$$\begin{aligned}
R_{i\overline{j}k\overline{l}} &:= R(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_i}) \\
&= g\left(\nabla^c_{\frac{\partial}{\partial z_i}} \nabla^c_{\frac{\partial}{\partial \overline{z_j}}} \frac{\partial}{\partial z_k} - \nabla^c_{\frac{\partial}{\partial \overline{z_j}}} \nabla^c_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_k} - \nabla^c_{[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}}]} \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \overline{z_l}}\right) \\
&= -\frac{\partial^2 g_{i\overline{j}}}{\partial z_k \partial \overline{z}_l} + \sum_{p,q=1}^{n} g^{q\overline{p}} \frac{\partial g_{i\overline{p}}}{\partial z_k} \frac{\partial g_{q\overline{j}}}{\partial \overline{z}_l},
\end{aligned}$$

where $i, j, k, l \in \{1, \ldots, n\}$.

The holomorphic sectional curvature with the unit direction η at $x \in M$ (i.e., $g_\omega(\eta, \eta) = 1$) is defined by

$$H(g)(x, \eta) = R(\eta, \overline{\eta}, \eta, \overline{\eta}) = R(v, Jv, Jv, v),$$

where v is the real tangent vector corresponding to η. We will often write $H(g)(x, \eta) = H(g)(\eta) = H(\eta)$. The Ricci tensor of a Kähler metric ω is defined by

$$\mathrm{Ric}(\omega) := -\sqrt{-1}\partial\overline{\partial} \log \det(g).$$

Given any complex manifold M, for each $p \in M$ and a tangent vector v at p, define the Carathéodory–Reiffen metric and the Kobayashi–Royden metric by

$$\gamma_M(p; v) := \sup \{|df(p)(v)|;\ f : M \to \mathbb{D}, f(p) = 0, f \text{ holomorphic}\},$$
$$\chi_M(p; v) := \inf \left\{\frac{1}{R};\ f : R\mathbb{D} \to M, f(0) = p, df(\tfrac{\partial}{\partial z}|_{z=0}) = v, f \text{ holomorphic}\right\},$$

respectively.

The Bergman metric is defined in terms of the Bergman kernel. Let $\Lambda^{(n,0)}M$ be the space of smooth complex differential $(n, 0)$-forms on M. For $\varphi, \psi \in \Lambda^{(n,0)}M$, define

$$\langle\varphi, \psi\rangle = (-1)^{n^2/2} \int_M \varphi \wedge \overline{\psi},$$

and

$$||\varphi|| = \sqrt{\langle\varphi, \varphi\rangle}.$$

Let $L^2_{(n,0)}$ be the completion of

$$\left\{\varphi \in \Lambda^{(n,0)}M; ||\varphi|| < +\infty\right\}$$

with respect to $|| \cdot ||$. Then $L^2_{(n,0)}$ is a separable Hilbert space with respect to the inner product $\langle\cdot, \cdot\rangle$.

Define $\mathcal{H} = \left\{\varphi \in L^2_{(n,0)}; \varphi \text{ is holomorphic}\right\}$. Suppose $\mathcal{H} \neq 0$. Let $\left\{e_j\right\}_{j\geq 0}$ be an orthonormal basis of $\mathcal{H}$ with respect to $\langle\cdot, \cdot\rangle$. Then the $2n$-form on $M \times M$, defined by

$$B(x, y) := \sum_{j\geq 0} e_j(x) \wedge \overline{e}_j(y), \qquad x, y \in M,$$

is called the Bergman kernel of M. Suppose for some point $p \in M$, we have $B(p, p) \neq 0$. Write $B(z, z) = b(z, z)dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$ in terms of local coordinates $(z_1, \cdots, z_n)$. Define

$$\omega_B(z) := \sqrt{-1}\partial\overline{\partial} \log b(z, z).$$

If the real $(1, 1)$-form ω_B is positive definite, we call the corresponding Hermitian metric g^B_M the Bergman metric. By definition, g^B_M is Kähler.

Lastly, the Kähler–Einstein metric ω_{KE} means the Kähler metric which is also the Einstein metric, and the Kähler–Einstein metric of the negative scalar curvature becomes an invariant metric.

We will use the following lemma to prove Theorem A:

Lemma 4 ([33, Lemma 19]) *Let (M, ω) be a Hermitian manifold such that the holomorphic sectional curvature has the upper bound $-\kappa < 0$. Then the Kobayashi–Royden metric satisfies*

$$\chi_M(x, v) \geq \sqrt{\frac{\kappa}{2}}|v|_\omega,$$

for each $x \in M, v \in T'_x M$.

3 Quasi-bounded geometry

In this section, we review some results from Sect. 2 in [33].

The notion of quasi-bounded geometry is introduced by Yau and Cheng ([9]). Let (M, ω) be an n-dimensional complete Kähler manifold. For a point $p \in M$, let $B_\omega(p; \rho)$ be the open geodesic ball centered at p in M of radius ρ; we omit the subscript ω if there is no peril of confusion. Denote by $B_{\mathbb{C}^n}(r)$ the open ball centered at the origin in $\mathbb{C}^n$ of radius r with respect to the standard metric $\omega_{\mathbb{C}^n}$.

An n-dimensional Kähler manifold (M, ω) is said to have *quasi-bounded geometry* if there exist two constants $r_2 > r_1 > 0$ such that for each point $p \in M$, there is a domain $U \subset \mathbb{C}^n$ and a nonsingular holomorphic map $\psi : U \to M$ satisfying

(1) $B_{\mathbb{C}^n}(r_1) \subset U \subset B_{\mathbb{C}^n}(r_2)$ and $\psi(0) = p$;

(2) there exists a constant $C > 0$ depending only on r_1, r_2, n such that

$$C^{-1}\omega_{\mathbb{C}^n} \leq \psi^*(\omega) \leq C\omega_{\mathbb{C}^n} \quad \text{on } U; \tag{3.1}$$

(3) for each integer $l \geq 0$, there exists a constant A_l depending only on l, n, r_1, r_2 such that

$$\sup_{x \in U} \left| \frac{\partial^{|\nu|+|\mu|} g_{i\bar{j}}}{\partial v^\mu \, \partial \overline{v}^\nu} \right| \leq A_l, \text{ for all } |\mu| + |\nu| \leq l, \tag{3.2}$$

where $g_{i\bar{j}}$ are the components of $\psi^*\omega$ on U in terms of the natural coordinates $(v^1, \ldots, v^n)$, and μ, ν are multiple indices with $|\mu| = \mu_1 + \cdots + \mu_n$. We call r_1 a *radius* of quasi-bounded geometry.

By applying the L^2-estimate, the following theorem is proved.

Theorem 5 ([33], Theorem 9) *Let (M, ω) be a complete Kähler manifold. Then the manifold (M, ω) has quasi-bounded geometry if and only if for each integer $q \geq 0$,* *there exists a constant $C_q > 0$ such that*

$$\sup_{p\in M} |\nabla^q R_m| \le C_q, \tag{3.3}$$

where $R_m = \{R_{i\bar{j}k\bar{l}}\}$ denotes the curvature tensor of ω. In this case, the radius of quasi-bounded geometry depends only on C_0 and the dimension of M.

Also, we will use the following lemma:

Lemma 6 ([33, Lemma 20]) *Suppose a complete Kähler manifold (M, ω) has quasi-bounded geometry. Then the Kobayashi–Royden metric satisfies*

$$\chi_M(x, v) \le C|v|_\omega,$$

for each $x \in M$, $v \in T'_x M$, where C depends only on the radius of quasi-bounded geometry of (M, ω).

4 The maximum principle and Shi's estimate on Kähler–Ricci flow

Let $(M, \widetilde{\omega})$ be an n-dimensional complete noncompact Kähler manifold. Suppose for some constant $T > 0$ there is a smooth solution $\omega(x, t) > 0$ for the evolution equation

$$\begin{cases} \frac{\partial}{\partial t} g_{\alpha\overline{\beta}}(x, t) = -4R_{\alpha\overline{\beta}}(x, t) & \text{on } M \times [0, T], \\ g_{\alpha\overline{\beta}}(x, 0) = \widetilde{g}_{\alpha\overline{\beta}}(x) & x \in M, \end{cases} \tag{4.1}$$

where $g_{\alpha\overline{\beta}}(x, t)$ and $\widetilde{g}_{\alpha\overline{\beta}}$ are the metric components of $\omega(x, t)$ and $\widetilde{\omega}$, respectively. Assume that the curvature $R_m(x, t) = \left\{R_{\alpha\overline{\beta}\gamma\overline{\delta}(x,t)}\right\}$ of $\omega(x, t)$ satisfies

$$\sup_{M\times[0,T]} |R_m(x, t)|^2 \le k_0 \tag{4.2}$$

for some constant $k_0 > 0$.

The following lemma is an extension of Lemma 15 in [33] to the case of complement of compact subset. Though the proof is similar, we provide some details to indicate where modifications are needed for the complement.

Lemma 7 *With the above assumptions, suppose a smooth tensor $\left\{W_{\alpha\overline{\beta}\gamma\overline{\delta}(x,t)}\right\}$ on M with complex conjugation $W_{\alpha\overline{\beta}\gamma\overline{\delta}(x,t)} = W_{\beta\overline{\alpha}\delta\overline{\gamma}(x,t)}$ satisfies*

$$\left(\frac{\partial}{\partial t} W_{\alpha\overline{\beta}\gamma\overline{\delta}(x,t)}\right) \eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta \le (\triangle W_{\alpha\overline{\beta}\gamma\overline{\delta}})\eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta + C_1|\eta|^4_{\omega(x,t)}, \tag{4.3}$$

for all $x \in M$, $\eta \in T'_x M$, $0 \le t \le T$, where $\triangle \equiv 2\, g^{\alpha\overline{\beta}}(x, t)(\nabla_{\overline{\beta}}\nabla_\alpha + \nabla_\alpha\nabla_{\overline{\beta}})$ and C_1 is a constant. Let

$$h(x, t) = \max\left\{W_{\alpha\overline{\beta}\gamma\overline{\delta}}\eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta;\ \eta \in T'_x M, |\eta|_{\omega(x,t)} = 1\right\},$$

for all $x \in M$ *and* $0 \leq t \leq T$. *For any compact subset* K *in* M, *suppose*

$$\sup_{x\in M, 0\leq t\leq T} |h(x,t)| \leq C_0, \tag{4.4}$$

$$\sup_{M\backslash K} h(x,0) \leq -\kappa, \tag{4.5}$$

for some constants $C_0 > 0$ *and* κ. *Then,*

$$h(x,t) \leq (8C_0\sqrt{nk_0} + C_1)t - \kappa,$$

for all $x \in M\backslash K$ *and* $0 \leq t \leq T$.

Proof Denote

$$C = 8C_0\sqrt{nk_0} + C_1 > 0. \tag{4.6}$$

Suppose

$$h(x_1, t_1) - Ct_1 + \kappa > 0, \tag{4.7}$$

for some $(x_1, t_1) \in M\backslash K \times [0, T]$. Then by (4.4) we have $t_1 > 0$. Under the conditions (4.1) and (4.2), it follows from [30] that there exists a function θ such that

$$0 < \theta(x,t) \leq 1, \text{ on } M \times [0,T], \tag{4.8}$$

$$\frac{\partial\theta}{\partial t} - \triangle_{\omega(x,t)}\theta + 2\theta^{-1}|\nabla\theta|^2_{\omega(x,t)} \leq -\theta \text{ on } M \times [0,T], \tag{4.9}$$

$$\frac{C_2^{-1}}{1 + d_0(x_0, x)} \leq \theta(x,t) \leq \frac{C_2}{1 + d_0(x_0, x)} \text{ on } M \times [0,T], \tag{4.10}$$

where x_0 is a fixed point in M, $d_0(x, y)$ is the geodesic distance between x and y with respect to $\omega(x, 0)$, and $C_2 > 0$ is a constant depending only on n, k_0 and T.

Let

$$m_0 = \sup_{M\backslash K, 0\leq t\leq T} ([h(x,t) - Ct + \kappa]\theta(x,t)).$$

Then $0 < m_0 \leq C_0 + |\kappa|$ by (4.4),(4.7), and (4.8). Denote

$$\Lambda = \frac{2C_2(C_0 + CT + |\kappa|)}{m_0} > 0.$$

Then, for any $x \in M\backslash K$ with $d_0(x_0, x) \geq \Lambda$, we have

$$|(h(x,t) - Ct + \kappa)\theta(x,t)| \leq \frac{C_2(C_0 + CT + |\kappa|)}{1 + d_0(x, x_0)} \leq \frac{m_0}{2}.$$

It follows that the function $(h - Ct + \kappa)\theta$ must attain its supremum m_0 on the compact set $\overline{B(x_0; \Lambda)} \times [0, T] \subset M\backslash K \times [0, T]$, where $\overline{B(x_0; r)}$ denotes the closure of the geodesic ball with respect to $\omega(x, 0)$ centered at x_0 of radius r. Let

$$f(x, \eta, t) = \frac{W_{\alpha\overline{\beta}\gamma\overline{\delta}}\eta^{\alpha}\overline{\eta}^{\beta}\eta^{\gamma}\overline{\eta}^{\delta}}{|\eta|^4_{\omega(x,t)}} - Ct + \kappa,$$

for all $(x, t) \in M\backslash K \times [0, T]$, $\eta \in T'_x M\backslash \{0\}$. Then there exist x_*, η_*, t_* with $x_* \in \overline{B(x_0; r)}$, $0 \le t_* \le T$, $\eta_* \in T'_{x_*} M$ and $|\eta_*|_{\omega(x_*,t_*)} = 1$, such that

$$m_0 = f(x_*, \eta_*, t_*)\theta(x_*, t_*) = \max_{\mathcal{S}_t \times [0,T]} (f\theta),$$

where $\mathcal{S}_t = \left\{(x, \eta) \in T'M; x \in M, \eta \in T'_x M, |\eta|_{\omega(x,t)} = 1\right\}$. Since $h(., 0)$ is a continuous function on M, either $x_* \in M\backslash K$ or $x_* \in \partial K$, $t_* > 0$ by (4.5). Now we extend η_* to a smooth vector field using the same argument as in the proof of Lemma 15 in [33]. Since $f\theta = f(x, \eta(x), t)\theta(x, t)$ attains its maximum at (x_*, t_*), we have

$$\tfrac{\partial}{\partial t}(f\theta) \ge 0,\ \nabla(f\theta) = 0,\ \triangle(f\theta) \le 0 \quad \text{at } (x_*, t_*). \tag{4.11}$$

From (4.11) and (4.9), one can see that at the point (x_*, t_*), we have

$$0 \le \tfrac{\partial}{\partial t}(f\theta) = -m_0 < 0$$

(for details, see [33]). This yields a contradiction and the proof is completed.

The following lemma is an extension of Lemma 13 in [33] to the case of complement of a compact subset.

Lemma 8 *Let (M, ω) be an n-dimensional complete noncompact Kähler manifold. Let K be a compact set in M such that*

$$-\kappa_2 \le H(\omega) \le -\kappa_1 < 0 \textit{ on } M\backslash K, \tag{4.12}$$

where $H(\omega)$ is the holomorphic sectional curvature and κ_1, κ_2 are positive constants. Then there exists another Kähler metric $\widetilde{\omega}$ such that

$$C^{-1}\omega \le \widetilde{\omega} \le C\omega \qquad \textit{on } M, \tag{4.13}$$

$$-\widetilde{\kappa_2} \le H(\widetilde{\omega}) \le -\widetilde{\kappa_1} < 0 \qquad \textit{on } M\backslash K, \tag{4.14}$$

$$\sup_{p\in M} |\widetilde{\nabla}^q \widetilde{R_m}| \le C_q \qquad \textit{on } M, \tag{4.15}$$

where $\widetilde{\nabla}^q$ denotes the q-th order covariant derivative of $\widetilde{R_m}$ with respect to $\widetilde{\omega}$, and the positive constants $C = C(n)$, $\widetilde{\kappa_j} = \widetilde{\kappa_j}(n, \kappa_1, \kappa_2)$, $j = 1, 2$, $C_q = C_q(n, q, \kappa_1, \kappa_2)$ depend only on the parameters in their parentheses.

The conditions (4.13) and (4.15) appear in [30, 33]. We provide below details for the pinching estimate.

Proof From the short time existence of the Kähler–Ricci flow [30], the equation (4.1) admits a smooth solution $\left\{g_{\alpha\overline{\beta}}(x,t)\right\}$ for all $0 \le t \le T$. The curvature $R_m(x,t)$ satisfies

$$\sup_{x\in M} |\nabla^q R_m(x,t)|^2 \le \frac{C(q,n,K)(\kappa_2-\kappa_1)^2}{t^q}, \qquad 0 < t \le \frac{\theta_0(n,K)}{\kappa_2-\kappa_1} \equiv T, \tag{4.16}$$

for each nonnegative integer q, where $C(q,n,k) > 0$ is a constant depending only on q, K and n, and $\theta_0(n,K) > 0$ is a constant depending only on n and K.

From the evolution equation of the curvature tensor (see [30, 33]), we have

$$\begin{aligned}\frac{\partial}{\partial t} R_{\alpha\overline{\beta}\gamma\overline{\delta}} &= 4\triangle R_{\alpha\overline{\beta}\gamma\overline{\delta}} + 4g^{\mu\overline{\nu}}g^{\rho\overline{\tau}}(R_{\alpha\overline{\beta}\mu\overline{\tau}}R_{\gamma\overline{\delta}\rho\overline{\nu}} + R_{\alpha\overline{\delta}\mu\overline{\tau}}R_{\gamma\overline{\beta}\rho\overline{\nu}} - R_{\alpha\overline{\nu}\gamma\overline{\tau}}R_{\mu\overline{\beta}\rho\overline{\delta}})\\ &\quad - 2g^{\mu\overline{\nu}}(R_{\alpha\overline{\nu}}R_{\mu\overline{\beta}\rho\overline{\tau}} + R_{\mu\overline{\beta}}R_{\alpha\overline{\nu}\rho\overline{\tau}} + R_{\gamma\overline{\nu}}R_{\alpha\overline{\beta}\mu\overline{\tau}} + R_{\mu\overline{\delta}}R_{\alpha\overline{\beta}\rho\overline{\nu}}),\end{aligned}$$

where $\triangle \equiv \triangle_{\omega(x,t)} = \frac{1}{2}g^{\alpha\overline{\beta}}(x,t)(\nabla_{\overline{\beta}}\nabla_\alpha + \nabla_\alpha\nabla_{\overline{\beta}})$. It follows that

$$\left(\frac{\partial}{\partial t} R_{\alpha\overline{\beta}\gamma\overline{\delta}}\right)\eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta \tag{4.17}$$

$$\le 4(\triangle R_{\alpha\overline{\beta}\gamma\overline{\delta}})\eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta + C_1(n)|\eta|^4_{g_{\alpha\overline{\beta}}}(x,t)|R_m(x,t)|^2_{\omega(x,t)} \tag{4.18}$$

$$\le 4(\triangle R_{\alpha\overline{\beta}\gamma\overline{\delta}})\eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta + \widetilde{C_1}(n,K)(\kappa_2-\kappa_1)^2|\eta|^4_{\omega(x,t)},$$

by (4.16) with $q = 0$. Let

$$H(x,\eta,t) = \frac{R_{\alpha\overline{\beta}\gamma\overline{\delta}})\eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta}{|\eta|^4_{\omega(x,t)}}.$$

Then by (4.12) and (4.16),

$$\begin{aligned}&H(\widetilde{\omega}) \le -\widetilde{\kappa_1} < 0 \text{ on } M\backslash K,\\ &|H(x,\eta,t)| \le |R_m(x,t)|_{\omega(x,t)} \le C_0(n,K)(\kappa_2-\kappa_1).\end{aligned}$$

To apply the maximum principle, let us denote

$$h(x,t) = \max\left\{H(x,\eta,t); |\eta|_{\omega(x,t)=1}\right\},$$

for all $x \in M$ and $0 \le t \le \frac{\theta(n,K)}{\kappa_2-\kappa_1}$. Then h with (4.17) satisfies the three conditions in Lemma 7. Then

$$H(x,\eta,t) \le h(x,t) \le -\frac{\kappa_1}{2} < 0,$$

for all $0 < t \leq t_0 := \min\left\{\frac{\kappa_1}{2\widetilde{C}_1(n,K)(\kappa_2-\kappa_1)^2}, \frac{\theta_0(n,K)}{\kappa_2-\kappa_1}\right\}$. Since the curvature tensor is bounded by (4.16) with $q = 0$, the complete Kähler metric $\omega(x,t) = \frac{\sqrt{-1}}{2} g_{\alpha\overline{\beta}}(x,t) dz^\alpha \wedge d\overline{z}^\beta$ is a desired metric for an arbitrary $t \in (0, t_0]$.

5 Generation of Kähler metrics with negative holomorphic sectional curvature

In this section, after establishing a proposition below, we prove Theorem 1.

Proposition 9 *Given an n-dimensional Kähler manifold (M, ω), assume that there exists a compact subset K in M such that the holomorphic sectional curvature of ω is negative outside of K, and M is biholomorphically and properly embedded into B_N, $N \geq n$, where B_N is the unit ball in $\mathbb{C}^N$. Then there exists a complete Kähler metric $\widetilde{\omega}$ whose holomorphic sectional curvature has a negative upper bound and $\widetilde{\omega} \geq \omega$.*

Proof From the holomorphic embedding $M \hookrightarrow B_N$, consider a Kähler metric of the form

$$\omega_m := m\omega_P + \omega, \quad m > 0,$$

where ω_P is the Poincaré metric of the unit ball B_N in $\mathbb{C}^N$. It is clear that $\omega_m \geq \omega$ for each $m > 0$. From the decreasing property of the holomorphic sectional curvature, ω_P restricted to M has a negative holomorphic sectional curvature [34]. From Lemma 4 of [34], we may assume that the holomorphic sectional curvature of ω_m is the Gaussian curvature on some embedded Riemann surfaces in M. Recall that for a Hermitian metric G on a Riemann surface, the holomorphic sectional curvature of G is the Gaussian curvature $H(g) = -\frac{1}{g}\frac{\partial^2 \log g}{\partial z \partial \overline{z}}$ of G for some positive smooth function $g = g(z, \overline{z})$. In this case, the holomorphic sectional curvature $H(G, t)$ becomes a real-valued function independent of the unit vector t. Thus we write $H(G)$ instead of $H(G, t)$.

From [25, Proposition 3.1], for any positive functions f and g with $m > 0$,

$$\begin{aligned} H(f+mg) &\leq \frac{f^2}{(f+mg)^2} H(f) + \frac{m^2 g^2}{(f+mg)^2} H(mg) \\ &= \frac{f^2}{(f+mg)^2} H(f) + \frac{m g^2}{(f+mg)^2} H(g). \end{aligned}$$

From here, we can deduce that $H(\omega_m)$ becomes negative on K by taking sufficiently large m. Since $H(\omega_m)$ is negative on $M \backslash K$, we are done.

Proof of Theorem 1 For the first statement, we fix a fundamental domain $\widetilde{M}$ and define a function $f : M \to \mathbb{C}$ by $f(z) := \frac{B(z)}{\omega_B^n(z)}$. Since the numerator and the denominator are smooth (n, n)-forms, the function f is well-defined and clearly smooth. Note that the

Bergman kernel and the Bergman metric are invariant under the automorphism group of M. Thus the boundedness assumption of f on $\widetilde{M}$ implies the boundedness of f on M, and we have a function f which is smooth and bounded on M satisfying $\mathrm{Ric}_{i\bar{j}} + g_{i\bar{j}} = f_{i\bar{j}}$ for each i, j, where we denote the Bergman metric in local coordinates by $(g_{i\bar{j}})$. Now we apply the main theorem in [6], and the conclusion follows.

The first part of the second statement follows from Lemma 4, Lemma 6 and Proposition 9 with the fact that for each $m > 0$,

$$\omega_B \leq \widetilde{\omega},$$

where $\widetilde{\omega}$ is defined in Proposition 9. For the second part of the case $N = n$, the metric $\widetilde{\omega}$ has the bounded curvature. Then one can solve the complex Monge–Ampere equation by following Wu–Yau's approach (see Lemma 31 and Theorem 3 in [33]).

Remark 10 When $N > n$, the holomorphic sectional curvature $\widetilde{\omega}$ does not need to be bounded below because of the presence of the second fundamental form (see [34]).

Remark 11 If (4.12) is replaced by

$$-\kappa_2 \leq H(\omega) \leq -\kappa_1 \text{ on } M \quad \text{for } \kappa_1 \in \mathbb{R},$$

then (4.13) and (4.15) still follow from the original Shi's argument. Combining it with Lemmas 6 and 8, we obtain a proof of the statement in Remark 3. Indeed, by applying Shi's estimate on Kähler–Ricci flow with the short-time existence, we can generate a complete Kähler metric ω such that any order of covariant derivatives of the curvature tensor is bounded, and ω is equivalent to the Bergman metric ω_B. Then by the characterization of quasi-bounded geometry of Wu–Yau [33], ω admits a quasi-bounded geometry, and the statement in Remark 3 follows from Lemma 6.

6 Domain $E_{p,\lambda}$

In this section, we consider the domain

$$E_{p,\lambda} = \{(x, y, z) \in \mathbb{C}^3; (|x|^{2p} + |y|^2)^{1/\lambda} + |z|^2 < 1\}, \qquad p, \lambda > 0,$$

and perform necessary computations to examine the comparison of invariant metrics through verification of the hypotheses in Theorem A.

First, we take a suitable compact set $K \subset E_{p,\lambda} \cup \partial E_{p,\lambda}$ that satisfies the conditions in Theorem A. Since any point $(x, y, z) \in \mathbb{C}^3$ can be realized as

$$|x| < r(z, y) = \left((1 - |z|^2)^\lambda - |y|^2\right)^{\frac{1}{2p}},$$

with a fixed pair (y, z), the point (x, y, z) can be mapped biholomorphically onto the form $(0, y, z)$ through the automorphism of one-dimensional disc with the radius $r(y, z)$ centered at the origin. Then using rotations, we can make the other two entries to have non-negative real-values. Since all these transformations are automorphisms of $E_{p,\lambda}$, we take the compact set:

$$K_1 = \overline{\{(0, y, z) \in E_{p,\lambda}; 0 \le x, y < 1\}},$$

where the closure is taken with respect to the usual topology of $\mathbb{C}^3$.

An explicit formula of Bergman kernel B on $E_{p,\lambda}$ is computed in [2]:

$$\begin{aligned} B((x,y,z),\overline{(x,y,z)}) = {} & \frac{\left((1-\nu_3)^\lambda-\nu_2\right)^{\frac{1}{p}-3}\nu_1{}^2(p-1)(\lambda(p-1)+p)}{(1-\nu_3)^{2-2\lambda}\pi^3p^2\left(\nu_1-((1-\nu_3)^\lambda-\nu_2)^{1/p}\right)^4} \\ & + \frac{(1-\nu_3)^{\lambda-2}\left((1-\nu_3)^\lambda-\nu_2\right)^{\frac{1}{p}-3}\nu_1{}^2(p-1)(\lambda-1)\nu_2p}{\pi^3p^2\left(\nu_1-((1-\nu_3)^\lambda-\nu_2)^{1/p}\right)^4} \\ & + \frac{\left((1-\nu_3)^\lambda-\nu_2\right)^{\frac{3}{p}-3}(p+1)\left((1-\nu_3)^\lambda(\lambda+\lambda p+p)+(\lambda-1)\nu_2p\right)}{(1-\nu_3)^{2-\lambda}\pi^3p^2\left(\nu_1-((1-\nu_3)^\lambda-\nu_2)^{1/p}\right)^4} \\ & - \frac{\left((1-\nu_3)^\lambda-\nu_2\right)^{\frac{2}{p}-3}2\nu_1\left((1-\nu_3)^\lambda(\lambda(p^2-2)+p^2)+(\lambda-1)\nu_2p^2\right)}{(1-\nu_3)^{2-\lambda}\pi^3p^2\left(\nu_1-((1-\nu_3)^\lambda-\nu_2)^{1/p}\right)^4}, \end{aligned} \tag{6.1}$$

where we set $\nu_1 := x\overline{x}$, $\nu_2 := y\overline{y}$ and $\nu_3 := z\overline{z}$.

We write

$$a = 1-\nu_3, \quad b = (1-\nu_3)^\lambda-\nu_2, \quad c = ((1-\nu_3)^\lambda-\nu_2)^{1/p}-\nu_1.$$

Then

$$\begin{aligned} B = {} & \frac{b^{\frac{1}{p}-3}\nu_1{}^2(p-1)(\lambda(p-1)+p)}{a^{2-2\lambda}\pi^3p^2c^4} + \frac{a^{\lambda-2}b^{\frac{1}{p}-3}\nu_1{}^2(p-1)(\lambda-1)\nu_2p}{\pi^3p^2c^4} \\ & + \frac{b^{\frac{3}{p}-3}(p+1)\left(a^\lambda(\lambda+\lambda p+p)+(\lambda-1)\nu_2p\right)}{a^{2-\lambda}\pi^3p^2c^4} \\ & - \frac{b^{\frac{2}{p}-3}2\nu_1\left(a^\lambda(\lambda(p^2-2)+p^2)+(\lambda-1)\nu_2p^2\right)}{a^{2-\lambda}\pi^3p^2c^4}. \end{aligned} \tag{6.2}$$

Write $D = a^2c^4$ and

$$\begin{aligned} N = {} & a^{2\lambda}b^{\frac{1}{p}-3}\nu_1{}^2(p-1)(\lambda(p-1)+p) + a^\lambda b^{\frac{1}{p}-3}\nu_1{}^2(p-1)(\lambda-1)\nu_2p \\ & + a^\lambda b^{\frac{3}{p}-3}(p+1)\left(a^\lambda(\lambda+\lambda p+p)+(\lambda-1)\nu_2p\right) \\ & - a^\lambda b^{\frac{2}{p}-3}2\nu_1\left(a^\lambda(\lambda(p^2-2)+p^2)+(\lambda-1)\nu_2p^2\right). \end{aligned}$$

Then

$$B = \frac{N}{\pi^3p^2D}. \tag{6.3}$$

Write

$$N_1 = a^{2\lambda} b^{\frac{1}{p}-3} v_1^2, \qquad N_2 = a^{\lambda} b^{\frac{1}{p}-3} v_1^2 v_2, \qquad N_3 = a^{2\lambda} b^{\frac{3}{p}-3},$$
$$N_4 = a^{\lambda} b^{\frac{3}{p}-3} v_2, \qquad N_5 = a^{2\lambda} b^{\frac{2}{p}-3} v_1, \qquad N_6 = a^{\lambda} b^{\frac{2}{p}-3} v_1 v_2,$$
$$u_1 = (p-1)(\lambda(p-1)+p), \qquad u_2 = p(p-1)(\lambda-1), \qquad u_3 = (p+1)(\lambda+\lambda p+p),$$
$$u_4 = p(p+1)(\lambda-1), \qquad u_5 = -2(\lambda(p^2-2)+p^2), \qquad u_6 = -2(\lambda-1)p^2.$$

Then

$$N = \sum_{i=1}^{6} u_i N_i.$$

Note that we have

$$u_1 + u_3 + u_5 = 6\lambda \quad \text{and} \quad u_2 + u_4 + u_6 = 0.$$

From the description of the Bergman kernel, we can check the pseudoconvexity of $E_{p,\lambda}$ for each $p, \lambda > 0$.

Proposition 12 *$E_{p,\lambda}$ is a pseudoconvex domain for each $p, \lambda > 0$.*

Proof [1] To show that $u = u_{p,\lambda} := \left(|x|^{2p} + |y|^2\right)^{\frac{1}{\lambda}} + |z|^2$ is a (bounded) plurisubharmonic exhaustion function of $E_{p,\lambda}$, it suffices to show that $v = v_{p,\lambda} := \left(|x|^{2p} + |y|^2\right)^{\frac{1}{\lambda}}$ is plurisubharmonic. To this end, consider

$$\log v = \frac{1}{\lambda} \log\left(e^{\psi_1} + e^{\psi_2}\right), \qquad \text{where } \psi_1 := 2p \log|x| \quad \text{and} \quad \psi_2 := 2\log|y|.$$

Now the plurisubharmonicity of $\log v$ follows from the fact that $\log\left(e^{\psi_1} + e^{\psi_2}\right)$ is always plurisubharmonic whenever ψ_1 and ψ_2 are plurisubharmonic, since we have

$$\begin{aligned}
&\frac{\partial^2}{\partial z \partial \overline{z}} \log\left(e^{\psi_1} + e^{\psi_2}\right) \\
&= \frac{1}{\left(e^{\psi_1} + e^{\psi_2}\right)^2} \left(e^{\psi_1+\psi_2} \left(\frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_2}{\partial z} \right)^2 + e^{\psi_1} \frac{\partial^2 \psi_1}{\partial z \partial \overline{z}} + e^{\psi_2} \frac{\partial^2 \psi_2}{\partial z \partial \overline{z}} \right) \geq 0.
\end{aligned}$$

From the plurisubharmonicity of $\log v$ it follows that $v = e^{\log v}$ is plurisubharmonic, as desired.

We are interested in behaviours of the metric and curvature components on the compact set $K_1 = \overline{\{(0, y, z) \in E_{\lambda,p}; 0 \leq y, z < 1\}}$. In what follows, we compute those components.

[1] This proof is suggested by an anonymous referee and replaces our original proof. We are grateful to the referee.

Recall the formula for the components of the Bergman metric

$$g_{i\bar{j}} = \frac{\partial^2 \log B}{\partial z_i \partial \overline{z_j}}, \quad i, j = 1, 2, 3,$$

where we set $(z_1, z_2, z_3) = (x, y, z)$. For $i = 1, 2, 3$, we write

$$\partial_i = \frac{\partial}{\partial z_i} \quad \text{and} \quad \overline{\partial}_i = \frac{\partial}{\partial \overline{z_i}}.$$

Proposition 13 *Each component of the Bergman metric $g_{i\bar{j}}$ at $(0, y, z) \in E_{p,\lambda}$, $0 \le y, z < 1$, is given as follows:*

$$\begin{aligned}
g_{1\bar{1}} &= \frac{1}{c} \cdot \frac{u_5 + u_6\delta}{u_3 + u_4\delta} + \frac{4}{c}, \\
g_{2\bar{2}} &= \frac{a^\lambda}{b^2}\left(\frac{1}{p} + 3\right) + \frac{a^\lambda}{b^2} \cdot \frac{u_3u_4(1-\delta)^2}{(u_3 + u_4\delta)^2}, \\
g_{2\bar{3}} = g_{3\bar{2}} &= \frac{\lambda yz}{a^{1-\lambda}b^2} \cdot \left(\frac{1}{p} + 3\right) + \frac{\lambda yz}{a^{1-\lambda}b^2} \cdot \frac{u_3u_4(1-\delta)^2}{(u_3 + u_4\delta)^2}, \\
g_{3\bar{3}} &= \frac{1 + \delta(\lambda z^2 - 1)}{a^{2-2\lambda}b^2} \cdot \frac{\lambda}{p} + \frac{\delta^2(2-2\lambda) + \delta(2\lambda^2 z^2 - 4) + \lambda + 2}{a^{2-2\lambda}b^2} \\
&\quad + \frac{\lambda\delta}{a^{2-2\lambda}b^2} \cdot \frac{u_3u_4(1+\delta^2)(1+\lambda z^2) + u_4^2\delta(1 + (\lambda z^2 - 1)\delta + \delta^2) + u_3^2(1+\lambda z^2)}{(u_3 + u_4\delta)^2}, \\
g_{i\bar{j}} &= 0 \ \text{otherwise,}
\end{aligned}$$

where we write $\delta := y^2/a^\lambda = y^2/(1-z^2)^\lambda$.

Proof All the formulas for $g_{i\bar{j}}$ are obtained from direct computations. For example, since

$$\begin{aligned}
&\overline{\partial}_1 D = -4a^2c^3x, && \overline{\partial}_1 N_1 = 2a^{2\lambda}b^{\frac{1}{p}-3}\nu_1 x, && \overline{\partial}_1 N_2 = 2a^\lambda b^{\frac{1}{p}-3}\nu_1 x\nu_2, \\
&\overline{\partial}_1 N_3 = 0, \quad \overline{\partial}_1 N_4 = 0, && \overline{\partial}_1 N_5 = a^{2\lambda}b^{\frac{2}{p}-3}x, && \overline{\partial}_1 N_6 = a^\lambda b^{\frac{2}{p}-3}x\nu_2,
\end{aligned}$$

and

$$\begin{aligned}
&\partial_1\overline{\partial}_1 D = -4a^2c^3 + 12a^2c^2\nu_1, && \partial_1\overline{\partial}_1 N_1 = 4a^{2\lambda}b^{\frac{1}{p}-3}\nu_1, && \partial_1\overline{\partial}_1 N_2 = 4a^\lambda b^{\frac{1}{p}-3}\nu_1\nu_2, \\
&\partial_1\overline{\partial}_1 N_3 = 0, \quad \partial_1\overline{\partial}_1 N_4 = 0, && \partial_1\overline{\partial}_1 N_5 = a^{2\lambda}b^{\frac{2}{p}-3}, && \partial_1\overline{\partial}_1 N_6 = a^\lambda b^{\frac{2}{p}-3}\nu_2,
\end{aligned}$$

we have

$$\begin{aligned}
g_{1\bar{1}} &= \frac{N(\partial_1\overline{\partial}_1 N) - (\partial_1 N)(\overline{\partial}_1 N)}{N^2} - \frac{D(\partial_1\overline{\partial}_1 D) - (\partial_1 D)(\overline{\partial}_1 D)}{D^2} \\
&\xrightarrow{(0,y,z)} \frac{\partial_1\overline{\partial}_1 N}{N} - \frac{\partial_1\overline{\partial}_1 D}{D} = \frac{u_5a^{2\lambda}b^{\frac{2}{p}-3} + u_6a^\lambda b^{\frac{2}{p}-3}y^2}{u_3a^{2\lambda}b^{\frac{3}{p}-3} + u_4a^\lambda b^{\frac{3}{p}-3}y^2} + \frac{4a^2c^3}{a^2c^4} \\
&= \frac{1}{c} \cdot \frac{u_5 + u_6\delta}{u_3 + u_4\delta} + \frac{4}{c},
\end{aligned}$$

where we use $c = b^{\frac{1}{p}}$ at $(0, y, z)$.

The other $g_{i\bar{j}}$ can be computed similarly, and we omit the details.

Remark 14 When $(0, y, z)$ approaches the boundary of K_1, we find that the limits of the metric components and those of curvature components cannot be determined. However, using δ introduced in the above proposition, we will be able to control the limit behaviors.

Write

$$g_{1\bar{1}} = \frac{1}{c} \cdot A_1, \quad g_{2\bar{2}} = \frac{a^{\lambda}}{b^2} \cdot A_2, \quad g_{2\bar{3}} = \frac{\lambda yz}{a^{1-\lambda}b^2} \cdot A_2, \quad g_{3\bar{3}} = \frac{1}{a^{2-2\lambda}b^2} \cdot A_3, \tag{6.4}$$

where

$$A_1 = \frac{u_5 + u_6\delta}{u_3 + u_4\delta} + 4, \qquad A_2 = \frac{1}{p} + 3 + \frac{u_3u_4(1-\delta)^2}{(u_3 + u_4\delta)^2},$$
$$A_3 = (1 + \delta(\lambda z^2 - 1)) \cdot \frac{\lambda}{p} + \delta^2(2 - 2\lambda) + \delta(2\lambda^2 z^2 - 4) + \lambda + 2$$
$$+ \lambda\delta \cdot \frac{u_3u_4(1+\delta^2)(1+\lambda z^2) + u_4^2\delta(1 + (\lambda z^2 - 1)\delta + \delta^2) + u_3^2(1 + \lambda z^2)}{(u_3 + u_4\delta)^2}.$$

Then

$$g_{2\bar{2}}g_{3\bar{3}} - g_{2\bar{3}}g_{3\bar{2}} = \frac{1}{a^{2-3\lambda}b^4}A_2(A_3 - \lambda^2\delta z^2 A_2) = \frac{1-\delta}{a^{2-3\lambda}b^4} \cdot A_2A_4 = \frac{A_2A_4}{a^{2-2\lambda}b^3}, \tag{6.5}$$

where we put $A_4 := (A_3 - \lambda^2\delta z^2 A_2)/(1-\delta)$ and use $1 - \delta = b/a^{\lambda}$. More explicitly, we have

$$A_4 = \frac{\delta^2 p^2(r-2)(r-1) + \delta p(r-1)(4pr + 4p + 3r) + p^2r^2 + 3p^2r + 2p^2 + 2pr^2 + 3pr + r^2}{p(\delta p(r-1) + pr + p + r)}.$$

Note that $0 \le \delta < 1$. Furthermore, as $(0, y, z) \in E_{p,\lambda}$ approaches the boundary, we have $\delta \to 1^-$. One sees that

$$\lim_{\delta\to 1^-} A_1 = \frac{4(2+p)}{1+2p}, \qquad \lim_{\delta\to 1^-} A_2 = 3 + \frac{1}{p} \quad \text{and} \quad \lim_{\delta\to 1^-} A_4 = \lambda\left(3 + \frac{1}{p}\right). \tag{6.6}$$

Lemma 15 *At* $(0, y, z) \in E_{p,\lambda}$, $0 \le y, z < 1$, *the ratio* $\dfrac{\det g_B}{B}$ *is bounded.*

Proof From (6.3), (6.4) and (6.5), we obtain

$$\frac{\det g_B}{B} = \frac{\frac{1}{c}A_1\frac{A_2A_4}{a^{2-2\lambda}b^3}}{\frac{N}{\pi^3p^2D}} = \frac{\pi^3p^2A_1A_2A_4a^2c^4}{ca^{2-2\lambda}b^3\cdot a^\lambda b^{\frac{3}{p}-3}(p+1)\left(a^\lambda(\lambda+\lambda p+p)+(\lambda-1)y^2p\right)}$$
$$= \frac{\pi^3p^2A_1A_2A_4}{(p+1)\left((\lambda+\lambda p+p)+(\lambda-1)p\delta\right)},$$

which is bounded.

Proposition 16 *The inverse metric of the Bergman metric $g_{i\bar{j}}$ at $(0, y, z) \in E_{p,\lambda}$, $0 \le y, z < 1$, are given as follows:*

$$g^{1\bar{1}} = \frac{c}{A_1}, \qquad g^{2\bar{2}} = \frac{b^2}{a^\lambda}\cdot\frac{A_3}{(1-\delta)A_2A_4} = \frac{bA_3}{A_2A_4},$$
$$g^{2\bar{3}} = g^{3\bar{2}} = -\frac{\lambda yza^{1-2\lambda}b^2}{(1-\delta)A_4} = -\frac{\lambda yza^{1-\lambda}b}{A_4}, \qquad g^{3\bar{3}} = \frac{a^{2-2\lambda}b^2}{(1-\delta)A_4} = \frac{a^{2-\lambda}b}{A_4},$$
$$g^{i\bar{j}} = 0 \ \text{ otherwise.}$$

Proof The formulas are obtained by taking the inverse matrix of the 3×3 matrix $(g_{i\bar{j}})_{i,j=1,2,3}$ calculated in Proposition 13. In particular, the determinant of the 2×2 block $(g_{i\bar{j}})_{i,j=2,3}$ is computed in (6.5). Also recall $1-\delta = b/a^\lambda$.

Through direct computations, we obtain the following for $(0, y, z) \in K_1$:

Here G_i are set to be the remaining factors after pulling out the factors involving a, b, c, y, z. Explicitly, we have

$$G_1 = \frac{4}{p} - \frac{(u_5+u_6\delta)((2p-3)u_4\delta+3(p-1)u_3+pu_4)}{p(u_3+\delta u_4)^2}$$
$$+\frac{2(p-1)u_6\delta+(3p-2)u_5+pu_6}{p(u_3+u_4\delta)},$$
$$G_2 = \frac{4\lambda}{p}+\frac{\lambda}{p}\cdot\frac{u_5+u_6\delta}{u_3+u_4\delta} - \frac{\lambda\delta(1-\delta)(u_4u_5-u_3u_6)}{(u_3+u_4\delta)^2}.$$

For simplicity, we do not present expressions for the other G_i's. Since $u_3+u_4\delta > 0$, one can see that G_i are bounded for $i = 1, 2, \ldots, 8$ as $\delta \to 1^-$.

Lemma 17 *We have*

$$G_4 = \lambda G_3.$$

If we define F_1 and F_2 by

$$F_1 := \frac{z^2}{1-\delta}(G_6-\lambda\delta G_5) \quad \text{and} \quad F_2 := \frac{1}{1-\delta}\left(G_8-\lambda\delta z^2 G_7\right),$$

then

$$\lim_{\delta\to 1^-} F_1 = \lambda\left(3+\frac{1}{p}\right) \quad \textit{and} \quad \lim_{\delta\to 1^-} F_2 = \frac{2\lambda^2(1+3p)}{p}.$$

Proof We verify the identities through direct computations with help of a computer algebra system.

Similarly, we obtain
Here H_i are the remaining factors; in particular, we have

$$H_1 = 8 + 4\cdot\frac{u_1+u_2\delta}{u_3+u_4\delta} - 2\cdot\frac{(u_5+u_6\delta)^2}{(u_3+u_4\delta)^2}.$$

We do not present explicit expressions for the other H_i's. Using $0 \le \delta < 1$ and $u_3+u_4\delta > 0$, one can check that H_i are bounded for $i = 1, 2, \ldots, 10$ as $\delta \to 1^-$.

Proposition 18 *Each curvature components of the Bergman metric at* $(0, y, z) \in E_{p,\lambda}$, $0 \le y, z < 1$, *is given by*

$$R_{1\bar{1}1\bar{1}} = \frac{1}{c^2}(-H_1) = \frac{1}{c^2}\cdot\widetilde{H}_1,$$

$$R_{1\bar{1}2\bar{2}} = R_{2\bar{1}1\bar{2}} = R_{1\bar{2}2\bar{1}} = R_{2\bar{2}1\bar{1}} = \frac{a^\lambda}{b^2c}\cdot\left(-H_2+\frac{\delta {G_1}^2}{A_1}\right) = \frac{a^\lambda}{b^2c}\cdot\widetilde{H}_2,$$

$$\begin{aligned}R_{1\bar{1}2\bar{3}} &= R_{1\bar{1}3\bar{2}} = R_{2\bar{1}1\bar{3}} = R_{1\bar{2}3\bar{1}} = R_{1\bar{3}2\bar{1}} = R_{2\bar{3}1\bar{1}} = R_{3\bar{1}1\bar{2}} = R_{3\bar{2}1\bar{1}}\\ &= \frac{yza^{\lambda-1}}{b^2c}\cdot\left(-H_3+\frac{G_1G_2}{A_1}\right) = \frac{yza^{\lambda-1}}{b^2c}\cdot\widetilde{H}_3,\end{aligned}$$

$$R_{1\bar{1}3\bar{3}} = R_{1\bar{3}3\bar{1}} = R_{3\bar{1}1\bar{3}} = R_{3\bar{3}1\bar{1}} = \frac{a^{2\lambda-2}}{b^2c}\cdot\left(-H_4+\frac{z^2G_2^2}{A_1}\right) = \frac{a^{2\lambda-2}}{b^2c}\cdot\widetilde{H}_4,$$

$$R_{2\bar{2}2\bar{2}} = \frac{a^{2\lambda}}{b^4}\cdot\left(-H_5+\frac{\delta G_3^2}{A_2}\right) = \frac{a^{2\lambda}}{b^4}\cdot\widetilde{H}_5,$$

$$R_{2\bar{2}2\bar{3}} = R_{2\bar{2}3\bar{2}} = R_{2\bar{3}2\bar{2}} = R_{3\bar{2}2\bar{2}} = \frac{yza^{2\lambda-1}}{b^4}\cdot\left(-H_6+\frac{\delta G_3G_5}{A_2}\right) = \frac{yza^{2\lambda-1}}{b^4}\cdot\widetilde{H}_6,$$

$$\begin{aligned}R_{2\bar{2}3\bar{3}} &= R_{2\bar{3}3\bar{2}} = R_{3\bar{2}2\bar{3}} = R_{3\bar{3}2\bar{2}}\\ &= \frac{a^{3\lambda-2}}{b^4}\cdot\left(-H_7+\frac{\delta^2z^2G_5^2}{A_2}+\frac{\delta(1-\delta)F_1^2}{A_4}\right) = \frac{a^{3\lambda-2}}{b^4}\cdot\widetilde{H}_7,\end{aligned}$$

$$R_{2\bar{3}2\bar{3}} = R_{3\bar{2}3\bar{2}} = \frac{a^{2\lambda-2}y^2z^2}{b^4}\cdot\left(-H_8+\frac{G_3G_7}{A_2}\right) = \frac{a^{2\lambda-2}y^2z^2}{b^4}\cdot\widetilde{H}_8,$$

$$\begin{aligned}R_{2\bar{3}3\bar{3}} &= R_{3\bar{2}3\bar{3}} = R_{3\bar{3}2\bar{3}} = R_{3\bar{3}3\bar{2}}\\ &= \frac{a^{3\lambda-3}yz}{b^4}\cdot\left(-H_9+\frac{\delta z^2G_5G_7}{A_2}+\frac{(1-\delta)F_1F_2}{A_4}\right) = \frac{a^{3\lambda-3}yz}{b^4}\cdot\widetilde{H}_9,\end{aligned}$$

Table 1 Formulas for $\partial_i g_{j\bar{k}}$

$\partial_1 g_{2\bar{1}} = \partial_2 g_{1\bar{1}} = \overline{\partial}_1 g_{1\bar{2}} = \overline{\partial}_2 g_{1\bar{1}} = \frac{y}{bc} G_1,$
$\partial_1 g_{3\bar{1}} = \partial_3 g_{1\bar{1}} = \overline{\partial}_1 g_{1\bar{3}} = \overline{\partial}_3 g_{1\bar{1}} = \frac{z}{a^{1-\lambda} bc} G_2,$
$\partial_2 g_{2\bar{2}} = \overline{\partial}_2 g_{2\bar{2}} = \frac{y a^{\lambda}}{b^3} G_3,$
$\partial_2 g_{2\bar{3}} = \overline{\partial}_2 g_{3\bar{2}} = \frac{y^2 z}{a^{1-\lambda} b^3} G_4,$
$\partial_2 g_{3\bar{2}} = \partial_3 g_{2\bar{2}} = \overline{\partial}_2 g_{2\bar{3}} = \overline{\partial}_3 g_{2\bar{2}} = \frac{y^2 z}{a^{1-\lambda} b^3} G_5,$
$\partial_2 g_{3\bar{3}} = \partial_3 g_{2\bar{3}} = \overline{\partial}_2 g_{3\bar{3}} = \overline{\partial}_3 g_{3\bar{2}} = \frac{y z^2}{a^{2-2\lambda} b^3} G_6,$
$\partial_3 g_{3\bar{2}} = \overline{\partial}_3 g_{2\bar{3}} = \frac{y z^2}{a^{2-2\lambda} b^3} G_7,$
$\partial_3 g_{3\bar{3}} = \overline{\partial}_3 g_{3\bar{3}} = \frac{z}{a^{3-3\lambda} b^3} G_8,$
$\partial_i g_{j\bar{k}} = \overline{\partial}_i g_{j\bar{k}} = 0$ otherwise.

$$R_{3\bar{3}3\bar{3}} = \frac{a^{4\lambda-4}}{b^4} \cdot \left(-H_{10} + \frac{\delta z^4 G_7^2}{A_2} + \frac{z^2(1-\delta) F_2^2}{A_4} \right) = \frac{a^{4\lambda-4}}{b^4} \cdot \widetilde{H}_{10},$$

$R_{i\bar{j}k\bar{l}} = 0$ *otherwise,*

where we define $\widetilde{H}_i$ *for* $i = 1, 2, \ldots, 10$ *for later use.*

Proof Recall that the components of curvature tensor R associated with g is given by

$$R_{i\bar{j}k\bar{l}} = -\partial_k \overline{\partial}_l g_{i\bar{j}} + \sum_{p,q=1}^{3} g^{q\bar{p}} (\partial_k g_{i\bar{p}})(\overline{\partial}_l g_{q\bar{j}}).$$

Thus the results follow from Tables 1 and 2 and Proposition 16.

Lemma 19 *We have*

$$\widetilde{H}_3 = \lambda \widetilde{H}_2, \quad \widetilde{H}_6 = \lambda \widetilde{H}_5, \quad \widetilde{H}_8 = \lambda \widetilde{H}_6 \quad \text{and} \quad \widetilde{H}_9 = 2\lambda \widetilde{H}_7 - \lambda^2 \delta z^2 \widetilde{H}_6.$$

If we define

$$\widetilde{F}_1 := \frac{1}{1-\delta} \left(\widetilde{H}_4 - \lambda \delta z^2 \widetilde{H}_3 \right), \qquad \widetilde{F}_2 := \frac{1}{1-\delta} \left(\widetilde{H}_7 - \lambda \delta z^2 \widetilde{H}_6 \right),$$
$$\widetilde{F}_3 = \frac{1}{(1-\delta)^2} \left(\widetilde{H}_{10} - 4\lambda^2 \delta z^2 \widetilde{H}_7 + 3\lambda^3 \delta^2 z^4 \widetilde{H}_6 \right),$$

then

$$\lim_{\delta \to 1^-} \widetilde{F}_1 = -\frac{4\lambda(2+p)}{p(1+2p)}, \quad \lim_{\delta \to 1^-} \widetilde{F}_2 = -\lambda \left(3 + \frac{1}{p} \right) \quad \text{and}$$
$$\lim_{\delta \to 1^-} \widetilde{F}_3 = -2\lambda^2 \left(3 + \frac{1}{p} \right). \tag{6.7}$$

Table 2 Formulas for $\partial_i\overline{\partial}_j g_{k\bar{l}}$

$\partial_1\overline{\partial}_1 g_{1\bar{1}} = \frac{1}{c^2}H_1,$

$\partial_1\overline{\partial}_1 g_{2\bar{2}} = \partial_1\overline{\partial}_2 g_{2\bar{1}} = \partial_2\overline{\partial}_1 g_{1\bar{2}} = \partial_2\overline{\partial}_2 g_{1\bar{1}} = \frac{a^\lambda}{b^2c}H_2,$

$\partial_1\overline{\partial}_1 g_{2\bar{3}} = \partial_1\overline{\partial}_3 g_{2\bar{1}} = \partial_2\overline{\partial}_1 g_{1\bar{3}} = \partial_2\overline{\partial}_3 g_{1\bar{1}} = \partial_1\overline{\partial}_1 g_{3\bar{2}} = \partial_1\overline{\partial}_2 g_{3\bar{1}} = \partial_3\overline{\partial}_1 g_{1\bar{2}} = \partial_3\overline{\partial}_2 g_{1\bar{1}} = \frac{yz}{a^{1-\lambda}b^2c}H_3,$

$\partial_1\overline{\partial}_1 g_{3\bar{3}} = \partial_1\overline{\partial}_3 g_{3\bar{1}} = \partial_3\overline{\partial}_1 g_{1\bar{3}} = \partial_3\overline{\partial}_3 g_{1\bar{1}} = \frac{1}{a^{2-2\lambda}b^2c}H_4,$

$\partial_2\overline{\partial}_2 g_{2\bar{2}} = \frac{a^{2\lambda}}{b^4}H_5,$

$\partial_2\overline{\partial}_2 g_{2\bar{3}} = \partial_2\overline{\partial}_3 g_{2\bar{2}} = \partial_2\overline{\partial}_2 g_{3\bar{2}} = \partial_3\overline{\partial}_2 g_{2\bar{2}} = \frac{yz}{a^{1-2\lambda}b^4}H_6,$

$\partial_2\overline{\partial}_2 g_{3\bar{3}} = \partial_2\overline{\partial}_3 g_{3\bar{2}} = \partial_3\overline{\partial}_2 g_{2\bar{3}} = \partial_3\overline{\partial}_3 g_{2\bar{2}} = \frac{1}{a^{2-3\lambda}b^4}H_7,$

$\partial_2\overline{\partial}_3 g_{2\bar{3}} = \partial_3\overline{\partial}_2 g_{3\bar{2}} = \frac{y^2z^2}{a^{2-2\lambda}b^4}H_8,$

$\partial_2\overline{\partial}_3 g_{3\bar{3}} = \partial_3\overline{\partial}_3 g_{2\bar{3}} = \partial_3\overline{\partial}_2 g_{3\bar{3}} = \partial_3\overline{\partial}_3 g_{3\bar{2}} = \frac{yz}{a^{3-3\lambda}b^4}H_9,$

$\partial_3\overline{\partial}_3 g_{3\bar{3}} = \frac{1}{a^{4-4\lambda}b^4}H_{10},$

$\partial_i\overline{\partial}_j g_{k\bar{l}} = 0$ otherwise.

Proof The identities are verified through direct computations and can be checked by a computer algebra system.

In order to see cancellations of factors involving a, b, c in the holomorphic sectional curvature, we apply the Gram–Schmidt process to determine an orthonormal frame X, Y, Z instead of using the global coordinate vector fields $\frac{\partial}{\partial z_i}$, $i = 1, 2, 3$. Indeed, let g be any Hermitian metric, and take the first unit vector field

$$X = \frac{\partial_1}{\sqrt{g_{1\bar{1}}}}. \tag{6.8}$$

Write $k_1 := \frac{1}{\sqrt{g_{1\bar{1}}}}$ so that $X = k_1\partial_1$. Then a vector field $\tilde{Y}$ which is orthogonal to X is given by

$$\tilde{Y} = \frac{\partial_2}{\sqrt{g_{2\bar{2}}}} - g\left(\frac{\partial_2}{\sqrt{g_{2\bar{2}}}}, X\right)X = a_1\partial_1 + a_2\partial_2,$$

where we put

$$a_1 := -\frac{g_{2\bar{1}}}{g_{1\bar{1}}\sqrt{g_{2\bar{2}}}} \quad \text{and} \quad a_2 := \frac{1}{\sqrt{g_{2\bar{2}}}}.$$

Since $g(\tilde{Y}, \tilde{Y}) = a_1\overline{a_1}g_{1\bar{1}} + a_1\overline{a_2}g_{1\bar{2}} + a_2\overline{a_1}g_{2\bar{1}} + a_2\overline{a_2}g_{2\bar{2}}$, we take

$$Y = \frac{\tilde{Y}}{\sqrt{g(\tilde{Y}, \tilde{Y})}} = \frac{a_1\partial_1 + a_2\partial_2}{\sqrt{a_1\overline{a_1}g_{1\bar{1}} + a_1\overline{a_2}g_{1\bar{2}} + a_2\overline{a_1}g_{2\bar{1}} + a_2\overline{a_2}g_{2\bar{2}}}} = t_1\partial_1 + t_2\partial_2, \tag{6.9}$$

where we put

$$t_i := \frac{a_i}{\sqrt{a_1\overline{a_1}g_{1\bar{1}} + a_1\overline{a_2}g_{1\bar{2}} + a_2\overline{a_1}g_{2\bar{1}} + a_2\overline{a_2}g_{2\bar{2}}}}, \quad i = 1, 2. \tag{6.10}$$

Similarly, consider

$$\tilde{Z} = p_1\partial_1 + p_2\partial_2 + p_3\partial_3,$$

where

$$\begin{aligned} p_1 &:= -\frac{g_{3\bar{1}}}{g_{1\bar{1}}\sqrt{g_{3\bar{3}}}} - \frac{t_1}{\sqrt{g_{3\bar{3}}}}(t_1 g_{3\bar{1}} + t_2 g_{3\bar{2}}), \\ p_2 &:= -\frac{t_2}{\sqrt{g_{3\bar{3}}}}(t_1 g_{3\bar{1}} + t_2 g_{3\bar{2}}), \qquad p_3 := \frac{1}{\sqrt{g_{3\bar{3}}}}. \end{aligned}$$

Normalizing $\tilde{Z}$ yields

$$Z = s_1\partial_1 + s_2\partial_2 + s_3\partial_3, \tag{6.11}$$

where

$$s_i := \frac{p_i}{\sqrt{\sum_{k,l=1}^{3} p_k p_l g_{k\bar{l}}}}, \quad i = 1, 2, 3.$$

These X, Y, Z are used in the following proposition which is the main result of this section.

Proposition 20 *At $(0, y, z) \in E_{p,\lambda}$, $0 \le y, z < 1$, the components of the holomorphic sectional curvature R are given by as follows.*

$$\begin{aligned} H(X) &= R(X, \bar{X}, X, \bar{X}) = \frac{\widetilde{H}_1}{A_1^2}, B(X, Y) = R(X, \bar{X}, Y, \bar{Y}) = \frac{\widetilde{H}_2}{A_1 A_2}, \\ H(Y) &= R(Y, \bar{Y}, Y, \bar{Y}) = \frac{\widetilde{H}_5}{A_2^2}, B(X, Z) = R(X, \bar{X}, Z, \bar{Z}) = \frac{\widetilde{F}_1}{A_1 A_4}, \\ H(Z) &= R(Z, \bar{Z}, Z, \bar{Z}) = \frac{\widetilde{F}_3}{A_4^2}, B(Y, Z) = R(Y, \bar{Y}, Z, \bar{Z}) = \frac{\widetilde{F}_2}{A_2 A_4}, \end{aligned}$$

$$\begin{aligned} R(X, \bar{X}, X, \bar{Y}) &= R(Y, \bar{Y}, Y, \bar{X}) = R(Z, \bar{Z}, Z, \bar{Y}) = R(Y, \bar{X}, Y, \bar{X}) = 0, \\ R(X, \bar{X}, X, \bar{Z}) &= R(Y, \bar{Y}, Y, \bar{Z}) = R(Z, \bar{Z}, Z, \bar{X}) = R(Z, \bar{X}, Z, \bar{X}) = 0, \\ R(X, \bar{X}, Y, \bar{Z}) &= R(Y, \bar{Y}, X, \bar{Z}) = R(Z, \bar{Z}, X, \bar{Y}) = R(Z, \bar{Y}, Z, \bar{Y}) = 0. \end{aligned}$$

Proof All the identities follow from Proposition 18 and Lemma 19. To illustrate the process, we compute $H(X)$, $B(X, Y)$ and $R(Y, \bar{Y}, Y, \bar{Z})$. Computations of the other components are similar.

Since $g_{2\bar{1}} = 0$ and $g_{3\bar{1}} = 0$, we have $a_1 = 0$, $t_1 = 0$, $p_1 = 0$ and $s_1 = 0$ on $(0, y, z)$. On the other hand,

$$t_2 = \frac{a_2}{\sqrt{a_2\overline{a_2}g_{2\bar{2}}}} = \frac{1}{\sqrt{g_{2\bar{2}}}}.$$

Thus, using (6.4), we obtain

$$H(Y) = t_2^4 R_{2\bar{2}2\bar{2}} = \frac{b^4}{a^{2\lambda}}\frac{1}{A_2^2} \cdot \frac{a^{2\lambda}}{b^4}\widetilde{H}_5 = \frac{\widetilde{H}_5}{A_2^2}.$$

Similarly,

$$B(X, Y) = k_1^2 t_2^2 R_{1\bar{1}2\bar{2}} = \frac{1}{g_{1\bar{1}}}\frac{1}{g_{2\bar{2}}} \cdot \frac{a^{\lambda}}{b^2 c} \cdot \widetilde{H}_2 = \frac{c}{A_1}\frac{b^2}{a^{\lambda}A_2}\frac{a^{\lambda}}{b^2 c}\widetilde{H}_2 = \frac{1}{A_1 A_2}\widetilde{H}_2.$$

To compute $R(Y, \bar{Y}, Y, \bar{Z})$, first observe

$$s_2 = -s_3 t_2^2 g_{3\bar{2}} = -s_3\frac{g_{3\bar{2}}}{g_{2\bar{2}}} = -s_3\frac{\lambda yz}{a}.$$

Thus it follows from Proposition 18 and Lemma 19 that

$$\begin{aligned} R(Y, \bar{Y}, Y, \bar{Z}) &= t_2^3 s_2 R_{2\bar{2}2\bar{2}} + t_2^3 s_3 R_{2\bar{2}2\bar{3}} = t_2^3\left(-s_3\frac{\lambda yz}{a}\right)\frac{a^{2\lambda}}{b^4}\widetilde{H}_5 + t_2^3 s_3\frac{yza^{2\lambda-1}}{b^4}\widetilde{H}_6 \\ &= \frac{t_2^3 s_3 a^{2\lambda-1}yz}{b^4}\left(-\lambda\widetilde{H}_5 + \widetilde{H}_6\right) = 0. \end{aligned}$$

Corollary 21 *The holomorphic sectional curvature near ∂K_1 is bounded for any $p, \lambda > 0$.*

Proof The assertion follows from (6.6) and (6.7) and the fact that G_i and H_i are bounded as $\delta \to 1^-$.

It is known [10] that the curvature tensor of the Bergman metric is bounded for $\lambda = 1$ and $p > 0$. The following proposition tells us that the same is true for any $p, \lambda > 0$.

Proposition 22 *The curvature tensor of the Bergman metric on $E_{p,\lambda}$ is bounded for any $p, \lambda > 0$.*

Proof The curvature tensor can be explicitly expressed in terms of the holomorphic sectional curvature H_{g_B}. Using the invariance of the Bergman metric, it suffices to show $H_{g_B} \leq C$ on ∂K_1 by some constant $C \in \mathbb{R}$. By Corollary 21, we are done.

Corollary 23 *For any $p, \lambda > 0$, there exist $C_0 > 0$ such that*

$$\chi_{E_{p,\lambda}}(p; v) \leq C_0\sqrt{\omega_B(v, v)} \qquad \text{for all } v \in T'_p E_{p,\lambda},\ p \in M,$$

and $C_1 > 0$ such that

$$\frac{1}{C_1}\omega_{KE}(v, v) \leq \omega_B(v, v) \leq C_1\omega_{KE}(v, v) \qquad \text{for all } v \in T' E_{p,\lambda}.$$

Proof The assertion immediately follows from Proposition 22 and Lemma 15.

Remark 24 For the third statement of Theorem A, in general, the holomorphic sectional curvature is not negatively pinched for $E_{p,\lambda}$. For example, when $\lambda = 1$ and $p = 1/5$, we have $\lim_{\delta\to 1^-} H(X) \approx 0.033 > 0$.

Lastly, we obtain interesting rigidity in the following proposition from direct computation of the Ricci curvature of the Bergman metric and we omit the proof.

Proposition 25 *The Bergman metric g_B on $E_{p,\lambda}$ is a Kähler–Einstein metric if and only if $\lambda = p = 1$.*

7 A lower bound of the integrated Carathéodory–Reiffen metric

In this last section, we prove the following theorem.

Theorem B *Let (M, g) be a simply-connected complete noncompact n-dimensional Kähler manifold whose Riemannian sectional curvature k of g satisfies $k \leq -a^2$ for some $a > 0$. We denote by d the geodesic distance on M, and by γ_M the Carathéodory–Reiffen metric on M. For any $p \geq 2$, the following are true.*

1. Let f be a holomorphic function from M to the unit disk $\mathbb{D}$ in $\mathbb{C}$. Then

$$\begin{aligned}\int_M \left|\int_M G(x, y)|\nabla f|^2(y)dy\right|^p dx \\ \leq \left(\frac{p}{(2n-1)a}\right)^p \int_M |f(x)|^p \gamma_M(x; \nabla f(x))^{\frac{p}{2}} dx,\end{aligned} \tag{7.1}$$

where $G(x, y)$ is the minimal positive Green's function on M.

2. If the Riemannian sectional curvature k of g further satisfies $-b^2 \leq k$ for some $b > 0$. Then there exists a constant $C(n) > 0$, which only depends on n, such that for any holomorphic function f from M to the unit disk $\mathbb{D}$, we have

$$\int_0^\infty \int_M \left(\int_M t^{-n} \exp\left[-\frac{d(x, y)^2}{2t} - \frac{(2n-1)^2 b^2 t}{8} - \frac{(2n-1)bd(x, y)}{2}\right](1 + bd(x, y))|\nabla f|^2(y)dy\right)^p dxdt$$

$$\leq C(n)\left(\frac{2\pi p}{(2n-1)a}\right)^p \int_M |f(x)|^p \gamma_M(x; \nabla f(x))^{\frac{p}{2}} dx. \tag{7.2}$$

The inequalities (7.1) and (7.2) can be interpreted as integrated gradient estimates of bounded holomorphic functions.

Although the lemmas below are known, we prove them here for tracking explicit constants for the proof of Theorem 7.

Let M be an n-dimensional complete noncompact, simply connected Riemannian manifold, and let $L^2(M)$ be the space of L^2-functions on M. Denote by $W^1(M)$ the Hilbert space consisting of L^2-functions whose gradient are also L^2, and by $W_0^1(M)$ the subspace in $W^1(M)$ which is the completion of the space $C_0^\infty(M)$ under $W^1(M)$-norm. When M is complete, we have $W^1(M) = W_0^1(M)$.

Lemma 26 ([29, Poincaré inequality]) *Let M be an n-dimensional complete noncompact, simply connected Riemannian manifold with sectional curvature $k \leq -a^2 < 0$. Then*

$$\int_M |u|^2 \leq \frac{4}{(n-1)^2 a^2} \int_M |\nabla u|^2, \qquad u \in W_0^1(M). \tag{7.3}$$

Proof Let $r(x) = d(p_0, x)$ be the distance function from a fixed point $p_0 \in M$. From the Rauch comparison theorem, we have

$$\triangle r \geq (n-1)a, \tag{7.4}$$

where $a > 0$.

Let Ω be the geodesic ball centered at p_0 with radius $R > 0$ in M. From the Green's theorem, we have for every $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |u|^2 \triangle r - \int_\Omega \nabla(|u|^2) \cdot \nabla r = \int_{b\Omega} |u|^2 d\sigma = 0,$$

where $d\sigma$ is the surface measure on $b\Omega$. We remark that r may not be smooth at p_0, but we can apply the Green's theorem to Ω minus a small ball of radius $\epsilon > 0$ around p_0 and let $\epsilon \to 0$. From (7.4) and $|\nabla r| = 1$, we have

$$(n-1)a\|u\|^2 \leq \int_\Omega |u|^2 \triangle r = \int_\Omega \nabla(|u|^2) \cdot \nabla r \leq \int_\Omega |\nabla(|u|^2)| \leq 2\|u\| \, \|\nabla u\|.$$

This gives

$$\|u\| \leq \frac{2}{(n-1)a} \|\nabla u\|, \qquad u \in C_0^\infty(\Omega).$$

Since $C_0^\infty(M)$ is dense in $W_0^1(M)$, we are done.

Let $\triangle_0$ denote the Laplace–Beltrami operator. We use Mckean's estimate [27] on the first eigenvalue of $\triangle_0$.

Lemma 27 ([27, Mckean's estimate]) *Let M be an n-dimensional complete noncompact, simply-connected Riemannian manifold with sectional curvature $k \le -a^2 < 0$. Then we have*

$$\lambda_1 \ge \frac{(n-1)^2 a^2}{4}, \tag{7.5}$$

where λ_1 is the smallest eigenvalue of $\triangle_0$.

Proof From Lemma 26, for every $u \in C_0^\infty(M)$,

$$(\triangle_0 u, u) = (du, du) = \int_\Omega |\nabla u|^2 \ge \frac{(n-1)^2 a^2}{4} \int_\Omega |u|^2.$$

The assertion follows.

Lemma 28 ([8, Cheng]) *Let M be an n-dimensional Riemannian manifold. Consider the first eigenvalue for the Dirichlet problem $\lambda_1(M) > 0$. Let Ω be a relatively compact domain of M such that $b\Omega$ is smooth. Let $f \in C^\infty(M)$ and let u be the solution of*

$$\begin{cases} \triangle u = \triangle f & \text{on } \Omega, \\ u = 0 & \text{on } b\Omega. \end{cases}$$

Then for any $p \ge 2$,

$$\int_\Omega |u|^p \le C_p \int_\Omega |\nabla f|^p, \tag{7.6}$$

where the constant C_p depends only on p and $\lambda_1(M)$.

Proof Assume that $p \ge 2$. Multiplying the equation by u^{p-1} and integrating it, we have

$$\begin{aligned} (p-1)\int_\Omega |\nabla u|^2 u^{p-2} &= (\nabla u, \nabla u^{p-1}) = (\nabla f, \nabla u^{p-1}) \\ &\le (p-1)\int_\Omega |\nabla f||\nabla u| u^{p-2} \\ &\le (p-1)\left(\int_\Omega |\nabla u|^2 u^{p-2}\right)^{1/2}\left(\int_\Omega |\nabla f|^2 u^{p-2}\right)^{1/2}. \end{aligned}$$

Thus we have

$$\frac{4}{p^2}\int_\Omega |\nabla u^{p/2}|^2 \le \int_\Omega |\nabla f|^2 u^{p-2} \le \left(\int_\Omega |u|^p\right)^{\frac{p-2}{p}} \left(\int_\Omega |\nabla f|^p\right)^{\frac{2}{p}}.$$

From (7.3), we obtain

$$\left(\frac{4\lambda_1}{p^2}\right)^{\frac{p}{2}} \int_\Omega |u|^p \leq \int_\Omega |\nabla f|^p.$$

The constant C_p depends only on p and λ_1. The general case can be proved similarly through multiplication by $(\operatorname{sgn} u)|u|^{p-1}$ and integration.

Proof of Theorem B From Lemma 27, M has the positive spectrum. It is a standard result that if the manifold has positive spectrum then there exists a positive symmetric Green's function G on M. Moreover, we can always take $G(x, y)$ to be the minimal Green's function constructed using exhaustion of compact subdomains. Hence

$$G(x, y) = \lim_{i\to\infty} G_i(x, y) > 0,$$

where G_i is the Dirichlet Green's function of a compact exhaustion $\{\Omega_i\}_i$ of M, and the limit is uniform on compact subsets of M.

Take any (bounded) holomorphic function $f : M \to \mathbb{D}$. For any relatively compact subdomain $\Omega \subset M$ with the smooth boundary $b\Omega$, we use f^2 in Lemma 28 and solving the Dirichlet boundary problem with the inequality

$$\left(g(\nabla f^2, \nabla f^2)(x)\right)^{\frac{p}{2}} = \left(4|f(x)|^2 df(\nabla f)(x)\right)^{\frac{p}{2}} \leq 2^p |f|^p(x)\gamma_M(x; \nabla f(x))^{\frac{p}{2}} \tag{7.7}$$

for any $x \in M$, and the condition $p \geq 2$ implies

$$\begin{aligned} \int_\Omega |u|^p &\leq \left(\frac{2p}{(2n-1)a}\right)^p \int_\Omega |f|^p \gamma_M(.; \nabla f)^{\frac{p}{2}} \\ &\leq \left(\frac{2p}{(2n-1)a}\right)^p \int_M |f|^p \gamma_M(.; \nabla f)^{\frac{p}{2}}, \end{aligned} \tag{7.8}$$

where u is the solution of

$$\begin{cases} \triangle u = 2|\nabla f|^2 & \text{on } \Omega, \\ u = 0 & \text{on } b\Omega, \end{cases} \tag{7.9}$$

and $a > 0$ is for the upper bound of the Riemannian sectional curvature $\leq -a^2 < 0$.

From the hypothesis $|f|^p \gamma_M(.; \nabla f)^{\frac{p}{2}} \in L^1(M)$ and from the exhaustion of compact subdomains, there exists $u \in C^\infty(M, \mathbb{R})$ such that

$$\int_M |u|^p < \infty,$$

and $\triangle u = 2|\nabla f|^2$ on M. Furthermore, the fact $\inf_{x \in M} \mathrm{Vol}\, B(x, r) > 0$ for any $r > 0$ implies that $u(x) \to 0$ as $d(p, x) \to \infty$ from some fixed point $p \in M$. Thus the Dirichlet problem is solvable and u can be represented by

$$u(x) = 2 \int_M G(x, y)|\nabla f|^2(y)dy, \tag{7.10}$$

which proves part (1).

For part (2), the positive minimal Green's function satisfies

$$G(x, y) = \int_0^\infty h_M(x, y, t)dt,$$

where we denote the heat kernel of the Laplace–Beltrami operator by $h_M(x, y, t)$. Hence (7.10) becomes

$$u(x) = 2 \int_0^\infty \int_M h_M(x, y, t)|\nabla f|^2(y)dydt. \tag{7.11}$$

We use the Cheeger and Yau's heat kernel comparison theorem [7]:

$$h_M(x, y, t) \geq h_{M_k}(d(x, y)), \tag{7.12}$$

where M_k is the space form with constant sectional curvature equal to k. From the two-sided estimate of Davies and Mandouvalos [15],

$$c(n)^{-1}h(t, d(x, y)) \leq h_{M_k}(d(x, y)) \leq c(n)h(t, d(x, y)), \tag{7.13}$$

where $c(n)$ depends only on n and

$$h(t, r) = (2\pi t)^{-n} \exp\left[-\frac{r^2}{2t} - \frac{(2n-1)^2b^2t}{8} - \frac{(2n-1)br}{2}\right](1 + br)\left(1 + br + \frac{b^2t}{2}\right)^{\frac{2n-1}{2}-1} \tag{7.14}$$

for $t, r > 0$, where $b > 0$ is for the lower bound of the Riemannian sectional curvature $\geq -b^2$.

Now combining (7.8) with (7.11), (7.12), (7.13), and (7.14) gives the desired inequality (7.2). This completes the proof.

We end this paper with an example for Theorem B.

Proposition 29 *In the case of unit disk $\mathbb{D}$ in $\mathbb{C}$, for each $p \geq 2$, we have*

$$2\pi \int_0^1 \left(\frac{1}{6} - \frac{R^2}{2}\ln R - \frac{R^4}{8}(4\ln R - 1) - \frac{R^6}{36}(6\ln R - 1)\right)^p R\, dR$$
$$\leq p^p \int_{\mathbb{D}} |z|^p \gamma_{\mathbb{D}}(z; \nabla z)^{\frac{p}{2}}.$$

Proof The Green function of the unit disk $\mathbb{D}$ in $\mathbb{C}$ has the following form:

$$G(x,y)=\frac{1}{2\pi}\ln\frac{|x-y|}{|x||y-\frac{x}{|x|^2}|}.$$

The function G satisfies $\triangle_x G(x,y)=\delta_y$ at fixed $y\in\mathbb{D}$ and $G(x,y)=0$ when $|x|=1$ and $|y|<1$. Since the gradient vector of $z\in\mathbb{D}$ with respect to the Poincaré metric is $(1-|z|^2)\frac{\partial}{\partial z}$, the integrand of the left-hand side of (7.1) is

$$\int_{|y|<1} G(x,y)(1-|y|^2)^2 dy. \tag{7.15}$$

Rewrite $G(x,y)=\frac{1}{4\pi}\ln\left(\frac{|x|^2|y-x/|x|^2|^2}{|x-y|^2}\right)$ and choose coordinates $x=(R,0)$ and $y=(r\cos\theta, r\sin\theta)$, then (7.15) becomes

$$\begin{aligned}&\frac{1}{4\pi}\int_0^1\int_0^{2\pi}\ln\left(\frac{1+r^2R^2-2rR\cos\theta}{R^2+r^2-2rR\cos\theta}\right)r(1-r^2)^2 d\theta dr\\ &=\frac{1}{4\pi}\int_0^1 r(1-r^2)^2\left(I(1,rR)-I(r,R)\right)dr,\end{aligned}$$

where $I(a,b):=\int_0^{2\pi}\ln(a^2+b^2-2ab\cos\theta)d\theta$. It is well-known that

$$I(a,b)=4\pi\max\{\ln|a|,\ln|b|\}.$$

Since $0\le r,R\le 1$, we have $I(1,rR)=0$. Thus the integral becomes

$$\begin{aligned}&-\int_0^1 r(1-r^2)^2\max\{\ln|r|,\ln|R|\}\,dr\\ &=-\ln R\int_0^R r(1-r^2)^2 dr-\int_R^1 r(1-r^2)^2\ln r dr\\ &=\frac{1}{6}-\frac{R^2}{2}\ln R-\frac{R^4}{8}(4\ln R-1)-\frac{R^6}{36}(6\ln R-1).\end{aligned}$$

Thus the left-hand side of (7.1) is

$$2\pi\int_0^1\left(\frac{1}{6}-\frac{R^2}{2}\ln R-\frac{R^4}{8}(4\ln R-1)-\frac{R^6}{36}(6\ln R-1)\right)^p R\,dR.$$

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References

1. Ahn, T., Gaussier, H., Kim, K.-T.: Positivity and completeness of invariant metrics. J. Geom. Anal. **26**(2), 1173–1185 (2016)
2. Beberok, T.: An explicit computation of the Bergman kernel function. Complex Var. Elliptic Equ. **60**(8), 1058–1067 (2015). https://doi.org/10.1080/17476933.2014.996214
3. Beberok, T.: Explicit formulas of the Bergman kernel for some Reinhardt domains. J. Geom. Anal. **26**(4), 2883–2892 (2016). https://doi.org/10.1007/s12220-015-9652-0
4. Catlin, D.W.: Estimates of invariant metrics on pseudoconvex domains of dimension two. Math. Z. **200**(3), 429–466 (1989). https://doi.org/10.1007/BF01215657
5. Chae, M., Cho, G., Gordina, M., Yang, G.: The Stochastic Schwarz lemma on Kähler manifolds by Couplings and its Applications, (2022), arXiv,
6. Chau, A.: Convergence of the Kähler–Ricci flow on noncompact Kähler manifolds. J. Differ. Geom. **66**(2), 211–232 (2004). (http://projecteuclid.org/euclid.jdg/1102538610)
7. Cheeger, J., Yau, S.T.: A lower bound for the heat kernel. Comm. Pure Appl. Math. **34**(4), 465–480 (1981). https://doi.org/10.1002/cpa.3160340404
8. Cheng, S.Y.: The Dirichlet problem at infinity for non-positively curved manifolds. Comm. Anal. Geom. **1**(1), 101–112 (1993). https://doi.org/10.4310/CAG.1993.v1.n1.a5
9. Cheng, S.Y., Yau, S.T.: On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation. Comm. Pure Appl. Math. **33**(4), 507–544 (1980). https://doi.org/10.1002/cpa.3160330404
10. Cho, G.: Invariant metrics on the complex ellipsoid. J. Geom. Anal. **31**(2), 2088–2104 (2021). https://doi.org/10.1007/s12220-019-00333-w
11. Cho, G.: Comparing the Carathéodory pseudo-distance and the Kähler–Einstein distance on complete Reinhardt domains. Asian J. Math. **26**(1), 37–44 (2022)
12. Cho, G.: Non-existence of complete kähler metric of negatively pinched holomorphic sectional curvature. Complex Anal. Syner. (2023)
13. Cho, G., Yuan, Y.: Bergman metric on the symmetrized bidisk and its consequences. Int. J. Math. **33**(10–11), Paper No. 225007116 (2022). https://doi.org/10.1142/S0129167X22500719
14. D'Angelo, J.P.: An explicit computation of the Bergman kernel function. J. Geom. Anal. **4**(1), 23–34 (1994). https://doi.org/10.1007/BF02921591
15. Davies, E.B., Mandouvalos, N.: Heat kernel bounds on hyperbolic space and Kleinian groups. Proc. Lond. Math. Soc. (3) **57**(1), 182–208 (1988). https://doi.org/10.1112/plms/s3-57.1.182
16. Diederich, K., Fornæ ss, J. E., Herbort, G.: Boundary behavior of the Bergman metric, bookComplex analysis of several variables (Madison, Wis., 1982), Proc. Sympos. Pure Math., 41, Amer. Math. Soc., Providence, RI, 59– 67, https://doi.org/10.1090/pspum/041/740872(1984)
17. Dinew, Ż: An example for the holomorphic sectional curvature of the Bergman metric. Ann. Polon. Math. **98**(2), 147–167 (2010). https://doi.org/10.4064/ap98-2-4
18. Forstnerič, F.: Holomorphic embeddings and immersions of Stein manifolds: a survey, Geometric complex analysis, Springer Proc. Math. Stat., 246, Springer, Singapore, 145–169, https://doi.org/10.1007/978-981-13-1672-2_11(2018)
19. Hahn, K.T.: On completeness of the Bergman metric and its subordinate metric. Proc. Nat. Acad. Sci. U.S.A. **73**(12), 4294 (1976). https://doi.org/10.1073/pnas.73.12.4294
20. Hahn, K.T.: On completeness of the Bergman metric and its subordinate metrics II. Pacific J. Math. **68**(2), 437–446 (1977)
21. Herbort, G.: An example of a pseudoconvex domain whose holomorphic sectional curvature of the Bergman metric is unbounded. Ann. Polon. Math. **92**(1), 29–39 (2007). https://doi.org/10.4064/ap92-1-3
22. Jarnicki, M., Pflug, P.: Invariant distances and metrics in complex analysis, extended, De Gruyter Expositions in Mathematics, Walter de Gruyter GmbH & Co. KG, Berlin 9, https://doi.org/10.1515/9783110253863(2013)

23. Kim, K.-T., Zhang, L.: On the uniform squeezing property of bounded convex domains in $\mathbb{C}^n$. Pacific J. Math. **282**(2), 341–358 (2016). https://doi.org/10.2140/pjm.2016.282.341
24. Kobayashi, S.: Geometry of bounded domains. Trans. Am. Math. Soc. **92**, 267–290 (1959). https://doi.org/10.2307/1993156
25. Kobayashi, S.: Hyperbolic manifolds and holomorphic mappings, Second, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2005), https://doi.org/10.1142/5936, An introduction,
26. Look, K.H.: Schwarz lemma and analytic invariants. Sci. Sinica **7**, 453–504 (1958)
27. McKean, H.P.: An upper bound to the spectrum of Δ on a manifold of negative curvature. J. Differ. Geometry **4**, 359–366 (1970). (http://projecteuclid.org/euclid.jdg/1214429509)
28. Park, J.-D.: Explicit formulas of the Bergman kernel for 3-dimensional complex ellipsoids. J. Math. Anal. Appl. **400**(2), 664–674 (2013). https://doi.org/10.1016/j.jmaa.2012.11.017
29. Schoen, R., Yau, S.-T.: Lectures on differential geometry, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA: Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu. Translated from the Chinese by Ding and S. Y, Cheng, With a preface translated from the Chinese by Kaising Tso (1994)
30. Shi, W.-X.: Ricci flow and the uniformization on complete noncompact Kähler manifolds. J. Differ. Geom. **45**(1), 94–220 (1997). (http://projecteuclid.org/euclid.jdg/1214459756)
31. Wu, D., Yau, S.-T.: Some negatively curved complex geometry, ICCM,
32. Wu, D., Yau, S.-T.: Negative holomorphic curvature and positive canonical bundle. Invent. Math. **204**(2), 595–604 (2016). https://doi.org/10.1007/s00222-015-0621-9
33. Wu, D., Yau, S.-T.: Invariant metrics on negatively pinched complete Kähler manifolds. J. Am. Math. Soc. **33**(1), 103–133 (2020). https://doi.org/10.1090/jams/933
34. Wu, H.: A remark on holomorphic sectional curvature. Indiana Univ. Math. J. **22**, 1103–1108 (1972/73). https://doi.org/10.1512/iumj.1973.22.22092
35. Wu, H.: Old and new invariant metrics on complex manifolds, (1993) Several complex variables (Stockholm, 1987/1988), Math. Notes, 38, Princeton Univ. Press, Princeton, NJ, 640– 682
36. Yeung, S.-K.: Geometry of domains with the uniform squeezing property. Adv. Math. **221**(2), 547–569 (2009). https://doi.org/10.1016/j.aim.2009.01.002

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PROBABILISTIC METHOD TO FUNDAMENTAL GAP PROBLEMS ON THE SPHERE

GUNHEE CHO, GUOFANG WEI, AND GUANG YANG

ABSTRACT. We provide a probabilistic proof of the fundamental gap estimate for Schrödinger operators in convex domains on the sphere, which extends the probabilistic proof of F. Gong, H. Li, and D. Luo [Potential Anal. 44 (2016), pp. 423–442] for the Euclidean case. Our results further generalize the results achieved for the Laplacian by S. Seto, L. Wang, and G. Wei [J. Differential Geom. 112 (2019), pp. 347–389], as well as by C. He, G. Wei, and Qi S. Zhang [Amer. J. Math. 142 (2020), pp. 1161–1191]. The essential ingredient in our analysis is the reflection coupling method on Riemannian manifolds.

1. INTRODUCTION

In the context of a bounded smooth domain Ω within a Riemannian manifold, the eigenvalues of the Laplacian or more generally, Schrödinger operators on Ω subject to Dirichlet boundary conditions are ordered as follows:

$$\lambda_1 < \lambda_2 \leq \lambda_3 \cdots \to \infty.$$

The estimate of the gap between the first two eigenvalues, known as the fundamental (or mass) gap, denoted as

$$\Gamma(\Omega) = \lambda_2 - \lambda_1 > 0,$$

holds significant importance in both mathematics and physics, and has been a subject of active study.

In a noteworthy achievement, using two point maximum principle B. Andrews and J. Clutterbuck successfully proved the fundamental gap conjecture for convex domains in Euclidean space [1]. Namely, for Schrödinger operators with convex potential, we have $\Gamma(\Omega) \geq \frac{3\pi^2}{D^2}$ for convex domains $\Omega \subset \mathbb{R}^n$, where D is the diameter of the domain. More recently, the second author, along with coauthors, extended the same estimate to convex domains on the sphere for the Laplacian [6, 9, 17].

An essential step in proving the fundamental gap conjecture is the log-concavity estimate of the first eigenfunction. The following definition was firstly introduced by B. Andrews and J. Clutterbuck [1]. An even function $\tilde{V} \in C^1([-D/2, D/2])$ is called a modulus of concavity for $V \in C^1(\Omega)$, if for all $x, y \in \Omega, x \neq y$, one has

$$\langle \nabla V(y), \gamma'(y)\rangle - \langle \nabla V(x), \gamma'(x)\rangle \leq 2\tilde{V}'\left(\frac{\rho(x,y)}{2}\right), \tag{1.1}$$

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where γ represents the minimizing geodesic that connects x, y and ρ denotes the geodesic distance function. Intuitively, this says that V is "more concave" than $\tilde{V}$. If the inequality (1.1) is reversed

$$\langle \nabla V(y), \gamma'(y)\rangle - \langle \nabla V(x), \gamma'(x)\rangle \geq 2\tilde{V}'\left(\frac{\rho(x,y)}{2}\right), \tag{1.2}$$

we say $\tilde{V}$ is a modulus of convexity for V.

Let ϕ_1 be the first positive Dirichlet eigenfunction of the Schrödinger operator $-\Delta + V$ on Ω and $\bar{\phi}_1$ the first Dirichlet eigenfunction of the one-dimensional model given in (1.3). The key estimates in obtaining the fundamental gap lower bound is the following log-concavity estimate of the first eigenfunction.

$$\langle \nabla \log \phi_1(y), \gamma'(y)\rangle - \langle \nabla \log \phi_1(x), \gamma'(x)\rangle \leq 2 \log \bar{\phi}_1'(\rho(x,y)/2)$$

for any $x \neq y \in \Omega$. Such log-concavity estimate was proved for Schrödinger operators in convex domains $\Omega \subset \mathbb{R}^n$ in [1], and for the Laplacian in convex domains $\Omega \subset \mathbb{S}^n$ with diameter $D < \pi/2$ in [17]. The diameter restriction was removed in [9].

In this paper we give a probabilistic proof of this estimate. Denote by $\mathbb{M}_k^n$ the n-dimensional simply connected manifolds with constant sectional curvature k. Our first main result is the next

Theorem 1.1. *Suppose $\Omega \subset \mathbb{M}_k^n$ $(k \geq 0)$ is a bounded strictly convex domain with diameter $D < \pi/\sqrt{k}$. Let $\lambda_1, \phi_1 > 0$ be the first Dirichlet eigenvalue, eigenfunction of $-\Delta + V$ on Ω and $\bar{\lambda}_1, \bar{\phi}_1$ be the first Dirichlet eigenvalue, eigenfunction of the one-dimensional model*

$$-\frac{\partial^2}{\partial s^2} + (n-1)tn_k(s)\frac{\partial}{\partial s} + \tilde{V}(s), \ \ s \in [-D/2, D/2], \tag{1.3}$$

where tn_k is defined in (2.7). If $\tilde{V}$ is a modulus of convexity of V, and

$$tn_k\left[(2\lambda_1 - V(x) - V(y)) - 2(\bar{\lambda}_1 - \tilde{V}(\rho(x,y)/2))\right] \geq 0, \tag{1.4}$$

then $\log \bar{\phi}_1$ is a modulus of concavity of $\log \phi_1$.

Note that $tn_k = 0$ when $k = 0$, so (1.4) is automatically satisfied for $\mathbb{R}^n$. When $k = 1$ and $V = 0$, by choosing $\tilde{V} = 0$, (1.4) becomes $\lambda_1 \geq \bar{\lambda}_1$, which holds by Lemma 3.12 in [17]. Therefore the above theorem recovers the logconcavity estimates in [1, 9, 17], extends the results in [9, 17] for the Laplacian operator to Schrödinger operators. Condition (1.4) indicates interesting difference between $\mathbb{R}^n$ and $\mathbb{S}^n$ for Schrödinger operators. In fact it is not clear if [17, Remark 1.4] holds as stated.

As mentioned before, our argument is based on the reflection coupling method (see for instance, [5, 10, 13]), which has been used in [8] to give a probabilistic proof for convex domain in the Euclidean space. Some of our computations are similar to that of [9, 17], but the whole proof is much simplified. Namely, the parabolic analysis in [9] is not needed at all. Also we do not need the initial analysis or log-concavity assumption of the first eigenfunction as in [8]. It is also worth mentioning that coupling methods have been used to derive lower bounds for the first non-zero Neumann eigenvalue [2–4].

With the log-concavity estimate, another ingredient for getting the fundamental gap estimate is the following gap comparison of [17, Theorem 4.1]. We extend it to Schrödinger operators with a probabilistic proof, which simplifies the original proof.

Theorem 1.2. *Let (M^n, g) be a complete Riemannian manifold with Ricci curvature lower bound $(n-1)k, k \in \mathbb{R}$ and $\Omega \subset M^n$ be a convex domain with diameter $D > 0$ ($D < \pi/\sqrt{k}$ if $k > 0$). Let λ_1, λ_2 be the first two Dirichlet eigenvalues of the Schrödinger operator $-\Delta + V$ on Ω and ϕ_1 be the corresponding first eigenfunction. Let $\bar{\lambda}_1, \bar{\lambda}_2$ be the first two Dirichlet eigenvalues of the one-dimensional model (1.3) on $[-D/2, D/2]$ and $\tilde{\phi}_1$ be the corresponding first eigenfunction. If $\log \tilde{\phi}_1$ is a modulus of concavity of $\log \phi_1$, then*

$$\Gamma(\Omega) = \lambda_2 - \lambda_1 \geq \bar{\lambda}_2(n, k, D) - \bar{\lambda}_1(n, k, D).$$

The above two theorems immediately give the following.

Corollary 1.1. *Let Ω be a strictly convex domain in $\mathbb{M}_k$ ($k \geq 0$) with diameter $D < \pi/\sqrt{k}$. Let $\lambda_i (i = 1, 2)$ be the first two Dirichlet eigenvalues of the Schrödinger operator $-\Delta + V$ on Ω. If $\tilde{V}$ is a modulus of convexity of V and (1.4) holds, then*

$$\Gamma(\Omega) = \lambda_2 - \lambda_1 \geq \bar{\lambda}_2(n, k, D) - \bar{\lambda}_1(n, k, D),$$

where $\bar{\lambda}_i(n, k, D)(i = 1, 2)$ are the first two Dirichlet eigenvalues of the operator (1.3) on $[-D/2, D/2]$.

This fundamental gap estimate recovers the estimate in [1], and extends the estimate in [9, 17] to Schrödinger operators.

The rest of this paper is structured as follows. Section 2 provides some necessary background materials for both Riemannian geometry and stochastic analysis. We define and study two diffusion couplings in Section 3, which will be used in our later proofs. Section 4 is devoted to the proof of the log-concavity of the first eigenfunction, and a proof for the gap comparison is given in Section 5.

2. Preliminaries

2.1. **Brownian motion and diffusion in Euclidean space.** In this subsection, we gather the essential materials from the classical stochastic analysis. These materials here are well-known; we include them so that readers without any probability background can acquire intuitions and basic understandings. Some excellent and comprehensive presentations of these topics can be found in [7, 12, 15, 16].

We fix a complete probability space $(A, \mathcal{F}, \mathbb{P})$. Here A is the sample space, $\mathcal{F}$ is a σ-field of subsets of A, and $\mathbb{P}$ a complete probability measure on $(A, \mathcal{F})$.

Definition 2.1 (Real Brownian motion)**.** A real-valued process $\{W_t\}_{t \geq 0}$ defined on A is said to be a standard Brownian motion if

(1) $W_0 = 0$.
(2) For each $t > 0$, W_t is Gaussian with mean 0 and variance t.
(3) W_t has independent increments; that is, $W_t - W_s$ and $W_u - W_v$ are independent whenever $[s, t] \cap [u, v] = \emptyset$.
(4) W_t is continuous as a function of $t \in [0, +\infty)$.

More generally, for any $n \geq 1$, we can define a standard Brownian motion as a $\mathbb{R}^n$-valued process whose components are independent standard real Brownian motions.

Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a family of increasing sub σ-field of $\mathcal{F}$. A process $\{X_t\}_{t\geq 0}$ is called $\mathcal{F}_t$-adapted if X_t is $\mathcal{F}_t$ measurable for any $t \geq 0$ (i.e. for any real numbers $a < b$, we have $\{\omega \in A : a < X_t(\omega) < b\} \in \mathcal{F}_t$).

Definition 2.2. A process $\{X_t\}_{t\geq 0}$ is said to be a $\mathcal{F}_t$-martingale if X_t is $\mathcal{F}_t$-adapted and integrable for every $t \geq 0$, and the conditional expectation $\mathbb{E}\left(X_t \mid \mathcal{F}_s\right)$ is equal to X_s almost surely, for all $0 \leq s \leq t$. In particular, $\mathbb{E}(X_t) = \mathbb{E}(X_0)$.

Brownian motion is the most classical example of martingales. The associated $\mathcal{F}_t$ is given by the σ-field generated by $\{\omega \in A : a < W_s(\omega) < b, \}$, where $0 \leq s \leq t$ and $a < b$ are any real numbers. We will call $\mathcal{F}_t$ the σ-field generated by $\{W_s\}_{0\leq s\leq t}$.

For any $t > 0$, a Brownian motion has unbounded variation on $[0, t]$. More precisely,

$$\lim_{|\Delta t_i|\to 0} \sum_i \left|W_{t_{i+1}} - W_{t_i}\right| = +\infty$$

almost surely, where $\{t_i\}_{i\geq 1}$ is any partition of $[0, t]$ with mesh Δt_i. As a result, it is in general impossible to define integrals against Brownian motions as Riemann-Stieltjes integrals. To properly define stochastic integrals, we will need the so called quadratic variation, which we describe now.

Suppose X_t is a square-integrable martingale (i.e. $\mathbb{E}(X_t^2) < +\infty$ for all $t \geq 0$), the quadratic variation of X_t is defined as

$$\langle X\rangle_t = \lim_{|\Delta t_i|\to 0} \sum_i \left|X_{t_{i+1}} - X_{t_i}\right|^2,$$

where the limit is taken in the probability measure $\mathbb{P}$. If two square integrable martingales X_t, Y_t are adapted to the same filtration $\mathcal{F}_t$, then their linear combinations are also square integrable martingales. We define their cross variation as

$$\langle X, Y\rangle_t = \frac{1}{4}\left(\langle X+Y\rangle_t - \langle X-Y\rangle_t\right).$$

The process $\langle X\rangle_t$ is always of bounded variation. In particular, we can find a locally bounded function $h(X)$ such that $d\langle X\rangle_t = h(X)_t dt$. We can give another characterization of standard Brownian motions through quadratic variation.

Proposition 2.3 (Levy's characterization)**.** *If X_t is a continuous square integrable martingale with $\langle X\rangle_t = t$ such that $\mathbb{P}(X_0 = 0) = 1$, then X_t is a standard real Brownian motion.*

Proposition 2.4. *Let Z_t be a $\mathcal{F}_t$-adapted process and X_t be a square integrable $\mathcal{F}_t$-martingale such that*

$$\int_0^T Z_s^2 d\langle X\rangle_s < +\infty.$$

Then, along any partitions of $[0, T]$, the limit

$$\lim_{|\Delta t_i|\to 0} \sum_i Z_{t_i}(X_{t_{i+1}} - X_{t_i})$$

exists in $L^2(A)$. This limit will be denoted as $\int_0^t Z_s dX_s$.

We can now define stochastic integration.

Definition 2.5 (Itô's integral)**.** The limit in Proposition 2.4 is called the Itô's integral of Z_t against X_t, and is denoted by $I_t(Z) = \int_0^t Z_s dX_s$. The quadratic variation and cross variation of Itô's integral are given as

$$\langle I(Z)\rangle_t = \int_0^t Z_s^2 d\langle X\rangle_s, \quad \langle I(Z), I(W)\rangle_t = \int_0^t Z_s W_s d\langle X\rangle_s.$$

Remark 2.6. It is important to note that in the previous approximating Riemann sum, we use the left point t_i for Z_s in each interval $[t_i, t_{i+1}]$. This special choice makes Itô's integrals martingales.

Another important type of stochastic integration is the following.

Definition 2.7. The Stratonovich integral of Z_t against X_t is defined to be

$$\int_0^t Z_s \circ dX_s := \int_0^t Z_s dX_s + \frac{1}{2}\langle Z, X\rangle_t.$$

The change of variable formula for stochastic integration is drastically different from the classical calculus.

Proposition 2.8 (Itô's formula)**.** *Let $f \in C^2(\mathbb{R}^n)$ and $\{X_t\}_{0\le t\le T}$ is a square integrable martingale. Then for any $0 \le t \le T$,*

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{i=1}^n \int_0^t \partial_i f(X_s) dX_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \partial^2_{i,j} f(X_s) d\langle X^i, X^j\rangle_s \\ &= f(X_0) + \sum_{i=1}^n \int_0^t \partial_i f(X_s) \circ dX_s^i. \end{aligned} \tag{2.1}$$

We see that in (2.1), there is an extra second derivative term, which does not appear in the classical calculus. Intuitively, this term corresponds to the second order term in the Taylor series expansion. Since martingales have finite quadratic variations, the sum of the second order terms in the Taylor expansion converges to a non-zero limit, which leads to the extra term in the Itô's formula.

Now we can formulate the next

Definition 2.9. Let X_t be a process on $\mathbb{R}^n$ and L a smooth second order differential operator on $\mathbb{R}^n$. We say X_t is the diffusion generated by L, if for any $f \in C_c^2(\mathbb{R}^n)$ the process

$$M_t^f := f(X_t) - f(X_0) + \int_0^t Lf(X_s) ds$$

is a martingale. Moreover, L is called the generator of X_t.

By Itô's formula (2.1), it is easily checked that that Brownian motion is the diffusion generated by $\frac{1}{2}\Delta_{\mathbb{R}^n}$.

A canonical example of diffusion is provided by stochastic differential equations (SDE) of the form

$$X_t = X_0 + \sum_{i=1}^n \int_0^t V_i(X_s) \circ dW_s^i + \int_0^t V_0(X_s) ds, \ 0 \le t \le T, \ X_0 \in \mathbb{R}^n, \tag{2.2}$$

where $\{V_i\}_{0\le i\le n}$ are smooth vector fields on $\mathbb{R}^n$ and W_t is a $\mathbb{R}^n$-valued Brownian motion. The following existence and uniqueness result is well-known.

Theorem 2.1. *Suppose $\{V_i\}_{0\le i\le n}$ are Lipschitz smooth vector fields on $\mathbb{R}^n$. Then for any $T>0$, (2.2) admits a unique solution adapted to the Brownian filtration $\mathcal{F}_t$.*

In addition, the generator of X_t defined above is given by the Hörmander type operator $L=\frac{1}{2}\sum_{i=1}^n V_i^2+V_0$.

Diffusion are also (strong) Markov processes. More specifically, we have

$$\mathbb{E}^{X_0}(X_{s+t}\mid \mathcal{F}_s)=\mathbb{E}^{X_s}(X_t),\ \forall s,t\ge 0.$$

Here the sup-scripts are used to indicate the starting points. We also emphasize that in the above equation s,t can be replaced by stopping times adapted to the underlying filtration. Intuitively, it says that to predict the behavior of X_{s+t} with the knowledge of $\{X_r,\ 0\le r\le s\}$ is the same as doing the prediction with knowledge of only X_s. In other words, for X_t, given the current position, the past and the future are independent. The Markov property of diffusion allows one to construct semi-groups from X_t. More specifically, for any bounded function $f\in C_c(\mathbb{R}^n)$

$$P_tf(x)=\mathbb{E}^x(f(X_t))$$

is a strong continuous semi-group, whose generator coincides with the generator of X_t.

2.2. **Coupling of diffusion in Euclidean space.** In this section, we review some basic facts of the coupling method of diffusion in Euclidean space. Let Z_t be a diffusion generated by a given elliptic operator L. We have the following

Definition 2.10. A couple diffusion $(X_t,Y_t)\in\mathbb{R}^n\times\mathbb{R}^n\cong\mathbb{R}^{2n}$ is said to be a coupling of Z_t, if both X_t and Y_t are diffusion generated by L.

Remark 2.11. This definition might look weird at first since one would have thought that we should have $(X_t,Y_t)=(Z_t,Z_t)$. However, the uniqueness we claimed for Z_t is in the sense of probability distribution. In other words, if X_t,Y_t,Z_t have the same starting point, then for any Borel set $B\in\mathbb{R}^n$, we always have

$$\mathbb{P}(X_t\in B)=\mathbb{P}(Y_t\in B)=\mathbb{P}(Z_t\in B),\ \forall t>0.$$

But this does not imply $X_t=Y_t=Z_t$. We can give a simple example. Let W_t be a standard Brownian motion on $\mathbb{R}^1$, then by symmetry $(W_t,-W_t)$ is a coupling of real Brownian motion. But clearly, W_t and $-W_t$ are not identical.

For our purpose, we will consider

$$X_t=X_0+\sum_{i=1}^n\int_0^t V_i(X_s)\circ dW_s^i+\int_0^t V_0(X_s)ds$$
$$Y_t=Y_0+\sum_{i=1}^n\int_0^t V_i(Y_s)\circ dB_s^i+\int_0^t V_0(Y_s)ds,$$

where (W_t,B_t) is a coupling of $\mathbb{R}^n$ Brownian motions. Obviously, (X_t,Y_t) is a coupling of diffusion generated by $L=\frac{1}{2}\sum_{i=1}^n V_i^2+V_0$. So, each coupling (W_t,B_t) of $\mathbb{R}^n$ Brownian motions gives us a coupling (X_t,Y_t). A special type of coupling that has attracted a lot of attentions is the so called Kendall-Cranston coupling, or reflection coupling, which was formally introduced in [5] and we now introduce.

We consider the orthogonal matrix

$$m'(x,y)=I_n-2\frac{(x-y)(x-y)^*}{|x-y|^2},\ x,y\in\mathbb{R}^n,x\neq y.$$

Multiplying $m'(x,y)$ with an vector v gives the reflection of v across the hyperplane perpendicular to the line segment that connects x and y.

Let W_t be a standard $\mathbb{R}^n$ Brownian motion, and define

$$X_t = X_0 + \sum_{i=1}^n \int_0^t V_i(X_s) \circ dW_s^i + \int_0^t V_0(X_s)ds \tag{2.3}$$

$$Y_t = Y_0 + \sum_{i=1}^n \int_0^t V_i(Y_s) \circ dB_s^i + \int_0^t V_0(Y_s)ds \tag{2.4}$$

$$B_t = \int_0^t m'(X_t, Y_t)dW_t. \tag{2.5}$$

Since $m'(X_s, Y_s)$ is orthogonal, by Levy's characterization, B_t is another Brownian motion adapted to $\sigma(W_t)$. Hence, the above-defined (X_t, Y_t) is a coupling. We will refer to $\tau := \inf\{t \geq 0 : \ X_t = Y_t\}$ as the coupling time. Now, the complete reflection coupling is give by (2.3)–(2.5) and $X_t = Y_t, \ \forall t \geq \tau$. Notice that we set $X_t = Y_t$ after the coupling time. By the strong Markov property, this does not change the generator of X_t and Y_t. The fact that these two diffusion stick together when t is sufficiently large is crucial to our analysis later.

2.3. **Diffusion on Riemannian manifolds.** Let M^n be a smooth compact Riemannian manifold of dimension $n > 1$ without boundary. We first make sense of SDE of the form

$$X_t = X_0 + \sum_{i=1}^n \int_0^t V_i(X_s) \circ dW_s^i + \int_0^t V_0(X_s)ds, \ 0 \leq t \leq T, \ X_0 \in M^n, \tag{2.6}$$

where $\{V_i\}_{0\leq i\leq n}$ are smooth vector fields on M^n and W_t is a $\mathbb{R}^n$-valued Brownian motion.

Definition 2.12. An adapted process X_t is said to be the solution to (2.6) if for any $f \in C^\infty(M^n)$, we have

$$f(X_t) = f(X_0) + \sum_{i=1}^n \int_0^t V_i f(X_s) \circ dW_s^i + \int_0^t V_0 f(X_s)ds, \ 0 \leq t \leq T.$$

By Whitney's embedding Theorem, M^n can be smoothly embeded into $\mathbb{R}^{2n}$ as a closed subset. We can also extend $\{V_i\}_{0\leq i\leq n}$ to smooth vector fields $\{\hat{V}_i\}_{0\leq i\leq n}$ on $\mathbb{R}^{2n}$. The existence and uniqueness of solution to (2.6) is guaranteed by Theorem 2.1 and the observation that X_t as a process in $\mathbb{R}^{2n}$ stays in M^n.

Diffusion on Riemannian manifolds is defined as a natural extension of the Definition 2.9.

Definition 2.13. Let X_t be a process on M^n and L a smooth second order differential operator on M^n. We say X_t is the diffusion generated by L, if for any $f \in C^2(M^n)$ the process

$$M_t^f := f(X_t) - f(X_0) + \int_0^t Lf(X_s)ds$$

is a martingale.

We are ready for the next

Definition 2.14. A process X_t on M^n is said to be a Brownian motion if the generator of X_t is $\frac{1}{2}\Delta$, where Δ is the Laplace-Beltrami operator on M^n.

In order to take full advantage of stochastic calculus, we would like to write a Brownian motion on M^n as the solution to a SDE on M^n (note that this task is trivial on $\mathbb{R}^n$). Thus, it is desirable to write the Laplace-Beltrami operator as a Hörmander type operator $L = \frac{1}{2}\sum_{i=1}^n V_i^2 + V_0$. However, there is no intrinsic way of achieving this on a general Riemannian manifold. We will employ the Eells-Elworthy-Malliavin approach (see chapter 3 of [10]), which we now describe.

Let $O(M^n)$ be the orthonormal frame bundle of M^n and $\pi : O(M^n) \to M^n$ the canonical projection. For any $u \in O(M^n)$, we denote $H_{e_i}(u) = H_i(u)$ the horizontal vector field such that $\pi_* H_i(u) = u \cdot e_i$, where $\{e_i\}_{1\le i\le n}$ is a basis of $\mathbb{R}^n$. Bochner's horizontal Laplacian on $O(M^n)$ is defined as $\Delta_{O(M^n)} = \sum_{i=1}^n H_i^2$. In particular, we have

Theorem 2.2. *Let $f \in C^\infty(M^n)$ and $\tilde{f} = f \circ \pi$ its lift to $O(M^n)$, then for any $u \in O(M^n)$*

$$\Delta_{O(M^n)}\tilde{f}(u) = \Delta f(\pi(u)).$$

With Bochner's horizontal Laplacian, a horizontal Brownian motion on $O(M^n)$ can be written as the solution to

$$\alpha_t = \alpha_0 + \sum_{i=1}^n \int_0^t H_i(\alpha_s) \circ dB_s^i,\ \alpha_0 \in O(M^n).$$

Its projection $X_t = \pi(\alpha_t)$ is a Brownian motion on M^n starts at $\pi(\alpha_0)$.

2.4. **Coupling of Brownian motions on Riemannian manifolds.** The definition of coupling on Riemannian manifold is exactly the same as on $\mathbb{R}^n$ as we introduced before. In order to construct a reflection coupling of Brownian motions on M^n, we only need to replace the orthogonal matrix $m'(x,y)$ by the so called mirror map $m(x,y) : T_xM^n \to T_yM^n$. For each $w \in T_xM^n$, $m(x,y)(w)$ is obtained by first parallel transport w along the unique minimizing geodesic from x to y, then reflect the resulting vector with respect to the hyperplane perpendicular to the geodesic at y. Clearly, $m(x,y)$ is an isometry and generalization of $m'(X,Y)$ that we used in $\mathbb{R}^n$. It should be pointed out that if x,y are conjugate points then $m(x,y)$ is undefined. This issue could be handled by an extension method (see [10, Theorem 6.62]).

The reflection coupling of Brownian motions on M^n is defined as

$$dU_t = \sum_{i=1}^n H_i(U_t) \circ dB_t^i, \qquad dV_t = \sum_{i=1}^n H_i(V_t) \circ dW_t^i,$$
$$X_t = \pi(U_t), \qquad Y_t = \pi(V_t), \qquad dW_t = V_t m(X_t, Y_t) U_t^{-1} dB_t.$$

2.5. **Boundary control of solutions of heat equation.** We recall from [17, Lemma 3.4] a general near boundary control of positive functions on a convex domain that vanish on the boundary.

Proposition 2.15. *Let Ω be a uniformly convex bounded domain in a Riemannian manifold M^n, and $u : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}$ a C^2 function such that u is positive on Ω, $u(\cdot,t) = 0$ and $\nabla u \neq 0$ on $\partial\Omega$. Given $T < \infty$, there exists $r_1 > 0$ such that $\operatorname{Hess}\log u|_{(x,t)} < 0$ whenever $d(x,\partial\Omega) < r_1$ and $t \in [0,T]$, and $C \in \mathbb{R}$ such that $\operatorname{Hess}\log u|_{(x,t)}(v,v) \leq C\|v\|^2$ for all $x \in \Omega$ and $t \in [0,T]$.*

We will also need the next

Proposition 2.16. *Let Ω be a bounded strictly convex domain in a Riemannian manifold with smooth boundary $\partial\Omega$. Denote by $\rho_{\partial\Omega} : \bar{\Omega} \to \mathbb{R}_+$the distance function to the boundary $\partial\Omega$, and N the unit inward normal vector field on $\partial\Omega$. Let $u_0 \in C^\infty(\bar{\Omega})$ be positive in Ω such that $u_0 = 0$ and $\nabla u_0 \neq 0$ on $\partial\Omega$. Let $u : \mathbb{R}_+ \times \bar{\Omega} \to \mathbb{R}_+$be a smooth solution to*

$$\begin{aligned} \frac{\partial u}{\partial t} &= (\Delta - V)u \quad \text{ in } \mathbb{R}_+ \times \Omega; \\ u &= 0 \quad \text{ on } \mathbb{R}_+ \times \partial\Omega \text{ and } u(0,\cdot) = u_0 \text{ in } \bar{\Omega}. \end{aligned}$$

Then the solution u verifies that

(1) *for any $T > 0, \theta_T := \inf_{[0,T]\times\partial\Omega} |\nabla u| > 0$;*
(2) *for every $x \in \partial\Omega, \nabla u(t,x) = |\nabla u(t,x)|N(x)$;*
(3) *$\lim_{\rho_{\partial\Omega}(x)\to 0} \frac{u(t,x)}{|\nabla u(t,x)|\rho_{\partial\Omega}(x)} = 1$ uniformly in $t \in [0,T]$;*
(4) *for any $T > 0$, there exists $C_T \geq 0$ such that $\operatorname{Hess}\log u|_{(t,x)}(y,y) \leq C_T|y|^2$ for all $t \in [0,T], x \in \Omega$ and $y \in T\Omega$ tangent vector at x.*

Proof. The first statement is a consequence of maximum principle for an elliptic operator and non-constancy of u_0. Since u behaves as a level set on $\mathbb{R}_+ \times \partial\Omega$, the second statement follows. The third statement follows from the following identification: for $r > 0$, we write $\partial_r\Omega = \{x \in \Omega : \rho_{\partial\Omega}(x) \leq r\}$ for the r-neighborhood of $\partial\Omega$. As [18], there exists $r_0 \in (0, D/2)$ such that $\rho_{\partial\Omega}$ is smooth on $\partial_{r_0}\Omega$. Then for any $x \in \partial_{r_0}\Omega$, there exists a unique $x' \in \partial\Omega$ such that $\rho_{\partial\Omega}(x) = |x - x'|$ (meaning that the minimal geodesic distance between x and x') and $\nabla\rho_{\partial\Omega}(x) = N(x')$. In particular, $\nabla\rho_{\partial\Omega} = N$ on the boundary $\partial\Omega$. The last statement follows from Proposition 2.15. □

2.6. **Jacobi fields and variations in the model spaces.** Here we recall the notations and variations from [17] which we will use later.

Let $\mathbb{M}^n_K$ be the n-dimensional simply connected manifold with constant sectional curvature K. Let $\operatorname{sn}_K(s)$ be the solution of

$$\operatorname{sn}_K''(s) + K\operatorname{sn}_K(s) = 0, \quad \operatorname{sn}_K(0) = 0, \quad \operatorname{sn}_K'(0) = 1,$$

$\mathrm{cs}_K(s) = \mathrm{sn}'_K(s)$, and $\mathrm{tn}_K(s) = K\frac{\mathrm{sn}_K(s)}{\mathrm{cs}_K(s)} = -\frac{\mathrm{cs}'_K(s)}{\mathrm{cs}_K(s)}$. Explicitly we have

$$\mathrm{sn}_K(s) = \begin{cases} \frac{1}{\sqrt{K}}\sin(\sqrt{K}s), & K>0 \\ s, & K=0 \\ \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}s) & K<0 \end{cases}$$

$$\mathrm{cs}_K(s) = \begin{cases} \cos(\sqrt{K}s), & K>0 \\ 1, & K=0 \\ \cosh(\sqrt{-K}s), & K<0 \end{cases}$$

and

$$\mathrm{tn}_K(s) = \begin{cases} \sqrt{K}\tan(\sqrt{K}s), & K>0 \\ 0, & K=0 \\ -\sqrt{-K}\tanh(\sqrt{-K}s). & K<0 \end{cases} \tag{2.7}$$

Let $x_0 \neq y_0$ in Ω with $\rho(x_0, y_0) = d_0$. Let $\gamma(s) : \left[-\frac{d_0}{2}, \frac{d_0}{2}\right] \mapsto \Omega$ be the normal minimal geodesic connecting x_0 to y_0. Associated to a vector $v \oplus w \in T_{x_0}\Omega \oplus T_{y_0}\Omega$, we use the variation $\eta(r,s)$ from [17, Page 362]. Let $\sigma_1(r)$ be the geodesic with $\sigma_1(0) = x_0, \frac{\partial}{\partial r}\sigma_1(0) = v, \sigma_2(r)$ be the geodesic with $\sigma_2(0) = y_0, \frac{\partial}{\partial r}\sigma_2(0) = w$, and $\eta(r,s), s \in \left[-\frac{d_0}{2}, \frac{d_0}{2}\right]$, be the minimal geodesic connecting $\sigma_1(r)$ and $\sigma_2(r)$, with $\eta(0,s) = \gamma(s)$. Namely $\eta(r,s) = \exp_{\sigma_1(r)} sV(r)$ for some $V(r)$. Since we are in a strictly convex domain, every two points are connected by a unique minimal geodesic, the variation $\eta(r,s)$ is smooth. Denote the variation field $\frac{\partial}{\partial r}\eta(r,s)$ by $J(r,s)$. Then $J(r,s)$ is the Jacobi field along s direction satisfying $J\left(r, -\frac{d_0}{2}\right) = v, J\left(r, \frac{d_0}{2}\right) = w$. Denote $J(s) = J(0,s)$. Note that with this parametrization, in general, for fixed $r, \eta(r,s)$ is not unit speed when $s \neq 0$.

Denote the unit vector of $\frac{\partial}{\partial s}\eta(r,s)$ by

$$T(r,s) = \frac{\eta'}{\|\eta'\|}. \tag{2.8}$$

For space with constant sectional curvature $\mathbb{M}^n_K$, it is shown in [17] that

$$\nabla_r T(r,s)|_{r=0} = J'(s), \tag{2.9}$$

$$\nabla_r\nabla_r T|_{r=0} = -\left\|J'(s)\right\|^2 e_n. \tag{2.10}$$

Let $\{e_i\}, i = 1, \ldots, n$ be an orthonormal basis of T_{x_0} with $e_n = \gamma'(0)$, parallel translate e_i along γ to y_0. In $\mathbb{M}^n_k$, we have explicit formulas for the Jacobi fields with given boundary values. For each $1 \leq i \leq n-1$, with boundary values 0 and e_i, the Jacobi fields are $Q_i(s) = \frac{\mathrm{sn}_K\left(\frac{d_0}{2}+s\right)}{\mathrm{sn}_K(d_0)}e_i(s)$; for boundary values e_i and 0, $Q_i(s) = \frac{\mathrm{sn}_K\left(-s+\frac{d_0}{2}\right)}{\mathrm{sn}_K(d_0)}e_i(s)$; and for e_i and e_i, $Q_i(s) = \frac{\mathrm{cs}_K(s)}{\mathrm{cs}_K\left(\frac{d_0}{2}\right)}e_i(s)$.

3. Diffusion couplings and related properties

In this section, we introduce two different couple diffusion and derive some useful properties for our use later. We first consider the following (X_t, Y_t) on the product manifold $M^n \times M^n$. This coupling will be used to prove the gap comparison. This coupling was also used in the Euclidean case in [8]. We consider

$$dU_t = \sum_{i=1}^{n} \sqrt{2} H_{e_i}(U_t) \circ dB_t^i + 2H_{\nabla \log \phi_1}(U_t)dt, \tag{3.1}$$

$$dV_t = \sum_{i=1}^{n} \sqrt{2} H_{e_i}(V_t) \circ dW_t^i + 2H_{\nabla \log \phi_1}(V_t)dt, \tag{3.2}$$

$$dW_t = V_t m_{X_t Y_t} U_t^{-1} dB_t, \tag{3.3}$$

$$X_t = \pi(U_t), \qquad Y_t = \pi(V_t), \tag{3.4}$$

where $m(X_t, Y_t)$ is the mirror map defined in Section 2.4 and H_{e_i} is the horizontal vector field we defined in Section 2.3. It is easily checked that (X_t, Y_t) is a diffusion coupling in the sense of Kendall-Cranston [10,13].

Remark 3.1. The main motivation behind this definition is that both X_t, Y_t are generated by the operator L that appears in the ground state equation (5.2) in Section 5, which is essential in proving the gap comparison. It should be pointed out that with the Neumann boundary condition, ϕ_1 is always constant. Therefore, $\nabla\phi_1 = 0$ and the coupling (X_t, Y_t) given above becomes a coupling of Brownian motions, which is the canonical diffusion to study the first non-zero Neumann eigenvalue (see [10] chapter 6).

Our first task is to prove that for arbitrary starting points $X_0, Y_0 \in \Omega$, the processes X_t, Y_t will stay in Ω for all $t \geq 0$.

Lemma 3.2. *Let Ω be a bounded strictly convex domain in a Riemannian manifold with smooth boundary $\partial\Omega$. If $X_0, Y_0 \in \Omega$, then $X_t, Y_t \in \Omega$ for all $t \geq 0$.*

Proof. First we choose a smooth function $r : \bar{\Omega} \to \mathbb{R}$ such that $r(x) > 0$ for all $x \in \Omega$ and for some $d_0 \in (0, D)$,

$$r(x) = \rho_{\partial\Omega}(x), \ \forall x \in \{p \in \Omega \mid \rho_{\partial\Omega}(p) < d_0\},$$

where $\rho_{\partial\Omega}(x)$ denotes the geodesic distance between x and the boundary $\partial\Omega$. By our definition and compactness (see for instance, Theorem 2.2 of [18]), it is always possible to find $c_0 > 0$, such that $\Delta r(x) \geq -c_0$ for all $x \in \Omega$.

Since we only need to know the behavior of X_t near the boundary, it suffices to show that $r(X_t) > 0$ for all $t \geq 0$. By the same arguments in [5, Theorem 3] or [11, Theorem 6.6.2], we have

$$dr(X_t) = \sqrt{2} d\beta_t + \left[\Delta r(X_t) + 2\nabla \log \phi_1(r)(X_t)\right] dt,$$

where β_t is a standard Brownian motion. Since $e^{-\lambda_1 t}\phi_1$ solves the heat equation in Proposition 2.16, $\phi_1(x)$ verifies (2) (3) of Proposition 2.16. Hence, we can find $d \in (0, d_0)$ such that

$$\Delta r(x) + 2\nabla \log \phi_1(r)(x) \geq 2\nabla \log \phi_1(r)(x) - c_0 \geq \frac{1}{r(x)},$$

for all $x \in \Omega$ with $\rho_{\partial\Omega(x)} < d$. So we get, when $\rho_{\partial\Omega}(X_t) < d$,

$$dr(X_t) > \sqrt{2} d\beta_t + \frac{1}{r(X_t)} dt,$$

which implies $r(X_t) > 0$. Our proof is complete. $\square$

Recall that τ is the coupling time of (X_t, Y_t). Our next result shows that τ is finite almost surely.

Lemma 3.3. *Let (M^n, g) be a complete Riemannian manifold with Ricci curvature lower bound $(n-1)k, k \in \mathbb{R}$ and $\Omega \subset M^n$ be a convex domain with diameter $D > 0$. Then, for any starting points $X_0, Y_0 \in \Omega$, the coupling (X_t, Y_t) is successful in the sense that $\tau < \infty$ almost surely.*

Proof. Let $\rho_t = \rho(X_t, Y_t)$ be the geodesic distance between X_t, Y_t. It suffices to prove ρ_t hits 0 in finite time almost surely. Let $\{e_i\}_{1\le i\le n}$ be an orthonormal frame at X_t and we parallel transport them along the unique minimizing geodesic that connects X_t, Y_t. We further assume that e_n is the tangential direction. By Itô's formula (see section 6.6 of [10] for a similar computation)

$$d\rho_t = 2\sqrt{2}d\beta_t + \left(\sum_{i=1}^n I(Q_i, Q_i) + 2\langle \nabla \log \phi_1(Y_t), \gamma'(Y_t)\rangle - 2\langle \nabla \log \phi_1(X_t), \gamma'(X_t)\rangle\right) dt,$$

where β_t is a Brownian motion independent from B_t and Q_i is the Jacobi field associated with any variation whose tangent vector at (X_t, Y_t) is given by $e_i \oplus m(X_t, Y_t)e_i$. The index form of a vector field X (with not necessarily vanishing endpoints) along a geodesic γ is defined by

$$I(\gamma, X, X) := \int_0^T \left(\langle \nabla_{\gamma'} X, \nabla_{\gamma'} X\rangle - \langle R(\gamma', X)X, \gamma'\rangle\right) dt,$$

where ∇ is the Levi-Civita connection and R is the Riemann curvature tensor of a given manifold. We will use Lemma 1 from [14] to verify that ρ_t hits 0 in finite time. We have

$$\begin{aligned}\theta_t :&= 2\left(\langle \nabla \log \phi_1(Y_t), \gamma'(Y_t)\rangle - \langle \nabla \log \phi_1(X_t).\gamma'(X_t)\rangle\right) + \sum_{i=1}^n I(Q_i, Q_i)\\ &= 2\int_0^{\rho_t} \nabla^2 \log \phi_1(\gamma'(s), \gamma'(s))ds + \sum_{i=1}^n I(Q_i, Q_i).\end{aligned}$$

By Index lemma, we can find $C_1(k) > 0$ such that

$$\frac{1}{2}\sum_{i=1}^n I(Q_i, Q_i) \le -(n-1)tn_k(\rho_t/2) \le C_1.$$

Consequently

$$\theta_t \le 2\int_0^{\rho_t} \nabla^2 \log \phi_1(\gamma'(s), \gamma'(s))ds + C_1.$$

Let

$$C_2 = \sup\{\mathrm{Hess}\, \phi_1(p)(v, v) : p \in \bar{\Omega},\ v \in T_pM^n,\ |v| = 1\}.$$

By (4) of Proposition 2.16, C_2 is finite. We introduce

$$\eta(r) = \begin{cases} \frac{1}{4}C_2 \cdot r + C_1 & r \le D\\ -1 & r > D.\end{cases}$$

Since X_t, Y_t never leave Ω, $\rho_t \le D$ for all $t \ge 0$ and we have $\theta_t \le 8\eta(\rho_t)$ for all $t \ge 0$. It follows that the assumptions of Lemma 1 of [14] are verified. We conclude that ρ_t hits 0 in finite time almost surely. The proof is complete. $\square$

Next we introduce another important pair of diffusion (X_t', Y_t'), which will be used in establishing the log-concavity estimate for the first eigenfunction of $-\Delta+V$. Let us consider

$$
\begin{aligned}
(3.5)\quad dU_t' &= \sum_{i=1}^n \sqrt{2}H_{e_i}(U_t')\circ dB_t^i + 2H_{\nabla\log\phi_1}(U_t')dt + 2tn_k\left(\frac{\tilde\rho(U_t',V_t')}{2}\right)\tilde\gamma'(t)dt,\\
dV_t' &= \sum_{i=1}^n \sqrt{2}H_{e_i}(V_t')\circ dW_t^i + 2H_{\nabla\log\phi_1}(V_t')dt - 2tn_k\left(\frac{\tilde\rho(U_t',V_t')}{2}\right)\tilde\gamma'(t)dt,\\
dW_t &= V_t' m_{X_tY_t}(U_t')^{-1}dB_t,\\
X_t' &= \pi(U_t'), \qquad Y_t' = \pi(V_t').
\end{aligned}
$$

Here $\tilde\rho$ is the lift of ρ, which is defined as $\tilde\rho(U,V) = \rho\left(\pi(U),\pi(V)\right)$, and $\tilde\gamma$ is the horizontal lift of the minimizing geodesic that goes from X_t' to Y_t'. We note that X_t', Y_t' cannot be defined separately since the drift term depends on both X_t' and Y_t'. For this reason, (X_t', Y_t') is not a coupling of diffusion. We define $\tau' = \inf_{t\geq 0}\{\rho(X_t', Y_t') = 0\}$. With a slight abuse of terminology, we will still call τ' the coupling time. As before, we define $X_t' = Y_t'$ when $t \geq \tau'$.

Remark 3.4. There is no intuitive explanation of the extra term, compared to (X_t, Y_t), added in the definition of (X_t', Y_t'); it was what came out of our computations. However, it is interesting to note that when $k = 0$, the couple diffusion (X_t', Y_t') coincides with the coupling (X_t, Y_t). We can thus regard (X_t', Y_t') as a non-trivial generalization of (X_t, Y_t) that has incorporated more information about the curvature of the underlying manifold.

One can argue in the same way as we did in Lemma 3.2 to justify the following.

Lemma 3.5. *Under the same assumptions of Lemma 3.2, if $X_0', Y_0' \in \Omega$, then $X_t', Y_t' \in \Omega$ for all $t \geq 0$.*

Let us observe that, (X_t', Y_t'), although do not form a usual coupling, will also meet in finite time. In particular, we can give a comparison between τ and τ' based on k.

Lemma 3.6. *As in the setting of Lemma 3.3, we have $\tau' < \infty$ almost surely. Moreover, we have $\tau' \leq \tau$ when $k \geq 0$ and $\tau > \tau'$ when $k < 0$ almost surely.*

Proof. Since $\nabla_{\gamma'(X_t')}\rho(X_t', Y_t') = -\nabla_{\gamma'(Y_t')}\rho(X_t', Y_t') = -1$, we have by Itô's formula

$$
d\rho(X_t', Y_t') = 2\sqrt{2}d\beta_t + \left(\sum_{i=1}^n I(Q_i,Q_i) + F(X_t', Y_t') - 4tn_k\left(\frac{\rho(X_t',Y_t')}{2}\right)\right)dt,
$$

where

$$
F(X_t, Y_t) = \langle \nabla\log\phi_1(Y_t), \gamma'(Y_t)\rangle - \langle \nabla\log\phi_1(X_t), \gamma'(X_t)\rangle,
$$

and β_t is a standard Brownian motion. Since $tn_k\left(\rho(X_t', Y_t')/2\right)$ is uniformly bounded in Ω, the exact same argument of Lemma 3.3 applies to $\rho(X_t', Y_t')$ and $\tau' < +\infty$ follows.

For our second claim, we observe that $tn_k\left(\rho(X_t',Y_t')/2\right) \geq 0$ when $k \geq 0$ and $tn_k\left(\rho(X_t',Y_t')/2\right) < 0$ when $k < 0$. Hence from the diffusion comparison theorem we deduce that $\rho(X_t,Y_t) \geq \rho(X_t',Y_t')$ when $k \geq 0$ and $\rho(X_t,Y_t) < \rho(X_t',Y_t')$ when $k < 0$. The proof is complete. $\square$

4. Log-concavity of the first eigenfunction

In this section we establish a log-concavity estimate of the first eigenfunction of the Schrödinger operator $-\Delta + V$ on Ω, Theorem 1.1. Our analysis in this section relies on the diffusion (X_t', Y_t') that we introduced in (3.5).

We pick two distinct points $x, y \in \Omega$ and let them be the starting points of X_t', Y_t' respectively. Denote $\xi_t = \rho(X_t', Y_t')/2$. Let γ be the normal minimal geodesic that goes from X_t' to Y_t' with $\gamma(-\xi_t) = X_t'$ and $\gamma(\xi_t) = Y_t'$. We choose $\{e_i\}_{1\leq i\leq n}$ an orthonormal basis at X_t with $e_n = \gamma'(X_t')$ and parallel transport $\{e_i\}_{1\leq i\leq n}$ along γ. Let

$$E_i = e_i \oplus e_i,\ 1 \leq i \leq n-1,\ E_n = e_n \oplus -e_n.$$

We define

$$F_t = \langle \nabla \log \phi_1(Y_t'), \gamma'(Y_t')\rangle - \langle \nabla \log \phi_1(X_t'), \gamma'(X_t')\rangle. \tag{4.1}$$

One crucial step of our method is to find the SDE of F_t. The essence is to use Itô's formula and the geodesic variations as in [17] (see Section 2.6) to compute the first two orders of covariant derivatives.

The following is our key formula.

Proposition 4.1. *Let λ_1 be the first eigenvalue of $-\Delta + V$ on a convex domain $\Omega \subset \mathbb{M}_k^n$ and $\omega = \log \phi_1$. Then the SDE of F_t is given by*

$$\begin{aligned} dF_t &= d\{martingale\} \\ &\quad + \{\langle \nabla V(Y_t'), e_n\rangle - \langle \nabla V(X_t'), e_n\rangle\}\, dt \\ &\quad + (n-1)(K - tn_k^2(\xi_t))\, \{\langle \nabla\omega(Y_t'), e_n\rangle - \langle \nabla\omega(X_t'), e_n\rangle\}\, dt \\ &\quad + 2tn_k(\xi_t)\left[\langle \nabla\omega(Y_t'), e_n\rangle^2 + \langle \nabla\omega(X_t'), e_n\rangle^2 + 2\lambda_1 - V(X_t') - V(Y_t')\right] dt \\ &\quad + \frac{2}{sn_k(2\xi_t)} \sum_{i=1}^{n-1} \left(\langle \nabla\omega(Y_t'), e_i\rangle - \langle \nabla\omega(X_t'), e_i\rangle\right)^2 dt. \end{aligned} \tag{4.2}$$

Remark 4.2. It is possible to give the precise expression for the martingale part of dF_t, which, however, is irrelevant to our purpose. We write it as it is now to save space.

Proof. Recall we use the notation $\xi_t = \rho(X_t', Y_t')/2$. By Itô's formula (2.1), we have

$$\begin{aligned} dF_t &= \sum_{i=1}^n \nabla_{e_i\oplus 0} F_t(X_t', Y_t') \circ d(X_t')^i + \nabla_{0\oplus e_i} F_t(X_t', Y_t') \circ d(Y_t')^i \\ &= d\{\text{martingale}\} + \{\sum_{i=1}^n \nabla_{E_i}\nabla_{E_i} F_t(X_t', Y_t')\}dt \\ &\qquad + \{2\nabla \log\phi_1(X_t')(F_t) + 2\nabla\log\phi_1(Y_t')(F_t) + 2tn_k\left(\xi_t\right) \nabla_{e_n\oplus 0} F_t \\ &\qquad\quad - 2tn_k\left(\xi_t\right) \nabla_{0\oplus e_n} F_t\}dt. \end{aligned}$$

We begin with first derivative terms. Use the variation from Section 2.6 for the vector $\nabla\omega(X_t')\oplus 0$, denote $\eta(r,s)$ the variation, and T the unit variation field, see (2.8).

$$\nabla \log\phi_1(X_t')(F_t) = \langle \nabla\omega(Y_t'), \nabla_{\frac{\partial}{\partial r}} T(r,s)\rangle|_{r=0,s=\xi_t} - \left(\langle \nabla_{\nabla\omega}\nabla\omega(r,s), T(r,s)\rangle + \langle\nabla\omega(X_t'), \nabla_{\frac{\partial}{\partial r}}T(r,s)\rangle\right)|_{r=0,s=-\xi_t}$$

By [17, Page 363]

$$\nabla_{\frac{\partial}{\partial r}}T(r,s)|_{r=0} = -\langle\gamma'(s), J'(s)\rangle e_n + J'(s),$$

where $J(s)$ is the Jacobi field along γ with $J(-\xi_t) = \nabla\omega(X_t')$, $J(\xi_t)=0$. For $\mathbb{M}_k^n$,

$$J(s) = \left(-\frac{\langle\nabla\omega(X_t'), e_n\rangle}{2\xi_t}s + \frac{\langle\nabla\omega(X_t'), e_n\rangle}{2}\right)e_n + \sum_{i=1}^{n-1}\langle\nabla\omega(X_t'), e_i\rangle\frac{sn_k(\xi_t - s)}{sn_k(2\xi_t)}e_i.$$

Hence

$$J'(s) = -\frac{\langle\nabla\omega(X_t'), e_n\rangle}{2\xi_t}e_n - \sum_{i=1}^{n-1}\langle\nabla\omega(X_t'), e_i\rangle\frac{cn_k(\xi_t - s)}{sn_k(2\xi_t)}e_i,$$

$$\nabla_{\frac{\partial}{\partial r}}T(r,s)|_{r=0} = -\sum_{i=1}^{n-1}\langle\nabla\omega(X_t'), e_i\rangle\frac{cn_k(\xi_t - s)}{sn_k(2\xi_t)}e_i.$$

Plug these in, we have

$$\begin{aligned}&\nabla\log\phi_1(X_t')(F_t)\\ &= \sum_{i=1}^{n-1}\left(\langle\nabla\omega(Y_t'), e_i\rangle\langle\nabla\omega(X_t'), e_i\rangle\frac{-1}{sn_k(2\xi_t)} + \langle\nabla\omega(X_t'), e_i\rangle^2\frac{cn_k(2\xi_t)}{sn_k(2\xi_t)}\right)\\ &\quad - \langle\nabla_{\nabla\omega}\nabla\omega(r,s), \gamma'(X_t')\rangle|_{r=0,s=-\xi_t}.\end{aligned}$$

Similarly

$$\begin{aligned}&\nabla\log\phi_1(Y_t')(F_t)\\ &= \sum_{i=1}^{n-1}\left(\langle\nabla\omega(Y_t'), e_i\rangle\langle\nabla\omega(X_t'), e_i\rangle\frac{-1}{sn_k(2\xi_t)} + \langle\nabla\omega(Y_t'), e_i\rangle^2\frac{cn_k(2\xi_t)}{sn_k(2\xi_t)}\right)\\ &\quad + \langle\nabla_{\nabla\omega}\nabla\omega(r,s), \gamma'(Y_t')\rangle|_{r=0,s=\xi_t}.\end{aligned}$$

Together with the fact that $\nabla_{\nabla\omega}\nabla\omega = \frac{1}{2}\nabla\|\nabla\omega\|^2$, we get

$$\begin{aligned}&2\nabla\log\phi_1(X_t')(F_t) + 2\nabla\log\phi_1(Y_t')(F_t)\\ &= \langle\nabla\|\nabla\omega(Y_t')\|^2, \gamma'(Y_t')\rangle - \langle\nabla\|\nabla\omega(X_t')\|^2, \gamma'(X_t')\rangle\\ &\quad + 2\sum_{i=1}^{n-1}\langle\nabla\omega(Y_t'), e_i\rangle\left(\frac{cs_k(2\xi_t)}{sn_k(2\xi_t)}\langle\nabla\omega(Y_t'), e_i\rangle - \frac{1}{sn_k(2\xi_t)}\langle\nabla\omega(X_s'), e_i\rangle\right)\\ &\quad - 2\sum_{i=1}^{n-1}\langle\nabla\omega(X_t'), e_i\rangle\left(\frac{1}{sn_k(2\xi_t)}\langle\nabla\omega(Y_t'), e_i\rangle - \frac{cs_k(2\xi_t)}{sn_k(2\xi_t)}\langle\nabla\omega(X_t'), e_i\rangle\right).\end{aligned} \tag{4.3}$$

For the other first order term, we have

$$\begin{aligned}&2tn_k(\xi_t)\nabla_{e_n\oplus 0}F_t - 2tn_k(\xi_t)\nabla_{0\oplus e_n}F_t\\ &= -2tn_k(\xi_t)\langle\nabla_{e_n}\nabla\omega(Y_t'),e_n\rangle - 2tn_k(\xi_t)\langle\nabla_{e_n}\nabla\omega(X_t'),e_n\rangle.\end{aligned}\tag{4.4}$$

For the second order derivative, we argue in the exact same way as equation (3.18) of [17] and get

$$\begin{aligned}&\sum_{i=1}^n \nabla_{E_i}\nabla_{E_i}F_t(X_t',Y_t') = \langle\Delta\nabla\omega(Y_t'),e_n\rangle - \langle\Delta\nabla\omega(X_t'),e_n\rangle\\ &\quad - 2tn_k(\xi_t)\sum_{i=1}^{n-1}\left(\langle\nabla_{e_i}\nabla\omega(Y_s'),e_i\rangle + \langle\nabla_{e_i}\nabla\omega(X_t'),e_i\rangle\right)\\ &\quad - (n-1)tn_k^2(\xi_t)\left(\langle\nabla\omega(Y_t'),e_n\rangle - \langle\nabla\omega(X_t'),e_n\rangle\right)\\ &= \langle\Delta\nabla\omega(Y_t'),e_n\rangle - \langle\Delta\nabla\omega(X_t'),e_n\rangle\\ &\quad - 2tn_k(\xi_t)\left(\Delta\omega(Y_t') + \Delta\omega(X_t')\right) + 2tn_k(\xi_t)\langle\nabla_{e_n}\nabla\omega(Y_t'),e_n\rangle\\ &\quad + 2tn_k(\xi_t)\langle\nabla_{e_n}\nabla\omega(X_t'),e_n\rangle\\ &\quad - (n-1)tn_k^2(\xi_t)\left(\langle\nabla\omega(Y_t'),e_n\rangle - \langle\nabla\omega(X_t'),e_n\rangle\right).\\ &= \langle\Delta\nabla\omega(Y_t'),e_n\rangle - \langle\Delta\nabla\omega(X_t'),e_n\rangle\\ &\quad + 2tn_k(\xi_t)\left(2\lambda_1 + \|\nabla\omega(Y_t')\|^2 + \|\nabla\omega(X_t')\|^2 - V(X_t') - V(Y_t')\right)\\ &\quad + 2tn_k(\xi_t)\langle\nabla_{e_n}\nabla\omega(Y_t'),e_n\rangle + 2tn_k(\xi_t)\langle\nabla_{e_n}\nabla\omega(X_t'),e_n\rangle\\ &\quad - (n-1)tn_k^2(\xi_t)\left(\langle\nabla\omega(Y_t'),e_n\rangle - \langle\nabla\omega(X_t'),e_n\rangle\right).\end{aligned}\tag{4.5}$$

Here, we used $\Delta\omega = -\lambda_1 - \|\nabla\omega\|^2 + V$ in the last equation. Finally, note that $\nabla\|\nabla\omega\|^2 = -\nabla\Delta\omega + \nabla V$. Then, equation (4.2) follows from adding up (4.3) (4.4) (4.5), the Bochner–Weitzenböck formula

$$\Delta\nabla\omega - \nabla\Delta\omega = Ric(\nabla\omega,\cdot) = (n-1)k\langle\nabla\omega,\cdot\rangle,$$

and the identity $tn_k(\xi_t) = \frac{1-cs_k(2\xi_t)}{sn_k(2\xi_t)}$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First, we pick $0 < D < D' < \pi/\sqrt{k}$ and continuously extend $\tilde{V}$ to an even function on $[-D'/2, D'/2]$. Let $\bar{\phi}_1$ be the first eigenfunction of the 1-dimensional model on $[-D'/2, D'/2]$. Moreover, let $\{Q_i\}_{1\le i\le n}$ be the Jacobi fields of the geodesic variations $\eta(r,s)$ associated with the vector fields $\{E_i\}_{1\le i\le n}$, which we introduced at the beginning of this section. For simplicity, we denote $\Psi = (\log\bar{\phi}_1)'$.

By Itô's formula (2.1) and the second variation formula,

$$\begin{aligned}&d\Psi(\xi_t) = d\{\text{martingale}\}\\ &\qquad + \{\frac{1}{2}\sum_{i=1}^n I(Q_i,Q_i)\Psi' + F_t\Psi' + \Psi'' - 2tn_k(\xi_t)\Psi'\}dt.\end{aligned}\tag{4.6}$$

For $\mathbb{M}_k^n$, we have

$$\frac{1}{2}\sum_{i=1}^n I(Q_i,Q_i) = -(n-1)tn_k(\xi_t).\tag{4.7}$$

On the other hand, a direct computation (similar to Lemma 2.7 of [17]) gives that

$$\Psi''+2\Psi'\Psi-tn_k\left((n+1)\Psi'+2\bar\lambda_1+2\Psi^2-2\tilde V\right)-\tilde V'-(n-1)(K-tn_k^2)\Psi=0. \tag{4.8}$$

Combining estimates (4.8)(4.7) with (4.2) (4.6) gives

$$\begin{aligned}
d\left(F_t-2\Psi(\xi_t)\right)&=d\{\text{martingale}\}+4tn_k\left(\lambda_1-\bar\lambda_1\right)\\
&\quad+\left((n-1)(K-tn_k^2)-2\Psi'\right)(F_t-2\Psi)-4tn_k(\xi_t)\Psi^2dt\\
&\quad+\left\{\langle\nabla V(Y_t'),e_n\rangle-\langle\nabla V(X_t'),e_n\rangle-2\tilde V'(\xi_t)\right\}\\
&\quad+2tn_k(\xi_t)(2\tilde V(\xi_t)-V(X_t')-V(Y_t'))dt\\
&\quad+2tn_k(\xi_t)\left(\langle\nabla\omega(Y_t'),e_n\rangle^2+\langle\nabla\omega(X_t'),e_n\rangle^2\right)dt\\
&\quad+\frac{2}{sn_k(2\xi_t)}\sum_{i=1}^{n-1}\left(\langle\nabla\omega(Y_t'),e_i\rangle-\langle\nabla\omega(X_t'),e_i\rangle\right)^2dt\\
&\ge d\{\text{martingale}\}+\left((n-1)(K-tn_k^2(\xi_t))-2\Psi'\right)(F_t-2\Psi)dt\\
&\quad+tn_k(\xi_t)(F_t+2\Psi)(F_t-2\Psi)dt\\
&=d\{\text{martingale}\}\\
&\quad+\left((n-1)(K-tn_k^2(\xi_t))-2\Psi'+tn_k(\xi_t)(F_t+2\Psi)\right)(F_t-2\Psi)dt.
\end{aligned}\tag{4.9}$$

Here for the inequality we used the assumptions that $\tilde V$ is a modulus of convexity of V (1.2), (1.4), $k\ge 0$ (this is the only place $k\ge 0$ is used), and

$$\langle\nabla\omega(Y_t'),e_n\rangle^2+\langle\nabla\omega(X_t'),e_n\rangle^2\ge\frac{\left(\langle\nabla\omega(Y_t'),e_n\rangle-\langle\nabla\omega(X_t'),e_n\rangle\right)^2}{2}=\frac{F_t^2}{2}.$$

Next, we observe that (4.9) is equivalent to

$$\begin{aligned}
&d\left(e^{\int_0^t 2\Psi'-(n-1)(K-tn_k^2(\xi_s))-tn_k(\xi_s)(F_s+2\Psi)ds}(F_t-2\Psi(\xi_t))\right)\\
&\ge e^{\int_0^t 2\Psi'-(n-1)(K-tn_k^2(\xi_s))-tn_k(\xi_s)(F_s+2\Psi)ds}d\{\text{martingale}\}.
\end{aligned}$$

Let $N>0$, integrating from 0 to $\tau'\wedge N:=\min\{\tau',N\}$ gives

$$\begin{aligned}
&\left(e^{\int_0^{\tau'\wedge N} 2\Psi'-(n-1)(K-tn_k^2(\xi_s))-tn_k(\xi_s)(F_s+2\Psi)ds}(F_{\tau'\wedge N}-2\Psi(\xi_{\tau'\wedge N}))\right)\\
&\qquad\ge(F_0-2\Psi(\xi_0))+\text{Martingale}_{\tau'\wedge N}.
\end{aligned}$$

Since $\tau'\wedge N$ is bounded, the stopped martingale $\text{Martingale}_{\tau'\wedge N}$ in the above inequality is another martingale. We can thus take expectation and get

$$\begin{aligned}
F_0-2\Psi(\xi_0)&\le\mathbb{E}\left(e^{\int_0^{\tau'\wedge N} 2\Psi'-(n-1)(K-tn_k^2(\xi_s))-tn_k(\xi_s)(F_s+2\Psi)ds}(F_{\tau'\wedge N}-2\Psi(\xi_{\tau'\wedge N}))\right)\\
&\le\mathbb{E}\left(e^{\int_0^{\tau'\wedge N} 2\Psi'-(n-1)(K-tn_k^2(\xi_s))-tn_k(\xi_s)(F_s+2\Psi)ds}\left|F_{\tau'\wedge N}-2\Psi(\xi_{\tau'\wedge N})\right|\right).
\end{aligned}$$

By Fatou,

$$\begin{aligned}
&F_0-2\Psi(\xi_0)\\
&\le\mathbb{E}\left(\liminf_{N\to\infty}e^{\int_0^{\tau'\wedge N} 2\Psi'-(n-1)(K-tn_k^2(\xi_s))-tn_k(\xi_s)(F_s+2\Psi)ds}\left|F_{\tau'\wedge N}-2\Psi(\xi_{\tau'\wedge N})\right|\right).
\end{aligned}$$

Since $\tau' < \infty$ almost surely by Lemma 3.6, we have $\tau' \wedge N \to \tau'$ as N approaches infinity. On the other hand, $F_{\tau'} = \Psi(\xi_{\tau'}) = 0$. Thus,

$$\begin{aligned}
&\mathbb{E}\left(\liminf_{N\to\infty} e^{\int_0^{\tau'\wedge N} 2\Psi' - (n-1)(K - tn_k^2(\xi_s)) - tn_k(\xi_s)(F_s + 2\Psi)ds} \left|F_{\tau'\wedge N} - 2\Psi(\xi_{\tau'\wedge N})\right|\right) \\
&= \mathbb{E}\left(e^{\int_0^{\tau'} 2\Psi' - (n-1)(K - tn_k^2(\xi_s)) - tn_k(\xi_s)(F_s + 2\Psi)ds} \left|F_{\tau'} - 2\Psi(\xi_{\tau'})\right|\right) = 0.
\end{aligned}$$

We conclude that

$$F_0 - 2\Psi(\xi_0) = \langle \nabla \log \phi_1(y), \gamma'(y)\rangle - \langle \nabla \log \phi_1(x), \gamma'(x)\rangle - 2\Psi(\rho(x,y)/2) \le 0.$$

Finally, we send $D' \to D$. The proof is complete. □

Remark 4.3. Our proof should be compared with that of Theorem 3.2 of [8]. In particular, we achieved two important improvements besides the handling of all the curvature terms. First, we did not assume the potential V is convex. Our proof with optional stopping technique does not require Ψ' to be non-positive (see remark 3.3 of [8]). Second, our argument does not require the first eigenfunction ϕ_1 to be log-concave, which was crucial in the proof in [8].

Remark 4.4. It is worth mentioning that at the critical point where $F_t = 2\Psi(\xi_t)$, the quantity $2\Psi' - (n-1)(K - tn_k^2(\xi_t)) - tn_k(\xi_t)(F_t + 2\Psi)$, which appears in (4.9) and later as the integrand in the exponential function becomes $2\Psi' - 4tn_k(\xi_t)\Psi - (n-1)(K - tn_k^2(\xi_t))$. The latter appeared in Theorem 3.8 of [17] (see also Remark 3.9 in the same reference) and was used in the proof of Theorem 1.6 of [17]. The diameter restriction $D < \pi/(2\sqrt{k})$ was coming from the need to make this quantity non-positive. In our proof above, this sign restriction, and hence the diameter restriction, was removed thanks to the optional stopping technique.

5. Fundamental gap comparison

In this section, we prove Theorem 1.2. Let $\phi_1,\ \phi_2$ be the eigenfunctions associated with the first two eigenvalues of the Schrödinger operator $-\Delta + V$ on Ω respectively. Recall that the ground state transform

$$v_t = \frac{e^{-\lambda_2 t}\phi_2}{e^{-\lambda_1 t}\phi_1} = e^{-(\lambda_2 - \lambda_1)t}\frac{\phi_2}{\phi_1} \tag{5.1}$$

is a smooth solution to

$$\begin{aligned}
\partial_t v &= Lv = \Delta v + 2\langle \nabla \log \phi_1, \nabla v\rangle \\
v(0,\cdot) &= \phi_2/\phi_1
\end{aligned} \tag{5.2}$$

on $\mathbb{R}^+ \times \bar{\Omega}$. We will use the coupling X_t, Y_t, from (3.1)–(3.4).

Proof of Theorem 1.2. Let $\rho_t = \rho(X_t, Y_t)$ and $\xi_t = \rho_t/2$. By Itô's formula,

$$\begin{aligned}
d\rho_t &= 2\sqrt{2}d\beta_t \\
&\quad + \left(\sum_{i=1}^n I(Q_i, Q_i) + 2\langle \nabla \log \phi_1(Y_t), \gamma'(Y_t)\rangle - 2\langle \nabla \log \phi_1(X_t), \gamma'(X_t)\rangle\right) dt,
\end{aligned} \tag{5.3}$$

where β_t is a Brownian motion independent from B_t. Let $\tilde{\phi}_1, \tilde{\phi}_2$ be the first two eigenfunctions of the corresponding one-dimensional model. Define $\Phi = \tilde{\phi}_2/\tilde{\phi}_1$ and let

$$F_t = \langle \nabla \log \phi_1(Y_t), \gamma'(Y_t)\rangle - \langle \nabla \log \phi_1(X_t), \gamma'(X_t)\rangle.$$

With (5.3) at hand, we apply Itô's formula again and get

$$d\Phi(\xi_t) = \sqrt{2}\Phi'(\xi_t)d\beta_t + \left(\frac{1}{2}\Phi'(\xi_t)\sum_{i=1}^{n} I(Q_i,Q_i) + \Phi'(\xi_t)F_t + \Phi''(\xi_t)\right)dt.$$

By our assumption, $\log\tilde{\phi}_1$ is a modulus of concavity of $\log\phi_1$. Thus, we have $F_t \leq 2(\log\tilde{\phi}_1)'(\xi_t)$. Moreover, by Index lemma

$$\frac{1}{2}\sum_{i=1}^{n} I(Q_i,Q_i) \leq -(n-1)tn_k(\xi_t).$$

As a result,

(5.4)
$$d\Phi(\xi_t) \leq \sqrt{2}\Phi'(\xi_t)d\beta_t + \Big(-\Phi'(\xi_t)(n-1)tn_k(\xi_t) + \Phi'(\xi_t)2(\log\tilde{\phi}_1)'(\xi_t) + \Phi''(\xi_t)\Big)\,dt.$$

It is easily checked that equation (2.7) of [17] still holds for the Schrödinger operator $-\Delta + V$. Hence, we have

$$\Phi''(s) - \Phi'(s)(n-1)tn_k(s) + \Phi'(s)2(\log\tilde{\phi}_1)'(s) = -(\bar{\lambda}_2 - \bar{\lambda}_1)\Phi(s). \tag{5.5}$$

Combining (5.4) and (5.5) gives

$$d\Phi(\xi_t) \leq \sqrt{2}\Phi'(\xi_t)d\beta_t - (\bar{\lambda}_2 - \bar{\lambda}_1)\Phi(\xi_t)dt,$$

which is equivalent to

$$d\left(e^{(\bar{\lambda}_2-\bar{\lambda}_1)t}\Phi(\xi_t)\right) \leq \sqrt{2}e^{(\bar{\lambda}_2-\bar{\lambda}_1)t}\Phi'(\xi_t)d\beta_t.$$

Integrating and taking expectation gives

$$\mathbb{E}\left(e^{(\bar{\lambda}_2-\bar{\lambda}_1)t}\Phi(\xi_t)\right) \leq \Phi(\xi_0),$$

or equivalently

$$\mathbb{E}\Phi(\xi_t) \leq e^{-(\bar{\lambda}_2-\bar{\lambda}_1)t}\Phi(\xi_0). \tag{5.6}$$

By Lemma 3.2, we know that the above inequality holds for all $t \geq 0$. Finally, by the definition (5.1) and Itô's formula, we have $v_t(x) = \mathbb{E}(v_0(X_t))$. Since v_0 is Lipschitz on $\bar{\Omega}$, we can find $K > 0$ such that

$$\begin{aligned} |v_t(x) - v_t(y)| &= |\mathbb{E}(v_0(X_t) - v_0(Y_t))| \leq \mathbb{E}\,|v_0(X_t) - v_0(Y_t)| \\ &\leq K\mathbb{E}\rho(X_t,Y_t) = 2K\mathbb{E}\xi_t. \end{aligned} \tag{5.7}$$

Since $\tilde{V}$ is even and (5.5) is the same as equation (2.7) of [17], Lemma 2.3 of [17] applies to Φ. Thus, Φ is increasing on $[0, D/2]$ with $\Phi'(0) > 0$; from which we deduce that there exists $c_1 > 0$, such that $\Phi(s) \geq c_1 s$ on $[0, D/2]$. Hence, we have from (5.6), that

$$c_1\mathbb{E}\xi_t \leq \mathbb{E}(\Phi(\xi_t)) \leq e^{-(\tilde{\lambda}_2-\tilde{\lambda}_1)t}\Phi(\xi_0).$$

Putting it together with (5.7) and the definition of v_t gives

$$e^{-(\lambda_2-\lambda_1)t}\,|v_0(x) - v_0(y)| = |v_t(x) - v_t(y)| \leq \frac{2K}{c_1}e^{-(\bar{\lambda}_2-\bar{\lambda}_1)t}\Phi(\xi_0).$$

Since v_0 is not constant function, we can find $x \neq y$ such that $|v_0(x) - v_0(y)| > 0$. Our result follows by sending $t \to \infty$. $\square$

Remark 5.1. This proof is in the flavor of parabolic proof of Theorem 4.1 in [17], but simplifies as we do not need to use the preservation of modulus of continuity, Theorem 4.6 in [17].

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References

[1] Ben Andrews and Julie Clutterbuck, *Proof of the fundamental gap conjecture*, J. Amer. Math. Soc. **24** (2011), no. 3, 899–916, DOI 10.1090/S0894-0347-2011-00699-1. MR2784332

[2] Fabrice Baudoin, Gunhee Cho, and Guang Yang, *A note on first eigenvalue estimates by coupling methods in Kähler and quaternion Kähler manifolds*, Electron. Commun. Probab. **27** (2022), Paper No. 12, 8, DOI 10.1214/22-ecp452. MR4386037

[3] Mu Fa Chen and Feng Yu Wang, *Application of coupling method to the first eigenvalue on manifold*, Sci. China Ser. A **37** (1994), no. 1, 1–14. MR1308707

[4] Mufa Chen and Fengyu Wang, *General formula for lower bound of the first eigenvalue on Riemannian manifolds*, Sci. China Ser. A **40** (1997), no. 4, 384–394, DOI 10.1007/BF02911438. MR1450586

[5] M. Cranston, *Gradient estimates on manifolds using coupling*, J. Funct. Anal. **99** (1991), no. 1, 110–124, DOI 10.1016/0022-1236(91)90054-9. MR1120916

[6] Xianzhe Dai, Shoo Seto, and Guofang Wei, *Fundamental gap estimate for convex domains on sphere—the case $n = 2$*, Comm. Anal. Geom. **29** (2021), no. 5, 1095–1125, DOI 10.4310/CAG.2021.v29.n5.a3. MR4349140

[7] Rick Durrett, *Probability—theory and examples*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 49, Cambridge University Press, Cambridge, 2019. Fifth edition of [MR1068527], DOI 10.1017/9781108591034. MR3930614

[8] Fuzhou Gong, Huaiqian Li, and Dejun Luo, *A probabilistic proof of the fundamental gap conjecture via the coupling by reflection*, Potential Anal. **44** (2016), no. 3, 423–442, DOI 10.1007/s11118-015-9476-3. MR3489850

[9] Chenxu He, Guofang Wei, and Qi S. Zhang, *Fundamental gap of convex domains in the spheres*, Amer. J. Math. **142** (2020), no. 4, 1161–1191, DOI 10.1353/ajm.2020.0033. MR4124117

[10] Elton P. Hsu, *Stochastic analysis on manifolds*, Graduate Studies in Mathematics, vol. 38, American Mathematical Society, Providence, RI, 2002, DOI 10.1090/gsm/038. MR1882015

[11] Elton P. Hsu and Karl-Theodor Sturm, *Maximal coupling of Euclidean Brownian motions*, Commun. Math. Stat. **1** (2013), no. 1, 93–104, DOI 10.1007/s40304-013-0007-5. MR3197874

[12] Ioannis Karatzas and Steven E. Shreve, *Brownian motion and stochastic calculus*, 2nd ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991, DOI 10.1007/978-1-4612-0949-2. MR1121940

[13] Wilfrid S. Kendall, *Nonnegative Ricci curvature and the Brownian coupling property*, Stochastics **19** (1986), no. 1-2, 111–129, DOI 10.1080/17442508608833419. MR864339

[14] Torgny Lindvall and L. C. G. Rogers, *Coupling of multidimensional diffusions by reflection*, Ann. Probab. **14** (1986), no. 3, 860–872. MR841588

[15] Bernt Øksendal, *Stochastic differential equations*, 6th ed., Universitext, Springer-Verlag, Berlin, 2003. An introduction with applications, DOI 10.1007/978-3-642-14394-6. MR2001996

[16] Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, 3rd ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999, DOI 10.1007/978-3-662-06400-9. MR1725357

[17] Shoo Seto, Lili Wang, and Guofang Wei, *Sharp fundamental gap estimate on convex domains of sphere*, J. Differential Geom. **112** (2019), no. 2, 347–389, DOI 10.4310/jdg/1559786428. MR3960269

[18] Feng-Yu Wang, *Gradient estimates and the first Neumann eigenvalue on manifolds with boundary*, Stochastic Process. Appl. **115** (2005), no. 9, 1475–1486, DOI 10.1016/j.spa.2005.04.004. MR2158015

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Vanishing results from Lichnerowicz Laplacian on complete Kähler manifolds and applications

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ABSTRACT

In this paper, we show several rigidity results for harmonic (p, q)-forms in complete Kähler manifolds. We also give several applications to study non-compact Kähler manifolds with parallel Bochner tensor or quaternion Kähler manifolds. Our results are natural extensions of Petersen and Wink's results in [13,15] in the setting of complete, non-compact Kähler manifolds.

1. Introduction

This paper is a next study conducted by the authors (see [3]) in the Kähler manifold setting.

Denote $\Delta_L = \nabla^*\nabla + c\,\mathrm{Ric}$ as Lichnerowicz Laplacian for $c > 0$. The precise description of Δ_L and all necessary definitions and notations are described in Section 2. As given in Section 2, let $\mathfrak{R}_{|\mathfrak{u}(m)}$ be Kähler curvature operator, then the first result of this paper is stated as follows.

Theorem 1.1. *Let (M, g) be a complete, non-compact Kähler manifold. Assume that $g(\mathfrak{R}_{|\mathfrak{u}(m)}(T^{\mathfrak{u}}), \overline{T}^{\mathfrak{u}})$ is nonnegative for every harmonic $(0, k)$-tensor T. Then every harmonic tensor T (with respect to the Lichnerowicz Laplacian) is parallel if $|T| \in L^Q(M)$ for some $Q \geq 2$.*

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With a presence of a negative lower bound of $g(\mathfrak{R}_{|\mathfrak{u}(m)}(T^{\mathfrak{u}}), \overline{T}^{\mathfrak{u}})$, we need some more assumption that the weighted Poincaré inequality holds ([8, Def 0.1]): for M^n an n-dimensional complete Riemannian manifold, we say that M satisfies a weighted Poincaré inequality with a nonnegative weight function ρ on M, if the inequality

$$\int_M \rho(x)\phi^2(x)dV \leq \int_M |\nabla\phi|^2 dV$$

is valid for any compactly supported smooth function $\phi \in C_0^\infty(M)$. We put two additional hypotheses on ρ,

$$\liminf_{x\to\infty} \rho(x) > 0, \tag{1.1}$$

and

$$M \text{ is nonparabolic,} \tag{1.2}$$

i.e., there exists a symmetric positive Green's function $G(x, y)$ for the Laplacian acting on L^2 functions (otherwise, we say M is parabolic). All assumptions on a weight function ρ can be regarded as the generalization of the positivity condition of the first Dirichlet eigenvalue $\lambda_1(M)$ [8]. We refer the interested readers to [8] for several examples regarding Riemannian manifolds with weighted Poincaré inequality and also see [4] to the study of minimal hypersurfaces. Some other examples are given by Minerbe in [12]. Moreover, Kähler manifolds with weighted Poincaré inequality are also investigated in [9,11].

Our second result is formulated as follows.

Theorem 1.2. *Let (M, g) be a connected complete non-compact Kähler manifold. Assume that M satisfies a weighted Poincaré inequality with a nonnegative weight function ρ with (1.1) and (1.2), and also $g(\mathfrak{R}_{|\mathfrak{u}(m)}(T^{\mathfrak{u}}), \overline{T}^{\mathfrak{u}}) \geqslant -\kappa\rho|T|^2$ for all $(0, k)$-tensors T, where $\kappa \geq 0$ is given. Then every harmonic tensor T (with respect to the Lichnerowicz Laplacian) vanishes provided that $|T| \in L^Q(M), Q \geq 2$ and $0 \leqslant \kappa < \frac{4(Q-1)}{cQ^2}$.*

As a consequence of Theorem 1.2, we have the following classification theorem which is motivated by the work of Bryant [1] and Kamishima [6] in classifying compact and complete Bochner flat manifolds.

Theorem 1.3. *Suppose that (M, g) is a complete Kähler non-compact manifold of complex dimension n satisfying that the Bochner tensor is divergence free. If M satisfies a weighted Poincaré inequality and*

$$\mu_1 + \ldots + \mu_{\lfloor \frac{n+1}{2} \rfloor} + \frac{1 + (-1)^n}{4}\mu_{\lfloor \frac{n+1}{2} \rfloor + 1} \geq -k\rho$$

then the Bochner tensor vanishes provided that its L^Q-norm is finite and $0 \leq k < \frac{Q-1}{Q^2}, Q \geq 2$. Consequently, if we assume furthermore that M is the complete Bochner flat geometry as in [6] then M must be one of the following forms:

1. *Complex Euclidean geometry* $(\mathbb{C}^n \rtimes \mathrm{U}(n), \mathbb{C}^n, g_{\mathbb{C}}, J_{\mathbb{C}})$,
2. *Product of complex hyperbolic and projective geometry* $(\mathrm{PU}(m, 1) \times \mathrm{PU}(n - m + 1), \mathbb{H}^m_{\mathbb{C}} \times \mathbb{CP}^{n-m}, g_{\mathbb{H}} \times g_{\mathbb{CP}}, J_{\mathbb{H}} \times J_{\mathbb{CP}}), m = 0, \ldots, n$,
3. *Intransitive Kähler geometry* $(\mathbb{C}^{n-k} \rtimes \mathrm{U}(n - k)) \times \mathrm{U}(\ell_1, \ldots, \ell_m), \mathbb{C}^n, \hat{g}_a, J_{\mathbb{C}})$, $k \geq 1$.

Moreover, if the first two cases hold and M is irreducible then M is of constant holomorphic sectional curvature. Consequently, M is $\mathbb{C}^n$, $\mathbb{CP}^n$, $\mathbb{B}^n$, or their quotients. Here we refer the reader to [6] for the notation of Bochner flat geometry and models.

Combining the vanishing results as in Theorem 1.2 and a classification of Kähler-Ricci solitons with vanishing Bochner tensor [16], we obtain the following proposition.

Proposition 1.4. *Suppose that (M,g) is a complete gradient Kähler-Ricci soliton of complex dimension n satisfying that the Bochner tensor is divergence free. If M satisfies a weighted Poincaré inequality and*

$$\mu_1+\ldots+\mu_{\lfloor\frac{n+1}{2}\rfloor}+\frac{1+(-1)^n}{4}\mu_{\lfloor\frac{n+1}{2}\rfloor+1}\geq -k\rho$$

then the Bochner tensor vanishes provided that its L^Q-norm is finite and $0\leq k<\frac{Q-1}{Q^2}, Q\geq 2$. Consequently, M is $\mathbb{C}^n$, $\mathbb{CP}^n$, $\mathbb{B}^n$, or their quotients.

The analogous theorem of the non-compact quaternion manifolds is also obtained.

Theorem 1.5. *Let (M,g) be a connected complete non-compact quaternion Kähler manifold. Assume that M satisfies a weighted Poincaré inequality with a nonnegative weight function ρ with (1.1) and (1.2), and also $g(\mathfrak{R}_{|\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}(T^{\mathfrak{sp}}),\overline{T}^{\mathfrak{sp}})\geqslant -\kappa\rho|T|^2$ for all $(0,k)$-tensors T, where $\kappa\geq 0$ is given. Then every harmonic tensor T (with respect to the Lichnerowicz Laplacian) vanishes provided that $|T|\in L^Q(M), Q\geq 2$ and $0\leqslant\kappa<\frac{4(Q-1)}{cQ^2}$.*

The paper is organized as follows. In Section 2, we recall some basic important facts on Bochner techniques on Kähler manifolds which are inspired by the work of Petersen and Wink. In Section 3, we derive vanishing results and prove Theorems 1.1-1.2 and Theorem 1.5. Geometric applications are introduced in Section 4. In particular, we give a proof of Theorem 1.3 and Proposition 1.4.

2. Preliminaries

2.1. Tensors

We collect the necessary ingredients from [15, Section 1]. Let (V,g) be an n-dimensional Euclidean vector space. The metric g induces a metric on $\otimes^k V^*$ and $\wedge^k V$ in the way that if $\{e_i\}_{i=1,\cdots,n}$ is an orthonormal basis for V, then $\{e_{i_1}\wedge\cdots\wedge e_{i_k}\}_{1\leq i_1<\cdots<i_k\leq n}$ is an orthonormal basis for $\wedge^k V$.

$\wedge^2 V$ inheries a Lie algebra structure from $\mathfrak{so}(V)$, and the induced Lie algebra action on V is given by

$$(X\wedge Y)Z=g(X,Z)Y-g(Y,Z)X.$$

In particular, for $\Xi_\alpha,\Xi_\beta\in\wedge^2 V$,

$$(\Xi_\alpha)\Xi_\beta=[\Xi_\alpha,\Xi_\beta].$$

Definition 2.1. Let $V_{\mathbb{C}}=V\bigotimes_{\mathbb{R}}\mathbb{C}$. For a complex valued, $\mathbb{R}$-multilinear tensor T on V, namely, $T\in\bigotimes^r V_{\mathbb{C}}^*$ and $L\in\mathfrak{so}(V)$ set

$$LT(X_1,\ldots,X_r)=-\sum_{i=1}^r T(X_1,\ldots,LX_i,\ldots,X_r).$$

If $\mathfrak{g}\subset\mathfrak{so}(V)$ is a Lie subalgebra, we define $T^{\mathfrak{g}}\in\left(\bigotimes^r V_{\mathbb{C}}^*\right)\bigotimes_{\mathbb{R}}\mathfrak{g}$

$$g(L,T^{\mathfrak{g}}(X_1,\ldots,X_r))=LT(X_1,\ldots,X_r)$$

for all $L\in\mathfrak{g}\subset\mathfrak{so}(V)=\Lambda^2 V$.

A tensor $\mathrm{Rm} \in \otimes^4 V^*$ is an algebraic curvature tensor if

$$\mathrm{Rm}(X,Y,Z,W) = -\mathrm{Rm}(Y,X,Z,W) = -\mathrm{Rm}(X,Y,W,Z) = \mathrm{Rm}(Z,W,X,Y),$$
$$\mathrm{Rm}(X,Y,Z,W) + \mathrm{Rm}(Y,Z,X,W) + \mathrm{Rm}(Z,X,Y,W) = 0.$$

In particular, it induces the curvature operator $\mathfrak{R} : \wedge^2 V \to \wedge^2 V$ via

$$g(\mathfrak{R}(X \wedge Y), Z \wedge W) = \mathrm{Rm}(X,Y,Z,W).$$

The associated symmetric bilinear form is denoted by $R \in \mathrm{Sym}^2_{\mathrm{B}}(\wedge^2 V)$. We have

$$|\mathrm{Rm}|^2 = 4|R|^2.$$

The curvature operator $\mathfrak{R}$ of a Riemannian manifold (M,g) vanishes on the complement of the holonomy algebra $\mathfrak{hol}$. In particular, it induces $\mathfrak{R}_{|\mathfrak{hol}} : \mathfrak{hol} \to \mathfrak{hol}$ and the corresponding curvature tensor $R \in \mathrm{Sym}^2_{\mathrm{B}}(\mathfrak{hol})$. If $\mathfrak{hol} = \mathfrak{u}(m)$, then (M,g) is Kähler and the operator $\mathfrak{R}_{|\mathfrak{u}(m)} : \mathfrak{u}(m) \to \mathfrak{u}(m)$ is called Kähler curvature operator and the associated $R \in \mathrm{Sym}^2_{\mathrm{B}}(\mathfrak{u}(\mathrm{m}))$ is the Kähler curvature tensor. Every Kähler curvature tensor on a Kähler manifold (M,g,J) with the complex structure J satisfies

$$\mathrm{Rm}(X,Y,Z,W) = \mathrm{Rm}(JX,JY,Z,W) = \mathrm{Rm}(X,Y,JZ,JW).$$

If $\mathfrak{hol} = \mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$, then (M,g) is quaternion Kähler and the operator $\mathfrak{R}_{|\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)} : \mathfrak{sp}(m) \oplus \mathfrak{sp}(1) \to \mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$ is called quaternion Kähler curvature operator and the associated $R \in \mathrm{Sym}^2_{\mathrm{B}}(\mathfrak{sp}(\mathrm{m}) \oplus \mathfrak{sp}(1))$ is the quaternion Kähler curvature tensor.

Moreover, if $\mathfrak{R} : \mathfrak{g} \to \mathfrak{g}$ is a self-adjoint operator with orthonormal eigenbasis $\{\Xi_\alpha\}$ and corresponding eigenvalues $\{\lambda_\alpha\}$, then

$$\mathfrak{R}(T^{\mathfrak{g}}) = \mathfrak{R} \circ T^{\mathfrak{g}} = \sum_\alpha \mathfrak{R}(\Xi_\alpha) \otimes \Xi_\alpha T,$$

as a consequence, we have

$$g(\mathfrak{R}(T^{\mathfrak{g}}), \overline{T}^{\mathfrak{g}}) = \sum_\alpha \lambda_\alpha |\Xi_\alpha T|^2$$

and in particular,

$$|T^{\mathfrak{g}}|^2 = \sum_\alpha |\Xi_\alpha T|^2.$$

In case $\mathfrak{g} = \mathfrak{u}(n)$ and $\mathfrak{g} = \mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$, we will write $T^{\mathfrak{u}}$ and $T^{\mathfrak{sp}}$ to simplify notation respectively.

2.2. Lichnerowicz Laplacian and the Bochner technique

Let (M,g) be a n-dimensional Riemannian manifold and denote $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$ as $(1,3)$- Riemmanian curvature tensor. We denote by $\mathcal{T}^{(0,k)}(M)$ the vector bundle of $(0,k)$-tensors on M. Recall that the *Weitzenböck curvature operator* on a tensor $T \in \mathcal{T}^{(0,k)}(M)$ is defined by

$$\mathrm{Ric}(T)\left(X_1, \ldots, X_k\right) = \sum_{i=1}^k \sum_{j=1}^n \left(R\left(X_i, e_j\right) T\right)\left(X_1, \ldots, e_j, \ldots, X_k\right).$$

For $c > 0$ the Lichnerowicz Laplacian is given by

$$\Delta_L = \nabla^* \nabla + c\,\mathrm{Ric}\,.$$

A tensor $T \in \mathcal{T}^{(0,k)}(M)$ is called harmonic if $\Delta_L T = 0$.

Example 2.2. There are some important example of Lichnerowicz Laplacian for different $c > 0$.

(a) The Hodge Laplacian is a Lichnerowicz Laplacian for $c = 1$.
(b) For $c = \frac{1}{2}$ the Riemmannian curvature tensor Rm is harmonic if it is divergence free. The fact is that Rm is divergence free if and only if its Ricci tensor is a Codazzi tensor, in this case its scalar curvature is constant. If the manifold is Einstein, its Ricci tensor is always Codazzi. Therefore, the curvature tensors of Einstein manifolds are harmonic.

Proposition 2.3. *Let $\mathfrak{R} : \Lambda^2 TM \to \Lambda^2 TM$ denote the curvature operator of (M, g). If $\mathfrak{g} \subset \mathfrak{so}(m)$ denotes the holonomy algebra, then $\mathfrak{R}_{|\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}, \mathfrak{R}_{|\mathfrak{g}^\perp} = 0$ and*

$$g(\mathrm{Ric}(T), \overline{T}) = g(\mathfrak{R}_{|\mathfrak{g}}(T^{\mathfrak{g}}), \overline{T}^{\mathfrak{g}})$$

for every $T \in \bigotimes^r T^*_{\mathbb{C}} M$.

Recall the Bochner formula for a tensor $T \in \mathcal{T}^{(0,k)}(M)$

$$\Delta \frac{1}{2}|T|^2 = |\nabla T|^2 - g\left(\nabla^* \nabla T, \overline{T}\right).$$

Therefore, if T is harmonic, then $\nabla^* \nabla T = -c\,\mathrm{Ric}\,T$. Hence, this together with Proposition 2.3 implies

$$\Delta \frac{1}{2}|T|^2 = |\nabla T|^2 + c \cdot g(\mathrm{Ric}(T), \overline{T}).$$

Lemma 2.4. *Let (V, g) be a Euclidean vector space, $\mathfrak{g} \subset \mathfrak{so}(V)$ a Lie subalgebra and let $\mathfrak{R} : \mathfrak{g} \to \mathfrak{g}$ be self-adjoint with eigenvalues $\mu_1 \leq \ldots \leq \mu_{\dim \mathfrak{g}}$. Let $T \in \bigotimes^r V^*_{\mathbb{C}}$. Suppose there is $C \geq 1$ such that*

$$|LT|^2 \leq \frac{1}{C}|T^{\mathfrak{g}}|^2 |L|^2$$

for all $L \in \mathfrak{g}$. Let $1 \leq \ell \leq \lfloor C \rfloor$ be an integer and let $\kappa \leq 0$.

1. *If $\mu_1 + \ldots + \mu_\ell + (C - \ell)\mu_{\ell+1} \geq \kappa(\ell + 1)$, then $g(\mathfrak{R}(T^{\mathfrak{g}}), \overline{T}^{\mathfrak{g}}) \geq \frac{\kappa(\ell+1)}{C}|T^{\mathfrak{g}}|^2$,*
2. *if $\mu_1 + \ldots + \mu_\ell + (C - \ell)\mu_{\ell+1} > 0$, then $g(\mathfrak{R}(T^{\mathfrak{g}}), \overline{T}^{\mathfrak{g}}) > 0$, unless $T^{\mathfrak{g}} = 0$.*

Proof. See [13, Lemma 1.8]. □

Let $V = \mathbb{C}^n$ and consider the natural $U(n)$-action on V. Denote by

$$\Lambda^{p,0} V^* = \Lambda^p V^* = \mathrm{span}_{\mathbb{C}}\{dz^{i_1} \wedge \ldots \wedge dz^{i_p} | 1 \leq i_1 < \ldots < i_p \leq n\}$$

the space of complex linear p-forms, by

$$\Lambda^{0,q} V^* = \Lambda^q \overline{V^*} = \mathrm{span}_{\mathbb{C}}\{d\bar{z}^{i_1} \wedge \ldots \wedge d\bar{z}^{i_q} | 1 \leq i_1 < \ldots < i_q \leq n\}$$

the space of conjugate linear q-form, and by

$$\Lambda^{p,q}V^* = \Lambda^{p,0}V^* \otimes_{\mathbb{C}} \Lambda^{0,q}V^*$$

the space of (p,q)-forms.

For $0 \le k \le \min\{p,q\}$ set

$$V_k^{p,q} = \Lambda^{p-k,0}V^* \otimes_{\mathbb{C}} \mathrm{span}\{\Omega^k\} \otimes_{\mathbb{C}} \Lambda^{0,q-k}V^*,$$

where Ω is the Kähler form.

Definition 2.5. For $\varphi \in \Lambda^{p,q}V^*$, set

$$\mathring{\varphi} = \begin{cases} \varphi - \frac{g(\varphi,\Omega^p)}{|\Omega^p|}\Omega^p & \text{if } p = q \\ \varphi & \text{if } p \neq q \end{cases}$$

The following proposition calculates $\varphi^{\mathfrak{u}}$ in term of $\mathring{\varphi}$.

Proposition 2.6. *Let $k \le \min\{p,q\}$ and $\varphi \in \Lambda_k^{p,q}V^*$. We have*

$$|\varphi^{\mathfrak{u}}| = [2(p-k)(q-k) + (p+q-2k)((n+1)-(p+q-2k))]|\mathring{\varphi}|^2.$$

The next proposition allows us to estimate $|LT|^2$ for various types of tensors

Proposition 2.7. *Suppose that $\varphi \in V_k^{p,q}$. It follows that*

$$|L\varphi|^2 \le (p+q-2k)|L|^2|\mathring{\varphi}|^2$$

for all $L \in \mathfrak{u}(V)$.

Proof. See [13, Proposition 3.4]. □

For $k \le \min\{p,q\}$, if $p+q-2k \neq 0$, let

$$C_k^{p,q} = n + 1 - (p+q) + 2\frac{pq-k^2}{p+q-2k},$$

and

$$C^{p,q} = n + 1 - \frac{p^2+q^2}{p+q}.$$

Here, we can assume $p+q \le n$ due to Serre duality.

Proposition 2.8. *Let $k \le \min\{p,q\}$ with $p+q-2k > 0$. Let $\kappa \le 0$ and let $\mathfrak{R} : \mathfrak{u}(V) \to \mathfrak{u}(V)$ be a Kähler curvature operator with eigenvalues $\mu_1 \le \ldots \le \mu_{n^2}$. Let $\varphi \in \Lambda_k^{p,q}V^*$.*

1. *If $\mu_1 + \ldots + \mu_{\lfloor C_k^{p,q} \rfloor} + (C_k^{p,q} - \lfloor C_k^{p,q} \rfloor)\mu_{\lfloor C_k^{p,q} \rfloor + 1} \ge \kappa(\lfloor C_k^{p,q})\rfloor + 1)$ then*

$$g(\mathfrak{R}(\varphi^{\mathfrak{u}}), \overline{\varphi}^{\mathfrak{u}}) \ge \kappa(\lfloor C_k^{p,q} \rfloor + 1)(p+q-2k)|\mathring{\varphi}|^2.$$

2. *If* $\mu_1 + \ldots + \mu_{\lfloor C_k^{p,q} \rfloor} + (C_k^{p,q} - \lfloor C_k^{p,q} \rfloor)\mu_{\lfloor C_k^{p,q} \rfloor + 1} > 0$ *then*

$$g(\mathfrak{R}(\varphi^{\mathfrak{u}}), \overline{\varphi}^{\mathfrak{u}}) > 0$$

unless $\varphi = 0$.

3. *If* $\mu_1 + \ldots + \mu_{\lfloor C^{p,q} \rfloor} + (C^{p,q} - \lfloor C^{p,q} \rfloor)\mu_{\lfloor C^{p,q} \rfloor + 1} \geq \kappa(\lfloor C^{p,q}) \rfloor + 1)$ *then*

$$g(\mathfrak{R}(\varphi^{\mathfrak{u}}), \overline{\varphi}^{\mathfrak{u}}) \geq \kappa(n + 2 - |p - q|)(p + q)|\mathring{\varphi}|^2.$$

Proof. See [13, Proposition 3.6] and [13, Corollary 3.7]. □

We note that, if $\mu_1 + \ldots + \mu_{\lfloor C^{p,q} \rfloor} + (C^{p,q} - \lfloor C^{p,q} \rfloor)\mu_{\lfloor C^{p,q} \rfloor + 1} \geq \kappa(\lfloor C^{p,q}) \rfloor + 1)$, then by [13] (see the Proof of Theorem B-D), we have

$$g(\mathrm{Ric}\varphi, \overline{\varphi}) \geq \kappa(n + 2 - |p - q|)(p + q)|\mathring{\varphi}|^2.$$

Combining all above discussion, we have the following Bochner formula:

Lemma 2.9. *Let* (M, g) *be a complete non-compact Kähler manifold of complex dimension* n*, let* $\kappa \leq 0$*. Suppose that* φ *is a harmonic* (p, q)*-forms. If*

$$\mu_1 + \ldots + \mu_{\lfloor C^{p,q} \rfloor} + (C^{p,q} - \lfloor C^{p,q} \rfloor)\mu_{\lfloor C^{p,q} \rfloor + 1} \geq \kappa(\lfloor C^{p,q}) \rfloor + 1),$$

then we have

$$\Delta \frac{1}{2}|\varphi|^2 \geq |\nabla \varphi|^2 + \kappa(n + 2 - |p - q|)(p + q)|\mathring{\varphi}|^2.$$

Lemma 2.10. *Every algebraic Kähler curvature tensor* $R \in Sym^2_B(\mathfrak{u}(n))$ *satisfies*

$$|\mathfrak{R}^{\mathfrak{u}}|^2 = 4(n + 1)|\mathring{R}|^2 - 4|\mathring{\mathrm{Ric}}|^2.$$

In particular, $|\mathfrak{R}^{\mathfrak{u}}|^2 = 0$ *if and only if* R *has constant holomorphic sectional curvature.*

Proof. [15, Lemma 5.2]. □

2.3. Quaternion Kähler manifold

A Riemannian manifold with holonomy contained in $Sp(m) \cdot Sp(1), m \geq 2$ is called quaternion Kähler manifold. Locally there exist almost complex structures I, J, K such that $IJ = -JI = K$. For a local orthonormal frame $\{e_i, Ie_i, Je_i, Ke_i\}_{i=1,\cdots,m}$, consider

$$\omega_I = \sum_{i=1}^{m} e_i \wedge Ie_i + Je_i \wedge Ke_i,$$

$$\omega_J = \sum_{i=1}^{m} e_i \wedge Je_i + Ke_i \wedge Ie_i,$$

$$\omega_K = \sum_{i=1}^{m} e_i \wedge Ke_i + Ie_i \wedge Je_i.$$

It is straightforward to check that

$$g(IX,Y) = g(X\wedge Y,\omega_I), g(JX,Y) = g(X\wedge Y,\omega_J), g(KX,Y) = g(X\wedge Y,\omega_K).$$

The curvature operator of quaternionic projective space is given by

$$\begin{aligned}\mathfrak{R}_{\mathbb{HP}^m}(X\wedge Y) &= X\wedge Y + IX\wedge IY + JX\wedge JY + KX\wedge KY \\ &\quad + 2g(X\wedge Y,\omega_I)\omega_I + 2g(X\wedge Y,\omega_J)\omega_J + 2g(X\wedge Y,\omega_K)\omega_K.\end{aligned}$$

The curvature operator $R\in Sym^2_B(TM)$ of a quaternion Kähler manifold satisfies

$$R = \frac{scal}{16m(m+2)}R_{\mathbb{HP}^m} + R_0,$$

where R_0 is the hyper-Kähler component. Hyper-Kähler manifolds are necessarily Ricci-flat.

Lemma 2.11. *Let $m\geq 2$. An algebraic quaternion Kähler curvature tensor $R\in Sym^2_B(\mathfrak{sp}(m)\oplus\mathfrak{sp}(1))$ satisfies*

$$|\mathfrak{R}^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}|^2 = \frac{4}{3}(3m+4)|R_0|^2$$

In particular, $\mathfrak{R}^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)} = 0$ if and only if R has constant quaternionic sectional curvature.

Proof. [15, Corollary 4.5]. □

2.4. Kähler manifolds with divergence free Bochner tensors

Let (M,J,g) be a Kähler manifold of real dimension $2m$. Let $\omega(X,Y) = g(JX,Y)$ denote the Kähler form and $\rho(X,Y) = \mathrm{Ric}(JX,Y)$ denote the Ricci form. The trace-free Ricci tensor is $\overset{\circ}{\mathrm{Ric}} = \mathrm{Ric} - \frac{\mathrm{scal}}{2m}g$ and the primitive part of the Ricci form is $\rho_0 = \rho - \frac{\mathrm{scal}}{2m}\omega$.

The curvature tensor decomposes into a Kähler curvature tensor with constant holomorphic sectional curvature, a Kähler curvature tensor with trace-free Ricci curvature and the Bochner tensor,

$$\begin{aligned}\mathrm{Rm} =& \frac{\mathrm{scal}}{4m(m+1)}\left(\frac{1}{2}g \owedge g + \frac{1}{2}\omega\owedge\omega + 2\omega\otimes\omega\right) \\ &+ \frac{1}{2(m+2)}\left(\overset{\circ}{\mathrm{Ric}}\owedge g + \rho_0\owedge\omega + 2(\rho_0\otimes\omega + \omega\otimes\rho_0)\right) + B.\end{aligned}$$

The Bochner tensor is totally trace-free [17]. That is, if $e_1,\cdots,e_{2m}$ is an orthonormal basis of TM, then

$$\sum_{i=1}^{2m} B(e_i,Y,e_i,W) = \sum_{i=1}^{2m} B(e_i,Je_i,Z,W) = 0.$$

Proposition 2.12. *Let (M,g) be a Kähler manifold. If the Bochner tensor is divergence free, then it satisfies the second Bianchi identity and consequently*

$$\nabla^*\nabla B + c\,\mathrm{Ric}(B) = 0.$$

Proof. [13, Proposition 3.2]. □

3. Vanishing results

In this section, we assume that the curvature term $g(\mathfrak{R}_{|\mathfrak{g}}(T^{\mathfrak{g}}), \overline{T}^{\mathfrak{g}}) \geqslant -\kappa|T|^2$ for every harmonic tensor T and for some $\kappa \geqslant 0$. We note that in the rest of this paper, we always assume that $Q \geq 2$. For $\kappa = 0$, we give a proof of Theorem 1.1 as follows.

Proof of Theorem 1.1. For T is a harmonic tensor, recall that the Bochner formula for harmonic tensor T implies

$$\Delta \frac{1}{2}|T|^2 = |\nabla T|^2 + c \cdot g(\mathfrak{R}_{|\mathfrak{g}}(T^{\mathfrak{g}}), \overline{T}^{\mathfrak{g}}).$$

Since $g(\mathfrak{R}_{|\mathfrak{g}}(T^{\mathfrak{g}}), \overline{T}^{\mathfrak{g}}) \geqslant 0$, same arguments from [3] yield the proof. $\square$

Proof of Theorem 1.2, 1.5. Using the assumption $g(\mathfrak{R}_{|\mathfrak{g}}(T^{\mathfrak{g}}), \overline{T}^{\mathfrak{g}}) \geqslant -\kappa\rho|T|^2$, we obtain that

$$c\kappa \int\limits_M \rho\varphi^2|T|^{q+2} \geqslant 2\int\limits_M \varphi|T|^{q+1}\langle\nabla\varphi, \nabla|T|\rangle + (q+1)\int\limits_M \varphi^2|T|^q|\nabla|T||^2.$$

From same arguments from [3], for any $\kappa < \frac{4(Q-1)}{cQ^2}$ and $R > 0$, it holds that

$$\int\limits_M \varphi^2|T|^{Q-2}|\nabla|T||^2 \leqslant \frac{4C}{R^2}\int\limits_M |T|^Q,$$

where $C = C(\varepsilon, q) > 0$ and ε is sufficient small real number. Let $R \to \infty$ and since $|T| \in L^Q(M)$, then $|T|$ is constant on each connected component of M. From the hypothesis (1.2), the volume of M is infinite [8, Corollary 3.2]. By $|T| \in L^Q(M)$, it follows that $|T| = 0$. Therefore, $T \equiv 0$. The proof is complete. $\square$

Remark 3.1. If a refined Kato inequality

$$|\nabla T|^2 \geq (1+a)|\nabla|T||^2$$

holds true then as in [3], we can improve the upper bound of κ to be

$$\kappa < \frac{4(Q-1+a)}{cQ^2}.$$

Theorem 3.2. *Let (M, g) be a complete non-compact Kähler manifold of complex dimension n. If $p \neq q$ and*

$$\mu_1 + \ldots + \mu_{\lfloor C^{p,q}\rfloor} + (C^{p,q} - \lfloor C^{p,q}\rfloor)\mu_{\lfloor C^{p,q}\rfloor+1} \geq 0,$$

then every harmonic (p, q)-form is parallel.

In particular, if $\mu_1 + \ldots + \mu_{\lfloor C^{p,q}\rfloor} + (C^{p,q} - \lfloor C^{p,q}\rfloor)\mu_{\lfloor C^{p,q}\rfloor+1} > 0$, then there are no nontrivial (p, q)-harmonic forms with finite $L^Q (Q \geq 2)$-norm on M.

Proof. Let ω be a harmonic (p, q)-form with $|\omega| \in L^Q(M)$ for some $Q \geq 2$. Applying Proposition 2.9 and Proposition 2.7, we obtain that

$$|L\omega|^2 \leq (p+q-2k)|\overset{\circ}{\omega}|^2|L|^2 = \frac{1}{C_k^{p,q}}|\omega^{\mathfrak{u}}|^2|L|^2 \tag{3.1}$$

for all $L \in \mathfrak{so}(TM)$.

If $\mu_1 + \ldots + \mu_{\lfloor C^{p,q} \rfloor} + (C^{p,q} - \lfloor C^{p,q} \rfloor)\mu_{\lfloor C^{p,q} \rfloor + 1} \geq 0$, then the first conclusion of Lemma 2.4 implies

$$g(\Re(\hat{\omega}), \hat{\omega}) \geq 0.$$

An application of Theorem 1.1 to Hodge Laplacian yields that ω is parallel. Moreover, $|\omega|$ is constant. Now, we can follow the argument as in Theorem 1.1 to complete the proof. □

We note that if $p = q$ then by Definition 2.5, we have $\mathring{\omega} = \omega$ if ω and Ω are perpendicular. Hence, using the proof of Theorem 3.2, we obtain the following result.

Theorem 3.3. *Let (M, g) be a complete non-compact Kähler manifold of complex dimension n. If*

$$\mu_1 + \ldots + \mu_{\lfloor C^{p,p} \rfloor} + (C^{p,p} - \lfloor C^{p,p} \rfloor)\mu_{\lfloor C^{p,p} \rfloor + 1} \geq 0,$$

then every harmonic (p,p)-form ω is parallel provided that $\omega \perp \Omega$.

In particular, if $\mu_1 + \ldots + \mu_{\lfloor C^{p,p} \rfloor} + (C^{p,p} - \lfloor C^{p,p} \rfloor)\mu_{\lfloor C^{p,p} \rfloor + 1} > 0$, then there are no nontrivial (p,p)-harmonic forms ω with finite $L^Q (Q \geq 2)$-norm on M provided that $\omega \perp \Omega$.

Following [2], the above result has a reduced L^2 cohomology interpretation as follows. Let $\mathcal{H}^\ell(M)$ be the space of L^2 harmonic ℓ-forms, saying $\mathcal{H}^\ell(M) = \{\omega \in L^2(\Lambda^\ell T^*M) : d\omega = \delta\omega = 0\}$, where δ is the dual of the differential operator d and $Z_2^\ell(M)$ the kernel of the unbounded operator d acting on $L^2(\Lambda^\ell T^*M)$, or equivalently

$$Z_2^\ell(M) = \{\omega \in L^2(\Lambda^\ell T^*M) : d\omega = 0\}.$$

The space $\mathcal{H}^\ell(M)$ can be used to characterize the *reduced L^2 cohomology group* as follows

$$\mathcal{H}^\ell(M) = Z_2^\ell(M)/\overline{dC_0^\infty(L^2(\Lambda^{\ell-1}T^*M)},$$

where the closure is taken with respect to the L^2 topology. It is worth to note that the finiteness of $\dim\mathcal{H}^\ell(M)$ depends only on the geometry of ends ([10]). Observe that

$$\Lambda^\ell T^*M = \underset{p+q=\ell}{\otimes} \Lambda^{p,q}T^*M.$$

Therefore, Theorem 3.2 leads immediately to the following result.

Corollary 3.4. *Let $n \geq 3$ and let (M, g) be a complete non-compact n-dimensional Kähler manifold. Then every harmonic (p,q)-form ω with $|\omega| \in L^Q(M)$ for some $Q \geq 2$ is vanishing if the curvature tensor is $\lceil \frac{n}{2} \rceil$-nonnegative. In particular, every harmonic (p,q)-form ω with $|\omega| \in L^2(M)$ is vanishing, consequently, every reduced L^2 cohomology groups $\mathcal{H}^\ell(M)$ are trivial if ℓ is odd.*

Remark 3.5. We recall that a k-form ω is said to be a harmonic field if $(d + d^*)\omega = 0$, where d is the differential operator and d^* its dual operator. If M is compact then the harmonic fields coincide with the harmonic forms. When M is Kähler and ω is a (p,q)-form, it is easy to prove that ω is a harmonic field if and only if $\partial\omega = \partial^*\omega = \overline{\partial}\omega = \overline{\partial}^*\omega = 0$. We would like to mention that there is a refined Kato inequality for (p,q)-harmonic field ω (see [5]), namely, there exists a constant $D^{p,q} \geq 0$ such that

$$|\nabla\omega|^2 \geq \frac{1}{D^{p,q}}|\nabla|\omega||^2,$$

where

$$D^{p,q} = \begin{cases} \min\left\{\max\left\{\frac{2p+1}{2p+2}, \frac{2n-2p+1}{2n-2p+2}\right\}, \max\left\{\frac{2q+1}{2q+2}, \frac{2n-2q+1}{2n-2q+2}\right\}\right\}^2 & p, q \neq n \\ \frac{1}{2} & p = n \text{ or } q = n. \end{cases}$$

Hence, Remark 3.1 implies an improvement of the upper bound of κ as follows

$$0 < \kappa < \frac{4(Q + \frac{1}{D^{p,q}} - 3)}{cQ^2}.$$

Moreover, it is proved in Proposition 4.5 (see also Remark 3.12) in [5] that if $\omega \in L^2(M)$ then ω is a harmonic field if and only if ω is harmonic.

The next results with a general curvature condition are a direct consequence of Theorem 1.2 and the above Kato inequality.

Theorem 3.6. *Let (M, g) be a complete non-compact Kähler manifold of complex dimension n. Assume that M satisfies a weighted Poincaré inequality with a nonnegative weight function ρ with (1.1) and (1.2). Denote $\mu_1 \leq \ldots \leq \mu_{n^2}$ eigenvalues of the curvature operator of (M, g). For $p \neq q$, if*

$$\frac{\mu_1 + \ldots + \mu_{\lfloor C^{p,q} \rfloor} + (C^{p,q} - \lfloor C^{p,q} \rfloor)\mu_{\lfloor C^{p,q} \rfloor + 1}}{C^{p,q} + 1} \geq -\kappa\rho$$

then every (p, q)-harmonic field ω vanishes provided that $|\omega| \in L^Q(M)$ and

$$0 < \kappa < \frac{4\left(Q + \frac{1}{D^{p,q}} - 3\right)}{(n + 2 - |p - q|(p + q))Q^2}.$$

Proof. Since ω is a (p, q)-harmonic field, using the Bochner formula in Lemma 2.9, we obtain that

$$\Delta\frac{1}{2}|\varphi|^2 \geq |\nabla\varphi|^2 + \kappa(n + 2 - |p - q|)(p + q)\rho|\mathring{\varphi}|^2.$$

Following the proof of Theorem 1.2, Remark 3.1 and Remark 3.5, we complete the proof. □

Observe that if $\omega \in L^2(M)$ and ω is harmonic then ω is a harmonic field. Therefore, Remark 3.5 and Theorem 3.6 imply the following corollary.

Corollary 3.7. *Let (M, g) be a complete non-compact Kähler manifold of complex dimension n. Assume that M satisfies a weighted Poincaré inequality with a nonnegative weight function ρ with (1.1) and (1.2). Denote $\mu_1 \leq \ldots \leq \mu_{n^2}$ eigenvalues of the curvature operator of (M, g). For $p \neq q$, if*

$$\frac{\mu_1 + \ldots + \mu_{\lfloor C^{p,q} \rfloor} + (C^{p,q} - \lfloor C^{p,q} \rfloor)\mu_{\lfloor C^{p,q} \rfloor + 1}}{C^{p,q} + 1} \geq -\kappa\rho$$

then every L^2 harmonic (p, q)-form ω vanishes provided that

$$0 < \kappa < \frac{\frac{1}{D^{p,q}} - 1}{n + 2 - |p - q|(p + q)}.$$

When $p = q$, we also can obtain a vanishing result as follows.

Corollary 3.8. *Let (M,g) be a complete non-compact Kähler manifold of complex dimension n. Assume that M satisfies a weighted Poincaré inequality with a nonnegative weight function ρ with (1.1) and (1.2). Denote $\mu_1 \leq \ldots \leq \mu_{n^2}$ eigenvalues of the curvature operator of (M,g). If*

$$\frac{\mu_1 + \ldots + \mu_{\lfloor C^{p,p} \rfloor} + (C^{p,p} - \lfloor C^{p,p} \rfloor)\mu_{\lfloor C^{p,p} \rfloor + 1}}{C^{p,p}+1} \geq -\kappa\rho$$

then every L^2 harmonic (p,p)-form ω vanishes provided that $\omega \perp \Omega$ and

$$0 < \kappa < \frac{\frac{1}{D^{p,q}} - 1}{n+2}.$$

Finally, the above corollary infers following vanishing result for reduced L^2 cohomology groups.

Corollary 3.9. *Let (M,g) be a complete non-compact Kähler manifold of complex dimension n. Assume that M satisfies a weighted Poincaré inequality with a nonnegative weight function ρ with (1.1) and (1.2). Denote $\mu_1 \leq \ldots \leq \mu_{n^2}$ eigenvalues of the curvature operator of (M,g). For $p \neq q$, if*

$$\frac{\mu_1 + \ldots + \mu_{\lfloor C^{p,q} \rfloor} + (C^{p,q} - \lfloor C^{p,q} \rfloor)\mu_{\lfloor C^{p,q} \rfloor + 1}}{C^{p,q}+1} \geq -\kappa\rho$$

then every harmonic (p,q)-form ω, for all $1 \leq \ell \leq n-1$ vanishes provided that $|\omega| \in L^2(M)$ for some κ satisfying

$$\kappa < \frac{\frac{1}{D^{p,q}} - 1}{n+2-|p-q|(p+q)}.$$

Consequently, every reduced L^2-cohomology groups $\mathcal{H}^\ell(M)$ are trivial if ℓ is odd.

4. Geometric applications

Theorem 4.1. *Suppose that (M,g) is a complete non-compact Kähler-Einstein manifold of complex dimension $n \geq 4$. If M satisfies a weighted Poincaré inequality and*

$$\mu_1 + \ldots + \mu_{\lfloor \frac{n+1}{2} \rfloor} + \frac{1+(-1)^n}{4}\mu_{\lfloor \frac{n+1}{2} \rfloor + 1} \geq -k\rho$$

then M is Riemannian flat provided that its L^Q-norm is finite, where $0 \leq k < \frac{Q-1}{Q^2}$, $Q \geq 2$.

Proof. As we mentioned in Example 2.2.(b) that the curvature tensor of Einstein manifold is harmonic with respect to Lichnerowicz Laplacian $\Delta_L = \nabla^*\nabla + \frac{1}{2}\,\mathrm{Ric}$.

Since M is Einstein, $\overset{\circ}{\mathrm{Ric}} = 0$, so Lemma 2.10 follows that $|\mathrm{Rm}^{\mathfrak{u}}|^2 = 4(n+1)|\overset{\circ}{\mathrm{Rm}}|^2$ and thus Lemma 2.2 in [13] implies

$$|L\mathrm{Rm}|^2 \leq 8|\overset{\circ}{\mathrm{Rm}}|^2|L|^2 = \frac{2}{n+1}|\mathrm{Rm}^{\mathfrak{u}}|^2|L|^2, \tag{4.1}$$

for all $L \in \mathfrak{u}(n)$. By the assumption on the eigenvalues of the Kähler curvature operator and Lemma 2.4 implies

$$g(\mathrm{Ric}(\mathrm{Rm}), \overline{\mathrm{Rm}}) = g(\mathfrak{R}_{|\mathfrak{u}}(\mathrm{Rm}^{\mathfrak{u}}), \overline{\mathrm{Rm}^{\mathfrak{u}}}) \geq -\frac{2k\rho}{n+1}|\mathrm{Rm}^{\mathfrak{u}}|^2 \geq -8k\rho|\mathrm{Rm}|^2.$$

Here, we used $|\mathrm{Rm}^{\mathfrak{u}}|^2 = 4(n+1)|\overset{\circ}{\mathrm{Rm}}|^2 \leq 4(n+1)|\mathrm{Rm}|^2$ since M is Einstein. Applying Theorem 1.2, we obtain that Rm is vanishing. This means M is flat. □

Proof of Theorem 1.3. Combining Lemma 2.2 in [13] and Lemma 5.2 in [14], we have

$$|LB|^2 \leq 8|B|^2|L|^2 = \frac{2}{n+1}|B^{\mathfrak{u}}|^2|L|^2, \tag{4.2}$$

for all $L \in \mathfrak{u}(n)$. By the assumption on the eigenvalues of the Kähler curvature operator and Lemma 2.4 implies

$$g(\Re(B^{\mathfrak{u}}), B^{\mathfrak{u}}) \geq -\frac{2k}{n+1}|B^{\mathfrak{u}}|^2 \geq -8k\rho|B|^2.$$

Applying Theorem 1.2, B is vanishing. This means M is Bochner flat, hence, using the classification of Bochner flat geometry as in Theorem 5.7 in [6], we complete the proof of the first conclusion.

Now, suppose that M is either complex Euclidean geometry; or product of complex heperbolic and projective geometry. Then, M has constant scalar curvature. By Theorem 1.1 in [7], this together with the fact that M has vanishing Bochner tensor and is irreducible implies M is Kähler-Einstein. Hence, due to Theorem 1.2 in [16], M has constant holomorphic sectional curvature. The proof is complete. □

Recall that an n-dimensional Kähler manifold (M^n, g) is called a gradient Kähler-Ricci soliton if there is a real-valued smooth function f satisfying the soliton equation

$$R_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} f = \lambda g_{i\bar{j}},$$

for some constant $\lambda \in \mathbb{R}$ and such that ∇f is a holomorphic vector field, i.e. $\nabla_i \nabla_j f = 0$. Following the proof of Theorem 1.3, we obtain the below proposition.

Proof of Proposition 1.4. Since the Bochner tensor has finite L^Q-norm, it must be vanishing. Hence the proof now follows by Theorem 1.2 in [16]. □

The final application is a rigidity result on quaternionic Kähler manifolds.

Theorem 4.2. *Suppose that (M, g) is a complete non-compact quaternion Kähler manifold of complex dimension $4m \geq 8$. Let $\mu_1 \leq \ldots \leq \mu_{m(2m+1)+3}$ denote the eigenvalues of the corresponding quaternion Kähler curvature operator. Suppose that the scalar curvature of M is vanishing. If M satisfies a weighted Poincaré inequality and*

$$\mu_1 + \ldots + \mu_{\lfloor \frac{m+1}{2} \rfloor} + \frac{5 + 3.(-1)^m}{12}\mu_{\lfloor \frac{m+1}{2} \rfloor + 1} \geq -k\rho$$

then M is Riemannian flat provided that the curvature tensor R has finite L^Q-norm, where $0 \leq k < \frac{Q-1}{Q}$, $Q \geq 2$.

Proof. Quaternion Kähler manifolds in real dimension $4m \geq 8$ are Einstein. Hence the curvature tensor R is harmonic and thus satisfies the Bochner formula

$$\Delta \frac{1}{2}|R|^2 = |\nabla R|^2 + \frac{1}{2} \cdot g(\Re(R^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}), R^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}).$$

By Lemma 2.11 and Lemma 2.2 in [13], we have

$$|LR|^2 = |LR_0|^2 \leq 8|L|^2|R_0|^2 = \frac{6}{3m+4}|R^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}|^2|L|^2, \tag{4.3}$$

for all $L \in \mathfrak{sp}(m)\oplus\mathfrak{sp}(1)$. By the assumption on the eigenvalues of the quaternion Kähler curvature operator and Lemma 2.4 implies

$$g(\mathfrak{R}(R^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}), R^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}) \geq -\frac{-6k\rho}{3m+4}|R^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}|^2 \geq -8k\rho|R_0|^2 = -8k\rho|R|^2.$$

Here in the last inequality, we used the assumption that the scalar curvature is zero to infer $|R_0| = |R|$ (see the proof of Corollary 4.5 in [15]). Applying Theorem 1.2, $R = 0$. Hence, M must be flat. □

Remark 4.3. If $k = 0$, we always have

$$g(\mathfrak{R}(R^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}), R^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}) \geq -\frac{-6k\rho}{3m+4}|R^{\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)}|^2 = 0.$$

Hence, we can remove the condition on scalar curvature in the statement of Theorem 4.2.

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References

[1] R. Bryant, Bochner–Kähler metrics, J. Am. Math. Soc. 14 (3) (2001) 623–715.
[2] G. Carron, L^2 harmonics forms on non compact manifolds, in: Proceedings of the Centre for Mathematics and its Applications, 2002, pp. 49–59.
[3] G.H. Cho, N.T. Dung, T.Q. Huy, Rigidity results with new curvature conditions and applications, Preprint, arXiv:2111.07496v2.
[4] X. Cheng, D.T. Zhou, Manifolds with weighted Poincaré inequality and uniqueness of minimal hypersurfaces, Commun. Anal. Geom. (2009) 139–154.
[5] D. Cibotaru, P. Zhu, Refined Kato inequalities for harmonic fields on Kähler manifolds, Pac. J. Math. 1 (2012) 51–66.
[6] Y. Kamishima, Heisenberg, spherical CR-geometry and Bochner flat cocally conformal Kähler manifolds, Int. J. Geom. Methods Mod. Phys. 3 (5 & 6) (2006) 1089–1116.
[7] Jaeman Kim, On Kähler manifolds with harmonic Bochner curvature tensor, Ann. Glob. Anal. Geom. 35 (4) (2009) 339–343.
[8] P. Li, J.P. Wang, Weighted Poincaré inequality and rigidity of complete manifolds, Ann. Sci. Éc. Norm. Supér. (4) 39 (2006) 921–982.
[9] P. Li, J.P. Wang, Connectedness at infinity of complete Kähler manifolds, Am. J. Math. 131 (3) (2009) 771–817.
[10] J. Lott, L^2-cohomology of geometrically infinite hyperbolic 3-manifold, Geom. Funct. Anal. 7 (1997) 81–119.
[11] O. Munteanu, Two results on the weighted Poincar’e inequality on complete Kähler manifolds, Math. Res. Lett. 14 (6) (2007) 995–1008.
[12] V. Minerbe, Weighted Sobolev inequalities and Ricci flat manifolds, Geom. Funct. Anal. 18 (5) (2009) 1696–1749.
[13] P. Petersen, M. Wink, New curvature conditions for the Bochner technique, Invent. Math. 224 (2021) 33–54.
[14] P. Petersen, M. Wink, Vanishing and estimation results for Hodge numbers, J. Reine Angew. Math. 2021 (780) (2021) 197–219, https://doi.org/10.1515/crelle-2021-0036.

[15] P. Petersen, M. Wink, Tachibana-type theorems and special holonomy, Ann. Glob. Anal. Geom. 61 (2022) 847–868.
[16] Y. Su, K. Zhang, On the Kähler-Ricci solitons with vanishing Bochner-Weyl tensor, Acta Math. Sci. 32B (3) (2012) 1239–1244.
[17] Shun-ichi Tachibana, On the Bochner curvature tensor, Nat. Sci. Rep. Ochanomizu Univ. 18 (1967) 15–19.