Projective Space

Ji, Yong-hyeon

October 17, 2024

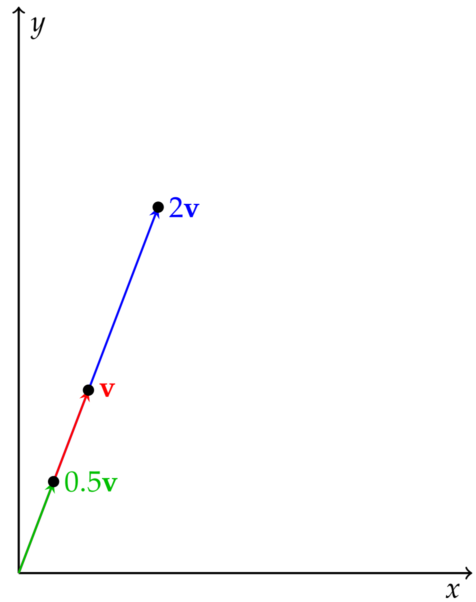
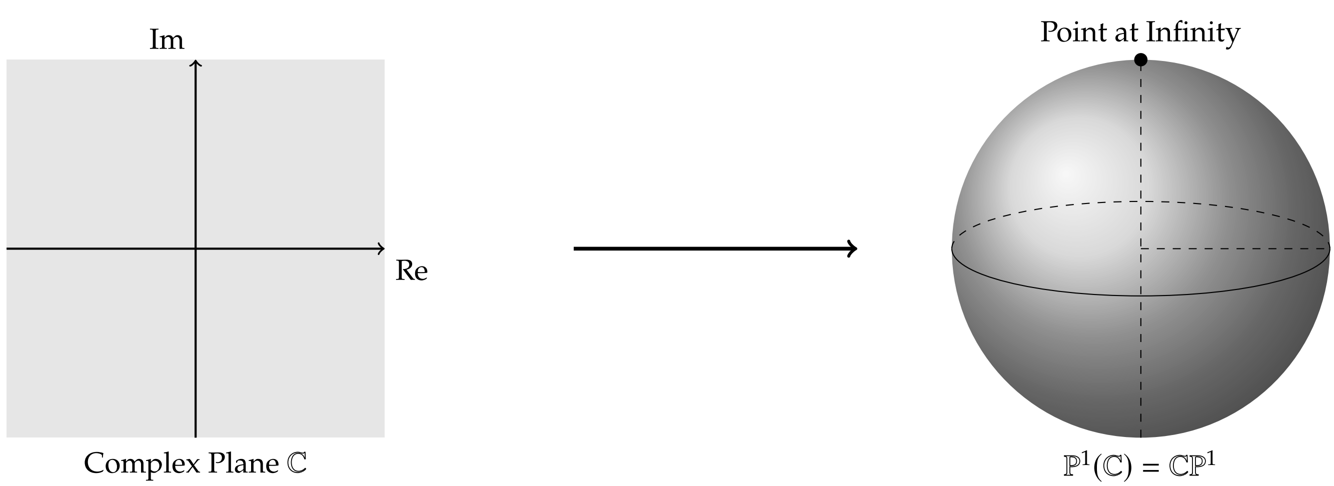


Figure 1: In $\mathbb{R}^2$ (or $\mathbb{C}^1$), vectors $\mathbf{v}$, $2\mathbf{v}$, and $0.5\mathbf{v}$ lie on the same line.



Projective Space

Definition. Let $\mathbb{K}$ be a field and $n \geq 0$. The **n-dimensional projective space** over $\mathbb{K}$, denoted $\mathbb{P}(\mathbb{K}^{n+1})$, is defined as the quotient:

$$\mathbb{P}(\mathbb{K}^{n+1}) = (\mathbb{K}^{n+1} \setminus \{0\})/\sim,$$

where $\sim$ is an equivalence relation defined on $\mathbb{K}^{n+1} \setminus \{0\}$ by

$$\mathbf{v} \sim \mathbf{w} \iff \exists \lambda \in \mathbb{K} \setminus \{0\} \text{ such that } \mathbf{v} = \lambda \mathbf{w}.$$

Remark. $\mathbb{P}^n(\mathbb{K})$ is the set of lines through the origin in $\mathbb{K}^{n+1}$. These lines represent equivalence classes of non-zero vectors in $\mathbb{K}^{n+1}$, where two vectors are equivalent if the y are scalar multiples of each other.

Note (Notation)**.** The projective space $\mathbb{P}(\mathbb{K}^{n+1})$ has dimension n.

- $\mathbb{P}(\mathbb{K}^{n+1}) = \mathbb{P}^n(\mathbb{K})$
- $\mathbb{P}(\mathbb{K}^{n+1}) = \mathbb{K}\mathbb{P}^n$