< Compute the sectional curvatures of hyperbolic 3-manifold >

Def) $\mathbb{H} = \{ (x^1, x^2, y) \in \mathbb{R}^3 \mid y > 0 \}$ endowed with the metric

$$g = \left(\frac{1}{y^2}\right) \cdot \left((dx^1)^2 + (dx^2)^2 + (dy)^2 \right).$$

$$[g_{ij}] = \begin{bmatrix} 1/y^2 & 0 & 0 \\ 0 & 1/y^2 & 0 \\ 0 & 0 & 1/y^2 \end{bmatrix} = \left(\frac{1}{y^2}\right) \cdot I_{3\times3} .$$

We hold $(\mathbb{H}, g)$. called poincaré half-space model.

Compute the christoffel symbols Γ^k_{ij}. $(i, j, k \leq 3)$

Using koszul's formula ;

$$\Gamma^k_{ij} = \frac{1}{2} \cdot g^{kl} \cdot \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$$

$$([g^{kl}] = \begin{bmatrix} y^2 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & y^2 \end{bmatrix} = y^2 \cdot I_{3\times3}.$$

(Next page)

which means $g^{kl} = \begin{cases} y^2 & k=l \\ 0 & \text{else.} \end{cases}$

$$\Gamma^k_{ij} = \frac{1}{2}\cdot(y^2)\cdot\left[\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right]$$

And if $(i.j.k)$ are all different. $g_{ij} = g_{jk} = g_{ik} = 0$ so $\Gamma^k_{ij} = 0$.

$$\Gamma^1_{11} = \frac{1}{2}\cdot(y^2)\left[\partial_1 g_{11}\right] = 0.$$

$$\Gamma^2_{11} = \frac{1}{2}(y^2)\left[2\cdot\partial_1 g_{12} - \partial_2 g_{11}\right] = 0$$

$$\Gamma^3_{11} = \frac{1}{2}(y^2)\cdot\left[2\cdot\partial_1 g_{13} - \partial_3 g_{11}\right] = \frac{1}{\cancel{2}}(\cancel{y^2})\cdot\left(\frac{\cancel{2}}{y\cancel{^2}}\right) = \underbrace{(1/y)}.$$

$$\Gamma^1_{12} = \frac{1}{2}(y^2)\left[\partial_1 \cancel{g_{12}} + \partial_2 g_{11} - \partial_1 \cancel{g_{12}}\right] = 0$$

$$\Gamma^2_{12} = \frac{1}{2}(y^2)\left[\partial_1 g_{22} + \partial_2 \cancel{g_{12}} - \partial_2 \cancel{g_{12}}\right] = 0$$

$$\Gamma^1_{13} = \frac{1}{2}(y^2)\left[\partial_1 \cancel{g_{13}} + \partial_3 g_{11} - \partial_1 \cancel{g_{13}}\right] = \underbrace{(-1/y)}$$

$$\Gamma^3_{13} = \frac{1}{2}(y^2)\left[\partial_1 g_{33} + \partial_3 \cancel{g_{13}} - \partial_3 \cancel{g_{13}}\right] = 0.$$

$$\Gamma^1_{22} = \frac{1}{2}(y^2)\left[\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}\right] = 0$$

$$\Gamma^2_{22} = \frac{1}{2}(y^2)\left[\partial_2 g_{22} + \partial_2 \cancel{g_{22}} - \partial_2 \cancel{g_{22}}\right] = 0$$

$$\Gamma^3_{22} = \frac{1}{2}(y^2)\left[\cancel{2\cdot\partial_2 g_{23}} - \partial_3 g_{22}\right] = \underbrace{(1/y)}$$

$$\Gamma^2_{23} = \frac{1}{2}(y^2)\left[\partial_2 \cancel{g_{23}} + \partial_3 g_{22} - \partial_2 \cancel{g_{23}}\right] = \underbrace{(-1/y)}$$

$$\Gamma^3_{23} = \frac{1}{2}(y^2)\left[\partial_2 g_{33} + \cancel{\partial_3 g_{23}} - \cancel{\partial_3 g_{23}}\right] = 0$$

$$\Gamma^1_{33} = \frac{1}{2}(y^2)\left[\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33}\right] = 0$$

$$\Gamma^2_{33} = \frac{1}{2}(y^2)\left[\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}\right] = 0$$

$$\Gamma^3_{33} = \frac{1}{2}y^2\left[\partial_3 g_{33} + \partial_3 \cancel{g_{33}} - \partial_3 \cancel{g_{33}}\right] = \underbrace{(-1/y)}$$

(Next page)

Sumarize them all,

$\Gamma^3_{11} = 1/y$, $\Gamma^3_{22} = 1/y$,

$\Gamma^1_{13} = \Gamma^1_{31} = -1/y$, $\Gamma^2_{23} = \Gamma^2_{32} = -1/y$. $\Gamma^3_{33} = -1/y$.

And the othe Γ^k_{ij} 's are all zero. (Note that $\Gamma^k_{ij} = \Gamma^k_{ji}$, symmetry).

Next. we compute the sectional curvatures $K(\partial_i, \partial_j)$.

$$K(\partial_1, \partial_3) = \frac{Rm(\partial_1, \partial_3, \partial_3, \partial_1)}{\|\partial_1\|^2 \|\partial_3\|^2 - \langle \partial_1, \partial_3 \rangle^2}$$

$$= (y^4) \cdot \langle R(\partial_1, \partial_3)\partial_3, \partial_1 \rangle$$ (we know $[g] = (1/y^2) \cdot I_{3\times 3}$).

$$R(\partial_1, \partial_3)\partial_3 = \nabla_{\partial_1} \nabla_{\partial_3} \partial_3 - \nabla_{\partial_3} \nabla_{\partial_1} \partial_3 - \nabla_{[\partial_1, \partial_3]} \partial_3 .$$

$$= \nabla_{\partial_1}\left(-\frac{1}{y} \cdot \partial_3\right) - \nabla_{\partial_3}\left(-\frac{1}{y} \partial_1\right)$$

$$= \left(-\frac{1}{y}\right) \cdot \left(-\frac{1}{y}\right) \partial_1 + \left(\left(-\frac{1}{y^2}\right) \partial_1 + \left(\frac{1}{y}\right)\left(-\frac{1}{y} \partial_1\right) \right)$$

$$= -\frac{1}{y^2} \partial_1 .$$

$$\Rightarrow K(\partial_1, \partial_3) = (y^4) \cdot \left\langle -\frac{1}{y^2} \partial_1, \partial_1 \right\rangle$$

$$= (-y^2) \cdot (1/y^2)$$

$$= -1 .$$

$\Rightarrow K(\partial_1, \partial_3) = -1$

We remained to compute $K(\partial_1 . \partial_2)$. $K(\partial_2 . \partial_3)$.
which gonna be in similar way.

$$K(\partial_1 . \partial_2) = \frac{Rm(\partial_1 . \partial_2 . \partial_2 . \partial_1)}{\|\partial_1\|^2 \|\partial_2\|^2 - \langle \partial_1 . \partial_2 \rangle^2}$$

$$= (y^4) \cdot \langle R(\partial_1 . \partial_2)\partial_2 . \partial_1 \rangle$$

$$R(\partial_1 . \partial_2)\partial_2 = \nabla_{\partial_1} \nabla_{\partial_2} \partial_2 - \nabla_{\partial_2} \nabla_{\partial_1} \partial_2 - \nabla_{[\partial_1, \partial_2]} \partial_2$$

$$= \nabla_{\partial_1}\left(\tfrac{1}{y} \partial_3\right) - \nabla_{\partial_2}(0)$$

$$= \left(\tfrac{1}{y}\right)\left(-\tfrac{1}{y}\partial_1\right)$$

$$= -\tfrac{1}{y^2} \partial_1 .$$

$$\Rightarrow K(\partial_1 . \partial_2) = (y^4) \langle -\tfrac{1}{y^2}\partial_1 . \partial_1 \rangle$$

$$= (y^4)\left(-\tfrac{1}{y^2}\right)\left(\tfrac{1}{y^2}\right)$$

$$= -1.$$

$$K(\partial_2 . \partial_3) = \frac{Rm(\partial_2 . \partial_3 . \partial_3 . \partial_2)}{\|\partial_2\|^2 \|\partial_3\|^2 - \langle \partial_2 . \partial_3 \rangle^2}$$

$$= (y^4) \langle R(\partial_2 \partial_3)\partial_3 . \partial_2 \rangle.$$

$$R(\partial_2 . \partial_3)\partial_3 = \nabla_{\partial_2} \nabla_{\partial_3} \partial_3 - \nabla_{\partial_3} \nabla_{\partial_2} \partial_3 - \nabla_{[\partial_2 . \partial_3]} \partial_3$$

$$= \nabla_{\partial_2}\left(-\tfrac{1}{y}\partial_3\right) - \nabla_{\partial_3}\left(-\tfrac{1}{y}\partial_2\right)$$

$$= \left(-\tfrac{1}{y}\right)\left(-\tfrac{1}{y}\partial_2\right) + \left(\left(-\tfrac{1}{y^2}\right)\partial_2 + \left(\tfrac{1}{y}\right)\left(-\tfrac{1}{y}\partial_2\right)\right)$$

$$= -\tfrac{1}{y^2}\partial_2 .$$

$$\Rightarrow K(\partial_2 . \partial_3) = (y^4) \langle -\tfrac{1}{y^2}\partial_2 . \partial_2 \rangle$$

$$= (y^4)\left(-\tfrac{1}{y^2}\right)\left(\tfrac{1}{y^2}\right)$$

$$= -1.$$