Claim: (M, g) : Riemannian manifold with Levi-Civita connection $\tilde{\nabla}$.

(N, h) : $N \subset M$, submanifold. h = induced metric.

∇ : Levi-Civita connection of (N, h).

$p \in N$, $\mathrm{prj}_{T_pN} : T_pM \to T_pN$ as $T_pN \subset T_pM$, g-orthogonal projection.

$X, Y \in \mathfrak{X}(N)$, $\tilde{X}, \tilde{Y} \in \mathfrak{X}(M)$ identical to X, Y around p.

$$\Rightarrow \mathrm{prj}_{T_pN}\left(\tilde{\nabla}_{\tilde{X}}\tilde{Y}\right)_p = \left(\nabla_X Y\right)_p \ ?$$

$(\tilde{\nabla}_{\tilde{X}}\tilde{Y})_p$ depends only on $\tilde{X}_p = X_p$, $\tilde{Y}_p = Y_p$.

$\therefore$ $\tilde{\nabla}_{\tilde{X}}\tilde{Y}\big|_N$ depends only on X, Y.

Define $\nabla : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)$ by

$$\nabla_X Y = \mathrm{prj}_{T_pN}\left(\tilde{\nabla}_{\tilde{X}}\tilde{Y}\right).$$

We will show ∇ is the Levi-Civita connection on N. The uniqueness of the Levi-Civita connection proves the claim.

1. ∇ is an affine connection.

prj_{T_pN} is linear. $\Rightarrow$ ∇ : affine connection

2. compatible with the metric

For $X, Y, Z \in \mathfrak{X}(N)$,

$$X \cdot h(Y,Z) = \tilde{X} \cdot g(\tilde{Y}, \tilde{Z})$$

$$= g(\tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z}) + g(\tilde{Y}, \tilde{\nabla}_{\tilde{X}} \tilde{Z})$$

Because $\tilde{Z} \in T_pN$, $g(\tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z}) = g(\mathrm{prj}_{T_pN} \tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z})$.

$$= g(\mathrm{prj}_{T_pN} \tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z}) + g(\tilde{Y}, \mathrm{prj}_{T_pN} \tilde{\nabla}_{\tilde{X}} \tilde{Z})$$

$$= h(\nabla_X Y, Z) + h(Y, \nabla_X Z).$$

3. torsion free.

$$\nabla_X Y - \nabla_Y X = \mathrm{prj}_{T_pN} \tilde{\nabla}_{\tilde{X}} \tilde{Y} - \mathrm{prj}_{T_pN} \tilde{\nabla}_{\tilde{Y}} \tilde{X}$$

$$= \mathrm{prj}_{T_pN} (\tilde{\nabla}_{\tilde{X}} \tilde{Y} - \tilde{\nabla}_{\tilde{Y}} \tilde{X})$$

$$= \mathrm{prj}_{T_pN} [\tilde{X}, \tilde{Y}]$$

$$= [X, Y] \leftarrow \quad \tilde{X} = X,\ \tilde{Y} = Y \text{ on } N.$$

$\therefore$ ∇ is the Levi-Civita connection on (N, h).