Note Title 06-09-2021

MA-222

Lec-16

Recall: Cyclic gp.

$H = \{a^n \mid n \in \mathbb{Z}\}$

Defn:

$$\underline{\langle S \rangle} = \left\{ s_1^{\alpha_1} \cdots s_r^{\alpha_r} \;\middle|\; \begin{array}{l} r \in \mathbb{N} \\ s_i \in S \\ \alpha_i \in \mathbb{Z} \end{array} \right\}$$

$S = \{a\}$

Then $\langle S \rangle = \underline{\langle a \rangle}$.

$HK \le G$ iff $HK = \underline{\langle H, K \rangle}$.

Defn: Order of an elt a in G.
is the <u>smallest</u> natural no n such that $a^n = e$. If no such n exists then we say a is of <u>infinite</u> order.

Equivalently $o(a) = o(\underline{\langle a \rangle})$ order of the cyclic subgp gen by a.

$G = \{e\}$. $o(e) = 1$ in any gp.

$o(a) = 1$ iff $a = e$.

$G = \mathbb{Z}$ $o(1) =$

$\exists ?\, n \in \mathbb{N}$ s.t. $\underbrace{1+1+ \cdots +1}_{n\text{-times}} = 0$

$\nexists$ any such n.

$\Rightarrow o(1)$ in $\mathbb{Z}$ is ∞.

$o(m) = \infty \quad \forall\, m \in \mathbb{Z}. \quad m \neq 0.$

$$\mathbb{Z}_n = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}.$$

$$\boxed{o(\bar{1}) = \underline{n}}$$

find n s.t.

$$\underbrace{\bar{1} + \bar{1} + \dots + \bar{1}}_{\underline{n}\text{ times}} = \bar{0}$$

is this smallest?

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}.$$

$$1 + 1 + 1 + 1 + 1 = 5 = \bar{0}$$

In $\underline{\underline{\mathbb{Z}_6}}$

$o(\bar{1}) = 6$ $\quad$ $o(\bar{4}) = 3$ $\quad$ $= ((1+1+\dots+1)$

$o(\bar{2}) = 3$ $\quad$ $o(\bar{5}) = 6$ $\quad$ $\underbrace{2+2+\dots+2}_{6\text{ times}} = 0$ $\quad$ $2+2+2=0$

$o(\bar{3}) = 2$ $\quad$ $o(\overline{-1})$

$$U_6 \qquad \text{identity} = 1. \qquad \xi = e^{2\pi i/6}$$

$$\{1, \xi, \xi^2, \xi^3, \xi^4, \xi^5\} \qquad \xi^6 = e^{2\pi i \, 6/6} = 1$$

$$o(\xi) = 6$$

$$o(\xi^2) = o(e^{2\pi i \, 2/6}) = 3 \qquad o(U_6) = 6$$

$$\xi^n = \xi^r$$

$$o(\xi^3) = 2 \qquad \langle \xi^2 \rangle \overset{?}{=} \langle \xi^4 \rangle$$

$$o(\xi^4) = 3$$

$$o(\xi^5) = 6$$

$$o(\xi^6) = o(1) = 1$$

$\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$

$o(1) = 12$
$o(11) = 12$
$\vdots$

$\mathbb{R}$. $o(x) = \infty$ $x \neq 0$

In $\mathbb{Q}$, $\exists$? $a \in \mathbb{Q}$ s.t. $o(a) < \infty$ & $a \neq 0$? ✓

H.W. Que: Find an infinite gp which all elts are of

* finite order?
* order 2?

X : set $\quad \mathcal{P}(X)$: power set

$A * B = A \Delta B$ symmetric diff. $(A \setminus B) \cup (B \setminus A)$

Ex: Show that $(\mathcal{P}(X), \Delta)$ is an Abe. gp.

- What is identity?
- What is inverse of A.
- What is order of A $(A \neq id)$
- Is $\mathcal{P}(X)$ is cyclic?

What are all the subgps of G?

Thm: Let G be cyclic gp of order n. Let a be a gen. of G

$G = \langle a \rangle$. Let $b \in G$. Write $b = a^r$.

Let $H = \langle b \rangle$. Then $o(H)$ is $\frac{n}{d}$ where

d is the gcd of (r, n).

$\langle a^r \rangle = \langle a^s \rangle$ iff $\gcd(r, n) = \gcd(s, n)$.

Cor:

Cor: $o(G) = n$ cyclic then G has $\varphi(n)$ no. of generators. In fact if a is a gen. then a^r is also a gen when $(r, n) = 1$. (What if $r > n$?)

Find all gen. of $\mathbb{Z}_{18}$, U_{18}

Is $\mathbb{Z}_{18}^*$ cyclic? If yes find all gen. → DCP.

$5\mathbb{Z} = \{0, \pm 5, \pm 10, \ldots\}$.

$A_1 = \{\ldots, -9, -4, 1, 6, 11, 16, \ldots\}$.

$A_2 = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$.

$A_3 = \{ \quad \}$

$A_4 = \{ \quad \}$

* They all are distinct & disjoint
* Their union is $\mathbb{Z}$
* Exactly one of them is a subgp of $\mathbb{Z}$.

Then all other sets can be "obtained" using this subgp by shifting/adding a fixed elt.