Note Title 03-09-2021

MA - 222

Lec - 15

Subgp of a cyclic gp is cyclic.

Coro: All subgps of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$.

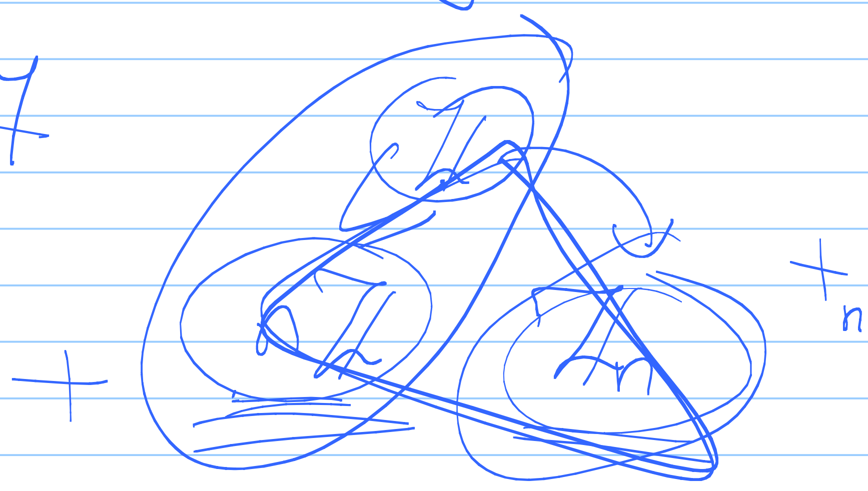
$$n\mathbb{Z} = \{ nk \mid k \in \mathbb{Z} \} \leq \mathbb{Z}$$

$H \leq \mathbb{Z}$. $\qquad H = \{0\} = 0\mathbb{Z}$.

$H \neq \{0\}$ then $\exists\, n \in \mathbb{N} \cap H$. .:

choose the smallest - say n_0.

then show that every elt of H is a multiple of n_0.



Structure of a cyclic gp.

finite

$\mathbb{Z}_n \longleftrightarrow G$ where $n = o(G)$.

$G = \{a^r \mid r \in \mathbb{Z}\}$

$a^r = a^{r+n}$

infinite

then G is "like" $\mathbb{Z}$

$\mathbb{Z} \longleftrightarrow G = \{a^n \mid n \in \mathbb{Z}\}$.

Subgps of a cyclic gp.

infinite $\underline{G_d} = \{(a^d)^n \mid n \in \mathbb{Z}\}$ $\forall d \in \mathbb{Z}$

$G_d \leq G$.

$$\underbrace{\underline{\langle a^2 \rangle}}_{H = \underline{G_2}} = \{1, a^2, a^{-2}, a^{\pm 4}, a^{\pm 6}, a^{\pm 8}, \dots\} \leq \underbrace{\langle a \rangle}_{G}$$

G : Finite cyclic gp. of order n.

Let $H \leq G$. Then $o(H) = \frac{n}{d}$ where d is divisor of n.

$\underset{\frac{n}{d}}{H_1}, \underset{\frac{n}{e}}{H_2} \leq G$.

$H_1 \leq H_2$ iff $e \mid d$.

Operations on subgps:

* $\underline{H_1, H_2} \leq G$ then $\underbrace{H_1 \cap H_2}$ $\leq G$.

$\underline{a, b} \in H_1 \cap H_2 \subseteq \underline{H_1} \,\&\, \underline{H_2}$

$\Rightarrow \underline{ab^{-1}} \in H_1 \cap H_2$

$\Rightarrow H_1 \cap H_2 \leq G$.

$\{H_\alpha \mid \underline{\alpha \in I}\}$ then $\bigcap_{\alpha \in I} H_\alpha \leq G$.

* $H_1 \cup H_2$ need not be a subgp.

$$2\mathbb{Z} \cup 3\mathbb{Z} = \{ n \in \mathbb{Z} \mid 2|n \text{ or } 3|n \}$$

* $H_1 \cup H_2$ is a subgp of G iff one is a subset of the other
$H_1 \subset H_2$ or $H_2 \subseteq H_1$

Ex.

* Product of subgps.

$\underline{H}, \underline{K}$ be subgps

Define $\underline{HK} = \{ \underline{h * k} \mid h \in H, k \in K \}$.

Thm/Remark: TFAE:

i) HK is a subgp of G.

ii) $HK = KH$

iii) $HK \subset KH$

iv) $KH \subset HK$.

v) HK is smallest subgp of G that contains both H & K. $\quad HK = \langle H, K \rangle$

Recall: cyclic subgp. gen by a.

$$H = \{a^n \mid n \in \mathbb{Z}\}.$$

Remark: H is the smallest subgp of G that contains a.

$$\mathcal{H} = \{H \leq G \mid a \in H\}$$

$$\bigcap_{H \in \mathcal{H}} H \leq G.$$

Defn: $S \subseteq G$. Then a subgp gen by S is the smallest subgp containing S.

Notn: $\langle S \rangle$

Thm/Remark: Let H, K be finite subgp of G such that $HK \leq G$ (i.e. $HK = KH$).

$$o(HK) = \frac{o(H) \cdot o(K)}{o(H \cap K)}.$$

Counting principle

$\langle S \rangle$ $\qquad$ $S = \emptyset$ then $\langle S \rangle = \{e\}$.

If $S \neq \emptyset$ then S contains all possible words formed by $\underbrace{\text{elts of } S}_{\text{alphabets}}$.

$s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_r^{\alpha_r}$ $\qquad$ $r \in \mathbb{N}$; $s_1, \ldots, s_r \in S$.

$\alpha_i \geq 0$.

$\underline{s_1 s_2 s_1 s_3^3 s_2^2 s_1 s_4}$