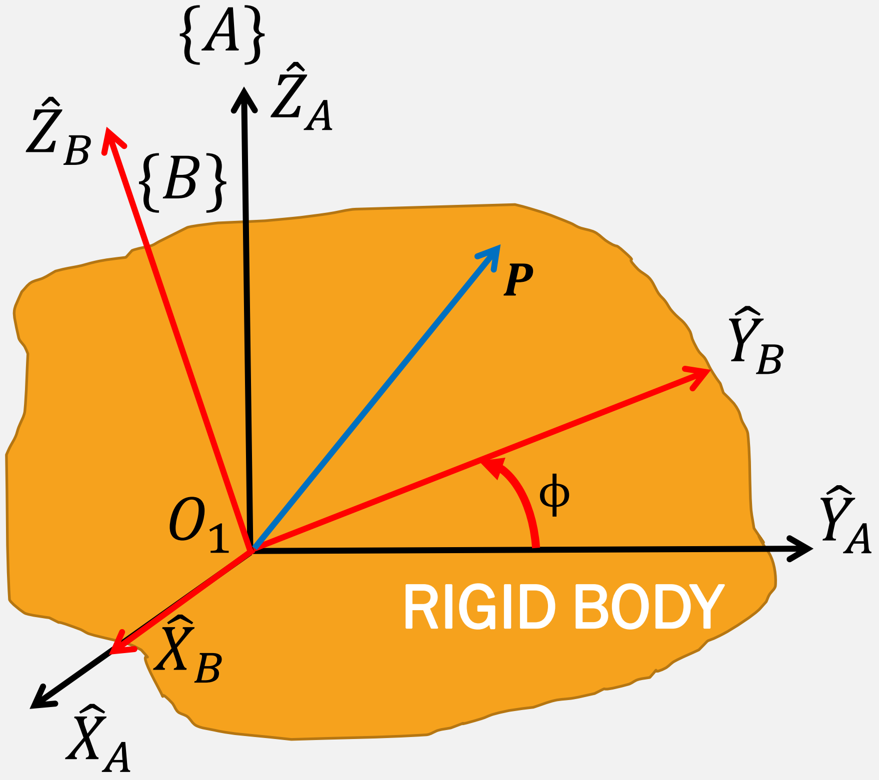
EXAMPLE – ROTATION AXIS IS X–AXIS



$$^{A}\boldsymbol{P} = [^{A}\widehat{\boldsymbol{X}}_B \quad ^{A}\widehat{\boldsymbol{Y}}_B \quad ^{A}\widehat{\boldsymbol{Z}}_B]\, ^{B}\boldsymbol{P}$$

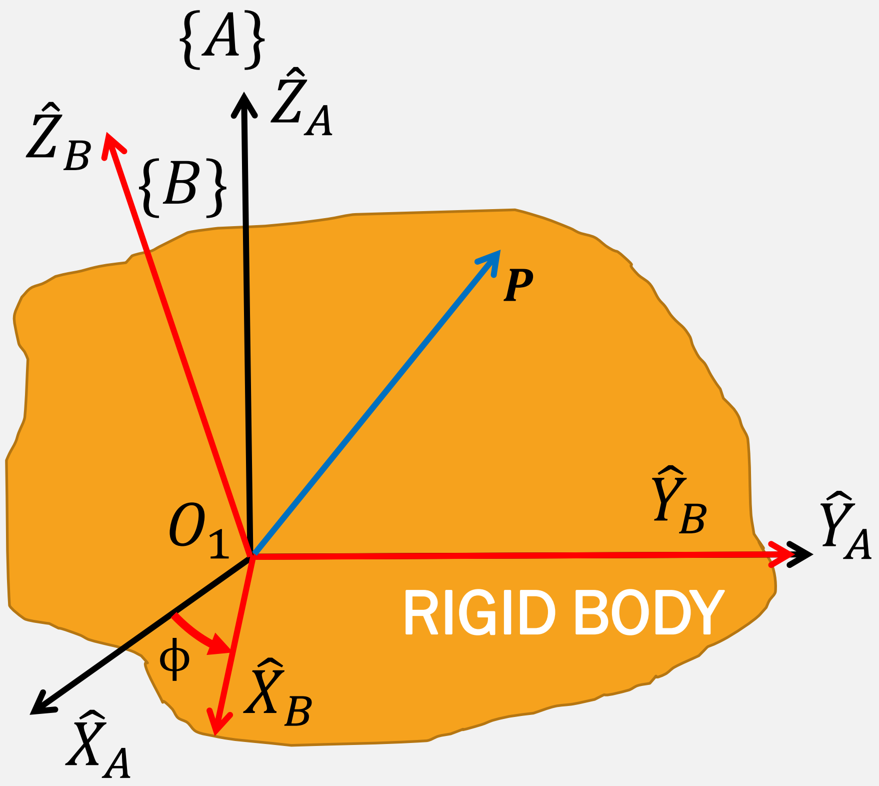
$$[^{A}\widehat{\boldsymbol{X}}_B \quad ^{A}\widehat{\boldsymbol{Y}}_B \quad ^{A}\widehat{\boldsymbol{Z}}_B] = \begin{bmatrix} \widehat{\boldsymbol{X}}_B \cdot \widehat{\boldsymbol{X}}_A & \widehat{\boldsymbol{Y}}_B \cdot \widehat{\boldsymbol{X}}_A & \widehat{\boldsymbol{Z}}_B \cdot \widehat{\boldsymbol{X}}_A \\ \widehat{\boldsymbol{X}}_B \cdot \widehat{\boldsymbol{Y}}_A & \widehat{\boldsymbol{Y}}_B \cdot \widehat{\boldsymbol{Y}}_A & \widehat{\boldsymbol{Z}}_B \cdot \widehat{\boldsymbol{Y}}_A \\ \widehat{\boldsymbol{X}}_B \cdot \widehat{\boldsymbol{Z}}_A & \widehat{\boldsymbol{Y}}_B \cdot \widehat{\boldsymbol{Z}}_A & \widehat{\boldsymbol{Z}}_B \cdot \widehat{\boldsymbol{Z}}_A \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}}_{\text{Rotation Matrix } {}^{A}_{B}[R] = [R(\widehat{X}, \phi)]}$$

What is rotation matrix for $[R(\widehat{Y}, \phi)]$ and $[R(\widehat{Z}, \phi)]$??

TRY ON YOUR OWN

EXAMPLE – ROTATION AXIS IS Y–AXIS

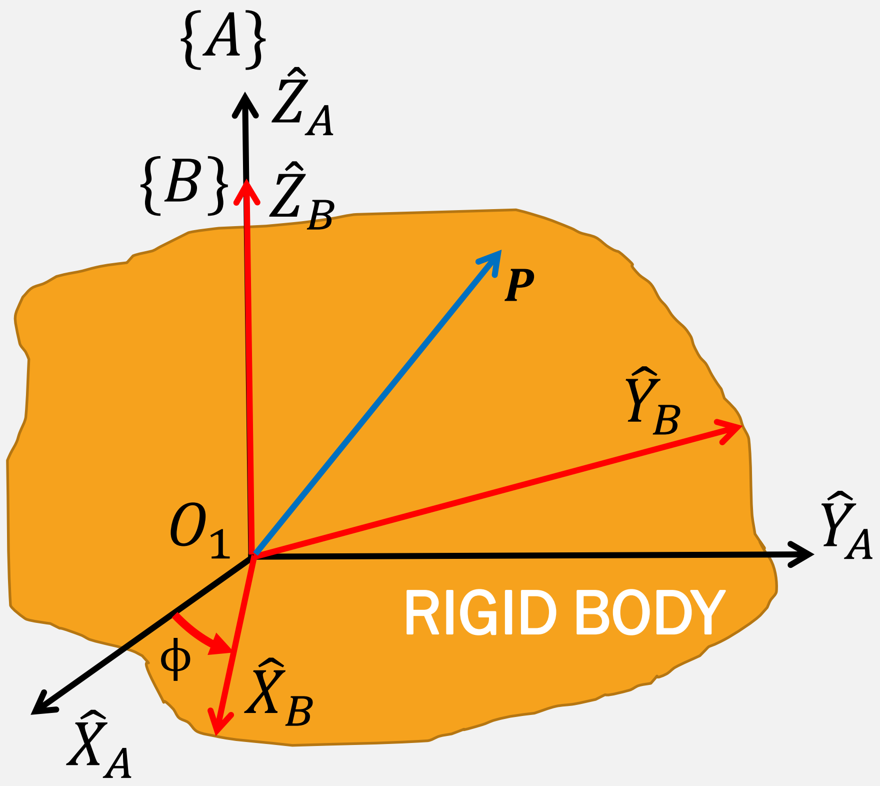


$$^{A}\boldsymbol{P} = [^{A}\widehat{\boldsymbol{X}}_B \quad ^{A}\widehat{\boldsymbol{Y}}_B \quad ^{A}\widehat{\boldsymbol{Z}}_B]\,^{B}\boldsymbol{P}$$

$$[^{A}\widehat{\boldsymbol{X}}_B \quad ^{A}\widehat{\boldsymbol{Y}}_B \quad ^{A}\widehat{\boldsymbol{Z}}_B] = \begin{bmatrix} \widehat{\boldsymbol{X}}_B \cdot \widehat{\boldsymbol{X}}_A & \widehat{\boldsymbol{Y}}_B \cdot \widehat{\boldsymbol{X}}_A & \widehat{\boldsymbol{Z}}_B \cdot \widehat{\boldsymbol{X}}_A \\ \widehat{\boldsymbol{X}}_B \cdot \widehat{\boldsymbol{Y}}_A & \widehat{\boldsymbol{Y}}_B \cdot \widehat{\boldsymbol{Y}}_A & \widehat{\boldsymbol{Z}}_B \cdot \widehat{\boldsymbol{Y}}_A \\ \widehat{\boldsymbol{X}}_B \cdot \widehat{\boldsymbol{Z}}_A & \widehat{\boldsymbol{Y}}_B \cdot \widehat{\boldsymbol{Z}}_A & \widehat{\boldsymbol{Z}}_B \cdot \widehat{\boldsymbol{Z}}_A \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}}_{\textbf{Rotation Matrix } {}^{A}_{B}[R] = [R(\widehat{Y}, \phi)]}$$

EXAMPLE – ROTATION AXIS IS Z–AXIS



$${}^{A}\boldsymbol{P} = \left[{}^{A}\widehat{\boldsymbol{X}}_B \quad {}^{A}\widehat{\boldsymbol{Y}}_B \quad {}^{A}\widehat{\boldsymbol{Z}}_B\right]{}^{B}\boldsymbol{P}$$

$$\left[{}^{A}\widehat{\boldsymbol{X}}_B \quad {}^{A}\widehat{\boldsymbol{Y}}_B \quad {}^{A}\widehat{\boldsymbol{Z}}_B\right] = \begin{bmatrix} \widehat{\boldsymbol{X}}_B \cdot \widehat{\boldsymbol{X}}_A & \widehat{\boldsymbol{Y}}_B \cdot \widehat{\boldsymbol{X}}_A & \widehat{\boldsymbol{Z}}_B \cdot \widehat{\boldsymbol{X}}_A \\ \widehat{\boldsymbol{X}}_B \cdot \widehat{\boldsymbol{Y}}_A & \widehat{\boldsymbol{Y}}_B \cdot \widehat{\boldsymbol{Y}}_A & \widehat{\boldsymbol{Z}}_B \cdot \widehat{\boldsymbol{Y}}_A \\ \widehat{\boldsymbol{X}}_B \cdot \widehat{\boldsymbol{Z}}_A & \widehat{\boldsymbol{Y}}_B \cdot \widehat{\boldsymbol{Z}}_A & \widehat{\boldsymbol{Z}}_B \cdot \widehat{\boldsymbol{Z}}_A \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Rotation Matrix ${}^{A}_{B}[\boldsymbol{R}] = [\boldsymbol{R}(\widehat{\boldsymbol{Z}}, \boldsymbol{\phi})]$

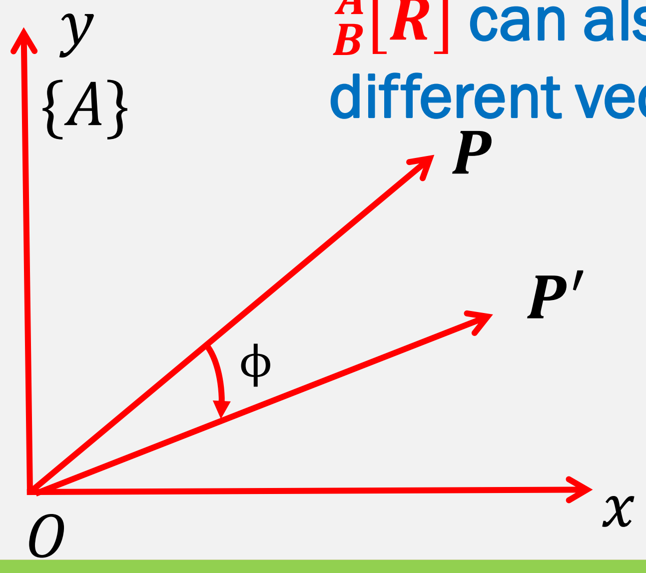
UNDERSTANDING THE ROTATION MATRIX

MATRIX ${}^{A}_{B}[R]$ can be thought of as an operator that, acting on the fixed system, transforms it into the body system

Symbolically, it may be written as $(P)' = [R]P$

Matrix [R] operating on the components of a vector in the fixed system yields the components of the vector in the body system

Parenthesis have been placed around P to make it clear that the same vector is involved on both sides of the equation. ONLY COMPONENTS HAVE CHANGED, WHILE VECTOR REMAINS UNCHANGED



${}^{A}_{B}[R]$ can also be thought of as an operator acting on vector P, changing it to a different vector P', with both vectors expressed in the same coordinate system

In 2-D, instead of rotating coordinate system counterclockwise, P is rotated clockwise by an angle ϕ **to a new vector P'. Components of new vector will then be related to components of old vector by same equations describing coordinate transformation**

SOME PROPERTIES OF ROTATION MATRICES

- The columns of a rotation matrix are unit vectors, orthogonal to one another
- The determinant of a rotation is matrix is +1. (WHY !! MORE ON THIS LATER)
- Inverse of Rotation matrix equals it's transpose - columns of ${}^{A}_{B}[\boldsymbol{R}]$ are the unit vectors of $\{B\}$ written in $\{A\}$, the rows of ${}^{A}_{B}[\boldsymbol{R}]$ are the unit vectors of $\{A\}$ written in $\{B\}$.
- One of its eigenvalues is +1. The other two are complex conjugate pairs of the form $e^{\pm i\phi}$, where $\phi = cos^{-1}\frac{r_{11}+r_{22}+r_{33}-1}{2}$
- Rotations generally do not commute
- The eigenvector corresponding to the unity eigenvalue is given by

$$\hat{k} = (1/2\sin\phi)[r_{32} - r_{23}, r_{13} - r_{31}, r_{21} - r_{12}]^T$$

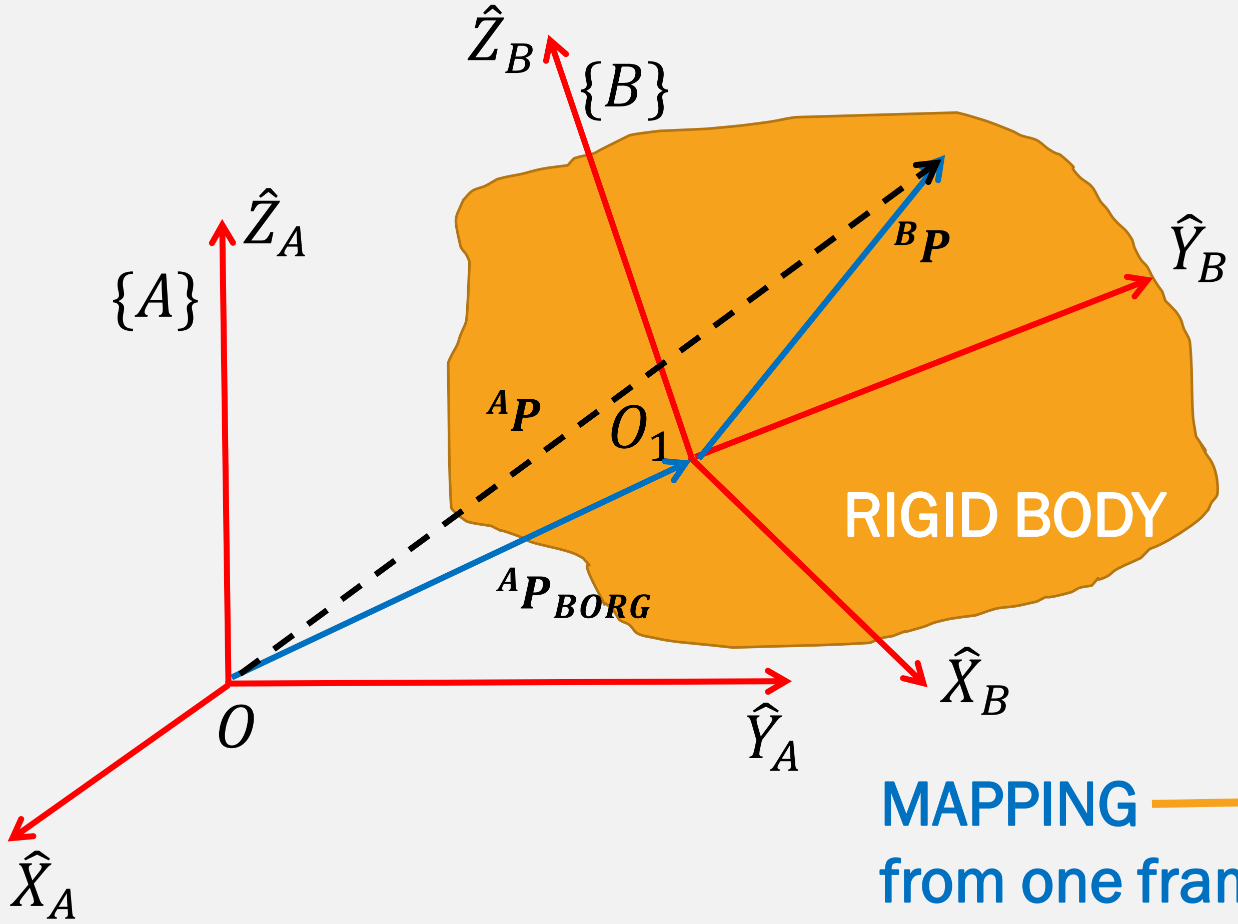
ANY QUESTIONS SO FAR ???

MODULE 1 – LECTURE 3

SPATIAL DESCRIPTIONS (CONTINUED….)

CHANGING DESCRIPTIONS FROM FRAME TO FRAME

PURE TRANSLATION



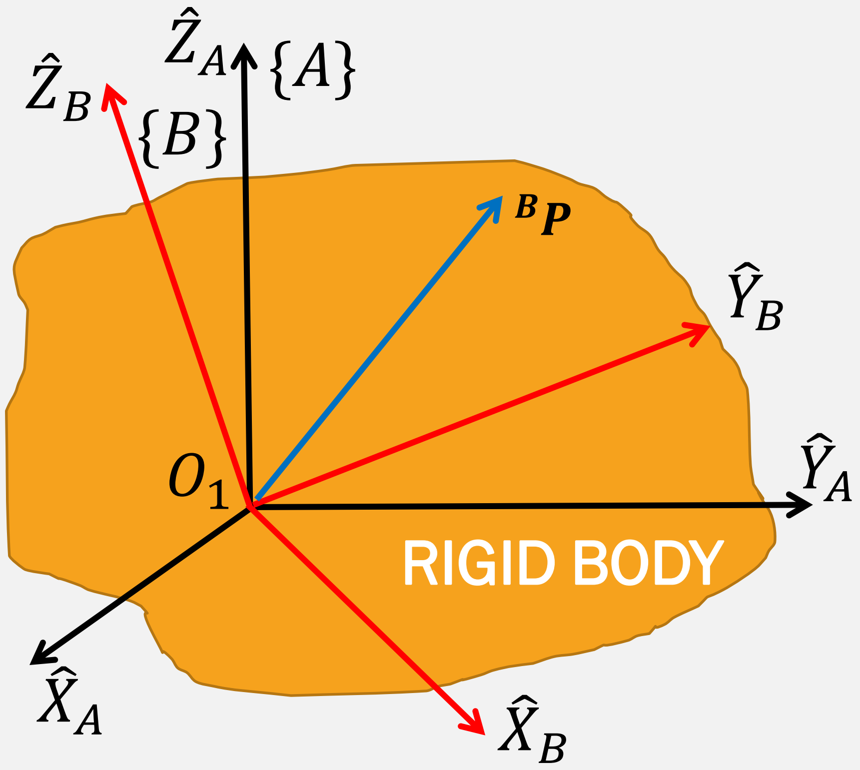
{B} differs from {A} only by a translation, which is given by ${}^{A}\boldsymbol{P}_{BORG}$, a vector that locates the origin of {B} relative to {A}

Since both vectors are defined relative to frames of the same orientation, we calculate the description of point *P* relative to {A} as

$$ {}^{A}\boldsymbol{P} = {}^{B}\boldsymbol{P} + {}^{A}\boldsymbol{P}_{BORG} $$

MAPPING → changing the description from one frame to another. The quantity itself, namely a point in space is NOT changed.

PURE ROTATION

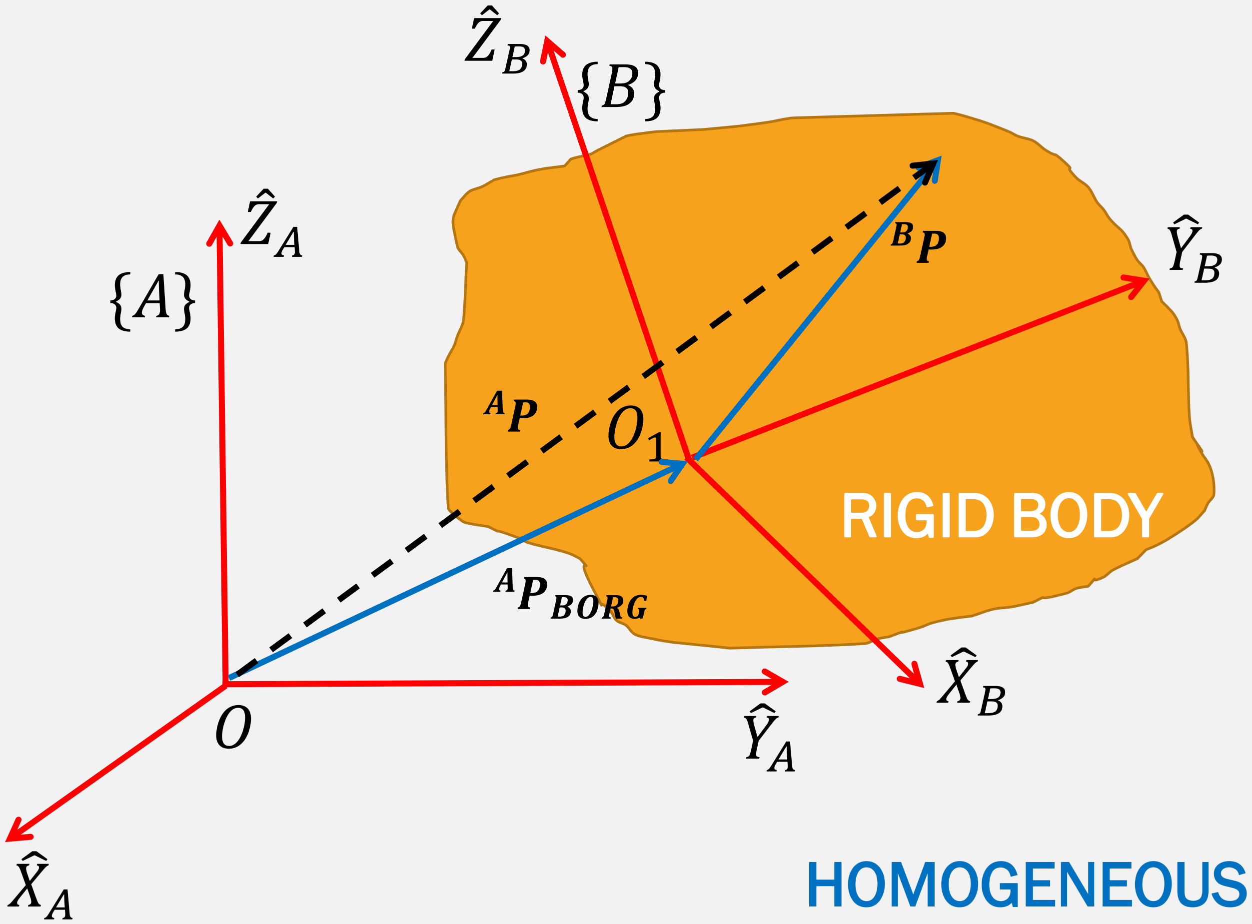


In order to calculate ${}^{A}\boldsymbol{P}$, consider that the components of any vector are simply the projections of that vector onto the unit directions of its frame. The projection is calculated as the vector dot product. Hence, the components of ${}^{A}\boldsymbol{P}$ may be calculated as

$$
\begin{aligned}
{}^{A}\boldsymbol{P}_x &= {}^{B}\widehat{\mathbf{X}}_A \cdot {}^{B}\boldsymbol{P} \\
{}^{A}\boldsymbol{P}_y &= {}^{B}\widehat{\mathbf{Y}}_A \cdot {}^{B}\boldsymbol{P} \\
{}^{A}\boldsymbol{P}_z &= {}^{B}\widehat{\mathbf{Z}}_A \cdot {}^{B}\boldsymbol{P}
\end{aligned}
$$

${}^{B}\widehat{\mathbf{X}}_A, {}^{B}\widehat{\mathbf{Y}}_A. {}^{B}\widehat{\mathbf{Z}}_A$ ***are the rows of*** ${}^{A}_{B}[\boldsymbol{R}]$ $\Rightarrow$ ${}^{A}\boldsymbol{P} = {}^{A}_{B}[\boldsymbol{R}]\,{}^{B}\boldsymbol{P}$

MAPPINGS INVOLVING GENERAL FRAMES



$${}^A\boldsymbol{P} = {}^A_B[\boldsymbol{R}]\,{}^B\boldsymbol{P} + {}^A\boldsymbol{P}_{BORG}$$

$${}^A\boldsymbol{P} = {}^A_B[\boldsymbol{T}]\,{}^B\boldsymbol{P}$$

We define a 4×4 matrix operator and use 4×1 position vectors to get

$$\begin{bmatrix} {}^A\boldsymbol{P} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^A_B[\boldsymbol{R}] & {}^A\boldsymbol{P}_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}}_{{}^A_B[\boldsymbol{T}]} \begin{bmatrix} {}^B\boldsymbol{P} \\ 1 \end{bmatrix}$$

HOMOGENEOUS TRANSFORMATION → ${}^A_B[\boldsymbol{T}]$

SOME PROPERTIES OF TRANSFORMATION MATRICES

- ${}^{A}_{B}[T] = \begin{bmatrix} {}^{A}_{B}[R] & & & {}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Upper left 3×3 matrix is identity matrix $\rightarrow$ Pure translation.
- Top right 3×1 vector is zero $\rightarrow$ Pure rotation.
- The inverse of ${}^{A}_{B}[T]$ can be found as ${}^{A}_{B}[T]^{-1} = \begin{bmatrix} {}^{A}_{B}[R]^{T} & & & -{}^{A}_{B}[R]^{T}\,{}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Two of its eigenvalues are +1 and +1. The other two are complex conjugate pairs of the form $e^{\pm i\phi}$, where $\phi = cos^{-1}\frac{r_{11}+r_{22}+r_{33}-1}{2}$
- Transformations generally do not commute
- The eigenvectors corresponding to the repeated unity eigenvalue are obtained from

$$\begin{bmatrix} {}^{A}_{B}[R]^{T}-[I] & & & {}^{A}P_{BORG} \\ 0 & 0 & 0 & 0 \end{bmatrix} X = 0$$

- Greater details can be found at Ghosal A., "*Robotics: Fundamental Concepts and Analysis*", Oxford University Press, 2006.

SUCCESSIVE ROTATIONS

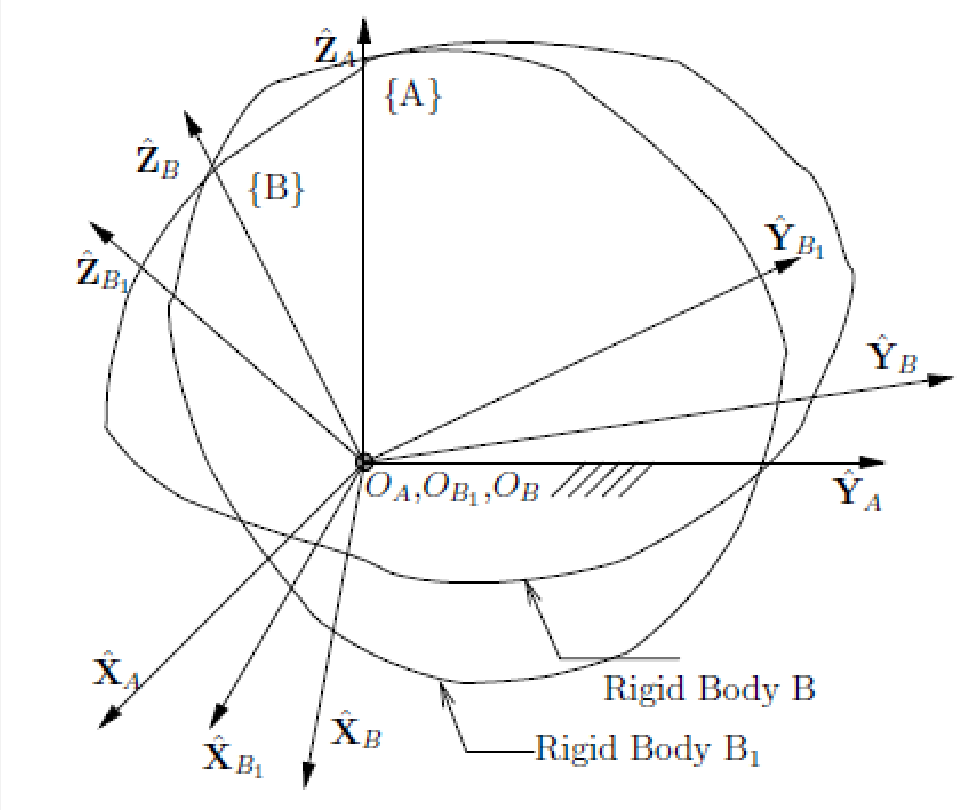


Figure from Ghosal, 2006

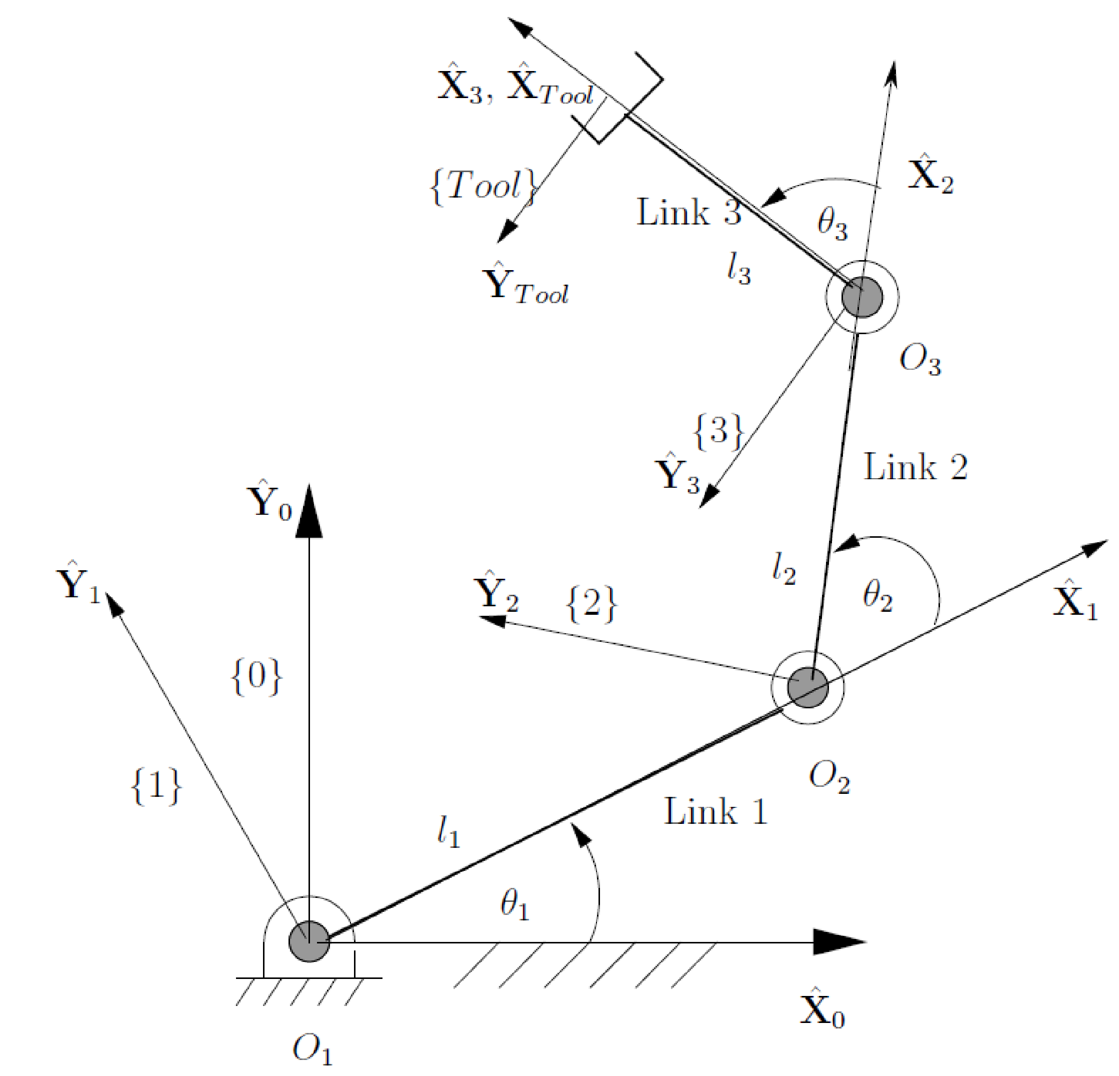
Two successive rotations:

- Initially B is coincident with $\{A\}$.
- First rotation relative to $\{A\}$. After first rotation $\{A\} \rightarrow \{B_1\}$
- Second rotation relative to $\{B_1\}$. After second rotation $\{B_1\} \rightarrow \{B\}$
- Resultant Rotation: ${}^{A}_{B}[\boldsymbol{R}] = {}^{A}_{B_1}[\boldsymbol{R}]\,{}^{B_1}_{B}[\boldsymbol{R}]$
- Resultant on ***n*** rotations: ${}^{A}_{B}[\boldsymbol{R}] = {}^{A}_{B_1}[\boldsymbol{R}]\,{}^{B_1}_{B_2}[\boldsymbol{R}] \ldots {}^{B_{n-1}}_{B}[\boldsymbol{R}]$
- Need to note that Matrix multiplication is NOT commutative.

Can be extended for case of transformation $\{A\} \rightarrow \{B_1\} \ldots \rightarrow \{\mathrm{B}\}$. ${}^{A}_{B}[\boldsymbol{T}] = {}^{A}_{B_1}[\boldsymbol{T}]\,{}^{B_1}_{B_2}[\boldsymbol{T}] \ldots {}^{B_{n-1}}_{B}[\boldsymbol{T}]$

$$
{}^{A}_{B}\boldsymbol{T} = \begin{bmatrix} {}^{A}_{B_1}[\boldsymbol{R}]\,{}^{B_1}_{B}[\boldsymbol{R}] & {}^{A}_{B_1}[\boldsymbol{R}]\,{}^{B_1}\boldsymbol{P}_{BORG} + {}^{A}\boldsymbol{P}_{B_1ORG} \\ \boldsymbol{0 \quad 0 \quad 0} & \boldsymbol{1} \end{bmatrix}
$$

EXAMPLE 1 (TRY IN CLASS)

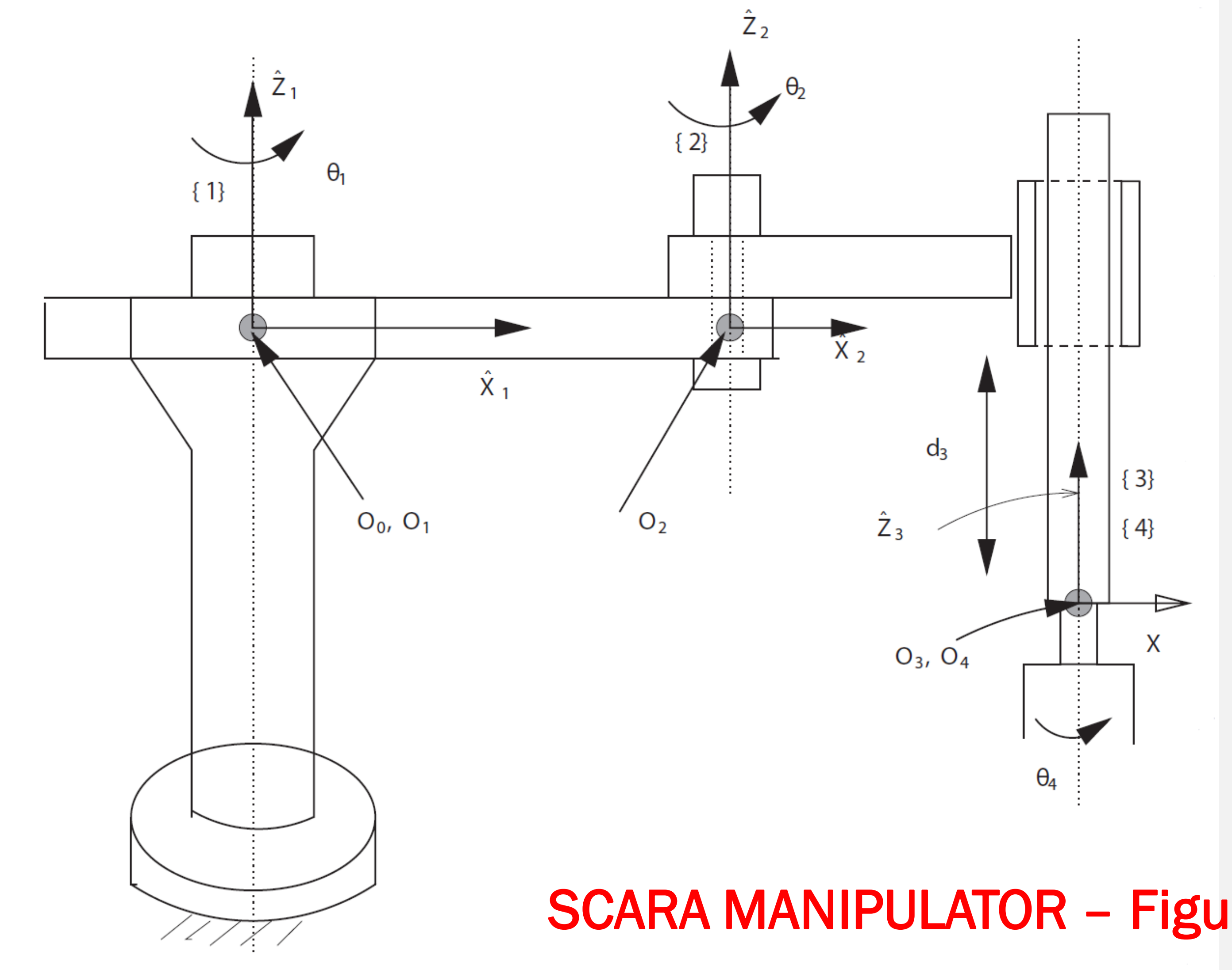


Find ${}^{0}_{Tool}[T]$

$$
\begin{aligned}
{}^{0}_{Tool}[T] &= {}^{0}_{3}[T]\;{}^{3}_{Tool}[T] \\
&= {}^{0}_{1}[T]\;{}^{1}_{2}[T]\;{}^{2}_{3}[T]\;{}^{3}_{Tool}[T]
\end{aligned}
$$

3R Manipulator – Figure from Ghosal, 2006

EXAMPLE 2 (IN CLASS)



Find ${}^{0}_{4}[T]$

$${}^{0}_{4}[T] = {}^{0}_{1}[T]\,{}^{1}_{2}[T]\,{}^{2}_{3}[T]\,{}^{3}_{4}[T]$$

SCARA MANIPULATOR – Figure from Ghosal, 2006

ANY QUESTIONS SO FAR ???

MODULE 1 – LECTURE 4

DESCRIPTIONS OF ORIENTATION

$$\begin{bmatrix} \widehat{X}_B \cdot \widehat{X}_A & \widehat{Y}_B \cdot \widehat{X}_A & \widehat{Z}_B \cdot \widehat{X}_A \\ \widehat{X}_B \cdot \widehat{Y}_A & \widehat{Y}_B \cdot \widehat{Y}_A & \widehat{Z}_B \cdot \widehat{Y}_A \\ \widehat{X}_B \cdot \widehat{Z}_A & \widehat{Y}_B \cdot \widehat{Z}_A & \widehat{Z}_B \cdot \widehat{Z}_A \end{bmatrix}$$

We have seen that the nine directions cosines are NOT independent quantities.

$$\begin{aligned} |\widehat{X}| &= 1 & \qquad \widehat{X} \cdot \widehat{Y} &= 0 \\ |\widehat{Y}| &= 1 & \qquad \widehat{X} \cdot \widehat{Z} &= 0 \\ |\widehat{Z}| &= 1 & \qquad \widehat{Y} \cdot \widehat{Z} &= 0 \end{aligned}$$

SIX relations expressing orthogonality conditions reduce independent quantities to three

In order to characterize motion of a rigid body, there is additional requirement matrix elements must satisfy beyond orthogonality

We have seen that ${}^{A}_{B}[R]$ is real and orthogonal, i.e. determinant of any orthogonal matrix can be ± 1

Can determinant of ${}^{A}_{B}[R]$ be -1 ??

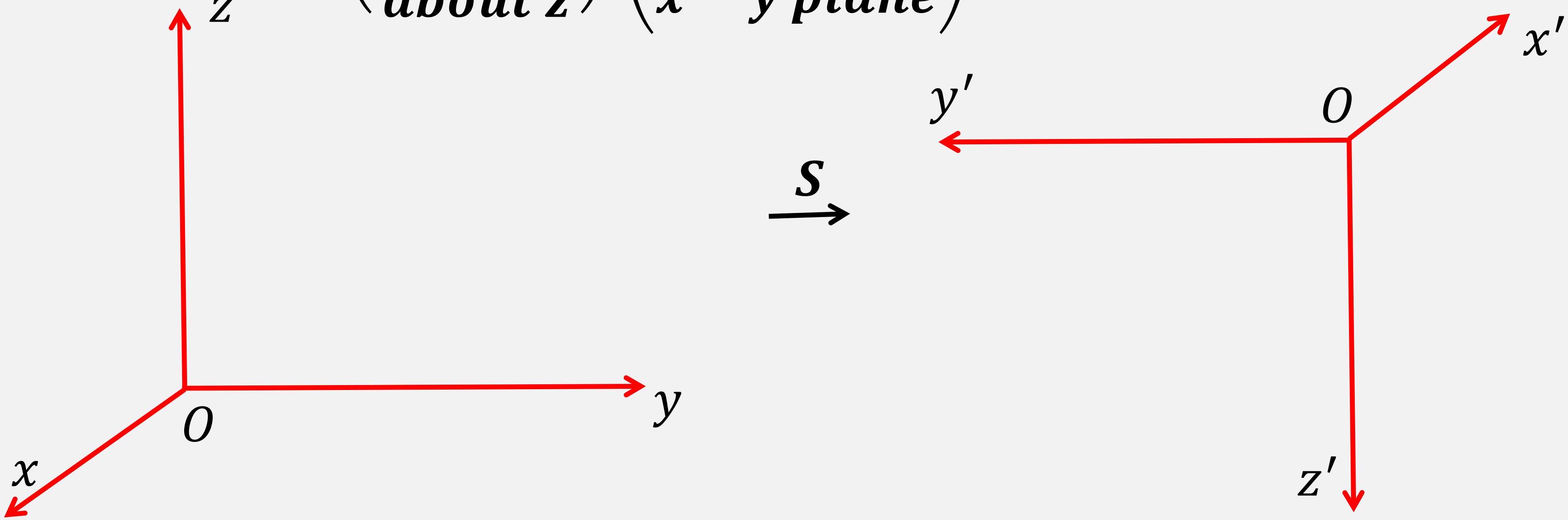
Consider the simplest 3×3 matrix with determinant −1

$$S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -1$$

Transformation S changes the sign of each of the coordinate axes.

<u>This is known as Inversion of coordinate axes.</u>

$$\begin{pmatrix} Rotate \\ by\ 180^0 \\ about\ z \end{pmatrix} \begin{pmatrix} Reflect \\ in\ the \\ x-y\ plane \end{pmatrix} = Inversion$$



OBSERVATIONS ON IMPLICATIONS OF INVERSION

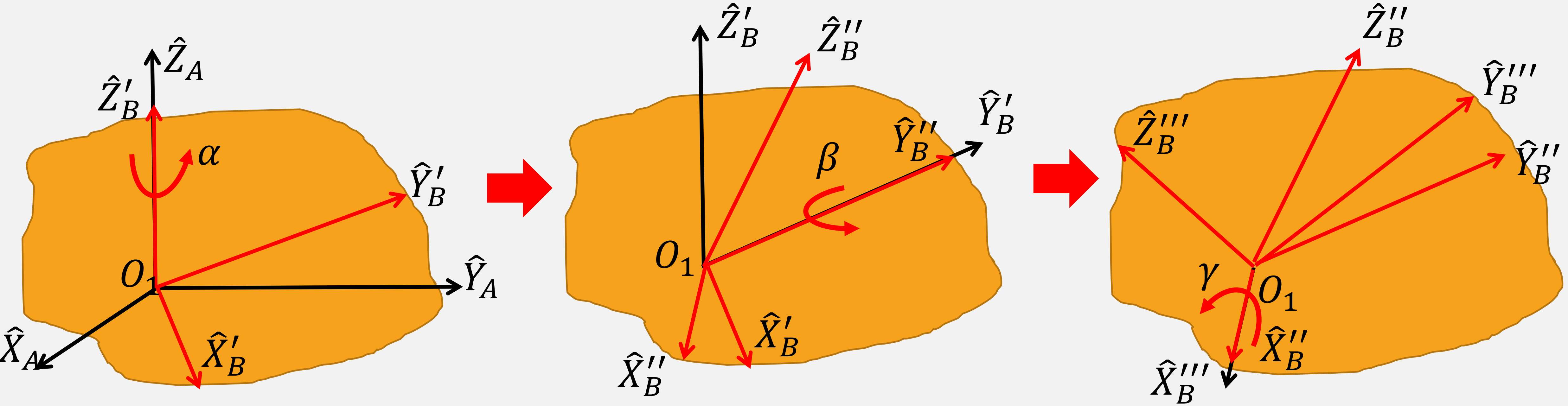
- Inversion of right handed system into left handed one cannot be accomplished by any rigid change in the orientation of the coordinate axes.
- "INVERSION" never corresponds to a physical displacement of a rigid body
- What is true for "**S**" is true for any matrix with determinant –1.
- Transformations representing rigid body motions must be limited to matrices having determinant +1 (PROPER ROTATION MATRICES).

In order to characterize motion of a rigid body, it is necessary to find three independent parameters that specify orientation of a rigid body in a manner such that the matrix of rotation has determinant +1.

Z–Y–X EULER ANGLES

One method of describing the body frame {B} is

Start with the frame coincident with a known reference frame *{A}*. Rotate *{B}* first about $\hat{Z}_B$ by an angle α, then about $\hat{Y}_B$ by an angle β and, finally, about $\hat{X}_B$ by an angle γ



Each rotation is performed about an axis of the moving system *{B}* – *hence the term "EULER ANGLES"*

Note that each rotation takes place about an axis whose location depends upon the preceding rotations

DERIVATION OF EQUIVALENT ROTATION MATRIX

$$ {}^{A}_{B}[R] = {}^{A}_{B'}[R]\,{}^{B'}_{B''}[R]\,{}^{B''}_{B'''}[R] $$

$$ {}^{A}_{B}[R]_{Z'Y'X'}(\alpha,\beta,\gamma) = [R]_Z(\alpha)\;[R]_Y(\beta)\;[R]_X(\gamma) $$

$$ = \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix} $$

$$ = \begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix} $$

INVERSE PROBLEM – EXTRACTING Z–Y–X EULER ANGLES FROM ROTATION MATRIX

There are nine elements of the rotation matrix and only three independent parameters to describe the rigid body, along with six dependencies

$$Let\ {}^{A}_{B}[R]_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}=\begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix}$$

1) $Solve\ for\ \cos\beta\ \ by\ taking\ \sqrt{r_{11}^2+r_{21}^2}$

2) $Solve\ for\ \beta\ as\ \beta = Atan2\left(-r_{31}, \sqrt{r_{11}^2+r_{21}^2}\right)$

3) $As\ long\ as\ \cos\beta \neq 0, \alpha = Atan2\left(\frac{r_{21}}{\cos\beta}, \frac{r_{11}}{\cos\beta}\right)$

4) $As\ long\ as\ \cos\beta \neq 0, \gamma = Atan2\left(\frac{r_{32}}{\cos\beta}, \frac{r_{33}}{\cos\beta}\right)$

where Atan2(y, x) is a two-argument arc tangent function. Also called “four quadrant arc-tangent”

Eg. $Atan2(-2.0,-2.0) = -135^0$

Whereas $Atan2(2.0, 2.0) = 45^0$

$It\ can\ be\ seen\ that$

$$-90^0 \leq \beta \leq 90^0$$

SPECIAL CASE OF $\beta = 90^0$

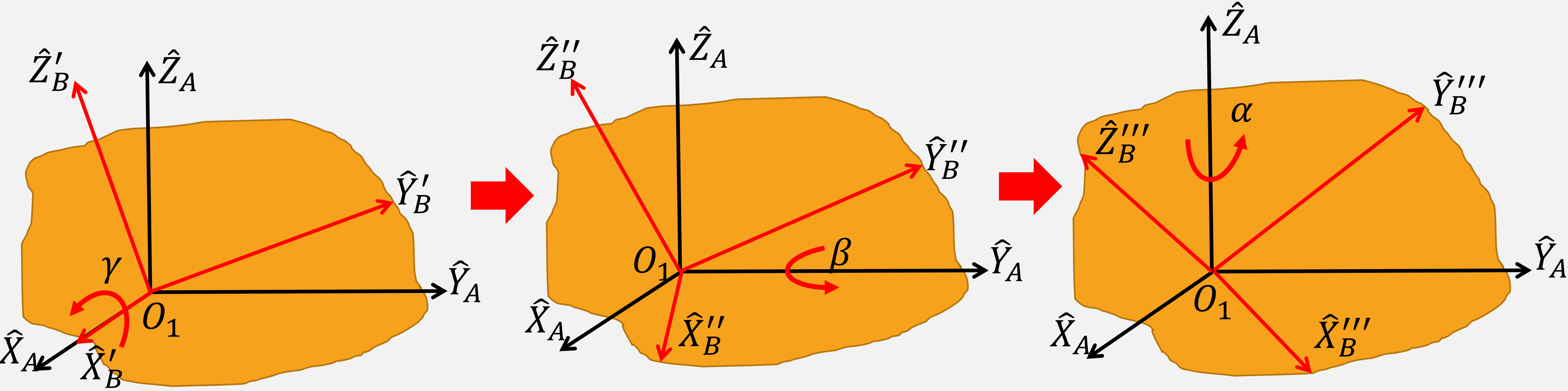
- Although a second solution exists, by using the positive square root in the formula for β for we always compute the single solution for which $-90^0 \leq \beta \leq 90^0$.
- This is usually a good practice, because we can then define one-to-one mapping functions between various representations of orientation.
- However, in some cases, calculating all solutions is important (MORE ON THIS WHEN WE STUDY INVERSE KINEMATICS)
- If $\beta = \pm 90^0$, then $\cos\beta = 0$ and the solution degenerates. In such cases, only sum or difference of $\alpha \; and \; \gamma$ can be computed. One possible convention is to choose $\alpha = 0.0$, whose results are given below:

1. $\beta = 90^0, \alpha = 0, \gamma = Atan2(r_{12}, r_{22})$
2. $\beta = -90^0, \alpha = 0, \gamma = -Atan2(r_{12}, r_{22})$

X–Y–Z FIXED ANGLES

Another method of describing the body frame {B} is

Start with the frame coincident with a known reference frame {A}. Rotate {B} first about $\hat{X}_A$ by an angle γ, then about $\hat{Y}_A$ by an angle β and, finally, about $\hat{Z}_A$ by an angle α



Each of the three rotations takes place about an axis in the fixed reference frame {A} – hence convention is known as X–Y–Z fixed angles

Sometimes this convention is referred to as roll, pitch, yaw angles – however, one must check proper terminology

DERIVATION OF EQUIVALENT ROTATION MATRIX

WHAT SHOULD BE THE ORDER OF ROTATIONS ??

IS THIS CORRECT ??

$$ {}^{A}_{B}[R]_{XYZ}(\gamma,\beta,\alpha) = [R]_Z(\alpha)\,[R]_Y(\beta)\,[R]_X(\gamma) $$

$$ = \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix} $$

$$ = \begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix} $$

SAME AS FOR Z-Y-X EULER ANGLES

Z–Y–Z EULER ANGLES

Another method of describing the body frame {B} is

Start with the frame coincident with a known reference frame *{A}*. Rotate *{B}* first about $\hat{Z}_B$ by an angle α, then about $\hat{Y}_B$ by an angle β and, finally, about $\hat{Z}_B$ by an angle γ

Each rotation is performed about an axis of the moving system *{B}*
– *Hence is EULER ANGLE DESCRIPTION*

Note that each rotation takes place about an axis whose location depends upon the preceding rotations

DERIVATION OF EQUIVALENT ROTATION MATRIX

$$^{A}_{B}[R] = {}^{A}_{B'}[R]\,{}^{B'}_{B''}[R]\,{}^{B''}_{B'''}[R]$$

$$^{A}_{B}[R]_{Z'Y'Z'}(\alpha,\beta,\gamma) = [R]_Z(\alpha)\ [R]_Y(\beta)\ [R]_Z(\gamma)$$

$$= \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} C\gamma & -S\gamma & 0 \\ S\gamma & C\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\alpha C\beta C\gamma - S\alpha S\gamma & -C\alpha C\beta S\gamma - S\alpha C\gamma & C\alpha S\beta \\ S\alpha C\beta C\gamma + C\alpha S\gamma & -S\alpha C\beta S\gamma + C\alpha C\gamma & S\alpha S\beta \\ -S\beta C\gamma & S\beta S\gamma & C\beta \end{bmatrix}$$

INVERSE PROBLEM – EXTRACTING X–Y–Z FIXED ANGLES FROM ROTATION MATRIX

$$ {}^{A}_{B}[R]_{Z'Y'Z'}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} C\alpha C\beta C\gamma - S\alpha S\gamma & -C\alpha C\beta S\gamma - S\alpha C\gamma & C\alpha S\beta \\ S\alpha C\beta C\gamma + C\alpha S\gamma & -S\alpha C\beta S\gamma + C\alpha C\gamma & S\alpha S\beta \\ -S\beta C\gamma & S\beta S\gamma & C\beta \end{bmatrix} $$

$$ If\ \sin\beta \neq 0 $$

$$ 1)\ Solve\ for\ \beta\ as\ \beta = Atan2\left(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right) $$

$$ 2)\alpha = Atan2\left(\frac{r_{23}}{\sin\beta}, \frac{r_{13}}{\sin\beta}\right) $$

$$ 3)\gamma = Atan2\left(\frac{r_{32}}{\sin\beta}, \frac{-r_{31}}{\sin\beta}\right) $$

SPECIAL CASE OF $\beta = 0^0\ or\ 180^0$

- Although a second solution exists, by using the positive square root in the formula for β for we always compute the single solution for which $0^0 \le \beta \le 180^0$.
- If $\beta = 0^0\ or\ 180^0$, then $\sin\beta = 0$ and the solution degenerates. In such cases, only sum or difference of $\alpha\ and\ \gamma$ can be computed. One possible convention is to choose $\alpha = 0.0$, whose results are given below:

1. $\beta = 0^0, \alpha = 0, \gamma = Atan2(-r_{12}, r_{11})$
2. $\beta = 180^0, \alpha = 0, \gamma = -Atan2(r_{12}, -r_{11})$

ANGLE SET CONVENTIONS

- Total of 24 angle-set conventions
- 12 conventions for fixed angle sets and 12 for euler angle sets
- No reason to favour one over another, but different authors use different conventions
- REFER APPENDIX B of Craig J.J., “Introduction to Robotics: Mechanics and Control” for entire list of angle-set conventions along with corresponding rotation matrices.

COMPUTING [R] FROM EULER ANGLES USING MATLAB

SYNTAX –

rotm = eul2rotm(eul)

rotm = eul2rotm(eul,sequence)

eul = [0 pi/2 0];

rotmZYX = eul2rotm(eul)

Command Window

New to MATLAB? See resources for Getting Started.

```
>> eul = [0 pi/2 0];
rotmZYX = eul2rotm(eul)

rotmZYX =

    0.0000         0    1.0000
         0    1.0000         0
   -1.0000         0    0.0000

fx >>
```

COMPUTING [R] FROM EULER ANGLES USING MATLAB

rotm = eul2rotm(eul) → Default order of sequence is Z-Y-X

rotm = eul2rotm(eul,sequence) → Specify sequence

eul = [0 pi/2 pi/2];
rotmZYX = eul2rotm(eul, 'ZYZ')

```
Command Window
New to MATLAB? See resources for Getting Started.
>> eul = [0 pi/2 pi/2];
rotmZYZ = eul2rotm(eul,'ZYZ')

rotmZYZ =

    0.0000   -0.0000    1.0000
    1.0000    0.0000         0
   -0.0000    1.0000    0.0000

fx >>
```

EXTRACTING EULER ANGLES USING MATLAB

SYNTAX –

eul = tform2eul(tform)

eul = tform2eul(tform, sequence)

Eg. – $^{A}_{B}[T] = \begin{bmatrix} 1 & 0 & 0 & 0.5 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -1.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

tform = [1 0 0 0.5; 0 -1 0 5; 0 0 -1 -1.2; 0 0 0 1];
eulZYX = tform2eul(tform)

Command Window

New to MATLAB? See resources for Getting Started.

```
>> tform = [1 0 0 0.5; 0 -1 0 5; 0 0 -1 -1.2; 0 0 0 1];
eulZYX = tform2eul(tform)

eulZYX =

         0         0    3.1416

fx >>
```

EXTRACTING EULER ANGLES USING MATLAB

eul = tform2eul(tform) → Default order of sequence is Z-Y-X

eul = tform2eul(tform, sequence) → Specify sequence

Eg. – $${}^{A}_{B}[T] = \begin{bmatrix} 1 & 0 & 0 & 0.5 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -1.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ tform = [1 0 0 0.5; 0 -1 0 5; 0 0 -1 -1.2; 0 0 0 1];
eulZYX = tform2eul(tform, 'ZYZ')

Command Window

New to MATLAB? See resources for Getting Started.

```
>> tform = [1 0 0 0.5; 0 -1 0 5; 0 0 -1 -1.2; 0 0 0 1];
>> eulZYX = tform2eul(tform,'ZYZ')

eulZYX =

         0   -3.1416    3.1416

fx >>
```

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