Note Title 03-08-2021

MA-222

Lec-3

$x + z = y + z \Rightarrow x = y$

$(x+z) - z = (y+z) - z \Rightarrow x = y$

$+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$(x, n) \mapsto x + n$

$x \neq n$

$$1 \cdot x = x$$

$$x + x = 2 \cdot x$$

$$x + x + x = 3 \cdot x = 2x + x$$

$$\underbrace{x + x + \cdots + x}_{n\text{-times}} = n \cdot x$$

$n \in \mathbb{N}$.
If $n \neq 1$
then n is a successor of something unique
$n-1$

$$\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$(n, x) \mapsto \underbrace{x + x + \cdots + x}_{n\text{-times}} = (((x + x) + x) + x) + \cdots) + x$$

$$n \cdot x = (n-1) \cdot x + x \quad \text{for } n \geq 2$$

$\cdot : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

such that

$(1, x) \longmapsto x$

$(n, x) \longmapsto (n-1) \cdot x + x \quad$ for $n \geq 2$

1 is only not S(anything)

$S(x) = S(y) \Rightarrow x = y$

Properties:

- Associative $\quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$
- Commutative
- Cancellation $\quad \left. \begin{array}{l} x \cdot z = y \cdot z \\ \Rightarrow x = y \end{array} \right\}$

Distributive law.

$$x\cdot(y+z) = x\cdot y + x\cdot z$$

$\mathbb{N}$ as ordered set.

$x, y \in \mathbb{N}$ $\quad$ $\underline{x > y}$ (equi. $y < x$)

if $\exists\, z \in \mathbb{N}$ s.t. $\quad x = y + z$.

Properties: $x > y$ & $y > z$ then $x > z$.

② Trichotomy $x, y \in \mathbb{N}$. Then exactly one of the following holds:

$$\left\{ \begin{array}{l} x > y \\ x = y \\ x < y \end{array} \right. \qquad \begin{array}{l} \geq \\ \\ \leq \end{array}$$

Thm: WOP (Well-ordering principle)
(Least elt property)

If S is a nonempty set of $\mathbb{N}$.
Then S has a least elt.
i.e. $\exists\, n \in S$ s.t. $\forall\, m \in S$, $m \geq n$.

Pf: Supp $S \subseteq \mathbb{N}$ does not have a least elt.
Let $T = \mathbb{N} \setminus S$

$T = \{\underbrace{1, 2, 3, 4, 5}, 10, 100, \}$

Claim: $T = \mathbb{N}$ $\Rightarrow S = \emptyset$.

Consider $\underline{U} = \{ \underline{n \in \mathbb{N}} \mid \text{if } 1 \le m \le n \text{ then } m \in T \}$

Further if $\underline{U = \mathbb{N}}$ then $\underline{T = \mathbb{N}}$.

Pf $\underline{1 \in S}$ then S has a least elt.

thus $1 \notin S \Rightarrow \underline{1 \in T} \Rightarrow \underline{1 \in U} \Rightarrow U \neq \emptyset$.

Supp. $\underline{n \in U} \Rightarrow \{1, 2, \ldots, n\} \in T$.

$\notin S$

If $m \in S \Rightarrow m > n$. i.e. $m \geq n+1$.

If $n+1 \in S$ then $n+1$ is the least elt of S

$\Rightarrow n+1 \notin S$

$\Rightarrow n+1 \in T \Rightarrow n+1 \in U$.

$\Rightarrow U = \mathbb{N} \Rightarrow T = \mathbb{N}$

$\Rightarrow S = \emptyset$. $\square$

T s.t.

$1 \in T$

$n \in T \Rightarrow s(n) \in T$

then $T = \mathbb{N}$.

Thm:

$a, b \in \mathbb{N}$ then $\exists\, n \in \mathbb{N}$ s.t.

$$na \geq b$$

Pf