MA-222

Note Title 06-08-2021

Lec-4

Thm: Archimedean Property.

$a, b \in \mathbb{N}$ $\exists\, n \in \mathbb{N}.$ $na \geq b$

Pf: Supp. not. i.e. $\forall n \in \mathbb{N}.$ $na < b.$

$\Rightarrow b - na > 0 \;\; \forall n \in \mathbb{N}.$

$S = \{ b - na \mid n \in \mathbb{N} \} \subseteq \mathbb{N}.$

$S \neq \emptyset.$

By WOP, S has a least elt. say n_0.

$n_0 = b - ma$ for some $m \in \mathbb{N}$.

$b - (m+1)a \in S$.

$b - (m+1)a = (b - ma) - a = \underbrace{n_0 - a}_{\in S} < n_0$.

contra to the leastness of n_0.

Thus $\exists n \in \mathbb{N}$ s.t. $na \geq b$. $\square$

Principle of mathematical induction

<u>Ver I</u>: Let $P(n)$ be a statement about a natural no. n.

(i) Suppose $P(1)$ is true.

(ii) Whenever $P(k)$ holds then $P(k+1)$ also holds.

Then $P(n)$ is true for all natural no. n.

Alt. ver. P(n) be that

$S = \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$

Supp. i) $1 \in S$.

ii) if $k \in S$ then $k+1 \in S$.

Then $S = \mathbb{N}$.

Ver 2 : (Extended P.M.I.)

$P(n)$: stmt.

$S = \{n \in \mathbb{N} \mid P(n) \text{ holds}\}$

Supp S has the following properties:

i) $\exists\, n_0 \in S$ ← base case

ii) Whenever $k \in S$ $(k \geq n_0)$ $\Rightarrow$ $k+1 \in S$. induction step.

Then $S = \{n_0, n_0+1, \ldots\} = \mathbb{N} - \{1, \ldots, n_0-1\}$.

Example:

$2^n < n!$ $\forall\, n \geq 4$

n	2^n	$n!$
1	2	1
2	4	4
3	8	6

WOP $\Longrightarrow$ PMI

Second PMI

$P(n)$ be a stmt.

$S = \{n \in \mathbb{N} \mid P(n) \text{ holds}\}$

S has the following properties:

a) $n_0 \in S$.

ii) If $n_0, n_0 + 2, \ldots, k \in S$ then $k+1 \in S$.

then $S = \mathbb{N}$.

$P(k) \Rightarrow P(k+1)$

$P(k-1)$

$P(k-2)$

$\vdots$

Double induction

$\mathbb{N} \times \mathbb{N}$

$P(m,n)$: be a stmt about m & n.

i) $P(m_0, n_0)$ be true.

ii) For every $a \geq m_0$, $P(a,n)$ is true $Q(n)$

then $P(a, n+1)$ is true. $Q(n+1)$

iii) For every $b \geq n_0$, $P(m,b) \Rightarrow P(m+1,b)$.

then P holds $\forall\, m, n \in \mathbb{N}$.
$m \geq m_0$, $n \geq n_0$

$P(m, n, d)$

Triple induction?

$\cancel{m^2 + n^2 - d^2 \geq 0}$

$P(n_1, n_2, \ldots, n_r)$

$\underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{r\text{-times.}}$

$\mathbb{C}$

$\mathbb{R}$

$\mathbb{Q}$

$\mathbb{Z}$

$\mathbb{N}$

$\tilde{\mathbb{Q}}$

$\mathbb{Z}$

$\mathbb{Z} \times \mathbb{Z}^* / \sim$

$\mathbb{N}$

$\{0\}$

$-\mathbb{N}$

whole number.

$i = \sqrt{-2}$

$(a, b) \sim (c, d)$

$ad = bc$

Geometer's Sketchpad

Cinderella

Galois theory

Field th

$ax - b$

$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$

$\mathbb{C}^n$

$\mathbb{Z}_p$

$S/\sim$

$a \in S$

$[a] = \{ b \in S \mid a \sim b \}$

$(1,1) \sim (2,2)$

$(2,4) \sim (1,2)$

$\frac{p}{q} = \frac{2p}{2q}$

$\frac{2}{4} \quad \frac{1}{2}$

$\frac{\mathbb{Z}}{\mathbb{Z}^*}$

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z},\ q \neq 0 \right\}$$

$(p, q) = 1$

$\frac{1}{2}$ $\quad$ $\frac{2}{4}$

$\frac{1}{-2}$ $\quad$ $\frac{-1}{2}$

$$\mathbb{Q} = \{ [(a, b)] \}$$