Indian Institute of Technology Bombay **A**

Department of Mathematics

MA 106: Linear Algebra : Quiz 1

Roll Number....................... **Name** .... **Div**....... **Batch**........

Date: 24 March 2023 Max. Marks 15

Duration : 8.00-9.00 am Weightage 100/3 %

Instructions:

(1) **Failure to follow these instructions will carry a penalty of 2 marks.**

(2) Enter your Division number, Tutorial batch, Roll number and signature on both pages of the question paper and on each page of your answer-book that displays solutions of three subjective questions.

(3) Try to write solutions of subjective questions on the first three pages of the answer-book. Use the remaining pages for rough calculations.

(4) Using Adobe scanner only, scan the two sides of the question paper first and then scan the pages that contain solutions to three subjective questions preferably on the first three pages.

(5) The scanned answer-book along with question paper must be uploaded on Moodle as a single file within 10 minutes during 9.00-9.10 am.

(6) You may not use any electronic devices, notes or books during the examination up to 9:00 AM.

(7) Answers for subjective questions unsupported by satisfactory reasoning will not be awarded any marks.

(8) No supplementary sheet will be provided.

(9) $\mathbb{R}$ denotes the set of all real numbers. Unless otherwise, all matrices and scalars are real.

(10) The marks for each question are mentioned at the end of the question.

Subjective questions

(1) (A) Using the REF of the coefficient matrix, find all real numbers a for which the following system of linear equations has a solution.

$$\begin{aligned} 3x - y + az &= 1 \\ 3x - y + z &= 5 \end{aligned}$$

(B) If a is such that the above system has a solution then write all the solutions. **[3]**

Answer:

(A) We write the given system as

$$\left[\begin{array}{ccc|c} 3 & -1 & a & 1 \\ 3 & -1 & 1 & 5 \end{array}\right].$$

Applying the row operation $R_2 \mapsto R_2 - R_1$ we get

$$\left[\begin{array}{ccc|c} 3 & -1 & a & 1 \\ 0 & 0 & 1-a & 4 \end{array}\right].$$

Hence, the given system has a solution if and only if $1 - a \neq 0$, i.e., $a \neq 1$. [1]

(B) From the solution of (A) above we know that the system has a solution if and only if $a \neq 1$. So. assume that $a \neq 1$. Then we get $(1-a)z = 4$, i.e., $z = \dfrac{4}{1-a}$.

We also have $3x - y + az = 1$. Hence, $3x - y + \frac{4a}{1-a} = 1$, i.e., $y = 3x + \frac{5a-1}{1-a}$. [1]

Therefore, the set of all solutions of the given system is as follows:

$$\left\{\left(x, 3x + \frac{5a-1}{1-a}, \frac{4}{1-a}\right) \in \mathbb{R}^3 \mid x \in \mathbb{R}\right\}.$$

[1]

[No marks for just mentioning that there are infinitely many solutions or that the solution set is a line if no justification is provided.]

(2) Find all 2×2 matrices such that $A^2 = I$. Show that there are infinitely many such matrices. [**3**]

Answer: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be such that $A^2 = I$. We have $A^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{bmatrix}$. Thus, we must have $a + d = 0$ or $b = c = 0$. [1]

If $b = c = 0$, then $a = \pm 1$, $b = \pm 1$.
If $a + d = 0$, then $A^2 = I$ if and only if $a^2 + bc = 1$.
Hence, the set of all 2×2 matrices satisfying $A^2 = I$ is as follows:

$$\left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a, b \in \{1, -1\}\right\} \bigcup \left\{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a^2 + bc = 1\right\}.$$

[1]

In particular, for every $n \in \mathbb{N}$, we have a matrix $A_n = \begin{bmatrix} 0 & 1/n \\ n & 0 \end{bmatrix}$ such that $A_n^2 = I$. This shows that there are infinitely many 2×2 matrices satisfying $A^2 = I$. [1]

(3) Prove that (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear points if and only if

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

[**3**]

Answer: The points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear *if and only if* there exists a line $ax + by + c = 0$ in $\mathbb{R}^2$ such that $ax_i + by_i + c = 0$ for $i = 1, 2, 3$ *if and only if* the system

$$\begin{aligned} x_1 a + y_1 b + c &= 0 \\ x_2 a + y_2 b + c &= 0 \\ x_3 a + y_3 b + c &= 0 \end{aligned}$$

has a nonzero solution *if and only if* the matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is not invertible *if and only if*

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

[3 marks if the entire solution is correct. Two marks if one implication is proved correctly.]

Alternate solution: Note that if the points $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$ are not distinct, then the result is clear, because in this case the given determinant is zero. So assume that A, B, C are distinct. Now, A, B, C are collinear if and only if the slopes of lines AB and BC are equal, i.e., $(y_2 - y_1)(x_3 - x_2) = (y_3 - y_2)(x_2 - x_1)$. [1]
This is equivalent to

$$x_3y_2 - x_3y_1 + x_2y_1 = x_2y_3 - x_1y_3 + x_1y_2$$

i.e.,

$$(x_1y_2 - x_2y_1) - (x_1y_3 - x_3y_1) + (x_2y_3 - x_3y_2).$$

[1]

or equivalently

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

[1]

*[Any other solution which uses some formula (e.g. formula for area of triangle in terms of determinant) without proving it will not be given more than 1 marks.]

MCQ type questions: Circle the right answer

(1) Consider the following two options:

(1) There exists A and B two $n \times n$ matrices for $n \geq 2$ such that $AB = 0$ but $BA \neq 0$.
(2) There exists C and D two $n \times n$ matrices for $n \geq 2$ such that $CD = I$ but $DC \neq I$.

Choose the correct response:
(A) Both options are true. (B) Only (1) is true. (C) Only (2) is true. (D) None can happen.
[1]

Answer: (B).

To show that (1) is true, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

To show that (2) is false, first note that if $CD = I$, then $\det(D) \neq 0$, and thus D is invertible. Now, $D = D(CD)$ gives $(DC - I)D = 0$. Hence, $0 = (DC - I)DD^{-1} = DC - I$.

(2) Let A be a square matrix of size $n \geq 2$ and A^2 is symmetric. Then

(A) A must be symmetric
(B) A must be skew symmetric
(C) A must be symmetric or skew-symmetric
(D) None of these.

[**1**]

Answer: (D).

Note that for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $A^2 = 0$.

(3) If a square matrix A is nilpotent, then:

(A) A is invertible (B) $\det A = 1$ (C) $I + A$ is invertible (D) None of the above.

[**1**]

Answer: (C).
If A is nilpotent, then $A^n = 0$ for some $n \in \mathbb{N}$. Hence, $I = (I+A)(I-A+A^2-A^3+\cdots+(-1)^n A^n)$.

(4) The statement that there is a 3×3 matrix whose row space and the null-space are lines in $\mathbb{R}^3$ is

(A) True (B) False (C) is true sometime but not always. (D) None of the above.

[**1**]

Answer: (B).

If the row space of a 3×3 matrix A is a line, then $A = \begin{bmatrix} \alpha a & \alpha b & \alpha c \\ \beta a & \beta b & \beta c \\ \gamma a & \gamma b & \gamma c \end{bmatrix}$, where $[a\ b\ c] \neq [0\ 0\ 0]$, and at least one of α, β, γ is nonzero. Hence, the null space of A is

$$\{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0\},$$

which is a plane in $\mathbb{R}^3$.

(5) The row canonical form for the matrix: $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 4 & 5 & 5 \end{bmatrix}$ is

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. [**1**]

Answer: (D).
Apply elementary row operations to get the row canonical form.

(6) If the row canonical form of a matrix A is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then the linear system of three equations in three unknowns: $AX = b$ must have

(A) Exactly one solution. (B) No solution (C) infinitely many solutions (D) Depends on b. [**1**]

Answer: (D).

To see that the solution set depends on b, take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and consider (1) $b = (0,0,0)^t$, (2) $b = (0,0,1)^t$.

(7) Let A be a 3×3 so that $A(x,y,z)^t = (z,x,y)^t$ for all $x,y,z \in \mathbb{R}$. Then

(A) $\det A = 1$ (B) $\det A = -1$ (C) $\det A = 0$ (D) $\det A$ can be both $1, -1$.

[**1**]

Answer: (A).
Let A_1, A_2, A_3 be columns of A. Then we have $A_1 = A(1,0,0)^t = (0,1,0)^t, A_2 = A(0,1,0)^t = (0,0,1)^t, A_3 = A(0,0,1)^t = (1,0,0)^t$; so that $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Therefore, $\det(A) = 1$.

(8) Let $\mathcal{C}(AB)$ and $\mathcal{C}(A)$ and $\mathcal{C}(B)$ denote the column space of AB, A, B where A and B are $n \times n$ matrices. Choose the right response:
(A) $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$ (B) $\mathcal{C}(A) \subseteq \mathcal{C}(AB)$ (C) $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ (D) $\mathcal{C}(B) \subseteq \mathcal{C}(AB)$ [**1**]

Answer: (A).

If $A_1, \ldots, A_n$ are columns of A, then $\mathcal{C}(A) = \{x_1A_1 + \cdots + x_nA_n \mid (x_1, \ldots, x_n) \in \mathbb{R}^n\}$, and $\mathcal{C}(AB) = \{x_1A_1 + \cdots + x_nA_n \mid (x_1, \ldots, x_n) \in \mathcal{C}(B)\}$. Since $\mathcal{C}(B) \subseteq \mathbb{R}^n$, it follows that $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$.

To show that (B) is false, take $A = I$ and $B = 0$.

To show that (C) is false, take $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

To show that (D) is false, take $A = 0$, $B = I$.

(9) Let r be a real number and A be a 3×3 real matrix. Then the number of values of r so that $rI + A$ (where I is the 3×3 identity matrix) is invertible is (A) atmost 2 (B) infinite (C) atmost 3 (D) atmost 1.
Answer: (B).
Note that $rI + A$ is invertible if and only if $\det(rI + A) \neq 0$. Since $\det(rI + A)$ is a degree 3 polynomial in r with real coefficients, it has at most 3 distinct roots. Thus, for all r except the roots of $\det(rI + A) = 0$, we have that $rI + A$ is invertible.

MA 106 Endsem Exam : Part A : Question Paper

Indian Institute of Technology Bombay

A

Roll No.: ....

Name: ....

Division.: D ☐

Tutorial: T ☐

📅 Apr. 19, 2023

🕓 8.30 - 10.30 AM

✓ 9.30 AM (Part A)

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

➔ There are 14 questions in Part A. Each question contains a **single correct answer** of one mark.
➔ You need to indicate your answer by bubbling the box in the OMR (Optical Mark Recognition) sheet. The right way to fill a box like ☐ is as ■. Use only a ball–point pen to fill the correct choice in the OMR sheet.
➔ **Answer provided in the OMR sheet will only be evaluated.** You can mark answer in the question paper for your reference but it will not be evaluated.
➔ The OMR sheet will be collected back at 9.30 AM.
➔ **Notation:** $\mathbb{R}^{m\times n}$: the set of all $m \times n$ real matrices, A^t: the transpose of A, E_λ: the eigenspace for an eigenvalue λ, $\chi_A(x)$: the characteristic polynomial of A, Null(A): the nullspace of A, $\mathcal{C}(A)$: the column space of A.

Q-1⟩ Consider V $= \mathbb{R}^{2\times 2}$ and the inner product $\langle A, B\rangle = \mathsf{tr}(A^tB)$. Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $p_I(J)$ denote the orthogonal projection of J along the identity matrix I. Then $p_I(J)$ is

■ A I ☐ B 0 ☐ C J ☐ D $2I$

Soln.: We have $P_I(J) = \dfrac{\langle I, J\rangle}{\langle I, I\rangle} I = \dfrac{tr(J)}{2} I = I.$ ✠

Q-2⟩ Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(u) = Au$ where $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}$. Then

■ A Null(T) is a line.

☐ B Null$(T) = \{0\}$.

☐ C Im(T) is a line.

☐ D Null(T) is a plane.

Soln.: Note that $\det A = 0$ and $\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} \neq 0$. Therefore, rank$(A) = 2$ and so nuility$(A) = 1$. ✠

Q-3⟩ Let $P_2(\mathbb{R})$ be the space of polynomials over $\mathbb{R}$ of degree ≤ 2. Define $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ by $T(f(x)) = (f(0), f(-1), f(1))^t$. Then rank$(T)$ is

☐ A 4 ☐ B 1 ☐ C 2 ■ D 3

Soln.: Let $f(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$. Then we have $f(-1) = a - b + c, f(0) = a, f(1) = a + b + c$. If $T(f) = 0$, then $a = b = c = 0$. This shows that f is one-one. Hence, T is an isomorphism. So, $rank(T) = 3$. ✠

Q-4⟩ Let A be a 3×3 real matrix and $A^2 = A$. Then $\dim(\text{Null}(A) \cap \mathcal{C}(A))$ is

☐ A 3 ☐ B 1 ☒ C 0 ☐ D 2

Soln.: If $v = Ax \in null(A) \cap \mathcal{C}(A)$, for some $x \in \mathbb{R}^3$, then $0 = Av = A^2x = Ax = v$. ✠

Q-5⟩ The number of 2×2 nilpotent real matrices is

☒ A infinite ☐ B 2 ☐ C 1 ☐ D 3

Soln.: For every $n \in \mathbb{Z}$ we have that $\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ is nilpotent. ✠

Q-6⟩ Let A be a 3×3 real matrix with $\mathtt{tr}(A) = 0$. Let $\chi_A(2) = \chi_A(3) = 0$. Then

☐ A $\chi_A(x) = (x-2)(x-3)$

☐ C $\chi_A(x) = (x-2)^2(x-3)$

☒ B $\chi_A(x) = (x+5)(x-2)(x-3)$

☐ D $\chi_A(x) = (x-2)(x-3)^2$

Soln.: Since $2, 3$ are roots of the $\chi_A(x)$, and since the sum of all the roots of $\chi_A(x) = tr(A) = 0$, we have that -5 is a root of $\chi_A(x)$. ✠

Q-7⟩ Consider positive real numbers a, b, c and matrix $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$. Let $\lambda = a + b + c$. Then

☐ A $\dim E_0 = 1,\ \dim E_\lambda = 1.$

☐ C $\dim E_0 = 2,\ \dim E_\lambda = 2.$

☒ B $\dim E_0 = 2,\ \dim E_\lambda = 1.$

☐ D $\dim E_0 = 1,\ \dim E_\lambda = 2.$

Soln.: We have $rank(A) = 2$ and $nullity(A) = 2$. So, 0 is an eigenvalue of A with $\dim(E_0) = 2$. Also, $A[1\ 1\ 1]^t = (a+b+c)[1\ 1\ 1]^t$. So, $\lambda = a + b + c$ is an eigenvalue of A. Since $\dim(E_0) + \dim(E_\lambda) = 3$, we get $\dim(E_\lambda) = 1$. ✠

Q-8⟩ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $(a-d)^2 + 4bc > 0$. Then

☐ A None of these answers are correct.

☐ B A is not diagonalizable over $\mathbb{R}$.

☐ C Eigenvalues of A are not real.

☒ D There is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

Soln.: Consider $\chi_A(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = x^2 - (a+d)x + (ad-bc)$.
Roots of $\chi_A(x)$ are

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right],$$

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right].$$

From the given condition we get $\lambda_1 \neq \lambda_2$. Thus, A is diagonalizable as the eigenvectors corresponding to λ_1 and λ_2 are linearly independent. ✠

Q-9⟩ Let $\mathrm{V} = \mathbb{R}^{2\times 2}$ and $P \in \mathrm{V}$ be a nonzero matrix. Consider the linear transformation $T : \mathrm{V} \to \mathrm{V}$ given by $T(A) = AP - PA$. Then

☐ A T is 1-1.

☒ B None of these answers are correct.

☐ C T is an isomorphism.

☐ D T is onto.

Soln.: Clearly $T(P) = 0$. Hence, T is not 1-1. Thus, T is not onto either. Hence, none of these answers are correct. ✠

Q-10⟩ If A is a 3×3 real matrix and $A^2 + I = 0$ then

☐ A $-1, i, -i$ are roots of $\chi_A(x)$.

☐ B $1, i, -i$ are roots of $\chi_A(x)$.

☐ C i and $-i$ are the only roots of $\chi_A(x)$.

☒ D There is no such matrix.

Soln.: Since $\chi_A(x)$ has degree 3, it has a real root r. As $A_I^2 = 0$, we get $r^2 = -1$. This is a contradiction. Therefore no such matrix exists. ✠

Q-11⟩ Let V be the vector space over $\mathbb{R}$ of 2×2 Hermitian matrices. Then dimension of V is

☒ A 4 ☐ B 2 ☐ C 1 ☐ D 3

Soln.: Any 2×2 Hermitian matrix A has the form $A = \begin{bmatrix} x & y+iz \\ y-iz & w \end{bmatrix}$ for some $x, y, z, w \in \mathbb{R}$. So,

$$A = x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that $\dim(V) = 4$. ✠

Q-12⟩ The conic described by the $5x^2 - 4xy + 8y^2 = 36$ is

☐ A a pair of lines. ☐ B a hyperbola. ☐ C a parabola. ☒ D an ellipse.

Soln.: The equation is

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 36.$$

Let $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$. Then $\chi_A(x) = \det(xI - A) = (x-4)(x-9)$. As the eigenvalues are 4 and 9, the equation in u, r plane is given by

$$4u^2 + 9r^2 = 36 \implies \frac{u^2}{9} + \frac{v^2}{1} = 1$$

This is as an ellipse. ✠

Q-13⟩ The rank of the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by $T(A) = PA$ where $A \in \mathbb{R}^{2\times 2}$ and $P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is

☒ A 2 ☐ B 3 ☐ C 1 ☐ D 4

Soln.: Clearly, $\text{Null}(T) = \{A \mid PA = 0\}$. It follows that

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} \in \text{Null}(T) \Leftrightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x & y \\ a & b \end{bmatrix} = 0 \Leftrightarrow \begin{matrix} x + 3a = 0 \\ y + 3b = 0 \end{matrix} \Leftrightarrow x = -3a, \quad y = -3b.$$

Therefore,

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} = a \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\dim \text{Null}(T) = 2$ and so $\text{rank}(T) = 2$. ✠

Q-14⟩ Let V $= \mathbb{R}^{3\times 3}$ and $\langle A, B \rangle = \mathtt{tr}(AB^t)$. Let D denote the subspace of diagonal matrices in V. Then $\dim D^\perp$ is

☒ A 6 ☐ B 9 ☐ C 5 ☐ D 3

Soln.: Since $\dim(D) = 3$ and $\dim(D) + \dim(V^\perp) = \dim(V) = 9$, we get that $\dim(V^\perp) = 6$. ✠

+1/5/56+

MA 106 Part A : Optical Mark Recognition (OMR) Sheet

Name: ....

Roll No.: ....

Division.: D ☐

Tutorial: T ☐

Stud. Sign.: ....

Invig. Sign.: ....

A

Bubble Your Rollnumber

....

☐ 0	☐ 0	☐ 0	☐ 0	☐ 0	☐ 0	☐ 0	☐ 0	☐ 0
☐ 1	☐ 1	☐ 1	☐ 1	☐ 1	☐ 1	☐ 1	☐ 1	☐ 1
☐ 2	☐ 2	☐ 2	☐ 2	☐ 2	☐ 2	☐ 2	☐ 2	☐ 2
☐ 3	☐ 3	☐ 3	☐ 3	☐ 3	☐ 3	☐ 3	☐ 3	☐ 3
☐ 4	☐ 4	☐ 4	☐ 4	☐ 4	☐ 4	☐ 4	☐ 4	☐ 4
☐ 5	☐ 5	☐ 5	☐ 5	☐ 5	☐ 5	☐ 5	☐ 5	☐ 5
☐ 6	☐ 6	☐ 6	☐ 6	☐ 6	☐ 6	☐ 6	☐ 6	☐ 6
☐ 7	☐ 7	☐ 7	☐ 7	☐ 7	☐ 7	☐ 7	☐ 7	☐ 7
☐ 8	☐ 8	☐ 8	☐ 8	☐ 8	☐ 8	☐ 8	☐ 8	☐ 8
☐ 9	☐ 9	☐ 9	☐ 9	☐ 9	☐ 9	☐ 9	☐ 9	☐ 9
		☐ B		☐ B				
		☐ D						

INSTRUCTIONS: ➜ Use only blue/black colored ball-point pen to make a bubble.
➜ The right way to fill a box like ☐ is as ■.
➜ The OMR sheet will be evaluated by using computer, so **no writing is allowed anywhere in the OMR sheet except dedicated places**.
➜ **No rewriting, no overwriting, no erasing, and no cancellation** are allowed.
➜ Before bubbling your rollnumber, first convert the rollnmber into nine characters by appending appropriate leading zero(s) at the beginning and bubble that nine characters rollnumber so that in the ith column you bubble the ith character only. For example, if your rollnumber is 22B1234 then first convert it into 0022B2134 and then bubble it accordingly.
➜ Each question has single correct answer, so bubbling more than one option in a single question will fetch zero mark.

Bubble the box corresponds to your correct choice.

Q-1⟩ ☒ A ☐ B ☐ C ☐ D
Q-2⟩ ☒ A ☐ B ☐ C ☐ D
Q-3⟩ ☐ A ☐ B ☐ C ☒ D
Q-4⟩ ☐ A ☐ B ☒ C ☐ D
Q-5⟩ ☒ A ☐ B ☐ C ☐ D
Q-6⟩ ☐ A ☒ B ☐ C ☐ D
Q-7⟩ ☐ A ☒ B ☐ C ☐ D
Q-8⟩ ☐ A ☐ B ☐ C ☒ D
Q-9⟩ ☐ A ☒ B ☐ C ☐ D
Q-10⟩ ☐ A ☐ B ☐ C ☒ D
Q-11⟩ ☒ A ☐ B ☐ C ☐ D
Q-12⟩ ☐ A ☐ B ☐ C ☒ D
Q-13⟩ ☒ A ☐ B ☐ C ☐ D
Q-14⟩ ☒ A ☐ B ☐ C ☐ D

MA 106 Endsem Exam : Part A : Question Paper
Indian Institute of Technology Bombay

B

Roll No.: ....

Name: ....

Division.: D ☐

Tutorial: T ☐

📅 Apr. 19, 2023

🕘 8.30 - 10.30 AM

✓ 9.30 AM (Part A)

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

➜ There are 14 questions in Part A. Each question contains a **single correct answer** of one mark.
➜ You need to indicate your answer by bubbling the box in the OMR (Optical Mark Recognition) sheet. The right way to fill a box like ☐ is as ■. Use only a ball–point pen to fill the correct choice in the OMR sheet.
➜ **Answer provided in the OMR sheet will only be evaluated.** You can mark answer in the question paper for your reference but it will not be evaluated.
➜ The OMR sheet will be collected back at 9.30 AM.
➜ **Notation:** $\mathbb{R}^{m\times n}$: the set of all $m \times n$ real matrices, A^t: the transpose of A, E_λ: the eigenspace for an eigenvalue λ, $\chi_A(x)$: the characteristic polynomial of A, $\text{Null}(A)$: the nullspace of A, $\mathcal{C}(A)$: the column space of A.

Q-1⟩ Let A be a 3×3 real matrix and $A^2 = A$. Then $\dim(\text{Null}(A) \cap \mathcal{C}(A))$ is

☐ A 1 ■ B 0 ☐ C 3 ☐ D 2

Soln.: If $v = Ax \in null(A) \cap \mathcal{C}(A)$, for some $x \in \mathbb{R}^3$, then $0 = Av = A^2x = Ax = v$. ✠

Q-2⟩ If A is a 3×3 real matrix and $A^2 + I = 0$ then

☐ A $-1, i, -i$ are roots of $\chi_A(x)$.

☐ B i and $-i$ are the only roots of $\chi_A(x)$.

☐ C $1, i, -i$ are roots of $\chi_A(x)$.

■ D There is no such matrix.

Soln.: Since $\chi_A(x)$ has degree 3, it has a real root r. As $A_I^2 = 0$, we get $r^2 = -1$. This is a contradiction. Therefore no such matrix exists. ✠

Q-3⟩ The conic described by the $5x^2 - 4xy + 8y^2 = 36$ is

☒ A an ellipse. ☐ B a parabola. ☐ C a pair of lines. ☐ D a hyperbola.

Soln.: The equation is

$$[x \quad y]\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 36.$$

Let $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$. Then $\chi_A(x) = \det(xI - A) = (x-4)(x-9)$. As the eigenvalues are 4 and 9, the equation in u, r plane is given by

$$4u^2 + 9r^2 = 36 \implies \frac{u^2}{9} + \frac{v^2}{1} = 1$$

This is as an ellipse. ✠

Q-4⟩ Consider positive real numbers a, b, c and matrix $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$. Let $\lambda = a + b + c$. Then

☐ A $\dim E_0 = 1,\ \dim E_\lambda = 2.$

☐ B $\dim E_0 = 2,\ \dim E_\lambda = 2.$

☒ C $\dim E_0 = 2,\ \dim E_\lambda = 1.$

☐ D $\dim E_0 = 1,\ \dim E_\lambda = 1.$

Soln.: We have $rank(A) = 2$ and $nullity(A) = 2$. So, 0 is an eigenvalue of A with $\dim(E_0) = 2$. Also, $A[1\ 1\ 1]^t = (a+b+c)[1\ 1\ 1]^t$. So, $\lambda = a + b + c$ is an eigenvalue of A. Since $\dim(E_0) + \dim(E_\lambda) = 3$, we get $\dim(E_\lambda) = 1$. ✠

Q-5⟩ Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(u) = Au$ where $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}$. Then

☐ A $\text{Null}(T) = \{0\}$.

☐ B $\text{Im}(T)$ is a line.

☒ C $\text{Null}(T)$ is a line.

☐ D $\text{Null}(T)$ is a plane.

Soln.: Note that $\det A = 0$ and $\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} \neq 0$. Therefore, $\text{rank}(A) = 2$ and so $\text{nuility}(A) = 1$. ✠

Q-6⟩ Let A be a 3×3 real matrix with $\text{tr}(A) = 0$. Let $\chi_A(2) = \chi_A(3) = 0$. Then

☐ A $\chi_A(x) = (x-2)(x-3)$

☐ B $\chi_A(x) = (x-2)(x-3)^2$

☒ C $\chi_A(x) = (x+5)(x-2)(x-3)$

☐ D $\chi_A(x) = (x-2)^2(x-3)$

Soln.: Since $2, 3$ are roots of the $\chi_A(x)$, and since the sum of all the roots of $\chi_A(x) = tr(A) = 0$, we have that -5 is a root of $\chi_A(x)$. ✠

Q-7⟩ The number of 2×2 nilpotent real matrices is

☐ A 1 ☐ B 3 ☐ C 2 ☒ D infinite

Soln.: For every $n \in \mathbb{Z}$ we have that $\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ is nilpotent. ✠

Q-8⟩ Let V be the vector space over $\mathbb{R}$ of 2×2 Hermitian matrices. Then dimension of V is

☐ A 2 ☐ B 1 ☒ C 4 ☐ D 3

Soln.: Any 2×2 Hermitian matrix A has the form $A = \begin{bmatrix} x & y+iz \\ y-iz & w \end{bmatrix}$ for some $x, y, z, w \in \mathbb{R}$. So,

$$A = x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that $\dim(V) = 4$. ✠

Q-9⟩ Consider $\mathrm{V} = \mathbb{R}^{2\times 2}$ and the inner product $\langle A, B\rangle = \mathsf{tr}(A^t B)$. Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $p_I(J)$ denote the orthogonal projection of J along the identity matrix I. Then $p_I(J)$ is

☐ A $2I$ ☐ B 0 ☐ C J ☒ D I

Soln.: We have $P_I(J) = \dfrac{\langle I, J\rangle}{\langle I, I\rangle} I = \dfrac{tr(J)}{2} I = I$. ✠

Q-10⟩ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $(a-d)^2 + 4bc > 0$. Then

☒ A There is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

☐ B Eigenvalues of A are not real.

☐ C A is not diagonalizable over $\mathbb{R}$.

☐ D None of these answers are correct.

Soln.: Consider $\chi_A(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = x^2 - (a+d)x + (ad - bc)$.
Roots of $\chi_A(x)$ are

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right],$$

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right].$$

From the given condition we get $\lambda_1 \neq \lambda_2$. Thus, A is diagonalizable as the eigenvectors corresponding to λ_1 and λ_2 are linearly independent. ✠

Q-11⟩ Let $\mathrm{V} = \mathbb{R}^{2\times 2}$ and $P \in \mathrm{V}$ be a nonzero matrix. Consider the linear transformation $T : \mathrm{V} \to \mathrm{V}$ given by $T(A) = AP - PA$. Then

☒ A None of these answers are correct.

☐ B T is an isomorphism.

☐ C T is onto.

☐ D T is 1-1.

Soln.: Clearly $T(P) = 0$. Hence, T is not 1-1. Thus, T is not onto either. Hence, none of these answers are correct. ✠

Q-12⟩ Let $P_2(\mathbb{R})$ be the space of polynomials over $\mathbb{R}$ of degree ≤ 2. Define $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ by $T(f(x)) = (f(0), f(-1), f(1))^t$. Then rank$(T)$ is

☐ A 1 ☐ B 2 ☐ C 4 ☒ D 3

Soln.: Let $f(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$. Then we have $f(-1) = a - b + c, f(0) = a, f(1) = a + b + c$. If $T(f) = 0$, then $a = b = c = 0$. This shows that f is one-one. Hence, T is an isomorphism. So, $rank(T) = 3$. ✠

Q-13⟩ The rank of the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by $T(A) = PA$ where $A \in \mathbb{R}^{2\times 2}$ and $P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is

☐ A 4 ☐ B 1 ☐ C 3 ☒ D 2

Soln.: Clearly, $\mathrm{Null}(T) = \{A \mid PA = 0\}$. It follows that

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} \in \mathrm{Null}(T) \Leftrightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x & y \\ a & b \end{bmatrix} = 0 \Leftrightarrow \begin{matrix} x + 3a = 0 \\ y + 3b = 0 \end{matrix} \Leftrightarrow x = -3a, \quad y = -3b.$$

Therefore,

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} = a \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\dim \mathrm{Null}(T) = 2$ and so $\mathrm{rank}(T) = 2$. ✠

Q-14⟩ Let $\mathrm{V} = \mathbb{R}^{3\times 3}$ and $\langle A, B \rangle = \mathrm{tr}(AB^t)$. Let D denote the subspace of diagonal matrices in V. Then $\dim D^\perp$ is

☐ A 5 ☐ B 9 ☒ C 6 ☐ D 3

Soln.: Since $\dim(D) = 3$ and $\dim(D) + \dim(V^\perp) = \dim(V) = 9$, we get that $\dim(V^\perp) = 6$. ✠

MA 106 Part A : Optical Mark Recognition (OMR) Sheet

Name: ....

Roll No.: ....

Division.: D ☐

Tutorial: T ☐

Stud. Sign.: ....

Invig. Sign.: ....

B

Bubble Your Rollnumber

....

0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9
		B		B				
		D						

INSTRUCTIONS: ➜ Use only blue/black colored ball-point pen to make a bubble.
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Bubble the box corresponds to your correct choice.

Q-1⟩	A	[✓ B]	C	D	Q-8⟩	A	B	[✓ C]	D
Q-2⟩	A	B	C	[✓ D]	Q-9⟩	A	B	C	[✓ D]
Q-3⟩	[✓ A]	B	C	D	Q-10⟩	[✓ A]	B	C	D
Q-4⟩	A	B	[✓ C]	D	Q-11⟩	[✓ A]	B	C	D
Q-5⟩	A	B	[✓ C]	D	Q-12⟩	A	B	C	[✓ D]
Q-6⟩	A	B	[✓ C]	D	Q-13⟩	A	B	C	[✓ D]
Q-7⟩	A	B	C	[✓ D]	Q-14⟩	A	B	[✓ C]	D

MA 106 Endsem Exam : Part A : Question Paper
Indian Institute of Technology Bombay

C

Roll No.: ....

Name: ....

Division.: D ☐

Tutorial: T ☐

📅 Apr. 19, 2023

🕒 8.30 - 10.30 AM

✓ 9.30 AM (Part A)

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

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Q-1⟩ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $(a-d)^2 + 4bc > 0$. Then

☐ A None of these answers are correct.

☐ B A is not diagonalizable over $\mathbb{R}$.

☐ C Eigenvalues of A are not real.

■ D There is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

Soln.: Consider $\chi_A(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = x^2 - (a+d)x + (ad-bc)$.
Roots of $\chi_A(x)$ are

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right],$$

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right].$$

From the given condition we get $\lambda_1 \neq \lambda_2$. Thus, A is diagonalizable as the eigenvectors corresponding to λ_1 and λ_2 are linearly independent. ✠

Q-2⟩ Consider V $= \mathbb{R}^{2\times 2}$ and the inner product $\langle A, B\rangle = \mathrm{tr}(A^tB)$. Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $p_I(J)$ denote the orthogonal projection of J along the identity matrix I. Then $p_I(J)$ is

☐ A 0 ■ B I ☐ C J ☐ D $2I$

Soln.: We have $P_I(J) = \frac{\langle I, J\rangle}{\langle I, I\rangle} I = \frac{tr(J)}{2} I = I.$ ✠

Q-3⟩ Let A be a 3×3 real matrix and $A^2 = A$. Then $\dim(\mathrm{Null}(A) \cap \mathcal{C}(A))$ is

☒ A 0   ☐ B 3   ☐ C 2   ☐ D 1

Soln.: If $v = Ax \in null(A) \cap \mathcal{C}(A)$, for some $x \in \mathbb{R}^3$, then $0 = Av = A^2x = Ax = v$. ✠

Q-4⟩ The rank of the linear transformation $T : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ given by $T(A) = PA$ where $A \in \mathbb{R}^{2\times2}$ and $P = \begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix}$ is

☐ A 4   ☐ B 1   ☒ C 2   ☐ D 3

Soln.: Clearly, $\mathrm{Null}(T) = \{A \mid PA = 0\}$. It follows that

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} \in \mathrm{Null}(T) \Leftrightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x & y \\ a & b \end{bmatrix} = 0 \Leftrightarrow \begin{matrix} x + 3a = 0 \\ y + 3b = 0 \end{matrix} \Leftrightarrow x = -3a, \quad y = -3b.$$

Therefore,

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} = a\begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\dim \mathrm{Null}(T) = 2$ and so $\mathrm{rank}(T) = 2$. ✠

Q-5⟩ If A is a 3×3 real matrix and $A^2 + I = 0$ then

☐ A $1, i, -i$ are roots of $\chi_A(x)$.

☐ C $-1, i, -i$ are roots of $\chi_A(x)$.

☒ B There is no such matrix.

☐ D i and $-i$ are the only roots of $\chi_A(x)$.

Soln.: Since $\chi_A(x)$ has degree 3, it has a real root r. As $A_I^2 = 0$, we get $r^2 = -1$. This is a contradiction. Therefore no such matrix exists. ✠

Q-6⟩ Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(u) = Au$ where $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}$. Then

☐ A $\mathrm{Null}(T)$ is a plane.

☐ C $\mathrm{Im}(T)$ is a line.

☒ B $\mathrm{Null}(T)$ is a line.

☐ D $\mathrm{Null}(T) = \{0\}$.

Soln.: Note that $\det A = 0$ and $\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} \neq 0$. Therefore, $\mathrm{rank}(A) = 2$ and so $\mathrm{nuility}(A) = 1$. ✠

Q-7⟩ The conic described by the $5x^2 - 4xy + 8y^2 = 36$ is

☐ A a pair of lines. ☒ B an ellipse. ☐ C a hyperbola. ☐ D a parabola.

Soln.: The equation is

$$[x \quad y]\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 36.$$

Let $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$. Then $\chi_A(x) = \det(xI - A) = (x-4)(x-9)$. As the eigenvalues are 4 and 9, the equation in u, r plane is given by

$$4u^2 + 9r^2 = 36 \implies \frac{u^2}{9} + \frac{v^2}{1} = 1$$

This is as an ellipse. ✠

Q-8⟩ Consider positive real numbers a, b, c and matrix $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$. Let $\lambda = a + b + c$. Then

☐ A $\dim E_0 = 1,\ \dim E_\lambda = 2.$

☒ B $\dim E_0 = 2,\ \dim E_\lambda = 1.$

☐ C $\dim E_0 = 2,\ \dim E_\lambda = 2.$

☐ D $\dim E_0 = 1,\ \dim E_\lambda = 1.$

Soln.: We have $rank(A) = 2$ and $nullity(A) = 2$. So, 0 is an eigenvalue of A with $\dim(E_0) = 2$. Also, $A[1\ 1\ 1]^t = (a + b + c)[1\ 1\ 1]^t$. So, $\lambda = a + b + c$ is an eigenvalue of A. Since $\dim(E_0) + \dim(E_\lambda) = 3$, we get $\dim(E_\lambda) = 1$. ✠

Q-9⟩ The number of 2×2 nilpotent real matrices is

☒ A infinite ☐ B 1 ☐ C 3 ☐ D 2

Soln.: For every $n \in \mathbb{Z}$ we have that $\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ is nilpotent. ✠

Q-10⟩ Let V be the vector space over $\mathbb{R}$ of 2×2 Hermitian matrices. Then dimension of V is

☐ A 1 ☒ B 4 ☐ C 3 ☐ D 2

Soln.: Any 2×2 Hermitian matrix A has the form $A = \begin{bmatrix} x & y + iz \\ y - iz & w \end{bmatrix}$ for some $x, y, z, w \in \mathbb{R}$. So,

$$A = x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that $\dim(V) = 4$. ✠

Q-11⟩ Let $\mathrm{V} = \mathbb{R}^{2\times 2}$ and $P \in \mathrm{V}$ be a nonzero matrix. Consider the linear transformation $T : \mathrm{V} \to \mathrm{V}$ given by $T(A) = AP - PA$. Then

☐ A T is 1-1.

☒ B None of these answers are correct.

☐ C T is onto.

☐ D T is an isomorphism.

Soln.: Clearly $T(P) = 0$. Hence, T is not 1-1. Thus, T is not onto either. Hence, none of these answers are correct. ✠

Q-12⟩ Let A be a 3×3 real matrix with $\mathsf{tr}(A) = 0$. Let $\chi_A(2) = \chi_A(3) = 0$. Then

☐ A $\chi_A(x) = (x-2)(x-3)^2$

☐ B $\chi_A(x) = (x-2)^2(x-3)$

☐ C $\chi_A(x) = (x-2)(x-3)$

☒ D $\chi_A(x) = (x+5)(x-2)(x-3)$

Soln.: Since $2, 3$ are roots of the $\chi_A(x)$, and since the sum of all the roots of $\chi_A(x) = tr(A) = 0$, we have that -5 is a root of $\chi_A(x)$. ✠

Q-13⟩ Let $P_2(\mathbb{R})$ be the space of polynomials over $\mathbb{R}$ of degree ≤ 2. Define $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ by $T(f(x)) = (f(0), f(-1), f(1))^t$. Then rank$(T)$ is

☐ A 1 ☐ B 2 ☐ C 4 ☒ D 3

Soln.: Let $f(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$. Then we have $f(-1) = a - b + c, f(0) = a, f(1) = a + b + c$. If $T(f) = 0$, then $a = b = c = 0$. This shows that f is one-one. Hence, T is an isomorphism. So, $rank(T) = 3$. ✠

Q-14⟩ Let $\mathrm{V} = \mathbb{R}^{3\times 3}$ and $\langle A, B\rangle = \mathsf{tr}(AB^t)$. Let D denote the subspace of diagonal matrices in V. Then $\dim D^\perp$ is

☒ A 6 ☐ B 9 ☐ C 5 ☐ D 3

Soln.: Since $\dim(D) = 3$ and $\dim(D) + \dim(V^\perp) = \dim(V) = 9$, we get that $\dim(V^\perp) = 6$. ✠

+3/5/46+

MA 106 Part A : Optical Mark Recognition (OMR) Sheet

Name: ....

Roll No.: ....

Division.: D ☐

Tutorial: T ☐

Stud. Sign.: ....

Invig. Sign.: ....

Bubble Your Rollnumber

....

☐0	☐0	☐0	☐0	☐0	☐0	☐0	☐0	☐0
☐1	☐1	☐1	☐1	☐1	☐1	☐1	☐1	☐1
☐2	☐2	☐2	☐2	☐2	☐2	☐2	☐2	☐2
☐3	☐3	☐3	☐3	☐3	☐3	☐3	☐3	☐3
☐4	☐4	☐4	☐4	☐4	☐4	☐4	☐4	☐4
☐5	☐5	☐5	☐5	☐5	☐5	☐5	☐5	☐5
☐6	☐6	☐6	☐6	☐6	☐6	☐6	☐6	☐6
☐7	☐7	☐7	☐7	☐7	☐7	☐7	☐7	☐7
☐8	☐8	☐8	☐8	☐8	☐8	☐8	☐8	☐8
☐9	☐9	☐9	☐9	☐9	☐9	☐9	☐9	☐9
		☐B		☐B				
		☐D						

C

INSTRUCTIONS: ➔ Use only blue/black colored ball-point pen to make a bubble.
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Bubble the box corresponds to your correct choice.

Q-1⟩ ☐A ☐B ☐C ■D
Q-2⟩ ☐A ■B ☐C ☐D
Q-3⟩ ■A ☐B ☐C ☐D
Q-4⟩ ☐A ☐B ■C ☐D
Q-5⟩ ☐A ■B ☐C ☐D
Q-6⟩ ☐A ■B ☐C ☐D
Q-7⟩ ☐A ■B ☐C ☐D
Q-8⟩ ☐A ■B ☐C ☐D
Q-9⟩ ■A ☐B ☐C ☐D
Q-10⟩ ☐A ■B ☐C ☐D
Q-11⟩ ☐A ■B ☐C ☐D
Q-12⟩ ☐A ☐B ☐C ■D
Q-13⟩ ☐A ☐B ☐C ■D
Q-14⟩ ■A ☐B ☐C ☐D

MA 106 Endsem Exam : Part A : Question Paper

Indian Institute of Technology Bombay

D

Roll No.: ....

Name: ....

Division.: D ☐

Tutorial: T ☐

📅 Apr. 19, 2023

🕘 8.30 - 10.30 AM

✓ 9.30 AM (Part A)

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

➜ There are 14 questions in Part A. Each question contains a **single correct answer** of one mark.
➜ You need to indicate your answer by bubbling the box in the OMR (Optical Mark Recognition) sheet. The right way to fill a box like ☐ is as ■. Use only a ball–point pen to fill the correct choice in the OMR sheet.
➜ **Answer provided in the OMR sheet will only be evaluated.** You can mark answer in the question paper for your reference but it will not be evaluated.
➜ The OMR sheet will be collected back at 9.30 AM.
➜ **Notation:** $\mathbb{R}^{m\times n}$: the set of all $m \times n$ real matrices, A^t: the transpose of A, E_λ: the eigenspace for an eigenvalue λ, $\chi_A(x)$: the characteristic polynomial of A, Null(A): the nullspace of A, $\mathcal{C}(A)$: the column space of A.

Q-1⟩ The number of 2×2 nilpotent real matrices is

☒ A infinite ☐ B 3 ☐ C 2 ☐ D 1

Soln.: For every $n \in \mathbb{Z}$ we have that $\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ is nilpotent. ✠

Q-2⟩ Let $P_2(\mathbb{R})$ be the space of polynomials over $\mathbb{R}$ of degree ≤ 2. Define $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ by $T(f(x)) = (f(0), f(-1), f(1))^t$. Then rank($T$) is

☐ A 4 ☒ B 3 ☐ C 2 ☐ D 1

Soln.: Let $f(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$. Then we have $f(-1) = a - b + c$, $f(0) = a$, $f(1) = a + b + c$. If $T(f) = 0$, then $a = b = c = 0$. This shows that f is one-one. Hence, T is an isomorphism. So, $rank(T) = 3$. ✠

Q-3⟩ The rank of the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by $T(A) = PA$ where $A \in \mathbb{R}^{2\times 2}$ and $P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is

☐ A 1 ☐ B 4 ☐ C 3 ☒ D 2

Soln.: Clearly, $\text{Null}(T) = \{A \mid PA = 0\}$. It follows that

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} \in \text{Null}(T) \Leftrightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x & y \\ a & b \end{bmatrix} = 0 \Leftrightarrow \begin{matrix} x + 3a = 0 \\ y + 3b = 0 \end{matrix} \Leftrightarrow x = -3a, \quad y = -3b.$$

Therefore,

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} = a \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\dim \text{Null}(T) = 2$ and so $\text{rank}(T) = 2$. ✠

Q-4⟩ Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(u) = Au$ where $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}$. Then

☐ A $\text{Null}(T) = \{0\}$.

☐ B $\text{Im}(T)$ is a line.

☐ C $\text{Null}(T)$ is a plane.

☒ D $\text{Null}(T)$ is a line.

Soln.: Note that $\det A = 0$ and $\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} \neq 0$. Therefore, $\text{rank}(A) = 2$ and so $\text{nuility}(A) = 1$. ✠

Q-5⟩ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $(a-d)^2 + 4bc > 0$. Then

☐ A None of these answers are correct.

☐ B Eigenvalues of A are not real.

☐ C A is not diagonalizable over $\mathbb{R}$.

☒ D There is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

Soln.: Consider $\chi_A(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = x^2 - (a+d)x + (ad-bc)$.

Roots of $\chi_A(x)$ are

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right],$$

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right].$$

From the given condition we get $\lambda_1 \neq \lambda_2$. Thus, A is diagonalizable as the eigenvectors corresponding to λ_1 and λ_2 are linearly independent. ✠

Q-6⟩ Let A be a 3×3 real matrix with $\mathsf{tr}(A) = 0$. Let $\chi_A(2) = \chi_A(3) = 0$. Then

☐ A $\chi_A(x) = (x-2)(x-3)^2$

☒ B $\chi_A(x) = (x+5)(x-2)(x-3)$

☐ C $\chi_A(x) = (x-2)^2(x-3)$

☐ D $\chi_A(x) = (x-2)(x-3)$

Soln.: Since $2, 3$ are roots of the $\chi_A(x)$, and since the sum of all the roots of $\chi_A(x) = tr(A) = 0$, we have that -5 is a root of $\chi_A(x)$. ✠

Q-7⟩ Consider $\mathrm{V} = \mathbb{R}^{2\times 2}$ and the inner product $\langle A, B\rangle = \mathsf{tr}(A^t B)$. Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $p_I(J)$ denote the orthogonal projection of J along the identity matrix I. Then $p_I(J)$ is

☒ A I ☐ B 0 ☐ C $2I$ ☐ D J

Soln.: We have $P_I(J) = \dfrac{\langle I, J\rangle}{\langle I, I\rangle} I = \dfrac{tr(J)}{2} I = I.$ ✠

Q-8⟩ Let $\mathrm{V} = \mathbb{R}^{2\times 2}$ and $P \in \mathrm{V}$ be a nonzero matrix. Consider the linear transformation $T : \mathrm{V} \to \mathrm{V}$ given by $T(A) = AP - PA$. Then

☐ A T is onto.

☐ B T is 1-1.

☒ C None of these answers are correct.

☐ D T is an isomorphism.

Soln.: Clearly $T(P) = 0$. Hence, T is not 1-1. Thus, T is not onto either. Hence, none of these answers are correct. ✠

Q-9⟩ Let $\mathrm{V} = \mathbb{R}^{3\times 3}$ and $\langle A, B\rangle = \mathsf{tr}(AB^t)$. Let D denote the subspace of diagonal matrices in V. Then $\dim D^\perp$ is

☐ A 5 ☐ B 9 ☐ C 3 ☒ D 6

Soln.: Since $\dim(D) = 3$ and $\dim(D) + \dim(V^\perp) = \dim(V) = 9$, we get that $\dim(V^\perp) = 6$. ✠

Q-10⟩ Consider positive real numbers a, b, c and matrix $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$. Let $\lambda = a + b + c$. Then

☐ A $\dim E_0 = 1,\ \dim E_\lambda = 2.$

☐ B $\dim E_0 = 1,\ \dim E_\lambda = 1.$

☐ C $\dim E_0 = 2,\ \dim E_\lambda = 2.$

☒ D $\dim E_0 = 2,\ \dim E_\lambda = 1.$

Soln.: We have $rank(A) = 2$ and $nullity(A) = 2$. So, 0 is an eigenvalue of A with $\dim(E_0) = 2$. Also, $A[1\ 1\ 1]^t = (a+b+c)[1\ 1\ 1]^t$. So, $\lambda = a + b + c$ is an eigenvalue of A. Since $\dim(E_0) + \dim(E_\lambda) = 3$, we get $\dim(E_\lambda) = 1$. ✠

Q-11⟩ If A is a 3×3 real matrix and $A^2 + I = 0$ then

☐ A $-1, i, -i$ are roots of $\chi_A(x)$.

☐ B i and $-i$ are the only roots of $\chi_A(x)$.

☐ C $1, i, -i$ are roots of $\chi_A(x)$.

☒ D There is no such matrix.

Soln.: Since $\chi_A(x)$ has degree 3, it has a real root r. As $A_I^2 = 0$, we get $r^2 = -1$. This is a contradiction. Therefore no such matrix exists. ✠

Q-12⟩ Let V be the vector space over $\mathbb{R}$ of 2×2 Hermitian matrices. Then dimension of V is

☒ A 4 ☐ B 3 ☐ C 2 ☐ D 1

Soln.: Any 2×2 Hermitian matrix A has the form $A = \begin{bmatrix} x & y+iz \\ y-iz & w \end{bmatrix}$ for some $x, y, z, w \in \mathbb{R}$. So,

$$A = x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that $\dim(V) = 4$. ✠

Q-13⟩ Let A be a 3×3 real matrix and $A^2 = A$. Then $\dim\left(\mathrm{Null}(A) \cap \mathcal{C}(A)\right)$ is

☐ A 1 ☐ B 3 ☐ C 2 ☒ D 0

Soln.: If $v = Ax \in null(A) \cap \mathcal{C}(A)$, for some $x \in \mathbb{R}^3$, then $0 = Av = A^2x = Ax = v$. ✠

Q-14⟩ The conic described by the $5x^2 - 4xy + 8y^2 = 36$ is

☐ A a pair of lines. ☐ B a hyperbola. ☒ C an ellipse. ☐ D a parabola.

Soln.: The equation is

$$[x \quad y]\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 36.$$

Let $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$. Then $\chi_A(x) = \det(xI - A) = (x-4)(x-9)$. As the eigenvalues are 4 and 9, the equation in u, r plane is given by

$$4u^2 + 9r^2 = 36 \implies \frac{u^2}{9} + \frac{v^2}{1} = 1$$

This is as an ellipse. ✠

MA 106 Part A : Optical Mark Recognition (OMR) Sheet

Name: ....

Roll No.: ....

Division.: D ☐

Tutorial: T ☐

Stud. Sign.: ....

Invig. Sign.: ....

D

Bubble Your Rollnumber

....

☐ 0	☐ 0	☐ 0	☐ 0	☐ 0	☐ 0	☐ 0	☐ 0	☐ 0
☐ 1	☐ 1	☐ 1	☐ 1	☐ 1	☐ 1	☐ 1	☐ 1	☐ 1
☐ 2	☐ 2	☐ 2	☐ 2	☐ 2	☐ 2	☐ 2	☐ 2	☐ 2
☐ 3	☐ 3	☐ 3	☐ 3	☐ 3	☐ 3	☐ 3	☐ 3	☐ 3
☐ 4	☐ 4	☐ 4	☐ 4	☐ 4	☐ 4	☐ 4	☐ 4	☐ 4
☐ 5	☐ 5	☐ 5	☐ 5	☐ 5	☐ 5	☐ 5	☐ 5	☐ 5
☐ 6	☐ 6	☐ 6	☐ 6	☐ 6	☐ 6	☐ 6	☐ 6	☐ 6
☐ 7	☐ 7	☐ 7	☐ 7	☐ 7	☐ 7	☐ 7	☐ 7	☐ 7
☐ 8	☐ 8	☐ 8	☐ 8	☐ 8	☐ 8	☐ 8	☐ 8	☐ 8
☐ 9	☐ 9	☐ 9	☐ 9	☐ 9	☐ 9	☐ 9	☐ 9	☐ 9
		☐ B		☐ B				
		☐ D						

Instructions: ➜ Use only blue/black colored ball-point pen to make a bubble.
➜ The right way to fill a box like ☐ is as ■.
➜ The OMR sheet will be evaluated by using computer, so **no writing is allowed anywhere in the OMR sheet except dedicated places**.
➜ **No rewriting, no overwriting, no erasing, and no cancellation** are allowed.
➜ Before bubbling your rollnumber, first convert the rollnmber into nine characters by appending appropriate leading zero(s) at the beginning and bubble that nine characters rollnumber so that in the ith column you bubble the ith character only. For example, if your rollnumber is 22B1234 then first convert it into 0022B2134 and then bubble it accordingly.
➜ Each question has single correct answer, so bubbling more than one option in a single question will fetch zero mark.

Bubble the box corresponds to your correct choice.

Q-1⟩	■ A	☐ B	☐ C	☐ D	Q-8⟩	☐ A	☐ B	■ C	☐ D
Q-2⟩	☐ A	■ B	☐ C	☐ D	Q-9⟩	☐ A	☐ B	☐ C	■ D
Q-3⟩	☐ A	☐ B	☐ C	■ D	Q-10⟩	☐ A	☐ B	☐ C	■ D
Q-4⟩	☐ A	☐ B	☐ C	■ D	Q-11⟩	☐ A	☐ B	☐ C	■ D
Q-5⟩	☐ A	☐ B	☐ C	■ D	Q-12⟩	■ A	☐ B	☐ C	☐ D
Q-6⟩	☐ A	■ B	☐ C	☐ D	Q-13⟩	☐ A	☐ B	☐ C	■ D
Q-7⟩	■ A	☐ B	☐ C	☐ D	Q-14⟩	☐ A	☐ B	■ C	☐ D

MA 106 Endsem Exam : Part B : Question-cum-Answer Booklet

A

Indian Institute of Technology Bombay

Exam Time: 8.30 AM - 10.30 AM

Roll No.: ....

Name: ....

Division.: D ☐

Tutorial: T ☐

Stud. Sign.: ....

Invig. Sign.: ....

Question	Points	Score
1	4	
2	4	
3	4	
4	4	
Total:	16	

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

- **Failure to follow instructions will carry a penalty of 2 marks.**
- There are 4 pages and 4 questions in Part B, for a total of 16 marks.
- Write your name, roll number, division number, tutorial batch number, and signature in space provided.
- Answers for subjective questions unsupported by satisfactory reasoning will not be awarded any marks.
- **Answer for subjective question have to be written in the spaces provided in the question-cum-answer booklet only.** Supplementary sheet can be used only for rough works.

1. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation so that $M_B^B(f) = M_C^C(f)$ for any two bases B and C of $\mathbb{R}^2$. Show that there exists $\lambda \in \mathbb{R}$ such that $f(u) = \lambda u$, for all $u \in \mathbb{R}^2$.

Solution: We have $M_B^B(f) = M_B^C(I)M_C^C(f)M_C^B(I)$ for all bases B, C of $\mathbb{R}^2$. [1]

Let $A = M_B^B(f) = M_C^C(f) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then from the above equation, for all invertible matrices P, we have $PA = AP$. [1]

Take $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to get

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

Thus, $b = c$ and $a = d$. [1]

Take $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ to get

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a & b \\ a+b & a+b \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & b \\ a+b & a \end{bmatrix}.$$

Thus, $a + b = a = b + a$, i.e., $b = 0$. So, $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. Hence, $f(u) = Au = au$ for all $u \in \mathbb{R}^2$. [1]

2. Let $V = \mathbb{R}^{2\times 2}$ and $T : V \to V$ be defined as $T(A) = A^t$ for all $A \in V$.
   (a) Find eigenvalues and eigenspaces of T.
   (b) Show that T is diagonalizable.

Solution: The subspace U of 2×2 symmetric matrices is 3-dimensional since we have

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for all $a, b, c \in \mathbb{R}$. The subspace W of skew-symmetric matrices is 1-dimensional since we have $\begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix} = c\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ for all $c \in \mathbb{R}$. Note that $T(A) = A$ for all $A \in U$ and $T(B) = -B$ for all $B \in W$. Hence $U \subseteq E_1$ and $W \subseteq E_{-1}$. [1]

Since $E_1 \cap E_{-1} = \{0\}$ and $\dim(E_1) + \dim(E_{-1}) = 4$, $V = E_1 \oplus E_{-1}$. So, T is diagonalizable. [1]

Thus, E_1= subspace of all symmetric 2×2 matrices [1]

and E_{-1}= subspace of all skew-symmetric 2×2 matrices. [1]

3. Let $V = \mathbb{R}^{2\times 2}$ with inner product $\langle A, B\rangle = \mathsf{tr}(A^tB)$ for $A, B \in V$. Define $T : V \to V$ as $T(A) = PA$ where P is a 2×2 nonzero real symmetric matrix. Show that T is diagonalizable. Show that the sets of eigenvalues of T and P are equal. Find a basis of eigenvectors of T.

Solution: For all $A, B \in V$ we have $\langle T(A), B\rangle = tr((PA)^tB) = tr(A^tPB) = \langle A, T(B)\rangle$. So, T is self-adjoint, and hence it is diagonalizable. [1]
If λ is an eigenvalue of T, then $T(A) = PA = \lambda A$ for some $0 \neq A \in V$. So, $(P - \lambda I)A = 0$. If $P - \lambda I$ is invertible, then $A = 0$, which is a contradiction. Hence λ is an eigenvalue of P. [1]
Conversely, if λ is an eigenvalue of P with eigenvector u, then $Pu = \lambda u$. Hence $P[u\ u] = \lambda[u\ u]$. So, λ is an eigenvalue of T. [1]
Let $\{u, v\}$ be a basis of eigenvectors of P with $Pu = \lambda u$ and $Pv = \mu v$. Then $P[u\ 0] = \lambda[u\ 0]$, $P[0\ u] = \lambda[0\ u]$ and $P[v\ 0] = \mu[v\ 0]$, $P[0\ v] = \mu[0\ u]$. Also, if for $a, b, c, d \in \mathbb{R}$ we have $a[u\ 0] + b[0\ u] + c[v\ 0] + d[0\ v] = [0\ 0]$, then we get $au + cv = 0$ and $bu + dv = 0$, which forces $a = c = 0 = b = d$ as $\{u, v\}$ is linearly independent. Thus, $\{[u\ 0], [0\ u], [v\ 0], [0\ v]\}$ is a basis of eigenvectors of T. [1]

Alternate way to show the diagonalizibility of T: Let $P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Consider the basis $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ of V. Then with respect to this basis, the matrix of T is $\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ b & 0 & c & 0 \\ 0 & b & 0 & c \end{bmatrix}$, which is symmetric. Thus, T is diagonalizable.

Note: The matrix representation of T should be a 4×4 matrix. It should not be a 2×2 matrix.

4. Let $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$. Find eigenvalues and eigenspaces of A and an orthogonal matrix S such that $S^tAS = \text{diag}(\lambda, \mu)$. Using this equation find a matrix B with positive eigenvalues such that $B^2 = A$.

Solution:

We have $\chi_A(x) = \det(xI - A) = \begin{vmatrix} x-5 & -4 \\ -4 & x-5 \end{vmatrix} = x^2 - 10x + 25 - 16 = x^2 - 10x + 9 = (x-9)(x-1)$.
Therefore, $\lambda = 9$ and $\mu = 1$ are the eigenvalues of A. [1]

Now, $(A - 9I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives $\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e., $x = y$.

Similarly, $(A - I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e., $x = -y$.

Hence, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 9$, and we have $E_1 = L(\{(1,1)\})$. Also, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector for $\mu = 1$, and we have $E_{-1} = L(\{(1,-1)\})$. [1]

Therefore, for the orthogonal matrix $S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ we have

$$S^tAS = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} = D.$$

[1]

Hence, $A = SDS^t = (S\sqrt{D}S^t)(S\sqrt{D}S^t)$, where $\sqrt{D} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A = B^2$, for $B = S\sqrt{D}S^t = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and B has positive eigenvalues 3, 1. [1]