Axiomatic Approach to the Determinant Function

- Recall the formula for determinants of $k \times k$ matrices for $k = 1, 2, 3$:

$$\det[a] = a, \quad \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1 \ \text{ and}$$

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1.$$

- Our approach to determinants of $n \times n$ matrices is via their properties (rather than via an explicit formula as above).
- Let d be a function that associates a scalar $d(A) \in \mathbb{R}$ with every $n \times n$ matrix A with entries in $\mathbb{R}$. We use the following notation. If the columns of A are $A_1, A_2, \ldots, A_n$, we write $d(A) = d([A_1, A_2, \ldots, A_n])$ simply as $d(A_1, A_2, \ldots, A_n)$. Thus d can be regarded as a function on $\mathbb{R}^{n \times 1} \times \cdots \times \mathbb{R}^{n \times 1}$, the n-fold product of $n \times 1$ column vectors.

Axioms for Determinant Function

(i) d is called **multilinear** if for each $j = 1, 2, \ldots, n$, scalars α, β and $n \times 1$ column vectors $A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n, B, C$, we have

$$d(A_1, \ldots, A_{j-1}, \alpha B + \beta C, A_{j+1}, \ldots, A_n) = \\ \alpha\, d(A_1, \ldots, A_{j-1}, B, A_{j+1}, \ldots, A_n) + \beta\, d(A_1, \ldots, A_{j-1}, C, A_{j+1}, \ldots, A_n).$$

(ii) d is called **alternating** if $d(A_1, A_2, \ldots, A_n) = 0$ whenever $A_i = A_j$ for some $i \neq j$, $i, j = 1, \ldots, n$.

(iii) d is called **normalized** if $d(I) = d(e_1, e_2, \ldots, e_n) = 1$, where e_j is the jth basic column vector, $j = 1, \ldots, n$.

(iv) A normalized, alternating and multillinear function d on the set of all $n \times n$ matrices is called a **determinant function** of order n.

Formula for Determinants of 2×2 Matrices

Suppose d is an alternating multilinear normalized function on 2×2 matrices $A = [A_1, A_2]$. We show that

$$d(A_1, A_2) = d\left(\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{21}a_{12}.$$

Write the first column as $A_1 = a_{11}e_1 + a_{21}e_2$ and the second column as $A_2 = a_{12}e_1 + a_{22}e_2$. Since $d(e_1, e_2) + d(e_2, e_1) = d(e_1 + e_2, e_1 + e_2) = 0$,

$$\begin{aligned} d(A_1, A_2) &= d(a_{11}e_1 + a_{21}e_2, a_{12}e_1 + a_{22}e_2) \\ &= a_{11}d(e_1, a_{12}e_1 + a_{22}e_2) + a_{21}d(e_2, a_{12}e_1 + a_{22}e_2) \\ &= a_{11}\left[a_{12}d(e_1, e_1) + a_{22}d(e_1, e_2)\right] \\ &\quad +a_{21}\left[a_{12}d(e_2, e_1) + a_{22}d(e_2, e_2)\right] \\ &= a_{11}a_{12}d(e_1, e_1) + a_{11}a_{22}d(e_1, e_2) \\ &\quad +a_{21}a_{12}d(e_2, e_1) + a_{21}a_{22}d(e_2, e_2) \\ &= (a_{11}a_{22} - a_{21}a_{12})d(e_1, e_2) = a_{11}a_{22} - a_{21}a_{12}. \end{aligned}$$

Exercise: Can you do this for 3×3 matrices $A = [A_1, A_2, A_3]$?

Properties of the Determinant Function

We show that there can be only one determinant function of order n.

Lemma: Suppose that $d(A_1, A_2, \ldots, A_n)$ is a multilinear alternating function on the set of all $n \times n$ matrices. Then
(a) If some $A_j = \mathbf{0}$, then $d(A_1, A_2, \ldots, A_n) = 0$.
(b) $d(A_1, A_2, \ldots, A_j, A_{j+1}, \ldots A_n) = -d(A_1, A_2, \ldots, A_{j+1}, A_j, \ldots, A_n)$.
(c) $d(A_1, A_2, \ldots, A_i, \ldots, A_j, \ldots, A_n) = -d(A_1, A_2, \ldots, A_j, \ldots, A_i, \ldots, A_n)$.

Proof: (a) Let $A_j = \mathbf{0}$. By the multilinearity,
$d(A_1, A_2, \ldots, A_j, \ldots, A_n) = d(A_1, A_2, \ldots, \mathbf{0} - \mathbf{0}, \ldots, A_n)$
$= d(A_1, A_2, \ldots, \mathbf{0}, \ldots, A_n) - d(A_1, A_2, \ldots, \mathbf{0}, \ldots, A_n) = 0.$

(b) Put $A_j = B, A_{j+1} = C$. By the alternating property,

$$\begin{aligned}
0 &= d(A_1, A_2, \ldots, B + C, B + C, \ldots, A_n) \\
&= d(A_1, A_2, \ldots, B, B + C, \ldots, A_n) + d(A_1, A_2, \ldots, C, B + C, \ldots, A_n) \\
&= d(A_1, A_2, \ldots, B, C, \ldots, A_n) + d(A_1, A_2, \ldots, C, B, \ldots, A_n).
\end{aligned}$$

Hence $d(A_1, A_2, \ldots, B, C, \ldots, A_n) = -d(A_1, A_2, \ldots, C, B, \ldots, A_n)$.

(c) follows from (b) by $2(j - i) - 1$ transpositions.

Uniqueness of the Determinant Function

Lemma: Suppose f is a multilinear alternating function on $n \times n$ matrices and $f(e_1, e_2, \ldots, e_n) = 0$. Then f is identically zero.

Proof: Let $A = [a_{ij}]$ be an $n \times n$ matrix with columns $A_1, \ldots, A_n$. For $j = 1, \ldots, n$, write A_j as

$$A_j = a_{1j}e_1 + a_{2j}e_2 + \cdots + a_{nj}e_n.$$

Since f is multilinear, we obtain

$$f(A_1, \ldots, A_n) = \sum_h a_{h(1)1}a_{h(2)2} \cdots a_{h(n)n}\, f(e_{h(1)}, e_{h(2)}, \ldots, e_{h(n)}),$$

where the sum is over all functions $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Since f is alternating, we obtain

$$f(A_1, \ldots, A_n) = \sum_h a_{h(1)1}a_{h(2)2} \cdots a_{h(n)n}\, f(e_{h(1)}, e_{h(2)}, \ldots, e_{h(n)}),$$

where the sum is now over all bijections $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Uniqueness of the Determinant Function (continued)

By using part (c) of the previous lemma, we see that

$$f(A_1, \ldots, A_n) = \sum_h \pm a_{h(1)1} a_{h(2)2} \cdots a_{h(n)n} \, f(e_1, e_2, \ldots, e_n),$$

where the sum is over all bijections $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. Since $f(e_1, \ldots, e_n) = 0$, we obtain $f(A) = 0$. $\square$

Theorem: Let f be an alternating multilinear function of order n, and let d be a determinant function of order n. Then for every $n \times n$ matrix $A = [A_1, A_2, \ldots, A_n]$, we have

$$f(A_1, A_2, \ldots, A_n) = d(A_1, A_2, \ldots, A_n) f(e_1, e_2, \ldots, e_n).$$

In particular, if f is also a determinant function, then

$$f(A_1, A_2, \ldots, A_n) = d(A_1, A_2, \ldots, A_n).$$

Proof of the Uniqueness of the Determinant Function

Proof: Consider the function g given by

$$g(A_1, A_2, \ldots, A_n) = f(A_1, A_2, \ldots, A_n) - d(A_1, A_2, \ldots, A_n) f(e_1, e_2, \ldots, e_n).$$

Since f and d are alternating and multilinear, so is g. Since

$$g(e_1, e_2, \ldots, e_n) = 0,$$

the result follows from the previous lemma. □

Notation: We shall denote the determinant of A by det A.

Setting $\det[a] = a$ gives the existence for $n = 1$.

Assume that we have proved the existence of the determinant function $\det$ of order $n-1$. The determinant function of order n can be defined in terms of the determinant function of order $n-1$ as follows.

Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from an $n \times n$ matrix A by deleting the ith row and jth column of A for $i, j = 1, \ldots, n$.

Existence of the Determinant Function

Theorem The function f on $n \times n$ matrices $A = [a_{ij}]$ given by

$$f(A) := a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}$$

is multilinear, alternating, and normalized, and hence it is the determinant function of order n. **(Expansion by the first row)**

Proof: Fix $j \in \{1, \ldots, n-1\}$. Suppose the columns A_j and A_{j+1} of A are equal. Then A_{1i} has two equal columns except when $i = j, j+1$.

By induction, $\det(A_{1i}) = 0$ if $i \neq j$ and if $i \neq j+1$. Thus

$$f(A) = (-1)^{j+1} a_{1j} \det(A_{1j}) + (-1)^{j+2} a_{1(j+1)} \det(A_{1(j+1)}).$$

Since $A_j = A_{j+1}$, we have $a_{1j} = a_{1j+1}$ and $A_{1j} = A_{1j+1}$. Thus $f(A) = 0$. Therefore the function f is alternating.

Let $A = I = [e_1, e_2, \ldots, e_n]$. Then $a_{12} = \cdots = a_{1n} = 0$, so that

$$f(A) = 1 \det(A_{11}) = \det(e_1, e_2, \ldots, e_{n-1}) = 1.$$

The multilinear property of the function f can be similarly proved.

Determinants of Triangular and Elementary Matrices

In a similar manner, we have the **expansion by row** $k = 2, \dots n$:

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det A_{kj} \quad \text{for } A = [a_{ij}].$$

Theorem: (i) Let U be an upper triangular or a lower triangular matrix. Then $\det U$ is equal to the product of the diagonal entries of U.

(ii) Let $E := I + \alpha E_{ij}$ be an elementary matrix of the type I, where $\alpha \in \mathbb{R}$ and $i \neq j$. Then $\det E = 1$.

(iii) Let $E := I + E_{ij} + E_{ji} - E_{ii} - E_{jj}$ be an elementary matrix of type II, where $i \neq j$. Then $\det E = -1$.

(iv) Let $E := I + (\alpha - 1)E_{ii}$ be an elementary matrix of type III, where $0 \neq \alpha \in \mathbb{R}$. Then $\det E = \alpha$.

Proof: (i) Let $U = [u_{ij}]$ be upper triangular. Arguing as in the lemma used for proving the uniqueness of the determinant function, we obtain

$$\det U = \sum_h \pm u_{h(1)1} u_{h(2)2} \cdots u_{h(n)n},$$

where the sum is over all bijections $h : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$.

Since U is upper triangular the only choice of the bijection h yeilding a nonzero term is the identity bijection (and this gives a plus sign).

(ii) follows from part (i).

(iii) E is obtained from the identity matrix by exchanging columns i and j. The result follows since the determinant function is alternating.

(iv) follows form part (i). $\square$

Theorem: (Multiplicativity of det) For any $n \times n$ matrices A and B,

$$\det(AB) = \det A \det B.$$

Proof: Since $AB = A[B_1, \ldots, B_n] = [AB_1, \ldots, AB_n]$, we need prove

$$\det(AB_1, AB_2 \ldots, AB_n) = \det A \det B.$$

Keep A fixed, and for an arbitrary $n \times n$ matrix B, define

$$f(B_1, B_2, \ldots, B_n) = \det(AB_1, AB_2, \ldots, AB_n).$$

$\det(AB) = \det A \det B$

The function f is alternating, since $f(B_1, \ldots, B_i, \ldots, B_i, \ldots, B_n) = \det(AB_1, \ldots, AB_i, \ldots, AB_i, \ldots, AB_n) = 0$.

To show that the function f is also multilinear, let C and D be $n \times 1$ column vectors, and let $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} f(B_1, \ldots, \alpha C + \beta D, \ldots, B_n) &= \det(AB_1, \ldots, A(\alpha C + \beta D), \ldots, AB_n) \\ &= \det(AB_1, \ldots, \alpha AC + \beta AD, \ldots, AB_n) \\ &= \alpha \det(AB_1, \ldots, AC, \ldots, AB_n) \\ &+ \beta \det(AB_1, \ldots, AD, \ldots, AB_n) \\ &= \alpha f(B_1, \ldots, C, \ldots B_n) + \beta f(B_1, \ldots, D, \ldots, B_n) \end{aligned}$$

Since f is alternating and multilinear, an earlier theorem gives

$f(B_1, B_2, \ldots, B_n) = \det(B_1, \ldots, B_n) f(e_1, e_2, \ldots, e_n)$.

But $f(e_1, e_2, \ldots, e_n) = \det(Ae_1, \ldots, Ae_n) = \det(A_1, \ldots, A_n) = \det A$. Thus $\det(AB_1, AB_2, \ldots, AB_n) = \det(B_1, \ldots, B_n) \det A = \det A \det B$, as desired.

Determinant and Invertibility

Lemma: (i) Let A be a square matrix. Then A is invertible if and only if $\det A \neq 0$, and in that case,

$$\det A^{-1} = \frac{1}{\det A}.$$

(ii) Suppose A and B are square matrices such that $AB = I$. Then A and B are invertible and $B = A^{-1}$.

Proof: (i) First, suppose A is invertible. Then by the last theorem, $(\det A)(\det A^{-1}) = \det AA^{-1} = \det I = 1$. Thus $\det A \neq 0$ in this case. Now, suppose A is not invertible. Then there is a nonzero column vector x such that $Ax = 0$. Hence a linear combination of the column vectors of A having at least one nonzero coefficient is equal to the zero column vector. Thus some column of A is a linear combination of other columns of A. It now follows from the multilinearity and alternating properties of the detetminant function that $\det A = 0$.

(ii) $\det A \det B = \det(AB) = \det(I) = 1$. So $\det A \neq 0$ and A is invertible. Now $B = (A^{-1}A)B = A^{-1}(AB) = A^{-1}$ is also invertible.

Determinant of the Transpose of a Matrix

Theorem: For any $n \times n$ matrix A,

$$\det A^{\mathsf{T}} = \det A.$$

Proof: If $A = (A^{\mathsf{T}})^{\mathsf{T}}$ is not invertible, then A^{T} is also not invertible, and $\det A = 0 = \det A^{\mathsf{T}}$. Now suppose A is invertible. Then the row canonical form of A is I, and so there are elementary matrices $E_1, \ldots, E_k$ such that

$$A = E_1 \cdots E_k, \quad \text{and so} \quad \det A = \det E_1 \cdots \det E_k.$$

Note that the transpose of an elementary matrix is also an elementary matrix of the same type, and has the same determinant. Hence $\det A^{\mathsf{T}} = \det(E_k^{\mathsf{T}} \cdots E_1^{\mathsf{T}}) = \det E_k^t \cdots \det E_1^{\mathsf{T}} = \det E_k \cdots \det E_1 = \det A \quad \square$.

Corollary: The determinant is a multilinear, alternating and normalized function of the rows of a square matrix, and so for $A = [a_{ij}]$, we have the following **expansion by column** k, where $k = 1, \ldots, n$:

$$\det A = \sum_{i=1}^{n} (-1)^{k+i} a_{ik} \det A_{ik}.$$

Computation of Determinant by Gauss-Jordan Method

Let A be an $n \times n$ matrix. Reduce A to its row canonical form (rcf) $\widetilde{U}$ by elementary row operations (E.R.O.s) of type I (denoted by $R_i \longleftrightarrow R_j$), of type II (denoted by $R_i + \alpha R_j$ with $i \neq j$), and of type III (denoted by $\alpha_i R_i$ with $\alpha_i \neq 0$), that is, by premultiplying A by elementary matrices $E_1, \ldots, E_s$ of types I, II, III. Then

$$\widetilde{U} = E_s \cdots E_1 A.$$

If r of the elementary matrices $E_1, \ldots, E_k$ are of type I, and p of them are of type III with nonzero multipliers $\alpha_1, \ldots, \alpha_p$, then

$$\det \widetilde{U} = (-1)^r \alpha_1 \cdots \alpha_p \det A.$$

This follows from the result about the determinants of elementary matrices and from the multiplicativity of the determinant function proved earlier.

If $u_{11}, \ldots, u_{nn}$ are the diagonal entries of $\widetilde{U}$, then $\det \widetilde{U} = u_{11} \cdots u_{nn}$.

Hence

$$\det A = (-1)^r (\alpha_1 \cdots \alpha_p)^{-1} u_{11} \cdots u_{nn}.$$

Expansion of Determinant using Permutations

Besides the axiomatic approach and the inductive definition, there is another way to define the determinant of a square matrix at once using the notions of a permutation and the sign of a permutation.

Let n be a positive integer. By a **permutation** of $\{1, 2, \ldots, n\}$ we mean a bijection from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$. The set of all permutations of $\{1, 2, \ldots, n\}$ is denoted by S_n. Note that S_n has $n!$ elements. Let $\sigma \in S_n$. A pair (i, j), where $i, j \in \{1, 2, \ldots, n\}$, is called an **inversion** of σ if $i < j$ and $\sigma(i) > \sigma(j)$. The **sign** of σ is defined by

$$\operatorname{sgn}(\sigma) = (-1)^r \quad \text{where } r \text{ is the number of inversions in } \sigma$$

We now define the **determinant** of an $n \times n$ matrix $A = (a_{j,k})$ by

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

It is not difficult to see that $\det A$ defined as above is a multilinear, alternating and normalized function, and hence by the results proved earlier, it coincides with the previous definitions of $\det A$.