MA 106 - LINEAR ALGEBRA. End-Term Examination

February 23, 2019 | Time : 16:00 to 18:00 | Max Marks : 30

Name :

CODE: Alpha - 312

Roll No. ☐☐☐ ☐☐☐ ☐☐☐

Div: D ☐ Tut: T ☐

Signature of invigilator :

INSTRUCTIONS:

1. Write your name, roll number, division and tutorial batch above and in the answer book. Get the signature of any invigilator after writing these. Answer sheets unsigned by an invigilator will not be graded. **Submit both the question paper as well as the answer book after the exam**.
2. Fill up **True or False** (Qn. 1) and **tick mark your answer for the Multiple Choice Questions** (Qn. 2) in the question paper itself. Exactly one answer is correct for the multiple choice questions. **Write the answer for the Fill up questions (Qn. 3) in the space given in the question paper itself**. If you need to change any answer, make sure that the previous answer is unambiguously scratched out. **Each ambiguity will fetch you $-1/2$.**
3. Use/possession of mobile phones/communication devices/calculators are STRICTLY PROHIBITED.

Objective Type Questions

Qn 1. True or False. 1 mark for the correct answer, -1/2 for wrong answers

(a) Let V be a vector space over $\mathbb{R}$ and let $\mathbf{v}_1, \mathbf{v}_2 \in V$ be linearly independent vectors. Define $\mathbf{u}_1 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$ and $\mathbf{u}_2 = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2$ for fixed $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Then, $\mathsf{Linear\_span}(\mathbf{v}_1, \mathbf{v}_2) = \mathsf{Linear\_span}(\mathbf{u}_1, \mathbf{u}_2)$. **Answer:** ☐

False.

(b) If A is a 5×8 matrix with $\mathsf{nullity}(A) = 3$, then the linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b} \in \mathbb{R}^5$. **Answer:** ☐

True.

(c) If A and B are row equivalent, then they have the same characteristic polynomial.
Answer: ☐

False.

Qn 2. Multiple choice. 1 mark for the correct answer, -1/3 for wrong answers

(a) Let V be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following are real vector spaces? Here, $\mathbb{Q}$ denotes the set of rational numbers.

$S : \{f \in V : f(1/2) \in \mathbb{Q}\}$ and $T : \{f \in V : f(1/2) = f(1)\}$.

(A) Only S (B) Only T (C) Both S and T (D) Neither S nor T

Answer: (B).

(b) Define maps $P, Q : \mathbb{R}^3 \to \mathbb{R}^3$ by $P(x, y, z) = (x - 1, z, y)$ and $Q(x, y, z) = (|x|, y - z, -y)$. Then, which of the two above maps is linear?

(A) Only P (B) Only Q (C) Both P and Q (D) Neither P nor Q

Answer: (D).

(c) Let V be the vector space of polynomial functions $f : (-1,1) \to \mathbb{R}$. Define the inner product $\langle f, g\rangle = \int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1-t^2}}dt$. Which of the following sets of vectors are orthogonal in V?

$P = \{t, 2t^2 - 1\}$ $\quad\quad$ $Q = \{t, 4t^3 - 3t\}$

(A) Only P (B) Only Q (C) Both P and Q (D) Neither P nor Q

Answer: (C).

Qn 3 Fill up. 2 marks each, no negative marks

(a) The eigenvectors of $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ are − − − − − − − − − − − − −.

They are $[1, \iota]^t$ and $[\iota, 1]^t$, where $\iota^2 = -1$. Sometimes students give scaled versions of these like $[x, x\iota]^t$ and $[y\iota, y]^t$. These are also correct. One mark for each eigenvector.

(b) The normal form of the quadratic form $q(x, y, z) = xy + yz + xz$ is − − − − − − − − − − − −.

The matrix corresponding to this form is $Q = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$. Its eigenvalues are clearly $1, -1/2$ and $-1/2$. [1]

Thus the normal form is $q(x, y, z) = u^2 - 1/2(v^2 + w^2)$ where u, v, w are linear combinations of x, y, z. [1]

1 mark has been deducted for incorrect scaling of the coefficients in the normal form(i.e. if your answer was correct upto a scalar factor the correct answer, for example: $\sqrt{3}(u^2 - \frac{1}{2}v^2 - \frac{1}{2}w^2)$).

Subjective Type Questions

INSTRUCTIONS: **CODE: Alpha - 312**

1. **Write your name, roll number, division, tutorial batch and CODE in the answer book.** Number the pages of your answer books and supplements and fill in the index on the cover page of your answer book. **Failure to do so will attract a penalty of 2 marks.**

2. The Subjective type questions need to be answered in the **Answer book**. In the Subjective part, do **NOT** answer any part of any question multiple times. Scratch out attempts and ensure that **each part of each question is answered at most once**. If a question or a part of a question is answered multiple times with more than one attempts left unscratched, then the graders can choose which of the answers they want to grade.

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1. (a) Let $A = \begin{bmatrix} 4 & -2 & 0 & 0 & 2 \\ -2 & 4 & 2 & 2 & -4 \\ 0 & 2 & 4 & 0 & -2 \\ 4 & 0 & 4 & 0 & 0 \\ 2 & -4 & -2 & -2 & 4 \end{bmatrix}$. Find the determinant rank of A. Justify your answer. [2]

   A itself can be seen to be singular and hence $\text{rank}(A) \leq 4$. [1]
   Check the determinant of all 4×4 suitable submatrices which can be obtained using the linear dependency of rows and columns. In all the cases, determinant is 0 and hence $\text{rank}(A) \leq 3$. [0.5]
   Rank is 3 as the principal submatrix obtained by deleting rows 4,5 and cols 4,5 has non zero det. Some other 3×3 submatrix might be given. Check the determinant for correctness. [0.5]
   Any alternate correct answer is also given full credits

   (b) Let A be a real symmetric matrix with characteristic polynomial $p(t) = t^3 + t^2 - t + 1$. Find a diagonal matrix that is similar to A. Justify your answer. [2]

   Can check that $p(t) = (t-1)(t+1)^2$. [0.5]
   Since A is a real symmetric matrix, by Spectral Theorem, A is diagonalisable. [1]
   Thus, $\mathsf{Diag}(1, -1, -1)$ or any permutation of $1, -1, -1$ is ok. [0.5]
   Any corrrect solution to the version with a typo also gets full credit.

2. (a) Let $\mathbf{u}_1 = [1, -1, 1, -1]^t$, $\mathbf{u}_2 = [1, 1, 3, -1]^t$ and $\mathbf{u}_3 = [1, 1, 1, 1]^t$. Give an orthonormal basis for $\mathsf{Linear\_span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. [2]

   Can check that the output of G-S after scaling to get unit vectors will be $\mathbf{v}_1 = (1/2)[1, -1, 1, -1]^t$, $\mathbf{v}_2 = (1/\sqrt{2})[0, 1, 1, 0]^t$ and $\mathbf{v}_3 = (1/\sqrt{2})[1, 0, 0, 1]^t$, half marks each for $\mathbf{v}_1$ and $\mathbf{v}_2$ and one mark for $\mathbf{v}_3$. If someone writes only orthogonal vectors, then $1/2$ marks for each correct orthogonal vector. [2]

   (b) Find a unit vector $\mathbf{v} \in \mathbb{R}^4$ that is orthogonal to $\mathsf{Linear\_span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. [2]

   A unit vector orthogonal to all of these is $\mathbf{v} = \pm(1/2)[-1, -1, 1, 1]^t$. The negative of this vector or scalar multiples without being in reduced form are valid answers. If the resultant vector is correct but not unit, then 1.5 marks will be awarded. [2]

   (c) Let $\mathbf{v}$ be your answer obtained in part (b) above. **Using Bessel's inequality**, check if $\mathbf{w} = [5, -5, 9, -5]^t \in \mathsf{Linear\_span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v})$. [2]

   As $\mathsf{Linear\_span}(\mathbf{u}_1, \mathbf{u}_2) = \mathsf{Linear\_span}(\mathbf{v}_1, \mathbf{v}_2)$, $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v})$ is an orthonormal basis for $\mathsf{Linear\_span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v})$. [1]
   Can check that $\langle \mathbf{w}, \mathbf{w} \rangle = 156$. Further, $\langle \mathbf{w}, \mathbf{v}_1 \rangle^2 = 144$, $\langle \mathbf{w}, \mathbf{v}_2 \rangle^2 = 8$ and $\langle \mathbf{w}, \mathbf{v} \rangle^2 = 4$. Since $156 = 144 + 8 + 4$, we get $\mathbf{w} \in \mathsf{Linear\_span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v})$. [1]
   For a solution not using Bessel's inquality, award zero marks.

3. Let V be the vector space of 2×2 real, symmetric matrices and let W be the vector space of all 2×2 real matrices. Define $T : V \to W$ by $T\left(\begin{bmatrix} a & b \\ b & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

Consider the ordered basis $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right\}$ of V and the ordered basis $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ of W. With respect to $\mathcal{B}_1, \mathcal{B}_2$, find the matrix representation of T. Further, find $\mathsf{rank}(T)$. [5]

Call the elements of $\mathcal{B}_1$ as E_1, E_2, E_3 and the elements of $\mathcal{B}_2$ as F_1, F_2, F_3, F_4.

$T(E_1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ [0.5]

$= -F_1 - F_2 - F_3 + 4F_4$ [0.5]

$T(E_2) = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$ [0.5]

$= 3F_1 - 5F_2 + 7F_3 - 4F_4$ [0.5]

$T(E_3) = \begin{bmatrix} 5 & 4 \\ 11 & 10 \end{bmatrix}$ [0.5]

$= F_1 - 7F_2 + F_3 + 10F_4$ [0.5]

Thus the matrix required is $P = \begin{bmatrix} -1 & 3 & 1 \\ -1 & -5 & -7 \\ -1 & 7 & 1 \\ 4 & -4 & 10 \end{bmatrix}$ [1]

Can check that $\mathsf{rank}(P) = 3$. [1]

If the entries of the matrix are found correct but the matrix is written in a wrong way (like upside down or as a 3×4 matrix) and rank is found correct, award zero marks for the matrix but full marks for the rank.

If the entries of the matrix are not correct, award zero marks for both matrix as well as the rank.

4. (a) Let A, B be real matrices of order 9×7 and 4×3 respectively. Show that there exists a non-zero 7×4 real matrix X such that $AXB = 0$. [2]

Assume that the entries of X are 28 variables. Write out $AXB = 0$ in full. We will get a homogenous system of 27 equations that have 28 variables. [1]

Since the homogenous system has strictly more variables than equations, there exists a non-zero solution. [1]

Remark Many people have solved it in different ways. For the correct solution they have got the full marks.

(b) If A is any $m \times n$ complex matrix with null space $\mathsf{Null}(A)$ and range space $\mathsf{Range}(A)$, then prove that $\mathsf{Null}(A^*) = (\mathsf{Range}(A))^\perp$ and $\mathsf{Range}(A^*) = (\mathsf{Null}(A))^\perp$. What do you infer from this about $\mathsf{rank}(A)$ and $\mathsf{rank}(A^*)$? Justify your answer. [3]

Let $A_{m \times n}$ be a complex matrix. If $\mathbf{u} \in \mathsf{Null}(A^*)$, then, $\mathbf{u} \in \mathbb{C}^m$ and $A^*\mathbf{u} = \mathbf{0}_{n \times 1}$. Thus for all $\mathbf{v} \in \mathbb{C}^n$, we have $0 = \langle A^*\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A\mathbf{v} \rangle$. We have shown that for all $\mathbf{v} \in \mathbb{C}^n$, $\langle \mathbf{u}, A\mathbf{v} \rangle = 0$. Thus, $\mathbf{u} \in (\mathsf{Range}(A))^\perp$. This shows that $\mathsf{Null}(A^*) \subseteq (\mathsf{Range}(A))^\perp$. The argument is reversible (how?) and shows the reverse containment. Thus, $\mathsf{Null}(A^*) = (\mathsf{Range}(A))^\perp$. [1]

Let $\mathbf{w} \in \mathsf{Range}(A^*)$. Thus there exists $\mathbf{x}$ with $A^*\mathbf{x} = \mathbf{w}$ and let $\mathbf{u} \in \mathsf{Null}(A)$, thus $A\mathbf{u} = \mathbf{0}$. Consider $\langle \mathbf{w}, \mathbf{u} \rangle = \langle A^*\mathbf{x}, \mathbf{u} \rangle = \langle \mathbf{x}, A\mathbf{u} \rangle = 0$. Thus $\mathbf{w} \in (\mathsf{Null}(A))^{\perp}$ and this shows $\mathsf{Range}(A^*) \subseteq (\mathsf{Null}(A))^{\perp}$. The reverse containment is similarly proved (how?). [1]

This shows that $\text{rank}(A^*) = \dim(\mathsf{Range}(A^*)) = \dim(\mathsf{Null}(A)^{\perp}) = n - \mathsf{nullity}(A) = \text{rank}(A)$. [1]

Remark

- We have not given marks to anyone who either made the incorrect relation between $\text{rank}(A)$ and $\text{rank}(A^*)$ or just stated that they are equal without any proper justification.
- For the other parts of the problem the marking scheme is same.