Indian Institute of Technology Bombay

Department of Mathematics

MA 106: Linear Algebra : Final Examination

Date: 25 February 2020 Max. Marks 32

Duration : 8.30-10.30 am Weightage 64 %

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1. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

   (a) Find the roots of the characteristic polynomial of A.

   (b) Show that $T_A : \mathbb{C}^2 \to \mathbb{C}^2$ given by $T_A(v) = Av$ for all $v \in \mathbb{C}^2$ is diagonalizable.

   (c) Show that $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T_A(v) = Av$ for all $v \in \mathbb{R}^2$ is not diagonalizable. [4]

   Solution. (a) The characteristic polynomial of A is $P_A(x) = \det(xI - A) = x^2 - 2x + 2$. Hence the roots are $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$. [**1 mark**]
   (b) Since $\lambda_1 \neq \lambda_2$ the eigenvectors for λ_1 and λ_2 are linearly independent. These vectors form a basis of $\mathbb{C}^2$. Hence T_A is diagonalisable. [**2 marks**]
   (c) Since A has no real eigenvalue, T_A is not diagonalizable. [**1 mark**]

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. Suppose that $M_B^B(T) = M_C^C(T)$ for any two bases B and C of $\mathbb{R}^2$. Show that there exists a $\lambda \in \mathbb{R}$ such that $T(v) = \lambda v$ for all $v \in \mathbb{R}^2$. [4]

   Solution. Since $A := M_B^B(T) = M_B^C M_C^C(T) M_C^B$. Let $P = M_C^B$. Then $AP = PA$ for all invertible matrices P since any invertible matrix represents M_C^B for some bases B and C. [**1 mark**]
   Taking $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we see that $b = c$ and $a = d$. [**1 mark**]
   Now take $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ to see that $b = 0$. [**1 mark**].
   Hence A is a scalar matrix. Thus there is a $\lambda \in \mathbb{R}$ so that $T(v) = \lambda v$ for all $v \in \mathbb{R}^2$. [**1 mark**]

3. Let $V = P_2(\mathbb{R})$ be the vector space of real polynomials of degrees at most 2. Consider V with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$ and determine an orthogonal basis of V by applying the Gram-Schmidt orthogonalization process on the basis $\{1, x, x^2\}$ of V.

   Solution. To determine an orthogonal basis $\{w_1, w_2, w_3\}$ of V, set $w_1 = 1$. Then

$$||w_1||^2 = ||1||^2 = \int_{-1}^{1} 1dt = 2$$

[**1 mark**]

$$\begin{aligned} w_2 &= x - \frac{\langle x, 1 \rangle}{\|1\|^2} 1 \\ &= x - \frac{1}{2} \int_{-1}^{1} tdt = x \end{aligned}$$

[1 mark]

and

$$||w_2||^2 = \int_{-1}^{1} t^2 dt = \frac{2}{3}$$

[1 mark]

$$\begin{aligned} w_3 &= x^2 - \langle x^2, 1\rangle \frac{1}{2} - \langle x^2, x\rangle \frac{x}{(2/3)} \\ &= x^2 - \frac{1}{2}\int_{-1}^{1} t^2 dt - \frac{3}{2}x \int_{-1}^{1} t^3 dt \\ &= x^2 - \frac{1}{3} \end{aligned}$$

[1 mark]

Thus $\left\{1, x, x^2 - \frac{1}{3}\right\}$ is an orthogonal basis of V.

4. Let V be an n-dimensional real inner product space, W be a k-dimensional subspace of V and $1 < k < n$. Let $P_W : V \to V$ be the orthogonal projection onto W.

   (a) Show that P_W is a self-adjoint operator.

   (b) Find the eigenvalues and the corresponding eigenspaces of P_W.

   (c) Find the characteristic polynomial of P_W. [4]

   Solution. (a) Let $v = x + y$ where $x \in W, y \in W^{\perp}$. Then $P_W(v) = x$. Let $u = z + w$ where $z \in W, w \in W^{\perp}$. Then P_W is self-adjoint due to the equations:

   $$\langle P_w(v), u\rangle = \langle x, z + w\rangle = \langle x, z\rangle \text{ and } \langle v, P_W(u)\rangle = \langle x + y, z\rangle = \langle x, z\rangle.$$

   [1 mark]

   (b) Since $P_W(w) = w = 1w$ for all $w \in W$, 1 is an eigenvalue of P_W and the corresponding eigenspace V_1 contains W. On the other hand any $u \in W^{\perp}$ satisfies $P_W(u) = 0 = 0u$. Hence 0 is another eigenvalue of P_W and the corresponding eigenspace V_0 contains $W^{\perp}$. **[1 mark]**

   Since $\dim V_0 \geq \dim W^{\perp} = n - k$ and $\dim V_1 \geq \dim W = k$ and $V_0 \cap V_1 = \{0\}$, we see that $\dim V_0 = n - k$ and $\dim V_1 = k$ and hence $0, 1$ are the only eigenvalues of P_W and $V_1 = W$, $V_0 = W^{\perp}$ are the corresponding eigenspaces.

   [1 mark]

   (c) Since the sum of two geometric multiplicities is n, the algebraic multiplicity of 1 is k and that of 0 is $n - k$. Hence $p(x) = (x - 1)^k x^{n-k}$ is the characteristic polynomial of P_W. **[1 mark]**.

5. Identify the locus of the equation $2x_1^2 - 4x_1x_2 - x_2^2 - 4x_1 + 10x_2 - 13 = 0$ and find the unit vectors determining the corresponding principal axes in $\mathbb{R}^2$. [4]

Solution. We write the equation in matrix form as

$$[x_1, x_2] \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [-4, 10] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 13 = 0.$$

Let $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$ and $t = 1/\sqrt{5}$. The eigenvalues of A are $\lambda_1 = 3, \lambda_2 = -2$. An orthonormal set of eigenvectors is $\{u_1 = t[2, -1]^t, u_2 = t[1, 2]^t\}$. [**1 mark**]

Now write $U = t \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

The transformed equation becomes

$$3y_1^2 - 2y_2^2 - 4t(2y_1 + y_2) + 10t(-y_1 + 2y_2) - 13 = 0$$

$$\implies 3y_1^2 - 2y_2^2 - 18ty_1 + 16ty_2 - 13 = 0.$$

[**1 mark**]

Complete the square to get $3(y_1 - 3t)^2 - 2(y_2 - 4t)^2 = 12$. Therefore

$$\frac{(y_1 - 3t)^2}{4} - \frac{(y_2 - 4t)^2}{6} = 1.$$

This represents a hyperbola with center $(3t, 4t)$ in the y_1y_2-plane. [**1 mark**]

The unit vectors in the direction of axes y_1, y_2 have the coordinates$[1, 0]^t$, $[0, 1]^t$ resp. in y_1y_2-plane. Thus it follows that the same unit vectors have x_1, x_2 coordinates determined by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = U \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ where $U = t \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$.

It follows that the eigenvectors u_1 and u_2 are the directions of positive y_1 and y_2 axes. [**1 mark**]

6. Let M and N be two $n \times n$ real matrices such that $M^n = N^n = 0$ but $M^{n-1} \neq 0 \neq N^{n-1}$. Let $T_M, T_N : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear maps defined as $T_M(v) = Mv$ and $T_N(v) = Nv$ for all $v \in \mathbb{R}^n$.

   (a) Show that there exist vectors $u, v \in \mathbb{R}^n$ such that the sets $B = \{u, Mu, \ldots, M^{n-1}u\}$ and $C = \{v, Nv, \ldots, N^{n-1}v\}$ are two bases of $\mathbb{R}^n$.

   (b) Write down the matrices $M_B^B(T_M)$ and $M_C^C(T_N)$.

   (c) Conclude that there exists an $n \times n$ invertible real matrix P such that $P^{-1}MP = N$. [4]

Solution. Since $M^{n-1} \neq 0 \neq N^{n-1}$, there exist vectors $u, v \in \mathbb{R}^n$ such that $M^{n-1}u \neq 0 \neq N^{n-1}v$.

[**1 mark**]

Note that

$$a_1u + a_2Mu + \cdots + a_nM^{n-1}u = 0 \implies a_1M^{n-1}u = 0 \implies a_1 = 0.$$

Similarly,

$$a_2Mu + \cdots + a_nM^{n-1}u = 0 \implies a_1M^{n-1}u = 0 \implies a_2 = 0.$$

By using induction we see that the sets $B = \{u, Mu, \ldots, M^{n-1}u\}$ and $C = \{v, Nv, \ldots, N^{n-1}v\}$ containing n vectors in $\mathbb{R}^n$ are linearly independent subsets of $\mathbb{R}^n$ and hence form bases of $\mathbb{R}^n$.

[**1 mark**]

$$M_B^B(T_M) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad M_C^C(T_N) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

[**1 mark**]

Since $M_B^B(T_M) = M_C^C(T_N)$, there exist $n \times n$ invertible real matrices P_1, P_2 such that $P_1^{-1}MP_1 = P_2^{-1}NP_2$ and this implies that for $P = P_1P_2^{-1}$, $P^{-1}MP = N$. [**1 mark**]

7. Let $\mathbb{R}^{3\times 3}$ be the real vector space of all 3×3 real matrices with the usual vector addition and scalar multiplication. Let $H_3 = \{A = (a_{ij}) \in \mathbb{R}^{3\times 3} \mid a_{ij} = a_{rs}, \text{ if } i + j = r + s\}$.

(a) Show that H_3 is a subspace of $\mathbb{R}^{3\times 3}$.

(b) Determine a basis of H_3. [4]

Solution. (a) For $A = (a_{ij}), B = (b_{ij}) \in H_3$ and $p, q \in \mathbb{F}$, we find that $pA + qB = (pa_{ij} + qb_{ij}) = (c_{ij})$. Now for $i + j = r + s$, $c_{ij} = pa_{ij} + qb_{ij} = pa_{rs} + qb_{rs} = c_{rs}$ and hence $pA + qB \in H_3$. Thus H_3 is a subspace of $\mathbb{F}^{3\times 3}$. [**1 mark**]

If $A = (a_{ij}) \in H_3$ then $a_{12} = a_{21}$, $a_{13} = a_{22} = a_{31}$, $a_{23} = a_{32}$ and other two entries a_{11}, a_{33} can be assigned any values in $\mathbb{F}$. We get that

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{31} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

[**1 mark**]

$$= a_{11}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{31}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$+ a_{32}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + a_{33}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus for $S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$, $L(S) = H_3$.

[**1 mark**]

Now, $a_1 \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + a_2 \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} + a_3 \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} + a_4 \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} + a_5 \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} = 0 \implies$
$\begin{pmatrix}a_1&a_2&a_3\\a_2&a_3&a_4\\a_3&a_4&a_5\end{pmatrix} = 0 \implies a_1 = a_2 = a_3 = a_4 = a_5 = 0$ and hence S is linearly independent. [**1 mark**]

Thus S is a basis for H_3.

8. Let $A = \begin{bmatrix}1&3&-2\\2&5&-7\\0&1&-4\end{bmatrix}$.

(a) Using elementary row-operations show that A is invertible and find A^{-1} using the same method.

(b) Write down the row canonical form (RCF) of A. [4]

Solution. Note that A is invertible if and only if the RCF of A is the identity matrix I.

[**1 mark**]

If we consider the matrix $[A|I]$ and transform it to its RCF $[I|B]$ using the elementary row-operations, then I is the RCF of A and B is A^{-1}.

[**1 mark**]

$$[A|I] = \left[\begin{array}{ccc|ccc}1&3&-2&1&0&0\\2&5&-7&0&1&0\\0&1&-4&0&0&1\end{array}\right] \quad R_2 - 2R_1 \longrightarrow \left[\begin{array}{ccc|ccc}1&3&-2&1&0&0\\0&-1&-3&-2&1&0\\0&1&-4&0&0&1\end{array}\right]$$

$$R_3 + R_2 \longrightarrow \left[\begin{array}{ccc|ccc}1&3&-2&1&0&0\\0&-1&-3&-2&1&0\\0&0&-7&-2&1&1\end{array}\right] -1 \times R_2 \ \& \ -1/7 \times R_3 \longrightarrow$$

$$\left[\begin{array}{ccc|ccc}1&3&-2&1&0&0\\0&1&3&2&-1&0\\0&0&1&2/7&-1/7&-1/7\end{array}\right]$$

[**1 mark**]

$$\left.\begin{array}{l}R_2 - 3R_3\\R_1 + 2R_3\end{array}\right\} \rightarrow \left[\begin{array}{ccc|ccc}1&3&0&11/7&-2/7&-2/7\\0&1&0&8/7&-4/7&3/7\\0&0&1&2/7&-1/7&-1/7\end{array}\right]$$

$$R_1 - 3R_2 \longrightarrow \left[\begin{array}{ccc|ccc}1&0&0&-13/7&10/7&-11/7\\0&1&0&8/7&-4/7&3/7\\0&0&1&2/7&-1/7&-1/7\end{array}\right] = [I|A^{-1}].$$

It follows that $A^{-1} = \begin{bmatrix}-13/7&10/7&-11/7\\8/7&-4/7&3/7\\2/7&-1/7&-1/7\end{bmatrix}$ [**1 mark**]

and $RCF(A) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.