Indian Institute of Technology Bombay

Department of Mathematics

MA 106: Linear Algebra : Test-1

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1. Let $\mathcal{P}_3 = \{a+bx+cx^2+dx^3 \mid a,b,c,d \in \mathbb{R}\}$ be the vector space of all real polynomials of degrees at most 3. Let $\mathcal{S} = \{f(x) \in \mathcal{P}_3 \mid f(0) = f(1)\}$. Show that $\mathcal{S}$ is a subspace of $\mathcal{P}_3$ and find a basis of $\mathcal{S}$. [3]

   Solution. If $a, b \in \mathbb{R}$ and $p(x), q(x) \in V := \mathcal{P}_3$ with $p(0) = p(1)$ and $q(0) = q(1)$. Then $ap(0) + bq(0) = ap(1) + bq(1)$. Therefore $ap(x) + bq(x) \in \mathcal{S}$. Hence $\mathcal{S}$ is a subspace of V. **(1 mark).**

   Let $p(x) = a+bx+cx^2+dx^3$. Then $p(0) = p(1) \iff b+c+d = 0 \iff d = -b-c$. Hence $p(x) = a + bx + cx^2 + (-b-c)x^3 = a + b(x - x^3) + c(x^2 - x^3)$. Hence $\mathcal{S} = L(1, x - x^3, x^2 - x^3)$. **(1 mark).**

   Let $p(x) = a + b(x - x^3) + c(x^2 - x^3) = 0$ for $a, b, c \in \mathbb{R}$. Then $p(x) = a + bx + cx^2 + (-b-c)x^3 = 0$. Hence $a = b = c = 0$. Thus $1, x - x^2, x - x^3$ are linearly independent and hence they form a basis of $\mathcal{S}$. Thus $\dim \mathcal{S} = 3$. **(1 mark).**

2. Show that an $n \times n$ real matrix $A = [A_1\ A_2\ \cdots\ A_n]$ is invertible if and only if the column vectors $A_1, A_2, \ldots, A_n$ of A are linearly independent in $\mathbb{R}^n$. [2]

   Solution. Let A be invertible and $x_1A_1 + \cdots + x_nA_n = 0$ for $x_1, \ldots, x_n \in \mathbb{R}$. Let $x = (x_1, x_2, \ldots, x_n)^t$. Then $Ax = 0$. As A is invertible, $x = 0$. Hence $A_1, \ldots, A_n$ are linearly independent. **(1 mark).**

   Conversely let $A_1, A_2, \ldots, A_n$ be linearly independent. To show that A is invertible, it is enough to show that $Ax = 0 \implies x = 0$ for any $x \in \mathbb{R}^n$. Let $x = (x_1, \ldots, x_n)^t$. Then $Ax = x_1A_1 + \cdots + x_nA_n = 0 \implies x_1 = x_2 = \ldots = x_n = 0$ since $A_1, \ldots, A_n$ are linearly independent. **(1 mark).**

3. Find a basis of the vector space of all 3×3 real symmetric matrices. [3]

   Solution. Let V be the vector space of all real symmetric 3×3 matrices. Let A 3×3 real symmetric matrix A can be written as

$$A = \begin{bmatrix} x & y & z \\ y & u & v \\ z & v & w \end{bmatrix} = x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$+u \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + v \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + w \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \textbf{(1 mark).}$$

The six matrices in the RHS in the above equation are symmetric. This equation shows that these six matrices span V. **(1 mark).**

If they are linearly dependent then we set the RHS of the above equation equal to 0. This will then imply that these scalars $x, y, z, u, v, w = 0$. Hence these six matrices are linearly independent. Hence these six matrices form a basis of the given vector space. **(1 mark).**

4. For given $r, s, t \in \mathbb{R}$ and distinct $a, b, c \in \mathbb{R}$, find all possible polynomials $p(x)$ of degrees at most 2 which satisfy the conditions $p(a) = r,\ p(b) = s,\ p(c) = t$. [3]

Solution. Let $p(x) = u + vx + wx^2$. The equations $p(a) = r, p(b) = s$ and $p(c) = t$ translate into the equations

$$\begin{aligned} p(a) &= u + va + wa^2 = r \\ p(b) &= u + vb + wb^2 = s \\ p(c) &= u + vc + wc^2 = t \quad (\textbf{1 mark}). \end{aligned}$$

These can be written in the matrix form as

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} \quad (\textbf{1 mark}).$$

Since a, b, c are distinct, the 3×3 matrix in the above equation is invertible as its deteminant is nonzero. Hence u, v, w are uniquely determined. Hence there is only one such polynomial. **(1 mark).**