# MA 106 Endsem Exam : Part A : Question Paper
## Indian Institute of Technology Bombay

A

Roll No.: ......................................

Name: ..........................................

Division.: D ☐

Tutorial: T ☐

📅 Apr. 19, 2023

🕘 8.30 - 10.30 AM

✓ 9.30 AM (Part A)

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

➔ There are 14 questions in Part A. Each question contains a **single correct answer** of one mark.

➔ You need to indicate your answer by bubbling the box in the OMR (Optical Mark Recognition) sheet. The right way to fill a box like ☐ is as ■. Use only a ball–point pen to fill the correct choice in the OMR sheet.

➔ **Answer provided in the OMR sheet will only be evaluated.** You can mark answer in the question paper for your reference but it will not be evaluated.

➔ The OMR sheet will be collected back at 9.30 AM.

➔ **Notation:** $\mathbb{R}^{m\times n}$: the set of all $m \times n$ real matrices, $A^t$: the transpose of $A$, $E_\lambda$: the eigenspace for an eigenvalue $\lambda$, $\chi_A(x)$: the characteristic polynomial of $A$, Null$(A)$: the nullspace of $A$, $\mathcal{C}(A)$: the column space of A.

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Q-1⟩ Consider V $= \mathbb{R}^{2\times 2}$ and the inner product $\langle A, B\rangle = \mathsf{tr}(A^tB)$. Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $p_I(J)$ denote the orthogonal projection of $J$ along the identity matrix $I$. Then $p_I(J)$ is

■ A $I$ ☐ B $0$ ☐ C $J$ ☐ D $2I$

Soln.: We have $P_I(J) = \dfrac{\langle I, J\rangle}{\langle I, I\rangle} I = \dfrac{tr(J)}{2} I = I.$ ✠

Q-2⟩ Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(u) = Au$ where $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}$. Then

■ A Null$(T)$ is a line.

☐ B Null$(T) = \{0\}$.

☐ C Im$(T)$ is a line.

☐ D Null$(T)$ is a plane.

Soln.: Note that $\det A = 0$ and $\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} \neq 0$. Therefore, rank$(A) = 2$ and so nuility$(A) = 1$. ✠

Q-3⟩ Let $P_2(\mathbb{R})$ be the space of polynomials over $\mathbb{R}$ of degree $\leq 2$. Define $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ by $T(f(x)) = (f(0), f(-1), f(1))^t$. Then rank$(T)$ is

☐ A 4 ☐ B 1 ☐ C 2 ■ D 3

Soln.: Let $f(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$. Then we have $f(-1) = a - b + c, f(0) = a, f(1) = a + b + c$. If $T(f) = 0$, then $a = b = c = 0$. This shows that $f$ is one-one. Hence, $T$ is an isomorphism. So, $rank(T) = 3$. ✠

Q-4⟩ Let $A$ be a $3 \times 3$ real matrix and $A^2 = A$. Then $\dim(\text{Null}(A) \cap \mathcal{C}(A))$ is

☐ A 3 ☐ B 1 ☑ C 0 ☐ D 2

Soln.: If $v = Ax \in null(A) \cap \mathcal{C}(A)$, for some $x \in \mathbb{R}^3$, then $0 = Av = A^2x = Ax = v$. ✠

Q-5⟩ The number of $2 \times 2$ nilpotent real matrices is

☑ A infinite ☐ B 2 ☐ C 1 ☐ D 3

Soln.: For every $n \in \mathbb{Z}$ we have that $\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ is nilpotent. ✠

Q-6⟩ Let $A$ be a $3 \times 3$ real matrix with $\mathtt{tr}(A) = 0$. Let $\chi_A(2) = \chi_A(3) = 0$. Then

☐ A $\chi_A(x) = (x-2)(x-3)$

☑ B $\chi_A(x) = (x+5)(x-2)(x-3)$

☐ C $\chi_A(x) = (x-2)^2(x-3)$

☐ D $\chi_A(x) = (x-2)(x-3)^2$

Soln.: Since $2, 3$ are roots of the $\chi_A(x)$, and since the sum of all the roots of $\chi_A(x) = tr(A) = 0$, we have that $-5$ is a root of $\chi_A(x)$. ✠

Q-7⟩ Consider positive real numbers $a, b, c$ and matrix $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$. Let $\lambda = a + b + c$. Then

☐ A $\dim E_0 = 1,\ \dim E_\lambda = 1.$

☑ B $\dim E_0 = 2,\ \dim E_\lambda = 1.$

☐ C $\dim E_0 = 2,\ \dim E_\lambda = 2.$

☐ D $\dim E_0 = 1,\ \dim E_\lambda = 2.$

Soln.: We have $rank(A) = 2$ and $nullity(A) = 2$. So, $0$ is an eigenvalue of $A$ with $\dim(E_0) = 2$. Also, $A[1\ 1\ 1]^t = (a+b+c)[1\ 1\ 1]^t$. So, $\lambda = a+b+c$ is an eigenvalue of $A$. Since $\dim(E_0) + \dim(E_\lambda) = 3$, we get $\dim(E_\lambda) = 1$. ✠

Q-8⟩ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $(a-d)^2 + 4bc > 0$. Then

☐ A None of these answers are correct.

☐ B $A$ is not diagonalizable over $\mathbb{R}$.

☐ C Eigenvalues of $A$ are not real.

☒ D There is a basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.

Soln.: Consider $\chi_A(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = x^2 - (a+d)x + (ad-bc)$.
Roots of $\chi_A(x)$ are

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right],$$

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right].$$

From the given condition we get $\lambda_1 \neq \lambda_2$. Thus, $A$ is diagonalizable as the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ are linearly independent. ✠

Q-9⟩ Let $\mathrm{V} = \mathbb{R}^{2\times 2}$ and $P \in \mathrm{V}$ be a nonzero matrix. Consider the linear transformation $T : \mathrm{V} \to \mathrm{V}$ given by $T(A) = AP - PA$. Then

☐ A $T$ is 1-1.

☒ B None of these answers are correct.

☐ C T is an isomorphism.

☐ D $T$ is onto.

Soln.: Clearly $T(P) = 0$. Hence, $T$ is not 1-1. Thus, $T$ is not onto either. Hence, none of these answers are correct. ✠

Q-10⟩ If $A$ is a $3\times 3$ real matrix and $A^2 + I = 0$ then

☐ A $-1, i, -i$ are roots of $\chi_A(x)$.

☐ B $1, i, -i$ are roots of $\chi_A(x)$.

☐ C $i$ and $-i$ are the only roots of $\chi_A(x)$.

☒ D There is no such matrix.

Soln.: Since $\chi_A(x)$ has degree 3, it has a real root $r$. As $A_I^2 = 0$, we get $r^2 = -1$. This is a contradiction. Therefore no such matrix exists. ✠

Q-11⟩ Let V be the vector space over $\mathbb{R}$ of $2\times 2$ Hermitian matrices. Then dimension of V is

☒ A 4 ☐ B 2 ☐ C 1 ☐ D 3

Soln.: Any $2\times 2$ Hermitian matrix $A$ has the form $A = \begin{bmatrix} x & y+iz \\ y-iz & w \end{bmatrix}$ for some $x, y, z, w \in \mathbb{R}$. So,

$$A = x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that $\dim(V) = 4$. ✠

Q-12$\rangle$ The conic described by the $5x^2 - 4xy + 8y^2 = 36$ is

☐ A a pair of lines. ☐ B a hyperbola. ☐ C a parabola. ☑ D an ellipse.

Soln.: The equation is

$$[x \quad y]\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 36.$$

Let $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$. Then $\chi_A(x) = \det(xI - A) = (x-4)(x-9)$. As the eigenvalues are 4 and 9, the equation in $u, r$ plane is given by

$$4u^2 + 9r^2 = 36 \implies \frac{u^2}{9} + \frac{v^2}{1} = 1$$

This is as an ellipse. ✠

Q-13$\rangle$ The rank of the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by $T(A) = PA$ where $A \in \mathbb{R}^{2\times 2}$ and $P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is

☑ A 2 ☐ B 3 ☐ C 1 ☐ D 4

Soln.: Clearly, $\mathrm{Null}(T) = \{A \mid PA = 0\}$. It follows that

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} \in \mathrm{Null}(T) \Leftrightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x & y \\ a & b \end{bmatrix} = 0 \Leftrightarrow \begin{matrix} x + 3a = 0 \\ y + 3b = 0 \end{matrix} \Leftrightarrow x = -3a, \quad y = -3b.$$

Therefore,

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} = a\begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\dim \mathrm{Null}(T) = 2$ and so $\mathrm{rank}(T) = 2$. ✠

Q-14$\rangle$ Let V $= \mathbb{R}^{3\times 3}$ and $\langle A, B\rangle = \mathrm{tr}(AB^t)$. Let $D$ denote the subspace of diagonal matrices in V. Then $\dim D^{\perp}$ is

☑ A 6 ☐ B 9 ☐ C 5 ☐ D 3

Soln.: Since $\dim(D) = 3$ and $\dim(D) + \dim(V^{\perp}) = \dim(V) = 9$, we get that $\dim(V^{\perp}) = 6$. ✠

# MA 106 Part A : Optical Mark Recognition (OMR) Sheet

Name: ........................................

Roll No.: ....................................

Division.: D ☐

Tutorial: T ☐

Stud. Sign.: ...........

Invig. Sign.: ...........

A

**Bubble Your Rollnumber**

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|---|---|---|---|---|---|---|---|---|
| ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 |
| ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 |
| ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 |
| ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 |
| ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 |
| ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 |
| ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 |
| ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 |
| ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 |
| ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 |
| | | ☐ B | | ☐ B | | | | |
| | | ☐ D | | | | | | |

INSTRUCTIONS: ➔ Use only blue/black colored ball-point pen to make a bubble.

➔ The right way to fill a box like ☐ is as ■.

➔ The OMR sheet will be evaluated by using computer, so **no writing is allowed anywhere in the OMR sheet except dedicated places**.

➔ **No rewriting, no overwriting, no erasing, and no cancellation** are allowed.

➔ Before bubbling your rollnumber, first convert the rollnmber into nine characters by appending appropriate leading zero(s) at the beginning and bubble that nine characters rollnumber so that in the *i*th column you bubble the *i*th character only. For example, if your rollnumber is 22B1234 then first convert it into 0022B2134 and then bubble it accordingly.

➔ Each question has single correct answer, so bubbling more than one option in a single question will fetch zero mark.

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Bubble the box corresponds to your correct choice.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Q-1⟩ | ■ A | ☐ B | ☐ C | ☐ D | Q-8⟩ | ☐ A | ☐ B | ☐ C | ■ D |
| Q-2⟩ | ■ A | ☐ B | ☐ C | ☐ D | Q-9⟩ | ☐ A | ■ B | ☐ C | ☐ D |
| Q-3⟩ | ☐ A | ☐ B | ☐ C | ■ D | Q-10⟩ | ☐ A | ☐ B | ☐ C | ■ D |
| Q-4⟩ | ☐ A | ☐ B | ■ C | ☐ D | Q-11⟩ | ■ A | ☐ B | ☐ C | ☐ D |
| Q-5⟩ | ■ A | ☐ B | ☐ C | ☐ D | Q-12⟩ | ☐ A | ☐ B | ☐ C | ■ D |
| Q-6⟩ | ☐ A | ■ B | ☐ C | ☐ D | Q-13⟩ | ■ A | ☐ B | ☐ C | ☐ D |
| Q-7⟩ | ☐ A | ■ B | ☐ C | ☐ D | Q-14⟩ | ■ A | ☐ B | ☐ C | ☐ D |

# MA 106 Endsem Exam : Part A : Question Paper
# Indian Institute of Technology Bombay

B

Roll No.: ....................................

Name: ......................................

Division.: D ☐

Tutorial: T ☐

Apr. 19, 2023

8.30 - 10.30 AM

9.30 AM (Part A)

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

- There are 14 questions in Part A. Each question contains a **single correct answer** of one mark.
- You need to indicate your answer by bubbling the box in the OMR (Optical Mark Recognition) sheet. The right way to fill a box like ☐ is as ■. Use only a ball–point pen to fill the correct choice in the OMR sheet.
- **Answer provided in the OMR sheet will only be evaluated.** You can mark answer in the question paper for your reference but it will not be evaluated.
- The OMR sheet will be collected back at 9.30 AM.
- **Notation:** $\mathbb{R}^{m\times n}$: the set of all $m \times n$ real matrices, $A^t$: the transpose of $A$, $E_\lambda$: the eigenspace for an eigenvalue $\lambda$, $\chi_A(x)$: the characteristic polynomial of $A$, $\text{Null}(A)$: the nullspace of $A$, $\mathcal{C}(A)$: the column space of A.

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Q-1⟩ Let $A$ be a $3 \times 3$ real matrix and $A^2 = A$. Then $\dim(\text{Null}(A) \cap \mathcal{C}(A))$ is

☐ A 1 ■ B 0 ☐ C 3 ☐ D 2

Soln.: If $v = Ax \in null(A) \cap \mathcal{C}(A)$, for some $x \in \mathbb{R}^3$, then $0 = Av = A^2x = Ax = v$. ✠

Q-2⟩ If $A$ is a $3 \times 3$ real matrix and $A^2 + I = 0$ then

☐ A $-1, i, -i$ are roots of $\chi_A(x)$.

☐ B $i$ and $-i$ are the only roots of $\chi_A(x)$.

☐ C $1, i, -i$ are roots of $\chi_A(x)$.

■ D There is no such matrix.

Soln.: Since $\chi_A(x)$ has degree 3, it has a real root $r$. As $A_I^2 = 0$, we get $r^2 = -1$. This is a contradiction. Therefore no such matrix exists. ✠

Q-3⟩ The conic described by the $5x^2 - 4xy + 8y^2 = 36$ is

☒ A an ellipse. ☐ B a parabola. ☐ C a pair of lines. ☐ D a hyperbola.

Soln.: The equation is

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 36.$$

Let $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$. Then $\chi_A(x) = \det(xI - A) = (x-4)(x-9)$. As the eigenvalues are 4 and 9, the equation in $u, r$ plane is given by

$$4u^2 + 9r^2 = 36 \implies \frac{u^2}{9} + \frac{v^2}{1} = 1$$

This is as an ellipse. ✠

Q-4⟩ Consider positive real numbers $a, b, c$ and matrix $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$. Let $\lambda = a + b + c$. Then

☐ A $\dim E_0 = 1, \ \dim E_\lambda = 2.$

☐ B $\dim E_0 = 2, \ \dim E_\lambda = 2.$

☒ C $\dim E_0 = 2, \ \dim E_\lambda = 1.$

☐ D $\dim E_0 = 1, \ \dim E_\lambda = 1.$

Soln.: We have $rank(A) = 2$ and $nullity(A) = 2$. So, 0 is an eigenvalue of $A$ with $\dim(E_0) = 2$. Also, $A[1 \ 1 \ 1]^t = (a + b + c)[1 \ 1 \ 1]^t$. So, $\lambda = a + b + c$ is an eigenvalue of $A$. Since $\dim(E_0) + \dim(E_\lambda) = 3$, we get $\dim(E_\lambda) = 1$. ✠

Q-5⟩ Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(u) = Au$ where $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}$. Then

☐ A $\text{Null}(T) = \{0\}$.

☐ B $\text{Im}(T)$ is a line.

☒ C $\text{Null}(T)$ is a line.

☐ D $\text{Null}(T)$ is a plane.

Soln.: Note that $\det A = 0$ and $\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} \neq 0$. Therefore, $\text{rank}(A) = 2$ and so $\text{nuility}(A) = 1$. ✠

Q-6⟩ Let $A$ be a $3 \times 3$ real matrix with $\text{tr}(A) = 0$. Let $\chi_A(2) = \chi_A(3) = 0$. Then

☐ A $\chi_A(x) = (x-2)(x-3)$

☐ B $\chi_A(x) = (x-2)(x-3)^2$

☒ C $\chi_A(x) = (x+5)(x-2)(x-3)$

☐ D $\chi_A(x) = (x-2)^2(x-3)$

Soln.: Since $2, 3$ are roots of the $\chi_A(x)$, and since the sum of all the roots of $\chi_A(x) = tr(A) = 0$, we have that $-5$ is a root of $\chi_A(x)$. ✠

Q-7⟩ The number of $2 \times 2$ nilpotent real matrices is

☐ A 1 ☐ B 3 ☐ C 2 ☒ D infinite

Soln.: For every $n \in \mathbb{Z}$ we have that $\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ is nilpotent. ✠

Q-8⟩ Let V be the vector space over $\mathbb{R}$ of $2 \times 2$ Hermitian matrices. Then dimension of V is

☐ A 2 ☐ B 1 ☒ C 4 ☐ D 3

Soln.: Any $2 \times 2$ Hermitian matrix $A$ has the form $A = \begin{bmatrix} x & y+iz \\ y-iz & w \end{bmatrix}$ for some $x, y, z, w \in \mathbb{R}$. So,

$$A = x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that $\dim(V) = 4$. ✠

Q-9⟩ Consider V $= \mathbb{R}^{2\times 2}$ and the inner product $\langle A, B\rangle = \mathrm{tr}(A^t B)$. Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $p_I(J)$ denote the orthogonal projection of $J$ along the identity matrix $I$. Then $p_I(J)$ is

☐ A $2I$ ☐ B 0 ☐ C $J$ ☒ D $I$

Soln.: We have $P_I(J) = \dfrac{\langle I, J\rangle}{\langle I, I\rangle} I = \dfrac{tr(J)}{2} I = I.$ ✠

Q-10⟩ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $(a-d)^2 + 4bc > 0$. Then

☒ A There is a basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.

☐ B Eigenvalues of $A$ are not real.

☐ C $A$ is not diagonalizable over $\mathbb{R}$.

☐ D None of these answers are correct.

Soln.: Consider $\chi_A(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = x^2 - (a+d)x + (ad - bc).$
Roots of $\chi_A(x)$ are

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right],$$

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right].$$

From the given condition we get $\lambda_1 \neq \lambda_2$. Thus, $A$ is diagonalizable as the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ are linearly independent. ✠

Q-11⟩ Let V $= \mathbb{R}^{2\times 2}$ and $P \in$ V be a nonzero matrix. Consider the linear transformation $T :$ V $\to$ V given by $T(A) = AP - PA$. Then

☒ A None of these answers are correct. ☐ C $T$ is onto.

☐ B T is an isomorphism. ☐ D $T$ is 1-1.

Soln.: Clearly $T(P) = 0$. Hence, $T$ is not 1-1. Thus, $T$ is not onto either. Hence, none of these answers are correct. ✠

Q-12⟩ Let $P_2(\mathbb{R})$ be the space of polynomials over $\mathbb{R}$ of degree $\leq 2$. Define $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ by $T(f(x)) = (f(0), f(-1), f(1))^t$. Then rank($T$) is

☐ A 1 ☐ B 2 ☐ C 4 ☒ D 3

Soln.: Let $f(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$. Then we have $f(-1) = a - b + c$, $f(0) = a$, $f(1) = a + b + c$. If $T(f) = 0$, then $a = b = c = 0$. This shows that $f$ is one-one. Hence, $T$ is an isomorphism. So, $rank(T) = 3$. ✠

Q-13⟩ The rank of the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by $T(A) = PA$ where $A \in \mathbb{R}^{2\times 2}$ and $P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is

☐ A 4 ☐ B 1 ☐ C 3 ☒ D 2

Soln.: Clearly, $\mathrm{Null}(T) = \{A \mid PA = 0\}$. It follows that

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} \in \mathrm{Null}(T) \Leftrightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x & y \\ a & b \end{bmatrix} = 0 \Leftrightarrow \begin{matrix} x + 3a = 0 \\ y + 3b = 0 \end{matrix} \Leftrightarrow x = -3a, \quad y = -3b.$$

Therefore,

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} = a\begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\dim \mathrm{Null}(T) = 2$ and so $\mathrm{rank}(T) = 2$. ✠

Q-14⟩ Let V $= \mathbb{R}^{3\times 3}$ and $\langle A, B\rangle = \mathrm{tr}(AB^t)$. Let $D$ denote the subspace of diagonal matrices in V. Then $\dim D^{\perp}$ is

☐ A 5 ☐ B 9 ☒ C 6 ☐ D 3

Soln.: Since $\dim(D) = 3$ and $\dim(D) + \dim(V^{\perp}) = \dim(V) = 9$, we get that $\dim(V^{\perp}) = 6$. ✠

# MA 106 Part A : Optical Mark Recognition (OMR) Sheet

Name: ........................................

Roll No.: ....................................

Division.: D ☐

Tutorial: T ☐

Stud. Sign.: ...........

Invig. Sign.: ...........

Bubble Your Rollnumber

........................................

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 |
| ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 |
| ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 |
| ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 |
| ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 |
| ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 |
| ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 |
| ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 |
| ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 |
| ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 |
| | | ☐ B | | ☐ B | | | | |
| | | ☐ D | | | | | | |

**B**

INSTRUCTIONS:
- Use only blue/black colored ball-point pen to make a bubble.
- The right way to fill a box like ☐ is as ■.
- The OMR sheet will be evaluated by using computer, so **no writing is allowed anywhere in the OMR sheet except dedicated places**.
- **No rewriting, no overwriting, no erasing, and no cancellation** are allowed.
- Before bubbling your rollnumber, first convert the rollnmber into nine characters by appending appropriate leading zero(s) at the beginning and bubble that nine characters rollnumber so that in the $i$th column you bubble the $i$th character only. For example, if your rollnumber is 22B1234 then first convert it into 0022B2134 and then bubble it accordingly.
- Each question has single correct answer, so bubbling more than one option in a single question will fetch zero mark.

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Bubble the box corresponds to your correct choice.

Q-1⟩ ☐ A ■ B ☐ C ☐ D

Q-2⟩ ☐ A ☐ B ☐ C ■ D

Q-3⟩ ■ A ☐ B ☐ C ☐ D

Q-4⟩ ☐ A ☐ B ■ C ☐ D

Q-5⟩ ☐ A ☐ B ■ C ☐ D

Q-6⟩ ☐ A ☐ B ■ C ☐ D

Q-7⟩ ☐ A ☐ B ☐ C ■ D

Q-8⟩ ☐ A ☐ B ■ C ☐ D

Q-9⟩ ☐ A ☐ B ☐ C ■ D

Q-10⟩ ■ A ☐ B ☐ C ☐ D

Q-11⟩ ■ A ☐ B ☐ C ☐ D

Q-12⟩ ☐ A ☐ B ☐ C ■ D

Q-13⟩ ☐ A ☐ B ☐ C ■ D

Q-14⟩ ☐ A ☐ B ■ C ☐ D

# MA 106 Endsem Exam : Part A : Question Paper
# Indian Institute of Technology Bombay

C

Roll No.: ........................................

Name: ........................................

Division.: D ☐

Tutorial: T ☐

Apr. 19, 2023

8.30 - 10.30 AM

9.30 AM (Part A)

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

➔ There are 14 questions in Part A. Each question contains a **single correct answer** of one mark.
➔ You need to indicate your answer by bubbling the box in the OMR (Optical Mark Recognition) sheet. The right way to fill a box like ☐ is as ■. Use only a ball–point pen to fill the correct choice in the OMR sheet.
➔ **Answer provided in the OMR sheet will only be evaluated.** You can mark answer in the question paper for your reference but it will not be evaluated.
➔ The OMR sheet will be collected back at 9.30 AM.
➔ **Notation:** $\mathbb{R}^{m\times n}$: the set of all $m \times n$ real matrices, $A^t$: the transpose of $A$, $E_\lambda$: the eigenspace for an eigenvalue $\lambda$, $\chi_A(x)$: the characteristic polynomial of $A$, Null$(A)$: the nullspace of $A$, $\mathcal{C}(A)$: the column space of A.

---

Q-1⟩ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $(a-d)^2 + 4bc > 0$. Then

☐ A None of these answers are correct.

☐ B $A$ is not diagonalizable over $\mathbb{R}$.

☐ C Eigenvalues of $A$ are not real.

■ D There is a basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.

Soln.: Consider $\chi_A(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = x^2 - (a+d)x + (ad-bc)$.
Roots of $\chi_A(x)$ are

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right],$$

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right].$$

From the given condition we get $\lambda_1 \neq \lambda_2$. Thus, $A$ is diagonalizable as the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ are linearly independent. ✠

Q-2⟩ Consider V $= \mathbb{R}^{2\times 2}$ and the inner product $\langle A, B\rangle = \mathrm{tr}(A^tB)$. Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $p_I(J)$ denote the orthogonal projection of $J$ along the identity matrix $I$. Then $p_I(J)$ is

☐ A $0$ ■ B $I$ ☐ C $J$ ☐ D $2I$

Soln.: We have $P_I(J) = \frac{\langle I, J\rangle}{\langle I, I\rangle} I = \frac{tr(J)}{2} I = I.$ ✠

Q-3⟩ Let $A$ be a $3 \times 3$ real matrix and $A^2 = A$. Then $\dim(\mathrm{Null}(A) \cap \mathcal{C}(A))$ is

☒ A 0 ☐ B 3 ☐ C 2 ☐ D 1

Soln.: If $v = Ax \in null(A) \cap \mathcal{C}(A)$, for some $x \in \mathbb{R}^3$, then $0 = Av = A^2x = Ax = v$. ✠

Q-4⟩ The rank of the linear transformation $T : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ given by $T(A) = PA$ where $A \in \mathbb{R}^{2\times2}$ and $P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is

☐ A 4 ☐ B 1 ☒ C 2 ☐ D 3

Soln.: Clearly, $\mathrm{Null}(T) = \{A \mid PA = 0\}$. It follows that

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} \in \mathrm{Null}(T) \Leftrightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x & y \\ a & b \end{bmatrix} = 0 \Leftrightarrow \begin{matrix} x + 3a = 0 \\ y + 3b = 0 \end{matrix} \Leftrightarrow x = -3a, \quad y = -3b.$$

Therefore,

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} = a\begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\dim \mathrm{Null}(T) = 2$ and so $\mathrm{rank}(T) = 2$. ✠

Q-5⟩ If $A$ is a $3 \times 3$ real matrix and $A^2 + I = 0$ then

☐ A $1, i, -i$ are roots of $\chi_A(x)$.

☐ C $-1, i, -i$ are roots of $\chi_A(x)$.

☒ B There is no such matrix.

☐ D $i$ and $-i$ are the only roots of $\chi_A(x)$.

Soln.: Since $\chi_A(x)$ has degree 3, it has a real root $r$. As $A_I^2 = 0$, we get $r^2 = -1$. This is a contradiction. Therefore no such matrix exists. ✠

Q-6⟩ Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(u) = Au$ where $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}$. Then

☐ A $\mathrm{Null}(T)$ is a plane.

☐ C $\mathrm{Im}(T)$ is a line.

☒ B $\mathrm{Null}(T)$ is a line.

☐ D $\mathrm{Null}(T) = \{0\}$.

Soln.: Note that $\det A = 0$ and $\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} \neq 0$. Therefore, $\mathrm{rank}(A) = 2$ and so $\mathrm{nuility}(A) = 1$. ✠

Q-7⟩ The conic described by the $5x^2 - 4xy + 8y^2 = 36$ is

☐ A a pair of lines. ☒ B an ellipse. ☐ C a hyperbola. ☐ D a parabola.

Soln.: The equation is

$$[x \quad y]\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 36.$$

Let $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$. Then $\chi_A(x) = \det(xI - A) = (x-4)(x-9)$. As the eigenvalues are 4 and 9, the equation in $u, r$ plane is given by

$$4u^2 + 9r^2 = 36 \implies \frac{u^2}{9} + \frac{v^2}{1} = 1$$

This is as an ellipse. ✠

Q-8⟩ Consider positive real numbers $a, b, c$ and matrix $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$. Let $\lambda = a + b + c$. Then

☐ A $\dim E_0 = 1, \ \dim E_\lambda = 2.$

☒ B $\dim E_0 = 2, \ \dim E_\lambda = 1.$

☐ C $\dim E_0 = 2, \ \dim E_\lambda = 2.$

☐ D $\dim E_0 = 1, \ \dim E_\lambda = 1.$

Soln.: We have $rank(A) = 2$ and $nullity(A) = 2$. So, 0 is an eigenvalue of $A$ with $\dim(E_0) = 2$. Also, $A[1\ 1\ 1]^t = (a + b + c)[1\ 1\ 1]^t$. So, $\lambda = a + b + c$ is an eigenvalue of $A$. Since $\dim(E_0) + \dim(E_\lambda) = 3$, we get $\dim(E_\lambda) = 1$. ✠

Q-9⟩ The number of $2 \times 2$ nilpotent real matrices is

☒ A infinite ☐ B 1 ☐ C 3 ☐ D 2

Soln.: For every $n \in \mathbb{Z}$ we have that $\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ is nilpotent. ✠

Q-10⟩ Let V be the vector space over $\mathbb{R}$ of $2 \times 2$ Hermitian matrices. Then dimension of V is

☐ A 1 ☒ B 4 ☐ C 3 ☐ D 2

Soln.: Any $2 \times 2$ Hermitian matrix $A$ has the form $A = \begin{bmatrix} x & y + iz \\ y - iz & w \end{bmatrix}$ for some $x, y, z, w \in \mathbb{R}$. So,

$$A = x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that $\dim(V) = 4$. ✠

Q-11⟩ Let V $= \mathbb{R}^{2\times 2}$ and $P \in$ V be a nonzero matrix. Consider the linear transformation $T :$ V $\to$ V given by $T(A) = AP - PA$. Then

☐ A $T$ is 1-1.

☒ B None of these answers are correct.

☐ C $T$ is onto.

☐ D T is an isomorphism.

Soln.: Clearly $T(P) = 0$. Hence, $T$ is not 1-1. Thus, $T$ is not onto either. Hence, none of these answers are correct. ✠

Q-12⟩ Let $A$ be a $3 \times 3$ real matrix with $\mathsf{tr}(A) = 0$. Let $\chi_A(2) = \chi_A(3) = 0$. Then

☐ A $\chi_A(x) = (x-2)(x-3)^2$

☐ B $\chi_A(x) = (x-2)^2(x-3)$

☐ C $\chi_A(x) = (x-2)(x-3)$

☒ D $\chi_A(x) = (x+5)(x-2)(x-3)$

Soln.: Since $2, 3$ are roots of the $\chi_A(x)$, and since the sum of all the roots of $\chi_A(x) = tr(A) = 0$, we have that $-5$ is a root of $\chi_A(x)$. ✠

Q-13⟩ Let $P_2(\mathbb{R})$ be the space of polynomials over $\mathbb{R}$ of degree $\leq 2$. Define $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ by $T(f(x)) = (f(0), f(-1), f(1))^t$. Then rank$(T)$ is

☐ A 1 ☐ B 2 ☐ C 4 ☒ D 3

Soln.: Let $f(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$. Then we have $f(-1) = a - b + c, f(0) = a, f(1) = a + b + c$. If $T(f) = 0$, then $a = b = c = 0$. This shows that $f$ is one-one. Hence, $T$ is an isomorphism. So, $rank(T) = 3$. ✠

Q-14⟩ Let V $= \mathbb{R}^{3\times 3}$ and $\langle A, B\rangle = \mathsf{tr}(AB^t)$. Let $D$ denote the subspace of diagonal matrices in V. Then $\dim D^\perp$ is

☒ A 6 ☐ B 9 ☐ C 5 ☐ D 3

Soln.: Since $\dim(D) = 3$ and $\dim(D) + \dim(V^\perp) = \dim(V) = 9$, we get that $\dim(V^\perp) = 6$. ✠

# MA 106 Part A : Optical Mark Recognition (OMR) Sheet

Name: ........................................

Roll No.: ......................................

Division.: D ☐

Tutorial: T ☐

Stud. Sign.: ...........

Invig. Sign.: ...........

C

Bubble Your Rollnumber

........................................

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| | | B | | B | | | | |
| | | D | | | | | | |

INSTRUCTIONS: ➔ Use only blue/black colored ball-point pen to make a bubble.

➔ The right way to fill a box like ☐ is as ■.

➔ The OMR sheet will be evaluated by using computer, so **no writing is allowed anywhere in the OMR sheet except dedicated places**.

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➔ Each question has single correct answer, so bubbling more than one option in a single question will fetch zero mark.

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Bubble the box corresponds to your correct choice.

Q-1⟩ A B C [✓ D]

Q-2⟩ A [✓ B] C D

Q-3⟩ [✓ A] B C D

Q-4⟩ A B [✓ C] D

Q-5⟩ A [✓ B] C D

Q-6⟩ A [✓ B] C D

Q-7⟩ A [✓ B] C D

Q-8⟩ A [✓ B] C D

Q-9⟩ [✓ A] B C D

Q-10⟩ A [✓ B] C D

Q-11⟩ A [✓ B] C D

Q-12⟩ A B C [✓ D]

Q-13⟩ A B C [✓ D]

Q-14⟩ [✓ A] B C D

# MA 106 Endsem Exam : Part A : Question Paper

# Indian Institute of Technology Bombay

D

Roll No.: ........................................

Name: ........................................

Division.: D ☐

Tutorial: T ☐

📅 Apr. 19, 2023

🕗 8.30 - 10.30 AM

✓ 9.30 AM (Part A)

READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

➜ There are 14 questions in Part A. Each question contains a **single correct answer** of one mark.
➜ You need to indicate your answer by bubbling the box in the OMR (Optical Mark Recognition) sheet. The right way to fill a box like ☐ is as ■. Use only a ball–point pen to fill the correct choice in the OMR sheet.
➜ **Answer provided in the OMR sheet will only be evaluated.** You can mark answer in the question paper for your reference but it will not be evaluated.
➜ The OMR sheet will be collected back at 9.30 AM.
➜ **Notation:** $\mathbb{R}^{m\times n}$: the set of all $m \times n$ real matrices, $A^t$: the transpose of $A$, $E_\lambda$: the eigenspace for an eigenvalue $\lambda$, $\chi_A(x)$: the characteristic polynomial of $A$, $\text{Null}(A)$: the nullspace of $A$, $\mathcal{C}(A)$: the column space of A.

---

Q-1⟩ The number of $2 \times 2$ nilpotent real matrices is

■ A infinite ☐ B 3 ☐ C 2 ☐ D 1

Soln.: For every $n \in \mathbb{Z}$ we have that $\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ is nilpotent. ✠

Q-2⟩ Let $P_2(\mathbb{R})$ be the space of polynomials over $\mathbb{R}$ of degree $\leq 2$. Define $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ by $T(f(x)) = (f(0), f(-1), f(1))^t$. Then rank$(T)$ is

☐ A 4 ■ B 3 ☐ C 2 ☐ D 1

Soln.: Let $f(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$. Then we have $f(-1) = a - b + c$, $f(0) = a$, $f(1) = a + b + c$. If $T(f) = 0$, then $a = b = c = 0$. This shows that $f$ is one-one. Hence, $T$ is an isomorphism. So, $rank(T) = 3$. ✠

Q-3⟩ The rank of the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by $T(A) = PA$ where $A \in \mathbb{R}^{2\times 2}$ and $P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ is

☐ A 1 ☐ B 4 ☐ C 3 ☒ D 2

Soln.: Clearly, $\mathrm{Null}(T) = \{A \mid PA = 0\}$. It follows that

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} \in \mathrm{Null}(T) \Leftrightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x & y \\ a & b \end{bmatrix} = 0 \Leftrightarrow \begin{matrix} x + 3a = 0 \\ y + 3b = 0 \end{matrix} \Leftrightarrow x = -3a, \quad y = -3b.$$

Therefore,

$$\begin{bmatrix} x & y \\ a & b \end{bmatrix} = \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} = a\begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\dim \mathrm{Null}(T) = 2$ and so $\mathrm{rank}(T) = 2$. ✠

Q-4⟩ Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(u) = Au$ where $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}$. Then

☐ A $\mathrm{Null}(T) = \{0\}$.

☐ B $\mathrm{Im}(T)$ is a line.

☐ C $\mathrm{Null}(T)$ is a plane.

☒ D $\mathrm{Null}(T)$ is a line.

Soln.: Note that $\det A = 0$ and $\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} \neq 0$. Therefore, $\mathrm{rank}(A) = 2$ and so $\mathrm{nuility}(A) = 1$. ✠

Q-5⟩ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $(a-d)^2 + 4bc > 0$. Then

☐ A None of these answers are correct.

☐ B Eigenvalues of $A$ are not real.

☐ C $A$ is not diagonalizable over $\mathbb{R}$.

☒ D There is a basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.

Soln.: Consider $\chi_A(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = x^2 - (a+d)x + (ad - bc)$.

Roots of $\chi_A(x)$ are

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right],$$

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}\right] = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right].$$

From the given condition we get $\lambda_1 \neq \lambda_2$. Thus, $A$ is diagonalizable as the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ are linearly independent. ✠

Q-6⟩ Let $A$ be a $3 \times 3$ real matrix with $\mathsf{tr}(A) = 0$. Let $\chi_A(2) = \chi_A(3) = 0$. Then

- ☐ A $\chi_A(x) = (x-2)(x-3)^2$
- ☒ B $\chi_A(x) = (x+5)(x-2)(x-3)$
- ☐ C $\chi_A(x) = (x-2)^2(x-3)$
- ☐ D $\chi_A(x) = (x-2)(x-3)$

Soln.: Since $2, 3$ are roots of the $\chi_A(x)$, and since the sum of all the roots of $\chi_A(x) = tr(A) = 0$, we have that $-5$ is a root of $\chi_A(x)$. ✠

Q-7⟩ Consider V $= \mathbb{R}^{2\times 2}$ and the inner product $\langle A, B\rangle = \mathsf{tr}(A^t B)$. Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $p_I(J)$ denote the orthogonal projection of $J$ along the identity matrix $I$. Then $p_I(J)$ is

☒ A $I$ ☐ B $0$ ☐ C $2I$ ☐ D $J$

Soln.: We have $P_I(J) = \dfrac{\langle I, J\rangle}{\langle I, I\rangle} I = \dfrac{tr(J)}{2} I = I.$ ✠

Q-8⟩ Let V $= \mathbb{R}^{2\times 2}$ and $P \in$ V be a nonzero matrix. Consider the linear transformation $T :$ V $\to$ V given by $T(A) = AP - PA$. Then

- ☐ A $T$ is onto.
- ☐ B $T$ is 1-1.
- ☒ C None of these answers are correct.
- ☐ D T is an isomorphism.

Soln.: Clearly $T(P) = 0$. Hence, $T$ is not 1-1. Thus, $T$ is not onto either. Hence, none of these answers are correct. ✠

Q-9⟩ Let V $= \mathbb{R}^{3\times 3}$ and $\langle A, B\rangle = \mathsf{tr}(AB^t)$. Let $D$ denote the subspace of diagonal matrices in V. Then $\dim D^{\perp}$ is

☐ A 5 ☐ B 9 ☐ C 3 ☒ D 6

Soln.: Since $\dim(D) = 3$ and $\dim(D) + \dim(V^{\perp}) = \dim(V) = 9$, we get that $\dim(V^{\perp}) = 6$. ✠

Q-10⟩ Consider positive real numbers $a, b, c$ and matrix $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$. Let $\lambda = a + b + c$. Then

- ☐ A $\dim E_0 = 1, \ \dim E_\lambda = 2.$
- ☐ B $\dim E_0 = 1, \ \dim E_\lambda = 1.$
- ☐ C $\dim E_0 = 2, \ \dim E_\lambda = 2.$
- ☒ D $\dim E_0 = 2, \ \dim E_\lambda = 1.$

Soln.: We have $rank(A) = 2$ and $nullity(A) = 2$. So, 0 is an eigenvalue of $A$ with $\dim(E_0) = 2$. Also, $A[1\ 1\ 1]^t = (a+b+c)[1\ 1\ 1]^t$. So, $\lambda = a + b + c$ is an eigenvalue of $A$. Since $\dim(E_0) + \dim(E_\lambda) = 3$, we get $\dim(E_\lambda) = 1$. ✠

Q-11⟩ If $A$ is a $3 \times 3$ real matrix and $A^2 + I = 0$ then

☐ A $-1, i, -i$ are roots of $\chi_A(x)$.

☐ B $i$ and $-i$ are the only roots of $\chi_A(x)$.

☐ C $1, i, -i$ are roots of $\chi_A(x)$.

☒ D There is no such matrix.

**Soln.:** Since $\chi_A(x)$ has degree 3, it has a real root $r$. As $A_I^2 = 0$, we get $r^2 = -1$. This is a contradiction. Therefore no such matrix exists. ✠

Q-12⟩ Let V be the vector space over $\mathbb{R}$ of $2 \times 2$ Hermitian matrices. Then dimension of V is

☒ A 4 ☐ B 3 ☐ C 2 ☐ D 1

**Soln.:** Any $2 \times 2$ Hermitian matrix $A$ has the form $A = \begin{bmatrix} x & y+iz \\ y-iz & w \end{bmatrix}$ for some $x, y, z, w \in \mathbb{R}$. So,

$$A = x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that $\dim(V) = 4$. ✠

Q-13⟩ Let $A$ be a $3 \times 3$ real matrix and $A^2 = A$. Then $\dim\left(\text{Null}(A) \cap \mathcal{C}(A)\right)$ is

☐ A 1 ☐ B 3 ☐ C 2 ☒ D 0

**Soln.:** If $v = Ax \in null(A) \cap \mathcal{C}(A)$, for some $x \in \mathbb{R}^3$, then $0 = Av = A^2x = Ax = v$. ✠

Q-14⟩ The conic described by the $5x^2 - 4xy + 8y^2 = 36$ is

☐ A a pair of lines. ☐ B a hyperbola. ☒ C an ellipse. ☐ D a parabola.

**Soln.:** The equation is

$$[x \quad y]\begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 36.$$

Let $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$. Then $\chi_A(x) = \det(xI - A) = (x-4)(x-9)$. As the eigenvalues are 4 and 9, the equation in $u, r$ plane is given by

$$4u^2 + 9r^2 = 36 \implies \frac{u^2}{9} + \frac{v^2}{1} = 1$$

This is as an ellipse. ✠

# MA 106 Part A : Optical Mark Recognition (OMR) Sheet

Name: ........................................

Roll No.: ....................................

Division.: D ☐

Tutorial: T ☐

Stud. Sign.: ...........

Invig. Sign.: ...........

D

Bubble Your Rollnumber

........................................

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 | ☐ 0 |
| ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 | ☐ 1 |
| ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 | ☐ 2 |
| ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 | ☐ 3 |
| ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 | ☐ 4 |
| ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 | ☐ 5 |
| ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 | ☐ 6 |
| ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 | ☐ 7 |
| ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 | ☐ 8 |
| ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 | ☐ 9 |
| | | ☐ B | | ☐ B | | | | |
| | | ☐ D | | | | | | |

INSTRUCTIONS: ➜ Use only blue/black colored ball-point pen to make a bubble.
➜ The right way to fill a box like □ is as ■.
➜ The OMR sheet will be evaluated by using computer, so **no writing is allowed anywhere in the OMR sheet except dedicated places**.
➜ **No rewriting, no overwriting, no erasing, and no cancellation** are allowed.
➜ Before bubbling your rollnumber, first convert the rollnmber into nine characters by appending appropriate leading zero(s) at the beginning and bubble that nine characters rollnumber so that in the *i*th column you bubble the *i*th character only. For example, if your rollnumber is 22B1234 then first convert it into 0022B2134 and then bubble it accordingly.
➜ Each question has single correct answer, so bubbling more than one option in a single question will fetch zero mark.

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Bubble the box corresponds to your correct choice.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Q-1⟩ | ■ A | ☐ B | ☐ C | ☐ D | Q-8⟩ | ☐ A | ☐ B | ■ C | ☐ D |
| Q-2⟩ | ☐ A | ■ B | ☐ C | ☐ D | Q-9⟩ | ☐ A | ☐ B | ☐ C | ■ D |
| Q-3⟩ | ☐ A | ☐ B | ☐ C | ■ D | Q-10⟩ | ☐ A | ☐ B | ☐ C | ■ D |
| Q-4⟩ | ☐ A | ☐ B | ☐ C | ■ D | Q-11⟩ | ☐ A | ☐ B | ☐ C | ■ D |
| Q-5⟩ | ☐ A | ☐ B | ☐ C | ■ D | Q-12⟩ | ■ A | ☐ B | ☐ C | ☐ D |
| Q-6⟩ | ☐ A | ■ B | ☐ C | ☐ D | Q-13⟩ | ☐ A | ☐ B | ☐ C | ■ D |
| Q-7⟩ | ■ A | ☐ B | ☐ C | ☐ D | Q-14⟩ | ☐ A | ☐ B | ■ C | ☐ D |