Day #4 Sums and Products

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ΔΣ Introduction

If I asked you what's the value of $1+2+3+4+5$, you'd probably be able to figure out that it is 15 by adding the numbers together. But, if I ask you the value of $1+2+3+4+5+6+7+8+9+10$, you'd probably shoot me a rightfully placed annoyed look. Sure, it's still possible to solve, but the example just became two times as difficult. Does that mean $1+2+3+4+5+...+96+97+98+99+100$ is ten times as hard to calculate as $1+2+3+4+5+6+7+8+9+10$? Not necessarily! Using some advanced techniques, we can derive a quick way to calculate the sum of the first n integers. In this handout, we will discuss the interesting properties of large addition problems, and look into similar properties with multiplication.

ΔΣ Warm-up

Problem 1 (*Warmup*) In the sum $1+2+3+4+5$, notice the **symmetry** if we focus on the number 3. The next term is $3+1=4$ and the term before is $3-1=2$. Can you leverage this symmetry to find a quick way to add $1+2+3+4+5+6+7+8+9+10$? What about

$$1+2+3+4+5+...+96+97+98+99+100?$$

ΔΣ Theory: Arithmetic Sequences and Series

Definition 1 (*Arithmetic Sequence*) A finite arithmetic sequence is a list of numbers of the form

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad a+4d, \quad ..., \quad a+(n-2)d, \quad a+(n-1)d$$

where a is the "staring number" and d is the "common difference."

An infinite arithmetic sequence is a list of numbers of the form

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad a+4d, \quad ...$$

Where the ... implies infinite continuation of the list.

Wow, that's a lot of variables... maybe we should try a few test cases to really see the arithmetic sequence in action. Let's say we want to make an arithmetic sequence with a starting term of 1 and a common difference of 1 to keep it simple. Then, we know from the form that

$$a=1, \quad a+d=1+1=2, \quad a+2d=1+2(1)=3, \quad a+3d=1+3(1)=4, \quad ...$$

It turns out, when we let $a=1$ and $d=1$, our sequence becomes the set of natural numbers. What if we let $a=5$ and $d=3$? Then, our sequence becomes

$$5, 8, 11, 14, 17, 20, 23, ...$$

In this case, we see that we start at 5 and then add our common difference of 3 to get to each next term.

Problem 2 (*Check Your Understanding*) Suppose I wanted to find the 13th term of an arithmetic sequence with a starting value of 12 and a common difference of 2. Can you make a shorthand equation to retrieve the answer? Hint: $2 \times 2 = 2 + 2 + 2 + 2$

Once you've figured out the equation for the problem before, we will want to find a general form: an equation to find the nth element of an arithmetic sequence with starting term a and common difference d. This way, we'll be able to plug in any numbers for our equation to get the answer quickly rather than individually calculating n terms of the sequence (which could be 1000000 if we so choose).

Proof 1 (*The nth term of an arithmetic sequence is $a + d(n-1)$*) I will show that the nth term of an arithmetic sequence with starting value a and common difference d is $a + d(n-1)$. Firstly, consider the arithmetic sequence

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad a+4d, \quad \ldots$$

The first term can be rewritten as $a + 0d$ and the second term as $a + 1d$. Rewriting our sum, we have

$$a+0d, \quad a+1d, \quad a+2d, \quad a+3d, \quad a+4d, \quad \ldots$$

and now we may see a pattern. For the first term, the coefficient of the common difference is $1-1$. For the second term, the common difference coefficient is $2-1$. For each nth term, the common difference coefficient is $n-1$ so the nth term is

$$a + d(n-1)$$

Sometimes, it's convenient to write an arithmetic sequence in "sequence form". That is, a shorthand way to write the sequence rather than writing a whole line of numbers.

Definition 2 (*Sequence Form of an Arithmetic Sequence*) The sequence form of an arithmetic sequence is given by

$$\{(a+d(n-1)\}_{n=1}^{\infty} \quad = \quad a, \quad a+d, \quad a+2d, \quad a+3d, \quad a+4d, \quad \ldots$$

The small subscript $n = 1$ is telling us the first number to plug in for n for $a + d(n-1)$. Indeed, when we do so, we get $a + d(1-1) = a$. This generates the first term. Now, we incriment n by 1 and plug in its new value, which is 2. Plugging in 2 for the second term, we get $a + d(2-1) = a + d$ which is the second term. In fact, for each value of n we plug in, we generate the nth element of the series because we already know that the nth term of an arithmetic sequence with starting value a and common difference d is $a + d(n-1)$.

The arithmetic sequence is merely a list of numbers. However, our original problems were involving sums of arithmetic sequences. In fact, the sum of an arithmetic sequence is called the arithmetic series and is defined as follows.

Definition 3 (*Finite Arithmetic Series*) A finite arithmetic series is a list of numbers in the form

$$a \quad + \quad a+d \quad + \quad a+2d \quad + \quad a+3d \quad + \quad a+4d \quad + \quad \ldots \quad + \quad a+d(n-1)$$

where a is the "staring number," d is the "common difference," and $a + d(n-1)$ denotes the final term of the sequence (the nth term)

Since we have a *finite* arithmetic series, surely we have an infinite arithmetic series...

Definition 4 (*Infinite Arithmetic Series*) An infinite arithmetic series is a list of numbers in the form

$$a \quad + \quad a+d \quad + \quad a+2d \quad + \quad a+3d \quad + \quad a+4d \quad + \quad \ldots$$

where a is the "staring number" and d is the "common difference."

Do you see the difference? With the finite arithmetic series, we have a "final term," but, there exists no such bound on an infinite arithmetic series. Regardless, here's where the interesting properties start to show. Let's start off with one of our original questions in a different format.

> **Problem 3** (*Check Your Understanding*) Let $S = 3+8+13+18+23+...+58$ be a finite arithmetic series. What is the simplified value of S?

In this case, let's start off by finding a, d, and n (for the number of terms there are). Clearly, $a = 3$ and we can find d by subtracting any two consecutive terms so $d = 8-3 = 5$. From here, we can use $58 = a+d(n-1)$ to solve for n. Simplifying, we see that $58 = 3 + 5(n - 1)$ so $n = 12$. Believe it or not, there are many different fabulous ways to solve this problem. I will show you one of my personal favorites. Remember, we control the solution, so I will start by adding two of the sums stacked on **two** of each other:

$$\begin{array}{ccccccccccccccccc}
2S & = & 3 & + & 8 & + & 13 & + & ... & + & 48 & + & 53 & + & 58 & & \\
 & + & 58 & + & 53 & + & 48 & + & ... & + & 13 & + & 8 & + & 3 & & \\
 & & & & & & & & & & & & & & & & \\
 & = & 63 & + & 63 & + & 63 & + & ... & + & 63 & + & 63 & + & 63 & & = 63(12) = 756
\end{array}$$

Now, since $2S = 756$, we can divide by 2 to get the value of the original sum, which is 378. Do you see what I did? I added twice the value of the sum, reveresed the order of the terms in the second sum so that each vertical pair added up to 63. Finally, I turned the sum into a multiplication problem and divided the result by 2 to get the original value. Try this technique to get the formula for the sum of the first n terms of a general arithmetic series:

> **Problem 4** (*Check Your Understanding*) Let $S = a + (a + d) + (a + 2d) + ... + (a + d[n - 1])$ be a finite arithmetic series. What is the closed form (simplified value) of S?
> Hint: use the same method of symmetry as used before!

Similar to before, we wish to find some sort of notation for a series. This way, we can write long sums using brief notation. Let's start by going over sigma notation. Consider the following sum: 1+2+3+4+...+97+98+99+100 (which is also the 100-term arithmetic series with $a = 1$, $d = 1$). As usual, it is implied that the three dots signify where you would write the rest of the integers of the sum. We can also rewrite this sum as

$$\sum_{n=1}^{100} n \quad \text{because} \quad \sum_{n=1}^{100} n = 1 + 2 + 3 + 4 + ... + 97 + 98 + 99 + 100$$

In this case, the $\sum$ symbol, which is called sigma, represents the operation of addition. When we say $n = 1$ at the bottom of the sigma, that implies our starting number (1). We will call n the 'index.' At the top, we define the number 100 as the upper bound, or, the place for the sum to stop. **In summation notation, the sum always starts with the index being equal to the lower bound and then for each term until the upper bound, the index increases by 1.** Let's take a look at SUM examples:

$$\sum_{n=1}^{10} 2n = 2(1) + 2(2) + 2(3) + ... + 2(8) + 2(9) + 2(10) = 110$$

$$\sum_{n=1}^{5} \frac{n^2+1}{n+1} = \frac{(1)^2+1}{(1)+1} + \frac{(2)^2+1}{(2)+1} + \frac{(3)^2+1}{(3)+1} + \frac{(4)^2+1}{(4)+1} + \frac{(5)^2+1}{(5)+1} = 13.9$$

$$\sum_{n=5}^{k} (n+1)(n-1) = ((5)+1)((5)-1) + ((6)+1)((6)-1) + ... + ((k-1)+1)((k-1)-1) + (k+1)(k-1)$$

In each example case, make sure you observe the action of the **index** since that's the changing part. If you truly understand these examples, then it is time to introduce the sigma notation of an arithmetic series using $\sum$ notation:

Definition 5 *(Sigma Notation of Arithmetic Series)* The sigma notation for a finite arithmetic sequence is given by

$$\sum_{i=1}^{n}(a+d[n-1]) = a+(a+d)+(a+2d)+...+(a+d[n-3])+(a+d[n-2])+(a+d[n-1]).$$

If the arithmetic sequence is infinite, then we have

$$\sum_{i=1}^{\infty}(a+d[n-1]) = a+(a+d)+(a+2d)+...$$

In this case, it's important not to be scared off by the variables. They are mere substitutions/replacements for what should be actual numbers.

ΔΣ Checkpoint #1

Problem #1. Given any two elements of an arithmetic sequence and their indices, find a formula to get the common difference

Problem #2. The 30th term of an arithmetic sequence is 19 and the 50th term is 44. What is the 40th term of the arithmetic sequence?

Problem #3. Represent the following three arithmetic series using sigma notation:

1.) $1+3+5+7+...+29$ 2.) $5+10+15+20+...$ 3.) $4+11+18+25+...+46$

Problem #4. Evaluate or compute the following sums

a.) $\sum_{n=1}^{10} n$ *b.)* $\sum_{n=1}^{10} 5$ *c.)* $\sum_{n=1}^{10} 5n$ *d.)* $(1-r)\sum_{k=0}^{2000} r^k$ *e.)* $\left(1-r^2\right)\sum_{k=0}^{1000} r^{2k}$

ΔΣ Theory: Summation Techniques

Since arithmetic series have many patterns, it's important to notice some algebraic tricks we can leverage to solve complex problems. Namely, observing and moving factors in these patterns leads to some nice results.

Consider the following sum:

$$\sum_{n=1}^{10} 2n = 2(1)+2(2)+2(3)+2(4)+2(5)+2(6)+2(7)+2(8)+2(9)+2(10).$$

Upon inspection, it is not hard to see that we can factor out a 2 to get $2(1+2+3+4+5+6+7+8+9+10)$. Thus,

$$\sum_{n=1}^{10} 2n = 2\sum_{n=1}^{10} n.$$

In general, we can factor out any constant value from the inside of a summation.

Result 1

$$\sum_{n=1}^{k} cn = c\sum_{n=1}^{k} n.$$

To take this to the extreme, consider the following sum

$$\sum_{i=1}^{3}\sum_{j=1}^{3} ij.$$

Expanding the inner term, we see that

$$\sum_{i=1}^{3}(1i+2i+3i)=\sum_{i=1}^{3}i(1+2+3)=(1+2+3)\sum_{i=1}^{3}i=\left(\sum_{i=1}^{3}i\right)\left(\sum_{j=1}^{3}j\right).$$

Problem 5 (*Check Your Understanding*) Show that

$$\sum_{a=1}^{n}\sum_{b=1}^{n}ab=\left(\sum_{a=1}^{n}a\right)\left(\sum_{b=1}^{n}b\right).$$

For an extra challenge, show that

$$\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n}abc=\left(\sum_{a=1}^{n}a\right)\left(\sum_{b=1}^{n}b\right)\left(\sum_{c=1}^{n}c\right).$$

$\Delta\Sigma$ Theory: Geometric Sequences and Series

A geometric sequence is similar to an arithmetic sequence. However, rather than each term differing by a value, each term is a fixed multiple of the previous. For instance, $4+12+36+108$ is a geometric sequence because $4\cdot 3=12$, $12\cdot 3=36$, and $36\cdot 3=108$. For each term, you can multiply by 3 to attain the next, or divide by 3 to attain the previous, (unless we're talking about the two end terms assuming it's finite).

Definition 6 (*Geometric Sequence*) A finite geometric sequence is a list of numbers in the form

$$a,\quad ar,\quad ar^2,\quad ar^3,\quad \ldots,\quad ar^{n-2},\quad ar^{n-1}$$

where a is the "starting number" and r is the "common ratio." Note that a geometric sequence can also be infinite, in which case we write

$$a,\quad ar,\quad ar^2,\quad ar^3,\quad \ldots$$

The question may arise, why don't we go up to ar^n for the finite geometric progression? This is because ar^{n-1} **is** the nth term of the sequence! Do you see why?

Problem 6 (*Check Your Understanding*) Show that the nth term of a geometric progression is given by ar^{n-1}.

Problem 7 (*Check Your Understanding*) What is the starting number and common ratios for the following geometric series?

$$\frac{1}{3},\quad \frac{1}{6},\quad \frac{1}{9},\quad \frac{1}{18}$$

In some cases, you may see the geometric sequence in sequence form. In this case, the geometric sequence $a+ar+ar^2+\ldots$ is given by $\{ar^{i-1}\}_{i=0}^{\infty}$ and $a+ar+ar^2+\ldots+ar^{n-2}+ar^{n-1}$ is given by $\{ar^{i-1}\}_{i=0}^{n}$.

Problem 8 (*Check Your Understanding*) Write the following in sequence notation

$$4,-2,1,-\frac{1}{2},\frac{1}{4},-\frac{1}{8},\ldots$$

A geometric series is just like an arithmetic series, except there is a common ratio instead of a common difference. Consider the sum $1+2+3+4+5$. In this case, the starting number is 1 and the common difference is 1. This is because we can add 1 to each number to get to the next.

Now consider the following sum $1+2+4+8+16$. In this case, there is no common difference because there is not one single number you can add to each element to attain the next. However, there is one number you can **multiply** to each term to get the next, which is 2. Notice that $1 \cdot 2 = 2$, $2 \cdot 2 = 4$, and $4 \cdot 2 = 8$. The common ratio is 2 because each term is 2 times the previous term.

Definition 7 (*Finite Geometric Series*) A finite geometric series is a summation of continuously scaled terms given by

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$$

Definition 8 (*Infinite Geometric Series*) An infinite geometric series is a summation of continuously scaled terms given by

$$a + ar + ar^2 + ar^3 + \ldots$$

Problem 9 (*Check Your Understanding*) What are the starting numbers and common ratios for the following geometric series?

$$5 + 25 + 125 + 625 \qquad 37 + 18.5 + 9.25 \qquad \frac{1}{4} + \frac{1}{5} + \frac{4}{25}$$

Now for another question. What is the value of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$? What about $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$? Surely there must be a more optimal way of calculation![1]

Proof 2 (*The closed form of a finite geometric progression is given by* $\frac{a(r^n-1)}{r-1}$) Let

$$S = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$$

and notice that

$$rS = ar + ar^2 + \ldots + ar^{n-1} + ar^n.$$

Therefore,

$$rS - S = ar^n - a \implies S(r-1) = a(r^n - 1) \implies S = \frac{a(r^n-1)}{r-1}$$

With just a bit of algebraic manipulation, we've found a way to calculate long sums in an easy way. But what about an infinite geometric series? Obviously, if $|r| \geq 1$, the infinite geometric series will tend to infinity, but, the case is different when the absolute value of the common ratio is less than 1.

Proof 3 (*The closed form of an infinite geometric series with* $|r| \leq 1$ *is given by* $\frac{a}{1-r}$) We know that the sum of a geometric sequence is given by $\frac{a(r^n-1)}{r-1}$. Expanding this fraction, we get

$$\frac{ar^n}{r-1} - \frac{a}{r-1}.$$

Notice that

$$\lim_{n\to\infty} \frac{ar^n}{r-1} = 0, \ \lim_{n\to\infty} \frac{a}{r-1} = \frac{a}{r-1}, \text{ so } \lim_{n\to\infty} \frac{ar^n}{r-1} - \frac{a}{r-1} = \frac{-a}{r-1} = \frac{a}{1-r}.$$

Because an infinite geometric series converges when $-1 < r < 1$, we know that the evaluation can always be given by $\frac{a}{1-r}$. Finally, to prove that we can only have a formula when $|r| < 1$, consider when $r < -1$. Clearly the series diverges because each next term alternates in sign and grows indefinitely.

[1] This is a powerful question to ask in the face of a complex math problem!

ΔΣ Checkpoint #2

Problem #1. Compute $\displaystyle\sum_{k=0}^{\infty} \frac{1+4^k}{5^k}$

Problem #2. Compute $\displaystyle\sum_{n=1}^{\infty} 4 \cdot \frac{1}{2^n}$

Problem #3 ($\star\star\star$)**.** Compute $\displaystyle\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} \frac{1}{2^{i+j}}\right)$.

ΔΣ Theory: Telescoping

One of the greatest feelings in math is when you get to cancel terms. You get to cross out unnecessary information and make your problem simpler. That's why **telescoping** is one of the most satisfying techniques to work with. The idea is to use a little bit of algebra in order to cancel a ton of terms into something simpler! For instance, let's try to compute the following:

$$\prod_{n=1}^{10} \frac{n}{n+1}.$$

First of all, what is Π? This is the "pi" symbol and it acts a lot like Σ. Here's a side by side comparison:

$$\sum_{n=1}^{3} n = 1+2+3 \qquad \text{while} \qquad \prod_{n=1}^{3} n = 1 \cdot 2 \cdot 3$$

Basically, the Π operator is just like the Σ operator, except the Π operator denotes continuous multiplication instead of continuous addition. Now, back to the problem. Let's just start out by writing out all of the terms, since the upper bound is not arbitrarily large. So, we have

$$\prod_{n=1}^{10} \frac{n}{n+1} = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\left(\frac{5}{6}\right)\left(\frac{6}{7}\right)\left(\frac{7}{8}\right)\left(\frac{8}{9}\right)\left(\frac{9}{10}\right)\left(\frac{10}{11}\right)$$

Notice that $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ because the twos cancel out. Also notice that $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$ because the twos and threes cancel out. See the pattern? For each term, the denominator of the previous is the numerator of the next, so we cancel all terms except the numerator of the first term and the denominator of the last term. So,

$$\prod_{n=1}^{10} \frac{n}{n+1} = \frac{1}{11}.$$

In fact, it's not hard to see that

$$\prod_{n=1}^{k} \frac{n}{n+1} = \frac{1}{k+1}.$$

This cancellation across terms in a series is known as **telescoping**, and what makes it so powerful is that you can manipulate terms into a telescope to find a closed form.

Result 2

$$\sum_{n=1}^{k} (f(n+1) - f(n)) = f(k+1) - f(1)$$

$$\prod_{n=1}^{k} \frac{n}{n+1} = \frac{1}{k+1}$$

ΔΣ Conclusion

That wraps it up for arithmetic and geometric sequences and series! While this document covers a lot of formulae related to these sequences and series, the goal is not to memorize the formulae. The goal is rather to truly come to understand this material intuitively so you can apply these concepts and tricks to other types of problems.

Remember, it's okay if you don't understand all of this content! Sometimes, in order to truly understand a theorem or result, it is best to plug in actual numbers for variables and watch the behavior of the result unfold.

Finally, you will understand this material 100x better if you learn about for loops in computer programming and then return, since indexing works the same way for both of these concepts. With that, enjoy the following practice problem we have designed!

ΔΣ Challenge Problems

Problem 10 (*Original*) Express the following fraction using sigma and pi notation

$$\frac{\frac{a}{b}+1+\frac{b}{a}+\ldots}{\ldots\cdot a\cdot a^2\cdot a^3}$$

Problem 11 (*Original*) Expand the following:

$$\frac{\sum_{\lambda=1}^{\gamma}\lambda^{\gamma-1}}{\prod_{\gamma=1}^{\zeta}\gamma^{\zeta+1}}$$

Problem 12 (*AIME*) Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9.

Problem 13 (*Original*) Compute the following:

$$\sum_{a=1}^{5}\sum_{b=1}^{a}\sum_{c=1}^{b}(1)+\sum_{i=1}^{100}\sum_{j=1}^{100}\sum_{k=1}^{100}(i \bmod 2)(j \bmod 2)(k \bmod 2)+\sum_{n=1}^{k}\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{k+1}$$

Problem 14 (*AIME*) The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

ΔΣ Further Reading

[1] Sequences and Series in the AMC and AIME by freeman66 + nikenissan