- **Submission of this exercise is NOT mandatory**. This exercise will not be reflected in the assessment. Also you are free to seek help from the internet and discuss it with others; even asking for help from a TA is fine.
- If you wish, you can send your answers to TA(kimjg1119@korea.ac.kr) by 23:59 on March 17, 2024, to request grading. Answers submitted after this time are not guaranteed- it depends on whether the teaching assistant is tired.
- Only PDF format is accepted. I **strongly recommend** using LaTeXor markdown with MathJax to write down your answer. If you are unfamiliar with these, the handwritten answer is accepted, but make sure your writing is readable.

빨간색은 채점을 위해 남겨주시면 감사하겠습니다 ㅠㅠ

Problem 1

A **Fibonacci sequence** is a sequence of integers in which each number is the sum of the two preceding ones. The sequence is like:

$$0, 1, 1, 2, 3, 5, 8, 13, 21 \ldots$$

Formally, the Fibonacci sequence is defined as follows.

$$F_0 = 0, \; F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \quad (\text{for } n \geq 2)$$

Problem 1.1

Prove the following theorem.

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

(basis case) $n=1$

$$F_1 = F_3 - 1$$
$$= F_1 + F_2 - 1$$
$$= 1 + F_1 + F_0 - 1$$
$$= 1$$

???

(Induction case) $\boxed{n=n}$ 일 때 식이 true 라고 가정하고

$$\sum_{i=1}^{n+1} F_i = \cancel{F_1 + F_2 \cdots + F_n} + F_{n+1}$$

Use sigma

$$= F_{n+2} - 1 + F_{n+1} \quad (\because \text{I.H.})$$
$$= F_{n+3} - 1$$

Problem 1.2

Prove the following theorem.

$$\sum_{i=1}^{n+1} F_i^2 = F_{n+1}F_{n+2}$$

(basis case) $n \to 0$

$$F_1^2 = F_1 F_2$$
$$1 = 1 \times 1 \quad (\text{True})$$

(Induction case)

$n=n$ 일때 해당 식이 true라고 가정하고

$n=n+1$ true임을 보이면됨

//

$$\sum_{i=1}^{n+2} F_i^2 = \cancel{F_1^2 + F_2^2 + \cdots + F_{n+1}^2} + F_{n+2}^2$$
$$= F_{n+1}F_{n+2} + F_{n+2}^2 \quad (\because \text{I.H.})$$
$$= F_{n+2}(F_{n+1} + F_{n+2})$$
$$= F_{n+2}F_{n+3}$$

Problem 2

A **parentheses string** p is a string(possibly empty) consisting of opening parenthesis '`(`' and closing parenthesis '`)`'. Also, each opening parenthesis should have a proper closing pair.

For example, below are the parentheses string:

- `()`
- `(())()`
- `((()))`

Otherwise, below are not:

- `(()`
- `(()))(`

Problem 2.1

Obviously, a set of parentheses strings is the **language**. Give a precise mathematical definition of this language.

$L = \{\epsilon, \text{'('}, \text{')'}\}$ Does "()" $\notin L$?

Problem 2.2

complement

Let L be the language defined in Problem 2.1. We define a new language K with

$$s_1 s_2 \dots s_n \in L \iff \overline{s_1 s_2 \dots s_n} \in K$$

when $\overline{s}$ is a conjugate of s; for example, $\overline{\texttt{()(())}} = \texttt{)())((}$.

Prove or disprove that $L = K^R$.

???

Problem 3

A **directed graph** is a pair $G = (V, E)$, where V is a set whose elements are called *vertices*, and E is a set of ordered pairs (x, y) of distinct vertices, formally,

$$E \subseteq \{(x, y) \mid (x, y) \in V \times V \wedge x \neq y\}.$$

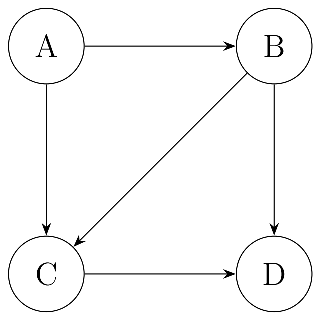

Figure 1: $V = \{A, B, C, D\}$ and $E = \{(A, B), (A, C), (B, D), (B, C), (C, D)\}$.

Hint: You may refer to the inductive definition of the directed graph.

Let (V, E) is a directed graph.

(Base case) $(\varnothing, \varnothing)$ is a directed graph.

(Inductive case 1) For $v \notin V$, $(V \cup \{v\}, E)$ is a directed graph.

(Inductive case 2) For $(x, y) \notin E$, $(V, E \cup \{(x, y)\})$ is a directed graph when $x \neq y$ and $x, y \in V$.

Problem 3.1

The possible number of different directed graphs of n vertices is $2^{n(n-1)}$. Prove it by **induction**(on integers).

Problem 3.2

Let $\text{in}(x)$ of the vertex x be the number of edges pointing to it, and $\text{out}(x)$ is the number of edges starting from it. For example, $\text{in}(B) = 1$ and $\text{out}(B) = 2$. Prove or disprove the theorem below using a **structural induction** or giving a **counterproof**.

$$\forall G = (V, E).\ \sum_{v \in V} \text{in}(v) = \sum_{v \in V} \text{out}(v)$$