

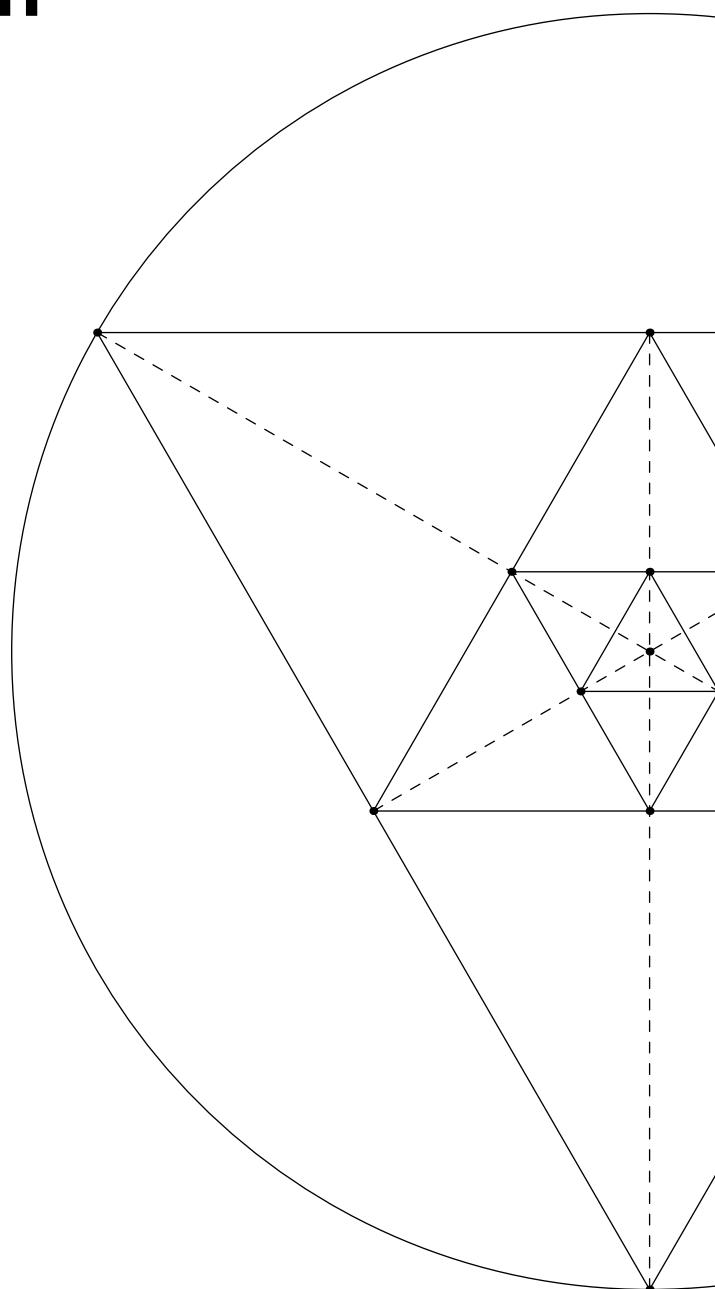
# A Geometric Approach To Matrices

## Answer Key

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## **Contents**

<b>1 Trigonometry Review</b>	<b>2</b>
<b>2 It's a Snap</b>	<b>9</b>
<b>3 From Snaps to Flips</b>	<b>14</b>
<b>4 Rotation and Reflection Groups</b>	<b>19</b>
<b>5 Infinite Groups</b>	<b>33</b>
<b>6 Geometry of Complex Numbers</b>	<b>43</b>
<b>7 Your Daily Dose of Vitamin <i>i</i></b>	<b>66</b>

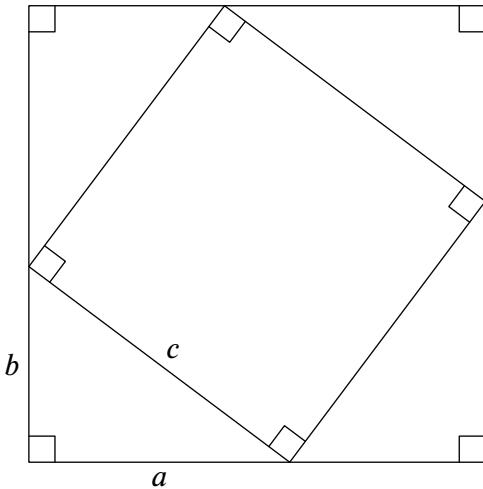


Figure 1: Scenario in Problem 1.

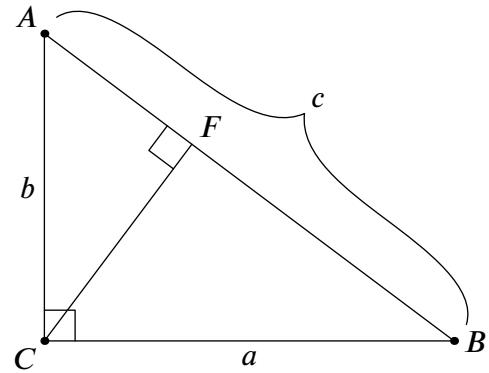


Figure 2: Scenario in Problem 2.

## 1 Trigonometry Review

1. Prove the Pythagorean theorem using “conservation of area.” Start with Figure 1.

In Figure 1, the larger square has side length  $a + b$ . The smaller, nested square has side length  $c$ . Four copies of the right triangle with side lengths  $a, b, c$  are placed around the square. We have

$$\begin{aligned}
 A_{\text{triangles}} + A_{\text{small sq.}} &= A_{\text{big sq.}} && [\text{Conservation of area}] \\
 4A_{\text{triangle}} + A_{\text{small sq.}} &= A_{\text{big sq.}} \\
 4\left(\frac{1}{2}ab\right) + c^2 &= (a+b)^2 && [\text{Areas of triangle, square}] \\
 2ab + c^2 &= a^2 + 2ab + b^2 && [\text{Expanding}] \\
 c^2 &= a^2 + b^2. && \text{Q.E.D.}
 \end{aligned}$$

2. Prove the Pythagorean theorem using a right triangle with an altitude drawn to its hypotenuse, making use of similar right triangles. This is shown in Figure 2.

Let  $h = CF$ , the length of the altitude to the hypotenuse.  $\triangle ACF \sim \triangle ABC$  by AA Similarity because they share an angle and both have a right angle. Therefore,  $\frac{AF}{AC} = \frac{AC}{AB}$ . Substituting named variables for these lengths, we get

$$\frac{AF}{b} = \frac{b}{c} \implies AF = \frac{b^2}{c}.$$

Applying the same logic to  $\triangle CFB$ , we get  $\triangle CFB \sim \triangle ABC$ , so  $\frac{BF}{BC} = \frac{BC}{AB}$ . Substituting, we get

$$\frac{BF}{a} = \frac{a}{c} \implies BF = \frac{a^2}{c}.$$

Since  $F$  is between  $A$  and  $B$ , we have  $AB = AF + FB$ ; substituting our found values for  $AF$  and  $FB$ , we get

$$\begin{aligned}
 c &= AB = AF + FB \\
 c &= \frac{b^2}{c} + \frac{a^2}{c} \\
 c^2 &= b^2 + a^2. && \text{Q.E.D.}
 \end{aligned}$$

3. Now you will prove the trig identities.

- (a) Draw and label a right triangle and a unit circle, then write trig definitions for cos, sin, tan, and sec in terms of your drawing.

The scenario is depicted in Figure 3. By the definition of sine and cosine, we have  $\sin \theta = AP$  and  $\cos \theta = OA$ . Since  $\triangle OAP \sim \triangle OPT$  by AA Similarity, we have  $\frac{TP}{OP} = \frac{AP}{OA}$ . Substituting known values, we get

$$\frac{TP}{1} = \frac{\sin \theta}{\cos \theta} \implies TP = \tan \theta.$$

Also,  $\triangle OAP \sim \triangle OKS$  by AA, so  $\frac{OS}{OK} = \frac{1}{\cos \theta}$ . Similarly, we have

$$\frac{OS}{1} = \frac{1}{\cos \theta} \implies OS = \sec \theta.$$

Finally, as an alternate interpretation of tan, we have  $\frac{KS}{OK} = \frac{AP}{OA}$ , so

$$\frac{KS}{1} = \frac{\sin \theta}{\cos \theta} \implies KS = \tan \theta.$$

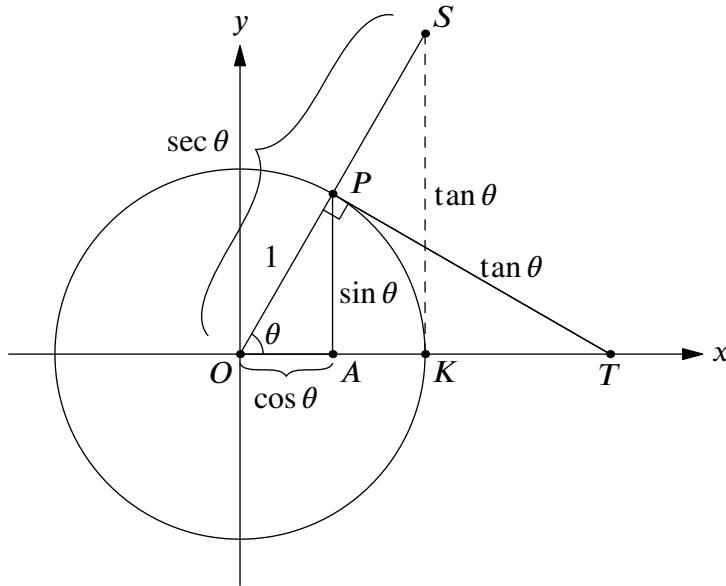


Figure 3: The right triangle and unit circle.

- (b) Use a right triangle and the definitions of sin and cos to find and prove a value for  $\sin^2 \theta + \cos^2 \theta$ .

Referring back to Figure 3, focus on  $\triangle OAP$ . It is a right triangle with side lengths  $a = \cos \theta$ ,  $b = \sin \theta$ , and  $c = 1$ . By the Pythagorean theorem, we have

$$\begin{aligned} OA^2 + AP^2 &= OP^2 && [\text{Pythagorean theorem}] \\ \cos^2 \theta + \sin^2 \theta &= 1^2 && [\text{Substitution}] \\ \sin^2 \theta + \cos^2 \theta &= 1 && [\text{Rearrange}] \end{aligned}$$

- (c) Use the picture of the unit circle in Figure 4 to find and prove a value for  $\cos(A - B)$ . Note that  $D_1$  and  $D_2$  are the same length because they subtend the same size arc of the circle. Set them equal and work through the algebra, using the distance formula and part (b) of this problem.

We have  $D_1 = D_2$ , so

$$\begin{aligned} D_1^2 &= D_2^2 \\ (\cos A - \cos B)^2 + (\sin A - \sin B)^2 &= (\cos(A - B) - 1)^2 + \sin^2(A - B) \\ \cos^2 A - 2 \cos A \cos B + \cos^2 B + \sin^2 A - 2 \sin A \sin B + \sin^2 B &= \cos^2(A - B) - 2 \cos(A - B) + \\ &\quad 1 + \sin^2(A - B) \end{aligned}$$

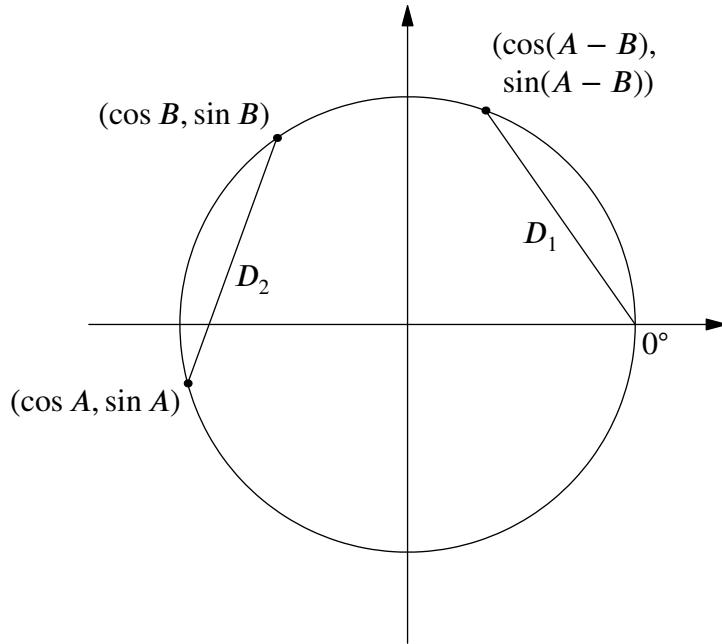


Figure 4: Scenario in Problem 3.

$$\begin{aligned}
 (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2 \sin A \sin B &= (\cos^2(A - B) + \sin^2(A - B)) + \\
 &\quad 1 - 2 \cos(A - B) \\
 1 + 1 - 2 \sin A \sin B - 2 \cos A \cos B &= 1 + 1 - 2 \cos(A - B) \\
 2 \sin A \sin B + 2 \cos A \cos B &= 2 \cos(A - B) \\
 \sin A \sin B + \cos A \cos B &= \cos(A - B). \tag*{Q.E.D.}
 \end{aligned}$$

4. Write down as many trig identities as you can. There's no need to prove all of these right now.

$\sin(A + B) =$	$\sin(A - B) =$	$\cos(A + B) =$
$\tan(A + B) =$	$\tan(A - B) =$	$\sin(2A) =$
$\cos(2A) =$	$\tan(2A) =$	$\sin\left(\frac{A}{2}\right) =$
$\cos\left(\frac{A}{2}\right) =$	$\tan\left(\frac{A}{2}\right) =$	

You should probably memorize these for convenience.

$$\begin{aligned}
 \sin(A + B) &= \sin A \cos B + \cos A \sin B \\
 \sin(A - B) &= \sin A \cos B - \cos A \sin B \\
 \cos(A + B) &= \cos A \cos B - \sin A \sin B \\
 \tan(A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
 \tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\
 \sin(2A) &= 2 \sin A \cos A \\
 \cos(2A) &= 2 \cos^2 A - 1 = 1 - 2 \sin^2 A = \cos^2 A - \sin^2 A \\
 \tan(2A) &= \frac{2 \tan A}{1 - \tan^2 A} \\
 \sin\left(\frac{A}{2}\right) &= \pm \sqrt{\frac{1 - \cos A}{2}}
 \end{aligned}$$

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos A}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A}$$

5. Let's review complex numbers and DeMoivre's theorem.

(a) Recall that you can write a complex number both in Cartesian and polar forms. Let

$$a + bi = (a, b) = (r \cos \theta, r \sin \theta) = r \cos \theta + i r \sin \theta.$$

What is  $r$  in terms of  $a$  and  $b$ ?

$r$  is just the distance to the origin from  $a+bi$ . Draw a right triangle as shown in Figure 5. By the pythagorean theorem,  $r = \sqrt{a^2 + b^2}$ .

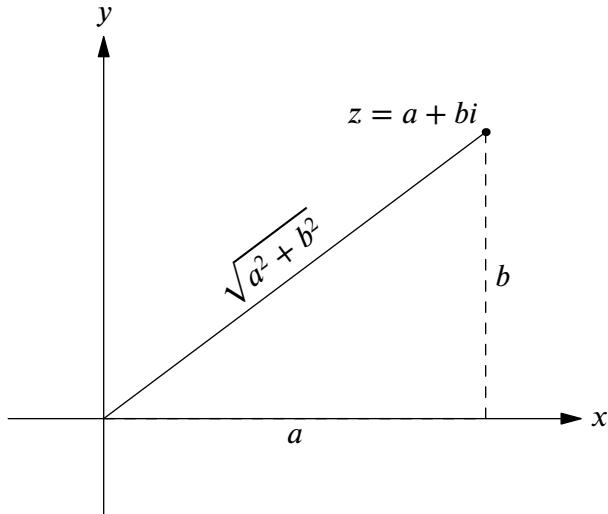


Figure 5:  $a + bi$  in the complex plane.

(b) Multiply  $(a + bi)(c + di)$  out using FOIL.

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + (bi)(di) \\ &= ac + (ad + bc)i - bd \\ &= ac - bd + (ad + bc)i.\end{aligned}$$

(c) Convert the two multiplicands<sup>1</sup> to polar form, noting that the two lengths and angles are different numbers. Call them  $r_1(\cos \theta + i \sin \theta)$  and  $r_2(\cos \phi + i \sin \phi)$ .

We have  $r_1 = \sqrt{a^2 + b^2}$  and  $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ <sup>2</sup>; similarly,  $r_2 = \sqrt{c^2 + d^2}$  and  $\phi = \tan^{-1}\left(\frac{d}{c}\right)$ .

(d) Multiply them, and use your results from Problems 3c and 3d to show that multiplying two complex numbers involves multiplying their lengths and adding their angles. This is DeMoivre's theorem!

$$\begin{aligned}r_1(\cos \theta + i \sin \theta)r_2(\cos \phi + i \sin \phi) &= r_1r_2(\cos \theta \cos \phi - \sin \theta \sin \phi + i(\sin \theta \cos \phi + \cos \theta \sin \phi)) \\ &= r_1r_2(\cos(\theta + \phi) + i \sin(\theta + \phi)).\end{aligned}$$

<sup>1</sup>This is the word for parts of a multiplication! So for example, if  $a \cdot b = c$ , then  $a$  and  $b$  are the multiplicands.

<sup>2</sup>You can get this from drawing a right triangle.

(e) Use part (d) to simplify  $(\sqrt{3} + i)^{18}$ .

We have  $\sqrt{3} + i = r(\cos \theta + i \sin \theta) = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$ .

$$\begin{aligned}
 (2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right))^{18} &= 2^{18} \cdot \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)^{18} \\
 &= 2^{18} \cdot \underbrace{\left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \cdots \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)}_{18 \text{ copies}} \\
 &= 2^{18} \cdot \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \underbrace{\left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \cdots \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)}_{16 \text{ copies}} \\
 &=: \\
 &= 2^{18} \cdot (\cos 3\pi + i \sin 3\pi) \\
 &= 2^{18} \cdot -1 \\
 &= -2^{18}.
 \end{aligned}$$

6. Here is a review of 2D rotation.

- (a) Remember that we can graph complex numbers as 2D ordered pairs in the complex plane. Now, consider the complex number  $z = \cos \theta + i \sin \theta$ , where  $\theta$  is fixed. What is the magnitude of  $z$ ?

We have

$$|z| = \sqrt{\cos^2 \theta + \sin^2 \theta} = \sqrt{1} = 1.$$

- (b) Multiplying  $z \cdot (x + yi)$  yields a rotation of the point  $(x, y)$  counterclockwise by the angle  $\theta$  around the origin. What if we wanted to rotate clockwise by  $\theta$  instead?

We can multiply by the conjugate of  $z$ , since

$$\bar{z} = \cos \theta - i \sin \theta = \cos -\theta + i \sin -\theta.$$

Thus, the operation is  $\bar{z} \cdot (x + yi)$  to rotate clockwise by  $\theta$ .

7. Rotate the following conics by (i)  $30^\circ$ , (ii)  $45^\circ$ , and (iii)  $\theta$ :

- (a)  $x^2 - y^2 = 1$

i.  $30^\circ$

We make the substitution  $x' = x \cos 30^\circ - y \sin 30^\circ = \frac{\sqrt{3}}{2}x - \frac{y}{2}$  and  $y' = x \sin 30^\circ + y \cos 30^\circ = \frac{x}{2} + \frac{\sqrt{3}}{2}y$ :

$$\begin{aligned}
 x'^2 - y'^2 &= 1 \\
 \left( \frac{\sqrt{3}}{2}x - \frac{y}{2} \right)^2 - \left( \frac{x}{2} + \frac{\sqrt{3}}{2}y \right)^2 &= 1 \\
 x^2/2 - \sqrt{3}xy - y^2/2 &= 1
 \end{aligned}$$

ii.  $45^\circ$

We make the substitution  $x' = x \cos 45^\circ - y \sin 45^\circ = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$  and  $y' = x \sin 45^\circ + y \cos 45^\circ = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$ :

$$\begin{aligned}
 x'^2 - y'^2 &= 1 \\
 \left( \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \right)^2 - \left( \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \right)^2 &= 1 \\
 -2xy &= 1.
 \end{aligned}$$

iii.  $\theta$

We make the substitution  $x' = x \cos \theta - y \sin \theta$  and  $y' = x \sin \theta + y \cos \theta$ :

$$\begin{aligned}x'^2 - y'^2 &= 1 \\(x \cos \theta - y \sin \theta)^2 - (x \sin \theta + y \cos \theta)^2 &= 1.\end{aligned}$$

(b)  $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

i.  $30^\circ$

We make the substitution  $x' = x \cos 30^\circ - y \sin 30^\circ = \frac{\sqrt{3}}{2}x - \frac{y}{2}$  and  $y' = x \sin 30^\circ + y \cos 30^\circ = \frac{x}{2} + \frac{\sqrt{3}}{2}y$ :

$$\begin{aligned}\frac{x'^2}{16} - \frac{y'^2}{9} &= 1 \\ \frac{\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)^2}{16} - \frac{\left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2}{9} &= 1 \\ \frac{1}{576}(11x^2 - 50\sqrt{3}xy - 39y^2) &= 1.\end{aligned}$$

ii.  $45^\circ$

We make the substitution  $x' = x \cos 45^\circ - y \sin 45^\circ = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$  and  $y' = x \sin 45^\circ + y \cos 45^\circ = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$ :

$$\begin{aligned}\frac{x'^2}{16} - \frac{y'^2}{9} &= 1 \\ \frac{\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)^2}{16} - \frac{\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{9} &= 1 \\ \frac{1}{288}(-x - 7y)(7x + y) &= 1\end{aligned}$$

iii.  $\theta$

We make the substitution  $x' = x \cos \theta - y \sin \theta$  and  $y' = x \sin \theta + y \cos \theta$ :

$$\begin{aligned}\frac{x'^2}{16} - \frac{y'^2}{9} &= 1 \\ \frac{(x \cos \theta - y \sin \theta)^2}{16} - \frac{(x \sin \theta + y \cos \theta)^2}{9} &= 1.\end{aligned}$$

(c)  $y^2 = 4Cx$

i.  $30^\circ$

We make the substitution  $x' = x \cos 30^\circ - y \sin 30^\circ = \frac{\sqrt{3}}{2}x - \frac{y}{2}$  and  $y' = x \sin 30^\circ + y \cos 30^\circ = \frac{x}{2} + \frac{\sqrt{3}}{2}y$ :

$$\begin{aligned}y'^2 &= 4Cx' \\ \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 &= 4C\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right).\end{aligned}$$

ii.  $45^\circ$

We make the substitution  $x' = x \cos 45^\circ - y \sin 45^\circ = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$  and  $y' = x \sin 45^\circ + y \cos 45^\circ = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$ :

$$\begin{aligned} y'^2 &= 4Cx' \\ \left( \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \right)^2 &= 4C \left( \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \right) \\ \frac{1}{2}(x+y)^2 &= 2C\sqrt{2}(x-y) \end{aligned}$$

iii.  $\theta$

We make the substitution  $x' = x \cos \theta - y \sin \theta$  and  $y' = x \sin \theta + y \cos \theta$ :

$$\begin{aligned} y'^2 &= 4Cx' \\ (x \cos \theta - y \sin \theta)^2 &= 4C(x \sin \theta + y \cos \theta). \end{aligned}$$

•	$I$	$A$	$B$	$C$	$D$	$E$
$I$						
$A$			$E$			
$B$						
$C$						
$D$						
$E$						

Figure 6: Unfilled 3-post snap group table.

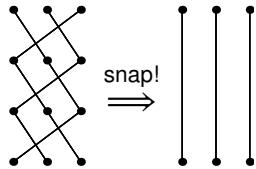


Figure 7:  $E \bullet E \bullet E = I$ ;  $E$  has period 3.

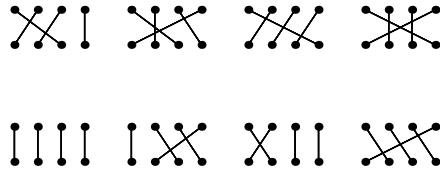


Figure 8: Some 4-post group elements.

## 2 It's a Snap

1. Fill out a  $6 \times 6$  table like the one in Figure 18, showing the results of each of the 36 possible snaps, where  $X \bullet Y$  is in  $X$ 's row and  $Y$ 's column.  $A \bullet B = E$  is done for you.

•	$I$	$A$	$B$	$C$	$D$	$E$
$I$	$I$	$A$	$B$	$C$	$D$	$E$
$A$	$A$	$I$	$E$	$D$	$C$	$B$
$B$	$B$	$D$	$I$	$E$	$A$	$C$
$C$	$C$	$E$	$D$	$I$	$B$	$A$
$D$	$D$	$B$	$C$	$A$	$E$	$I$
$E$	$E$	$C$	$A$	$B$	$I$	$D$

2. Would this table look different if you wrote the elements  $A$  through  $E$  in a different order?

Yes; here's an example:

•	$I$	$E$	$A$	$D$	$B$	$C$
$I$	$I$	$E$	$A$	$D$	$B$	$C$
$E$	$E$	$D$	$C$	$I$	$A$	$B$
$A$	$A$	$B$	$I$	$C$	$E$	$D$
$D$	$D$	$I$	$B$	$E$	$C$	$A$
$B$	$B$	$C$	$D$	$A$	$I$	$E$
$C$	$C$	$A$	$E$	$B$	$D$	$I$

3. Which of the elements is the **identity element**  $K$ , such that  $X \bullet K = K \bullet X = X$  for all  $X$ ?

The identity element is  $I$ , since  $I \bullet A = A \bullet I = A$ ,  $I \bullet B = B \bullet I = B$ , and so forth.

4. Does every element have an inverse; can you get to the identity element from every element using only one snap?

Yes you can. The inverses are shown below.

$$\begin{aligned} I &\leftrightarrow I \\ A &\leftrightarrow A \\ B &\leftrightarrow B \\ C &\leftrightarrow C \\ D &\leftrightarrow E \end{aligned}$$

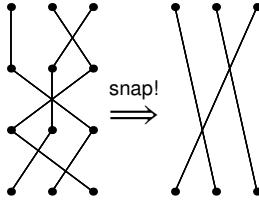


Figure 9: A  $4 \times 3$  grid of posts has a unique result after the snap operation.

Note that the inverse of an element  $X$  is denoted  $X^{-1}$ .

5.

- (a) Is the snap operation commutative (does  $X \bullet Y = Y \bullet X$  for all  $X, Y$ )?

No, the snap operation is not commutative. For example,  $A \bullet B = E$ , but  $B \bullet A = D$ .

- (b) Is the snap operation associative (does  $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$  for all  $X, Y, Z$ )?

Yes, the snap operation is associative. You can rationalize this as the fact that a  $4 \times 3$  grid of posts is snapped to a single configuration, regardless of which middle row you remove first. This is shown in Figure 9.

6.

- (a) For any elements  $X, Y$ , is there always an element  $Z$  so that  $X \bullet Z = Y$ ?

Yes, there is always a way to get from one element to another in one snap. You can prove this by construction. If element  $X$  connects  $n_1$  to  $n'_1$ ,  $n_2$  to  $n'_2$ , and  $n_3$  to  $n'_3$ , and element  $Y$  connects  $m_1$  to  $m'_1$ ,  $m_2$  to  $m'_2$ , and  $m_3$  to  $m'_3$ , then the solution  $Z$  to  $X \bullet Z = Y$  connects  $m_1$  to  $n_{m'_1}$ ,  $m_2$  to  $n_{m'_2}$ , and  $m_3$  to  $n_{m'_3}$ .

That's probably a bit hard to understand, but a more clever solution uses inverses. We multiply  $X$  by  $X^{-1}$ , then by  $Y$ :

$$X \bullet X^{-1} \bullet Y = Y.$$

But since every element has an inverse, and the snap operation is associative, we have

$$\begin{aligned} X \bullet (X^{-1} \bullet Y) &= Y \\ \implies Z &= X^{-1} \bullet Y. \end{aligned}$$

In this way, we have constructed the element  $Z$ .

- (b) For (a), is  $Z$  always unique?

Yes. To show this, we use a proof by contradiction. Suppose we have two solutions  $Z_1$  and  $Z_2$  so that  $Z_1 \neq Z_2$  and

$$\begin{aligned} X \bullet Z_1 &= Y \\ X \bullet Z_2 &= Y. \end{aligned}$$

We multiply to the left by  $Y^{-1}$ . Note that since the snap operation is not commutative, we need to multiply both sides on a specific side:

$$\begin{aligned} Y^{-1} \bullet X \bullet Z_1 &= Y^{-1} \bullet Y = I \\ Y^{-1} \bullet X \bullet Z_2 &= I \end{aligned}$$

So  $Z_1, Z_2$  are the inverses of  $Y^{-1} \bullet X$ . But the inverse of an element is unique; we've showed this by listing them all out! Thus,  $Z_1 = Z_2$ , contradicting our assumption and proving that  $Z$  is unique in  $X \bullet Z = Y$ .

7. If you constructed a  $5 \times 5$  table using only 5 of the snap elements, the table would not describe a group, because there would be entries in the table not in those 5. Therefore, a group must be **closed** under its operation; if  $X, Y \in G$  ( $\in$  means “is/are in”), then  $X \cdot Y \in G$  for all  $X, Y$ . Some subsets, however, do happen to be closed.

Write valid group tables using exactly 1, 2, and 3 elements from the snap group. These are known as **subgroups**.

Here are tables with 1, 2, and 3 elements:

$\bullet$	$I$	
	$I$	$A$
	$A$	$I$

$\bullet$	$I$	$A$	
$I$	$I$	$A$	
$A$	$A$	$I$	

$\bullet$	$I$	$D$	$E$
$I$	$I$	$D$	$E$
$D$	$D$	$E$	$I$
$E$	$E$	$I$	$D$

8. What do you guess is the complete definition of a mathematical group? (Hint: consider your answers to Problems 3–7.)

(Answers may vary.)

Definition of **group**: A group  $G$  is a set of elements together with a **binary operation** that meets the following criteria:

- (a) Identity: There is an element  $I \in G$  such that for all  $X \in G$ ,  $X \cdot I = I \cdot X = X$ .
- (b) Closure: If  $X, Y$  are elements of the group, then  $X \cdot Y$  is also an element of the group.
- (c) Invertibility: Each element  $X$  has an inverse  $X^{-1}$  such that  $X \cdot X^{-1} = X^{-1} \cdot X = I$ .
- (d) Associativity: For all elements  $X, Y$ , and  $Z$ ,  $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$ .

9. Notice that  $E \cdot E \cdot E = I$ . (See Figure 7.) This means that  $E$  has a **period** of 3 when acting upon itself. Which elements have a period of

- (a) 1?

$I$  is the only element with a period of 1, since  $I = I$ .

- (b) 2?

$A, B$ , and  $C$  have periods of 2, since for each  $X \in A, B, C$  we have  $X \cdot X = I$ .

- (c) 3?

$D$  and  $E$  have periods of 3, since for each  $Y \in D, E$  we have  $Y \cdot Y \neq I$ , but  $Y \cdot Y \cdot Y = I$ .

10. Answer the following with the 1, 2, and 4-post snap groups  $S_1$ ,  $S_2$  and  $S_4$ .

- (a) How many elements would there be?

$S_1$  has  $1! = 1$  elements.  $S_2$  has  $2! = 2$  elements.  $S_4$  has  $4! = 24$  elements.

- (b) Systematically draw and name them.

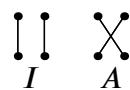


Figure 10: Elements of  $S_1$ .

Figure 11: Elements of  $S_2$ .

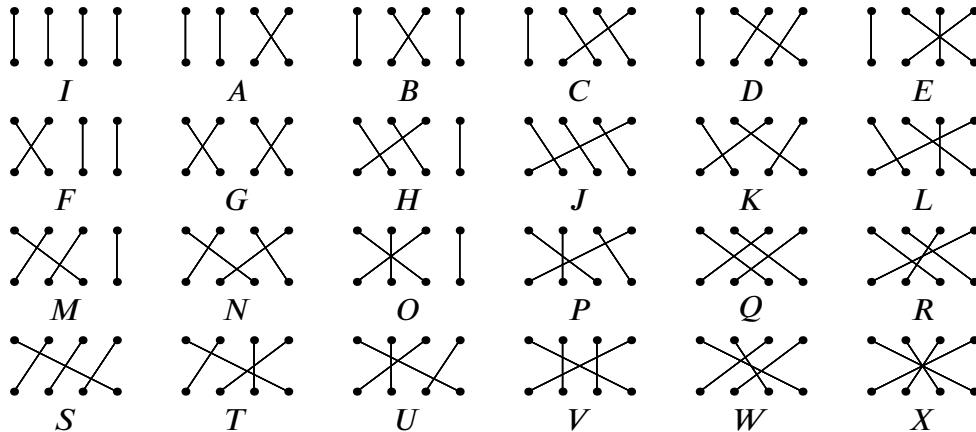


Figure 12: Elements of  $S_4$ .

- (c) Make a group table of these elements. For 4 posts, instead of creating the massive table, give the number of entries that table would have.

Here are group tables for  $S_1$  and  $S_2$ .

•	$I$
$I$	$I$

Figure 13: Group table for  $S_1$ .

•	$I$	$A$
$I$	$I$	$A$
$A$	$A$	$I$

Figure 14: Group table for  $S_2$ .

The table for  $S_4$  is given at the end of the section in Figure 17 for the curious.

- (d) What is the relationship between this new table and your original table?

Both  $S_1$ 's and  $S_2$ 's tables are subgroups of the original table for  $S_3$ . In turn,  $S_3$  is a subgroup of  $S_4$ .

11. Can you think of an easier way to generate a snap group table without drawing all the possible configurations?

(Answers may vary.)

One way to do it is to treat each element as a list of indices. For example,  $I$  is the ordered triple  $(1, 2, 3)$  because it takes column 1 to 1, 2 to 2, and 3 to 3.  $A$  is  $(1, 3, 2)$ , because it takes 1 to 1, 2 to 3, and 3 to 2.

This makes it a bit easier to calculate, because you can simply substitute indices for each configuration rather than make a drawing. It also makes it easy to write a program to calculate; this is actually how all the tables in this answer key were generated.

- 12.

- (a) How many elements would there be in the 5-post snap group?

There would be  $5! = 120$  elements in  $S_5$ .

- (b) How many entries would its table have?

There would be  $5!^2 = 14400$  entries in  $S_5$ 's table.

- (c) What possible periods would its elements have?

This is a more difficult question. We must ask what characteristics of an element determine its period.

If we observe the periodicity of an element with a pretty large period, say one from  $S_5$  with a period of 6, you can see how a large period can arise. This is shown in Figure 16.

We can split up this element into two components: a component with period 3 and one with period 2. Let's call these components  $C_3$  and  $C_2$ . After 2 steps, the  $C_3$  has not completed one period, even though  $C_2$ . After 3 steps,  $C_3$  has completed one period, but  $C_2$  has gone through  $\frac{3}{2}$ . It takes  $\text{lcm}(2, 3) = 6$  steps before both components "line up!"

All elements can be split up into some number of these cyclic components, even if it doesn't look like it at first glance. For example, the element from  $S_8$  shown in Figure 15 is actually two size 3 and size 2 components. It therefore has a period of  $\text{lcm}(2, 3, 3) = 6$ . Note that it does *not* have a period of  $2 \cdot 3 \cdot 3 = 18$ !

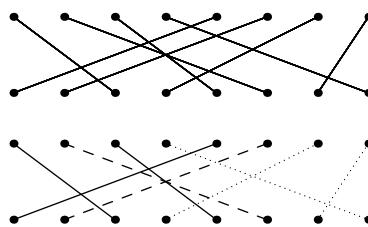


Figure 15: This element from  $S_8$  has components of size 2, 3, 3.

For  $S_5$ , we can split it up into components of size 1, 1, 1, 1, 1, giving period 1; components of size 1, 1, 1, 2, giving period 2; components of size 1, 1, 3, giving period 3; components of size 1, 4, giving period 4; a component of size 5, giving period 5; and component of size 1, 2, 3, giving period 6. Thus, periods 1, 2, 3, 4, 5, 6 are achievable.

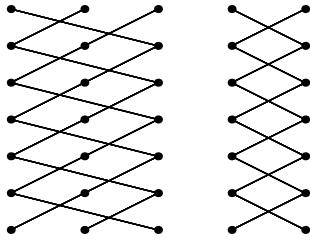


Figure 16: This element from  $S_5$  has a period of 6.

- (d) Extend your answers for a–c to  $M$  posts per row.

This is rather straightforward. There are (a)  $M!$  elements in the  $M$ -post snap group, and thus (b)  $M!^2$  elements in the corresponding group table. The possible periods are harder to calculate, but they can be generated like so:

Let integers  $x_i > 0$  and  $\sum_i x_i = M$ . In other words, the sum of all  $x_i$  is  $M$ . Then  $\text{lcm}(x_1, x_2, \dots, x_n)$  is a valid period; the least common multiple of all  $x_i$  is a possible period.

For fun: in set builder notation, we have the set of possible periods  $P_n$  for the  $n$ -post snap group as

$$P_n = \left\{ \text{lcm}(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{Z}^+ \wedge \sum_i x_i = n \right\}.$$

The maximum such period (i.e.  $\max P_n$ ) is actually known as Landau's function,  $g(n)$ .

13. As we learned, a **permutation** of some things is an order in which they can be arranged. What is the relationship between the set of permutations of  $m$  things and the  $m$ -post snap group?

We can make a pretty simple correspondence between a permutation of  $m$  things and an element of the  $m$ -post snap group. If we think back to the idea of treating each element of the group as a list of indices, the correspondence is obvious. For example,  $I$  is the ordered triple  $(1, 2, 3)$  because it takes column 1 to 1, 2 to 2, and 3 to 3.  $A$  is  $(1, 3, 2)$ , because it takes 1 to 1, 2 to 3, and 3 to 2. But each ordered triple is a permutation of  $1, 2, 3!$  This extends to any  $m$ .

.	$I$	$A$	$B$	$C$	$D$	$E$	$F$	$G$	$H$	$J$	$K$	$L$	$M$	$N$	$O$	$P$	$Q$	$R$	$S$	$T$	$U$	$V$	$W$	$X$	
$I$	$I$	$A$	$B$	$C$	$D$	$E$	$F$	$G$	$H$	$J$	$K$	$L$	$M$	$N$	$O$	$P$	$Q$	$R$	$S$	$T$	$U$	$V$	$W$	$X$	
$A$	$A$	$I$	$D$	$E$	$B$	$C$	$G$	$F$	$K$	$L$	$H$	$J$	$S$	$T$	$U$	$V$	$W$	$X$	$M$	$N$	$O$	$P$	$Q$	$R$	
$B$	$B$	$C$	$I$	$A$	$E$	$D$	$M$	$N$	$O$	$P$	$Q$	$R$	$F$	$G$	$H$	$J$	$K$	$L$	$T$	$S$	$W$	$X$	$U$	$V$	
$C$	$C$	$B$	$E$	$D$	$I$	$A$	$N$	$M$	$Q$	$R$	$O$	$P$	$T$	$S$	$W$	$X$	$U$	$V$	$F$	$G$	$H$	$J$	$K$	$L$	
$D$	$D$	$E$	$A$	$I$	$C$	$B$	$S$	$T$	$U$	$V$	$W$	$X$	$G$	$F$	$K$	$L$	$H$	$J$	$N$	$M$	$Q$	$R$	$O$	$P$	
$E$	$E$	$D$	$C$	$B$	$A$	$I$	$T$	$S$	$W$	$X$	$U$	$V$	$N$	$M$	$Q$	$R$	$O$	$P$	$G$	$F$	$K$	$L$	$H$	$J$	
$F$	$F$	$G$	$H$	$J$	$K$	$L$	$I$	$A$	$B$	$C$	$D$	$E$	$O$	$P$	$M$	$N$	$R$	$Q$	$U$	$V$	$S$	$T$	$X$	$W$	
$G$	$G$	$F$	$K$	$L$	$H$	$J$	$A$	$I$	$D$	$E$	$B$	$C$	$U$	$V$	$S$	$T$	$X$	$W$	$O$	$P$	$M$	$N$	$R$	$Q$	
$H$	$H$	$J$	$F$	$G$	$L$	$K$	$O$	$P$	$M$	$N$	$R$	$Q$	$I$	$A$	$B$	$C$	$D$	$E$	$V$	$U$	$X$	$W$	$S$	$T$	
$J$	$J$	$H$	$L$	$K$	$F$	$G$	$P$	$O$	$R$	$Q$	$M$	$N$	$V$	$U$	$X$	$W$	$S$	$T$	$I$	$A$	$B$	$C$	$D$	$E$	
$K$	$K$	$L$	$G$	$F$	$J$	$H$	$U$	$V$	$S$	$T$	$X$	$W$	$A$	$I$	$D$	$E$	$B$	$C$	$P$	$O$	$R$	$Q$	$M$	$N$	
$L$	$L$	$K$	$J$	$H$	$G$	$F$	$V$	$U$	$X$	$W$	$S$	$T$	$P$	$O$	$R$	$Q$	$M$	$N$	$A$	$I$	$D$	$E$	$B$	$C$	
$M$	$M$	$N$	$O$	$P$	$Q$	$R$	$B$	$C$	$I$	$A$	$E$	$D$	$H$	$J$	$F$	$G$	$L$	$K$	$W$	$X$	$T$	$S$	$V$	$U$	
$N$	$N$	$N$	$M$	$Q$	$R$	$O$	$P$	$C$	$B$	$E$	$D$	$I$	$A$	$W$	$X$	$T$	$S$	$V$	$U$	$H$	$J$	$F$	$G$	$L$	$K$
$O$	$O$	$P$	$M$	$N$	$R$	$Q$	$H$	$J$	$F$	$G$	$L$	$K$	$B$	$C$	$I$	$A$	$E$	$D$	$X$	$W$	$V$	$U$	$T$	$S$	
$P$	$P$	$O$	$R$	$Q$	$M$	$N$	$J$	$H$	$L$	$K$	$F$	$G$	$X$	$W$	$V$	$U$	$T$	$S$	$B$	$C$	$I$	$A$	$E$	$D$	
$Q$	$Q$	$R$	$N$	$M$	$P$	$O$	$W$	$X$	$T$	$S$	$V$	$U$	$C$	$B$	$E$	$D$	$I$	$A$	$J$	$H$	$L$	$K$	$F$	$G$	
$R$	$R$	$Q$	$P$	$O$	$N$	$M$	$X$	$W$	$V$	$U$	$T$	$S$	$J$	$H$	$L$	$K$	$F$	$G$	$C$	$B$	$E$	$D$	$I$	$A$	
$S$	$S$	$T$	$U$	$V$	$W$	$X$	$D$	$E$	$A$	$I$	$C$	$B$	$K$	$L$	$G$	$F$	$J$	$H$	$Q$	$R$	$N$	$M$	$P$	$O$	
$T$	$T$	$S$	$W$	$X$	$U$	$V$	$E$	$D$	$C$	$B$	$A$	$I$	$Q$	$R$	$N$	$M$	$P$	$O$	$K$	$L$	$G$	$F$	$J$	$H$	
$U$	$U$	$V$	$S$	$T$	$X$	$W$	$K$	$L$	$G$	$F$	$J$	$H$	$D$	$E$	$A$	$I$	$C$	$B$	$R$	$Q$	$P$	$O$	$N$	$M$	
$V$	$V$	$U$	$X$	$W$	$S$	$T$	$L$	$K$	$J$	$H$	$G$	$F$	$R$	$Q$	$P$	$O$	$N$	$M$	$D$	$E$	$A$	$I$	$C$	$B$	
$W$	$W$	$X$	$T$	$S$	$V$	$U$	$Q$	$R$	$N$	$M$	$P$	$O$	$E$	$D$	$C$	$B$	$A$	$I$	$L$	$K$	$J$	$H$	$G$	$F$	
$X$	$X$	$W$	$V$	$U$	$T$	$S$	$R$	$Q$	$P$	$O$	$N$	$M$	$L$	$K$	$J$	$H$	$G$	$F$	$E$	$D$	$C$	$B$	$A$	$I$	

Figure 17: Group table for  $S_4$ .

### 3 From Snaps to Flips

1. The six “operations” are considered **isometries**. Isometries are ways of mapping the triangle to itself, preserving shape and location. Are there any others for this triangle?

There are no other isometries for this triangle; our list of operations is complete. To see why, note that the vertices must exchange places. At most there is  $3! = 6$  ways to do this, so we have already achieved the maximum possible number of isometries.

.	$I$	$A$	$B$	$C$	$D$	$E$
$I$						
$A$						$B$
$B$						
$C$						
$D$						
$E$						

Figure 18: Unfilled  $D_3$  group table.

2. As with the snap group, we can make a group table for the flip group. Fill out a table like the one in Figure 18 in your notebook. Like the snap group table, the top row indicates what operation is done first and the left column indicates what’s done second, so that  $XY$  is in the  $X^{\text{th}}$  row and  $Y^{\text{th}}$  column.  $AD = B$  is done for you.

The completed table is shown in Figure 19.

3. What is the relationship between the tables for the snap group  $S_3$  and the flip group  $D_3$ ?

$D_3$ ’s table is  $S_3$ ’s table flipped over the top left–bottom right diagonal, and vice versa. Contrast  $D_3$  from Figure 19 to  $S_3$  in Figure 20. If these were matrices, one would be the transpose of the other: we’ll get to that later.

.	<i>I</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>I</i>	<i>I</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>A</i>	<i>A</i>	<i>I</i>	<i>D</i>	<i>E</i>	<i>B</i>	<i>C</i>
<i>B</i>	<i>B</i>	<i>E</i>	<i>I</i>	<i>D</i>	<i>C</i>	<i>A</i>
<i>C</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>I</i>	<i>A</i>	<i>B</i>
<i>D</i>	<i>D</i>	<i>C</i>	<i>A</i>	<i>B</i>	<i>E</i>	<i>I</i>
<i>E</i>	<i>E</i>	<i>B</i>	<i>C</i>	<i>A</i>	<i>I</i>	<i>D</i>

Figure 19: Completed  $D_3$  group table.

.	<i>I</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>I</i>	<i>I</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
<i>A</i>	<i>A</i>	<i>I</i>	<i>E</i>	<i>D</i>	<i>C</i>	<i>B</i>
<i>B</i>	<i>B</i>	<i>D</i>	<i>I</i>	<i>E</i>	<i>A</i>	<i>C</i>
<i>C</i>	<i>C</i>	<i>E</i>	<i>D</i>	<i>I</i>	<i>B</i>	<i>A</i>
<i>D</i>	<i>D</i>	<i>B</i>	<i>C</i>	<i>A</i>	<i>E</i>	<i>I</i>
<i>E</i>	<i>E</i>	<i>C</i>	<i>A</i>	<i>B</i>	<i>I</i>	<i>D</i>

Figure 20: Completed  $S_3$  group table from the last chapter.

4. Check your understanding by defining isomorphic in your own words.

(Answers may vary.)

Isomorphic means that two groups have the same structure. Isomorphic means that there is a correspondence between the elements of two groups so that the correspondence preserves the order. In the language of abstract algebra, an isomorphism between groups  $A$  and  $B$  exists if there is a homomorphism from  $A$  to  $B$  and from  $B$  to  $A$ .

5.

- (a) Make a table for only the rotations of  $D_3$ , a subgroup of  $D_3$ .

The table is shown below. Note that the identity element  $I$  is a rotation of 0.

.	<i>I</i>	<i>D</i>	<i>E</i>
<i>I</i>	<i>I</i>	<i>D</i>	<i>E</i>
<i>D</i>	<i>D</i>	<i>E</i>	<i>I</i>
<i>E</i>	<i>E</i>	<i>I</i>	<i>D</i>

Interestingly, this subgroup is a commutative group, also known as an abelian group.

- (b) Which subgroup of the snap group  $S_3$  is isomorphic to the subgroup in (a)?

The same elements (nominally) make the same subgroup:

.	<i>I</i>	<i>D</i>	<i>E</i>
<i>I</i>	<i>I</i>	<i>D</i>	<i>E</i>
<i>D</i>	<i>D</i>	<i>E</i>	<i>I</i>
<i>E</i>	<i>E</i>	<i>I</i>	<i>D</i>

6. What shape's dihedral group is isomorphic to

- (a) the two post snap group  $S_2$ ?

The dihedral group of a line segment is isomorphic to  $S_2$ . After all, you can only reflect it over its midpoint, which is the other element of  $S_2$  besides the identity. We can also think of this as permuting the two endpoints or vertices of a line segment.

- (b) the one post snap group  $S_1$ ?

The dihedral group of a point is isomorphic to  $S_1$ , because the only element is the identity element. This is permuting the one vertex of a point.

- (c) the four post snap group  $S_4$ ?

For this question we need to think 3 dimensions. There are four vertices to permute, but we can't do that on a square since diagonal points will remain on diagonals, as shown in Figure 21.

Instead, we choose the regular tetrahedron, so that there are no “diagonals”; every permutation is achievable. Note that rotations and reflections are now in 3 dimensional space, which is a bit difficult to visualize. A sample rotation is depicted in Figure 22.

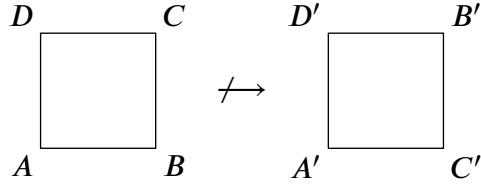


Figure 21: At right is a valid permutation of the vertices, but not a valid isometry of the square.

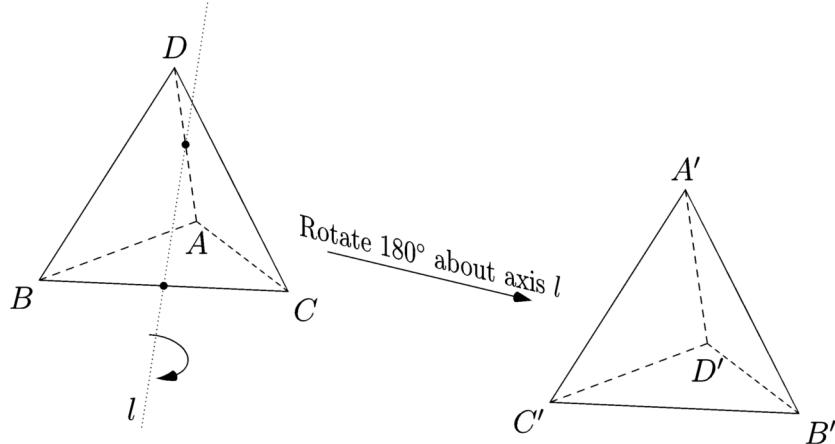


Figure 22: A rotation of the tetrahedron (orthographic view).

(d) the five post snap group  $S_5$ ?

This is isomorphic to the dihedral group of the 4-dimensional equivalent of the tetrahedron, also known as the regular 4-simplex. A projection is shown in Figure 23, but it cannot be faithfully represented on this paper.

7. Find a combination of  $A$  and  $D$  that yields  $C$ .

8. We call  $A$  and  $D$  **generators** of the group because every element of the group is expressible as some combination of  $A$ s and  $D$ s. For convenience, let's call  $A$  "f" since it's a flip, and call  $D$  "r" meaning a  $120^\circ$  rotation counterclockwise. Then, for example,  $fr^2$  is a rotation of  $2 \cdot 120^\circ = 240^\circ$ , followed by a flip across the  $A$  axis, equivalent to our original  $C$ . Make a new table using  $I$ ,  $r$ ,  $r^2$ ,  $f$ ,  $fr$ , and  $fr^2$  as elements, like the one in Figure 24. Note that the element order is different!

.	$I$	$r$	$r^2$	$f$	$fr$	$fr^2$
$I$						
$r$					$fr^2$	
$r^2$						
$f$						
$fr$						
$fr^2$						

Figure 24: Unfilled alternate  $D_3$  table.

The filled table is shown in Figure 25 below.

.	$I$	$r$	$r^2$	$f$	$fr$	$fr^2$
$I$	$I$	$r$	$r^2$	$f$	$fr$	$fr^2$
$r$	$r$	$r^2$	$I$	$fr^2$	$f$	$fr$
$r^2$	$r^2$	$I$	$r$	$fr$	$fr^2$	$f$
$f$	$f$	$fr$	$fr^2$	$I$	$r$	$r^2$
$fr$	$fr$	$fr^2$	$f$	$r^2$	$I$	$r$
$fr^2$	$fr^2$	$f$	$fr$	$r$	$r^2$	$I$

Figure 25: Completed alternate  $D_3$  table.

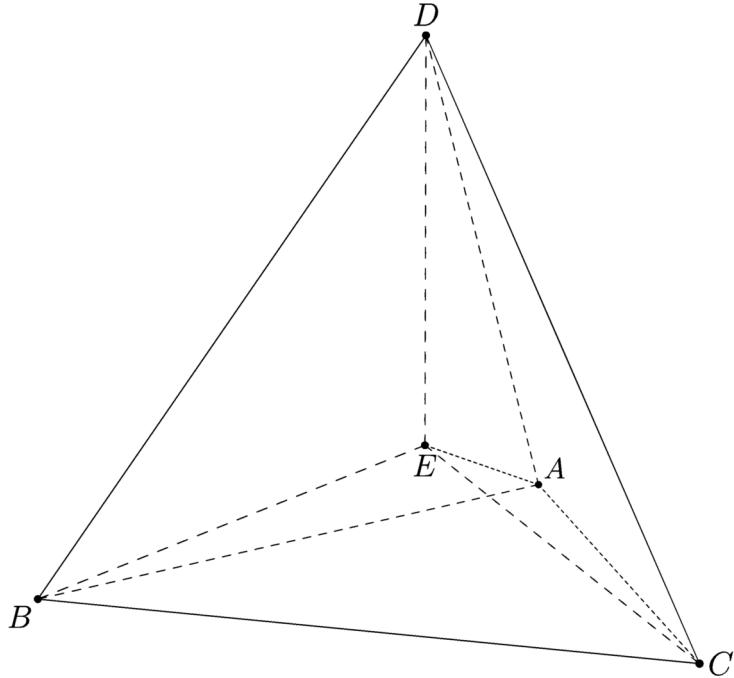


Figure 23: A 3D projection of the regular 4-simplex. In a true realization, every line segment here would be the same length.

Note that  $I = I$ ,  $A = f$ ,  $B = fr$ ,  $C = fr^2$ ,  $D = r$ , and  $E = r^2$ .

9. What other pairs of elements could you have used to generate the table?

You could also use any of the following pairs:  $\{A, E\}$ ,  $\{B, D\}$ ,  $\{B, E\}$ ,  $\{C, D\}$ ,  $\{C, E\}$ ,  $\{A, B\}$ ,  $\{B, C\}$ ,  $\{A, C\}$ . In essence, you can generate it with any rotation element and any reflection element, or with any two reflection elements.

10. You should notice the  $3 \times 3$  table of a group you've already described in the top-left corner of your table. What is it, and what are the two possible generators of this three-element group?

This is the cyclic group of order 3,  $C_3$ , also known as the rotation group of the equilateral triangle. The two possible generators are  $r$  and  $r^2$ .

11. Explain why each element of the flip group  $D_3$  has the period it has.

$I$  has a period of 1 because it is the identity.  $A, B, C$  have periods of 2 because they are reflections, so they are their own inverse transformation.  $D$  and  $E$  are rotations of a multiple of  $1/3$  of a turn. Since 3 is a prime, they take 3 iterations to resolve, and thus have period 3.

12. Some pairs of elements of the flip group are two-element subgroups. Which pairs are they?

These would be the pairs  $I, A$ ,  $I, B$ , and  $I, C$ , since  $A \cdot A = B \cdot B = C \cdot C = I$  so the subgroup is closed. These are shown in Figure 26.

.	$I$	$A$
$I$	$I$	$A$
$A$	$A$	$I$

.	$I$	$B$
$I$	$I$	$B$
$B$	$B$	$I$

.	$I$	$C$
$I$	$I$	$C$
$C$	$C$	$I$

Figure 26: The three two-element subgroups.

13. One of the elements forms a one-element subgroup. Which is it?

The element  $I$  forms the so-called trivial group, or the only group of order 1; this is shown in Figure 27. It is not very interesting.

.	$I$
$I$	$I$

Figure 27: The trivial group.

14. Addition of two numbers is a binary operation, while addition of three numbers is not. In logic,  $\wedge$  (and) and  $\vee$  (or) are binary operations, but  $\neg$  (not) is not. Define binary operation in your own words, and name some other binary operations.

(Answers may vary.)

A binary operation is an operation with two arguments.

Some binary operations:

- |                   |                    |                                 |                          |
|-------------------|--------------------|---------------------------------|--------------------------|
| 1. multiplication | 4. subtraction     | 7. bitwise OR                   | 10. function convolution |
| 2. exponentiation | 5. division        | 8. bitwise AND                  |                          |
| 3. addition       | 6. modulo operator | 9. snap operation ( $\bullet$ ) |                          |

15. In your original flip group table, what is

- (a) The identity element?

The identity element is  $I$ .

- (b) The inverse of  $A$ ?

The inverse of  $A$  is also  $A$ , since it is a reflection.

- (c) The inverse of  $E$ ?

The inverse of  $E$  is  $D$ , since  $240^\circ + 120^\circ \equiv 0^\circ$ .

## 4 Rotation and Reflection Groups

1. Notice that the original dihedral group had twice as many elements as the rotation group. Why?

(Answers may vary.)

There are a couple ways to think about this, but an intuitive way is to consider a “mirror world” of reflection and the “normal world” where the orientation is normal. Here, orientation is not absolute orientation, but the difference between clockwise and counterclockwise. For chemistry nerds, it is like chirality. Rotation preserves orientation, but reflection does not. Instead, it takes us between these two “worlds.” Thus, it allows twice the number of elements.

2. Make and justify a conjecture extending this observation to the dihedral groups of other shapes like rectangles, squares, hexagons, cubes, etc.

(Answers may vary.)

Conjecture: The dihedral groups of a shape has twice the order of its rotation group.

Informal Justification: A shape can be flipped or not, and it can have whatever rotational isometries applied to it whether it's flipped or not. Thus, the dihedral group allows for twice the number of elements as the rotation group.

3. Let  $r$  be a  $180^\circ$  rotation,  $x$  be a reflection over the  $x$ -axis, and  $y$  be a reflection over the  $y$ -axis. Write a table for the dihedral group of the rectangle, recalling that the allowed isometries are reflections and rotations. How does this table differ from the dihedral group of the equilateral triangle?

.	$I$	$r$	$x$	$y$
$I$	$I$	$r$	$x$	$y$
$r$	$r$	$I$	$y$	$x$
$x$	$x$	$y$	$I$	$r$
$y$	$y$	$x$	$r$	$I$

The table is shown above. The four elements are shown acting on a rectangle with “P” painted on it in Figure 28 to show the transformation a bit better.

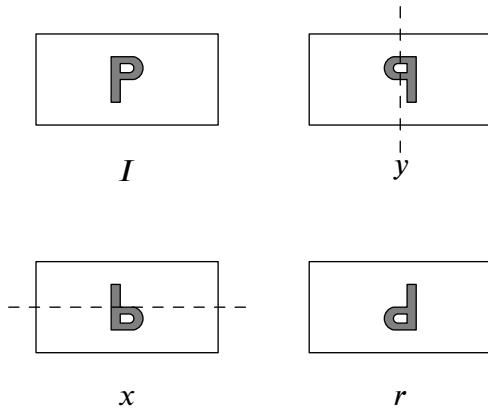


Figure 28: A rectangle AMBULATES and FLIPS around.

This differs from the dihedral group of the equilateral triangle,  $D_3$ , in several ways. The most obvious is that there are only 4 elements. Also, all elements besides  $I$  in this group have a period of 2, while  $D_3$  has two elements with a period of 3.

4. Write a table for the rotation group of the square, with 4 elements and 16 entries. Compare this table to problem 3.

The elements are  $I = r_0$ ,  $r = r_{90}$ ,  $r^2 = r_{180}$ , and  $r^3 = r_{270}$ . The table is shown below.

.	$I$	$r$	$r^2$	$r^3$
$I$	$I$	$r$	$r^2$	$r^3$
$r$	$r$	$r^2$	$r^3$	$I$
$r^2$	$r^2$	$r^3$	$I$	$r$
$r^3$	$I$	$r$	$r^2$	$r^3$

While this has the same order as the rectangle's dihedral group, it has a different structure. There are two elements with period 4 ( $r, r^3$ ) and one element with period 2 ( $r^2$ ).

For each of the following problems, find the following:

- Number of elements; this is known as the **order**
- If order  $< 10$ , the set of elements; otherwise, an explanation of how you know the order
- A smallest possible **generating set**; in other words, the list of elements which generate a group<sup>3</sup>
- Whether the group is **commutative**; in other words, whether its operation  $X \cdot Y$  doesn't care about the order of its operands ( $X$  and  $Y$ )

#### 5. Rectangle under rotation

- Number of elements

This group has two elements, the identity and the rotation of  $180^\circ$ .

- If order  $< 10$ , the set of elements; otherwise, an explanation of how you know the order

As stated, they are the identity  $I$  and the rotation  $r$  of  $180^\circ$ , as shown in Figure 29.

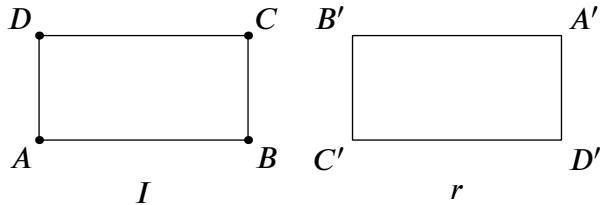


Figure 29: Rectangle under rotation.

- A smallest possible **generating set**

The smallest possible generating set is the singleton  $\{r\}$ .

- Whether the group is **commutative**

The group is commutative, since it's only comprised of rotations, which commute.

#### 6. Rectangle under reflection

We already considered this in problem 3.

- Number of elements

There are 4 elements in this group.

- If order  $< 10$ , the set of elements; otherwise, an explanation of how you know the order

The elements are the identity  $I$ , rotation  $r$  by  $180^\circ$ , reflection  $x$  over the  $x$ -axis, and reflection  $y$  over the  $y$ -axis.

- A smallest possible **generating set**

(Answers may vary.)

$\{r, x\}$ ,  $\{r, y\}$ , and  $\{x, y\}$  all generate the group. No single element, however, can generate the group.

- Whether the group is **commutative**

This group is commutative.

#### 7. Square under rotation

---

<sup>3</sup>There may be multiple generating sets of the same size.

Again, we have considered this group before.

- (a) Number of elements

There are 4 elements.

- (b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The elements are rotations  $I = r_0$ ,  $r = r_{90}$ ,  $r^2 = r_{180}$ , and  $r^3 = r_{270}$ .

- (c) A smallest possible **generating set**

(Answers may vary.)

Both  $\{r\}$  and  $\{r^3\}$  generate the group, because 1, 3 are coprime to 4.

- (d) Whether the group is **commutative**

The group is commutative, since it consists of all rotations.

## 8. Square under reflection

- (a) Number of elements

There are 8 elements in this group. We can quickly see this by noting that it is the dihedral group of the square, which has twice the order of the rotation group of the square. We just found that had 4 elements, and  $2 \cdot 4 = 8$ .

- (b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The elements are as follows:

Rotations  $I = r_0$ ,  $r = r_{90}$ ,  $r^2 = r_{180}$ , and  $r^3 = r_{270}$ ; reflections  $f$  = flip over the  $x$ -axis,  $fr = r$  then  $f$ ,  $fr^2$  and  $fr^3$ .

Recall that rotations can be generated by a sequence of two reflections.

Each of these elements is shown in Figure 30.

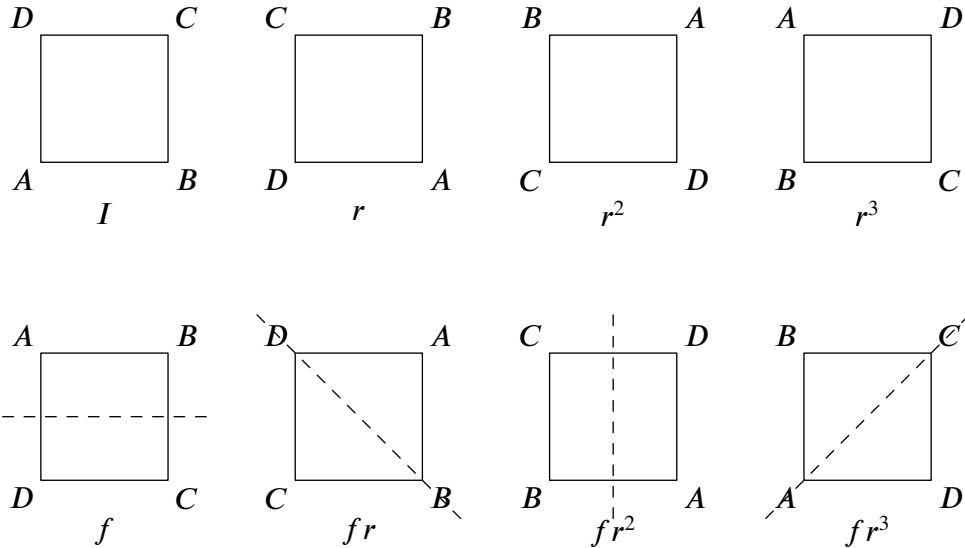


Figure 30: Reflections of a square.

- (c) A smallest possible **generating set**

(Answers may vary.)

Any pair of a rotation and flip will generate the set, except for  $\{r^2, fr^2\}$  and  $\{r^2, f\}$ ; these will produce the rectangle group instead. Any pair of two flips, except for  $\{f, fr^2\}$ , will also work. As an example of both of these categories, both  $\{r^2, fr^3\}$  and  $\{f, fr\}$  will generate the group.

(d) Whether the group is **commutative**

This group is not commutative. For example,  $fr = fr$ , but  $rf = fr^3$ .

9. Square prism under rotation

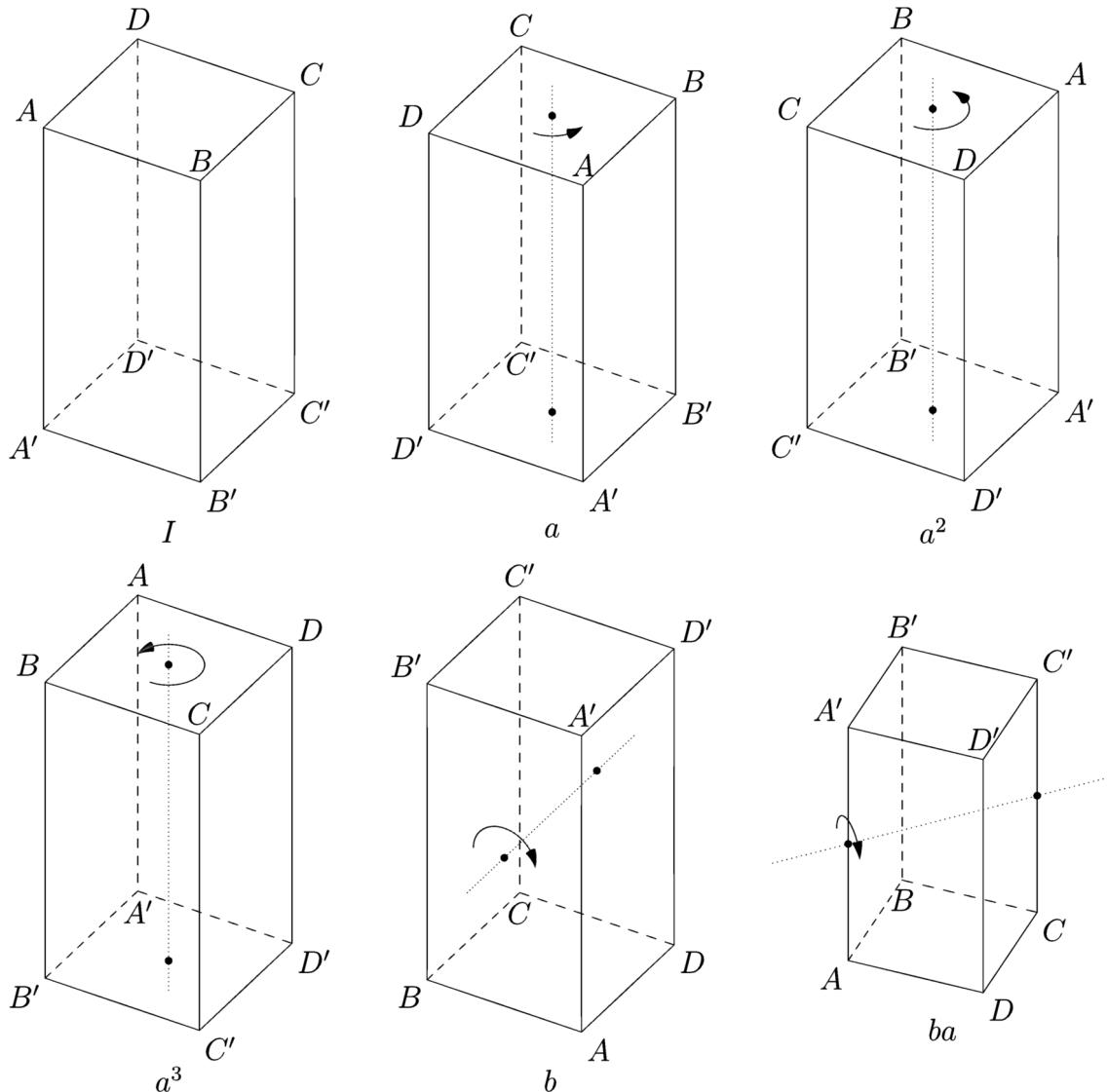
This group is isomorphic to the dihedral group of the square in Problem 8.

(a) Number of elements

This is a bit more difficult than the previous questions, because we need to understand what elements are possible. We can rotate the prism about its central axis, which is an action analogous to just rotating a square: 4 elements. But we can also rotate the prism  $180^\circ$  on an axis through the middle (pictures are shown in the next subpart). This switches the top square face with the bottom face, giving another 4 elements. In total, we have 8 elements.

(b) If order  $< 10$ , the set of elements; otherwise, an explanation of how you know the order

The set of elements are shown in Figure 31 below. Let  $a$  be a rotation of  $90^\circ$  counterclockwise—as viewed from the top—around the central axis, going through the centers of both square faces; let  $b$  be a rotation of  $180^\circ$  around an axis going through the centers of faces  $\square ABB'A'$  and  $\square DCC'D'$ .



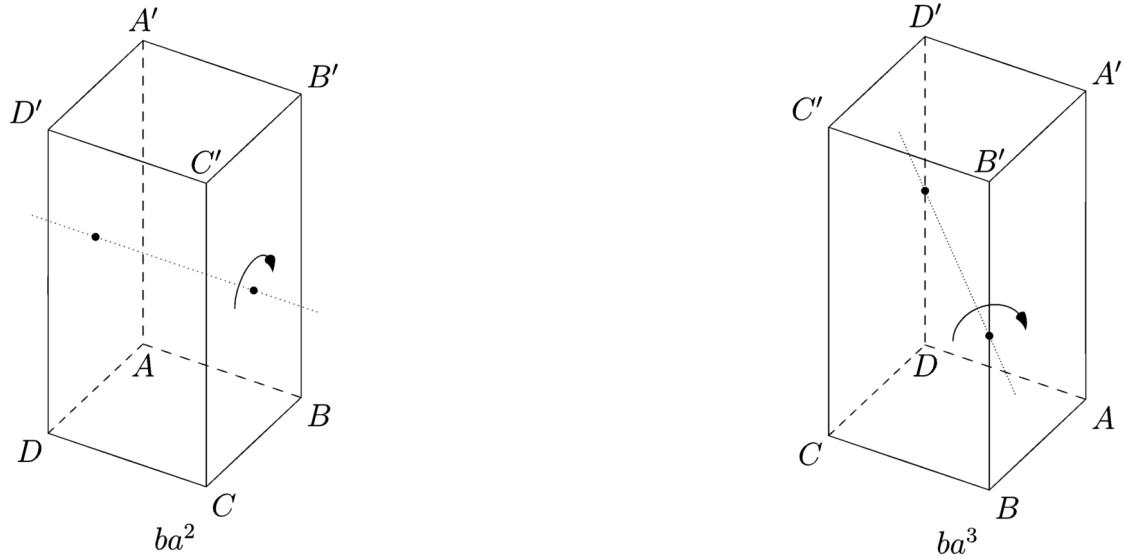


Figure 31: The elements of the rotation group of the rectangular prism.

(c) A smallest possible **generating set**

(Answers may vary.) The elements with  $b$  in their name are equivalent to the reflections in the dihedral group of the square. Thus, we need a “reflection”  $ba^n$  and a rotation  $a^m$ , or two separate reflections. All such pairs work except for  $\{a^2, ba^2\}$ ,  $\{a^2, b\}$  and  $\{b, ba^2\}$ . An example from each category:  $\{a, b\}$ ,  $\{b, ba\}$ .

(d) Whether the group is **commutative**

This group is not commutative. For example,  $ba = ba$ , but  $ab = ba^3$ .

10. Square prism under reflection

(a) Number of elements

If the previous group—the rotation group of the square prism—had 8 elements, then this group should have 16 elements.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

We know the order because the previous group has 8 elements, and dihedral groups have twice the number of elements of the rotation group, this group has 16 elements.

(c) A smallest possible **generating set**

(Answers may vary significantly.)

Since we could generate the previous group with (most) pairs of  $\{ba^n, a^m\}$ , or (most) pairs of  $\{a^n, a^m\}$ , we could just add another element  $c$  which is a true geometric reflection about, say, the midplane  $P$  between  $\square DCD'C'$  and  $\square ABB'A$  as shown in Figure 32.

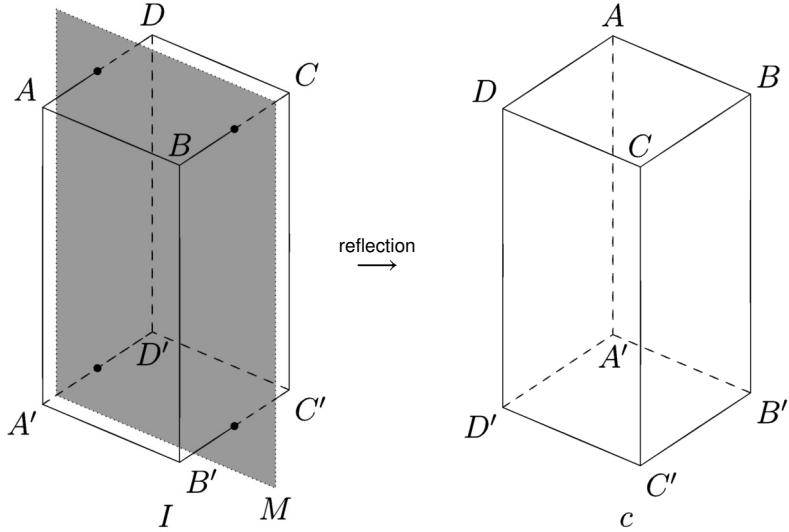


Figure 32: 3D reflection over the midplane  $M$  is  $c$ .

Thus,  $\{a, b, c\}$  can generate the group. You can prove that two generators are impossible, but the proof either requires making the group table or some more sophisticated abstract algebra. I will give the latter for those who are well-versed in group theory already, but it will probably be inaccessible to most.

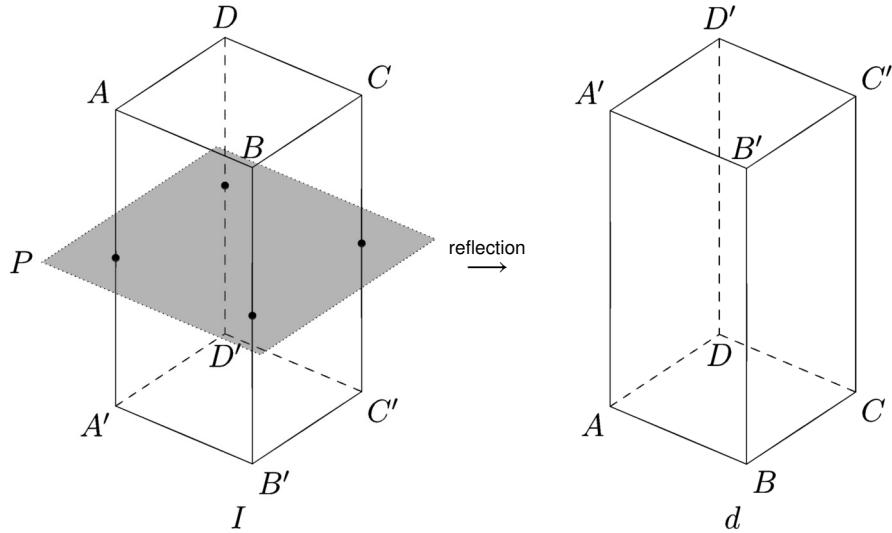


Figure 33:  $d$  is the reflection through midplane  $P$ .

The rotation group generated by  $\{a, b\}$  is  $D_4$ . Define a new element  $d$  which is the reflection through the midplane  $P$  between the two square faces.<sup>4</sup> This is crudely shown in Figure 33; I couldn't be bothered to make a nicer figure. Then the reflection group generated by  $\{d\}$  is  $Z_2$ . Furthermore, the operation sets  $\{a, b\}$  and  $\{d\}$  are separable, in that  $a^x b^y d = d a^x b^y$ <sup>5</sup>. Thus, the group  $G$  described in this problem is (isomorphic to) the direct product:

$$G \cong D_4 \times Z_2.$$

We wish to show that  $Z_2 \times Z_2 \times Z_2$  is a quotient of this group. That is, we wish to find a normal subgroup  $N$  such that

<sup>4</sup>For the curious,  $d = cba^2$ .

<sup>5</sup>This can be shown concretely by simply showing geometrically that  $ad = da$  and  $bd = db$ .

$$G/N = Z_2 \times Z_2 \times Z_2.$$

If this is true, then the minimal generating set of  $G$  has at least cardinality 3. All that remains is to find  $N$  and  $G/N$ .

It suffices to show that  $Z_2 \times Z_2 \triangleleft D_4$ , since then  $Z_2 \times Z_2 \times Z_2 \triangleleft D_4 \times Z_2$ . We have  $|Z_2 \times Z_2| = 2^2 = 4$ , so we want  $|D_4/N| = 4$ . We know  $|D_4| = 8$ , so by Lagrange's theorem,  $|N| = 2$ .

A normal subgroup of  $D_4$  is  $N = \{1, a^2\}$ . It is normal because for  $x \in \{0, 1, 2, 3\}$  and  $y \in \{0, 1\}$ :

$$\begin{aligned} (b^x a^y) a^2 (b^x a^y)^{-1} &= (b^x a^y) a^2 (a^{-y} b^{-x}) \\ &= b^x a^{2+y-y} b^{-x} \\ &= b^x a^2 b^{-x} \\ &= b^x b^{-x} a^2 \\ &= a^2 \in \{1, a^2\}. \end{aligned}$$

The corresponding quotient group is

$$D_4/N = \{\{1, a^2\}, \{a, a^3\}, \{b, ba^2\}, \{ba, ba^3\}\}.$$

We have the isomorphism  $\{b^x a^y, b^x a^{y+2}\} \leftrightarrow (x, y)$  under the operation of element-wise addition modulo 2. After all,

$$\{b^{x_1} a^{y_1}, b^{x_1} a^{y_1+2}\} \cdot \{b^{x_2} a^{y_2}, b^{x_2} a^{y_2+2}\} = \{b^{x_1+x_2} a^{y_1+y_2}, b^{x_1+x_2} a^{y_1+y_2+2}\}.$$

Therefore,

$$D_4/N \cong Z_2 \times Z_2,$$

so

$$Z_2 \times Z_2 \times Z_2 \triangleleft D_4 \times Z_2 = G.$$

Since the minimal generating set of  $Z_2 \times Z_2 \times Z_2$  is 3,  $G$ 's generating set is at least 3. But we've already found the set  $\{a, b, c\}$  which generates  $G$ <sup>6</sup>. Thus, it is minimal.

#### (d) Whether the group is **commutative**

As we found in the previous problem, the rotation group of the square prism is not commutative, and since that's a subgroup of this group, this group certainly isn't commutative.

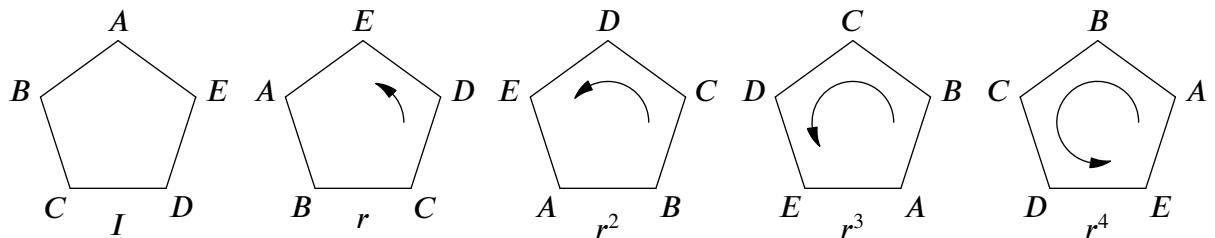
#### 11. Pentagon under rotation

##### (a) Number of elements

This is just the cyclic group of order 5, so there are 5 elements.

##### (b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The elements are rotations of  $I = r_0$ ,  $r = r_{72}$ ,  $r^2 = r_{144}$ ,  $r^3 = r_{216}$ ,  $r^4 = r_{288}$ . They are shown below.



Pentagons should always wear helmets, lest they want to damage their vertices.

##### (c) A smallest possible **generating set**

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<sup>6</sup>  $\{a, b, d\}$  also generates  $G$ .

Any rotation by itself  $\{r^n\}$  works, since 5 is a prime.

(d) Whether the group is **commutative**

The group is indeed commutative, since all operations are rotations.

12. Pentagon under reflection

(a) Number of elements

This is the dihedral group of the pentagon, which has  $2 \cdots 5 = 10$  elements.

(b) If order  $< 10$ , the set of elements; otherwise, an explanation of how you know the order

We know the order because it should have twice the number of elements as the rotation group, which has 5 elements, giving 10 elements total.

(c) A smallest possible **generating set**

We can either do a rotation and a reflection or two reflections. Since 5 is prime, all pairs work (unlike for the square). Let  $f$  is a flip over the vertical axis. Examples of each are  $\{r, f\}$  and  $\{f, fr\}$ .

(d) Whether the group is **commutative**

The group is not commutative. For example,  $fr = fr$ , but  $rf = fr^4$ .

13. Pentagonal prism under rotation

This is isomorphic to the dihedral group of the pentagon, which is Problem 12. The reason is the same as for Problem 9's dependence on 8, thus I will not explain it.

14. Pentagonal prism under reflection

This is akin to Problem 10.

(a) Number of elements

$$2 \cdot 10 = 20.$$

(b) If order  $< 10$ , the set of elements; otherwise, an explanation of how you know the order

We know the order because the rotation group of the pentagonal prism has 10 elements, so its dihedral group has 20 elements.

(c) A smallest possible **generating set**

If  $a$  is a rotation of  $72^\circ$  about the central axis,  $b$  is a rotation of  $180^\circ$  about a horizontal axis, and  $d$  is a reflection across the midplane between the two pentagonal faces, then  $\{a, b, d\}$  generates the set, since  $\{a, b\}$  generates all rotations and  $d$  turns them into their mirror images. But this isn't the right answer.

Are there any smaller generating sets? The previous trick asserting no using more advanced abstract algebra doesn't actually work<sup>7</sup>. We have  $ad = da$  and  $bd = db$  (you can verify this geometrically). So to have a two element subgroup we likely need something like  $a^n d$  and  $ba^m$  for some integers  $n, m$ , so that we can potentially generate  $a, b$  and  $d$ .

Let's try  $ad$  and  $b$ . Taking successive powers of  $ad$ , we get

$$\begin{aligned} ad &= ad \\ (ad)^2 &= a^2 \\ (ad)^3 &= a^3 d \\ (ad)^4 &= a^4 \\ (ad)^5 &= a^5 d = d \\ (ad)^6 &= a \end{aligned}$$

<sup>7</sup>If you understand it, it's because  $Z_2 \times Z_2 \times Z_2$  isn't a quotient of this group, since this group  $D_5 \times Z_2$  has order 20 which is not divisible by 8.

We've just generated  $d$  and  $a$  from  $ad$  alone! Since we have  $b$  already, we have created  $\{a, b, d\}$  from  $\{ad, b\}$ . Thus, the smallest generating set has size 2. (We can't have size 1 because then the group would be cyclic and thus commutative, which this group certainly isn't.)

This is a hard problem. Don't worry if you didn't get it.

(d) Whether the group is **commutative**

The dihedral group of the pentagon is a subgroup of this group, and is not commutative, so this group is not commutative.

15. Pentagonal pyramid under rotation

This is just isomorphic to the rotation group of the pentagon, or Problem 11.

16. Pentagonal pyramid under reflection

This is just isomorphic to the reflection group of the pentagon, or Problem 12.

17. Tetrahedron (triangular pyramid) under rotation

This is isomorphic to a subgroup of  $S_4$ , the snap group of order 24.

(a) Number of elements

The snap group includes reflections, but this does not: thus, this group has  $\frac{4!}{2} = 12$  elements.

(b) If order  $< 10$ , the set of elements; otherwise, an explanation of how you know the order

We know the order because this is the rotation group of a tetrahedron, and the reflection group of a tetrahedron has 24 elements, so this must have half that.

(c) A smallest possible **generating set**

Another difficult problem!

Let's figure out where the rotation axes actually are. There are 4 axes going through a vertex—let's call these *vertex axes*  $v_i$ . There are also 3 axes going through the midpoints of opposite edges: let's call these *edge axes*  $e_i$ . These axes are enumerated and shown in Figure 34.

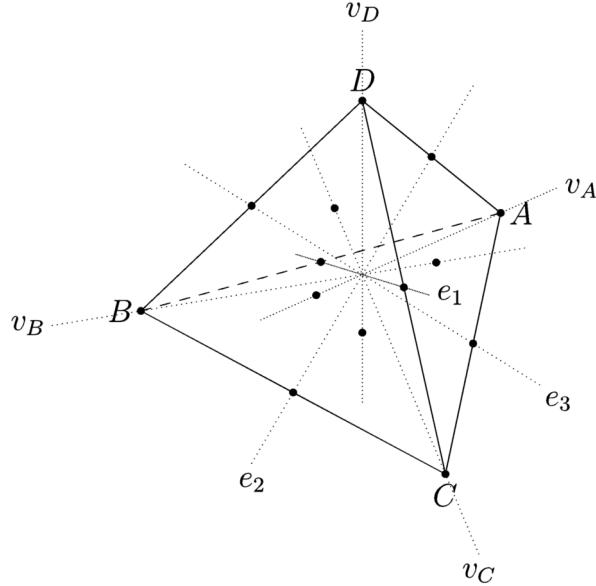


Figure 34: Regular tetrahedron's succulent rotation axes.

We can rotate by  $120^\circ$  or  $240^\circ$  (counterclockwise as viewed from the vertex) about any  $v_i$ , but only by  $180^\circ$  about any  $e_i$ . Along with the identity, this gives all  $2 \cdot 4 + 3 + 1 = 12$  elements.

To make manipulating these elements easier, treat them as moving elements in a list. We name this list with indices as shown in Figure 35. Thus, the identity element  $I$  is  $(A, B, C, D)$ . A rotation of  $240^\circ$  around  $v_A$  swaps vertices in positions  $(3 \quad 4)$  then  $(2 \quad 3)$ , so  $v_A = (A, D, B, C)$  as shown in Figure 36.

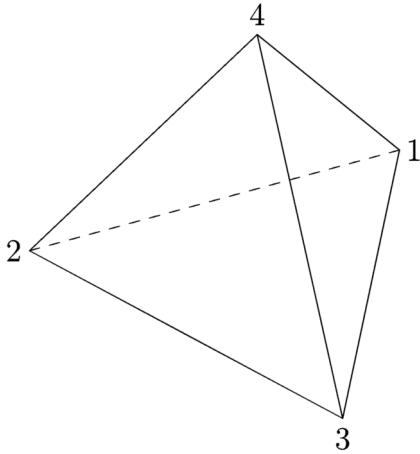


Figure 35: Regular tetrahedron's indices.

If we take a look at an edge rotation, say  $e_1$ , you will see it also swaps two vertices: in this case, (3 4) and (1 2). In general, any edge rotation or vertex rotation will swap two vertices—you can see this by plain symmetry or if you want, working it out for each rotation.

We now have a more abstract representation of this group: namely, it is the group of *even permutations* of  $(A, B, C, D)$ . Even permutations are permutations made by swapping two pairs at a time. For example,  $(B, A, D, C)$  is even, but  $(A, B, D, C)$  is not. The group operation is composing two permutations by chaining them together. Note that the identity,  $(A, B, C, D)$  is considered even, just as 0 is considered even.

One element is clearly not enough, because this group is not cyclic. Can we do it in two elements though?

Consider two vertex rotations, which cycle (without loss of generality) the first three vertices and the last three vertices. That is,  $a = (3, 1, 2, 4)$  and  $b = (1, 4, 2, 3)$ . Can we get every even permutation with combinations of  $a$  and  $b$ ? Let's try list them out:

$$\begin{aligned}
 a &= (3, 1, 2, 4), & a^2 &= (2, 3, 1, 4), & a^3 &= I = (1, 2, 3, 4), & b &= (1, 4, 2, 3), \\
 b^2 &= (1, 3, 4, 2), & b^3 &= I = a^3, & ab &= (2, 1, 4, 3), & ab^2 &= (4, 1, 3, 2), \\
 a^2b &= (4, 2, 1, 3), & a^2b^2 &= (3, 4, 1, 2), & b\bar{a} &= a^2b^2, & b^2a &= (3, 2, 4, 1), \\
 ba^2 &= (2, 4, 3, 1), & b^2\bar{a} &= ab, & b\bar{a}\bar{b} &= a^2, & b^2\bar{a}\bar{b} &= ba^2, \\
 b\bar{a}\bar{b} &= a^2b, & b^2ab^2 &= (4, 3, 2, 1) = ba^2b
 \end{aligned}$$

We have successfully generated all 12 elements with the set  $\{a, b\}$ . Thus, a two element generating set is sufficient! Interestingly, this means you can turn a tetrahedron however you want by holding it at two corners and twisting it with each.

For the curious, this group is known as the alternating group  $A_4$ .

#### (d) Whether the group is **commutative**

The group is clearly not commutative, since  $ab \neq ba$ .

### 18. Tetrahedron under reflection

This is just the snap group of order 4,  $S_4$ .

#### (a) Number of elements

As we found in the first problem,  $S_4$  has  $4! = 24$  elements.

#### (b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

As we found in the first problem,  $S_4$  has  $4! = 24$  elements.

#### (c) A smallest possible **generating set**

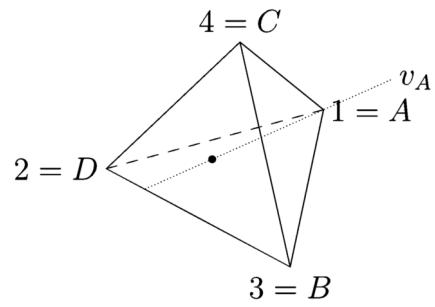


Figure 36:  $v_A = (A, D, B, C)$ .

This is tricky.

The obvious thing to do is keep  $\{a, b\}$  from the previous problem and add some reflection  $c$ . Then  $\{a, b, c\}$  has all 24 elements, since  $\{a, b\}$  makes 12 elements and  $c$  makes a copy of each “in the mirror world.” This is not, however, the right answer.

A generating of 2 elements is actually possible! There are several ways to see this, but I find a permutation argument easiest to follow.

$S_4$  is not just the reflection group of the tetrahedron, but also the group of all permutations of  $(1, 2, 3, 4)$ . Consider the permutation  $j = (4, 1, 2, 3)$ , which cycles all the elements, and the permutation  $k = (2, 1, 3, 4)$ , which swaps the first elements. Then

$$j = (4, 1, 2, 3), \quad j^2 = (3, 4, 1, 2), \quad j^3 = (2, 3, 4, 1), \quad j^4 = I = (1, 2, 3, 4).$$

We can flip any two adjacent elements (as well as the first and last elements) by doing the following:

1. Cycle using powers of  $j$  until the two elements in question are the first two elements.
2. Swap them with an application of  $k$ .
3. Cycle back to the starting position with powers of  $j$ .

In more mathematical terms, we can swap indices  $i$  and  $i + 1$ , where  $1 \leq i \leq 3$ , with the following element.

$$j^{i-1} k j^{5-i}.$$

Intuitively, if you can swap any two adjacent elements, you can make any permutations. The proof of this is pretty standard and outside the scope of this answer key.

For fun, let’s see what the elements  $j$  and  $k$  actually are, operating on the tetrahedron.

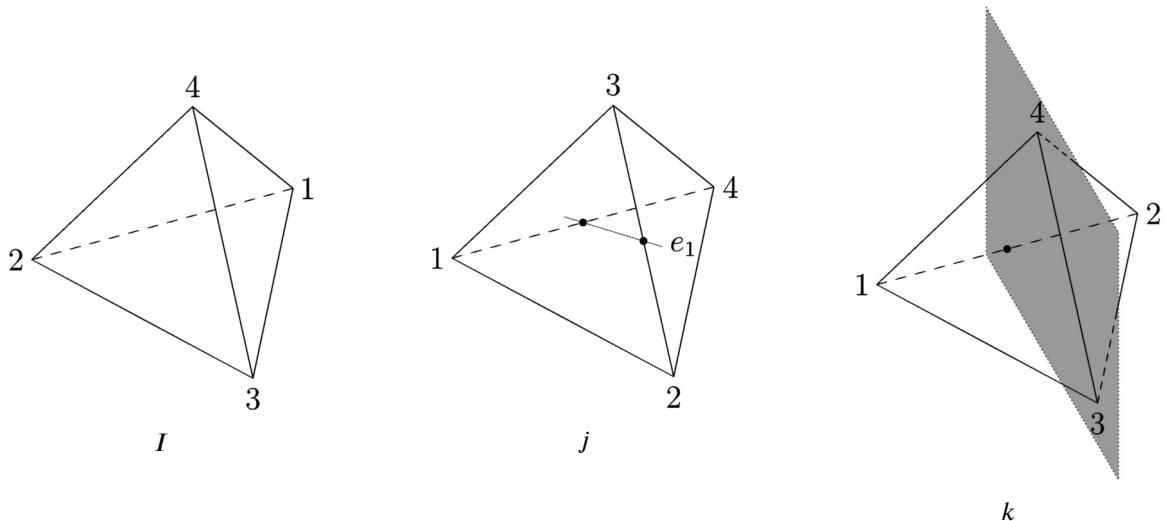


Figure 37: The two elements  $j$  and  $k$  generate the full symmetry group of the tetrahedron.

Thus, the true minimal generating set is  $\{j, k\}$  as described.

#### (d) Whether the group is **commutative**

This group is certainly not commutative, since the previous group from Problem 17 was not commutative and is a subgroup of this group.

#### 19. Cube under rotation

There are a couple ways to analyze this. My favorite one is to choose a face to make the top face, which can be done in 6 ways, then choose which rotation that face should be in, which can be done in 4 ways.

#### (a) Number of elements

Since we choose a front face in 6 ways, and its rotations in 4 ways, we have  $6 \cdot 4 = 24$  total rotations.

(b) If  $\text{order} < 10$ , the set of elements; otherwise, an explanation of how you know the order

The order is found above.

(c) A smallest possible **generating set**

This is tough until you make a key observation. If we label space-diagonally opposite vertices (that is, vertices which don't share a face) with the same number, as shown in Figure 38, then we can easily enumerate valid rotations.

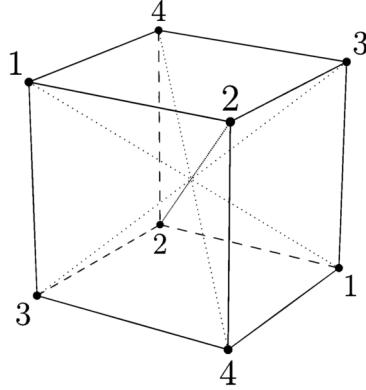


Figure 38: Marking opposite pairs of vertices.

The front face starts off saying “1,2,3,4.” I claim that the  $4!$  permutations of these four labels on the front face yields every rotation, and only rotations. This can be manually verified, but the higher-level argument isn’t too bad.

First, note that you will always see the numbers “1,2,3,4” in *some order* on the front face; you cannot see two of one number because all numbers are placed on diagonals of each other, and never share a side.

Second, note that the list of four numbers on the front face uniquely determines the other labels, since each has exactly one pair on the back face. For example, if there is a 3 in the closest corner to the camera, then there *must* be a 3 in the furthest corner of the camera.

Third, we demonstrate that the permutation of labels can always be represented as a rotation. There are six fundamentally different types of label squares under rotation *rotation*, as shown in Figure 39. But all appear somewhere as a face on the cube, as shown in Figure 40.

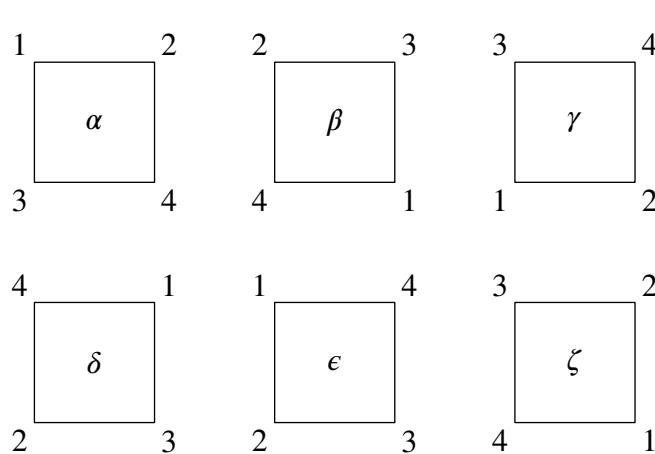


Figure 39: The six different labelings of a square.

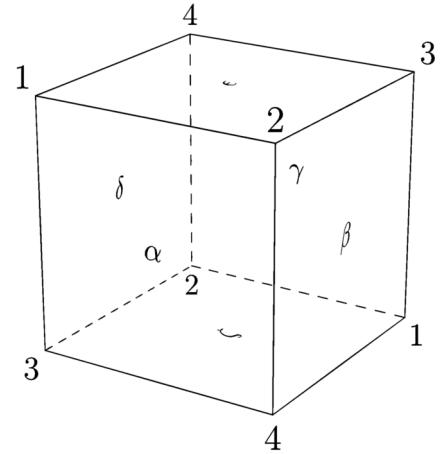


Figure 40: The six different labelings indeed appear on the cube!

We have demonstrated that every permutation of the front face labels 1. creates a unique orientation of the cube and 2. that orientation is a rotations. Since there are 24 unique permutations and 24 unique rotations, every rotation has exactly one corresponding permutation and vice versa. We can now construct an isomorphism! The set of label permutations under the operation of composing permutations (as we did

with the tetrahedron) and the set of rotations under the operation of composing rotations are *isomorphic*. In symbols,  $S_4 \cong G$ , our group.

So the group of rotations of a cube is actually  $S_4$ , the permutation group of 4 elements. I find this incredible.

Back to the main question: what is the minimal generating set? In the previous question, we found that the permutations  $(4, 1, 2, 3)$ —cycling all elements forward—and  $(2, 1, 3, 4)$ —swapping the first two elements—generating  $S_4$ . For the cube, those are two rotations  $a$  and  $b$  as shown in Figure 41.



Figure 41: The two rotations  $a$  and  $b$ .

(d) Whether the group is **commutative**

This group is not commutative, since  $S_4$  is not commutative.

20. Cube under reflection

(a) Number of elements

There are 24 elements in the rotation group of the cube, so naturally there are 48 elements in the reflection group.

(b) If  $\text{order} < 10$ , the set of elements; otherwise, an explanation of how you know the order

For each of the 24 rotations of the cube, there is also a reflected version over some plane. This gives  $2 \cdot 24 = 48$  total elements in this group.

(c) A smallest possible **generating set**

If  $c$  is a reflection about, say, the origin of the cube, then  $\{a, b, c\}$  (where  $a, b$  are the rotations from before) would generate the whole group, since  $\{a, b\}$  generates all rotations and  $\{c\}$  generates their respective reflections. But can we do it in two?

As usual it seems, the answer is yes! The proof is not mine, because I couldn't figure it out, but due to math.SE user **verret**. It does require some more advanced concepts, so it is probably inaccessible to most.

The group we've been analyzing is  $S_4 \times Z_2$ . Let  $S_4$  be permuting elements  $\{1, 2, 3, 4\}$  and  $Z_2$  be permuting elements  $\{5, 6\}$  (note that  $Z_2 = S_2$ ). Then given two elements  $g = (4, 1, 2, 3, 5, 6)$  in our notation, meaning that indices  $(1, 2, 3, 4)$  are cycled, and  $h = (3, 1, 2, 4, 6, 5)$ , meaning that indices  $(1, 2, 3)$  and  $(5, 6)$  are cycled, we can construct the group.

Note that  $h^2 = (2, 3, 1, 5, 6)$  is in  $S_4$ , since it does not permute indices 5, 6. It has a period of 3, and thus generates a subgroup of order 3. Furthermore,  $h^3$  only permutes  $(5, 6)$ . Furthermore,  $g$  is an element in  $S_4$  and has a period of 4. Thus, since  $\gcd(3, 4) = 1$ , by Lagrange's theorem we know that  $\{h^2, g\}$  generates a subgroup of  $S_4$  of at least order  $3 \cdot 4 = 12$ .

The only such subgroup, besides  $S_4$  itself, is the alternating group  $A_4$ . But  $g$  is outside of  $A_4$ , since it is an odd permutation:

$$(1, 2, [3, 4]) \rightarrow (1, [2, 4], 3) \rightarrow ([1, 4], 2, 3) \rightarrow (4, 1, 2, 3).$$

Thus,  $\{h^2, g\}$  does not generate  $A_4$ , and must generate  $S_4$ . Adding  $h^3$ , the generator for  $Z_2$ , to this set gives the full  $S_4, Z_2$ . The minimal generating set is therefore  $\{g, h\}$  as defined.

For the curious, using our vertex “labeling” convention as before, the elements  $g$  and  $h$  are shown in Figure 42.

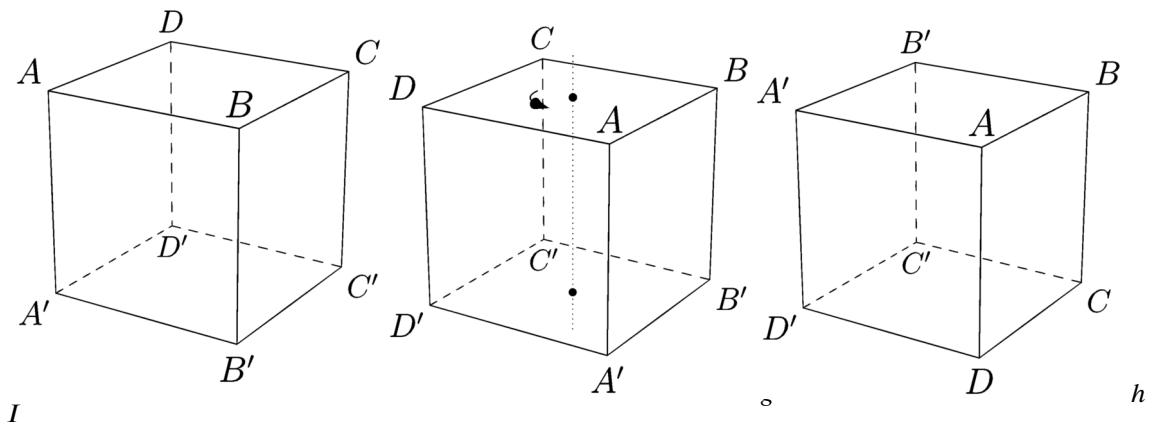


Figure 42: Elements  $g$  and  $h$ . Note that  $h$  is not solely a reflection about a mirror plane, but actually a combination of a rotation and reflection: a so-called rotoreflection!

(d) Whether the group is **commutative**

The subgroup of rotations of the cube,  $S_4$ , is not commutative, so this group is definitely not commutative.

## 5 Infinite Groups

Note: an **injection**  $f$  is a function taking  $A$  into  $B$  such that for all  $a \in A$ ,  $f(a) \in B$  and  $f(a)$  is unique. In other words, there are no two  $a_1, a_2 \in A$ ,  $a_1 \neq a_2$  such that  $f(a_1) = f(a_2)$ .

- Where have you come across the roots *iso-* and *-morphic* before?

(Answers may vary.)

*Iso-* is a root meaning equal. You might have seen it in isometry, isometric (paper), isomer, isosceles, isotonic, isotropy, and isotope. *Morph* means “form” or “shape.” You might have seen it in metamorphosis, amorphous, anthropomorphism, or morpheme.

- Can two groups be isomorphic if they have different orders?

No. Suppose we have groups  $A$  and  $B$  such that  $|A| > |B|$  ( $A$  is bigger than  $B$ ). Then we can't have a one-to-one correspondence between the elements of  $A$  and  $B$ , because there will always be elements in  $A$  without a “partner” in  $B$ . Thus, they cannot be isomorphic.

- The rotation group for the hexagon  $H$  has six elements: the identity, and rotations of  $\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$  radians. A rotation of  $\frac{\pi}{3}$  generates the group.

- Which other rotation generates the group?

The other rotation which generates the group is  $\frac{2\pi}{3}$ , because 5 is coprime with 6. This is necessary because otherwise a subgroup of the full  $C_6$  is formed. For example,  $\frac{2\pi}{3}$  generates

$$\left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right\},$$

which is merely  $C_3$ . Lame!

- What is the period of each element?

0 or  $I : 1$

$$\frac{\pi}{3} : 6$$

$$\frac{2\pi}{3} : 3$$

$$\pi : 2$$

$$\frac{4\pi}{3} : 3$$

$$\frac{5\pi}{3} : 6$$

- $H$  has the same number of elements as the dihedral group  $D_3$ .

- Are the two groups isomorphic? How do you know?

No, the two groups are not isomorphic, although they are the same size. An easy way to see this is that  $D_3$  has three reflections, which have period 2, but  $H$  only has one element of period 2.

- What is the period of each element of  $D_3$ ?

$I$  or  $I : 1$

$$r : 3$$

$$r^2 : 3$$

$$f : 2$$

$$fr : 2$$

$$fr^2 : 2$$

(c) What can you say if the sets of the periods of the elements of each group are not the same?

If the periods of each group can't be paired up, then the elements cannot be paired up either; after all, isomorphism is a structure-preserving operation. Thus, the two groups are not isomorphic.

(d) Which subgroups of the cyclic group  $C_6$  and  $D_3$  are isomorphic?

One is  $C_2$ , which is  $\{0, \pi\}$  in  $C_6$  and  $\{I, \text{any reflection}\}$  in  $D_3$ . The other non-trivial one is  $C_3$ , which is  $\left\{0, \frac{(3\pm 1)\pi}{3}\right\}$  in  $C_6$  and  $\{I, \text{any rotation}\}$  in  $D_3$ . Both also have the trivial subgroup  $\{I\}$  of just the identity element.

5. Could an infinite group be isomorphic to a finite group?

No, because their sizes are not the same; a one-to-one correspondence cannot be constructed.

6. Do you think all infinite groups are isomorphic to each other? Find a counterexample if you can.

Not all infinite groups are isomorphic. For example, the set of rotations about the origin has only one element of period 2, namely  $r_{180^\circ}$ . But the set of reflections about the origin has infinitely many elements of period 2. Both, however, are infinite in size.

7. Make guesses to the relative sizes of the following pairs of sets. You may use shorthand like  $|a| < |b|$ ,  $|a| > |b|$ ,  $|a| = |b|$ . After you have made your guesses, we will analyze some of the cases and you can find out how good your intuition was.

(Answers may vary, but the “correct” answers are shown.)

(a) natural numbers,  $\mathbb{N}$  vs. positive even numbers,  $2\mathbb{N}$

$$|\mathbb{N}| = |2\mathbb{N}|$$

(b) natural numbers,  $\mathbb{N}$  vs. positive rational numbers,  $\mathbb{Q}^+$

$$|\mathbb{N}| = |\mathbb{Q}^+|$$

(c) natural numbers,  $\mathbb{N}$  vs. real numbers between zero and one,  $[0, 1)$

$$|\mathbb{N}| < |[0, 1)|$$

(d) real numbers,  $\mathbb{R}$  vs. complex numbers,  $\mathbb{C}$

$$|\mathbb{R}| = |\mathbb{C}|$$

(e) real numbers,  $\mathbb{R}$  vs. points on a line

$$|\mathbb{R}| = |\text{points on a line}|$$

(f) points on a line vs. points on a line segment

$$|\text{points on a line}| = |\text{points on a line segment}|$$

(g) points on a line vs. points on a plane

$$|\text{points on a line}| = |\text{points on a plane}|$$

(h) rational numbers,  $\mathbb{Q}$  vs. Cantor set (look this up or ask your teacher)

$$|\mathbb{Q}| < |\mathcal{C}|$$

8. Now, please return to problem 7 and revise your answers. Justify each answer by producing a one-to-one correspondence, or showing the impossibility of doing so. Part (h) is an optional challenge.

(a) natural numbers,  $\mathbb{N}$  vs. positive even numbers,  $2\mathbb{N}$

This one is pretty straightforward. We have the following injection from  $\mathbb{N}$  to  $2\mathbb{N}$ :

$$s \in \mathbb{N} \rightarrow 2s \in 2\mathbb{N}.$$

We have the following injection from  $2\mathbb{N}$  to  $\mathbb{N}$ :

$$s \in \mathbb{N} \rightarrow \frac{s}{2} \in \mathbb{N}.$$

Since we can go both ways, we have  $|\mathbb{N}| = |2\mathbb{N}|$ , even though  $\mathbb{N} \subset 2\mathbb{N}$  ( $\mathbb{N}$  is a subset of  $2\mathbb{N}$ ).<sup>8</sup>

- (b) natural numbers,  $\mathbb{N}$  vs. positive rational numbers,  $\mathbb{Q}^+$

Surprisingly, we can make a one-to-one correspondence. If we list out the positive rationals in reduced form ( $\frac{p}{q}$  with  $p, q$  coprime), ordered by increasing denominator, we can create the correspondence:

$\mathbb{Q}^+$	0	1	1	2	1	3	1	2	3	4	...
	1	1	2	1	3	1	4	3	2	1	
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	...
$\mathbb{N}$	1	2	3	4	5	6	7	8	9	10	...

More details of this construction are given later in the textbook chapter. In any case,  $|\mathbb{Q}^+| = |\mathbb{N}|$ .

- (c) natural numbers,  $\mathbb{N}$  vs. real numbers between zero and one,  $[0, 1)$

A one-to-one correspondence cannot exist between these two sets, so  $|\mathbb{N}| < |[0, 1)|$ . The classic proof of this is Cantor's diagonal argument, which is given in the textbook.

- (d) real numbers,  $\mathbb{R}$  vs. complex numbers,  $\mathbb{C}$

This is a pretty tough problem to do in a logically sound way. The key is to represent complex numbers  $a + bi$  as the ordered pair  $(a, b)$  where  $a, b \in \mathbb{R}$ . The set of all  $(a, b)$  is denoted  $\mathbb{R}^2$ .

Here is the route we will take:

1. Construct a one-to-one correspondence between the interval  $[0, 1)$  and  $\mathbb{R}$ .
2. Use (1) to construct a similar correspondence between  $[0, 1)^2$  and  $\mathbb{R}^2$ . That is, we will construct a correspondence between ordered pairs of reals in  $[0, 1)$  and ordered pairs of any reals.
3. We find an injection from  $[0, 1)$  into  $[0, 1)^2$ .
4. We find an injection from  $[0, 1)^2$  into  $[0, 1)$ . This shows there is a one-to-one correspondence between  $[0, 1)$  and  $[0, 1)^2$ .
5. We "chain" the correspondences together:

$$\mathbb{R} \leftrightarrow [0, 1) \leftrightarrow [0, 1)^2 \leftrightarrow \mathbb{R}^2.$$

Step 1: The most straight forward way to do this is to show there is an injection from  $[0, 1)$  into  $\mathbb{R}$ , and vice versa.<sup>9</sup> We have  $f(x) = x$  as a straightforward injection from  $[0, 1)$  into  $\mathbb{R}$ , and

$$g(x) = \frac{1}{1 + e^{-x}}$$

as an injection  $g : \mathbb{R} \rightarrow [0, 1)$ . Thus, there exists a one-to-one correspondence  $H$  between  $\mathbb{R}$  and  $[0, 1)$ .

Step 2: If  $H$  is the function from Step 1, we have

$$J(a, b) = (H(a), H(b))$$

as a one-to-one correspondence between  $\mathbb{R}^2$  and  $[0, 1)^2$ .

Step 3: An injection from  $[0, 1)$  into  $[0, 1)^2$  is straightforward:

---

<sup>8</sup>We didn't explicitly state it because it's pretty intuitive, but this is using the Cantor-Schröder-Bernstein theorem (CSB).

<sup>9</sup>Again, this uses the Cantor-Schröder-Bernstein theorem.

$$k_1(x) = (x, 0).$$

Step 4: An injection from  $[0, 1]^2$  into  $[0, 1]$  is the more challenging portion. The basic idea is to interleave digits like so:

$$(0.123456789..., 0.314159265) \xrightarrow{k_2} 0.132134415569728695...$$

The main issue with this construction is that  $0.5 = 0.4999\dots$  gives two different outputs, so this mapping isn't even a function:

$$\begin{aligned} (0.5, 0.0) &\rightarrow 0.50 \\ (0.499\dots, 0.0) &\rightarrow 0.409090\dots \neq 0.50. \end{aligned}$$

The easiest thing to do here is arbitrarily choose one of these mappings. In particular, we represent a number with an infinite sequence of trailing zeroes  $a_1 a_2 \dots a_n 00000\dots$  with the numerically equivalent

$$0.a_1 a_2 \dots (a_n - 1)9999\dots$$

Now, our function  $k_2$  is a true injection, since  $k(a, b) \in [0, 1]$  for all  $(a, b) \in [0, 1]^2$  and  $k(a_1, b_1) \neq k(a_2, b_2)$  for  $(a_1, b_1) \neq (a_2, b_2)$ .

Step 5: We have constructed an injection  $k_1$  from  $[0, 1] \rightarrow [0, 1]^2$  and an injection  $k_2$  from  $[0, 1]^2 \rightarrow [0, 1]$ . Thus, there exists a one-to-one correspondence  $K$  between  $[0, 1]$  and  $[0, 1]^2$ .

We chain the correspondences, finally proving that there exists a one-to-one correspondence between  $\mathbb{R}$  and  $\mathbb{R}^2$ :

$$\mathbb{R} \xleftrightarrow{H} [0, 1] \xleftrightarrow{K} [0, 1]^2 \xleftrightarrow{J} \mathbb{R}^2.$$

$$\text{Thus, } |\mathbb{R}| = |\mathbb{R}^2|.$$

(e) real numbers,  $\mathbb{R}$  vs. points on a line

This is pretty straightforward if you think of points on a line as points on a number line. We arbitrarily choose a point on the line for 0 and a point for 1. In this regime, each point on the line corresponds with a unique real number. Thus,  $|\mathbb{R}| = |\text{points on a line}|$ .

(f) points on a line vs. points on a line segment

The simplest way to do this is, once again, to show there is an injection going both ways. We can go from segment  $\rightarrow$  line by observing that a segment is just a subset of a line. We can go from line to segment by representing each point as a real number  $\mathbb{R}$  as we already did, then taking the function

$$f(x) = \frac{1}{1 + e^{-x}}$$

which turns that point into a real number in the interval  $(0, 1)$ . This can be mapped onto the line segment by simply choosing one endpoint to be 0 and the other to be 1.

$$|\text{points on a line segment}| = |\text{points on a line}|$$

(g) points on a line vs. points on a plane

We can represent points on a line, as usual, with  $\mathbb{R}$ . We can represent points on a plane by arbitrarily choosing non-collinear points for  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$  and letting this be a coordinate space where points  $(a, b)$  are expressed as

$$a < 1, 0 > + b < 0, 1 > .$$

Note that the two vectors don't have to be perpendicular. This shows that we can represent points on a plane by  $\mathbb{R}^2$ . But we've already proved  $|\mathbb{R}^2| = |\mathbb{R}|$ ! Thus,

$$|\text{points on a plane}| = |\text{points on a line}|.$$

(h) rational numbers,  $\mathbb{Q}$  vs. Cantor set (look this up or ask your teacher)

The Cantor set  $\mathcal{C}$  is formed by iteratively deleting the open middle third of segments, starting with the unit segment. Formally, we can construct this like so<sup>10</sup>:

$$\begin{aligned} C_1 &= [0, 1] \\ C_n &= \frac{1}{3}C_{n-1} \cup \left( \frac{1}{3}C_{n-1} + \frac{2}{3} \right) \quad \text{for } n \geq 2 \\ \mathcal{C} &= \bigcap_{n=1}^{\infty} C_n. \quad (\text{Intersection of all intervals } C_n.) \end{aligned}$$

More intuitively, the construction is shown in Figure 43.

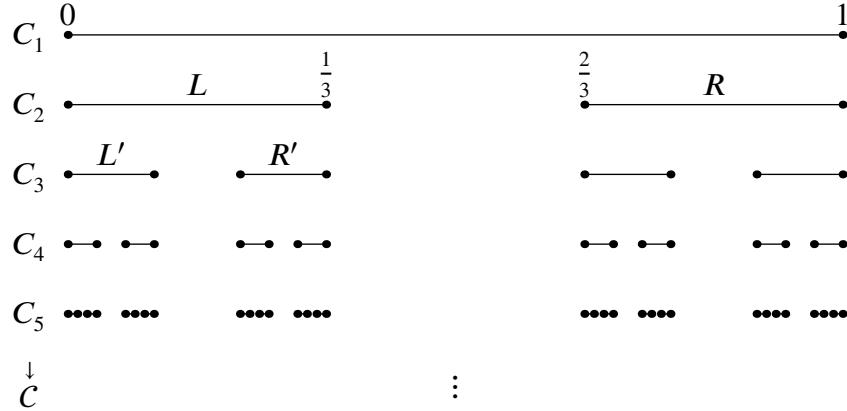


Figure 43: The construction of the Cantor set  $\mathcal{C}$ .

How do we attack this problem? It's not immediately clear how to tell whether a number  $x$  is in  $\mathcal{C}$ .<sup>11</sup> We should turn it into something we know how to deal with.

Consider how we would choose a random point in  $\mathcal{C}$ . Starting at  $C_1$ , we can either go to the left segment or the right segment (marked as  $L$  and  $R$  in Figure 43). If we choose  $L$ , then we once again have 2 choices: to go to  $L'$  or  $R'$ . This continues ad infinitum. Thus, we can correspond each element  $x \in \mathcal{C}$  with a binary number in the interval  $[0, 1]$ .<sup>12</sup>

As an example, suppose we choose segments in the sequence  $LRRLLLRLLLL\dots$ . Then the corresponding binary number is

$$0.0110001\bar{0}_2 = \frac{39}{128} \in [0, 1].$$

Thus, we have a one-to-one correspondence between the elements of  $\mathcal{C}$  and  $[0, 1]$ . The question is asking the relative sizes of  $\mathbb{Q}$  and  $\mathcal{C}$ . We already know (by Cantor's diagonal argument or otherwise) that  $|\mathbb{Q}| < |[0, 1]|$ . Therefore, we have

$$|\mathbb{Q}| < |[0, 1]| = |\mathcal{C}| \implies |\mathbb{Q}| < |\mathcal{C}|.$$

9. Here's a list of infinite sets, each with an operation. For each pair, answer: i. Does it form a group?  
ii. Which previous group(s) is it isomorphic to?

(a) natural numbers, addition

- i. Does it form a group?

Nope! It cannot satisfy the identity, since for  $x + I = I + x = I$  to be true for all  $x$  we need  $I = 0$ . If you're a fan of the standard ISO 80000, and include  $0 \in \mathbb{N}$ , then it still doesn't form a group, since it can't satisfy the invertibility property. For example, the inverse of 1 should be  $-1$  so that  $1 + (-1) = 0$ , but  $-1 \notin \mathbb{N}$ .

<sup>10</sup>This construction, taken literally, is actually scaling the set down by  $\frac{1}{3}$  and copying it at each step. The net result, however, is equivalent.

<sup>11</sup>A number  $x$  is in  $\mathcal{C}$  if and only if it has a ternary (base-3) representation consisting of only 0s and 2s. For example,  $1/27$  is in the Cantor set because  $1/27 = 0.001_3 = 0.000\bar{2}_3$ .

<sup>12</sup>Note it is inclusive because  $0.1111\dots_2 = 1$  and  $0.0000\dots_2 = 0$ .

ii. Which previous group(s) is it isomorphic to?

oof

(b) integers, addition

i. Does it form a group?

It does form a group. The identity element is 0, and it satisfies all necessary properties:

Identity:  $x + 0 = 0 + x = x$ .

Closure: If  $x, y \in \mathbb{Z}$ , then  $x + y \in \mathbb{Z}$ .

Associativity: We have  $x + (y + z) = (x + y) + z$  for all  $x, y, z \in \mathbb{Z}$ .

Inverse: The inverse of  $x$  is  $-x$ , since  $x + (-x) = (-x) + x = 0$ .

ii. Which previous group(s) is it isomorphic to?

None, I wonder why.

(c) even integers, addition

i. Does it form a group?

It does form a group. The identity element is 0, and it satisfies all necessary properties:

Identity:  $x + 0 = 0 + x = x$ .

Closure: If  $x, y \in 2\mathbb{Z}$ , then  $x + y = 2s + 2t = 2(s + t) \in 2\mathbb{Z}$ .

Associativity: We have  $x + (y + z) = (x + y) + z$  for all  $x, y, z \in 2\mathbb{Z}$ .

Inverse: The inverse of  $x$  is  $-x$ , since  $x + (-x) = (-x) + x = 0$ .

ii. Which previous group(s) is it isomorphic to?

It is isomorphic to integers under addition, because we can simply correspond  $2n \in 2\mathbb{Z}$  with  $n \in \mathbb{Z}$ . All the group structure is preserved, since  $2m + 2n \in 2\mathbb{Z}$  corresponds with  $m + n \in \mathbb{Z}$ .

(d) odd integers, addition

i. Does it form a group?

This does not form a group, since it cannot satisfy the identity property. There is no odd integer  $I$  such that  $x + I = I + x = x$ .

ii. Which previous group(s) is it isomorphic to?

oof

(e) rational numbers, addition

i. Does it form a group?

Yes, this forms a group with identity element 0. It satisfies all necessary properties:

Identity:  $\frac{p}{q} + 0 = 0 + \frac{p}{q} = \frac{p}{q}$ .

Closure: If  $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q}$ , then

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \in \mathbb{Q}.$$

Associativity: We have  $x + (y + z) = (x + y) + z$  for all  $x, y, z \in \mathbb{Q}$ .

Inverse: The inverse of  $\frac{p}{q}$  is  $-\frac{p}{q}$ , since

$$\frac{p}{q} + \left(-\frac{p}{q}\right) = \left(-\frac{p}{q}\right) + \frac{p}{q} = 0.$$

ii. Which previous group(s) is it isomorphic to?

None. It's not isomorphic to integers under addition because for each element  $x \in \mathbb{Q}$ , there exists an element  $y = \frac{x}{2} \in \mathbb{Q}$  such that

$$y + y = x.$$

This is impossible for any odd integers (analogously, for the group of even integers under addition, impossible for any elements not divisible by 4).

(f) real numbers, addition

i. Does it form a group?

Yes, this forms a group with identity element 0. It satisfies all necessary properties:

Identity:  $x + 0 = 0 + x = x$ .

Closure: If  $x, y \in \mathbb{R}$ , then  $x + y \in \mathbb{R}$ .

Associativity: We have  $x + (y + z) = (x + y) + z$  for all  $x, y, z \in \mathbb{R}$ .

Inverse: The inverse of  $x$  is  $-x$ , since  $x + (-x) = (-x) + x = 0$ .

ii. Which previous group(s) is it isomorphic to?

None. After all,  $\mathbb{R}$  is uncountable, while the groups we've seen so far are countable.

(g) complex numbers, addition

i. Does it form a group?

Yes, this forms a group with identity element  $0 = 0 + 0i$ . It satisfies all necessary properties:

Identity:  $x + 0 = 0 + x = x$ .

Closure: If  $x, y \in \mathbb{C}$ , then  $x + y \in \mathbb{C}$ .

Associativity: We have  $x + (y + z) = (x + y) + z$  for all  $x, y, z \in \mathbb{C}$ .

Inverse: The inverse of  $x$  is  $-x$ , since  $x + (-x) = (-x) + x = 0$ .

ii. Which previous group(s) is it isomorphic to?

Assuming the axiom of choice<sup>13</sup>, it is actually isomorphic to  $\mathbb{R}$  under addition. Since  $\mathbb{C}$  is uncountable, this is the only candidate.

Proving they are isomorphic is tough<sup>14</sup> without the introduction of vector spaces (specifically,  $\mathbb{Q}$ -vector spaces). I was originally going to put it in, but I couldn't get it under a satisfactory length. If you're really curious, check out <https://math.stackexchange.com/a/1511685/677124> for a mildly accessible view of the subject... if you already understand the basics of vector spaces. In summary, both  $\mathbb{R}$  and  $\mathbb{R}^2$  are vector spaces over the rational numbers  $\mathbb{Q}$ , and since  $|\mathbb{R}| = |\mathbb{R}^2|$  they are isomorphic as vector spaces. This also implies that they are isomorphic as additive groups.

(h) integers, multiplication

i. Does it form a group?

No, this does not form a group. The identity element would be 1, so that  $1 \cdot x = x \cdot 1 = x$ , but  $1 \cdot 0 = 0 \neq 1$ , so it cannot satisfy invertibility. Even if we removed 0, for any  $p \neq \pm 1$  there is no integer  $q$  such that  $pq = 1$ .

ii. Which previous group(s) is it isomorphic to?

oof

(i) integer powers of 2, multiplication

i. Does it form a group?

---

<sup>13</sup>The axiom of choice states that for every indexed family of sets  $(S_i)_{i \in I}$ , where  $S_i \neq \emptyset$ , there exists an indexed family of elements  $(x_i)_{i \in I}$  such that  $x_i \in S_i$  for all  $i \in I$ . Intuitively, this means that given a list of non-empty sets, you can select exactly one item from each set.

<sup>14</sup>In fact, it is impossible to construct an “explicit” isomorphism because  $\mathbb{R} \not\cong \mathbb{C}$  is consistent with the axiom of choice.

Yes! Let the group be called  $\mathcal{W} = \{2^x : x \in \mathbb{Z}\}$  for fun. The identity element is  $2^0 = 1$ . The group properties are satisfied:

Identity:  $2^x \cdot 1 = 1 \cdot 2^x = 2^x$ .

Closure:  $2^x \cdot 2^y = 2^{x+y} \in \mathcal{W}$ .

Associativity: We have  $2^x(2^y \cdot 2^z) = (2^x \cdot 2^y)2^z = 2^{x+y+z}$  for all  $x, y, z \in \mathbb{Z}$ .

Inverse: The inverse of  $2^x$  is  $2^{-x}$ , since  $2^x 2^{-x} = 2^{-x} 2^x = 2^0 = 1$ .

ii. Which previous group(s) is it isomorphic to?

It is isomorphic to integers under addition.  $2^n \in \mathcal{W}$  corresponds with  $n \in \mathbb{Z}$ , since we have

$$2^m \cdot 2^n = 2^{m+n} \leftrightarrow m + n.$$

(j) rational numbers, multiplication

i. Does it form a group?

The rational numbers under multiplication do not form a group, because if the identity element is 1, then  $1 \cdot 0 = 0 \neq 1$ , so it cannot satisfy invertibility.

ii. Which previous group(s) is it isomorphic to?

oof

(k) rational numbers excluding 0, multiplication

i. Does it form a group?

Yes! The rational numbers excluding 0, written  $\mathbb{Q} \setminus 0$ , form a group under multiplication with identity element 1.

1. The group properties are satisfied:

Identity:  $x \cdot 1 = 1 \cdot x = x$ .

Closure: If  $x, y \in \mathbb{Q} \setminus 0$ , then  $x + y \in \mathbb{Q} \setminus 0$ ; the product of two nonzero rational numbers is rational and nonzero.

Associativity: We have  $x(yz) = (xy)z$  for all  $x, y, z \in \mathbb{Q} \setminus 0$ .

Inverse: The inverse of  $x \in \mathbb{Q} \setminus 0$  is  $\frac{1}{x}$ , since  $x \left( \frac{1}{x} \right) = \left( \frac{1}{x} \right) x = 1$  and  $x \neq 0$ .

ii. Which previous group(s) is it isomorphic to?

It is not isomorphic to any. The only candidates are other groups with countably infinite order, aka addition of rational numbers and addition of integers.

It cannot be addition of rational numbers, because for every element  $k = \frac{p}{q}$  there exists an element  $r = \frac{p}{2q}$  such that  $k = 2r$ . This property is not true for rational numbers under multiplication, because the corresponding  $r$  would be  $\sqrt{\frac{p}{q}}$ , which is only valid for perfect (rational) squares  $\frac{p}{q}$ . For the same reason, it cannot be addition of integers, since odd integers also cannot have this property.

(l) complex numbers, multiplication

i. Does it form a group?

No, because 0 prevents the group from satisfying invertibility.

ii. Which previous group(s) is it isomorphic to?

oof

(m) rotation by a rational number of degrees

i. Does it form a group?

Yes! The identity is 0 and it can simply by thought of adding rationals modulo 360. The group properties are satisfied:

Identity:  $x \cdot 1 = 1 \cdot x = x$ .

Closure: If  $x, y \in \mathbb{Q}$  then  $x + y \in \mathbb{Q}$  and it

Associativity: We have  $x(yz) = (xy)z$  for all  $x, y, z \in \mathbb{Q} \setminus 0$ .

Inverse: The inverse of  $x \in \mathbb{Q} \setminus 0$  is  $\frac{1}{x}$ , since  $x \left( \frac{1}{x} \right) = \left( \frac{1}{x} \right) x = 1$  and  $x \neq 0$ .

ii. Which previous group(s) is it isomorphic to?

None. The easiest way to see this is that the element  $\frac{360}{n}^\circ$ , where  $n$  is an integer, has period  $n$ , so we can construct elements of arbitrary periods. No previous groups have elements of arbitrary periods.

(n) rotation by a rational number of radians

i. Does it form a group?

Yes! The identity element is 0 and it is totally equivalent to the rational numbers under addition. That's because for any rational radian rotations  $r_{p_1/q_1}$  and  $r_{p_2/q_2}$ , where  $p_1, q_1, p_2, q_2 \in \mathbb{Z}$ ,  $q_1, q_2 \neq 0$  and  $p_1/q_1 \neq p_2/q_2$ , we have  $r_{p_1/q_1} \cong r_{p_2/q_2}$ . The easiest way to understand this is that for two such rotations to be equal, there must be some integer  $k$  such that

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} + 2\pi k.$$

There are two cases to consider:  $k = 0$  and  $k \neq 0$ . If  $k = 0$ , then  $\frac{p_1}{q_1} = \frac{p_2}{q_2}$ , which violates our assumption. If  $k \neq 0$ , then since  $\pi$  is irrational, the RHS is irrational while the LHS is rational. Since an irrational and rational cannot be equal, such an integer  $k$  cannot exist and  $r_{p_1/q_1} \not\cong r_{p_2/q_2}$ .

The group properties are straightforward and identical to the rationals under addition.

ii. Which previous group(s) is it isomorphic to?

As explained, it is isomorphic to the rational numbers under addition.

(o) rotation by an integer number of radians

i. Does it form a group?

Yes. The identity element is 0 and it is equivalent to the integers under addition. We proceed in a similar method to the previous problem: consider two integer radian rotations  $r_a$  and  $r_b$  where  $a, b \in \mathbb{Z}$  and  $a \neq b$ .

Suppose  $r_a \cong r_b$ . Then there is some integer  $k$  such that

$$a = b + 2\pi k.$$

If  $k = 0$ , then  $a = b$ , contradicting our assumption that  $a \neq b$ . If  $k \neq 0$ , then the RHS is irrational while the LHS is rational. This is impossible to satisfy, so  $k$  does not exist and  $r_a \not\cong r_b$ .

The group properties are straightforward and identical to the integers under addition.

ii. Which previous group(s) is it isomorphic to?

As explained, it is isomorphic to the integers under addition.

10. Can an irrational number taken to an irrational power ever be rational? Consider the potential example

$a = \sqrt{2}^{\sqrt{2}}$ . To help you answer this question, let  $b = a^{\sqrt{2}}$ . Simplify  $b$ , and explain why we don't need to know whether  $a$  is rational or irrational.

We have

$$b = a^{\sqrt{2}} = \left( \sqrt{2}^{\sqrt{2}} \right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2.$$

We know  $\sqrt{2}$  is irrational. Suppose  $a$  is rational. Then we have an example of an irrational number raised to an irrational power which is rational! But if  $a$  is not rational, then  $b$  must be irrational, so  $b$  is an example of

an irrational number ( $a$ ) raised to an irrational power ( $\sqrt{2}$ ) which is rational! Thus, no matter the case, there exists an irrational number raised to an irrational power which is rational. Interestingly, we don't know whether  $a$  is rational or irrational, only that one of  $a, b$  satisfies the problem.

Logically, let  $A = a$  is rational and  $C = \text{there exists an irrational number raised to an irrational power which is rational}$ . Then  $A \rightarrow C$  and  $\neg A \rightarrow C$ . Since  $A \vee \neg A$ , we know that  $C \vee C$ , so  $C$  is true.

As an aside,  $a$  is the irrational one (and is in fact transcendental) by the Gelfond-Schneider theorem. Proving this theorem requires some pretty advanced analysis, yet we are able to derive related results with simple logic.

## 6 Geometry of Complex Numbers

- Explain why  $iz$  is perpendicular to  $z$ , without using DeMoivre's theorem.

Let  $z = a + bi$ . Then  $iz = i(a + bi) = -b + ai$ , which is the transformation  $(a, b) \rightarrow (-b, a)$ . Drawing this out on the 2D plane makes clear that the angle between the two points and the origin is  $90^\circ$ , simply by subtracting angles:  $(90^\circ + \theta) - \theta = 90^\circ$ . This is shown in Figure 44.

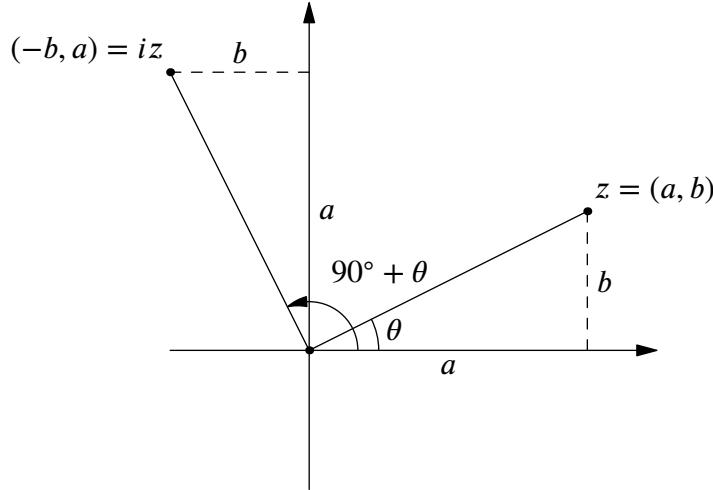


Figure 44:  $iz$  is perpendicular to  $z$  as long as  $z \neq 0$ .

You can also rationalize it by the fact that the lines through  $z/iz$  and the origin have slopes of  $\frac{b}{a}$  and  $-\frac{a}{b}$ , respectively, so they must be perpendicular. Also,  $\langle a, b \rangle \cdot \langle -b, a \rangle = 0$ .

- How does  $\text{Arg } \bar{z}$  relate to  $\text{Arg } z$ ? (Hint: symmetry!)

Again, let  $z = a + bi$ .  $\bar{z} = a - bi$  is flipped over the  $x$ -axis, since the imaginary part is negated. Thus,  $\text{Arg } \bar{z} = -\text{Arg } z$  due to congruent triangles formed by  $z$  and  $\bar{z}$ .<sup>15</sup> The geometric interpretation is shown in Figure 45.

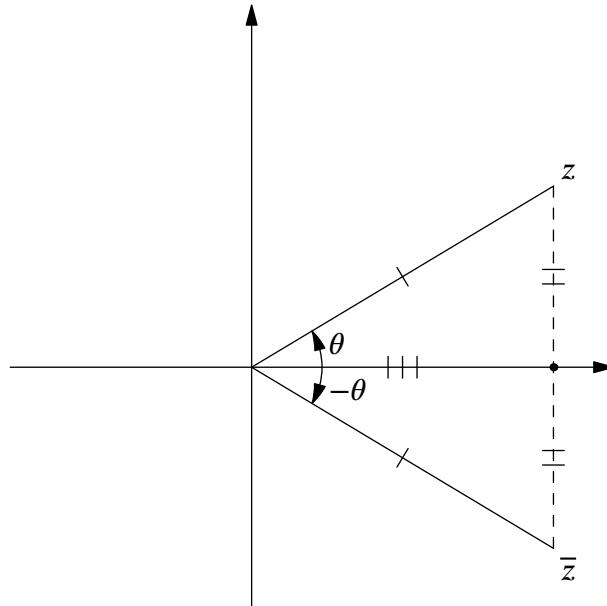


Figure 45:  $z$  and  $\bar{z}$  form congruent triangles, showing that  $\text{Arg } z = -\text{Arg } \bar{z}$ .

- Compute  $z\bar{z}$  and relate it to the cis form of  $z$ .

---

<sup>15</sup>If we are to be pedantic, we'd either use the multivalued function  $\arg$  or say this is modulo  $2\pi$ .

Once more, let  $z = a + bi$ . Then

$$z\bar{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

If  $z = r \operatorname{cis} \theta$ , then  $z\bar{z} = r^2$ . In other words, it is the square of the distance from  $z$  to the origin.

4. Explain, using a picture, why  $\tan(\operatorname{Arg} z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$ .

This is basically just an application of soh-cah-toa to a triangle in the complex plane. The details are shown in Figure 46.

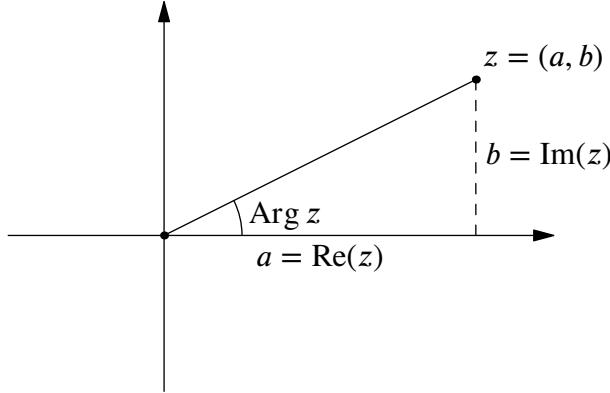


Figure 46:  $\tan(\operatorname{Arg} z) = \frac{b}{a} = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$ .

5. Divide  $\frac{a+bi}{c+di}$  by rationalizing the denominator.

$$\begin{aligned} \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} &= \frac{(a+bi)(c-di)}{c^2+d^2} \\ &= \frac{ac+bd+(bc-ad)i}{c^2+d^2}. \end{aligned}$$

6. Divide  $\frac{r_1 \operatorname{cis} \theta}{r_2 \operatorname{cis} \phi}$  using DeMoivre's theorem.

We don't have a rule yet for applying DeMoivre's theorem for division, but we can quickly derive it. We have

$$\begin{aligned} \frac{r_1 \operatorname{cis} \theta}{r_2 \operatorname{cis} \phi} \cdot \overline{\frac{\operatorname{cis} \phi}{\operatorname{cis} \phi}} &= \frac{r_1 \operatorname{cis} \theta \overline{\operatorname{cis} \phi}}{\underbrace{r_2 \operatorname{cis} \phi \operatorname{cis} \phi}_{=1}} && \text{Multiplying by conjugate} \\ &= \frac{r_1 \operatorname{cis} \theta \operatorname{cis}(-\phi)}{r_2} && \text{Using } \operatorname{Arg} z = -\operatorname{Arg} \bar{z} \\ &= \frac{r_1}{r_2} \operatorname{cis}(\theta - \phi). && \text{Use DeMoivre's theorem} \end{aligned}$$

7. Compare and contrast the methods of division in problems 5 and 6. Which is more convenient? Or does it depend on the circumstance?

Opinions may vary, but 6 is definitely faster to do if the dividend and divisor are already in cis form. 5 is likely convenient than converting from rectangular to cis, then back to rectangular.

8.

- (a) If  $z = r \operatorname{cis} \theta$ , what is  $\frac{1}{z}$ ?

As we hinted at in the previous problem,  $\frac{1}{z} = \frac{1}{r} \operatorname{cis}(-\theta)$ :

$$\begin{aligned}\frac{1}{r \operatorname{cis} \theta} \cdot \overline{\frac{\operatorname{cis} \theta}{\operatorname{cis} \theta}} &= \frac{\overline{\operatorname{cis} \theta}}{r \operatorname{cis} \theta \operatorname{cis} \theta} \\ &= \frac{\operatorname{cis}(-\theta)}{r |\operatorname{cis} \theta|^2} \\ &= \frac{1}{r} \operatorname{cis}(-\theta).\end{aligned}$$

- (b) Explain how this shows  $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$ , without having to rationalize the denominator. (Hint: use problems 3, 4, and 7.)

Let  $a+bi = r \operatorname{cis} \theta$ . We have

$$\begin{aligned}\frac{1}{r \operatorname{cis} \theta} &= \frac{1}{r} \operatorname{cis}(-\theta) \\ &= \frac{r \operatorname{cis}(-\theta)}{r^2} \\ &= \frac{a-bi}{a^2+b^2}.\end{aligned}$$

9. Compute  $(1+i)^{13}$ ; pencil, paper, and brains only. No calculators!

We have  $1+i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$ , since it forms a  $45^\circ$  angle with the  $x$ -axis. Applying DeMoivre's theorem,

$$\begin{aligned}(1+i)^{13} &= \left(\sqrt{2} \operatorname{cis} \frac{\pi}{4}\right)^{13} \\ &= (\sqrt{2})^{13} \operatorname{cis} \frac{13\pi}{4} \\ &= 64\sqrt{2} \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right) \\ &= 64\sqrt{2} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ &= 64(-1-i) \\ &= -64-64i.\end{aligned}$$

10. Compute  $\frac{(1+i\sqrt{3})^3}{(1-i)^2}$  without a calculator.

We convert to cis form and apply DeMoivre's theorem.

$$\begin{aligned}\frac{(1+i\sqrt{3})^3}{(1-i)^2} &= \frac{\left(2 \operatorname{cis} \frac{\pi}{3}\right)^3}{\left(\sqrt{2} \operatorname{cis} -\frac{\pi}{4}\right)^2} \\ &= \frac{8 \operatorname{cis}(-\pi)}{2 \operatorname{cis} -\frac{\pi}{2}} \\ &= \frac{8 \cdot -1}{2 \cdot -i} \\ &= \frac{4}{i} \cdot \frac{-i}{-i} \\ &= -4i.\end{aligned}$$

11. Draw  $\operatorname{cis} \left(\frac{\pi}{4}\right) + \operatorname{cis} \left(\frac{\pi}{2}\right)$ . Use your picture to prove an expression for  $\tan \left(\frac{3\pi}{8}\right)$ . (Hint: add them as vectors.)

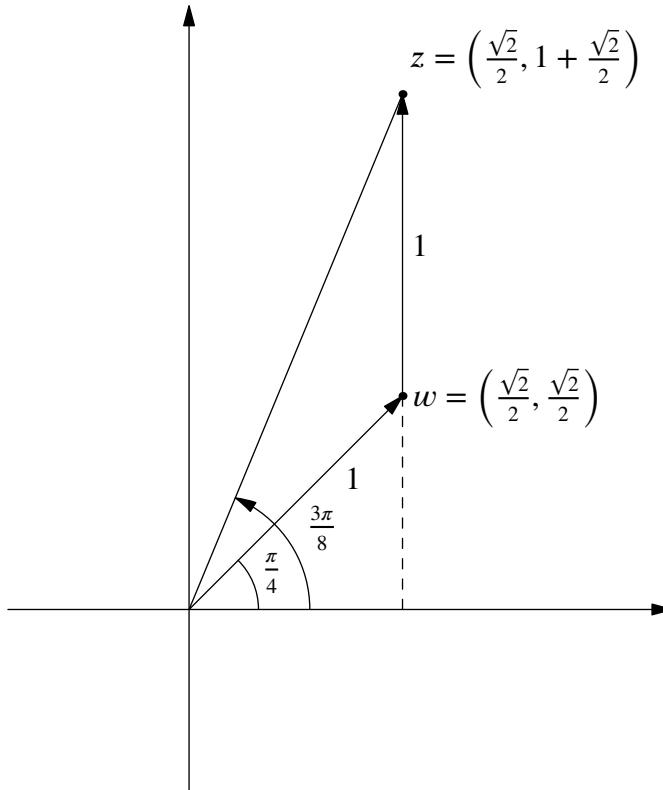


Figure 47: Addition of  $\text{cis}\left(\frac{\pi}{4}\right) + \text{cis}\left(\frac{\pi}{2}\right)$  as vectors.

The drawing is shown in Figure 47. The first vector, starting at the origin, is  $\text{cis}\frac{\pi}{4}$ . The second vector, starting at the endpoint of the first vector, is  $\text{cis}\frac{\pi}{2}$ . The origin, along with points  $w$  and  $z$ , form an isosceles triangle. Furthermore, the apex of this triangle, at  $w$ , has a measure of  $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$  radians. Thus, the base angles of the isosceles triangle are

$$\frac{\pi - \frac{3\pi}{4}}{2} = \frac{\pi}{8}.$$

Adding this with  $\frac{\pi}{4}$  shows that the angle  $z$  forms with the  $x$ -axis is

$$\frac{\pi}{8} + \frac{\pi}{4} = \frac{3\pi}{8},$$

our desired angle to analyze. We wish to find the tangent of this angle, which is just  $\tan(\text{Arg } z)$ . But we know how to compute that!

$$\begin{aligned} \tan(\text{Arg } z) &= \frac{\text{Im}(z)}{\text{Re}(z)} = \frac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2} + 1}{1} \\ \tan\left(\frac{3\pi}{8}\right) &= \sqrt{2} + 1. \end{aligned}$$

12. Solve  $z^3 = 1$ , and show that its solutions under the operation of multiplication form a group, isomorphic to the rotation group of the equilateral triangle. Write a group table!

There's numerous ways to solve this, but let's use  $\text{cis}$  form as usual. Let  $z = r \text{cis } \theta$ . Then

$$\begin{aligned}
z^3 &= r^3 \operatorname{cis} 3\theta = 1 \\
\implies r &= 1 \\
\operatorname{cis} 3\theta &= 1 \\
\cos 3\theta &= 1 \\
3\theta &= 2\pi k \quad \text{For } k \in \mathbb{Z} \\
\theta &= \frac{2\pi k}{3} \implies \theta \in \left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right\} \\
\implies z &\in \left\{ \operatorname{cis} 0, \operatorname{cis} \frac{2\pi}{3}, \operatorname{cis} \frac{4\pi}{3} \right\}.
\end{aligned}$$

Under multiplication, these three values of  $z$  indeed form a group isomorphic to the rotation group of the equilateral triangle,  $C_3$ . In particular,  $\operatorname{cis} 0$  is the identity,  $\operatorname{cis} \frac{2\pi}{3}$  is a rotation by  $120^\circ$  counterclockwise, and  $\operatorname{cis} \frac{4\pi}{3}$  is a rotation by  $240^\circ$  counterclockwise. Let  $I = \operatorname{cis} 0$ ,  $r = \operatorname{cis} \frac{2\pi}{3}$ , and  $r^2 = \operatorname{cis} \frac{4\pi}{3}$ . Then, we have the following group table:

.	$I$	$r$	$r^2$
$I$	$I$	$r$	$r^2$
$r$	$r$	$r^2$	$I$
$r^2$	$r^2$	$I$	$r$

13.

- (a) Find multiplication groups of complex numbers which are isomorphic to the rotation groups for
- a non-square rectangle

Since this rotation group is just the identity and a rotation of  $180^\circ$ , we can just choose the group  $\{-1, 1\}$  under multiplication. 1 is the identity, and  $-1 = \operatorname{cis} 180^\circ$  is the rotation.

- a regular hexagon.

We follow in the footsteps of the equivalent problem for the equilateral triangle. We have elements

$$\{\operatorname{cis} 0, \operatorname{cis} 60^\circ, \operatorname{cis} 120^\circ, \operatorname{cis} 180^\circ, \operatorname{cis} 240^\circ, \operatorname{cis} 300^\circ\}.$$

As should be obvious, these are rotations of  $0^\circ$ ,  $60^\circ$ ,  $120^\circ$ ,  $180^\circ$ ,  $240^\circ$ , and  $300^\circ$  respectively. Under multiplication, this is isomorphic to the rotation group of the hexagon,  $C_6$ .

- (b) Make a table for each group.

- a non-square rectangle

Let  $I$  be the identity and  $r$  be the rotation of  $180^\circ$ .

.	$I$	$r$
$I$	$I$	$r$
$r$	$r$	$I$

- a regular hexagon.

Let  $I$  be the identity and  $r$  be the rotation of  $60^\circ$ .  $r^n$  is defined in the natural way, by raising  $\operatorname{cis} 60^\circ$  to the power  $n$ .

.	$I$	$r$	$r^2$	$r^3$	$r^4$	$r^5$
$I$	$I$	$r$	$r^2$	$r^3$	$r^4$	$r^5$
$r$	$r$	$r^2$	$r^3$	$r^4$	$r^5$	$I$
$r^2$	$r^2$	$r^3$	$r^4$	$r^5$	$I$	$r$
$r^3$	$r^3$	$r^4$	$r^5$	$I$	$r$	$r^2$
$r^4$	$r^4$	$r^5$	$I$	$r$	$r^2$	$r^3$
$r^5$	$r^5$	$I$	$r$	$r^2$	$r^3$	$r^4$

(c) Compare the regular hexagon's group to the dihedral group of the equilateral triangle,  $D_3$ . Consider: how are they the same? How are they different? Is the difference fundamental?

The two groups are not isomorphic, although they are the same size; the difference is fundamental. The hexagon's rotation group,  $C_6$ , has elements of periods  $\{1, 2, 3, 3, 6, 6\}$ , while  $D_3$  has elements of periods  $\{1, 2, 2, 2, 3, 3\}$ . They do share some subgroups however: the trivial subgroup of just the identity, and the subgroups generated by  $r^2$  and by  $r^3$  in  $C_6$ , which are  $C_3$  and  $C_2$  respectively.

14. Which of the following sets is a group under (i) addition and (ii) multiplication?

(a)  $\{0\}$

This is a group under (i) addition, since it has an identity 0, is closed, has 0 as 0's inverse, and  $0 + (0 + 0) = (0 + 0) + 0$ . It also is a group under (ii) multiplication, for the same reasons.

(b)  $\{1\}$

This is not a group under (i) addition, since  $1 + 1 = 2 \notin \{1\}$ . It is a group under multiplication, though, since  $1 \cdot 1 = 1$  and all other properties are satisfied.

(c)  $\{0, 1\}$

This is not a group under (i) addition, since  $1 + 1 = 2 \notin \{0, 1\}$ . It is also not a group under (ii) multiplication. 0 can't be the identity, since  $1 \cdot 0 = 0 \neq 1$ . 1 also can't be the identity, since then 0 has no inverse  $K$  such that  $0 \cdot K = 1$ .

(d)  $\{-1, 1\}$

This is not a group under (i) addition, since  $1 + (-1) = 0 \notin \{\pm 1\}$ . It is a group under (ii) multiplication, since it satisfies the group properties.

Identity: 1 is the identity  
 Associativity: Multiplication is associative anyways  
 Invertibility: Each element is their own inverse  
 Closure:  $(\pm 1)(\pm 1) \in \{\pm 1\}$

(e)  $\{1, -1, i, -i\}$

This is not a group under (i) addition, since the sum of any two of the elements takes you out of the set. It is a group under (ii) multiplication, however. One easy way to see this is that  $1 = \text{cis } 0$ ,  $-1 = \text{cis } \pi$ ,  $i = \text{cis } \frac{\pi}{2}$ , and  $-i = \text{cis } \frac{3\pi}{2}$ , which are all rotations of multiples of  $90^\circ$ . In particular, it is isomorphic to  $C_4$ , the rotation group of the square.

(f)  $\{\text{naturals}\}$

This is not a group under (i) addition, because it cannot satisfy invertibility. There is no element  $X \in \mathbb{N}$  such that  $1 + X = I = 0$ , for example. This is also not a group under (ii) multiplication for the same reason.

(g)  $\{\text{integers}\}$

This is a group under (i) addition, because all the group properties are satisfied. The inverse of an element  $n$  is just  $-n$ , addition is associative, the identity is 0, and the sum of two integers is another integer. It is not a group under (ii) multiplication, because no numbers except  $\pm 1$  have integer multiplicative inverses.

(h)  $\{\text{rationals}\}, \mathbb{Q}$

This is a group under (i) addition with identity element 0. The inverse of an element  $\frac{p}{q}$  is  $-\frac{p}{q}$ , addition is associative, and the sum of two rational numbers is another rational number. It is not a group under (ii) multiplication, because no number is the multiplicative inverse of 0.

(i)  $\{\mathbb{Q} \text{ without zero}\}$

This is no longer a group under (i) addition, since the identity element needs to be 0. It is now, however, a group under (ii) multiplication, because all numbers have their inverses. Multiplication is associative, the inverse of  $\frac{p}{q}$  is  $\frac{q}{p}$ , and the product of two rationals is another rational.

(j)  $\{\text{complex numbers}\}, \mathbb{C}$

This is a group under (i) addition with identity element 0. The inverse of an element  $z$  is  $-z$ , addition is associative, and the sum of two complex numbers is another complex number. This is not a group under (ii) multiplication, because again, no number is the multiplicative inverse of 0.

(k)  $\{\mathbb{C} \text{ without zero}\}$

This is no longer a group under (i) addition, since the identity element needs to be 0. It is now, however, a group under (ii) multiplication, because all numbers have their inverses. Multiplication is associative, the inverse of  $z$  is  $\frac{1}{z}$ , and the product of two complex numbers is another complex number.

15. For what values  $a$  and  $b$  does  $\{a + b\sqrt{2}\}$  generate a group under (a) addition and (b) multiplication?  
Look for several answers.

(a) addition

Let  $x = a + b\sqrt{2}$ . At some point, we must generate the additive inverse of  $x$ , which is  $-x$ . This would also generate the identity, since  $x + (-x) = 0$ , and all group criteria would be satisfied. As we add  $x$  to itself, all we can do is scale it by some positive integer factor. Thus,

$$-x = nx$$

for  $n \in \mathbb{Z}^+$ . But this is not true for any  $x \in \mathbb{C}$  except 0. Thus, the only solution is  $a = b = 0 \rightarrow x = 0$ , which generates the trivial group  $\{0\}$  under addition.

(a) multiplication

Again, let  $x = a + b\sqrt{2}$ . We can only generate elements of the form  $x^n$ , where  $n \in \mathbb{Z}^+$ . We must eventually generate the identity 1, so we have  $x^n = 1$ . For real  $x$ , the only solution is  $x = 0$ . For complex  $x$ , we need to have  $x^n = 1$ . Thus, let  $x = r \text{ cis } \theta$ , and proceed:

$$\begin{aligned} x^n &= r^n \text{ cis } n\theta = 1 \\ \implies r &= 1 \\ n\theta &= 2\pi k \\ \theta &= \frac{2\pi k}{n}. \end{aligned}$$

for  $k \in \mathbb{Z}$ . Thus, the general complex solutions are  $x = \text{cis } \frac{2\pi k}{n}$  for  $k, n \in \mathbb{Z}$ ,  $\gcd(k, n) = 1$  and  $n \neq 0$ .

16. Prove that  $(r_1 \text{ cis } \theta)(r_2 \text{ cis } \phi) = r_1 r_2 \text{ cis}(\theta + \phi)$  using brute force and the angle-sum trig identities for cos and sin. Do you prefer this method or the one on the previous page? Which method gives you a better understanding of why DeMoivre's works?

$$\begin{aligned} (r_1 \text{ cis } \theta)(r_2 \text{ cis } \phi) &= r_1 r_2 (\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) \\ &= r_1 r_2 (\cos \theta \cos \phi + i \cos \theta \sin \phi + i \sin \theta \cos \phi - \sin \theta \sin \phi) \\ &= r_1 r_2 ((\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\cos \theta \sin \phi + \sin \theta \cos \phi)) \\ &= r_1 r_2 (\cos(\theta + \phi) + i \sin(\theta + \phi)) \\ &= r_1 r_2 \text{ cis}(\theta + \phi) \end{aligned}$$

I actually prefer this because it's kind of satisfying, but the previous way gives a better understanding of the underlying mechanics probably.

17. Find an identity for  $\sin 3\theta$  as we have done for cos. Most of the work is already done for you!

We already know that

$$\cos 3\theta + i \sin 3\theta = \text{cis } 3\theta = (c^3 - 3c^2) + i(3c^2 s - s^3),$$

where  $c = \cos \theta$  and  $s = \sin \theta$ . Equating imaginary parts, we have

$$\begin{aligned}\sin 3\theta &= 3c^2 s - s^3 \\ &= 3 \cos^2 \theta \sin \theta - \sin^3 \theta.\end{aligned}$$

18. Your friend's textbook says  $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$ , different from our identity. Who's right?

Both are right. Our identity is

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta.$$

Remembering that  $\sin^2 \theta = 1 - \cos^2 \theta$ , we can pretty easily change the form:

$$\begin{aligned}\cos^3 \theta - 3 \cos \theta \sin^2 \theta &= \cos^3 \theta - 3 \cos \theta(1 - \cos^2 \theta) \\ &= \cos^3 \theta + 3 \cos^3 \theta - 3 \cos \theta \\ &= 4 \cos^3 \theta - 3 \cos \theta.\end{aligned}$$

19. Now you can finish the rest of the proof.

If you need context for this answer, check out the relevant textbook section.

(a) Draw  $a, b, c, d, m, n$  approximately for the quadrilateral on the previous page.

The quadrilateral is shown in Figure 48.

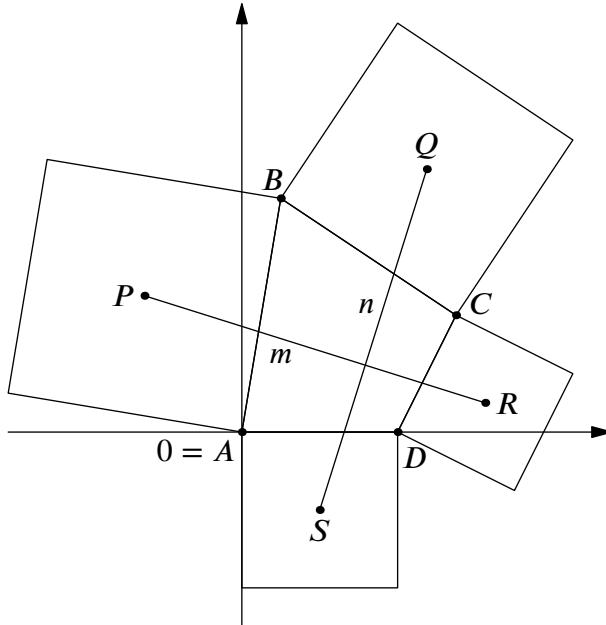


Figure 48: The quadrilateral to analyze.

The relative magnitudes and directions are shown in Figure 49 below. We find  $a, b, c, d$  from halving the sides of the quadrilateral.  $m$  and  $n$  are just the vectors from  $P$  to  $R$  and  $Q$  to  $S$ , respectively.

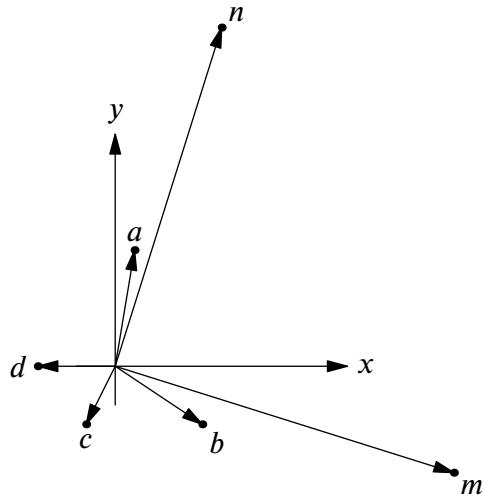


Figure 49: The relative magnitudes and directions of  $a, b, c, d, m, n$ .

(b) Why does showing  $n = \pm im$  prove the segments are (i) perpendicular and (ii) the same length?

They are (i) perpendicular because  $iz$  is perpendicular to  $z$  for all  $z \neq 0$ , and (ii) are the same length because  $|n| = |\pm im| = |im| = |m|$ .

(c) Explain why  $Q = 2a + b + ib$ .

The justification is geometric. We know that  $B = 2a$ , and we can get to the midpoint of  $\overline{BC}$  by adding  $b$ . Then, we go up to the center of the square on  $\overline{BC}$  by adding  $ib$ . This process is shown in Figure 50.

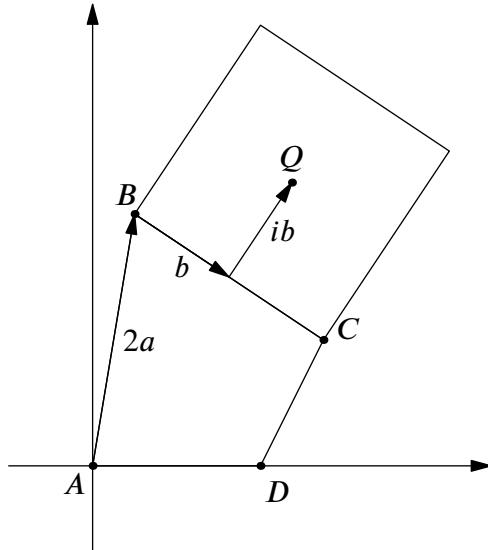


Figure 50:  $Q = 2a + b + ib$ .

(d) Find formulae for  $R$  and  $S$  in terms of  $c$  and  $d$ .

In a similar fashion, we have  $S = -d + id$  (note that it is  $-d$  because we are going counterclockwise now) and  $R = -2d - c + ic$ . The interpretations of these are shown in Figure 51 below.

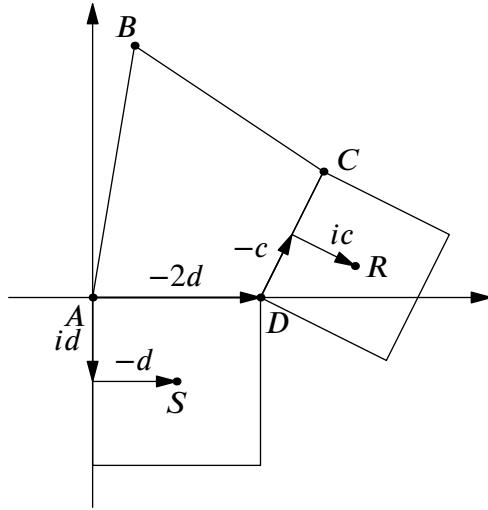


Figure 51:  $S = -d + id$  and  $R = -2d - c_i c$ .

(e) Find  $m$  and  $n$  in terms of  $a, b, c$ , and  $d$ .

We have  $m = R - P = (-2d - c + ic) - (a + ia)$  and  $n = Q - S = (2a + b + ib) - (-d + id)$ .

(f) Check that  $n - im = 0$ , using the fact that  $a + b + c + d = 0$ .

We evaluate straightforwardly:

$$\begin{aligned}
 (2a + b + ib + d - id) - i(-2d - c + ic - a - ia) &= 2a + b + ib + d - id + 2id + ic + c + ia - a \\
 &= a + b + c + d + ia + ib + ic + id \\
 &= (a + b + c + d)(1 + i) \\
 &= 0.
 \end{aligned}$$

20. In the previous problem, we drew squares outside a quadrilateral and connected their centers. Conjecture what happens if we draw equilateral triangles outside a triangle and connect their centers. Prove your conjecture using complex numbers.

We conjecture that following this construction leads to connecting together another equilateral triangle. An example, with the variables we'll use labeled, is shown in Figure 52.

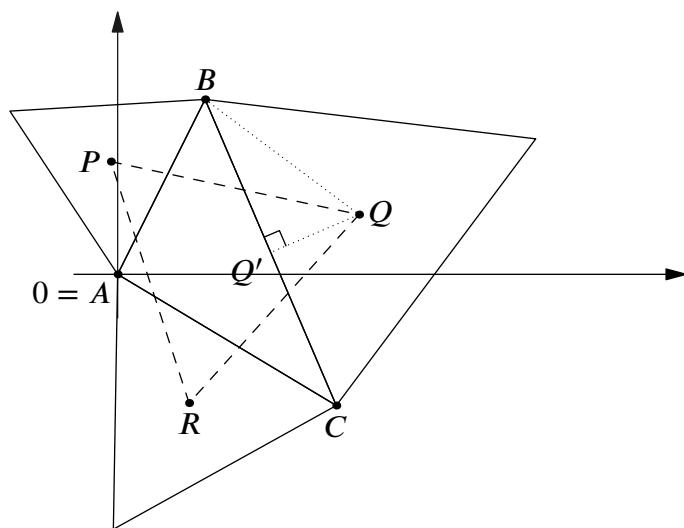


Figure 52: Equilateral triangles around a central, arbitrary triangle  $\triangle ABC$  with  $A$  at the origin.

Similar to the last problem, let  $A$ ,  $B$ , and  $C$  be numbers in the complex plane. Without loss of generality, let  $A = 0$  be the origin. Also, define  $a = \frac{B-A}{2}$ ,  $b = \frac{C-B}{2}$ , and  $c = \frac{A-C}{2}$  to be the vectors going halfway along each of  $\overrightarrow{AB}$ ,  $\overrightarrow{BC}$ , and  $\overrightarrow{CA}$ . Finally, let  $P$ ,  $Q$ , and  $R$  be the centers of the triangles on sides  $AB$ ,  $BC$ , and  $CA$  respectively.

Consider  $Q$  in the figure. It is on the  $60^\circ$  vertex of a 30-60-90 triangle  $\triangle BB'Q$ , outlined in dotted line. We know that  $\overline{BB'} = b$ . Thus, since  $BB' : B'Q = \sqrt{3} : 1$ ,

$$B'Q = \frac{|b|}{\sqrt{3}}.$$

Furthermore, since  $\overline{B'Q} \perp \overline{BB'}$ , we know that it is  $s \cdot ib$  for some real  $s$ . Combining these facts,

$$B'Q = \frac{ib}{|ib|} \frac{|b|}{\sqrt{3}} = \frac{ib}{\sqrt{3}}.$$

Since  $Q = \overrightarrow{AB} + \overrightarrow{BB'} + \overrightarrow{B'Q}$ , we have

$$Q = 2a + b + \frac{ib}{\sqrt{3}}.$$

With similar logic, we know that

$$\begin{aligned} P &= a + \frac{ia}{\sqrt{3}} \\ R &= -c + \frac{ic}{\sqrt{3}} \end{aligned}$$

Like with the quadrilateral, we know  $a + b + c = 0$ , since  $2(a + b + c) = 0$ . To prove the dashed triangle is indeed equilateral, we can just show that  $P - R = (Q - P)$  cis  $120^\circ$ . After all, if the vectors  $\overrightarrow{RP}$  and  $\overrightarrow{PQ}$  have an angle of  $120^\circ$  between them and they have the same magnitude,  $\triangle PQR$  is equilateral by SAS Congruence as shown in Figure 53. Substituting in our found values for  $P, Q, R$  in terms of  $a, b, c$ , we get

$$\begin{aligned}
P - R &= (Q - P) \text{ cis } 120^\circ \\
a + \frac{ia}{\sqrt{3}} - \left( -c + \frac{ic}{\sqrt{3}} \right) &= \left( 2a + b + \frac{ib}{\sqrt{3}} - \left( a + \frac{ia}{\sqrt{3}} \right) \right) \left( -\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \\
(a + c) + \frac{ia - ic}{\sqrt{3}} &= \left( a + b + \frac{ib - ia}{\sqrt{3}} \right) \left( -\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \\
&= -\frac{1}{2}a - \frac{1}{2}b - \frac{ib - ia}{2\sqrt{3}} + \frac{\sqrt{3}}{2}ia + \frac{\sqrt{3}}{2}ib + \frac{ib - ia}{2} \cdot i \\
&= \left( -\frac{1}{2}a - \frac{1}{2}b + \frac{ib - ia}{2} \cdot i \right) + \left( -\frac{ib - ia}{2\sqrt{3}} + \frac{\sqrt{3}}{2}ia + \frac{\sqrt{3}}{2}ib \right) \\
&= \left( -\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}b + \frac{1}{2}a \right) + \left( \frac{ia - ib + 3ia + 3ib}{2\sqrt{3}} \right) \\
&= (-b) + \left( \frac{4ia + 2ib}{2\sqrt{3}} \right) \\
&= (-b + a + b + c) + \left( \frac{i(2a + b)}{\sqrt{3}} \right) \\
&= (a + c) + \frac{i(2a + b - (a + b + c))}{\sqrt{3}} \\
(a + c) + \frac{ia - ic}{\sqrt{3}} &= (a + c) + \frac{ia - ic}{\sqrt{3}}
\end{aligned}$$

Tedious, but it worked.

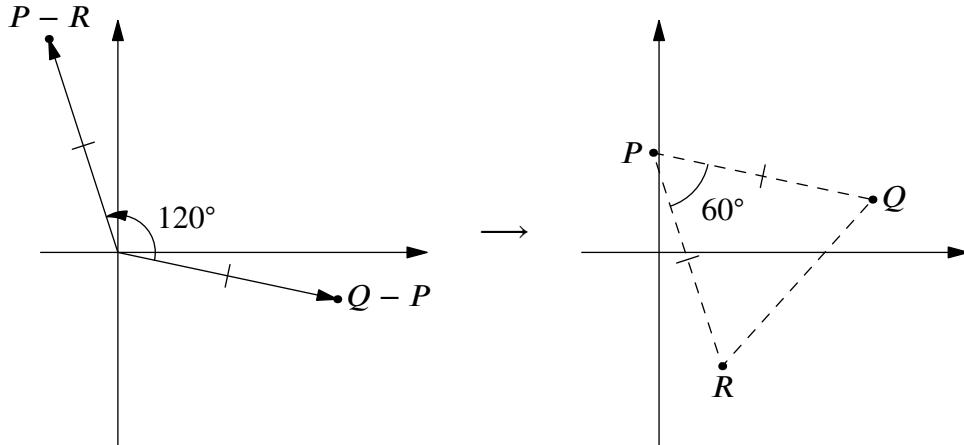


Figure 53: SAS Congruence lets us say that  $P - R = (Q - P) \text{ cis } 120^\circ$  is sufficient to prove the triangle  $\triangle PQR$  is equilateral.

21. The hard way to find an identity for  $\tan 3\theta$  is to divide the identity for  $\sin$  and  $\cos$  that we already found. Try this. Make sure your answer is in terms of  $\tan$  only!

We have found that  $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$  and  $\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$ . We set  $\tan \theta = \frac{\sin \theta}{\cos \theta}$  and evaluate:

$$\begin{aligned}
\tan \theta &= \frac{\sin \theta}{\cos \theta} = \frac{3 \cos^2 \theta \sin \theta - \sin^3 \theta}{\cos^3 \theta - 3 \cos \theta \sin^2 \theta} \\
&= \frac{\sin \theta}{\cos \theta} \cdot \frac{3 \cos^2 \theta - \sin^2 \theta}{\cos^2 \theta - 3 \sin^2 \theta} \cdot \frac{\frac{1}{\cos^2 \theta}}{\frac{1}{\cos^2 \theta}} \\
&= \tan \theta \cdot \frac{3 - \frac{\sin^2 \theta}{\cos^2 \theta}}{1 - \frac{3 \sin^2 \theta}{\cos^2 \theta}} \\
&= \tan \theta \cdot \frac{3 - \tan^2 \theta}{1 - 3 \tan^2 \theta} \\
&= \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}.
\end{aligned}$$

22. The easier way to get an identity for  $\tan 3\theta$  starts with setting  $z = 1 + i \tan \theta$ .

(a) Why is  $\operatorname{Arg} z = \theta$ ?

You can see this pretty quickly with a diagram, like in Figure 54. More algebraically, we have

$$\begin{aligned}
\tan(\operatorname{Arg} z) &= \operatorname{Im}(z) = \tan(\theta) \\
\implies \operatorname{Arg} z &= \theta.
\end{aligned}$$

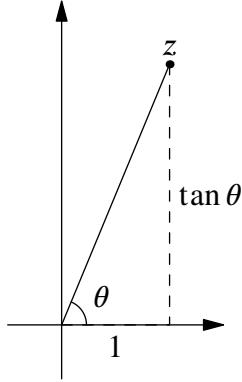


Figure 54:  $\operatorname{Arg} z = \tan^{-1} \left( \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right) = \theta$ .

(b) Why is  $\tan 3\theta = \frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)}$ ?

We have

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{\operatorname{Im}(\operatorname{cis} 3\theta)}{\operatorname{Re}(\operatorname{cis} 3\theta)}.$$

But since  $z$  makes an angle of  $\theta$  with the  $x$  axis, we can express it as  $r \operatorname{cis} \theta$  for some real  $r$ . Thus,

$$\frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)} = \frac{\operatorname{Im}(r^3 \operatorname{cis} 3\theta)}{\operatorname{Re}(r^3 \operatorname{cis} 3\theta)} = \frac{\operatorname{Im}(\operatorname{cis} 3\theta)}{\operatorname{Re}(\operatorname{cis} 3\theta)},$$

which matches the expression for  $\tan 3\theta$ .

(c) Use (b) to find an identity for  $\tan 3\theta$ .

We expand out  $z^3$  and factor into real and imaginary parts:

$$\begin{aligned}
z^3 &= (1 + i \tan \theta)^3 = 1^3 + 3i \tan \theta - 3 \tan^2 \theta - i \tan^3 \theta \\
&= (1 - 3 \tan^2 \theta) + i(3 \tan \theta - \tan^3 \theta).
\end{aligned}$$

Then we use our expression for  $\tan 3\theta$  in terms of  $z^3$ :

$$\begin{aligned}\tan 3\theta &= \frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)} \\ &= \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}.\end{aligned}$$

23. Find multiplication groups of complex numbers isomorphic to rotation groups for the

- (a) regular octagon

We choose complex numbers corresponding to rotations of  $0, 45^\circ, \dots, 315^\circ$ :

$$z = \left\{ \operatorname{cis} 0, \operatorname{cis} \frac{\pi}{4}, \operatorname{cis} \frac{\pi}{2}, \operatorname{cis} \frac{3\pi}{4}, \operatorname{cis} \pi, \operatorname{cis} \frac{5\pi}{4}, \operatorname{cis} \frac{3\pi}{2}, \operatorname{cis} \frac{7\pi}{4} \right\}.$$

- (b) regular pentagon.

We simply choose complex numbers corresponding to rotations of  $0, 72^\circ, \dots, 288^\circ$ :

$$z = \left\{ \operatorname{cis} 0, \operatorname{cis} \frac{2\pi}{5}, \operatorname{cis} \frac{4\pi}{5}, \operatorname{cis} \frac{6\pi}{5}, \operatorname{cis} \frac{8\pi}{5} \right\}.$$

24. Make tables for

- (a) the rotation group of the regular octagon.

There are 8 elements. If  $r$  is a rotation by  $45^\circ$ , then the elements are  $I, r, r^2, \dots, r^7$ . The group table is shown below.

.	$I$	$r$	$r^2$	$r^3$	$r^4$	$r^5$	$r^6$	$r^7$
$I$	$I$	$r$	$r^2$	$r^3$	$r^4$	$r^5$	$r^6$	$r^7$
$r$	$r$	$r^2$	$r^3$	$r^4$	$r^5$	$r^6$	$r^7$	$I$
$r^2$	$r^2$	$r^3$	$r^4$	$r^5$	$r^6$	$r^7$	$I$	$r$
$r^3$	$r^3$	$r^4$	$r^5$	$r^6$	$r^7$	$I$	$r$	$r^2$
$r^4$	$r^4$	$r^5$	$r^6$	$r^7$	$I$	$r$	$r^2$	$r^3$
$r^5$	$r^5$	$r^6$	$r^7$	$I$	$r$	$r^2$	$r^3$	$r^4$
$r^6$	$r^6$	$r^7$	$I$	$r$	$r^2$	$r^3$	$r^4$	$r^5$
$r^7$	$r^7$	$I$	$r$	$r^2$	$r^3$	$r^4$	$r^5$	$r^6$

- (b) the dihedral group of the square.

There are, once again,  $4 \cdot 2 = 8$  elements. Let  $r$  be a rotation by  $90^\circ$ , and  $f$  be a flip about say, the  $x$ -axis. The group table is shown below.

.	$I$	$r$	$r^2$	$r^3$	$f$	$fr$	$fr^2$	$fr^3$
$I$	$I$	$r$	$r^2$	$r^3$	$f$	$fr$	$fr^2$	$fr^3$
$r$	$r$	$r^2$	$r^3$	$I$	$fr^3$	$f$	$fr$	$fr^2$
$r^2$	$r^2$	$r^3$	$I$	$r$	$fr^2$	$fr^3$	$f$	$fr$
$r^3$	$r^3$	$I$	$r$	$r^2$	$fr$	$fr^2$	$fr^3$	$f$
$f$	$f$	$fr$	$fr^2$	$fr^3$	$I$	$r$	$r^2$	$r^3$
$fr$	$fr$	$fr^2$	$fr^3$	$f$	$r^3$	$I$	$r$	$r^2$
$fr^2$	$fr^2$	$fr^3$	$f$	$fr$	$r^2$	$r^3$	$I$	$r$
$fr^3$	$fr^3$	$f$	$fr$	$fr^2$	$r$	$r^2$	$r^3$	$I$

- (c) Is the difference between them fundamental?

Yes, the difference is fundamental, even though they have the same order. The easiest way to see this is that the latter group has 4 elements of order 2, but the former group has only 1 such element.

25. Which of the following tables defines a group? Why or why not?

\$	<i>I</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>I</i>	<i>I</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>A</i>	<i>A</i>	<i>C</i>	<i>D</i>	<i>B</i>	<i>I</i>
<i>B</i>	<i>B</i>	<i>I</i>	<i>C</i>	<i>D</i>	<i>A</i>
<i>C</i>	<i>C</i>	<i>D</i>	<i>A</i>	<i>I</i>	<i>B</i>
<i>D</i>	<i>D</i>	<i>B</i>	<i>I</i>	<i>A</i>	<i>C</i>

This table does not define a group, because it does not follow associativity. For example,  $(D\$A)\$A = B\$A = I$ , but  $D\$(A\$A) = D\$C = A$ .

#	<i>I</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>I</i>	<i>I</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>A</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>I</i>
<i>B</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>I</i>	<i>A</i>
<i>C</i>	<i>C</i>	<i>D</i>	<i>I</i>	<i>A</i>	<i>B</i>
<i>D</i>	<i>D</i>	<i>I</i>	<i>A</i>	<i>B</i>	<i>C</i>

This table is a group; in fact, it is a commutative group. The quickest way to see this is noting that it is (up to isomorphism) the cyclic group of order 5, where  $A = r$ ,  $B = r^2$ ,  $C = r^3$ , and  $D = r^4$ .

26. Which subsets of the complex numbers are groups under multiplication? After you list a few of each type, generalize.

Some simple examples:  $\{1\}$ ,  $\{\pm 1\}$ ,  $\{\pm 1, \pm i\}$ .

In general, we choose the  $n$ th roots of unity: the numbers of the form  $\text{cis} \frac{2\pi k}{n}$  for  $k \in \mathbb{Z}$ . Each rotation is a symmetry of the  $n$ -gon, and thus this set under multiplication is isomorphic to the cyclic group of order  $n$ .

27. Prove with a diagram that if  $|z| = 1$ , then  $\text{Im}\left(\frac{z}{(z+1)^2}\right) = 0$ .

To draw a diagram, we need to interpret these expressions as points on the complex plane.  $|z| = 1$  implies that  $z$  is 1 away from the origin.  $z + 1$  is  $z$ , translated right by 1 unit in the  $x$ -axis. Let  $z + 1 = r \text{ cis } \theta$ . Then  $(z + 1)^2 = r^2 \text{ cis } 2\theta$ , so it forms an angle of  $2\theta$  with the origin.

If the quotient  $\frac{z}{(z+1)^2}$  has no imaginary part, then  $(z + 1)^2$  is a real scalar times  $z$ . In other words, the two numbers have the same complex argument. Thus, we wish to prove that  $\text{Arg } z$  and  $\text{Arg}(z + 1)^2$  are equal.

The scenario is shown in Figure 55.

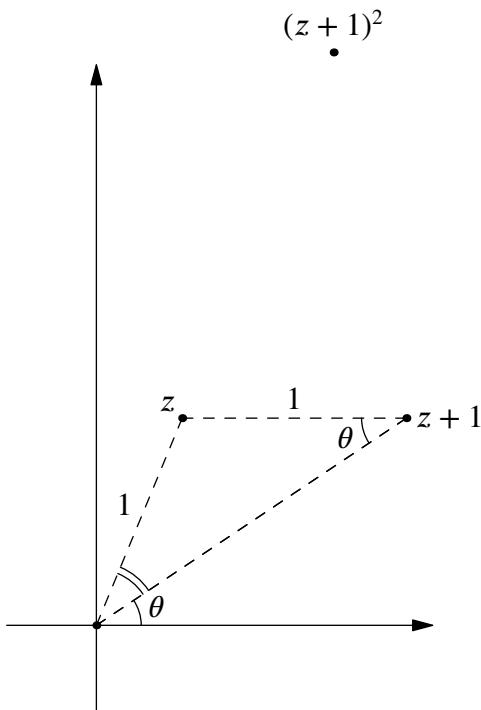


Figure 55: A graph of  $z$ ,  $z + 1$ , and  $(z + 1)^2$ .

As shown in the diagram,  $\text{Arg}(z + 1) = \theta$ . The triangle formed by  $O$ ,  $z$  and  $z + 1$  is isosceles, since it has two sides of length 1. Furthermore, it has a base angle of  $\theta$  by the Parallel Postulate. Thus, the angle marked with a double line is also  $\theta$ , and  $\text{Arg } z = 2\theta$ . But  $\text{Arg}(z + 1)^2 = 2\theta$ ! Thus, we have

$$\begin{aligned}\text{Im}\left(\frac{z}{(z+1)^2}\right) &= \text{Im}\left(\frac{r_1 \text{cis } 2\theta}{r_2 \text{cis } 2\theta}\right) \\ &= \text{Im}\left(\frac{r_1}{r_2} + 0i\right) \\ &= 0.\end{aligned}$$

This is not truly complete, because we have only considered  $z$  in the first quadrant. In this case, extending it to other locations of  $z$  is pretty trivial. Nonetheless, I provide an algebraic solution for fun.

We wish to show that  $(z + 1)^2 = kz$  for some real  $k$ . Express  $z$  as  $\text{cis } \theta$ . Then

$$(\text{cis } \theta + 1)^2 = \text{cis}^2 \theta + 2 \text{cis } \theta + 1.$$

Our supposed  $k$  is

$$\begin{aligned}k &= \frac{(z+1)^2}{z} = \frac{\text{cis}^2 \theta + 2 \text{cis } \theta + 1}{\text{cis } \theta} \\ &= \text{cis } \theta + 2 + \text{cis}(-\theta) \\ &= \text{cis } \theta + 2 + \overline{\text{cis } \theta} \\ &= 2 \text{Re}(\text{cis } \theta) + 2,\end{aligned}$$

which is indeed real. It's interesting what the scale factor actually is. Furthermore, since  $\text{Re}(\text{cis } \theta) = \cos \theta$ , we have a polar equation

$$|kz| = r = 2 \cos \theta + 2,$$

hinting that the path traced out by  $(z + 1)^2$  is in fact a cardioid! The cardioid produced is shown in Figure 56 below.

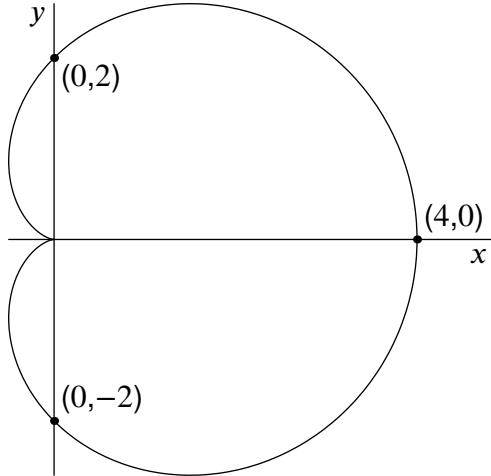


Figure 56: The cardioid produced by  $(z + 1)^2$  for  $|z| = 1$ .

It's all connected, guys.

28. Prove geometrically that if  $|z| = 1$ , then  $|1 - z| = \left|2 \sin\left(\frac{\text{Arg } z}{2}\right)\right|$ .

To prove this geometrically, we must again consider what the various expressions in the desired equation mean.  $|z| = 1$  means that  $z$  is distance 1 from the origin.  $1 - z$  is the reflection of  $z$  across the origin, then translated by 1 units right.

To geometrically interpret  $A = 2 \sin\left(\frac{\text{Arg } z}{2}\right)$ , we halve the angle  $\theta = \text{Arg } z$  and draw a vector of length 2; the imaginary component, or  $y$  height, of this new point is the desired quantity.

The diagram of all this is shown in Figure 57.

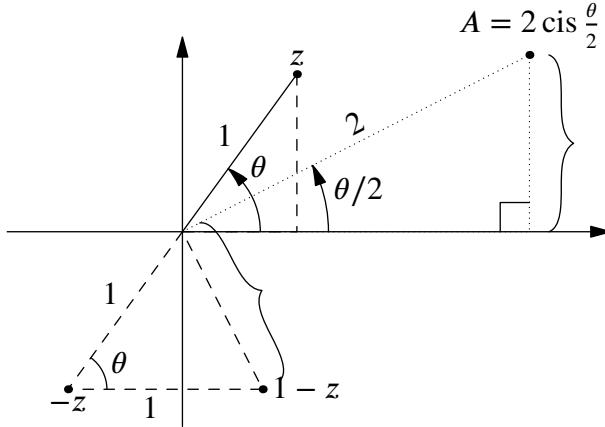


Figure 57: Graph of  $z$ ,  $1 - z$ , and  $\sin\left(\frac{\theta}{2}\right)$ .

We wish to show that the two lengths indicated in braces are equal. There's a couple of ways to do this; perhaps the most natural is to find a triangle congruent to the one formed by the origin,  $-z$ , and  $1 - z$ . This is the other dashed triangle shown in Figure 58.

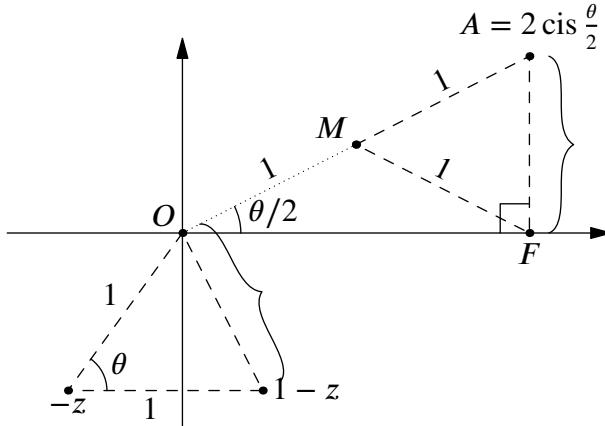


Figure 58: The succulent triangle,

Because the angles of a triangle sum to  $\pi$ , we know the angle  $\angle MAF$  is  $\pi - \frac{\theta}{2} - \frac{\pi}{2} = \frac{\pi-\theta}{2}$ . Furthermore,  $\triangle AMF$  is isosceles with an apex at  $M$ , since the midpoint of the hypotenuse of a right triangle is equidistant from all vertices. Thus,  $\angle MAF = \angle MFA$ , and we have

$$\angle AMF = \pi - \angle MAF - \angle MFA = \pi - \frac{\pi-\theta}{2} \cdot 2 = \theta.$$

Thus, by SAS Congruence, the two dashed triangles are congruent. Finally, pairing up the previously indicated sides, we have

$$|1 - z| = \overline{AF} = \left| \operatorname{Im}\left(2 \operatorname{cis} \frac{\theta}{2}\right) \right|,$$

as desired.

Technically, this proof is slightly incomplete, because some of these triangles do not exist as described for  $\theta \geq 90^\circ$ . You can extend it to these cases with no problem, but I'd also like to give a algebraic proof to show its perks.

By the half-angle identity,

$$\left| 2 \sin\left(\frac{\operatorname{Arg} z}{2}\right) \right| = \left| \pm 2 \sqrt{\frac{1 - \cos \operatorname{Arg} z}{2}} \right| = \sqrt{2(1 - \cos \operatorname{Arg} z)}.$$

Let  $z = a + bi = \operatorname{cis} \theta$ ; note that  $r = 1$  since  $|z| = 1$ . We know that  $\cos \operatorname{Arg} z = a$ . Then

$$\begin{aligned}
|1 - (a + bi)| &= |(1 - a) - bi| \\
&= \sqrt{b^2 + (1 - a)^2} \\
&= \sqrt{1 - a^2 + (1 - 2a + a^2)} \\
&= \sqrt{2 - 2a} \\
&= \sqrt{2(1 - a)} \\
&= \sqrt{2(1 - \cos \operatorname{Arg} z)}.
\end{aligned}$$

This matches our expression using half-angle for  $\left|2 \sin\left(\frac{\operatorname{Arg} z}{2}\right)\right|$ .

Personally, I strongly prefer the algebraic solution because it is quick, easy to understand, and truly complete. Nonetheless, the geometric solution gives a better idea of *why* the equation is true.

29.

- (a) Prove that if  $(z - 1)^{10} = z^{10}$ , then  $\operatorname{Re}(z) = \frac{1}{2}$ . (Hint: if two numbers are equal, they have the same magnitude.)

We do as the hint suggests. We know that  $|(z - 1)^{10}| = |z^{10}|$ . Expanding this out would be rough, but we can take the exponents out of the inside of the magnitude symbols<sup>16</sup>.

So  $|z - 1|^{10} = |z|^{10}$ . Since  $|n| \geq 0$ , we have  $|z - 1| = |z|$ .

Let  $z = a + bi$ . Then  $|a + bi - 1| = \sqrt{(a - 1)^2 + b^2}$  and  $|a + bi| = \sqrt{a^2 + b^2}$ . We set these equal and solve:

$$\begin{aligned}
\sqrt{(a - 1)^2 + b^2} &= \sqrt{a^2 + b^2} \\
(a - 1)^2 + b^2 &= a^2 + b^2 \\
(a - 1)^2 &= a^2 \\
a - 1 &= \pm a.
\end{aligned}$$

If  $a - 1 = a$ , then  $-1 = 0$ , which is dumb. Thus,  $a - 1 = -a$ , so  $a = \frac{1}{2}$  and indeed,  $\operatorname{Re}(z) = \frac{1}{2}$  as desired.

- (b) How many solutions does this equation have?

We have  $(z - 1)^{10} = z^{10}$ , so  $(z - 1)^{10} - z^{10} = P(z) = 0$ , where  $P$  is a polynomial of degree 9. Thus, by the Fundamental Theorem of Algebra, there are 9 solutions... if there aren't any repeated roots. So this is only truly complete if we know there are no roots which appear in the factorization twice or more. Unfortunately, I can't think of a way to do this without calculus<sup>17</sup>.

Let's start over. We should use the fact that  $\operatorname{Re}(z) = \frac{1}{2}$ . A simple diagram reveals that  $z$  and  $z - 1$  are symmetric about the  $y$ -axis, since  $\operatorname{Re}(z) = \operatorname{Re}\left(\frac{1}{2} + bi\right) = -\operatorname{Re}\left(\frac{1}{2} + bi - 1\right)$ . The diagram is shown in Figure 59.

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<sup>16</sup>This is true because  $|(r \operatorname{cis} \theta)^n| = |r^n \operatorname{cis} n\theta| = |r^n|$ .

<sup>17</sup>For that route, we simply check that  $P''(z) = 0$  for all solutions, which isn't pleasant until a clever rearrangement and substitution. Try it if you know how!

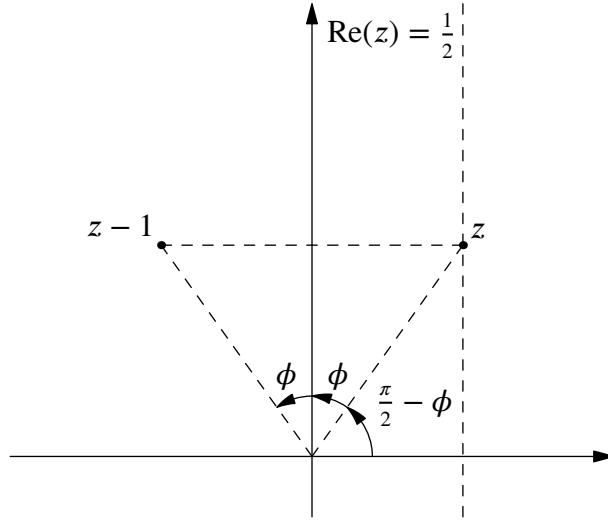


Figure 59:  $z$  and  $z - 1$ , residents of the complex plane.

Let  $z$  in the first quadrant make a angle  $\phi$  to the  $\pm y$ -axis. Note that we're using the  $y$ -axis, not the  $x$ -axis, for mathematical convenience. In general, for  $z$  in the first and fourth quadrants<sup>18</sup>, we have  $\text{Arg } z = \frac{\pi}{2} - \phi$  and  $\text{Arg}(z - 1) = \frac{\pi}{2} + \phi$ . Since  $|z| = |z - 1| = r$ , we have

$$z = r \text{cis} \left( \frac{\pi}{2} - \phi \right); z - 1 = r \text{cis} \left( \frac{\pi}{2} + \phi \right).$$

Since  $(z - 1)^{10} = z^{10}$ , we have

$$\begin{aligned} \left( r \text{cis} \left( \frac{\pi}{2} - \phi \right) \right)^{10} &= \left( r \text{cis} \left( \frac{\pi}{2} + \phi \right) \right)^{10} \\ r^{10} \text{cis}(5\pi - 10\phi) &= r^{10} \text{cis}(5\pi + 10\phi) \\ \text{cis}(5\pi - 10\phi) &= \text{cis}(5\pi + 10\phi) \\ 5\pi - 10\phi + 2\pi k &= 5\pi + 10\phi \quad \text{For some } k \in \mathbb{Z} \\ 20\phi &= 2\pi k \\ \phi &= \frac{2\pi k}{20}. \end{aligned}$$

To find all unique solutions, we restrict  $k$  to the range  $0 \leq k \leq 19$ ... wait, isn't that 20 solutions?

The issue is that  $z$  must be in the first or fourth quadrant, so that our premise  $|z| = |z - 1|$  is true. That means  $0 < \phi < \pi$ , a strict relation because  $\phi = 0$  or  $\phi = \pi$  only gives values along the  $y$ -axis, which does not intersect with  $\text{Re}(z) = 0$ . Solving this gives

$$\begin{aligned} 0 &< \frac{2\pi k}{20} < \pi \\ 0 &< \pi k < 10\pi \\ 0 &< k < 10 \\ k &\in \{1, 2, \dots, 8, 9\}, \end{aligned}$$

which is 9 solutions, in agreement with our polynomial argument.

30. I claim that  $e^{i\theta} = \cos \theta + i \sin \theta = \text{cis } \theta$ , for  $\theta$  in radians.

(a) Find  $e^{-it}$ .

$$e^{-it} = \cos(-t) + i \sin(-t) = \cos t - i \sin t.$$

(b) Find  $\frac{e^{i\theta} + e^{-i\theta}}{2}$ .

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<sup>18</sup>If  $z$  is in the fourth quadrant, then you'd define  $\phi$  as  $\pi + \text{angle to negative y-axis.}$ , where the angle is taken clockwise so it's positive.

$$\begin{aligned}\frac{e^{i\theta} + e^{-i\theta}}{2} &= \frac{\cos \theta + i \sin \theta + \cos \theta - i \sin \theta}{2} \\ &= \cos \theta.\end{aligned}$$

(c) Find  $\frac{e^{i\theta} - e^{-i\theta}}{2i}$ .

$$\begin{aligned}\frac{e^{i\theta} - e^{-i\theta}}{2i} &= \frac{\cos \theta + i \sin \theta - (\cos \theta - i \sin \theta)}{2i} \\ &= \sin \theta.\end{aligned}$$

30. (cont.) Use your new, complex definitions for cos and sin to find:

(d)  $\cos^2 \theta + \sin^2 \theta$

$$\begin{aligned}\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 + \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 &= \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 - \left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)^2 \\ &= \left(\frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i\theta} - e^{-i\theta}}{2}\right) \left(\frac{e^{i\theta} + e^{-i\theta}}{2} - \frac{e^{i\theta} - e^{-i\theta}}{2}\right) \\ &= (e^{-\theta})(e^{i\theta}) \\ &= e^{-\theta+i\theta} \\ &= e^0 \\ &= 1.\end{aligned}$$

That was expected.

(e)  $\tan \theta$

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} = \frac{\frac{e^{i\theta} - e^{-i\theta}}{2i}}{\frac{e^{i\theta} + e^{-i\theta}}{2}} \\ &= \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}.\end{aligned}$$

(f)  $\cos 2\theta$

$$\cos 2\theta = \frac{e^{2i\theta} + e^{-2i\theta}}{2}.$$

(g)  $\sin 2\theta$

$$\sin 2\theta = \frac{e^{2i\theta} - e^{-2i\theta}}{2i}.$$

(h) What kind of group is generated by  $\{e^{i\theta}, e^{-i\theta}\}$  under the operation of multiplication if  $\theta$  is an integer? A rational multiple of  $\pi$ ?

If  $\theta = 0$ , then the group is the trivial group of order 1. If  $\theta$  is any other integer, then a group isomorphic to the additive group of the integers is generated. We correspond  $e^{ik\theta}$  with the integer  $k$ , so that

$$e^{ik_1\theta} \cdot e^{ik_2\theta} = e^{i(k_1+k_2)\theta} \leftrightarrow k_1 + k_2.$$

If  $\theta$  is a rational multiple of  $\pi$ , say  $\frac{p}{q} \cdot 2\pi$  where  $\gcd(p, q) = 1$ , then we get (up to isomorphism) cyclic group of order  $q$ .

31. You've used the quadratic equation throughout high school, but there's also a cubic equation that finds the roots of any cubic. Let's derive it, starting with the cubic  $x^3 + bx^2 + cx + d = 0$ .

- (a) Make the substitution  $x = y - \frac{b}{3}$ . Combine like terms to create an equation of the form  $y^3 - 3py - 2q = 0$ , with  $p, q$  in terms of  $b, c$ , and  $d$ .

$$\begin{aligned} & \left(y - \frac{b}{3}\right)^3 + b\left(y - \frac{b}{3}\right)^2 + c\left(y - \frac{b}{3}\right) + d = 0 \\ & \left(y^3 - 3 \cdot \frac{by^2}{3} + 3 \cdot \frac{b^2y}{9} - \frac{b^3}{27}\right) + \left(by^2 - \frac{2b^2y}{3} + \frac{b^3}{9}\right) + \left(cy - \frac{bc}{3}\right) + d = 0 \\ & y^3 + (-b + b)y^2 + \left(\frac{b^2y}{3} - \frac{2b^2y}{3} + c\right)y + \left(-\frac{b^3}{27} + \frac{b^3}{9} - \frac{bc}{3} + d\right) = 0 \\ & y^3 + \left(c - \frac{b^2}{3}\right)y + \left(d - \frac{bc}{3} + \frac{2b^3}{27}\right) = 0 \\ & y^3 - 3 \underbrace{\left(\frac{b^2}{9} - \frac{c}{3}\right)}_p y - 2 \underbrace{\left(\frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}\right)}_q = 0. \end{aligned}$$

Thus,  $p = \frac{b^2}{9} - \frac{c}{3}$  and  $q = \frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}$ .

- (b) Rearrange this equation as  $y^3 = 3py + 2q$ .

$$y^3 = 3 \left(\frac{b^2}{9} - \frac{c}{3}\right)y + 2 \left(\frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}\right).$$

- (c) Make the substitution  $y = s + t$  into (b), and prove that  $y$  is a solution of the cubic in part (a) if  $st = p$  and  $s^3 + t^3 = 2q$ .

We substitute  $y = s + t$  and use the fact that  $st = p$  and  $s^3 + t^3 = 2q$  to simplify.

$$\begin{aligned} (s+t)^3 &= 3p(s+t) + 2q \\ s^3 + 3s^2t + 3st^2 + t^3 &= 3ps + 3pt + 2q \\ 3s(st) + 3t(st) + s^3 + t^3 &= 3ps + 3pt + 2q \\ 3sp + 3tp + 2q &= 3ps + 3pt + 2q \\ 0 &= 0. \end{aligned}$$

This checks out.

- (d) Eliminate  $t$  between these two equations to get a quadratic in  $s^3$ .

We have  $t^3 = 2q - s^3$ . Also,  $(st)^3 = p^3$ , so

$$\begin{aligned} (st)^3 &= s^3t^3 = s^3(2q - s^3) = p^3 \\ -(s^3)^2 + 2qs^3 - p^3 &= 0 \\ (s^3)^2 - 2qs^3 + p^3 &= 0. \end{aligned}$$

- (e) Solve this quadratic to find  $s^3$ . By symmetry, what is  $t^3$ ?

Let  $w = s^3$ . Then the above quadratic is  $w^2 - 2qw + p^3 = 0$ . The solutions are

$$s^3 = w = \frac{2q \pm \sqrt{4q^2 - 4p^3}}{2} = \frac{2q \pm 2\sqrt{q^2 - p^3}}{2} = q \pm \sqrt{q^2 - p^3}.$$

We have  $t^3 = 2q - s^3 = 2q - (q \pm \sqrt{q^2 - p^3}) = q \mp \sqrt{q^2 - p^3}$ . This inverted  $\pm$  sign,  $\mp$ , means that when  $s^3$ 's  $\pm$  sign is positive, the  $\mp$  sign is negative, and vice versa.

(f) Find a formula for  $y$  in terms of  $p$  and  $q$ . What about a formula for  $x$ ?

Taking cube roots of both sides of our expressions for  $t^3$  and  $s^3$ , we find that

$$s = \sqrt[3]{q \pm \sqrt{q^2 - p^3}},$$

$$t = \sqrt[3]{q \mp \sqrt{q^2 - p^3}}.$$

We must keep in mind, however, that over the complex numbers, cube rooting has 3 possible values. Thus, the three solutions for  $s$  and  $t$  are

$$s = \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \text{cis} \frac{2\pi k}{3},$$

$$t = \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \text{cis} \left(2\pi - \frac{2\pi k}{3}\right),$$

where  $k \in \{0, 1, 2\}$  and the cube root is taking its principal value. We multiply them by cis with these angles to preserve the fact that  $st = q$ , since otherwise it would produce another result:

$$\begin{aligned} \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \text{cis} \frac{2\pi k}{3} \cdot \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \text{cis} \left(2\pi - \frac{2\pi k}{3}\right) &= \sqrt[3]{(q \pm \sqrt{q^2 - p^3})(q \mp \sqrt{q^2 - p^3})} \cdot \text{cis} 2\pi \\ &= \sqrt[3]{q^2 - (q^2 + p^3)} \\ &= p. \end{aligned}$$

Thus, we have

$$y = s + t = \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \text{cis} \frac{2\pi k}{3} + \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \text{cis} \left(2\pi - \frac{2\pi k}{3}\right).$$

To get  $x$ , we substitute  $x = y - \frac{b}{3}$  to get

$$x = \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \text{cis} \frac{2\pi k}{3} + \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \text{cis} \left(2\pi - \frac{2\pi k}{3}\right) - \frac{b}{3}.$$

You could substitute our values of  $p, q$  in terms of  $b, c, d$  to get a monstrous equation for  $x$  in terms of only  $b, c, d \dots$  but no thanks.

(g) What if we started with  $ax^3 + bx^2 + cx + d = 0$ , with a coefficient in front of the  $x^3$  term as well? Can you come up with a formula for  $x$ ?

Sure! We divide through the equation by  $a$ :

$$\begin{aligned} \frac{ax^3 + bx^2 + cx + d}{a} &= 0 \\ \implies x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} &= 0. \end{aligned}$$

We can then attack this as we already did, setting  $b' = \frac{b}{a}$ ,  $c' = \frac{c}{a}$  and  $d' = \frac{d}{a}$ , then applying the formula.

32. Starting with the same cubic as in problem 31b.

(a) Let  $c = \cos \theta$ . Remember that  $\cos 3\theta = 4c^3 - 3c$ , as we proved. Substitute  $y = 2c\sqrt[3]{p}$  into  $y^3 = 3py + 2q$  to obtain  $4c^3 - 3c = \frac{q}{p^{3/2}}$ .

We substitute and proceed:

$$\begin{aligned}
y^3 &= 3py + 2q \\
(2c\sqrt{p})^3 &= 3p(2c\sqrt{p}) + 2q \\
8c^3 p^{3/2} &= 6cp^{3/2} + 2q \\
8c^3 p^{3/2} - 6cp^{3/2} &= 2q \\
4c^3 - 3c &= \frac{2q}{2p^{3/2}} = \frac{q}{p^{3/2}}.
\end{aligned}$$

- (b) Provided that  $q^2 \leq p^3$ , show that  $y = 2\sqrt{p} \cos\left(\theta + \frac{2\pi n}{3}\right)$ , where  $n$  is an integer. Why does this yield all three solutions?

This isn't actually hard. We know that  $\cos 3\theta = 4c^3 - 3c = \frac{q}{p^{3/2}}$ , so there are three possible values for  $\cos \theta = c$ . Namely, if  $\theta_0 = \cos^{-1} \frac{q}{p^{3/2}}$  is the principal value, then we also have unique solutions

$$\theta_1 = \frac{2\pi}{3} + \theta_0, \quad \theta_2 = \frac{4\pi}{3} + \theta_0,$$

because multiplying these by 3 to get  $\theta$  just adds a multiple of  $2\pi$ . Indeed, we have

$$c = \cos\left(\theta + \frac{2\pi n}{3}\right)$$

as a solution for any integer  $n$ . Substituting into the expression for  $y$ , we get

$$y = 2c\sqrt{p} = 2 \cos\left(\theta + \frac{2\pi n}{3}\right) \sqrt{p},$$

as desired. This yields all three solutions because, as we observed, the only unique values this makes are for  $n \in \{0, 1, 2\}$ .

Note that if  $q^2 > p^3$ , then this strategy actually still works, but you have to define  $\cos$  (and  $\cos^{-1}$ ) over a larger, complex domain. This is certainly possible though!

- (c) Explain how you would find  $\theta$  from  $p$  and  $q$ , and how we would use what we have found to solve an arbitrary cubic  $ax^3 + bx^2 + cx + d = 0$ .

We have  $\cos 3\theta = \frac{q}{p^{3/2}}$ , so

$$\theta = \cos^{-1} \frac{q}{p^{3/2}}.$$

The steps to solving an arbitrary cubic are the following:

1. Divide through by  $a$  to get a new cubic  $x^3 + b'x^2 + c'x + d' = 0$ .
2. Compute  $p = \frac{b'^2}{9} - \frac{c'}{3}$  and  $q = \frac{b'c'}{6} - \frac{b'^3}{27} - \frac{d'}{2}$ .
3. Compute  $\theta = \cos^{-1} \frac{q}{p^{3/2}}$ .
4. Substitute this value of  $\theta$  into  $x = y - \frac{b}{3} = 2\sqrt{p} \cos\left(\frac{1}{3}(\theta + 2\pi n)\right) - \frac{b}{3}$ , where  $n \in \{0, 1, 2\}$ .

These two problems were a doozy!

## 7 Your Daily Dose of Vitamin $i$

1. We will use complex numbers to find identities for  $\cot$ . Use Pascal's triangle to expand the following:

$$(a+b)^3$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a+b)^4$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

$$(a+b)^5$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

1. (cont.) Then substitute  $b = i = \sqrt{-1}$  and expand:

$$(d) (a+i)^3$$

$$(a+i)^3 = a^3 + 3a^2i + 3ai^2 + i^3 = a^3 + 3a^2i - 3a - i.$$

$$(e) (a+i)^4$$

$$(a+i)^4 = a^4 + 4a^3i + 6a^2i^2 + 4ai^3 + i^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1.$$

$$(f) (a+i)^5$$

$$(a+i)^5 = a^5 + 5a^4i + 10a^3i^2 + 10a^2i^3 + 5ai^4 + i^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i.$$

1. (cont.) Finally, substitute  $a = \cot \theta$  and expand:

$$(g) (\cot \theta + i)^3$$

$$(\cot \theta + i)^3 = a^3 + 3a^2i - 3a - i = (\cot^3 \theta - 3 \cot \theta) + i(3 \cot^2 \theta - 1).$$

$$(h) (\cot \theta + i)^4$$

$$(\cot \theta + i)^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1 = (\cot^4 \theta - 6 \cot^2 \theta + 1) + (4 \cot^3 \theta - 4 \cot \theta).$$

$$(i) (\cot \theta + i)^5$$

$$(\cot \theta + i)^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i = (\cot^5 \theta - 10 \cot^3 \theta + 5 \cot \theta) + i(5 \cot^4 \theta - 10 \cot^2 \theta + 1).$$

1. (cont.) Consider  $z = i + \cot \theta$ .

(j) Use the above results to find identities for (i)  $\cot 3\theta$ , (ii)  $\cot 4\theta$ , and (iii)  $\cot 5\theta$ .

i.  $\cot 3\theta$

Given the right triangle formed by  $z = i + \cot \theta$  in Figure 7, we have  $\tan(\text{Arg } z) = \frac{1}{\cot \theta} = \tan \theta$ , so  $\text{Arg } z = \theta$  and  $z = r \text{ cis } \theta$ .

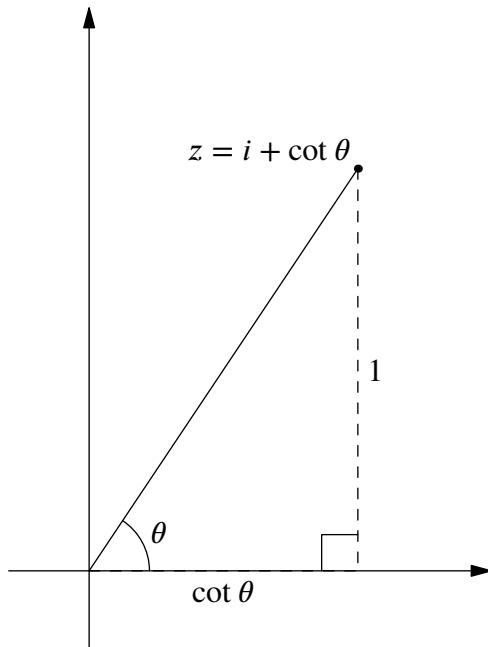


Figure 60:  $\text{Arg}(i + \cot \theta) = \theta$ .

Thus, we have

$$\begin{aligned}\cot 3\theta &= \frac{\cos 3\theta}{\sin 3\theta} \\ &= \frac{\text{Re}(\text{cis } 3\theta)}{\text{Im}(\text{cis } 3\theta)} \\ &= \frac{\text{Re}(r^3 \text{ cis } 3\theta)}{\text{Im}(r^3 \text{ cis } 3\theta)} \\ &= \frac{\text{Re}(z^3)}{\text{Im}(z^3)}.\end{aligned}$$

We substitute in our expression for  $z^3$ ,  $(\cot^3 \theta - 3 \cot \theta) + i(3 \cot^2 \theta - 1)$ :

$$\cot 3\theta = \frac{\cot^3 \theta - 3 \cot \theta}{3 \cot^2 \theta - 1}.$$

i.  $\cot 4\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}\cot 4\theta &= \frac{\cos 4\theta}{\sin 4\theta} \\ &= \frac{\text{Re}(\text{cis } 4\theta)}{\text{Im}(\text{cis } 4\theta)} \\ &= \frac{\text{Re}(r^4 \text{ cis } 4\theta)}{\text{Im}(r^4 \text{ cis } 4\theta)} \\ &= \frac{\text{Re}(z^4)}{\text{Im}(z^4)} \\ \cot 4\theta &= \frac{\cot^4 \theta - 6 \cot^2 \theta + 1}{4 \cot^3 \theta - 4 \cot \theta}.\end{aligned}$$

i.  $\cot 5\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}\cot 5\theta &= \frac{\cos 5\theta}{\sin 5\theta} \\ &= \frac{\text{Re}(\text{cis } 5\theta)}{\text{Im}(\text{cis } 5\theta)} \\ &= \frac{\text{Re}(r^5 \text{ cis } 5\theta)}{\text{Im}(r^5 \text{ cis } 5\theta)} \\ &= \frac{\text{Re}(z^5)}{\text{Im}(z^5)} \\ \cot 5\theta &= \frac{\cot^5 \theta - 10 \cot^3 \theta + 5 \cot \theta}{5 \cot^4 \theta - 10 \cot^2 \theta + 1}.\end{aligned}$$

(k) Graph  $z$ ,  $z^2$ ,  $z^3$ ,  $z^4$ , and  $z^5$ , with  $\theta \approx 75^\circ$ . What is your solution method?

To graph these, I first calculated the approximate magnitude of  $z$ , which is how many times each subsequent power will be scaled by. We have  $|1 + \cot 75^\circ| \approx 1.268$ , so we only need to scale by about  $\frac{5}{4}$  each time. Of course, we rotate by about  $75^\circ$  each time.

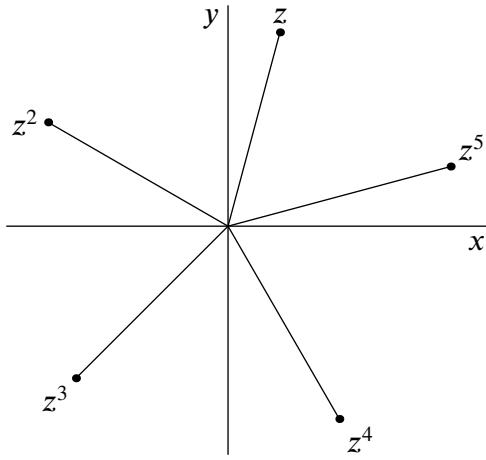


Figure 61: Graphs of  $z$ ,  $z^2$ ,  $z^3$ ,  $z^4$ , and  $z^5$ .

2. Compute  $(1+i)^n$  for  $n = 3, 4, 5, \dots$ . Can you find a general pattern?

We have

$$\begin{aligned} (1+i)^3 &= 1^3 + 3i - 3 - i &= -2 - 2i \\ (1+i)^4 &= 1^4 + 4i - 6 - 6i + 1 &= -4 - 2i \\ (1+i)^5 &= 1^5 + 5i - 10 - 10i + 5 + i &= -4 - 4i. \end{aligned}$$

We can find the pattern by representing  $1+i = \sqrt{2} \operatorname{cis} 45^\circ$ . This shows that it has period 8 and let's us find an expression for  $(1+i)^n$ :

$$(1+i)^n = \left( \sqrt{2} \operatorname{cis} 45^\circ \right)^n = 2^{n/2} \operatorname{cis} \left( \frac{n\pi}{4} \right).$$

3. Expand and graph  $\operatorname{cis}^n \theta$  for  $n = 2, 3, 4, \dots$

Let  $\cos \theta = c$  and  $\sin \theta = s$ . We have

$$\begin{aligned} (c+is)^2 &= c^2 + 2csi - s^2 = (c^2 - s^2) + i(2cs) \\ (c+is)^3 &= c^3 + 3c^2si - 3cs^2 - s^3i = (c^3 - 3cs^2) + i(3c^2s - s^3) \\ (c+is)^4 &= c^4 + 4c^3si - 6c^2s^2 - 4cs^3i + s^4 = (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3) \\ (c+is)^5 &= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i = (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5). \end{aligned}$$

The graphs of  $\operatorname{cis}^n \theta$  for  $\theta \approx 50^\circ$  are shown in Figure 62 below.

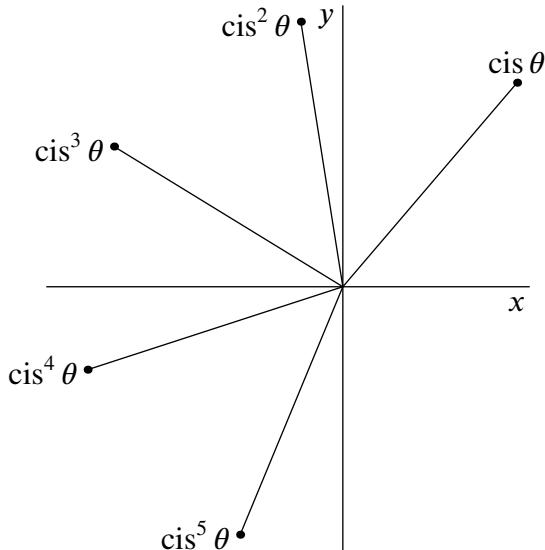


Figure 62: Graphs of  $\text{cis}^n \theta$  for  $\theta \approx 50^\circ$ .

- (a) Why is the real part  $\cos n\theta$  and the imaginary part  $\sin n\theta$ ?

By DeMoivre's theorem,  $\text{cis}^n \theta = \text{cis } n\theta$ , which by definition has  $\text{Im}(\text{cis } n\theta) = \cos n\theta$  and  $\text{Re}(\text{cis } n\theta) = \sin n\theta$ .

- (b) Use your results to write identities for  $\cos n\theta$  and  $\sin n\theta$  for  $n = 2, 3, 4, 5$ .

Here they are. Again, let  $\cos \theta = c$  and  $\sin \theta = s$ :

$$\begin{aligned}\cos 2\theta &= \text{Re}(\text{cis } 2\theta) = c^2 - s^2 \\ \cos 3\theta &= \text{Re}(\text{cis } 3\theta) = c^3 - 3cs^2 \\ \cos 4\theta &= \text{Re}(\text{cis } 4\theta) = c^4 - 6c^2s^2 + s^4 \\ \cos 5\theta &= \text{Re}(\text{cis } 5\theta) = c^5 - 10c^3s^2 + 5cs^4 \\ \sin 2\theta &= \text{Im}(\text{cis } 2\theta) = 2cs \\ \sin 3\theta &= \text{Im}(\text{cis } 3\theta) = 3c^2s - s^3 \\ \sin 4\theta &= \text{Im}(\text{cis } 4\theta) = 4c^3s - 4cs^3 \\ \sin 5\theta &= \text{Im}(\text{cis } 5\theta) = 5c^4s - 10c^2s^3 + s^5.\end{aligned}$$

4. Compute  $\cos 7^\circ + \cos 79^\circ + \cos 151^\circ + \cos 223^\circ + \cos 295^\circ$  without a calculator. (Hint: what does this have to do with complex numbers?)

These numbers look random, but a closer inspection reveals they are in arithmetic progression, with starting term 7 and increasing  $72^\circ$  each time. That's the rotation of a pentagon!

We rewrite this as the real component of a sum of cises, then manipulate and evaluate:

$$\begin{aligned}\cos 7^\circ + \cos 79^\circ + \cos 151^\circ + \cos 223^\circ + \cos 295^\circ &= \text{Re}(\text{cis } 7^\circ + \text{cis } 79^\circ + \text{cis } 151^\circ + \text{cis } 223^\circ + \text{cis } 295^\circ) \\ &= \text{Re}((\text{cis } 7^\circ)(\text{cis } 0^\circ + \text{cis } 72^\circ + \text{cis } 144^\circ + \text{cis } 216^\circ + \text{cis } 288^\circ)) \\ &= \text{Re}((\text{cis } 7^\circ)(0)) \\ &= \text{Re}(0) \\ &= 0.\end{aligned}$$

5. Factor the following:

- (a)  $x^6 - 1$  as a difference of squares

We substitute  $y = x^3$ , giving  $y^2 - 1 = (y + 1)(y - 1)$ . Substituting back in, we get

$$(x^3 + 1)(x^3 - 1).$$

(b)  $x^6 - 1$  as a difference of cubes

We now substitute  $y = x^2$ , giving  $y^3 - 1 = (y - 1)(y^2 + y + 1)$ . Substituting back in, we get

$$(x^2 - 1)(x^4 + x^2 + 1)$$

(c)  $x^4 + x^2 + 1$  over the real numbers

This one isn't as obvious. We substitute  $y = x^2$  to get  $y^2 + y + 1$  and find the quadratic's zeroes:

$$y = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}.$$

So it is irreducible over the reals.

(d)  $x^6 - 1$  completely

We already broke it down into  $(x^3 + 1)$  and  $(x^3 - 1)$ . Going further, we have  $x^3 + 1 = (x + 1)(x^2 - x + 1)$  and  $x^3 - 1 = (x - 1)(x^2 + x + 1)$ . To break apart the last two quadratics, we find their zeros:

$$x^2 - x + 1 = 0 \implies x = \frac{1 \pm i\sqrt{3}}{2} \implies \left(x - \frac{1 - i\sqrt{3}}{2}\right) \left(x - \frac{1 + i\sqrt{3}}{2}\right).$$

$$x^2 + x + 1 = 0 \implies x = \frac{-1 \pm i\sqrt{3}}{2} \implies \left(x + \frac{1 - i\sqrt{3}}{2}\right) \left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

Combining all these, we get the complete factorization over the complex numbers:

$$x^6 - 1 = (x + 1) \left(x - \frac{1 - i\sqrt{3}}{2}\right) \left(x - \frac{1 + i\sqrt{3}}{2}\right) (x - 1) \left(x + \frac{1 - i\sqrt{3}}{2}\right) \left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

(e)  $x^4 + x^2 + 1$  completely

We could do a lot of work again, or we could observe that  $x^4 + x^2 + 1 = \frac{x^6 - 1}{x^2 - 1} = \frac{x^6 - 1}{(x+1)(x-1)}$ . Removing the denominator's terms from our factorization of  $x^6 - 1$  we found in the last subproblem, we get

$$x^4 + x^2 + 1 = \left(x - \frac{1 - i\sqrt{3}}{2}\right) \left(x - \frac{1 + i\sqrt{3}}{2}\right) \left(x + \frac{1 - i\sqrt{3}}{2}\right) \left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

6. Let  $f(z) = \frac{z+1}{z-1}$ .

(a) Without a calculator, compute  $f^{2014}(z)$ .

This seems terrifying. Let's try computing  $f^2(z)$  and perhaps  $f^3(z)$ .

$$f^2(z) = \frac{f(z) + 1}{f(z) - 1} = \frac{\frac{z+1}{z-1} + 1}{\frac{z+1}{z-1} - 1} = \frac{\frac{2z}{z-1}}{\frac{2}{z-1}} = z.$$

Oh.

Since 2014 is even, we have  $f^{2014}(z) = (f^2)^{1007}(z) = z$ .

(b) What if you replace 2014 with the current year?

Let  $y$  be the current year. As I write this, it is 1492.

If  $y$  is even, then  $f^y(z) = (f^2)^{y/2}(z) = z$ . If  $y$  is odd, then  $f^y(z) = f((f^2)^{(y-1)/2}(z)) = f(z) = \frac{z+1}{z-1}$ .

7. Find  $\text{Im}((\text{cis } 12^\circ + \text{cis } 48^\circ)^6)$ .

These are some weird looking angles. Thinking back to some older problems, however, the resultant angle of the addition may be tractable. We draw a diagram, shown in Figure 63.

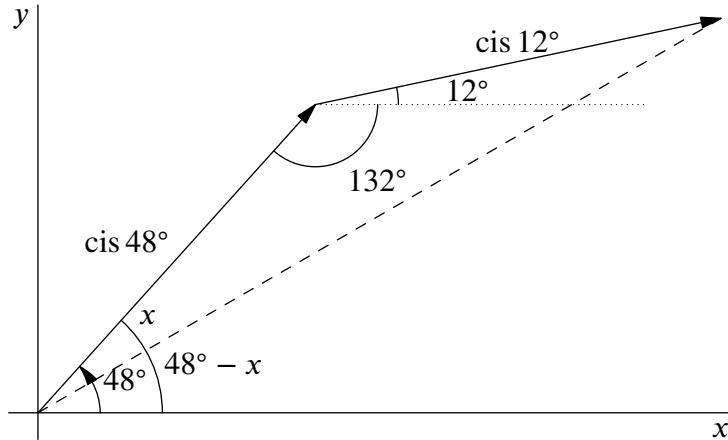


Figure 63: Adding  $\text{cis } 12^\circ + \text{cis } 48^\circ$ .

Consider the isosceles triangle. The apex has angle measure  $132^\circ + 12^\circ = 144^\circ$ , so the base angles are each  $x = \frac{180^\circ - 144^\circ}{2} = 18^\circ$ . But  $\text{Arg}(\text{cis } 12^\circ + \text{cis } 48^\circ) = 48^\circ - x = 30^\circ$ !

That's a familiar angle. Indeed, we have  $z = \text{cis } 12^\circ + \text{cis } 48^\circ = r \text{ cis } 30^\circ$  for some  $r$ . It doesn't really matter which  $r$ , because

$$\text{Im}((r \text{ cis } 30^\circ)^6) = \text{Im}(r^6 \text{ cis } 180^\circ) = \text{Im}(-r^6) = 0.$$

8. Let  $x$  satisfy the equation  $x + \frac{1}{x} = 2 \cos \theta$ .

(a) Compute  $x^2 + \frac{1}{x^2}$  in terms of  $\theta$ .

Squaring the left hand side will get us some terms that look close to what we want.

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}.$$

So  $x^2 + \frac{1}{x^2} = (2 \cos \theta)^2 - 2 = 4 \cos^2 \theta - 2 = 2(2 \cos^2 \theta - 1) = 2 \cos 2\theta$ . Huh.

(b) Compute  $x^n + \frac{1}{x^n}$  in terms of  $n$  and  $\theta$ .

We conjecture that this is equal to  $2 \cos n\theta$ . To do this, we let  $x = \text{cis } \frac{\theta}{n}$ , so  $x^n = \text{cis } \theta$ , and compute. That should give us some similar looking terms:

$$\begin{aligned} x^n + \frac{1}{x^n} &= \text{cis } \theta + \frac{1}{\text{cis } \theta} \\ &= \text{cis } \theta + \text{cis}(-\theta) \\ &= \text{cis } \theta + \overline{\text{cis } \theta} \\ &= 2 \text{Re}(\text{cis } \theta) \\ &= 2 \cos \theta. \end{aligned}$$

This proves the relationship.