3 From Snaps to Flips

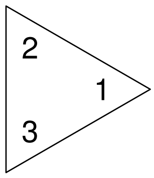

Figure 1: The paper triangle.

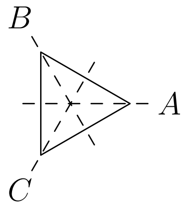

Figure 2: Its axes of reflection.

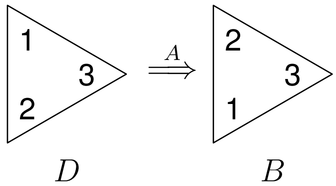

Figure 3: $AD = B$; notice the right-to-left evaluation.

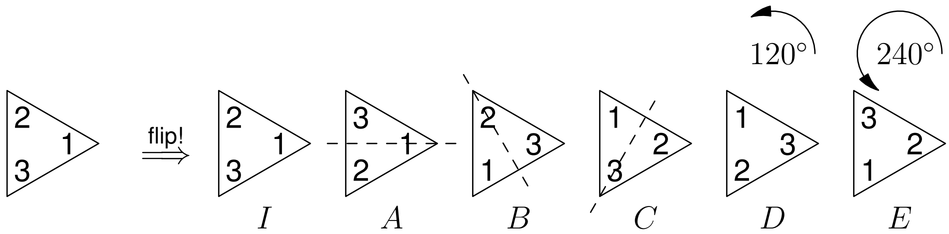

Figure 4: The six ending positions.

Please use a paper or cardboard triangle to help visualize the next concept. Cut out an equilateral triangle, label its front vertices 1, 2, and 3 as shown in Figure 1, and place it down in the shown orientation. Consider the possible ways to transform this triangle, while preserving the position of its shape on the plane. From this starting position, you can reflect the triangle over one of three axes: A, B, or C, as shown in Figure 2. You could also rotate the triangle $120°$ or $240°$ counterclockwise. The final possible positions are shown in Figure 4.

Notice that each position corresponds to a different transformation which preserves the triangle's location. For example, I means "leave the triangle alone," A means "flip the triangle about the A axis," and D means "rotate the triangle $120°$ counterclockwise." We can combine these operations to form other operations by sequencing them, which we write like multiplication. We evaluate them right-to-left as we did for the snap group. For example, $AD = B$, as shown in Figure 3.

These six positions, along with the operation $\cdot$ of transformation composition, form another group: the **dihedral group** of the equilateral triangle, or D_3. It is denoted D_3 because it is a dihedral group of a 3-sided regular polygon. If we split "dihedral" into "di-" and "-hedral," we see it means "two faces"; this etymology refers to the two faces of our paper triangle.

Dihedral groups exist for any polygonal figure; the dihedral group D_n is the group of symmetries, or **symmetry group**, of a n-sided regular polygon. Shapes in any number of dimensions have symmetry groups; the symmetry groups of plane figures are specifically called dihedral groups. For example, a pentagon's symmetry group is a dihedral group, while a cube's symmetry group is not. Let's study the properties of the dihedral group of the equilateral triangle.

1. Is the list of six operations complete? (Are there any other isometries of the equilateral triangle that preserve its shape and location?)

2. As with the snap group, we can make a group table for the dihedral group. Fill out a table like the one in Figure 5 in your notebook. Like the snap group table, the top row indicates what operation is done first and the left column indicates what's done second. In other words, XY is in the X's' row and Y's column. $AD = B$ is done for you.

$\cdot$	I	A	B	C	D	E
I						
A					B	
B						
C						
D						
E						

Figure 5: Unfilled D_3 group table.

3. What is the relationship between the tables for the snap group S_3 and the dihedral group D_3?

S_3 and D_3 are said to be **isomorphic**. Groups A with operation $\bullet$ and B with operation $\star$ are isomorphic if you can find a one-to-one correspondence between the two groups' elements, where the results of each group's operation on corresponding elements also correspond. This means we can find some pairing of elements between the two groups $A_1 \leftrightarrow B_1, A_2 \leftrightarrow B_2, \cdots, A_n \leftrightarrow B_n$ such that $A_j \bullet A_k = A_l \leftrightarrow B_j \star B_k = B_l$, for all j, k, l.

$\cdot$	I	r	r^2	f	fr	fr^2
I						
r				fr^2		
r^2						
f						
fr						
fr^2						

Figure 6: Unfilled alternate D_3 table.

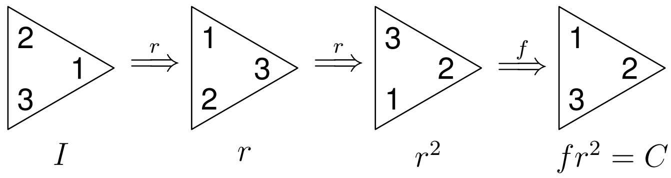

Figure 7: $fr^2 = C$. Again, notice the right-to-left evaluation.

4. Check your understanding by defining isomorphic in your own words.
5. (a) Make a table for only the rotations of D_3, a subgroup of D_3.
   (b) Which subgroup of the snap group S_3 is isomorphic to the subgroup in (a)?
6. What shape's symmetry group is isomorphic to the (a) two-post snap group S_2, (b) one-post group S_1, (c) four-post group S_4 (hint: it's not a square), and (d) five-post group S_5?
7. Find a combination of A and D that yields C.
8. We call A and D **generators** of the group because every element of the group is expressible as some combination of As and Ds. For convenience, let's call A "f" since it's a flip, and call D "r" meaning a $120°$ rotation counterclockwise. Then, for example, fr^2 is a rotation of $2 \cdot 120° = 240°$, followed by a flip across the A axis, equivalent to our original C (see Figure 7). Make a new table using I, r, r^2, f, fr, and fr^2 as elements, like the one in Figure 6. *Note that the element order is different!*
9. What other pairs of elements could you have used to generate the table?
10. Notice the 3×3 table of a group you've already described in the top-left corner of your table. What is it, and what are the two possible generators of this three-element group?
11. Explain why each element of the dihedral group D_3 has the period it has.
12. Some pairs of elements of the dihedral group are two-element subgroups. Which pairs are they?
13. One of the elements forms a one-element subgroup. Which is it?

A **group** G is a set of elements together with a **binary operation** that meets the following criteria:

(a) Identity: There is an element $I \in G$ such that for all $X \in G$, $X \bullet I = I \bullet X = X$.

(b) Closure: If X, Y are elements of the group, then $X \bullet Y$ is also an element of the group.

(c) Invertibility: Each element X has an inverse X^{-1} such that $X \bullet X^{-1} = X^{-1} \bullet X = I$.

(d) Associativity: For all elements X, Y, and Z, $X \bullet (Y \bullet Z) = (X \bullet Y) \bullet Z$.

14. The addition of two numbers is a binary operation, while the addition of three numbers is not. In logic, $\wedge$ (and) and $\vee$ (or) are binary operations, but $\neg$ (not) is not. Define binary operation in your own words, and name some other binary operations.
15. In your original dihedral group table, what is

    (a) the identity element? (b) the inverse of A? (c) the inverse of E?