7 Your Daily Dose of Vitamin i

1. We will use complex numbers to find identities for $\cot$. Use Pascal's triangle to expand the following:

(a) $(a+b)^3$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

(b) $(a+b)^4$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

(c) $(a+b)^5$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

1. (cont.) Then substitute $b = i = \sqrt{-1}$ and expand:

(d) $(a+i)^3$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + 3a^2i - 3a - i.$$

(e) $(a+i)^4$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1.$$

(f) $(a+i)^5$

$$(a+i)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i.$$

1. (cont.) Finally, substitute $a = \cot\theta$ and expand:

(g) $(\cot\theta + i)^3$

$$(\cot\theta + i)^3 = a^3 + 3a^2i - 3a - i = (\cot^3\theta - 3\cot\theta) + i(3\cot^2\theta - 1).$$

(h) $(\cot\theta + i)^4$

$$(\cot\theta + i)^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1 = (\cot^4\theta - 6\cot^2\theta + 1) + (4\cot^3\theta - 4\cot\theta).$$

(i) $(\cot\theta + i)^5$

$$(\cot\theta + i)^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i = (\cot^5\theta - 10\cot^3\theta + 5\cot\theta) + i(5\cot^4\theta - 10\cot^2\theta + 1).$$

1. (cont.) Consider $z = i + \cot\theta$.

(j) Use the above results to find identities for (i) $\cot 3\theta$, (ii) $\cot 4\theta$, and (iii) $\cot 5\theta$.

i. $\cot 3\theta$

Given the right triangle formed by $z = i + \cot\theta$ in Figure 7, we have $\tan(\operatorname{Arg} z) = \frac{1}{\cot\theta} = \tan\theta$, so $\operatorname{Arg} z = \theta$ and $z = r\operatorname{cis}\theta$.

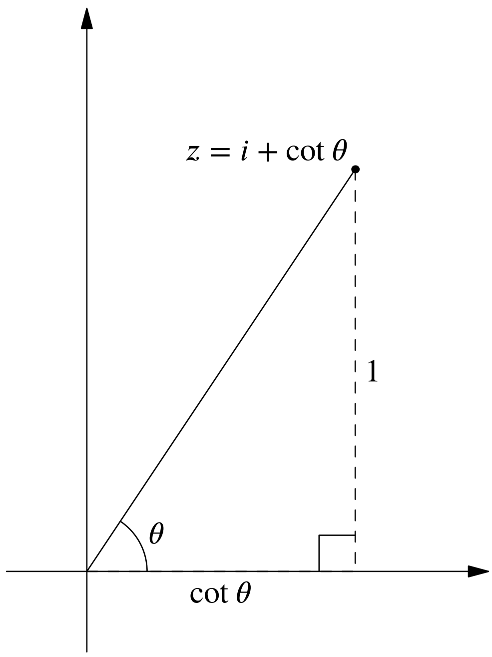

Figure 1: $\mathrm{Arg}(i + \cot\theta) = \theta$.

Thus, we have

$$\begin{aligned}
\cot 3\theta &= \frac{\cos 3\theta}{\sin 3\theta} \\
&= \frac{\mathrm{Re}(\operatorname{cis} 3\theta)}{\mathrm{Im}(\operatorname{cis} 3\theta)} \\
&= \frac{\mathrm{Re}(r^3 \operatorname{cis} 3\theta)}{\mathrm{Im}(r^3 \operatorname{cis} 3\theta)} \\
&= \frac{\mathrm{Re}(z^3)}{\mathrm{Im}(z^3)}.
\end{aligned}$$

We substitute in our expression for z^3, $(\cot^3\theta - 3\cot\theta) + i(3\cot^2\theta - 1)$:

$$\cot 3\theta = \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}.$$

i. $\cot 4\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}
\cot 4\theta &= \frac{\cos 4\theta}{\sin 4\theta} \\
&= \frac{\mathrm{Re}(\operatorname{cis} 4\theta)}{\mathrm{Im}(\operatorname{cis} 4\theta)} \\
&= \frac{\mathrm{Re}(r^4 \operatorname{cis} 4\theta)}{\mathrm{Im}(r^4 \operatorname{cis} 4\theta)} \\
&= \frac{\mathrm{Re}(z^4)}{\mathrm{Im}(z^4)} \\
\cot 4\theta &= \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}.
\end{aligned}$$

i. $\cot 5\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}
\cot 5\theta &= \frac{\cos 5\theta}{\sin 5\theta} \\
&= \frac{\mathrm{Re}(\operatorname{cis} 5\theta)}{\mathrm{Im}(\operatorname{cis} 5\theta)} \\
&= \frac{\mathrm{Re}(r^5 \operatorname{cis} 5\theta)}{\mathrm{Im}(r^5 \operatorname{cis} 5\theta)} \\
&= \frac{\mathrm{Re}(z^5)}{\mathrm{Im}(z^5)} \\
&= \frac{\cot^5\theta - 10\cot^3\theta + 5\cot\theta}{5\cot^4\theta - 10\cot^2\theta + 1}.
\end{aligned}$$

(k) Graph z, z^2, z^3, z^4, and z^5, with $\theta \approx 75°$. What is your solution method?

To graph these, I first calculated the approximate magnitude of z, which is how many times each subsequent power will be scaled by. We have $|1 + \cot 75°| \approx 1.268$, so we only need to scale by about $\frac{5}{4}$ each time. Of course, we rotate by about $75°$ each time.

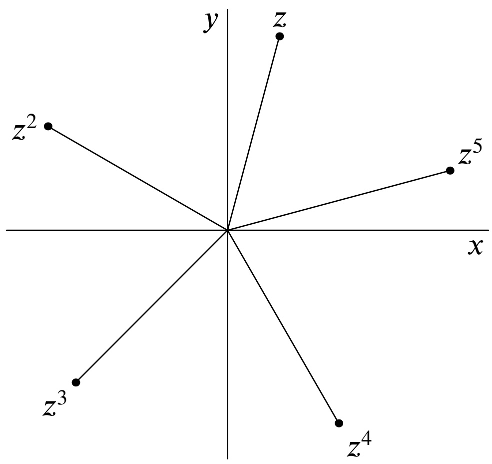

Figure 2: Graphs of z, z^2, z^3, z^4, and z^5.

2. Compute $(1+i)^n$ for $n = 3, 4, 5, \ldots$. Can you find a general pattern?

We have

$$\begin{aligned}
(1+i)^3 &= 1^3 + 3i - 3 - i &&= -2 - 2i\\
(1+i)^4 &= 1^4 + 4i - 6 - 6i + 1 &&= -4 - 2i\\
(1+i)^5 &= 1^5 + 5i - 10 - 10i + 5 + i &&= -4 - 4i.
\end{aligned}$$

We can find the pattern by representing $1 + i = \sqrt{2}45$. This shows that it has period 8 and let's us find an expression for $(1+1)^n$:

$$(1+i)^n = \left(\sqrt{2}45\right)^n = 2^{n/2} \operatorname{cis}\left(\frac{n\pi}{4}\right).$$

3. Expand and graph $\operatorname{cis}^n \theta$ for $n = 2, 3, 4, \ldots$.

Let $\cos\theta = c$ and $\sin\theta = s$. We have

$$\begin{aligned}
&(c+is)^2 = c^2 + 2csi - s^2 = (c^2 - s^2) + i(2cs)\\
&(c+is)^3 = c^3 + 3c^2si - 3cs^2 - s^3i = (c^3 - 3cs^2) + i(3c^2s - s^3)\\
&(c+is)^4 = c^4 + 4c^3si - 6c^2s^2 - 4cs^3i + s^4 = (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3)\\
&(c+is)^5 = c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i = (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5).
\end{aligned}$$

The graphs of $\operatorname{cis}^n \theta$ for $\theta \approx 50°$ are shown in Figure 3 below.

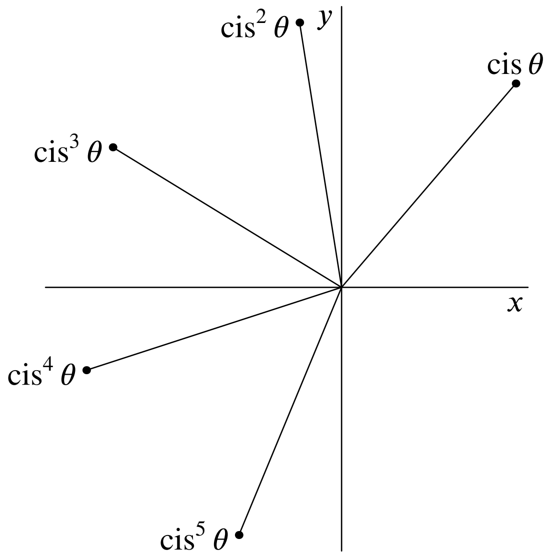

Figure 3: Graphs of $\text{cis}^n\,\theta$ for $\theta \approx 50°$.

(a) Why is the real part $\cos n\theta$ and the imaginary part $\sin n\theta$?

By DeMoivre's theorem, $\text{cis}^n\,\theta = \text{cis}\,n\theta$, which by definition has $\text{Im}(\text{cis}\,n\theta) = \cos n\theta$ and $\text{Re}(\text{cis}\,n\theta) = \sin n\theta$.

(b) Use your results to write identities for $\cos n\theta$ and $\sin n\theta$ for $n = 2, 3, 4, 5$.

Here they are. Again, let $\cos\theta = c$ and $\sin\theta = s$:

$$
\begin{aligned}
\cos 2\theta &= \text{Re}(\text{cis}\,2\theta) = c^2 - s^2 \\
\cos 3\theta &= \text{Re}(\text{cis}\,3\theta) = c^3 - 3cs^2 \\
\cos 4\theta &= \text{Re}(\text{cis}\,4\theta) = c^4 - 6c^2s^2 + s^4 \\
\cos 5\theta &= \text{Re}(\text{cis}\,5\theta) = c^5 - 10c^3s^2 + 5cs^4 \\
\sin 2\theta &= \text{Im}(\text{cis}\,2\theta) = 2cs \\
\sin 3\theta &= \text{Im}(\text{cis}\,3\theta) = 3c^2s - s^3 \\
\sin 4\theta &= \text{Im}(\text{cis}\,4\theta) = 4c^3s - 4cs^3 \\
\sin 5\theta &= \text{Im}(\text{cis}\,5\theta) = 5c^4s - 10c^2s^3 + s^5.
\end{aligned}
$$

4. **Compute $7 + 79 + 151 + 223 + 295$ without a calculator. (Hint: what does this have to do with complex numbers?)**

These numbers look random, but a closer inspection reveals they are in arithmetic progression, with starting term 7 and increasing 72° each time. That's the rotation of a pentagon!

We rewrite this as the real component of a sum of cises, then manipulate and evaluate:

$$
\begin{aligned}
7 + 79 + 151 + 223 + 295 &= \text{Re}(7 + 79 + 151 + 223 + 295) \\
&= \text{Re}((7)(0 + 72 + 144 + 216 + 288)) \\
&= \text{Re}((7)(0)) \\
&= \text{Re}(0) \\
&= 0.
\end{aligned}
$$

Note that going from the second to third step, we used the fact that the cis expressions are the vertices of a regular pentagon, which sum to 0. If you want to be more formal about it, a fun way to prove that $0 + 72 + 144 + 216 + 288 = 0$ is to set it to Ξ and calculate:

$$\begin{aligned}\Xi \cdot 72 &= (0+72+144+216+288)72\\ &= 72+144+216+288+360\\ &= 72+...+288+0\\ &= \Xi.\end{aligned}$$

If $\Xi \cdot$ (something that's not one) $= \Xi$, then Ξ must be 0.

5. Factor the following:

(a) $x^6 - 1$ as a difference of squares

We substitute $y = x^3$, giving $y^2 - 1 = (y+1)(y-1)$. Substituting back in, we get

$$(x^3+1)(x^3-1).$$

(b) $x^6 - 1$ as a difference of cubes

We now substitute $y = x^2$, giving $y^3 - 1 = (y-1)(y^2+y+1)$. Substituting back in, we get

$$(x^2-1)(x^4+x^2+1)$$

.

(c) $x^4 + x^2 + 1$ over the real numbers

This one isn't as obvious. We substitute $y = x^2$ to get $y^2 + y + 1$ and find the quadratic's zeroes: $y = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm i\sqrt{3}}{2}$.
So it is irreducible over the reals.

(d) $x^6 - 1$ completely

We already broke it down into (x^3+1) and (x^3-1). Going further, we have $x^3+1 = (x+1)(x^2-x+1)$ and $x^3-1 = (x-1)(x^2+x+1)$. To break apart the last two quadratics, we find their zeros:

$$x^2 - x + 1 = 0 \Longrightarrow x = \frac{1 \pm i\sqrt{3}}{2} \Longrightarrow \left(x - \frac{1-i\sqrt{3}}{2}\right)\left(x - \frac{1+i\sqrt{3}}{2}\right).$$

$$x^2 + x + 1 = 0 \Longrightarrow x = \frac{-1 \pm i\sqrt{3}}{2} \Longrightarrow \left(x + \frac{1-i\sqrt{3}}{2}\right)\left(x + \frac{1+i\sqrt{3}}{2}\right).$$

Combining all these, we get the complete factorization over the complex numbers:

$$x^6 - 1 = (x+1)\left(x - \frac{1-i\sqrt{3}}{2}\right)\left(x - \frac{1+i\sqrt{3}}{2}\right)(x-1)\left(x + \frac{1-i\sqrt{3}}{2}\right)\left(x + \frac{1+i\sqrt{3}}{2}\right).$$

(e) $x^4 + x^2 + 1$ completely

We could do a lot of work again, or we could observe that $x^4 + x^2 + 1 = \frac{x^6-1}{x^2-1} = \frac{x^6-1}{(x+1)(x-1)}$. Removing the denominator's terms from our factorization of $x^6 - 1$ we found in the last subproblem, we get

$$x^4 + x^2 + 1 = \left(x - \frac{1-i\sqrt{3}}{2}\right)\left(x - \frac{1+i\sqrt{3}}{2}\right)\left(x + \frac{1-i\sqrt{3}}{2}\right)\left(x + \frac{1+i\sqrt{3}}{2}\right).$$

6. Let $f(z) = \frac{z+1}{z-1}$.

(a) Without a calculator, compute $f^{2014}(z)$.

This seems terrifying. Let's try computing $f^2(z)$ and perhaps $f^3(z)$.

$$f^2(z) = \frac{f(z)+1}{f(z)-1} = \frac{\frac{z+1}{z-1}+1}{\frac{z+1}{z-1}-1} = \frac{\frac{2z}{z-1}}{\frac{2}{z-1}} = z.$$

Oh.
Since 2014 is even, we have $f^{2014}(z) = (f^2)^{1007}(z) = z$.

(b) What if you replace 2014 with the current year?

Let y be the current year. As I write this, it is 1492.
If y is even, then $f^y(z) = (f^2)^{y/2}(z) = z$. If y is odd, then $f^y(z) = f((f^2)^{(y-1)/2}(z)) = f(z) = \frac{z+1}{z-1}$.

7. Find $\mathrm{Im}\left((12+48)^6\right)$.

These are some weird looking angles. Thinking back to some older problems, however, the resultant angle of the addition may be tractable. We draw a diagram, shown in Figure 4.

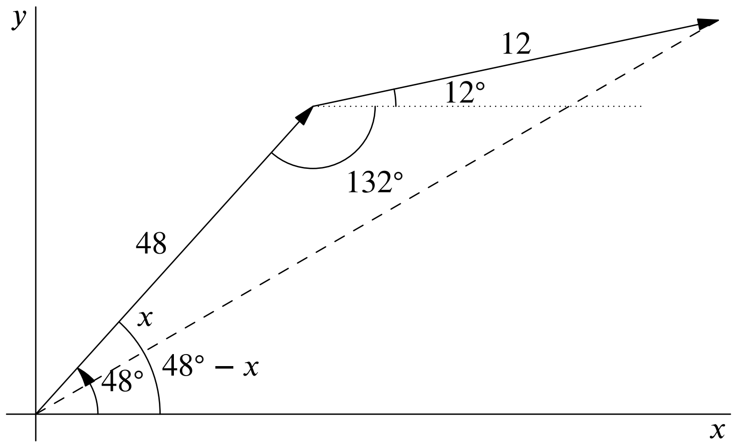

Figure 4: Adding $12+48$.

Consider the isosceles triangle. The apex has angle measure $132° + 12° = 144°$, so the base angles are each $x = \frac{180°-144°}{2} = 18°$. But $\mathrm{Arg}(12+48) = 48° - x = 30°$!

That's a familiar angle. Indeed, we have $z = 12 + 48 = r30$ for some r. It doesn't really matter which r, because

$$\mathrm{Im}((r30)^6) = \mathrm{Im}(r^6 180) = \mathrm{Im}(-r^6) = 0.$$

8. Let x satisfy the equation $x + \frac{1}{x} = 2\cos\theta$.

(a) Compute $x^2 + \frac{1}{x^2}$ in terms of θ.

Squaring the left hand side will get us some terms that look close to what we want.

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}.$$

So $x^2 + \frac{1}{x^2} = (2\cos\theta)^2 - 2 = 4\cos^2\theta - 2 = 2(2\cos^2\theta - 1) = 2\cos 2\theta$. Huh.

(b) Compute $x^n + \frac{1}{x^n}$ in terms of n and θ.

It's unclear how to start, so we might as well try to compute $x^3 + \frac{1}{x^3}$ in the same way.

$$\begin{aligned}
\left(x+\frac{1}{x}\right)^3 &= \\
x^3+3x+\frac{3}{x}+\frac{1}{x^3} &= 8\cos^3\theta \\
x^3+3\underbrace{\left(x+\frac{1}{x}\right)}_{2\cos\theta}+\frac{1}{x^3} &= 8\cos^3\theta \\
x^3+\frac{1}{x^3} &= 8\cos^3\theta-6\cos\theta.
\end{aligned}$$

Now what? The astute among you may recognize that $8\cos^3\theta - 6\cos\theta = 2\cos 3\theta$, at which point you could make a conjecture (and could jump ahead). But suppose we didn't find that.

We know that $x^2+\frac{1}{x^2} = 2\cos 2\theta$. By analogy, if we make the substitution $y = x^2$ and $\phi = 2\theta$, we get that $y+\frac{1}{y} = 2\cos\phi$, and thus $y^2+\frac{1}{y^2} = 2\cos 2\phi \implies x^4+\frac{1}{x^4} = 2\cos 4\theta$. In general,

$$x^{2^m}+\frac{1}{x^{2^m}} = 2\cos 2^m\theta.$$

The exponent on x, in this case 2^m, is the same as the multiple of θ. Pretty sus. We've solved the problem for n which are powers of two, but we conjecture that the relationship holds for all integers n. To be explicit, we want to show that

$$x^n+\frac{1}{x^n} = 2\cos n\theta.$$

There's a couple of ways to do it. But seeing x^n and $\cos n\theta$ in the same place immediately recalls exponentiating $\operatorname{cis}\theta$. So, let's try rewriting the problem a bit by entering into the complex plane. Let $x = r\operatorname{cis}\phi$, which we really should have done earlier. Then we're given that $r\operatorname{cis}\phi + \frac{1}{r\operatorname{cis}\phi} = 2\cos\theta$. Working further,

$$\begin{aligned}
2\cos\theta &= r\operatorname{cis}\phi + \frac{1}{r}\operatorname{cis}(-\phi) \\
&= r\cos\phi + \frac{1}{r}\cos\phi + i(r\sin\phi - \frac{1}{r}\sin\phi) \\
&= \left(r+\frac{1}{r}\right)\cos\phi + i\left(r-\frac{1}{r}\right)\sin\phi
\end{aligned}$$

The imaginary part needs to be zero, since the left hand side is real. So either $r = \frac{1}{r}$ or $\sin\phi = 0$. Let's examine each case. In the first case, $r = 1$ (it can't be -1 since $r \geq 0$. In the second case, we have $\cos\phi = \pm 1$, and substituting, we get

$$2\cos\theta = \pm\left(r+\frac{1}{r}\right).$$

That's not helpful... except now we know that r is real and ≥ 0. Considering $r+\frac{1}{r}$, we see that it approaches ∞ as $r \to 0$. What range of values does it make? Graphing it shows that it has a range of $[2,\infty)$, reaching its minimum at $r = 1$. Another way to prove this is via AM-GM with $x = 2r$ and $y = \frac{2}{r}$:

$$\frac{x+y}{2} \geq \sqrt{xy} \implies r+\frac{1}{r} \geq \sqrt{2r\left(\frac{2}{r}\right)} = 2.$$

But the range of $2\cos\theta$ is $[-2,2]$, and the only possible value of the equation is the intersection of their ranges, aka 2. So $r = 1$ no matter what. That's damn useful, because then

$$\begin{aligned}
2\cos\theta &= \left(r+\frac{1}{r}\right)\cos\phi + i\left(r-\frac{1}{r}\right)\sin\phi && \text{From before} \\
&= 2\cos\phi.
\end{aligned}$$

So $\cos\phi = \cos\theta$. We wish to find an expression for $x^n + \frac{1}{x^n}$.

$$\begin{aligned}
x^n + \frac{1}{x^n} &= r^n \operatorname{cis}^n \phi + \frac{1}{r^n} \operatorname{cis}^n(-\phi) \\
&= r^n(\cos n\phi + \sin n\phi) + \frac{1}{r^n}(\cos n\phi - \sin n\phi) \\
&= \cos n\phi + \sin n\phi + \cos n\phi - \sin n\phi \\
&= 2\cos n\phi \\
&= 2\cos n\theta.
\end{aligned}$$

Note that in the last step, we have to be careful, but cosine does have this property. Anyway, that's a gg, QED.