12 Composite Mappings of the Plane

1. **For Problems a through e, fill in the blank.**

   (a) **We start by finding the images of our points under the $-90°$ rotation.**

   i. **Find the matrix R which results in a $-90°$ rotation.**

That's just $R = \begin{bmatrix} \cos -90° & -\sin -90° \\ \sin -90° & \cos -90° \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

   ii. **Multiply R by our unit vectors and point (u, v):**

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} 0 & 1 & v \\ -1 & 0 & -u \end{bmatrix}.$$

   (b) **Next, we reflect those intermediate image points over the line $y = 0$.**

   i. **Find the matrix S which does this.**

We want to flip the y coordinate, so $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

   ii. **Multiply S by the result of Problem ii.**

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & v \\ -1 & 0 & -u \end{bmatrix} = \begin{bmatrix} 0 & 1 & v \\ 1 & 0 & u \end{bmatrix}.$$

   (c) **You should notice that the net result of the two transformations taken together is a reflection over the line $y = x$. Which matrix represents this transformation?**

The matrix that represents a reflection over $y = x$ is $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   (d) **Notice that what we did to achieve this mapping was**

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix},$$

   where we multiplied the two rightmost matrices first but didn't use the associative property to multiply the two leftmost matrices first. See what happens when you multiply the two left hand matrices together:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

   Look familiar?

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This is our matrix M from part (c)!

   (e) **See what happens when you reverse the order of multiplication:**

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(f)

i. What transformation does this new matrix result in?

This is a reflection about the line $y = -x$. After all, we have the mappings $(1,0) \to (0,-1)$ and $(0,1) \to (-1,0)$:

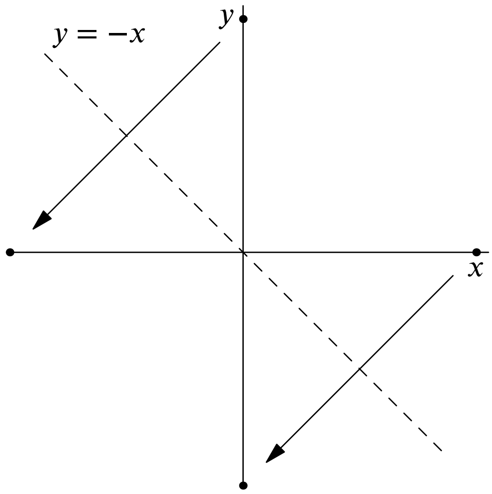


Figure 1: The mapping is a reflection over $y = -x$.

ii. How is a reflection followed by a rotation different from a rotation followed by a reflection? Visualize this by following what happens to a point under both sets of transformations.

A reflection followed by a rotation and a rotation followed by a reflection are both (at least in 2D) reflections overall. In our example, the first ordering of the matrices is the rotation followed by the reflection (recall we're working from right to left), and the second ordering is the reflection followed by the rotation.

In a reflection followed by a rotation clockwise (a.k.a. our rotation of $-90°$), the line of reflection is moved clockwise by half the (positive) angle of rotation. In a rotation clockwise followed by a reflection, the line of reflection is moved *counterclockwise* by half the angle of rotation.

(g) Notice that we apply the transformations from right to left. If you wanted to read from left to right, what would you have to change about the way you wrote the mapping matrices, the vectors representing points, and the order of the matrices?

We would have to reverse the order of the matrices, write the vectors representing points as row vectors, and transpose the matrices. For example, here is our original notation:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ -a \end{bmatrix},$$

and here is it in a left to right format:

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -b & -a \end{bmatrix}.$$

(h) How does our convention for ordering transformation matrices compare...

i. ... to the convention for writing composite functions, like $f(g(x))$?

It is similar in order to composing functions, because these both are evaluated sequentially from the right to the left.

ii. ... to the "followed by" convention we used for "From Snaps to Flips?"

We evaluated the flip elements from right to left, so it is like our transformation matrices.

iii. ... to the "from _ to _" convention for transportation matrices?

We wrote transportation matrices with the destinations between on top (as columns), and the origins on the left side (as rows). Thus, they are evaluated from left to right, unlike our transformation matrices.

2. There are two, infinite classes of matrices which comprise all isometries of the plane which keep the origin fixed. These are the rotation matrix and reflection matrix. Let's look first at the rotation matrix and make sure that it really always works the way it should.

(a) What is the result of a rotation by an angle θ followed by one of ϕ?

It is a rotation by $\theta + \phi$.

(b) Multiply their rotation matrices: $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}.$

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix}.$$

(c) Use the angle addition formulae to simplify your answer.

$$\begin{bmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix} = \begin{bmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{bmatrix}.$$

(d) Should the result be the same if you reverse the order of rotation?

Yes, since rotation is (unlike most other planar transformations) commutative.

(e) What happens to the points $(1,0)$, $(0,1)$, and (x,y) when you operate on them with the rotation matrix?

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix}.$$

We multiply as directed:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x\cos\theta - y\sin\theta \\ \sin\theta & \cos\theta & x\sin\theta + y\cos\theta \end{bmatrix}.$$

3. Now let's check for the generalized reflection matrix.

(a) Take the matrix which results in a reflection over the line $y = x\tan\frac{\theta}{2}$ and reflect over that line twice:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}.$$

(b) Simplify your answer and explain the result.

$$\begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This happens because a reflection is its own inverse. How boring!

(c) Let's do a reflection over the line $y = \tan\frac{\theta}{2}$ followed by a reflection over the line $y = \tan\frac{\phi}{2}$:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}.$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi + \sin\theta\sin\phi & \cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi - \cos\theta\sin\phi & \sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}.$$

(d) Simplify your answer using the angle addition formulae, and interpret.

$$\begin{bmatrix}\cos\theta\cos\phi+\sin\theta\sin\phi & \cos\theta\sin\phi-\sin\theta\cos\phi\\ \sin\theta\cos\phi-\cos\theta\sin\phi & \sin\theta\sin\phi+\cos\theta\cos\phi\end{bmatrix}=\begin{bmatrix}\cos(\theta-\phi) & \sin(\phi-\theta)\\ \sin(\theta-\phi) & \cos(\theta-\phi)\end{bmatrix}$$
$$=\begin{bmatrix}\cos(\theta-\phi) & -\sin(\theta-\phi)\\ \sin(\theta-\phi) & \cos(\theta-\phi)\end{bmatrix}.$$

This is rotation by $\theta-\phi$ counterclockwise.

(e) Does it make a difference which reflection comes first? Do the matrix multiplication to confirm your answer.

In general, reflections aren't commutative, so it probably will make a difference.

$$\begin{bmatrix}\cos\phi & \sin\phi\\ \sin\phi & -\cos\phi\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}=\begin{bmatrix}\cos\phi\cos\theta+\sin\phi\sin\theta & \cos\phi\sin\theta-\sin\phi\cos\theta\\ \sin\phi\cos\theta-\cos\phi\sin\theta & \sin\phi\sin\theta+\cos\phi\cos\theta\end{bmatrix}$$
$$=\begin{bmatrix}\cos(\phi-\theta) & -\sin(\phi-\theta)\\ \sin(\phi-\theta) & \cos(\phi-\theta)\end{bmatrix}.$$

This is different from the original. In fact, it is the opposite rotation (and thus the matrix's inverse).

We didn't really have to do the matrix multiplication. We could have substituted the rather confusing $\theta=\phi'$ and $\phi=\theta'$ into the original expression and gotten the same result. Oh well.

4. We've found specific matrices which map the plane in the following ways:

- **identity;**
- **rotation about the origin by θ;**
- **reflection over a line $y=x\tan\frac{\theta}{2}$;**
- **size change by some factor centered at the origin;**
- **stretching along a specific line through the origin by some factor;**
- **shearing perpendicular to a specific line through the origin by some factor.**

We want to generalize those ideas. What does each of the following matrices do? Be quantitative by specifying angle, equation of line, and/or factor:

(a) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

This is the identity matrix; it does nothing.

(b) $\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}$

This matrix scales (or dilates) through the origin by a factor of a.

(c) $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

This matrix rotates counterclockwise by θ.

(d) $\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

This is initially rather mysterious, but if we recall that $\cos\theta=\cos-\theta$ and $\sin-\theta=-\sin\theta$, we realize that this matrix can be rewritten:

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}\cos-\theta & -\sin-\theta\\ \sin-\theta & \cos-\theta\end{bmatrix}.$$

This makes it clear that it is a rotation clockwise by θ.

(e) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

This is a stretch (or squish) along the x-axis by a factor of a.

(f) $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$

This is a stretch (or squish) along the y-axis by a factor of a.

(g) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

This is a shear along the x-axis by a factor of a.

(h) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

This is a shear along the y-axis by a factor of a.

(i) $\begin{bmatrix} a & b \\ ca & cb \end{bmatrix}$

Again, this seems a bit foreign. Multiplying it by a point (u, v), though, we see its true meaning:

$$\begin{bmatrix} a & b \\ ca & cb \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au + bv \\ c(au + bv) \end{bmatrix}.$$

Letting $t = au + bv$, we see that this parametrizes the line $t \begin{bmatrix} 1 \\ c \end{bmatrix}$. In standard form, this is $cx - y = 0$; in sane person's form, this is $y = cx$.

(j) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

This is a negation of the x coordinate, or a reflection about the y-axis.

(k) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

As we found earlier, this is a reflection about the line $y = -x$.

(l) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

Another rather foreign one. We notice that it is similar to the rotation matrix by 2θ, but the right column is negated. As a matrix multiplication, we have

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, this matrix is a reflection about the x-axis, followed by a rotation of 2θ counterclockwise. We know a reflection followed by a rotation is a reflection, so what axis is this reflection about?

Well, an easy way to find out is to note which points remain *fixed* after the matrix transformation, as these will be precisely the points on the line of reflection. Let the point be $(c, 1)$, since then (tc, t) gives every point on the line of reflection. We wish to solve the system of equations

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} c \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ 1 \end{bmatrix}.$$

Multiplying out the left side, we get

$$\begin{bmatrix} c\cos 2\theta + \sin 2\theta \\ c\sin 2\theta - \cos 2\theta \end{bmatrix} = \begin{bmatrix} c \\ 1 \end{bmatrix}.$$

Equating corresponding parts, we get the system of equations

$$\begin{cases} c\cos 2\theta + \sin 2\theta = c \\ c\sin 2\theta - \cos 2\theta = 1 \end{cases}.$$

The second equation gives us $c = \frac{1+\cos 2\theta}{\sin 2\theta}$. By the double angle formula, this is

$$c = \frac{1 + 2\cos^2\theta - 1}{2\cos\theta\sin\theta} = \frac{2\cos^2\theta}{2\cos\theta\sin\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

Thus, the line can be parameterized as $(t\cot\theta, t)$. In standard form, this is $x - y\cot\theta = 0$; in sane person's form, this is $y = x\tan\theta$. Thus, this matrix is a reflection over the line $y = x\tan\theta$, which is the line $\theta = \theta$ in polar coordinates. That's some unfortunate notation, but I hope you get what I mean.

5. **What matrix/transformation undoes each of a through l? For instance, matrix x is a rotation of θ. It is undone by a rotation of $-\theta$.**

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The inverse of the identity matrix is itself: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

The inverse is a scaling by $\frac{1}{a}$: $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix}$.

(c) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

This matrix rotates counterclockwise by θ. Thus, the inverse is a rotation by $-\theta$—or a rotation clockwise by θ—which is $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (the subject of the next problem).

(d) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

This is a rotation clockwise by θ, so the inverse is a matrix rotating counterclockwise by θ: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, the subject of the previous problem.

(e) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

This is a stretch (or squish) along the x-axis by a factor of a. Thus, the inverse is a stretch along the x-axis by a factor of $\frac{1}{a}$: $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}$.

(f) $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$

This is a stretch (or squish) along the y-axis by a factor of a. Thus, the inverse is a stretch along the y-axis by a factor of $\frac{1}{a}$: $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a} \end{bmatrix}$.

(g) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

This is a shear along the x-axis by a factor of a. Thus, the inverse is a shear along the x-axis by a factor of $-a$: $\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$.

(h) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

This is a shear along the y-axis by a factor of a. Thus, the inverse is a shear along the y-axis by a factor of $-a$: $\begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}$.

(i) $\begin{bmatrix} a & b \\ ca & cb \end{bmatrix}$

This matrix doesn't have an inverse, because multiple points can be mapped to the same point. For example, if $a = b = c = 1$, then the matrix is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then for example, both $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \end{bmatrix}$ are mapped to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(j) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

This is a negation of the x coordinate, or a reflection about the y-axis. Since it's a reflection, the inverse is itself: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(k) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

As we found earlier, this is a reflection about the line $y = -x$. Since it's a reflection, the inverse is itself: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(l) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

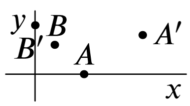
We found that this matrix is a reflection over the line $y = x \tan \theta$. Since it's a reflection, it is its own inverse: $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

6. **In this problem, you will observe the effects of multiplying two or more matrices. Do the following matrix multiplications, graph the preimage $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and image, then identify the transformations and their order. Note the effect of order on the outcome!**

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} = \begin{bmatrix} 2.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix}.$$

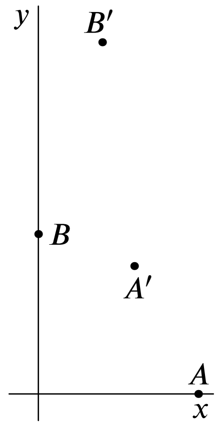
This is a rotation of $\tan^{-1} \frac{4}{3} \approx 53.13°$, followed by a shear along the x-axis by a factor of 2.



(b) $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ 0.8 & 2.2 \end{bmatrix}.$$

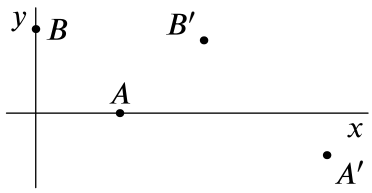
This is a shear along the x-axis by a factor of 2, followed by a rotation of $\tan^{-1} \frac{4}{3} \approx 53.13°$. The order does change the outcome as compared with the previous problem.



(c) $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} & 2 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

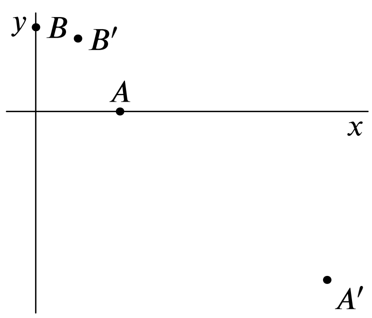
This is a rotation of $-30°$, followed by a stretch along the x-axis by a factor of 4.



(d) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} & \frac{1}{2} \\ -2 & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

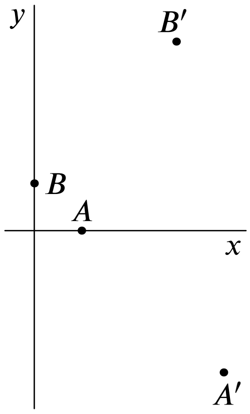
This is a stretch along the x-axis by a factor of 4, followed by a rotation of $60°$.



(e) $\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$$\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}.$$

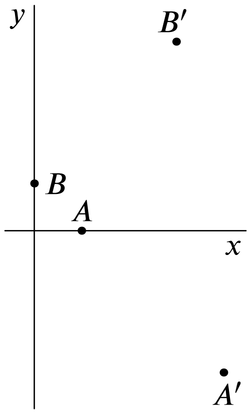
This is a dilation by a factor of 5, followed by a rotation of $\tan^{-1} -\frac{3}{4} \approx -36.87°$.



(f) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}.$$

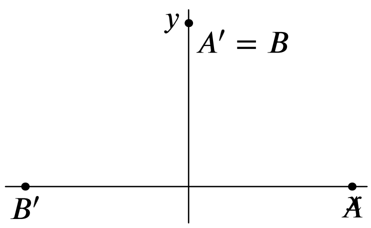
This is a rotation of $\tan^{-1} -\frac{3}{4} \approx -36.87°$, followed by a dilation by a factor of 5. In this case, order doesn't matter.



(g) $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}\begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$

$$\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}\begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

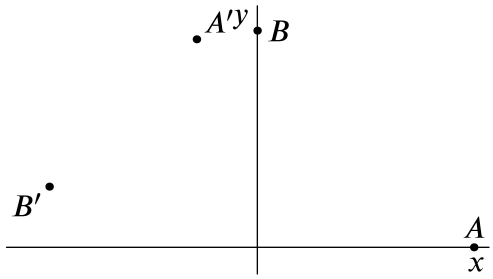
This is a rotation of $\tan^{-1} \frac{3}{4} \approx 36.87°$, followed by a rotation by $\tan^{-1} \frac{4}{3} \approx 53.13°$.



(h) $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

$$\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

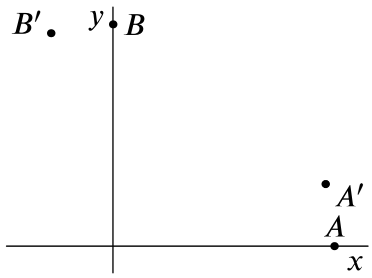
This is a reflection about the line $\theta = \frac{1}{2}\tan^{-1}\frac{4}{3} \approx 26.57°$, followed by the same reflection, which yields the identity.



(i) $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}$

$$\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix} = \begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix}.$$

This is a reflection about the line $\theta = \frac{1}{2}\tan^{-1}\frac{3}{4} \approx 18.43°$, followed by a reflection about the line $\theta = \frac{1}{2}\tan^{-1}\frac{4}{3} \approx 26.57°$.



7. **A linear mapping f is one in which all lines are mapped to lines and the origin remains a fixed point. Algebraically, $f\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = xf\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right) + yf\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right)$. I claim that we can build any linear mapping of the plane by multiplying together some combination of the matrices from Problem 4. Only two classes of matrix, however, are necessary; all other matrices are products or examples of these. Which two classes of matrix do you think comprise the minimum set from which the others can be composed? Be able to justify your choice.**

(Answers may vary.) Stretches and shears seem like the most general classes of transformations, but they cannot change orientation like reflection. All together, these three classes of transformations can generate all linear transformations. We could extend our definition of "stretch" to include negative scaling factors, which would include reflections as well.

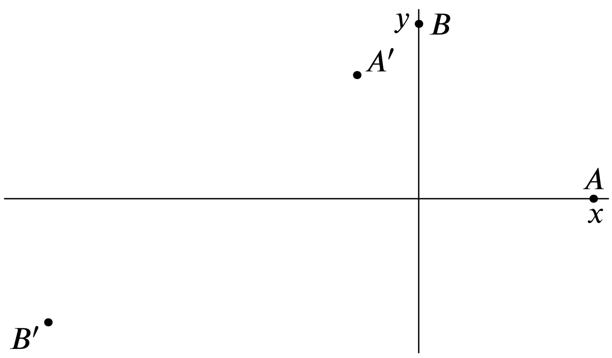
8. **Write matrix products that perform the following mappings. Do the indicated multiplication and graph the preimage and image when applied to $(1, 0)$ and $(0, 1)$.**

   (a) Rotation by $135°$ followed by a shear by a factor of $\frac{1}{2}$ *perpendicular* to the y-axis

The matrix multiplication is as follows:

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos 135° & -\sin 135° \\ \sin 135° & \cos 135° \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{4} & -\frac{3\sqrt{2}}{4} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

Note that a shear *perpendicular* to the y-axis is *parallel* with the x-axis. The graph is below.

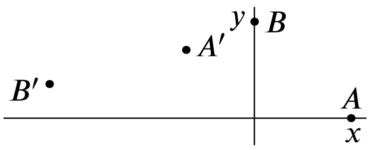


(b) Same transformations as in (a), but reversed

The matrix multiplication is as follows:

$$\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{4} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} \end{bmatrix}.$$

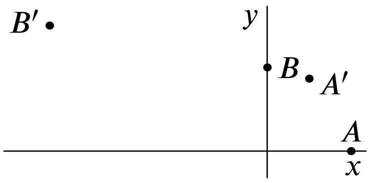
The graph is below.



(c) Stretch in the y direction by a factor of 3 followed by a rotation of $60°$

The matrix multiplication is as follows:

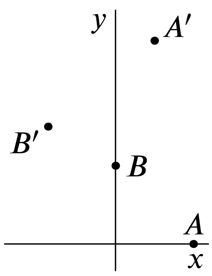
$$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix}.$$



(d) Same transformations as in (c), but reversed

The matrix multiplication is as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix}.$$



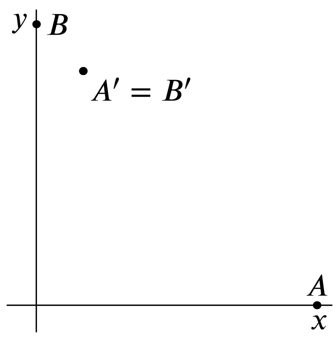
(e) Projection onto the line $y = 5x$

We might think that any old matrix like $\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$, which takes every point to a point on the line $y = 5x$, may work. This matrix, however, is not a *projection*. In a projection to a line, the image of a point is the foot of the altitude from the point to the line.

To map every point onto $y = 5x$, the matrix must be of the form $c \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$ for some real constant c. To make it a true projection, we choose a point that's already on $y = 5x$ and note that it must map to itself. In this case, we'll choose $(1, 5)$. then

$$c \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$
$$c \begin{bmatrix} 6 \\ 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$
$$\implies c = \frac{1}{6}.$$

Thus, the transformation matrix is $\begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{5}{6} \end{bmatrix}$.



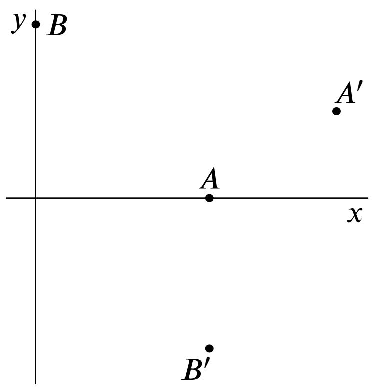
(f) Reflection over $\theta = \frac{\pi}{12}$ followed by a stretch in the x direction by a factor of 2

We recall that a reflection over the line $\theta = \phi$ is the matrix $\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$. Substituting $\phi = \frac{\pi}{12}$ yields the matrix

$$\begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & -\cos\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

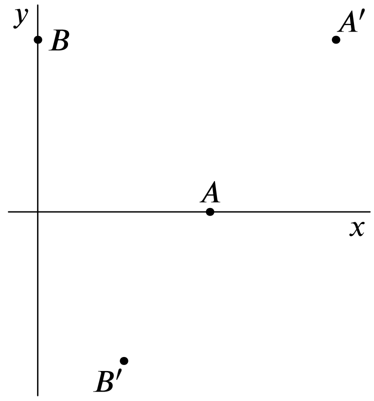
Stretching in the x direction by a factor of 2 is just $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, the total transformation matrix is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 1 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$



(g) **Same transformations as in (f), but reversed**

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \frac{1}{2} \\ 1 & -\frac{\sqrt{3}}{2} \end{bmatrix}$$



9. **Write a set of matrices which undoes Problems a to g. You will find one of them impossible to undo; explain why.**

(a) **Rotation by $135°$ followed by a shear by a factor of $\frac{1}{2}$ *perpendicular* to the y-axis**

We shear by $-\frac{1}{2}$ along the x-axis, then rotate $-135°$.

$$\begin{bmatrix} \cos -135° & -\sin -135° \\ \sin -135° & \cos -135° \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

(b) **Same transformations as in (a), but reversed**

We rotate $-135°$, then shear by $-\frac{1}{2}$ along the x-axis.

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

(c) **Stretch in the y direction by a factor of 3 followed by a rotation of $60°$**

We rotate by $-60°$, then stretch by $\frac{1}{3}$ in the y direction.

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \cos -60° & -\sin -60° \\ \sin -60° & \cos -60° \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{6} & \frac{1}{6} \end{bmatrix}.$$

(d) **Same transformations as in (c), but reversed**

We stretch by $1/3$ in the y direction, then rotate by $-60°$.

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{2} & \frac{1}{6} \end{bmatrix}.$$

(e) **Projection onto the line $y = 5x$**

This matrix doesn't have an inverse, because multiple points map to the same point. For example, both $(0, 0)$ and $(5, -1)$ project to $(0, 0)$

(f) **Reflection over $\theta = \frac{\pi}{12}$ followed by a stretch in the x direction by a factor of 2**

We stretch by a factor of $\frac{1}{2}$ in the x direction, then apply our old reflection matrix:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{1}{4} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

(g) Same transformations as in (f), but reversed

We first apply our old reflection matrix, then stretch by a factor of $\frac{1}{2}$ in the x direction:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

10.

(a) Find the height of the parallelogram in Figure 2 in terms of b and a trig function in terms of φ.

We see that $\sin\varphi = h/b$, so $h = b\sin\varphi$.

(b) Find the area of the parallelogram in terms of a, b, and φ.

Let A be the area of the parallelogram. We have $A = ah$, and using information from the previous problem, we know that $A = ab\sin\varphi$.

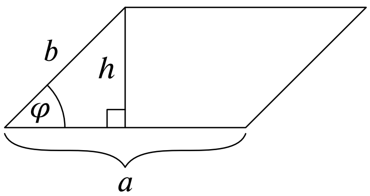


Figure 2: A parallelogram.

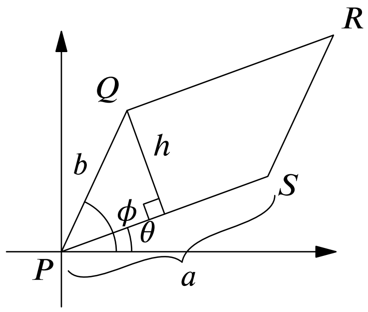


Figure 3: The parallelogram in the xy plane.

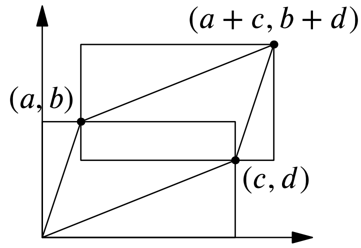


Figure 4: Scenario for Problem 12.

11. In Figure 3, we have put our parallelogram onto the xy plane so that a makes an angle of θ with the x axis and b makes an angle of ϕ with the x axis. Thus, $\varphi = \phi - \theta$.

(a) Rewrite the equation for the area of the parallelogram in terms of θ and ϕ.

We substitute $\varphi = \phi - \theta$ to find that $A = ab\sin(\phi - \theta)$.

(b) Find the x and y coordinates of P, Q, R, S in terms of a, b, ϕ, θ.

The vector $\overrightarrow{PS}$ is just $\langle a\cos\theta, a\sin\theta\rangle$, and the vector $\overrightarrow{PQ}$ is just $\langle b\cos\phi, b\sin\phi\rangle$. Thus, we have the following coordinates for P, Q, R, S:

$$\begin{aligned} P &= (0,0) \\ Q &= P + \overrightarrow{PQ} = (b\cos\phi, b\sin\phi) \\ R &= P + \overrightarrow{PQ} + \overrightarrow{PS} = (a\cos\theta + b\cos\phi, a\sin\theta + b\sin\phi) \\ S &= P + \overrightarrow{PS} = (a\cos\theta, a\sin\theta) \end{aligned}$$

(c) Write a matrix so that the first column contains the coordinates of S and the second column contains the coordinates of Q. This matrix maps the plane.

The matrix is $M = \begin{bmatrix} a\cos\theta & b\cos\phi \\ a\sin\theta & b\sin\phi \end{bmatrix}$, with S in the first column and Q in the second. Thus, $(1,0)$ is mapped to S and $(0,1)$ is mapped to Q.

(d) Your matrix has two diagonals. One rises from left to right and the other descends from left to right. Subtract the product of the entries of the descending diagonal from the product of those of the ascending diagonal.

This is the determinant of the matrix M in the previous answer, which can be computed as

$$\det M = \underbrace{(a\cos\theta)(b\sin\phi)}_{\text{descending diagonal}} - \underbrace{(b\cos\phi)(a\sin\theta)}_{\text{ascending diagonal}} = ab\cos\theta\sin\phi - ab\cos\phi\sin\theta.$$

(e) Use angle addition formulas to simplify your answer.

$$ab\cos\theta\sin\phi - ab\cos\phi\sin\theta = ab(\cos\theta\sin\phi - \cos\phi\sin\theta) = ab\sin(\phi-\theta).$$

(f) You should find some relationship between your answers to problems 11a and 11d. What is it?

The answers are equal!

(g) The difference of the products of the two diagonals of a 2×2 matrix is called the determinant of the matrix, written $\det\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] = ad - bc$. What does it measure?

It measures the area (in terms of magnitude) of the parallelogram formed by the two vectors $\langle a, b\rangle$ and $\langle c, d\rangle$. The determinant corresponds to the area of the parallelogram.

(h) Find a matrix which produces a rotation. What is its determinant?

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is such a matrix. It has a determinant of $(0)(0) - (-1)(1) = 1$.

(i) Find a matrix that produces a reflection.

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is such a matrix.

iv. What is the absolute value of its determinant?

The absolute value of its determinant is $|(-1)(1) - (0)(0)| = |-1| = 1$.

v. How does its determinant differ from that of a rotation matrix?

It is negative, while a rotation matrix's determinant is positive.

vi. What property is not conserved under reflection?

Orientation (or "handedness," "chirality," whatever you want to call it) is not preserved.

(j) Find a matrix that produces a dilation.

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is such a matrix.

vii. What is its determinant?

It has a determinant of $(2)(2) - (0)(0) = 4$.

viii. What does the size of its determinant indicate?

This size of its determinant indicates the amount by which areas will be scaled by. If it is negative, than the orientation is changed. If it is zero, then the transformation is degenerate; it's a mapping to a line or the origin.

12. Here is another way to think about the area of the image of the unit square under a linear transformation. First, we use the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ to transform the unit square into a parallelogram. Then, we graph the image.

(a) There are three rectangles and four triangles in Figure 4. Find the dimensions and the area of each one. You can use this information to figure out the area of the parallelogram in terms of a, b, c, and d. Write a sentence or equation explaining how you can use the seven areas to find the area of the parallelogram.

Possible sentence: The area of the parallelogram is the sum of the areas of the two big rectangles, minus the sum of all the areas shaded in bricks. Possible equation: $A_{\text{parallelogram}} = A_{\text{large rectangles}} - A_{\text{triangles}} - A_{\text{small rectangle}}$.

(b) Carry out the algebra to find the area.

The large rectangles are each c units wide and b units tall, so they have a total area of $2bc$. There are two sets of two congruent triangles here. We can combine each set together to create two rectangles with an area of ab and cd, respectively. Finally, the small rectangle has area $(c-a)(b-d)$. The area of the parallelogram is

$$2bc - (ab + cd) - (c - a)(b - d) = 2bc - (ab + cd) - (bc + ad - ab - cd) = bc - ad.$$

(c) Calculate the determinant of the matrix.

The determinant is $\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$.

(d) What is the relationship between the determinant of the matrix and the area of its associated parallelogram?

The determinant of the matrix is the area of its associated parallelogram negated. In mathematical terms, $A_{parallelogram} = bc - ad = -(ad - bc) = -\det \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

(e) Consider what happens if (a, b) and (c, d) switch places in the graph.

i. How would the area you calculated be different?

The new area would be $ad - bc$, and thus the negative of what it was before!

ii. What property would now be preserved by the transformation?

Orientation (or chirality or handedness whatever) would now be preserved.

iii. What isometry would have been included in any composition of simple transformations yielding the mapping?

Any such composition would require a reflection, because all the other simple transformations do not flip.

iv. What would be true of the determinant?

The determinant would be positive, since it is now equal to the (necessarily positive) area of the parallelogram.

(f)

i. What does a reversal of the orientation of figure in its image say about the determinant of the transformation matrix?

It says that the determinant is negative.

ii. What does that same property of the determinant imply that a transformational matrix does?

It implies that a transformation matrix reverses the orientation of a figure.

iii. What isometry reverses orientation?

Reflections reverse orientation; rotations don't.

(g)

i. What would have happened to the parallelogram if we replaced c, d in the matrix with kc, kd for some $k > 0$, so that the transformation matrix is $\begin{bmatrix} a & kc \\ b & kd \end{bmatrix}$?

The parallelogram would become lengthened in the direction of (c, d), by a factor of k. See the diagram:

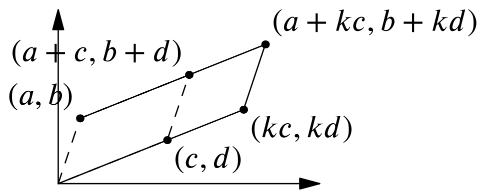


Figure 5: The parallelogram gets scaled by a factor of k. In this case, $k = 1.7$.

ii. What would its area be?

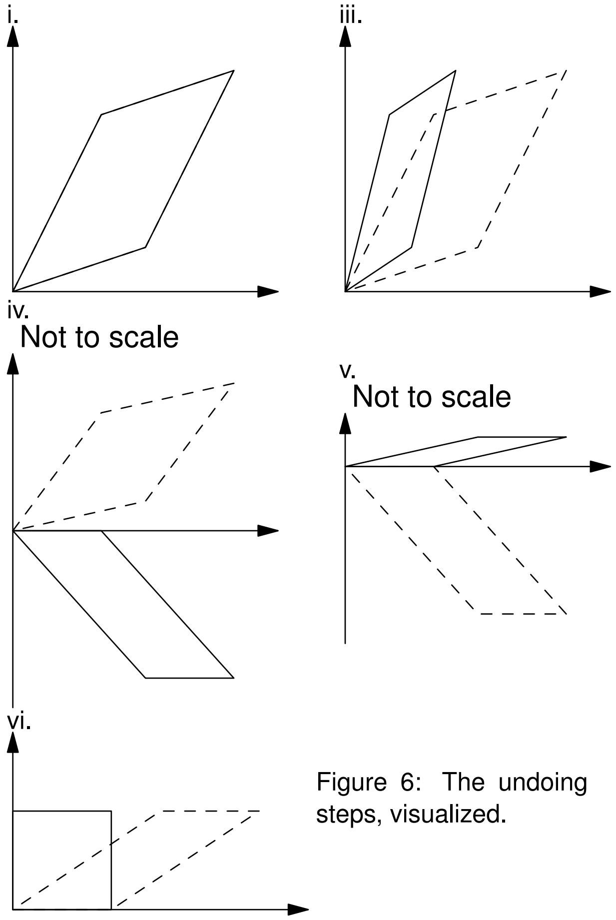
Its area would be scaled by a factor of k; it would be $k(bc - ad)$.

iii. What would the determinant of the matrix be?

The determinant of the matrix would be $k(ad - bc)$.

iv. What if $[\, b \; d \,] = r [\, a \; c \,]$? That is, what if the second row of the matrix was a linear multiple of the first row?

In this case, the the two vectors constructing the parallelogram are collinear. Thus, the transformation collapses to a line.



i. The determinant is not 0, so we continue.

ii. $a \neq 0$, so we continue.

iii. $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 4 & 1 \end{bmatrix}$.

iv. $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -5 \end{bmatrix}$.

v. $\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$.

vi. $\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Figure 6: The undoing steps, visualized.

13. Look at Figure 6 and describe the transformation in each step.

iii. We stretch by a factor of $\frac{1}{2}$ in the x direction. iv. We shear by a factor of -4 in the y direction (perpendicular to the x direction). v. We stretch by a factor of $-\frac{1}{5}$ in the y direction. vi. We shear by a factor of $-\frac{3}{2}$ in the x direction.

14.

(a) How do you undo a shear in the x direction? $\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) How do you undo a stretch along the x-axis? $\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} \frac{1}{x} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) How do you undo a shear in the y direction? $\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(d) How do you undo a stretch along the y-axis? $\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{y} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

15. **Now let's put this all together. Undo each of the operations in turn, until only matrix $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ remains on the left side. Remember that what you do on the left side of the expression must also be done to the right side, so on the right side you will see the basic operations from which $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ is composed. Order is important!**

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$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

These are the matrices all filled in:

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{ad-bc}{a} \end{bmatrix}\begin{bmatrix} 1 & \frac{c}{a} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

16. **Each step in the decomposition of $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$ is explained below.**

    (i) Stretch along the x-axis by factor of $\frac{1}{3}$.

$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -5 \end{bmatrix}$$

    (ii) Shear perpendicular to the x-axis by -2

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{23}{3} \end{bmatrix}$$

    (iii) Stretch along y-axis by $-\frac{3}{23}$

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{23} \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{23}{3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}$$

    (iv) Shear perpendicular to the y-axis by $-\frac{4}{3}$

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

    Taken all together, the decomposition is:

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{23} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}.$$

    Therefore:

$$\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{23}{3} \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}.$$

    What does each matrix do?

Going from right to left, the first matrix shears by $\frac{4}{3}$ in the x direction (perpendicular to the y-axis). The second matrix stretches by $-\frac{23}{3}$ in the y direction. The third matrix shears by 2 in the y direction (perpendicular to the x-axis). The fourth and final matrix stretches by 3 in the x direction.

17. **Here is another way that you could have decomposed the above matrix.**

$$\overset{\textbf{i}}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{13}{23} \end{bmatrix}}\overset{\textbf{ii}}{\begin{bmatrix} 1 & -\frac{2}{23} \\ 0 & 1 \end{bmatrix}}\overset{\textbf{iii}}{\begin{bmatrix} \frac{1}{\sqrt{13}} & 0 \\ 0 & \frac{1}{\sqrt{13}} \end{bmatrix}}\overset{\textbf{iv}}{\begin{bmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}}\overset{\textbf{v}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

    (a) Identify what matrices i through v each do.

i scales by $\frac{13}{23}$ along the y-axis. ii shears by $-\frac{2}{23}$. iii scales by $\frac{1}{\sqrt{13}}$. iv rotates by $\tan^{-1}\frac{2}{3}$. v reflects over the x-axis (since the y coordinate is being flipped).

17. (cont.) Next, we undo this sequence of operations by working backwards.

$$\overset{\text{i}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \overset{\text{ii}}{\begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}} \overset{\text{iii}}{\begin{bmatrix} \sqrt{13} & 0 \\ 0 & \sqrt{13} \end{bmatrix}} \overset{\text{iv}}{\begin{bmatrix} 1 & \frac{2}{23} \\ 0 & 1 \end{bmatrix}} \overset{\text{v}}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{23}{13} \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}.$$

(b) Explain what happens at each matrix, i through v.

i reflects over the x-axis, since the y coordinate is being flipped. ii rotates by $-\tan^{-1}\frac{2}{3}$. iii scales by $\sqrt{13}$. iv shears by $\frac{2}{23}$ along the x-axis. Finally, v scales by $\frac{23}{13}$.

18. Find a set of basic transformations which is equivalent to each of the following matrices.

(a) $\begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix}$

(Answers may vary.)
Using the method we described, we first stretch along the x-axis by $\frac{1}{12}$:

$$\begin{bmatrix} \frac{1}{12} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{3} \\ 5 & 15 \end{bmatrix}.$$

We then shear along the y-axis by -5:

$$\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & \frac{35}{3} \end{bmatrix}.$$

We then stretch along the y-axis by $\frac{3}{35}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{35} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix}.$$

Finally, we shear by $-\frac{2}{3}$ along the x-axis:

$$\begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{35} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now we undo the operations in turn:

$$\underbrace{\begin{bmatrix} 12 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{35}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{35} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Longrightarrow \begin{bmatrix} 12 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{35}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix}.$$

(b) $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$

(Answers may vary.)
Using the method we described, we first stretch along the x-axis by $\frac{1}{3}$:

$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 4 & 7 \end{bmatrix}.$$

We then shear along the y-axis by -4:

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & -25 \end{bmatrix}.$$

We then stretch along the y-axis by $-\frac{1}{25}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{25} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}.$$

Finally, we shear by -8 along the x-axis:

$$\begin{bmatrix} 1 & -8 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{25} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We undo the operations in turn:

$$\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -25 \end{bmatrix}\begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 & -8 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{25} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -25 \end{bmatrix}\begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}.$$

(c) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

This matrix can't be decomposed with our usual method! That's because it lacks an inverse transformation, since it's projecting to a line (namely, the line $y = 2x$).

Let's think a bit more laterally here. What of our operations take the whole plane to a line? Shears don't, rotations and reflections don't, but stretches with a factor of 0 can. For example, we can take everything to the x-axis with the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, we should be able to decompose our matrix as

$$A\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

for some matrices A, B. If we want to take the x-axis to the line $y = 2x$, which is the job of matrix A (remember, right to left!), we'll need A to be a shear by a factor of 2 along the y-axis. (See Figure 7.) Thus,

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

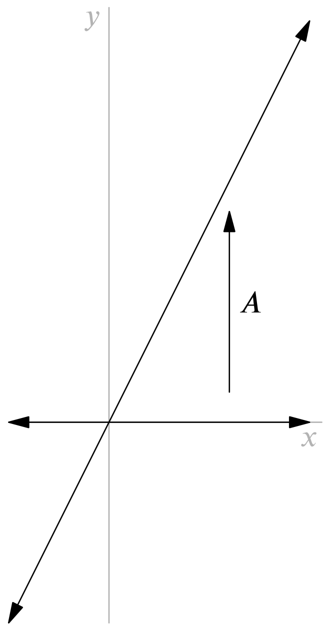

Figure 7: The shearing action of A on the x-axis.

To find B, we solve the matrix equation. We have

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}.$$

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}.$$

Interesting! c, d can be any real numbers, because their effect is nullified by the mapping to a line. But $a = 2$ and $b = 3$. Let's just choose $c = 0$ and $d = 1$ for simplicity of decomposition. We have

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = B$$

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We undo the new matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} = B.$$

Thus, our full decomposition is

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}}_{B} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}.$$

In words, this is a shear of $\frac{3}{2}$ along the x-axis, followed by a stretch of a factor of 2 along the x-axis, a stretch of a factor of 0 along the y-axis, and a shear of 2 along the y-axis.

As an aside, here's what happens if you try using the usual method. We first stretch along the x-axis by $\frac{1}{2}$, then shear along the y-axis by -2:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 2 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}.$$

Here, our problem arises. We cannot make the bottom-right corner 1.

19. One of the matrices in Problem 18 is a projection onto a line.

(a) Which matrix is it?

Problem (c): $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ is the matrix.

(b) What line does it project onto?

It projects onto the line $y = 2x$.

(c) If you try to decompose this matrix to the identity matrix, what happens? Why?

You get stuck, because it lacks an inverse! The details are given as an aside at the end of that problem.

20. Onto what line does $\left[\begin{smallmatrix} a & b \\ 2a & 2b \end{smallmatrix}\right]$ map the plane? Solve for a and b such that the matrix projects perpendicular onto the line. Hint: The projection of a point already on the line should not move. Choose a suitable point and solve a system of two equations with two unknowns.

Multiply this matrix by some vector (x, y). We find that it is mapped onto

$$\begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ 2ax + 2by \end{bmatrix},$$

which falls on the line $y = 2x$. To project perpendicular on the line, we do as the problem suggests. Consider the point $(1, 2)$, which is on the line. The image could generally be any point on the line, but if the transformation is a *projection*, then the point should not move under the transformation. Thus,

$$\begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The left side expands out to $\begin{bmatrix} a + 2b \\ 2a + 4b \end{bmatrix}$. Thus, as long as $a+2b = 1$, the matrix will be a projection onto $y = 2x$. A simple example is $a = b = \frac{1}{3}$, so a possible matrix is

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

21. Use Problem 20 to decompose $\left[\begin{smallmatrix} 2 & 3 \\ 4 & 6 \end{smallmatrix}\right]$ into a projection to a line followed by a size change.

We see that this matrix is of the form $M = \left[\begin{smallmatrix} a & b \\ 2a & 2b \end{smallmatrix}\right]$, but is not quite a projection since $(a, b) = (2, 3)$ and $a + 2b = 8 \neq 1$. If we scale the matrix by $\frac{1}{8}$, we then do get a projection. Thus, the decomposition is

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix},$$

where the second matrix is the projection and the first matrix is to scale it back up to our original.

22. Decompose $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right]$ into a projection perpendicular to a line followed by a size change.

This matrix maps onto the line $y = 3x$, since it's of the form $\begin{bmatrix} a & b \\ 3a & 3b \end{bmatrix}$, but is not a projection. We try a similar method as the last problem to find the matrix of an actual projection, choosing the point $(1, 3)$:

$$\begin{bmatrix} a & b \\ 3a & 3b \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} a + 3b \\ 3a + 9b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

So $a + 3b = 1$. We see that $1 + 3(2) = 7$, so we scale down the matrix by a factor of 7, giving

$$\begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{6}{7} \end{bmatrix},$$

so the decomposition is

$$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{6}{7} \end{bmatrix}.$$

23. Write matrices which project onto the following lines:

(a) $y = x$

The matrix will be of the form $\begin{bmatrix} a & b \\ a & b \end{bmatrix}$. We choose the fixed point $(1, 1)$:

$$\begin{bmatrix} a & b \\ a & b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a+b \\ a+b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So $a + b = 1$. For simplicity, we choose $a = b = \frac{1}{2}$, giving the matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(b) $y = 5x$

The matrix will be of the form $\begin{bmatrix} a & b \\ 5a & 5b \end{bmatrix}$. We choose the fixed point $(1, 5)$:

$$\begin{bmatrix} a & b \\ 5a & 5b \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} a+5b \\ 5a+25b \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

So $a + 5b = 1$. For simplicity, we choose $a = b = \frac{1}{6}$, giving the matrix

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{5}{6} \end{bmatrix}.$$

(c) $y = mx$

This is pretty easy to generalize. The matrix will be of the form $\begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$. We choose the fixed point $(1, m)$:

$$\begin{bmatrix} a & b \\ ma & mb \end{bmatrix} \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}$$

$$\begin{bmatrix} a+mb \\ ma+m^2b \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}.$$

So $a + mb = 1$. For simplicity, we choose $a = b = \frac{1}{m+1}$, giving the matrix

$$\begin{bmatrix} \frac{1}{m+1} & \frac{1}{m+1} \\ \frac{m}{m+1} & \frac{m}{m+1} \end{bmatrix}.$$