

6 Geometry of Complex Numbers

1. Explain why iz is perpendicular to z , without using DeMoivre's theorem.

Let $z = a + bi$. Then $iz = i(a + bi) = -b + ai$, which is the transformation $(a, b) \rightarrow (-b, a)$. Drawing this out on the 2D plane makes clear that the angle between the two points and the origin is 90° , simply by subtracting angles: $(90^\circ + \theta) - \theta = 90^\circ$. This is shown in Figure 1.



Figure 1: iz is perpendicular to z as long as $z \neq 0$.

You can also explain it by observing that the lines through z/iz and the origin have slopes of $\frac{b}{a}$ and $-\frac{a}{b}$, respectively, so they must be perpendicular. In terms of vectors, $\langle a, b \rangle \cdot \langle -b, a \rangle = 0$, with \cdot being the dot product.

2. How does $\text{Arg } \bar{z}$ relate to $\text{Arg } z$? (Hint: symmetry!)

Again, let $z = a + bi$. $\bar{z} = a - bi$ is z flipped over the x -axis, since the imaginary part is negated. Thus, $\text{Arg } \bar{z} = -\text{Arg } z$ due to congruent triangles formed by z and \bar{z} .¹¹ The geometric interpretation is shown in Figure 2.

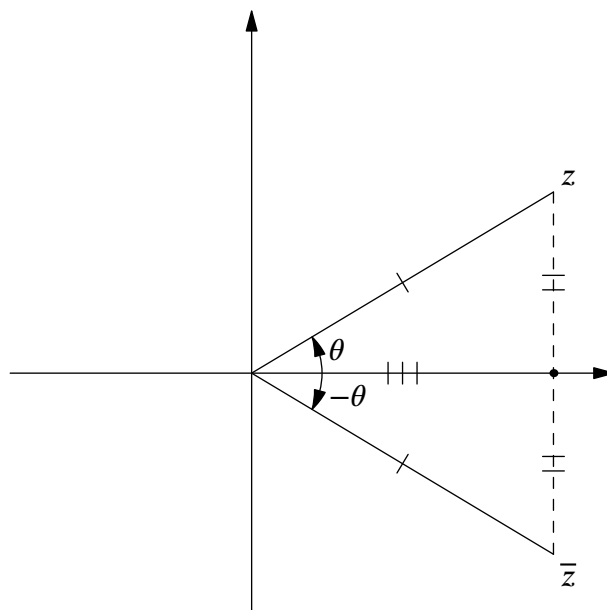


Figure 2: z and \bar{z} form congruent triangles, showing that $\text{Arg } z = -\text{Arg } \bar{z}$.

¹¹The pedants may criticize my use of the equality symbol here; take this equality as modulo 2π if you must. Note that you may eventually learn that \arg is a multivalued function.

3. Compute $z\bar{z}$ and relate it to the cis form of z .

Once more, let $z = a + bi$. Then

$$z\bar{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

If $z = r \operatorname{cis} \theta$, then $z\bar{z} = r^2$. In other words, it is the square of the distance from z to the origin.

4. Explain, using a picture, why $\tan(\operatorname{Arg} z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$.

This is basically just an application of soh-cah-toa to a triangle in the complex plane. The details are shown in Figure 3.



Figure 3: $\tan(\operatorname{Arg} z) = \frac{b}{a} = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$.

5. Divide $\frac{a+bi}{c+di}$ by rationalizing the denominator.

$$\begin{aligned} \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} &= \frac{(a+bi)(c-di)}{c^2+d^2} \\ &= \frac{ac+bd+(bc-ad)i}{c^2+d^2}. \end{aligned}$$

6. Divide $\frac{r_1 \operatorname{cis} \theta}{r_2 \operatorname{cis} \phi}$ using DeMoivre's theorem.

We don't have a rule yet for applying DeMoivre's theorem for division, but we can quickly derive it. We have

$$\begin{aligned} \frac{r_1 \operatorname{cis} \theta}{r_2 \operatorname{cis} \phi} \cdot \frac{\overline{\operatorname{cis} \phi}}{\overline{\operatorname{cis} \phi}} &= \frac{r_1 \operatorname{cis} \theta \overline{\operatorname{cis} \phi}}{r_2 \underbrace{\operatorname{cis} \phi \overline{\operatorname{cis} \phi}}_{=1}} && \text{Multiplying by conjugate} \\ &= \frac{r_1 \operatorname{cis} \theta \operatorname{cis}(-\phi)}{r_2} && \text{Using } \operatorname{Arg} z = -\operatorname{Arg} \bar{z} \\ &= \frac{r_1}{r_2} \operatorname{cis}(\theta - \phi). && \text{Use DeMoivre's theorem} \end{aligned}$$

7. Compare and contrast the methods of division in Problems 5 and 6. Which is more convenient? Or does it depend on the circumstance?

Opinions may vary, but Problem 5's method is definitely faster to do if the dividend and divisor are already in cis form. Problem 6's is likely more convenient than converting from rectangular to cis, then back to rectangular.

8.

(a) If $z = r \operatorname{cis} \theta$, what is $\frac{1}{z}$?

As we hinted at in the previous problem, $\frac{1}{z} = \frac{1}{r} \operatorname{cis}(-\theta)$:

$$\begin{aligned} \frac{1}{r \operatorname{cis} \theta} \cdot \frac{\overline{\operatorname{cis} \theta}}{\operatorname{cis} \theta} &= \frac{\overline{\operatorname{cis} \theta}}{r \operatorname{cis} \theta \operatorname{cis} \theta} \\ &= \frac{\operatorname{cis}(-\theta)}{r |\operatorname{cis} \theta|^2} \\ &= \frac{1}{r} \operatorname{cis}(-\theta). \end{aligned}$$

(b) Explain how this shows $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$, without having to rationalize the denominator. (Hint: use problems 3, 4, and 7.)

Let $a + bi = r \operatorname{cis} \theta$. We have

$$\begin{aligned} \frac{1}{r \operatorname{cis} \theta} &= \frac{1}{r} \operatorname{cis}(-\theta) \\ &= \frac{r \operatorname{cis}(-\theta)}{r^2} \\ &= \frac{a - bi}{a^2 + b^2}. \end{aligned}$$

9. Compute $(1 + i)^{13}$; pencil, paper, and brains only. No calculators!

We have $1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$, since it forms a 45° angle with the x -axis. Applying DeMoivre's theorem,

$$\begin{aligned} (1 + i)^{13} &= \left(\sqrt{2} \operatorname{cis} \frac{\pi}{4} \right)^{13} \\ &= \left(\sqrt{2} \right)^{13} \operatorname{cis} \frac{13\pi}{4} \\ &= 64\sqrt{2} \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) \\ &= 64\sqrt{2} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \\ &= 64(-1 - i) \\ &= -64 - 64i. \end{aligned}$$

10. Compute $\frac{(1+i\sqrt{3})^3}{(1-i)^2}$ without a calculator.

We convert to cis form and apply DeMoivre's theorem.

$$\begin{aligned} \frac{(1 + i\sqrt{3})^3}{(1 - i)^2} &= \frac{\left(2 \operatorname{cis} \left(\frac{\pi}{3} \right) \right)^3}{\left(\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right) \right)^2} \\ &= \frac{8 \operatorname{cis}(\pi)}{2 \operatorname{cis} \left(-\frac{\pi}{2} \right)} \\ &= \frac{8 \cdot -1}{2 \cdot -i} \\ &= \frac{4}{i} \cdot \frac{-i}{-i} \\ &= -4i. \end{aligned}$$

11. Draw $\text{cis}\left(\frac{\pi}{4}\right) + \text{cis}\left(\frac{\pi}{2}\right)$. Use your picture to prove an expression for $\tan\left(\frac{3\pi}{8}\right)$. (Hint: add them as vectors.)

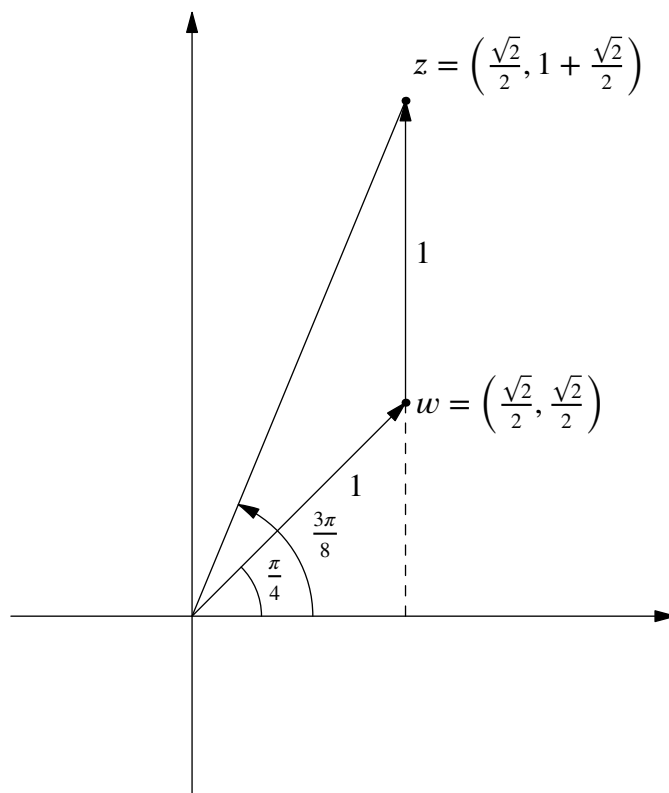


Figure 4: Addition of $\text{cis}\left(\frac{\pi}{4}\right) + \text{cis}\left(\frac{\pi}{2}\right)$ as vectors.

The drawing is shown in Figure 4. The first vector, starting at the origin, is $\text{cis}\frac{\pi}{4}$. The second vector, starting at the endpoint of the first vector, is $\text{cis}\frac{\pi}{2}$. The origin, along with points w and z , form an isosceles triangle. Furthermore, the apex of this triangle, at w , has an argument of $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$ radians. Thus, the base angles of the isosceles triangle are

$$\frac{\pi - \frac{3\pi}{4}}{2} = \frac{\pi}{8}.$$

Adding this with $\frac{\pi}{4}$ shows that the angle z forms with the x -axis is

$$\frac{\pi}{8} + \frac{\pi}{4} = \frac{3\pi}{8},$$

our desired angle to analyze. We wish to find the tangent of this angle, which is just $\tan(\text{Arg } z)$. But we know how to compute that!

$$\begin{aligned} \tan(\text{Arg } z) &= \frac{\text{Im}(z)}{\text{Re}(z)} = \frac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2} + 1}{1} \\ \tan\left(\frac{3\pi}{8}\right) &= \sqrt{2} + 1. \end{aligned}$$

12. Solve $z^3 = 1$, and show that its solutions under the operation of multiplication form a group, isomorphic to the rotation group of the equilateral triangle. Write a group table!

There's numerous ways to solve this, but let's use cis form as usual. Let $z = r \operatorname{cis} \theta$. Then

$$\begin{aligned}
 z^3 &= r^3 \operatorname{cis} 3\theta = 1 \\
 \implies r &= 1 \\
 \operatorname{cis} 3\theta &= 1 \\
 \cos 3\theta &= 1 \\
 3\theta &= 2\pi k && \text{For } k \in \mathbb{Z} \\
 \theta &= \frac{2\pi k}{3} \implies \theta \in \left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right\} \\
 \implies z &\in \left\{ \operatorname{cis} 0, \operatorname{cis} \frac{2\pi}{3}, \operatorname{cis} \frac{4\pi}{3} \right\}.
 \end{aligned}$$

Under multiplication, these three values of z indeed form a group isomorphic to the rotation group of the equilateral triangle, C_3 . In particular, $\operatorname{cis} 0$ is the identity, $\operatorname{cis} \frac{2\pi}{3}$ is a rotation by 120° counterclockwise, and $\operatorname{cis} \frac{4\pi}{3}$ is a rotation by 240° counterclockwise. Let $I = \operatorname{cis} 0$, $r = \operatorname{cis} \frac{2\pi}{3}$, and $r^2 = \operatorname{cis} \frac{4\pi}{3}$. Then, we have the following group table:

\cdot	I	r	r^2
I	I	r	r^2
r	r	r^2	I
r^2	r^2	I	r

13.

(a) Find multiplication groups of complex numbers which are isomorphic to the rotation groups for

i. a non-square rectangle

Since this rotation group is just the identity and a rotation of 180° , we can just choose the group $\{-1, 1\}$ under multiplication. 1 is the identity, and $-1 = \operatorname{cis} 180^\circ$ is the rotation.

ii. a regular hexagon.

We follow in the footsteps of the equivalent problem for the equilateral triangle. We have elements

$$\left\{ \operatorname{cis} 0, \operatorname{cis} 60^\circ, \operatorname{cis} 120^\circ, \operatorname{cis} 180^\circ, \operatorname{cis} 240^\circ, \operatorname{cis} 300^\circ \right\}.$$

These are indeed rotations of $0, 60^\circ, 120^\circ, 180^\circ, 240^\circ$, and 300° , respectively. This set under multiplication is isomorphic to the rotation group of the hexagon, C_6 .

(b) Make a table for each group.

i. a non-square rectangle

Let I be the identity and r be the rotation of 180° .

\cdot	I	r
I	I	r
r	r	I

ii. a regular hexagon.

Let I be the identity and r be the rotation of 60° . r^n is defined in the natural way, by raising $\operatorname{cis} 60^\circ$ to the power n .

\cdot	I	r	r^2	r^3	r^4	r^5
I	I	r	r^2	r^3	r^4	r^5
r	r	r^2	r^3	r^4	r^5	I
r^2	r^2	r^3	r^4	r^5	I	r
r^3	r^3	r^4	r^5	I	r	r^2
r^4	r^4	r^5	I	r	r^2	r^3
r^5	r^5	I	r	r^2	r^3	r^4

(c) Compare the regular hexagon's group to the dihedral group of the equilateral triangle, D_3 . Consider: how are they the same? How are they different? Is the difference fundamental?

The two groups are not isomorphic, although they are the same size; the difference is fundamental. The hexagon's rotation group, C_6 , has elements of periods $\{1, 2, 3, 3, 6, 6\}$, while D_3 has elements of periods $\{1, 2, 2, 2, 3, 3\}$. They do share some subgroups however: the trivial subgroup of just the identity, and the subgroups generated by r^2 and by r^3 in C_6 , which are C_3 and C_2 respectively.

14. Which of the following sets is a group under (i) addition and (ii) multiplication?

(a) $\{0\}$

This is a group under (i) addition, since it has an identity 0, is closed, has 0 as 0's inverse, and $0 + (0 + 0) = (0 + 0) + 0$. It also is a group under (ii) multiplication, for the same reasons.

(b) $\{1\}$

This is not a group under (i) addition, since $1 + 1 = 2 \notin \{1\}$. It is a group under multiplication, though, since $1 \cdot 1 = 1$ and all other properties are satisfied.

(c) $\{0, 1\}$

This is not a group under (i) addition, since $1 + 1 = 2 \notin \{0, 1\}$. It is also not a group under (ii) multiplication. 0 can't be the identity, since $1 \cdot 0 = 0 \neq 1$. 1 also can't be the identity, since then 0 has no inverse K such that $0 \cdot K = 1$.

(d) $\{-1, 1\}$

This is not a group under (i) addition, since $1 + (-1) = 0 \notin \{\pm 1\}$. It is a group under (ii) multiplication, since it satisfies the group properties:

1. Identity: 1 is the identity
2. Associativity: Multiplication is associative
3. Invertibility: Each element is its own inverse
4. Closure: $(\pm 1)(\pm 1) \in \{\pm 1\}$

(e) $\{1, -1, i, -i\}$

This is not a group under (i) addition, since the sum of any two of the elements takes you out of the set. It is a group under (ii) multiplication, however. One easy way to see this is that $1 = \text{cis } 0$, $-1 = \text{cis } \pi$, $i = \text{cis } \frac{\pi}{2}$, and $-i = \text{cis } \frac{3\pi}{2}$, which are all rotations of multiples of 90° . In particular, it is isomorphic to C_4 , the rotation group of the square.¹²

(f) $\{\text{naturals}\}$

This is not a group under (i) addition, because it cannot satisfy invertibility. There is no element $X \in \mathbb{N}$ such that $1 + X = I = 0$, for example. This is also not a group under (ii) multiplication for the same reason.

(g) $\{\text{integers}\}$

This is a group under (i) addition, because all the group properties are satisfied. The inverse of an element n is just $-n$, addition is associative, the identity is 0, and the sum of two integers is another integer. It is not a group under (ii) multiplication, because no numbers except ± 1 have integer multiplicative inverses.

(h) $\{\text{rationals}\}, \mathbb{Q}$

¹²Note that the *only* sets of complex numbers that form multiplication groups isomorphic to C_n ($n \geq 1$) are $\left\{ \text{cis } \frac{2\pi m}{n} \right\}$ ($0 \leq m < n$). It's fun to prove this!

This is a group under (i) addition with identity element 0. The inverse of an element $\frac{p}{q}$ is $-\frac{p}{q}$, addition is associative, and the sum of two rational numbers is another rational number. It is not a group under (ii) multiplication, because no number is the multiplicative inverse of 0.

(i) $\{\mathbb{Q} \text{ without zero}\}$

This is no longer a group under (i) addition, since the identity element needs to be 0. It is now, however, a group under (ii) multiplication, because all numbers have their inverses. Multiplication is associative, the inverse of $\frac{p}{q}$ is $\frac{q}{p}$, and the product of two rationals is another rational.

(j) $\{\text{complex numbers}\}, \mathbb{C}$

This is a group under (i) addition with identity element 0. The inverse of an element z is $-z$, addition is associative, and the sum of two complex numbers is another complex number. This is not a group under (ii) multiplication, because again, no number is the multiplicative inverse of 0.

(k) $\{\mathbb{C} \text{ without zero}\}$

This is no longer a group under (i) addition, since the identity element needs to be 0. It is now, however, a group under (ii) multiplication, because all numbers have their inverses. Multiplication is associative, the inverse of z is $\frac{1}{z}$, and the product of two complex numbers is another complex number.

15. Prove that $(r_1 \operatorname{cis} \theta)(r_2 \operatorname{cis} \phi) = r_1 r_2 \operatorname{cis}(\theta + \phi)$ using brute force and the angle-sum trig identities for \cos and \sin . Do you prefer this method or the one on the previous page? Which method gives you a better understanding of why DeMoivre's works?

$$\begin{aligned}(r_1 \operatorname{cis} \theta)(r_2 \operatorname{cis} \phi) &= r_1 r_2 (\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) \\&= r_1 r_2 (\cos \theta \cos \phi + i \cos \theta \sin \phi + i \sin \theta \cos \phi - \sin \theta \sin \phi) \\&= r_1 r_2 ((\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\cos \theta \sin \phi + \sin \theta \cos \phi)) \\&= r_1 r_2 (\cos(\theta + \phi) + i \sin(\theta + \phi)) \\&= r_1 r_2 \operatorname{cis}(\theta + \phi)\end{aligned}$$

(Opinions may vary.) I actually prefer this because it's kind of satisfying, but the previous way likely gives a better understanding of the underlying mechanics.

16. Find an identity for $\sin 3\theta$ as we have done for \cos . Most of the work is already done for you!

We already know that

$$\cos 3\theta + i \sin 3\theta = \operatorname{cis} 3\theta = (c^3 - 3c^2s) + i(3c^2s - s^3),$$

where $c = \cos \theta$ and $s = \sin \theta$. Equating imaginary parts, we have

$$\begin{aligned}\sin 3\theta &= 3c^2s - s^3 \\&= 3 \cos^2 \theta \sin \theta - \sin^3 \theta.\end{aligned}$$

17. Your friend's textbook says $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$, different from our identity. Who's right?

Both are right. Our identity is

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta.$$

Remembering that $\sin^2 \theta = 1 - \cos^2 \theta$, we can pretty easily change the form:

$$\begin{aligned}\cos^3 \theta - 3 \cos \theta \sin^2 \theta &= \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta) \\&= \cos^3 \theta + 3 \cos^3 \theta - 3 \cos \theta \\&= 4 \cos^3 \theta - 3 \cos \theta.\end{aligned}$$

18. Now you can finish the rest of the proof.

If you need context for this answer, check out the relevant textbook section.

(a) Draw a, b, c, d, m, n approximately for the quadrilateral on the previous page.

The quadrilateral is shown in Figure 5.

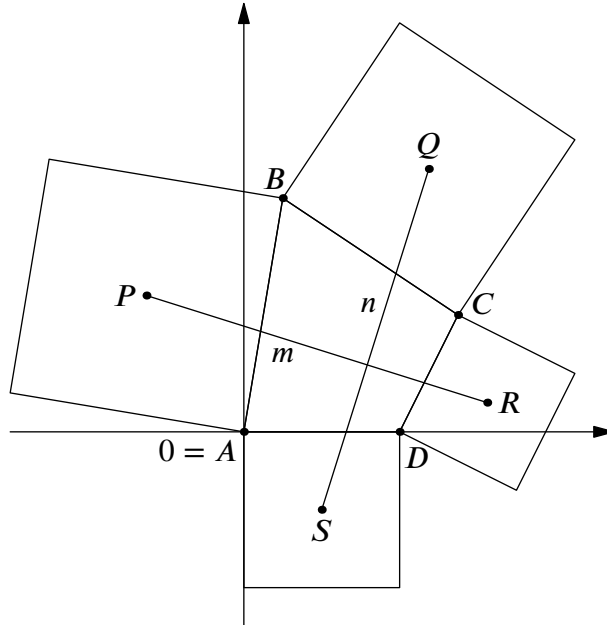


Figure 5: The quadrilateral to analyze.

The relative magnitudes and directions are shown in Figure 6 below. We find a, b, c, d from halving the sides of the quadrilateral. m and n are just the vectors from P to R and Q to S , respectively.

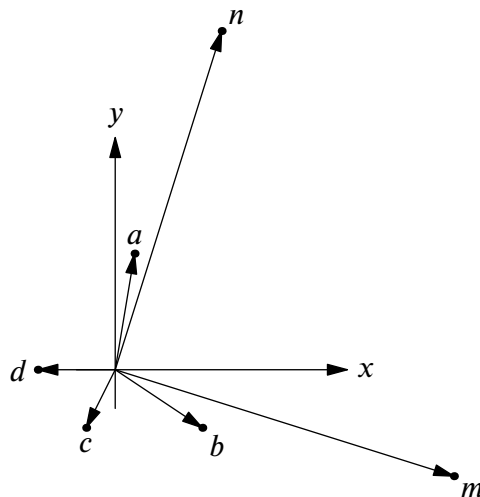


Figure 6: The relative magnitudes and directions of a, b, c, d, m, n .

(b) Why does showing $n = \pm im$ prove the segments are (i) perpendicular and (ii) the same length?

They are (i) perpendicular because iz is perpendicular to z for all $z \neq 0$, and (ii) are the same length because $|n| = |\pm im| = |im| = |m|$.

(c) Explain why $Q = 2a + b + ib$.

The justification is geometric. We know that $B = 2a$, and we can get to the midpoint of \overline{BC} by adding b . Then, we go up to the center of the square on \overline{BC} by adding ib . This process is shown in Figure 7.



Figure 7: $Q = 2a + b + ib$.

(d) Find formulae for R and S in terms of c and d .

In a similar fashion, we have $S = -d + id$ (note that it is $-d$ because we are going counterclockwise now) and $R = -2d - c + ic$. The interpretations of these are shown in Figure 8 below.



Figure 8: $S = -d + id$ and $R = -2d - c + ic$.

(e) Find m and n in terms of a, b, c , and d .

We have $m = R - P = (-2d - c + ic) - (a + ia)$ and $n = Q - S = (2a + b + ib) - (-d + id)$.

(f) Check that $n - im = 0$, using the fact that $a + b + c + d = 0$.

We evaluate straightforwardly:

$$\begin{aligned}
 (2a + b + ib + d - id) - i(-2d - c + ic - a - ia) &= 2a + b + ib + d - id + 2id + ic + c + ia - a \\
 &= a + b + c + d + ia + ib + ic + id \\
 &= (a + b + c + d)(1 + i) \\
 &= 0.
 \end{aligned}$$

19. In the previous problem, we drew squares outside a quadrilateral and connected their centers. Conjecture what happens if we draw equilateral triangles outside a triangle and connect their centers. Prove your conjecture using complex numbers.

We conjecture that following this construction leads to connecting together another equilateral triangle. An example, with the variables we'll use labeled, is shown in Figure 9.



Figure 9: Equilateral triangles around a central, arbitrary triangle $\triangle ABC$ with A at the origin.

Similar to the last problem, let A , B , and C be numbers in the complex plane. Without loss of generality, let $A = 0$ be the origin. Also, define $a = \frac{B-A}{2}$, $b = \frac{C-B}{2}$, and $c = \frac{A-C}{2}$ to be the vectors going halfway along each of \overline{AB} , \overline{BC} , and \overline{CA} . Finally, let P , Q , and R be the centers of the triangles on sides AB , BC , and CA respectively.

Consider Q in the figure. It is on the 60° vertex of a 30-60-90 triangle $\triangle BB'Q$, outlined in dotted line. We know that $\overline{BB'} = b$. Thus, since $BB' : B'Q = \sqrt{3} : 1$,

$$B'Q = \frac{|b|}{\sqrt{3}}.$$

Furthermore, since $\overline{B'Q} \perp \overline{BB'}$, we know that it is $s \cdot ib$ for some real s . Combining these facts,

$$B'Q = \frac{ib}{|ib|} \frac{|b|}{\sqrt{3}} = \frac{ib}{\sqrt{3}}.$$

Since $Q = \overline{AB} + \overline{BB'} + \overline{B'Q}$, we have

$$Q = 2a + b + \frac{ib}{\sqrt{3}}.$$

With similar logic, we know that

$$P = a + \frac{ia}{\sqrt{3}}$$

$$R = -c + \frac{ic}{\sqrt{3}}$$

Like with the quadrilateral, we know $a + b + c = 0$, since $2(a + b + c) = 0$. To prove the dashed triangle is indeed equilateral, we can just show that $P - R = (Q - P) \text{cis } 120^\circ$. After all, if the vectors \overline{RP} and \overline{PQ} have an angle of 120° between them and they have the same magnitude, $\triangle PQR$ is equilateral by SAS Congruence as shown in Figure 10. Substituting in our found values for P , Q , R in terms of a , b , c , we get

$$\begin{aligned}
P - R &= (Q - P) \operatorname{cis} 120^\circ \\
a + \frac{ia}{\sqrt{3}} - \left(-c + \frac{ic}{\sqrt{3}}\right) &= \left(2a + b + \frac{ib}{\sqrt{3}} - \left(a + \frac{ia}{\sqrt{3}}\right)\right) \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
(a + c) + \frac{ia - ic}{\sqrt{3}} &= \left(a + b + \frac{ib - ia}{\sqrt{3}}\right) \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
&= -\frac{1}{2}a - \frac{1}{2}b - \frac{ib - ia}{2\sqrt{3}} + \frac{\sqrt{3}}{2}ia + \frac{\sqrt{3}}{2}ib + \frac{ib - ia}{2} \cdot i \\
&= \left(-\frac{1}{2}a - \frac{1}{2}b + \frac{ib - ia}{2} \cdot i\right) + \left(-\frac{ib - ia}{2\sqrt{3}} + \frac{\sqrt{3}}{2}ia + \frac{\sqrt{3}}{2}ib\right) \\
&= \left(-\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}b + \frac{1}{2}a\right) + \left(\frac{ia - ib + 3ia + 3ib}{2\sqrt{3}}\right) \\
&= (-b) + \left(\frac{4ia + 2ib}{2\sqrt{3}}\right) \\
&= (-b + a + b + c) + \left(\frac{i(2a + b)}{\sqrt{3}}\right) \\
&= (a + c) + \frac{i(2a + b - (a + b + c))}{\sqrt{3}} \\
(a + c) + \frac{ia - ic}{\sqrt{3}} &= (a + c) + \frac{ia - ic}{\sqrt{3}}
\end{aligned}$$

Tedious, but it worked.



Figure 10: SAS Congruence lets us say that $P - R = (Q - P) \operatorname{cis} 120^\circ$ is sufficient to prove the triangle $\triangle PQR$ is equilateral.

20. The hard way to find an identity for $\tan 3\theta$ is to divide the identity for \sin and \cos that we already found. Try this. Make sure your answer is in terms of \tan only!

We have found that $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$ and $\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$. We set $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and evaluate:

$$\begin{aligned}
\tan \theta &= \frac{\sin \theta}{\cos \theta} = \frac{3 \cos^2 \theta \sin \theta - \sin^3 \theta}{\cos^3 \theta - 3 \cos \theta \sin^2 \theta} \\
&= \frac{\sin \theta}{\cos \theta} \cdot \frac{3 \cos^2 \theta - \sin^2 \theta}{\cos^2 \theta - 3 \sin^2 \theta} \cdot \frac{\frac{1}{\cos^2 \theta}}{\frac{1}{\cos^2 \theta}} \\
&= \tan \theta \cdot \frac{3 - \frac{\sin^2 \theta}{\cos^2 \theta}}{1 - \frac{3 \sin^2 \theta}{\cos^2 \theta}} \\
&= \tan \theta \cdot \frac{3 - \tan^2 \theta}{1 - 3 \tan^2 \theta} \\
&= \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}.
\end{aligned}$$

21. The easier way to get an identity for $\tan 3\theta$ starts with setting $z = 1 + i \tan \theta$.

(a) Why is $\text{Arg } z = \theta$?

You can see this pretty quickly with a diagram, like in Figure 11. More algebraically, we have

$$\begin{aligned}
\tan(\text{Arg } z) &= \text{Im}(z) = \tan(\theta) \\
\Rightarrow \text{Arg } z &= \theta.
\end{aligned}$$



Figure 11: $\text{Arg } z = \tan^{-1} \left(\frac{\tan \theta}{1} \right) = \theta$.

(b) Why is $\tan 3\theta = \frac{\text{Im}(z^3)}{\text{Re}(z^3)}$?

We have

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{\text{Im}(\text{cis } 3\theta)}{\text{Re}(\text{cis } 3\theta)}.$$

But since z makes an angle of θ with the x -axis, we can express it as $r \text{cis } \theta$ for some real r . Thus,

$$\frac{\text{Im}(z^3)}{\text{Re}(z^3)} = \frac{\text{Im}(r^3 \text{cis } 3\theta)}{\text{Re}(r^3 \text{cis } 3\theta)} = \frac{\text{Im}(\text{cis } 3\theta)}{\text{Re}(\text{cis } 3\theta)},$$

which matches the expression for $\tan 3\theta$.

(c) Use (b) to find an identity for $\tan 3\theta$.

We expand out z^3 and factor into real and imaginary parts:

$$\begin{aligned}
z^3 &= (1 + i \tan \theta)^3 = 1^3 + 3i \tan \theta - 3 \tan^2 \theta - i \tan^3 \theta \\
&= (1 - 3 \tan^2 \theta) + i(3 \tan \theta - \tan^3 \theta).
\end{aligned}$$

Then we use our expression for $\tan 3\theta$ in terms of z^3 :

$$\begin{aligned}\tan 3\theta &= \frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)} \\ &= \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}.\end{aligned}$$

22. Find multiplication groups of complex numbers isomorphic to rotation groups for the

(a) regular octagon.

We choose complex numbers corresponding to rotations of $0, 45^\circ, \dots, 315^\circ$:

$$z = \left\{ \operatorname{cis} 0, \operatorname{cis} \frac{\pi}{4}, \operatorname{cis} \frac{\pi}{2}, \operatorname{cis} \frac{3\pi}{4}, \operatorname{cis} \pi, \operatorname{cis} \frac{5\pi}{4}, \operatorname{cis} \frac{3\pi}{2}, \operatorname{cis} \frac{7\pi}{4} \right\}.$$

(b) regular pentagon.

We simply choose complex numbers corresponding to rotations of $0, 72^\circ, \dots, 288^\circ$:

$$z = \left\{ \operatorname{cis} 0, \operatorname{cis} \frac{2\pi}{5}, \operatorname{cis} \frac{4\pi}{5}, \operatorname{cis} \frac{6\pi}{5}, \operatorname{cis} \frac{8\pi}{5} \right\}.$$

23. Make tables for

(a) the rotation group of the regular octagon.

There are 8 elements. If r is a rotation by 45° , then the elements are I, r, r^2, \dots, r^7 . The group table is shown below.

\cdot	I	r	r^2	r^3	r^4	r^5	r^6	r^7
I	I	r	r^2	r^3	r^4	r^5	r^6	r^7
r	r	r^2	r^3	r^4	r^5	r^6	r^7	I
r^2	r^2	r^3	r^4	r^5	r^6	r^7	I	r
r^3	r^3	r^4	r^5	r^6	r^7	I	r	r^2
r^4	r^4	r^5	r^6	r^7	I	r	r^2	r^3
r^5	r^5	r^6	r^7	I	r	r^2	r^3	r^4
r^6	r^6	r^7	I	r	r^2	r^3	r^4	r^5
r^7	r^7	I	r	r^2	r^3	r^4	r^5	r^6

(b) the dihedral group of the square.

There are, once again, $4 \cdot 2 = 8$ elements. Let r be a rotation by 90° , and f be a flip about say, the x -axis. The group table is shown below.

\cdot	I	r	r^2	r^3	f	fr	fr^2	fr^3
I	I	r	r^2	r^3	f	fr	fr^2	fr^3
r	r	r^2	r^3	I	fr^3	f	fr	fr^2
r^2	r^2	r^3	I	r	fr^2	fr^3	f	fr
r^3	r^3	I	r	r^2	fr	fr^2	fr^3	f
f	f	fr	fr^2	fr^3	I	r	r^2	r^3
fr	fr	fr^2	fr^3	f	r^3	I	r	r^2
fr^2	fr^2	fr^3	f	fr	r^2	r^3	I	r
fr^3	fr^3	f	fr	fr^2	r	r^2	r^3	I

(c) Is the difference between them fundamental?

Yes, the difference is fundamental, even though they have the same order. The easiest way to see this is that the latter group has 4 elements of order 2, but the former group has only 1 such element.

24. Which of the following tables defines a group? Why or why not?

(d)

$\$$	I	A	B	C	D
I	I	A	B	C	D
A	A	C	D	B	I
B	B	I	C	D	A
C	C	D	A	I	B
D	D	B	I	A	C

This table does not define a group, because it does not follow associativity. For example, $(D\$A)\$A = B\$A = I$, but $D\$(A\$A) = D\$C = A$.

(e)

$\#$	I	A	B	C	D
I	I	A	B	C	D
A	A	B	C	D	I
B	B	C	D	I	A
C	C	D	I	A	B
D	D	I	A	B	C

This table is a group; in fact, it is a commutative group. The quickest way to see this is noting that it is (up to isomorphism) the cyclic group of order 5, where $A = r$, $B = r^2$, $C = r^3$, and $D = r^4$.

25. Name some subsets of the complex numbers that are groups under multiplication. I can name an infinite number of both finite and infinite groups with this property, so after you list a few of each type, try to generalize.

Some simple examples: $\{1\}$, $\{\pm 1\}$, $\{\pm 1, \pm i\}$.

In general, we choose the n th roots of unity: the numbers of the form $\text{cis } \frac{2\pi k}{n}$ for $k \in \mathbb{Z}$. Each rotation is a symmetry of the n -gon, and thus this set under multiplication is isomorphic to the cyclic group of order n .

26. Prove with a diagram that if $|z| = 1$, then $\text{Im} \left(\frac{z}{(z+1)^2} \right) = 0$.

To draw a diagram, we need to interpret these expressions as points on the complex plane. $|z| = 1$ implies that z is 1 away from the origin. $z + 1$ is z , translated right by 1 unit in the x -axis. Let $z + 1 = r \text{cis } \theta$. Then $(z + 1)^2 = r^2 \text{cis } 2\theta$, so it forms an angle of 2θ with the origin.

If the quotient $\frac{z}{(z+1)^2}$ has no imaginary part, then $(z + 1)^2$ is a real scalar times z . In other words, the two numbers have the same complex argument. Thus, we wish to prove that $\text{Arg } z$ and $\text{Arg } (z + 1)^2$ are equal.

The scenario is shown in Figure 12.



Figure 12: A graph of z , $z + 1$, and $(z + 1)^2$.

As shown in the diagram, $\text{Arg}(z + 1) = \theta$. The triangle formed by O , z and $z + 1$ is isosceles, since it has two sides of length 1. Furthermore, it has a base angle of θ by the parallel postulate. Thus, the angle marked with a double line is also θ , and $\text{Arg } z = 2\theta$. But $\text{Arg}(z + 1)^2 = 2\theta$! Thus, we have

$$\begin{aligned}\text{Im}\left(\frac{z}{(z+1)^2}\right) &= \text{Im}\left(\frac{r_1 \text{cis } 2\theta}{r_2 \text{cis } 2\theta}\right) \\ &= \text{Im}\left(\frac{r_1}{r_2} + 0i\right) \\ &= 0.\end{aligned}$$

This is not truly complete, because we have only considered z in the first quadrant. In this case, extending it to other locations of z is pretty trivial. Nonetheless, I provide an algebraic solution for fun.

We wish to show that $(z + 1)^2 = kz$ for some real k . Express z as $\text{cis } \theta$. Then

$$(\text{cis } \theta + 1)^2 = \text{cis}^2 \theta + 2 \text{cis } \theta + 1.$$

Our supposed k is

$$\begin{aligned}k &= \frac{(z+1)^2}{z} = \frac{\text{cis}^2 \theta + 2 \text{cis } \theta + 1}{\text{cis } \theta} \\ &= \text{cis } \theta + 2 + \text{cis}(-\theta) \\ &= \text{cis } \theta + 2 + \overline{\text{cis } \theta} \\ &= 2 \text{Re}(\text{cis } \theta) + 2,\end{aligned}$$

which is indeed real. It's interesting what the scale factor actually is. Furthermore, since $\text{Re}(\text{cis } \theta) = \cos \theta$, we have a polar equation

$$|kz| = r = 2 \cos \theta + 2,$$

hinting that the path traced out by $(z + 1)^2$ is in fact a cardioid! The cardioid produced is shown in Figure 13 below.



Figure 13: The cardioid produced by $(z + 1)^2$ for $|z| = 1$.

It's all connected, guys.

27. Prove geometrically that if $|z| = 1$, then $|1 - z| = \left|2 \sin\left(\frac{\text{Arg } z}{2}\right)\right|$.

To prove this geometrically, we must again consider what the various expressions in the desired equation mean. $|z| = 1$ means that z is distance 1 from the origin. $1 - z$ is the reflection of z across the origin, then translated by 1 units right.

To geometrically interpret $A = 2 \sin\left(\frac{\text{Arg } z}{2}\right)$, we halve the angle $\theta = \text{Arg } z$ and draw a vector of length 2; the imaginary component, or y height, of this new point is the desired quantity.

The diagram of all this is shown in Figure 14.



Figure 14: Graph of z , $1 - z$, and $\sin\left(\frac{\theta}{2}\right)$.

We wish to show that the two lengths indicated in braces are equal. There's a couple of ways to do this; perhaps the most natural is to find a triangle congruent to the one formed by the origin, $-z$, and $1 - z$. This is the other dashed triangle shown in Figure 15.



Figure 15: The succulent triangle,

Because the angles of a triangle sum to π , we know the angle $\angle MAF$ is $\pi - \frac{\theta}{2} - \frac{\pi}{2} = \frac{\pi - \theta}{2}$. Furthermore, $\triangle AMF$ is isosceles with an apex at M , since the midpoint of the hypotenuse of a right triangle is equidistant from all vertices. Thus, $\angle MAF = \angle MFA$, and we have

$$\angle AMF = \pi - \angle MAF - \angle MFA = \pi - \frac{\pi - \theta}{2} \cdot 2 = \theta.$$

Thus, by SAS Congruence, the two dashed triangles are congruent. Finally, pairing up the previously indicated sides, we have

$$|1 - z| = \overline{AF} = \left| \operatorname{Im}\left(2 \operatorname{cis} \frac{\theta}{2}\right) \right|,$$

as desired.

Technically, this proof is slightly incomplete, because some of these triangles do not exist as described for $\theta \geq 90^\circ$. You can extend it to these cases with no problem, but I'd also like to give an algebraic proof to show its perks.

By the half-angle identity,

$$\left| 2 \sin\left(\frac{\operatorname{Arg} z}{2}\right) \right| = \left| \pm 2 \sqrt{\frac{1 - \cos \operatorname{Arg} z}{2}} \right| = \sqrt{2(1 - \cos \operatorname{Arg} z)}.$$

Let $z = a + bi = \operatorname{cis} \theta$; note that $r = 1$ since $|z| = 1$. We know that $\cos \operatorname{Arg} z = a$. Then

$$\begin{aligned}
|1 - (a + bi)| &= |(1 - a) - bi| \\
&= \sqrt{b^2 + (1 - a)^2} \\
&= \sqrt{1 - a^2 + (1 - 2a + a^2)} \\
&= \sqrt{2 - 2a} \\
&= \sqrt{2(1 - a)} \\
&= \sqrt{2(1 - \cos \operatorname{Arg} z)}.
\end{aligned}$$

This matches our expression using half-angle for $\left|2 \sin \left(\frac{\operatorname{Arg} z}{2}\right)\right|$.

I have a slight preference for the algebraic solution because it is quick, easy to understand, and mathematically complete. Nonetheless, the geometric solution gives a better idea of *why* the equation is true.

28.

(a) Prove that if $(z - 1)^{10} = z^{10}$, then $\operatorname{Re}(z) = \frac{1}{2}$. (Hint: if two numbers are equal, they have the same magnitude.)

We do as the hint suggests. We know that $|(z - 1)^{10}| = |z^{10}|$. Expanding this out would be rough, but we can take the exponents out of the inside of the magnitude symbols¹³.

So $|z - 1|^{10} = |z|^{10}$. Since $|n| \geq 0$, we have $|z - 1| = |z|$.

Let $z = a + bi$. Then $|a + bi - 1| = \sqrt{(a - 1)^2 + b^2}$ and $|a + bi| = \sqrt{a^2 + b^2}$. We set these equal and solve:

$$\begin{aligned}
\sqrt{(a - 1)^2 + b^2} &= \sqrt{a^2 + b^2} \\
(a - 1)^2 + b^2 &= a^2 + b^2 \\
(a - 1)^2 &= a^2 \\
a - 1 &= \pm a.
\end{aligned}$$

If $a - 1 = a$, then $-1 = 0$, which is dumb. Thus, $a - 1 = -a$, so $a = \frac{1}{2}$ and indeed, $\operatorname{Re}(z) = \frac{1}{2}$ as desired.

(b) How many solutions does this equation have?

We have $(z - 1)^{10} = z^{10}$, so $(z - 1)^{10} - z^{10} = P(z) = 0$, where P is a polynomial of degree 9. Thus, by the Fundamental Theorem of Algebra, there are 9 solutions... if there aren't any repeated roots. So this is only truly complete if we know there are no roots which appear in the factorization twice or more. Unfortunately, I can't think of a way to do this without calculus.¹⁴

Let's start over. We should use the fact that $\operatorname{Re}(z) = \frac{1}{2}$. A simple diagram reveals that z and $z - 1$ are symmetric about the y -axis, since $\operatorname{Re}(z) = \operatorname{Re}\left(\frac{1}{2} + bi\right) = -\operatorname{Re}\left(\frac{1}{2} + bi - 1\right)$. The diagram is shown in Figure 16.

¹³This is true because $|(r \operatorname{cis} \theta)^n| = |r^n \operatorname{cis} n\theta| = |r^n|$.

¹⁴For that route, we simply check that $P''(z) \neq 0$ for all solutions, which isn't pleasant until a clever rearrangement and substitution. Try it if you know how!



Figure 16: z and $1 - z$, residents of the complex plane.

Let z in the first quadrant make a angle ϕ to the $\pm y$ -axis. Note that we're using the y -axis, not the x -axis, for mathematical convenience. In general, for z in the first and fourth quadrants¹⁵, we have $\text{Arg } z = \frac{\pi}{2} - \phi$ and $\text{Arg}(z - 1) = \frac{\pi}{2} + \phi$. Since $|z| = |z - 1| = r$, we have

$$z = r \text{cis} \left(\frac{\pi}{2} - \phi \right); \quad z - 1 = r \text{cis} \left(\frac{\pi}{2} + \phi \right).$$

Since $(z - 1)^{10} = z^{10}$, we have

$$\begin{aligned} \left(r \text{cis} \left(\frac{\pi}{2} - \phi \right) \right)^{10} &= \left(r \text{cis} \left(\frac{\pi}{2} + \phi \right) \right)^{10} \\ r^{10} \text{cis}(5\pi - 10\phi) &= r^{10} \text{cis}(5\pi + 10\phi) \\ \text{cis}(5\pi - 10\phi) &= \text{cis}(5\pi + 10\phi) \\ 5\pi - 10\phi + 2\pi k &= 5\pi + 10\phi && \text{For some } k \in \mathbb{Z} \\ 20\phi &= 2\pi k \\ \phi &= \frac{2\pi k}{20}. \end{aligned}$$

To find all unique solutions, we restrict k to the range $0 \leq k \leq 19$... wait, isn't that 20 solutions?

The issue is that z must be in the first or fourth quadrant, so that our premise $|z| = |z - 1|$ is true. That means $0 < \phi < \pi$, a strict relation because $\phi = 0$ or $\phi = \pi$ only gives values along the y -axis, which does not intersect with $\text{Re}(z) = 0$. Solving this gives

$$\begin{aligned} 0 &< \frac{2\pi k}{20} < \pi \\ 0 &< \pi k < 10\pi \\ 0 &< k < 10 \\ k &\in \{1, 2, \dots, 8, 9\}, \end{aligned}$$

which is 9 solutions, in agreement with our polynomial argument.

29. I claim that $e^{i\theta} = \cos \theta + i \sin \theta = \text{cis } \theta$, for θ in radians.

(a) Find e^{-it} .

$$e^{-it} = \cos(-t) + i \sin(-t) = \cos t - i \sin t.$$

(b) Find $\frac{e^{i\theta} + e^{-i\theta}}{2}$.

¹⁵If z is in the fourth quadrant, then you'd define ϕ as $\pi +$ angle to negative y -axis, where the angle is taken clockwise so it's positive.

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{\cos \theta + i \sin \theta + \cos \theta - i \sin \theta}{2} \\ = \cos \theta.$$

(c) Find $\frac{e^{i\theta} - e^{-i\theta}}{2i}$.

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{\cos \theta + i \sin \theta - (\cos \theta - i \sin \theta)}{2i} \\ = \sin \theta.$$

30. Use your new, complex definitions for \cos and \sin to find:

(a) $\cos^2 \theta + \sin^2 \theta$

$$\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 + \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 - \left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)^2 \\ = \left(\frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i\theta} - e^{-i\theta}}{2}\right) \left(\frac{e^{i\theta} + e^{-i\theta}}{2} - \frac{e^{i\theta} - e^{-i\theta}}{2}\right) \\ = (e^{-i\theta})(e^{i\theta}) \\ = e^{-i\theta+i\theta} \\ = e^0 \\ = 1.$$

That was expected.

(b) $\tan \theta$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{e^{i\theta} - e^{-i\theta}}{2i}}{\frac{e^{i\theta} + e^{-i\theta}}{2}} \\ = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}.$$

(c) $\cos 2\theta$

$$\cos 2\theta = \frac{e^{2i\theta} + e^{-2i\theta}}{2}.$$

(d) $\sin 2\theta$

$$\sin 2\theta = \frac{e^{2i\theta} - e^{-2i\theta}}{2i}.$$

(e) What kind of group is generated by $\{e^{i\theta}, e^{-i\theta}\}$ under the operation of multiplication if θ is an integer? A rational multiple of π ?

If $\theta = 0$, then the group is the trivial group of order 1. If θ is any other integer, then a group isomorphic to the additive group of the integers is generated. We pair up $e^{ik\theta}$ with the integer k , so that

$$e^{ik_1\theta} \cdot e^{ik_2\theta} = e^{i(k_1+k_2)\theta} \leftrightarrow k_1 + k_2.$$

If θ is a rational multiple of π , say $\frac{p}{q} \cdot 2\pi$ where $\gcd(p, q) = 1$, then we get (up to isomorphism) cyclic group of order q .

31. You've used the quadratic equation throughout high school, but there's also a cubic equation that finds the roots of any cubic. Let's derive it, starting with the cubic $x^3 + bx^2 + cx + d = 0$.

(a) Make the substitution $x = y - \frac{b}{3}$. Combine like terms to create an equation of the form $y^3 - 3py - 2q = 0$, with p, q in terms of b, c , and d .

$$\begin{aligned} & \left(y - \frac{b}{3}\right)^3 + b\left(y - \frac{b}{3}\right)^2 + c\left(y - \frac{b}{3}\right) + d = 0 \\ & \left(y^3 - 3 \cdot \frac{by^2}{3} + 3 \cdot \frac{b^2y}{9} - \frac{b^3}{27}\right) + \left(by^2 - \frac{2b^2y}{3} + \frac{b^3}{9}\right) + \left(cy - \frac{bc}{3}\right) + d = 0 \\ & y^3 + (-b + b)y^2 + \left(\frac{b^2y}{3} - \frac{2b^2y}{3} + c\right)y + \left(-\frac{b^3}{27} + \frac{b^3}{9} - \frac{bc}{3} + d\right) = 0 \\ & y^3 + \left(c - \frac{b^2}{3}\right)y + \left(d - \frac{bc}{3} + \frac{2b^3}{27}\right) = 0 \\ & y^3 - 3\underbrace{\left(\frac{b^2}{9} - \frac{c}{3}\right)}_p y - 2\underbrace{\left(\frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}\right)}_q = 0. \end{aligned}$$

Thus, $p = \frac{b^2}{9} - \frac{c}{3}$ and $q = \frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}$. This type of cubic equation—in which the x^3 coefficient is 1 and the x^2 coefficient is 0—is known as a *depressed cubic*. Make of that what you will.

(b) Rearrange this equation as $y^3 = 3py + 2q$.

$$y^3 = 3\left(\frac{b^2}{9} - \frac{c}{3}\right)y + 2\left(\frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}\right).$$

(c) Make the substitution $y = s + t$ into (b), and prove that y is a solution of the cubic in part (a) if $st = p$ and $s^3 + t^3 = 2q$.

We substitute $y = s + t$ and use the fact that $st = p$ and $s^3 + t^3 = 2q$ to simplify.

$$\begin{aligned} (s + t)^3 &= 3p(s + t) + 2q \\ s^3 + 3s^2t + 3st^2 + t^3 &= 3ps + 3pt + 2q \\ 3s(st) + 3t(st) + s^3 + t^3 &= 3ps + 3pt + 2q \\ 3sp + 3tp + 2q &= 3ps + 3pt + 2q \\ 0 &= 0. \end{aligned}$$

This checks out.

(d) Eliminate t between these two equations to get a quadratic in s^3 .

We have $t^3 = 2q - s^3$. Also, $(st)^3 = p^3$, so

$$\begin{aligned} (st)^3 &= s^3t^3 = s^3(2q - s^3) = p^3 \\ -(s^3)^2 + 2qs^3 - p^3 &= 0 \\ (s^3)^2 - 2qs^3 + p^3 &= 0. \end{aligned}$$

(e) Solve this quadratic to find s^3 . By symmetry, what is t^3 ?

Let $w = s^3$. Then the above quadratic is $w^2 - 2qw + p^3 = 0$. The solutions are

$$s^3 = w = \frac{2q \pm \sqrt{4q^2 - 4p^3}}{2} = \frac{2q \pm 2\sqrt{q^2 - p^3}}{2} = q \pm \sqrt{q^2 - p^3}.$$

We have $t^3 = 2q - s^3 = 2q - (q \pm \sqrt{q^2 - p^3}) = q \mp \sqrt{q^2 - p^3}$. This inverted \pm sign, \mp , means that when s^3 's \pm sign is positive, the \mp sign is negative, and vice versa.

(f) Find a formula for y in terms of p and q . What about a formula for x ?

Taking cube roots of both sides of our expressions for t^3 and s^3 , we find that

$$s = \sqrt[3]{q \pm \sqrt{q^2 - p^3}},$$

$$t = \sqrt[3]{q \mp \sqrt{q^2 - p^3}}.$$

We must keep in mind, however, that over the complex numbers, taking cube roots outputs 3 possible values. Thus, the three solutions for s and t are

$$s = \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \text{cis} \frac{2\pi k}{3},$$

$$t = \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \text{cis} \left(2\pi - \frac{2\pi k}{3} \right),$$

where $k \in \{0, 1, 2\}$ and the cube root is taking its principal value. We multiply them by cis with these angles to preserve the fact that $st = q$, since otherwise it would produce another result:

$$\begin{aligned} \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \text{cis} \frac{2\pi k}{3} \cdot \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \text{cis} \left(2\pi - \frac{2\pi k}{3} \right) &= \sqrt[3]{(q \pm \sqrt{q^2 - p^3})(q \mp \sqrt{q^2 - p^3})} \cdot \text{cis} 2\pi \\ &= \sqrt[3]{q^2 - (q^2 - p^3)} \\ &= p. \end{aligned}$$

Thus, we have

$$y = s + t = \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \text{cis} \frac{2\pi k}{3} + \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \text{cis} \left(2\pi - \frac{2\pi k}{3} \right).$$

To get x , we substitute $x = y - \frac{b}{3}$ to get

$$x = \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \text{cis} \frac{2\pi k}{3} + \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \text{cis} \left(2\pi - \frac{2\pi k}{3} \right) - \frac{b}{3}.$$

You could substitute our values of p, q in terms of b, c, d to get a monstrous equation for x in terms of only $b, c, d \dots$ but no thanks.

(g) What if we started with $ax^3 + bx^2 + cx + d = 0$, with a coefficient in front of the x^3 term as well? Can you come up with a formula for x ?

Sure! We divide through the equation by a :

$$\frac{ax^3 + bx^2 + cx + d}{a} = 0$$

$$\implies x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0.$$

We can then attack this as we already did, setting $b' = \frac{b}{a}$, $c' = \frac{c}{a}$ and $d' = \frac{d}{a}$, then applying the formula.

32. Starting with the same cubic as in problem 31b.

- (a) Let $c = \cos \theta$. Remember that $\cos 3\theta = 4c^3 - 3c$, as we proved. Substitute $y = 2c\sqrt{p}$ into $y^3 = 3py + 2q$ to obtain $4c^3 - 3c = \frac{q}{p^{3/2}}$.

We substitute and proceed:

$$\begin{aligned} y^3 &= 3py + 2q \\ (2c\sqrt{p})^3 &= 3p(2c\sqrt{p}) + 2q \\ 8c^3 p^{3/2} &= 6cp^{3/2} + 2q \\ 8c^3 p^{3/2} - 6cp^{3/2} &= 2q \\ 4c^3 - 3c &= \frac{2q}{2p^{3/2}} = \frac{q}{p^{3/2}}. \end{aligned}$$

- (b) Provided that $q^2 \leq p^3$, show that $y = 2\sqrt{p} \cos\left(\frac{1}{3}(\theta + 2\pi n)\right)$, where n is an integer. Why does this yield all three solutions?

This isn't actually hard. We know that $\cos 3\theta = 4c^3 - 3c = \frac{q}{p^{3/2}}$, so there are three possible values for $\cos \theta = c$. Namely, if $\theta_0 = \frac{1}{3} \cos^{-1} \frac{q}{p^{3/2}}$ is the principal value, then we also have unique solutions

$$\theta_1 = \frac{2\pi}{3} + \theta_0, \theta_2 = \frac{4\pi}{3} + \theta_0,$$

because multiplying these by 3 to get 3θ just adds a multiple of 2π . Indeed, we have

$$c = \cos \frac{1}{3}(\theta + 2\pi n)$$

as a solution for any integer n . Substituting into the expression for y , we get

$$y = 2c\sqrt{p} = 2 \cos\left(\frac{1}{3}(\theta + 2\pi n)\right) \sqrt{p},$$

as desired. This yields all three solutions because, as we observed, the only unique values this makes are for $n \in \{0, 1, 2\}$.

Note that if $q^2 > p^3$, then this strategy actually still works, but you have to define \cos (and \cos^{-1}) over a larger, complex domain. This is certainly possible though!

- (c) Explain how you would find θ from p and q , and how we would use what we have found to solve an arbitrary cubic $ax^3 + bx^2 + cx + d = 0$.

We have $\cos 3\theta = \frac{q}{p^{3/2}}$, so

$$\theta = \cos^{-1} \frac{q}{p^{3/2}}.$$

The steps to solving an arbitrary cubic are the following:

1. Divide through by a to get a new cubic $x^3 + b'x^2 + c'x + d' = 0$.
2. Compute $p = \frac{b'^2}{9} - \frac{c'}{3}$ and $q = \frac{b'c'}{6} - \frac{b'^3}{27} - \frac{d'}{2}$.
3. Compute $\theta = \cos^{-1} \frac{q}{p^{3/2}}$.
4. Substitute this value of θ into $x = y - \frac{b}{3} = 2\sqrt{p} \cos\left(\frac{1}{3}(\theta + 2\pi n)\right) - \frac{b}{3}$, where $n \in \{0, 1, 2\}$.

These two problems were a doozy! This method of solving the cubic was known to French mathematician François Viète in the late 16th century. It also has a very pretty geometric interpretation. Wikipedia gives excellent details here: https://en.wikipedia.org/wiki/Cubic_equation\#Geometric_interpretation_of_the_roots.