13 Inverses

Now that we've talked about matrix multiplication quite a bit, it's time to start talking about matrix "division." We've seen how matrix multiplication transforms the plane; since division is the opposite of multiplication, division will "untransform" the plane, putting all the points back where they started. But first, we'll review division in a few more familiar contexts.

1. (a) With real numbers, one of the important purposes of division is that it lets you solve equations like $ax = b$ for x. Solve this by division (difficult!).
   (b) If division didn't exist, you could still solve this equation by multiplication. The number you'd multiply by is called the "**multiplicative inverse**" of a. What is the property that defines this special number?
   (c) The multiplicative inverse of a is often written a^{-1}. Why does this notation make sense?
2. (a) For fixed a, b, you might think that the equation $ax = b$ has only one solution, but sometimes it can have zero or infinitely many. Give an example of both cases.
   (b) How does the existence of a unique solution relate to the idea of multiplicative invertibility?
   (c) Are there any other possible numbers of solutions?
3. (a) Define "one-to-one" function.
   (b) Is $f(x) = ax$ a one-to-one function for all real a? (Hint: look for the silly exception(s)!)
4. Would your answers to the previous numbers change if you were talking about complex numbers instead of just real numbers? Why or why not?

Now, consider "**clock arithmetic**," which deals with integers 0 through 11 as hours on the clock: 0 replaces 12 for mathematical convenience. Numbers "wrap" around as the remainder when they are divided by 12, so for example $13 \to 1$ and $25 \to 1$. We use the congruency sign $\equiv$ to denote clock arithmetic, instead of $=$.

As some basic examples, $7 + 7 \equiv 2$, because "7 hours after 7 o'clock is 2 o'clock." This is shown in Figure 1. In addition, $7 \times 7 \equiv 1$, because 49 has remainder 1 when divided by 12. In more formal language, this clock arithmetic is actually called "arithmetic **modulo** 12."

5. In the following problems, x can be any integer from 0 to 11.
   (a) Find all solutions of $5x \equiv 7$ in clock arithmetic.
   (b) Find all solutions of $2x \equiv 6$ in clock arithmetic.
   (c) Find all solutions of $6x \equiv 6$ in clock arithmetic.
   (d) Find all solutions of $2x \equiv 7$ in clock arithmetic.
   (e) For integers a, b, what are all possible numbers of solutions that $ax \equiv b$ can have in clock arithmetic?

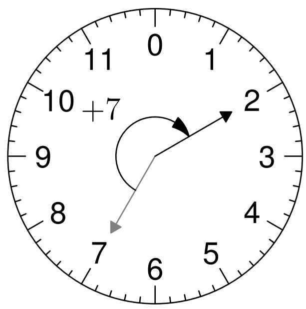

Figure 1: $7 + 7 \equiv 2$ on the clock.

6. How does the number of solutions to $ax \equiv b$ relate to the idea of multiplicative inverse? (Hint: you can try solving for $a = 5, 7, 11$ and $b = 1$. What numbers would be 5^{-1}, 7^{-1}, 11^{-1} in clock arithmetic?)
7. How does this all relate to groups?
   (a) The clock numbers are a group under clock addition. Name that group!
   (b) They are not a group under clock multiplication. Why?
   (c) A subset of four of the clock numbers form a group under the operation of clock multiplication. Find them, and write a group table.
   (d) Describe this group. What is the inverse of each element?
   (e) What symmetry group is it isomorphic to?
8. If the numbers on an advanced Mars clock went from 0 to 4,

(a) They would form a group under addition. Make a group table!

(b) What group is this isomorphic to?

(c) A subset of four of these numbers forms a group under multiplication. Find them and write a group table.

(d) Describe this multiplication group.

(e) What symmetry group is it isomorphic to?

Now we've seen how "division" (multiplicative inversion, or just inversion for short) works in a few more familiar situations, as well as the weird world of clock arithmetic. Let's see what happens with matrices! In the situation $AX = B$ we've been studying, X is a column vector representing a point in the plane, A is a 2×2 matrix describing a transformation, and B is the image of point X. Just like in the previous cases, we'll treat A and B as known and X as the unknown.

9. (a) Find all solutions (x, y) of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, by multiplying out the left side and rewriting this as a system of equations.

   (b) Find all solutions (x, y) of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

   (c) Find all solutions (x, y) of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$.

   (d) What are all possible numbers of solutions that $AX = B$ can have, where A, B are 2×2 and 2×1 matrices respectively and $X = \begin{bmatrix} x \\ y \end{bmatrix}$? Use your knowledge of the properties of systems of equations.

10. Now, let's relate the two 2×2 matrices from the previous problem to the transformations we know.

   (a) Contrast the mapping properties of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

   (b) Find the determinants of these matrices. What do you notice?

   (c) When is $f(X) = AX$ a one-to-one function? That is, in mapping the plane, when does each point in the image have exactly one preimage?

   (d) Compare how you find the number of solutions of the real number equation $ax = b$ with how you find the number of solutions of the matrix equation $AX = B$.

11. Let $K = \begin{bmatrix} 5 & 7 \\ 8 & -3 \end{bmatrix}$.

   (a) Find all solutions to $K \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$.

   (b) If we knew a matrix which was the inverse of K, written K^{-1}, we could write the following equation:

$$K^{-1}K \begin{bmatrix} x \\ y \end{bmatrix} = K^{-1} \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

   What would the left side reduce to?

Note that evaluating the right side would only entail simple matrix multiplication. This would save you a bit of work. If you were dealing with a system of three equations and three unknowns, however, it would save a lot work. You wouldn't even want to touch a system with 6, let alone the systems of hundreds or thousands that appear in today's computer problems!

Our problem simplifies as follows: how do we find the inverse of a 2×2 matrix? We've actually already done this. On page 36, you read the following string of matrices:

$$\overbrace{\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The indicated matrices, to the left of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, together constitute $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$.

12. Consider the following matrix inverses:

$$\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{23} & \frac{4}{23} \\ \frac{2}{23} & -\frac{5}{23} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}.$$

(a) Look for a pattern in these inverses.

(b) Describe the inverse of an arbitrary matrix: $\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1} = \frac{1}{\quad}\begin{bmatrix} & \\ & \end{bmatrix}$. Use the word determinant in your answer.

(c) We've been writing the inverse of matrix A as A^{-1}. Why does this notation make sense?

13. Now, see what happens when you multiply the following matrices:

(a) $-\frac{1}{2}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$

(b) $\frac{1}{71}\begin{bmatrix} 5 & 7 \\ 8 & -3 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 8 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} a & c \\ b & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

(d) $\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

14. For another approach to finding the inverse of a matrix, solve the following for w, x, y, z in terms of a, b, c, d by converting the matrix equations into a set of four linear equations:

$$\begin{bmatrix} w & y \\ x & z \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Gauss-Jordan elimination is an effective method we use to find inverses. If a matrix A is invertible, then as we've seen with matrix decomposition, there is a set of steps to reduce it to the identity matrix. Therefore, we have some set of elementary matrices E_i such that

$$E_n E_{n-1} \cdots E_2 E_1 A = I.$$

Multiplying by A^{-1} on the right, we get

$$E_n E_{n-1} \cdots E_2 E_1 I = A^{-1}.$$

Before, these elementary matrices were stretches, shears and the like. We will restrict ourselves to matrices like $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, which multiply a row by a, or matrices like $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, which adds a multiple of the first row to the second. That's because instead of thinking of a matrix, we simply have to think "multiply this row by a" or "add these rows together," content that they are valid matrix multiplications. Some valid row operations are as follows:

- Swapping rows i and j
- Adding row i to j
- Multiplying row i by a constant $a \neq 0$
- Adding a times row i to j
- Adding multiple rows to another
- Taking multiple rows, multiplying them by any nonzero coefficients, and adding them to another

This last operation is the most powerful and encompasses most of the previous ones. To apply our multiplications easily, we **augment** a matrix A by juxtaposing it with the identity matrix. Also, we write what row operation we actually did on the left after each step. As shown in the first equation, we have multiplied all the elementary matrices once A has become I. But in that process, I will have become A^{-1}!

Choosing the steps is a bit of an art, which takes practice. Let's see Gauss-Jordan elimination in action on the matrix $A = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$.

We first augment the matrix like so:

$$[\ A \mid I\ ] = \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right].$$

The line is there just so we remember this is no ordinary matrix, but two matrices joined together for convenience. Again, we want to turn the left side into the identity matrix.

We need a 1 in the bottom right corner, so we can add the top row to the bottom row to get -1:

$$\left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right] \implies R_2 = R_1 + R_2 \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{array}\right].$$

We can then get 1 by multiplying the bottom row by -1. The rest of the steps are shown/justified as well:

$$\implies R_2 = -R_2 \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$$\text{Want } -1 \text{ in the top left corner} \implies R_1 = R_1 + 4R_2 \left[\begin{array}{cc|cc} -1 & 0 & -3 & -4 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$$\text{Turn } -1 \text{ into } 1 \implies R_1 = -R_1 \left[\begin{array}{cc|cc} 1 & 0 & 3 & 4 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$$\text{Get rid of } -1 \text{ in bottom left corner} \implies R_2 = R_1 + R_2 \left[\begin{array}{cc|cc} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right].$$

Observe! We have found that $A^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Let's try this new technique out on some systems of equations.

15. Rewrite each system of equations in matrix form. Use your calculator to calculate a matrix inverse, solve the system, and finally, check your answer. Remember to make clear in your work when you have used a calculator.

    (a) $\begin{cases} 2x + 3y &= 5 \\ 4x + 5y &= 7 \end{cases}$

    (b) $\begin{cases} 37x + 12y &= 65 \\ 93x + 40y &= 156 \end{cases}$

    (c) $\begin{cases} 2x + 5y + 3z &= 5 \\ 3x + 2y + 4z &= 7 \\ 13x + 16y + 18z &= 4 \end{cases}$

    (d) $\begin{cases} w + 2x + 3y + 4z &= 7 \\ 3w - x - 2y - 5z &= 5 \\ 5w + 3x - y - 4z &= 3 \\ 7w + 9x + 5y - 2z &= 2 \end{cases}$

    (e) $\begin{cases} 2x + 5y + 2z &= 1 \\ 3x + 2y + 4z &= 1 \\ 13x + 16y + 18z &= 5 \end{cases}$

    (f) When can you use matrix inverses to solve a system of equations?

16. You can fit a polynomial to any set of points in the plane, so long as the points pass the Vertical Line Test.

    (a) What is the least degree polynomial through

        i. One point? ii. Two points? iii. Three points? iv. n points?

    (b) Find a polynomial of least degree that passes through $(0, 3)$, $(1, 5)$, $(2, -3)$, $(3, 4)$, and $(4, 7)$.