

7 Your Daily Dose of Vitamin i

1. We will use complex numbers to find identities for \cot . Use Pascal's triangle to expand the following:

(a) $(a + b)^3$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

(b) $(a + b)^4$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

(c) $(a + b)^5$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

1. (cont.) Then substitute $b = i = \sqrt{-1}$ and expand:

(d) $(a + i)^3$

$$(a + i)^3 = a^3 + 3a^2i + 3ab^2 + b^3 = a^3 + 3a^2i - 3a - i.$$

(e) $(a + i)^4$

$$(a + i)^4 = a^4 + 4a^3i + 6a^2b^2 + 4ab^3 + b^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1.$$

(f) $(a + i)^5$

$$(a + i)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i.$$

1. (cont.) Finally, substitute $a = \cot \theta$ and expand:

(g) $(\cot \theta + i)^3$

$$(\cot \theta + i)^3 = a^3 + 3a^2i - 3a - i = (\cot^3 \theta - 3 \cot \theta) + i(3 \cot^2 \theta - 1).$$

(h) $(\cot \theta + i)^4$

$$(\cot \theta + i)^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1 = (\cot^4 \theta - 6 \cot^2 \theta + 1) + i(4 \cot^3 \theta - 4 \cot \theta).$$

(i) $(\cot \theta + i)^5$

$$(\cot \theta + i)^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i = (\cot^5 \theta - 10 \cot^3 \theta + 5 \cot \theta) + i(5 \cot^4 \theta - 10 \cot^2 \theta + 1).$$

1. (cont.) Consider $z = i + \cot \theta$.

(j) Use the above results to find identities for (i) $\cot 3\theta$, (ii) $\cot 4\theta$, and (iii) $\cot 5\theta$.

i. $\cot 3\theta$

Given the right triangle formed by $z = i + \cot \theta$ in Figure 7, we have $\tan(\text{Arg } z) = \frac{1}{\cot \theta} = \tan \theta$, so $\text{Arg } z = \theta$ and $z = r \text{cis } \theta$.



Figure 1: $\text{Arg}(i + \cot \theta) = \theta$.

Thus, we have

$$\begin{aligned}\cot 3\theta &= \frac{\cos 3\theta}{\sin 3\theta} \\ &= \frac{\text{Re}(\text{cis } 3\theta)}{\text{Im}(\text{cis } 3\theta)} \\ &= \frac{\text{Re}(r^3 \text{cis } 3\theta)}{\text{Im}(r^3 \text{cis } 3\theta)} \\ &= \frac{\text{Re}(z^3)}{\text{Im}(z^3)}.\end{aligned}$$

We substitute in our expression for z^3 , $(\cot^3 \theta - 3 \cot \theta) + i(3 \cot^2 \theta - 1)$:

$$\cot 3\theta = \frac{\cot^3 \theta - 3 \cot \theta}{3 \cot^2 \theta - 1}.$$

i. $\cot 4\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}\cot 4\theta &= \frac{\cos 4\theta}{\sin 4\theta} \\ &= \frac{\text{Re}(\text{cis } 4\theta)}{\text{Im}(\text{cis } 4\theta)} \\ &= \frac{\text{Re}(r^4 \text{cis } 4\theta)}{\text{Im}(r^4 \text{cis } 4\theta)} \\ &= \frac{\text{Re}(z^4)}{\text{Im}(z^4)} \\ \cot 4\theta &= \frac{\cot^4 \theta - 6 \cot^2 \theta + 1}{4 \cot^3 \theta - 4 \cot \theta}.\end{aligned}$$

i. $\cot 5\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}\cot 5\theta &= \frac{\cos 5\theta}{\sin 5\theta} \\ &= \frac{\text{Re}(\text{cis } 5\theta)}{\text{Im}(\text{cis } 5\theta)} \\ &= \frac{\text{Re}(r^5 \text{cis } 5\theta)}{\text{Im}(r^5 \text{cis } 5\theta)} \\ &= \frac{\text{Re}(z^5)}{\text{Im}(z^5)} \\ &= \frac{\cot^5 \theta - 10 \cot^3 \theta + 5 \cot \theta}{5 \cot^4 \theta - 10 \cot^2 \theta + 1}.\end{aligned}$$

(k) Graph z , z^2 , z^3 , z^4 , and z^5 , with $\theta \approx 75^\circ$. What is your solution method?

To graph these, I first calculated the approximate magnitude of z , which is how many times each subsequent power will be scaled by. We have $|1 + \cot 75^\circ| \approx 1.268$, so we only need to scale by about $\frac{5}{4}$ each time. Of course, we rotate by about 75° each time.



Figure 2: Graphs of z , z^2 , z^3 , z^4 , and z^5 .

2. Compute $(1 + i)^n$ for $n = 3, 4, 5, \dots$. Can you find a general pattern?

We have

$$\begin{aligned}(1 + i)^3 &= 1^3 + 3i - 3 - i &= -2 - 2i \\(1 + i)^4 &= 1^4 + 4i - 6 - 6i + 1 &= -4 - 2i \\(1 + i)^5 &= 1^5 + 5i - 10 - 10i + 5 + i &= -4 - 4i.\end{aligned}$$

We can find the pattern by representing $1 + i = \sqrt{2} \text{cis } 45^\circ$. This shows that it has period 8 and let's us find an expression for $(1 + i)^n$:

$$(1 + i)^n = \left(\sqrt{2} \text{cis } 45^\circ \right)^n = 2^{n/2} \text{cis} \left(\frac{n\pi}{4} \right).$$

3. Expand and graph $\text{cis}^n \theta$ for $n = 2, 3, 4, \dots$

Let $\cos \theta = c$ and $\sin \theta = s$. We have

$$\begin{aligned}(c + is)^2 &= c^2 + 2csi - s^2 = (c^2 - s^2) + i(2cs) \\(c + is)^3 &= c^3 + 3c^2si - 3cs^2 - s^3i = (c^3 - 3cs^2) + i(3c^2s - s^3) \\(c + is)^4 &= c^4 + 4c^3si - 6c^2s^2 - 4cs^3i + s^4 = (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3) \\(c + is)^5 &= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i = (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5).\end{aligned}$$

The graphs of $\text{cis}^n \theta$ for $\theta \approx 50^\circ$ are shown in Figure 3 below.



Figure 3: Graphs of $\text{cis}^n \theta$ for $\theta \approx 50^\circ$.

(a) Why is the real part $\cos n\theta$ and the imaginary part $\sin n\theta$?

By DeMoivre's theorem, $\text{cis}^n \theta = \text{cis } n\theta$, which by definition has $\text{Im}(\text{cis } n\theta) = \cos n\theta$ and $\text{Re}(\text{cis } n\theta) = \sin n\theta$.

(b) Use your results to write identities for $\cos n\theta$ and $\sin n\theta$ for $n = 2, 3, 4, 5$.

Here they are. Again, let $\cos \theta = c$ and $\sin \theta = s$:

$$\cos 2\theta = \text{Re}(\text{cis } 2\theta) = c^2 - s^2$$

$$\cos 3\theta = \text{Re}(\text{cis } 3\theta) = c^3 - 3cs^2$$

$$\cos 4\theta = \text{Re}(\text{cis } 4\theta) = c^4 - 6c^2s^2 + s^4$$

$$\cos 5\theta = \text{Re}(\text{cis } 5\theta) = c^5 - 10c^3s^2 + 5cs^4$$

$$\sin 2\theta = \text{Im}(\text{cis } 2\theta) = 2cs$$

$$\sin 3\theta = \text{Im}(\text{cis } 3\theta) = 3c^2s - s^3$$

$$\sin 4\theta = \text{Im}(\text{cis } 4\theta) = 4c^3s - 4cs^3$$

$$\sin 5\theta = \text{Im}(\text{cis } 5\theta) = 5c^4s - 10c^2s^3 + s^5.$$

4. Compute $\cos 7^\circ + \cos 79^\circ + \cos 151^\circ + \cos 223^\circ + \cos 295^\circ$ without a calculator. (Hint: what does this have to do with complex numbers?)

These numbers look random, but a closer inspection reveals they are in arithmetic progression, with starting term 7 and increasing 72° each time. That's the rotation of a pentagon!

We rewrite this as the real component of a sum of cises, then manipulate and evaluate:

$$\begin{aligned} \cos 7^\circ + \cos 79^\circ + \cos 151^\circ + \cos 223^\circ + \cos 295^\circ &= \text{Re}(\text{cis } 7^\circ + \text{cis } 79^\circ + \text{cis } 151^\circ + \text{cis } 223^\circ + \text{cis } 295^\circ) \\ &= \text{Re}((\text{cis } 7^\circ)(\text{cis } 0^\circ + \text{cis } 72^\circ + \text{cis } 144^\circ + \text{cis } 216^\circ + \text{cis } 288^\circ)) \\ &= \text{Re}((\text{cis } 7^\circ)(0)) \\ &= \text{Re}(0) \\ &= 0. \end{aligned}$$

5. Factor the following:

(a) $x^6 - 1$ as a difference of squares

We substitute $y = x^3$, giving $y^2 - 1 = (y + 1)(y - 1)$. Substituting back in, we get

$$(x^3 + 1)(x^3 - 1).$$

(b) $x^6 - 1$ as a difference of cubes

We now substitute $y = x^2$, giving $y^3 - 1 = (y - 1)(y^2 + y + 1)$. Substituting back in, we get

$$(x^2 - 1)(x^4 + x^2 + 1)$$

(c) $x^4 + x^2 + 1$ over the real numbers

This one isn't as obvious. We substitute $y = x^2$ to get $y^2 + y + 1$ and find the quadratic's zeroes:

$$y = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}.$$

So it is irreducible over the reals.

(d) $x^6 - 1$ completely

We already broke it down into $(x^3 + 1)$ and $(x^3 - 1)$. Going further, we have $x^3 + 1 = (x + 1)(x^2 - x + 1)$ and $x^3 - 1 = (x - 1)(x^2 + x + 1)$. To break apart the last two quadratics, we find their zeros:

$$x^2 - x + 1 = 0 \implies x = \frac{1 \pm i\sqrt{3}}{2} \implies \left(x - \frac{1 - i\sqrt{3}}{2}\right)\left(x - \frac{1 + i\sqrt{3}}{2}\right).$$

$$x^2 + x + 1 = 0 \implies x = \frac{-1 \pm i\sqrt{3}}{2} \implies \left(x + \frac{1 - i\sqrt{3}}{2}\right)\left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

Combining all these, we get the complete factorization over the complex numbers:

$$x^6 - 1 = (x + 1)\left(x - \frac{1 - i\sqrt{3}}{2}\right)\left(x - \frac{1 + i\sqrt{3}}{2}\right)(x - 1)\left(x + \frac{1 - i\sqrt{3}}{2}\right)\left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

(e) $x^4 + x^2 + 1$ completely

We could do a lot of work again, or we could observe that $x^4 + x^2 + 1 = \frac{x^6 - 1}{x^2 - 1} = \frac{x^6 - 1}{(x + 1)(x - 1)}$. Removing the denominator's terms from our factorization of $x^6 - 1$ we found in the last subproblem, we get

$$x^4 + x^2 + 1 = \left(x - \frac{1 - i\sqrt{3}}{2}\right)\left(x - \frac{1 + i\sqrt{3}}{2}\right)\left(x + \frac{1 - i\sqrt{3}}{2}\right)\left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

6. Let $f(z) = \frac{z+1}{z-1}$.

(a) Without a calculator, compute $f^{2014}(z)$.

This seems terrifying. Let's try computing $f^2(z)$ and perhaps $f^3(z)$.

$$f^2(z) = \frac{f(z) + 1}{f(z) - 1} = \frac{\frac{z+1}{z-1} + 1}{\frac{z+1}{z-1} - 1} = \frac{\frac{2z}{z-1}}{\frac{2}{z-1}} = z.$$

Oh.

Since 2014 is even, we have $f^{2014}(z) = (f^2)^{1007}(z) = z$.

(b) What if you replace 2014 with the current year?

Let y be the current year. As I write this, it is 1492.

If y is even, then $f^y(z) = (f^2)^{y/2}(z) = z$. If y is odd, then $f^y(z) = f((f^2)^{(y-1)/2}(z)) = f(z) = \frac{z+1}{z-1}$.

7. Find $\text{Im}((\text{cis } 12^\circ + \text{cis } 48^\circ)^6)$.

These are some weird looking angles. Thinking back to some older problems, however, the resultant angle of the addition may be tractable. We draw a diagram, shown in Figure 4.



Figure 4: Adding $\text{cis } 12^\circ + \text{cis } 48^\circ$.

Consider the isosceles triangle. The apex has angle measure $132^\circ + 12^\circ = 144^\circ$, so the base angles are each $x = \frac{180^\circ - 144^\circ}{2} = 18^\circ$. But $\text{Arg}(\text{cis } 12^\circ + \text{cis } 48^\circ) = 48^\circ - x = 30^\circ$!

That's a familiar angle. Indeed, we have $z = \text{cis } 12^\circ + \text{cis } 48^\circ = r \text{cis } 30^\circ$ for some r . It doesn't really matter which r , because

$$\text{Im}((r \text{cis } 30^\circ)^6) = \text{Im}(r^6 \text{cis } 180^\circ) = \text{Im}(-r^6) = 0.$$

8. Let x satisfy the equation $x + \frac{1}{x} = 2 \cos \theta$.

(a) Compute $x^2 + \frac{1}{x^2}$ in terms of θ .

Squaring the left hand side will get us some terms that look close to what we want.

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}.$$

$$\text{So } x^2 + \frac{1}{x^2} = (2 \cos \theta)^2 - 2 = 4 \cos^2 \theta - 2 = 2(2 \cos^2 \theta - 1) = 2 \cos 2\theta. \text{ Huh.}$$

(b) Compute $x^n + \frac{1}{x^n}$ in terms of n and θ .

We conjecture that this is equal to $2 \cos n\theta$. To do this, we let $x = \text{cis } \frac{\theta}{n}$, so $x^n = \text{cis } \theta$, and compute. That should give us some similar looking terms:

$$\begin{aligned} x^n + \frac{1}{x^n} &= \text{cis } \theta + \frac{1}{\text{cis } \theta} \\ &= \text{cis } \theta + \text{cis}(-\theta) \\ &= \text{cis } \theta + \overline{\text{cis } \theta} \\ &= 2 \text{Re}(\text{cis } \theta) \\ &= 2 \cos \theta. \end{aligned}$$

This proves the relationship.