

7 Your Daily Dose of Vitamin i

1. We will use complex numbers to find identities for \cot . Use Pascal's triangle to expand the following:

(a) $(a + b)^3$

(b) $(a + b)^4$

(c) $(a + b)^5$

Then substitute $b = i = \sqrt{-1}$ and expand:

(d) $(a + i)^3$

(e) $(a + i)^4$

(f) $(a + i)^5$

Finally, substitute $a = \cot \theta$ and expand:

(g) $(\cot \theta + i)^3$

(h) $(\cot \theta + i)^4$

(i) $(\cot \theta + i)^5$

Consider $z = i + \cot \theta$.

(j) Use the above results to find identities for (i) $\cot 3\theta$, (ii) $\cot 4\theta$, and (iii) $\cot 5\theta$.

(k) Graph z, z^2, z^3, z^4 , and z^5 , with $\theta \approx 75^\circ$. What method did you use?

2. Compute $(1 + i)^n$ for $n = 3, 4, 5, \dots$. Can you find a general pattern?

3. Expand and graph $\operatorname{cis}^n \theta$ for $n = 2, 3, 4, \dots$ and $\theta \approx 50^\circ$.

(a) Why is the real part $\cos n\theta$ and the imaginary part $\sin n\theta$?

(b) Use your results to write identities for $\cos n\theta$ and $\sin n\theta$ for $n = 2, 3, 4, 5$.

4. Compute $\cos 7^\circ + \cos 79^\circ + \cos 151^\circ + \cos 223^\circ + \cos 295^\circ$ without a calculator. (Hint: What does this have to do with complex numbers?)

5. Factor the following:

(a) $x^6 - 1$ as a difference of squares;

(d) $x^6 - 1$ completely;

(b) $x^6 - 1$ as a difference of cubes;

(e) $x^4 + x^2 + 1$ completely.

(c) $x^4 + x^2 + 1$ over the real numbers;

6. Let $f(z) = \frac{z+1}{z-1}$.

(a) Without a calculator, compute $f^{2020}(z)$.

(b) What if you replace 2020 with the current year?

7. Find $\operatorname{Im}((\operatorname{cis} 12^\circ + \operatorname{cis} 48^\circ)^6)$.

8. Let x satisfy the equation $x + \frac{1}{x} = 2 \cos \theta$.

(a) Compute $x^2 + \frac{1}{x^2}$ in terms of θ .

(b) Compute $x^n + \frac{1}{x^n}$ in terms of n and θ .