



Figure 1: Graph of  $P(z)$  from 0 to 1

The roots of the polynomial  $P(z) = (z - 1)^{10} - z^{10}$  have interesting structure, as we've seen. If we didn't have the inspiration (or, let's be honest, the instruction from Mr. Herreshoff) to represent  $z$  in its polar form  $r \operatorname{cis} \theta$ , what could we do instead to find how many roots it has, and where they are? This is a fun exercise in algebra, and more importantly, an exercise in thinking like a mathematician. I make no claim to completeness or efficiency, but hopefully you'll find instructive the thought process of a reasonably mathematically inclined high schooler.

First of all, we note that  $P(z)$  has degree 9, since the degree 10 term of  $(z - 1)^{10}$  is canceled out. Thus it has at most 9 roots, by the Fundamental Theorem of Algebra. We could try expanding out  $P(z)$  and analyzing it further. But before that messiness, are there any real roots? There's clearly a real root at  $z = 0.5$ . Are there any others? Well, there are many valid ways to verify that there aren't, one of which is graphing it (see Figure ??). That's what I did. Another way is to examine the function in certain domains  $z > 0.5$  and  $z < 0.5$  and use inequalities to show the function is either positive or negative—and thus nonzero—in those ranges.

Okay, so  $P(0.5) = 0$ . Nice. But what about the (up to) 8 other roots just chillin' somewhere in the complex plane? We know little about those. Graphing the polynomial made me notice something: It looks relatively simple, kind of like  $x^3$ . It's monotonic, meaning it always decreases over time; it has no humps.



Figure 2: Roots of  $P(z)$  must be symmetric about  $\operatorname{Re}(z) = \frac{1}{2}$ .

And it looks to be somewhat symmetric about  $x = \frac{1}{2}$ . To make it clearer, let's consider  $P(z)$  and its reflection about  $x = \frac{1}{2}$ , which is  $P(1 - z)$ .

$$\begin{aligned}
 P(1 - z) &= (1 - z - 1)^{10} - (1 - z)^{10} && \text{Definition of } P(z) \\
 &= (-z)^{10} - (1 - z)^{10} && \text{Simplifying} \\
 &= z^{10} - (z - 1)^{10} && \text{Raising to an even power} \\
 &= -P(z)
 \end{aligned}$$

Indeed, the graph is antisymmetric about  $(0.5, 0)$ , meaning that it flips sign but looks the same. This identity,  $P(1 - z) = -P(z)$ , holds for all  $z$ .

One of the most immediate consequences is that the roots of  $P(z)$  are symmetric about  $\frac{1}{2}$ . To see this, suppose  $z$  is a root of  $P(z)$ . Then  $P(1 - z) = -P(z) = 0$ , so  $1 - z$  is also a root of  $P(z)$ , which is  $z$ 's reflection. See Figure ?? for a visualization.

A root is either on the line of symmetry or off it. Let's see what happens when we substitute  $z = 0.5 + \xi i$ , which is a point on the line of symmetry. I love that Greek letter  $\xi$ . Yes please.

$$\begin{aligned}
P(z) &= P(0.5 + \xi i) \\
&= (\xi i + 0.5 - 1)^{10} - (\xi i + 0.5)^{10} && \text{Substitution} \\
&= (\xi i - 0.5)^{10} - (\xi i + 0.5)^{10}
\end{aligned}$$

Hm. We have our original solution  $\xi = 0$  corresponding to  $z = 0.5$ , but are there any other solutions of this form? Well, we can make things a bit easier by multiplying everything through by  $2^{10}$  and substituting  $\omega = 2\xi$ :

$$\begin{aligned}
2^{10} \cdot P(z) &= 2^{10} \cdot ((\xi i - 0.5)^{10} - (\xi i + 0.5)^{10}) && \text{Substitution} \\
&= (2\xi i - 1)^{10} - (2\xi i + 1)^{10} \\
&= (\omega i - 1)^{10} - (\omega i + 1)^{10} = Q(\omega)
\end{aligned}$$

Glorious. Note that the roots of  $Q(\omega)$  correspond with roots of  $P(z)$ , with  $z = \frac{1}{2} + \frac{\omega i}{2}$ . How can we find the roots of  $Q(\omega)$ ? Well, we can use the binomial theorem to expand. For the  $n$ th term in the expansion, each term contributes a certain multiple of  $(\omega i)^n$ :

$$\begin{aligned}
\sum_{n=0}^{10} \binom{10}{n} \cdot (\omega i)^n \cdot (-1)^{10-n} - \sum_{n=0}^{10} \binom{10}{n} \cdot (\omega i)^n \cdot (1)^{10-n} \\
&= \sum_{n=0}^{10} \binom{10}{n} (\omega i)^n ((-1)^{10-n} - 1) \\
&= \sum_{n=1, \text{ odd}}^9 \binom{10}{n} i^n \omega^n && \text{Because } ((-1)^{10-n} - 1) \text{ is 0 for even } n \\
&= i \left( \binom{10}{1} \omega - \binom{10}{3} \omega^3 + \binom{10}{5} \omega^5 - \binom{10}{7} \omega^7 + \binom{10}{9} \omega^9 \right) \\
&= i \cdot R(\omega)
\end{aligned}$$

We only care about the roots, so we ignore  $i$  and graph  $R(\omega)$  as in Figure ??.

Well, there's nine roots. So using the transformation  $z = \frac{1}{2} + \frac{\omega i}{2}$ , those correspond to the nine roots of our original equation, graphed in Figure ??. That wasn't too bad.

You'll notice that the roots are symmetric about the real axis. Recall that in the complex plane, roots come in conjugate pairs. In other words, if  $P(a + bi) = 0$ , then  $P(\overline{a + bi}) = P(a - bi) = 0$ . The real root(s) conjugate to themselves, and the rest of the roots form four pairs.

But what if we don't have a graphing calculator? And what other information can we find about these roots?



Figure 3: Graph of  $R(\omega)$ .

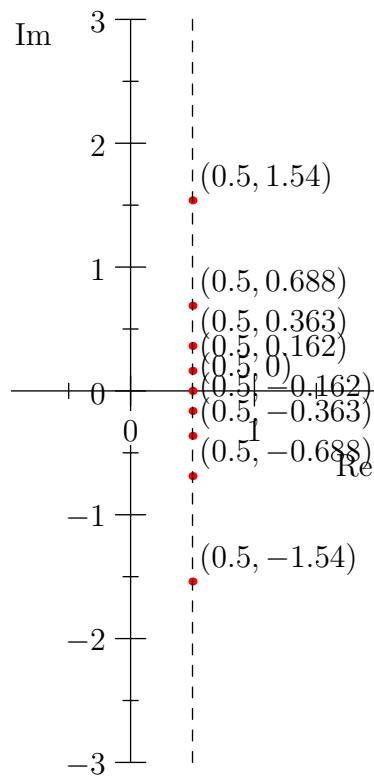


Figure 4: The roots of  $P(z)$  in the complex plane.