7 Your Daily Dose of Vitamin i

1. **We will use complex numbers to find identities for $\cot$. Use Pascal's triangle to expand the following:**

   (a) $(a+b)^3$

   $$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

   (b) $(a+b)^4$

   $$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

   (c) $(a+b)^5$

   $$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

1. **(cont.) Then substitute $b = i = \sqrt{-1}$ and expand:**

   (d) $(a+i)^3$

   $$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + 3a^2i - 3a - i.$$

   (e) $(a+i)^4$

   $$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1.$$

   (f) $(a+i)^5$

   $$(a+i)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i.$$

1. **(cont.) Finally, substitute $a = \cot\theta$ and expand:**

   (g) $(\cot\theta + i)^3$

   $$(\cot\theta + i)^3 = a^3 + 3a^2i - 3a - i = (\cot^3\theta - 3\cot\theta) + i(3\cot^2\theta - 1).$$

   (h) $(\cot\theta + i)^4$

   $$(\cot\theta + i)^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1 = (\cot^4\theta - 6\cot^2\theta + 1) + (4\cot^3\theta - 4\cot\theta).$$

   (i) $(\cot\theta + i)^5$

   $$(\cot\theta + i)^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i = (\cot^5\theta - 10\cot^3\theta + 5\cot\theta) + i(5\cot^4\theta - 10\cot^2\theta + 1).$$

1. **(cont.) Consider $z = i + \cot\theta$.**

   (j) Use the above results to find identities for (i) $\cot 3\theta$, (ii) $\cot 4\theta$, and (iii) $\cot 5\theta$.

   i. $\cot 3\theta$

Given the right triangle formed by $z = i + \cot\theta$ in Figure 7, we have $\tan(\operatorname{Arg} z) = \frac{1}{\cot\theta} = \tan\theta$, so $\operatorname{Arg} z = \theta$ and $z = r\operatorname{cis}\theta$.

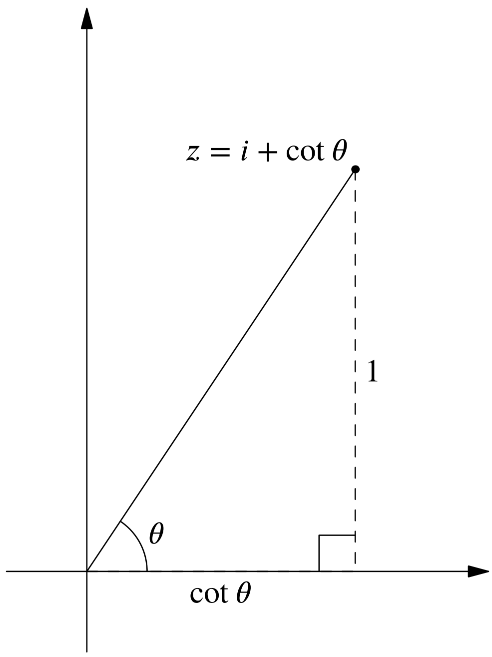

Figure 1: $\operatorname{Arg}(i + \cot\theta) = \theta$.

Thus, we have

$$\begin{aligned}
\cot 3\theta &= \frac{\cos 3\theta}{\sin 3\theta} \\
&= \frac{\operatorname{Re}(\operatorname{cis} 3\theta)}{\operatorname{Im}(\operatorname{cis} 3\theta)} \\
&= \frac{\operatorname{Re}(r^3 \operatorname{cis} 3\theta)}{\operatorname{Im}(r^3 \operatorname{cis} 3\theta)} \\
&= \frac{\operatorname{Re}(z^3)}{\operatorname{Im}(z^3)}.
\end{aligned}$$

We substitute in our expression for z^3, $(\cot^3\theta - 3\cot\theta) + i(3\cot^2\theta - 1)$:

$$\cot 3\theta = \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}.$$

i. $\cot 4\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}
\cot 4\theta &= \frac{\cos 4\theta}{\sin 4\theta} \\
&= \frac{\operatorname{Re}(\operatorname{cis} 4\theta)}{\operatorname{Im}(\operatorname{cis} 4\theta)} \\
&= \frac{\operatorname{Re}(r^4 \operatorname{cis} 4\theta)}{\operatorname{Im}(r^4 \operatorname{cis} 4\theta)} \\
&= \frac{\operatorname{Re}(z^4)}{\operatorname{Im}(z^4)} \\
\cot 4\theta &= \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}.
\end{aligned}$$

i. $\cot 5\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}
\cot 5\theta &= \frac{\cos 5\theta}{\sin 5\theta} \\
&= \frac{\operatorname{Re}(\operatorname{cis} 5\theta)}{\operatorname{Im}(\operatorname{cis} 5\theta)} \\
&= \frac{\operatorname{Re}(r^5 \operatorname{cis} 5\theta)}{\operatorname{Im}(r^5 \operatorname{cis} 5\theta)} \\
&= \frac{\operatorname{Re}(z^5)}{\operatorname{Im}(z^5)} \\
&= \frac{\cot^5\theta - 10\cot^3\theta + 5\cot\theta}{5\cot^4\theta - 10\cot^2\theta + 1}.
\end{aligned}$$

(k) Graph z, z^2, z^3, z^4, and z^5, with $\theta \approx 75°$. What is your solution method?

To graph these, I first calculated the approximate magnitude of z, which is how many times each subsequent power will be scaled by. We have $|1 + \cot 75°| \approx 1.268$, so we only need to scale by about $\frac{5}{4}$ each time. Of course, we rotate by about $75°$ each time.

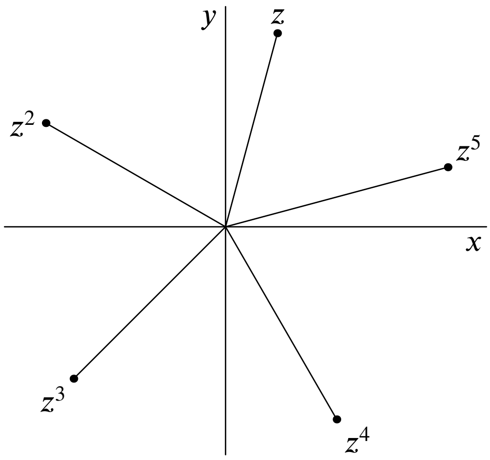

Figure 2: Graphs of z, z^2, z^3, z^4, and z^5.

2. Compute $(1+i)^n$ for $n = 3, 4, 5, \ldots$. Can you find a general pattern?

We have

$$\begin{aligned}
(1+i)^3 &= 1^3 + 3i - 3 - i &&= -2 - 2i \\
(1+i)^4 &= 1^4 + 4i - 6 - 6i + 1 &&= -4 - 2i \\
(1+i)^5 &= 1^5 + 5i - 10 - 10i + 5 + i &&= -4 - 4i.
\end{aligned}$$

We can find the pattern by representing $1 + i = \sqrt{2}\operatorname{cis} 45°$. This shows that it has period 8 and let's us find an expression for $(1+1)^n$:

$$(1+i)^n = \left(\sqrt{2}\operatorname{cis} 45°\right)^n = 2^{n/2}\operatorname{cis}\left(\frac{n\pi}{4}\right).$$

3. Expand and graph $\operatorname{cis}^n \theta$ for $n = 2, 3, 4, \ldots$.

Let $\cos\theta = c$ and $\sin\theta = s$. We have

$$\begin{aligned}
(c+is)^2 &= c^2 + 2csi - s^2 = (c^2 - s^2) + i(2cs) \\
(c+is)^3 &= c^3 + 3c^2si - 3cs^2 - s^3i = (c^3 - 3cs^2) + i(3c^2s - s^3) \\
(c+is)^4 &= c^4 + 4c^3si - 6c^2s^2 - 4cs^3i + s^4 = (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3) \\
(c+is)^5 &= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i = (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5).
\end{aligned}$$

The graphs of $\operatorname{cis}^n\theta$ for $\theta \approx 50°$ are shown in Figure 3 below.

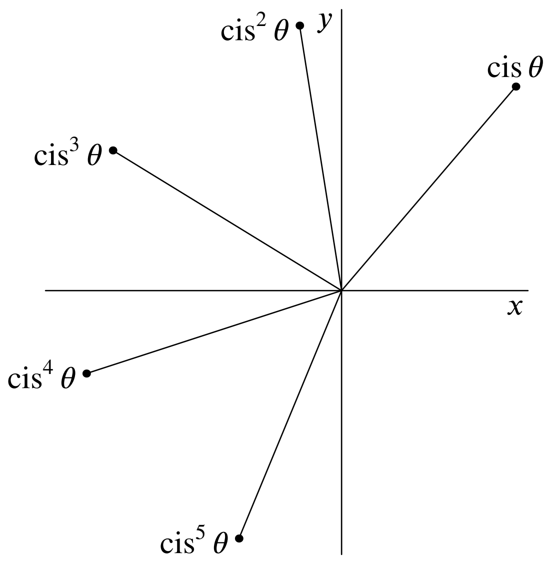

Figure 3: Graphs of $\operatorname{cis}^n\theta$ for $\theta \approx 50°$.

(a) Why is the real part $\cos n\theta$ and the imaginary part $\sin n\theta$?

By DeMoivre's theorem, $\operatorname{cis}^n\theta = \operatorname{cis} n\theta$, which by definition has $\operatorname{Im}(\operatorname{cis} n\theta) = \cos n\theta$ and $\operatorname{Re}(\operatorname{cis} n\theta) = \sin n\theta$.

(b) Use your results to write identities for $\cos n\theta$ and $\sin n\theta$ for $n = 2, 3, 4, 5$.

Here they are. Again, let $\cos\theta = c$ and $\sin\theta = s$:

$$\begin{aligned}
\cos 2\theta &= \operatorname{Re}(\operatorname{cis} 2\theta) = c^2 - s^2 \\
\cos 3\theta &= \operatorname{Re}(\operatorname{cis} 3\theta) = c^3 - 3cs^2 \\
\cos 4\theta &= \operatorname{Re}(\operatorname{cis} 4\theta) = c^4 - 6c^2s^2 + s^4 \\
\cos 5\theta &= \operatorname{Re}(\operatorname{cis} 5\theta) = c^5 - 10c^3s^2 + 5cs^4 \\
\sin 2\theta &= \operatorname{Im}(\operatorname{cis} 2\theta) = 2cs \\
\sin 3\theta &= \operatorname{Im}(\operatorname{cis} 3\theta) = 3c^2s - s^3 \\
\sin 4\theta &= \operatorname{Im}(\operatorname{cis} 4\theta) = 4c^3s - 4cs^3 \\
\sin 5\theta &= \operatorname{Im}(\operatorname{cis} 5\theta) = 5c^4s - 10c^2s^3 + s^5.
\end{aligned}$$

4. Compute $\cos 7° + \cos 79° + \cos 151° + \cos 223° + \cos 295°$ without a calculator. (Hint: what does this have to do with complex numbers?)

These numbers look random, but a closer inspection reveals they are in arithmetic progression, with starting term 7 and increasing 72° each time. That's the rotation of a pentagon!

We rewrite this as the real component of a sum of cises, then manipulate and evaluate:

$$\begin{aligned}
\cos 7° + \cos 79° + \cos 151° + \cos 223° + \cos 295° &= \operatorname{Re}(\operatorname{cis} 7° + \operatorname{cis} 79° + \operatorname{cis} 151° + \operatorname{cis} 223° + \operatorname{cis} 295°) \\
&= \operatorname{Re}((\operatorname{cis} 7°)(\operatorname{cis} 0° + \operatorname{cis} 72\circ + \operatorname{cis} 144° + \operatorname{cis} 216° + \operatorname{cis} 288°)) \\
&= \operatorname{Re}((\operatorname{cis} 7°)(0)) \\
&= \operatorname{Re}(0) \\
&= 0.
\end{aligned}$$

5. Factor the following:

(a) $x^6 - 1$ as a difference of squares

We substitute $y = x^3$, giving $y^2 - 1 = (y+1)(y-1)$. Substituting back in, we get

$$(x^3+1)(x^3-1).$$

(b) $x^6 - 1$ as a difference of cubes

We now substitute $y = x^2$, giving $y^3 - 1 = (y-1)(y^2+y+1)$. Substituting back in, we get

$$(x^2-1)(x^4+x^2+1)$$

.

(c) $x^4 + x^2 + 1$ over the real numbers

This one isn't as obvious. We substitute $y = x^2$ to get $y^2 + y + 1$ and find the quadratic's zeroes: $y = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm i\sqrt{3}}{2}$.
So it is irreducible over the reals.

(d) $x^6 - 1$ completely

We already broke it down into (x^3+1) and (x^3-1). Going further, we have $x^3+1 = (x+1)(x^2-x+1)$ and $x^3-1 = (x-1)(x^2+x+1)$. To break apart the last two quadratics, we find their zeros:

$$x^2 - x + 1 = 0 \Longrightarrow x = \frac{1\pm i\sqrt{3}}{2} \Longrightarrow \left(x - \frac{1-i\sqrt{3}}{2}\right)\left(x - \frac{1+i\sqrt{3}}{2}\right).$$

$$x^2 + x + 1 = 0 \Longrightarrow x = \frac{-1\pm i\sqrt{3}}{2} \Longrightarrow \left(x + \frac{1-i\sqrt{3}}{2}\right)\left(x + \frac{1+i\sqrt{3}}{2}\right).$$

Combining all these, we get the complete factorization over the complex numbers:

$$x^6 - 1 = (x+1)\left(x - \frac{1-i\sqrt{3}}{2}\right)\left(x - \frac{1+i\sqrt{3}}{2}\right)(x-1)\left(x + \frac{1-i\sqrt{3}}{2}\right)\left(x + \frac{1+i\sqrt{3}}{2}\right).$$

(e) $x^4 + x^2 + 1$ completely

We could do a lot of work again, or we could observe that $x^4 + x^2 + 1 = \frac{x^6-1}{x^2-1} = \frac{x^6-1}{(x+1)(x-1)}$. Removing the denominator's terms from our factorization of $x^6 - 1$ we found in the last subproblem, we get

$$x^4 + x^2 + 1 = \left(x - \frac{1-i\sqrt{3}}{2}\right)\left(x - \frac{1+i\sqrt{3}}{2}\right)\left(x + \frac{1-i\sqrt{3}}{2}\right)\left(x + \frac{1+i\sqrt{3}}{2}\right).$$

6. Let $f(z) = \frac{z+1}{z-1}$.

(a) Without a calculator, compute $f^{2014}(z)$.

This seems terrifying. Let's try computing $f^2(z)$ and perhaps $f^3(z)$.

$$f^2(z) = \frac{f(z)+1}{f(z)-1} = \frac{\frac{z+1}{z-1}+1}{\frac{z+1}{z-1}-1} = \frac{\frac{2z}{z-1}}{\frac{2}{z-1}} = z.$$

Oh.
Since 2014 is even, we have $f^{2014}(z) = (f^2)^{1007}(z) = z$.

(b) What if you replace 2014 with the current year?

Let y be the current year. As I write this, it is 1492.
If y is even, then $f^y(z) = (f^2)^{y/2}(z) = z$. If y is odd, then $f^y(z) = f((f^2)^{(y-1)/2}(z)) = f(z) = \frac{z+1}{z-1}$.

7. Find $\operatorname{Im}\left((\operatorname{cis}12^\circ + \operatorname{cis}48^\circ)^6\right)$.

These are some weird looking angles. Thinking back to some older problems, however, the resultant angle of the addition may be tractable. We draw a diagram, shown in Figure 4.

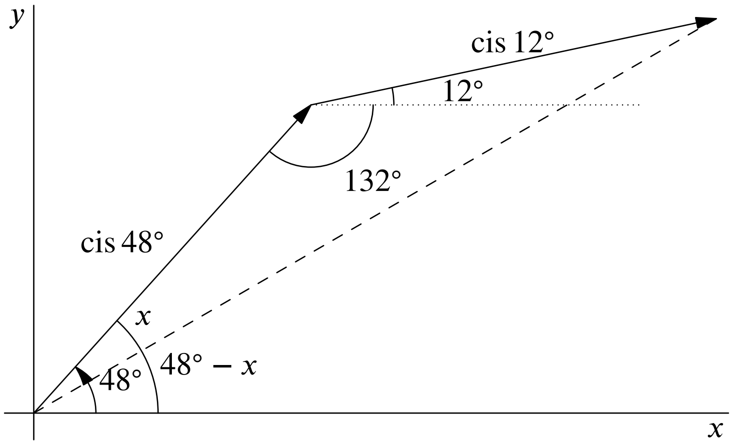

Figure 4: Adding $\operatorname{cis} 12° + \operatorname{cis} 48°$.

Consider the isosceles triangle. The apex has angle measure $132° + 12° = 144°$, so the base angles are each $x = \frac{180°-144°}{2} = 18°$. But $\operatorname{Arg}(\operatorname{cis} 12° + \operatorname{cis} 48°) = 48° - x = 30°$!

That's a familiar angle. Indeed, we have $z = \operatorname{cis} 12° + \operatorname{cis} 48° = r \operatorname{cis} 30°$ for some r. It doesn't really matter which r, because

$$\operatorname{Im}((r \operatorname{cis} 30°)^6) = \operatorname{Im}(r^6 \operatorname{cis} 180°) = \operatorname{Im}(-r^6) = 0.$$

8. Let x satisfy the equation $x + \frac{1}{x} = 2\cos\theta$.

(a) Compute $x^2 + \frac{1}{x^2}$ in terms of θ.

Squaring the left hand side will get us some terms that look close to what we want.

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}.$$

So $x^2 + \frac{1}{x^2} = (2\cos\theta)^2 - 2 = 4\cos^2\theta - 2 = 2(2\cos^2\theta - 1) = 2\cos 2\theta$. Huh.

(b) Compute $x^n + \frac{1}{x^n}$ in terms of n and θ.

We conjecture that this is equal to $2\cos n\theta$. To do this, we let $x = \operatorname{cis}\frac{\theta}{n}$, so $x^n = \operatorname{cis}\theta$, and compute. That should give us some similar looking terms:

$$\begin{aligned}
x^n + \frac{1}{x^n} &= \operatorname{cis}\theta + \frac{1}{\operatorname{cis}\theta} \\
&= \operatorname{cis}\theta + \operatorname{cis}(-\theta) \\
&= \operatorname{cis}\theta + \overline{\operatorname{cis}\theta} \\
&= 2\operatorname{Re}(\operatorname{cis}\theta) \\
&= 2\cos\theta.
\end{aligned}$$

This proves the relationship.