3 From Snaps to Flips

1. **Is the list of six operations complete? (Are there any other isometries of the equilateral triangle that preserve its shape and location?)**

There are no other isometries for this triangle; our list of operations is complete. To see why, note that the vertices must exchange places. At most there is $3! = 6$ ways to do this, so we have already achieved the maximum possible number of isometries.

$\cdot$	I	A	B	C	D	E
I						
A					B	
B						
C						
D						
E						

Figure 1: Unfilled D_3 group table.

2. **As with the snap group, we can make a group table for the dihedral group. Fill out a table like the one in Figure 1 in your notebook. Like the snap group table, the top row indicates what operation is done first and the left column indicates what's done second. In other words, XY is in X's row and Y's column. $AD = B$ is done for you.**

The completed table is shown in Figure 2.

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	D	E	B	C
B	B	E	I	D	C	A
C	C	D	E	I	A	B
D	D	C	A	B	E	I
E	E	B	C	A	I	D

Figure 2: Completed D_3 group table.

$\bullet$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	E	D	C	B
B	B	D	I	E	A	C
C	C	E	D	I	B	A
D	D	B	C	A	E	I
E	E	C	A	B	I	D

Figure 3: Completed S_3 group table from the last chapter.

3. **What is the relationship between the tables for the snap group S_3 and the dihedral group D_3?**

D_3's table is S_3's table flipped over the top-left–bottom-right diagonal, and vice versa. Contrast D_3 from Figure 2 to S_3 in Figure 3. If these were matrices, one would be the transpose of the other: we'll get to that later.

4. **Check your understanding by defining isomorphic in your own words.**

(Answers may vary.)
Isomorphic groups have:

- the same order (size);
- the same structure;
- a one-to-one correspondence between the elements of the groups;
- and group operations preserved by the correspondence.

5. (a) **Make a table for only the rotations of D_3, a subgroup of D_3.**

The table is shown below. Note that the identity element I is a rotation of 0. Interestingly, this subgroup is a commutative group (discussed in the next chapter), also known as an abelian group.

$\cdot$	I	D	E
I	I	D	E
D	D	E	I
E	E	I	D

(b) Which subgroup of the snap group S_3 is isomorphic to the subgroup in (a)?

The same elements (nominally) make the same subgroup:

$\cdot$	I	D	E
I	I	D	E
D	D	E	I
E	E	I	D

6. What shape's dihedral group is isomorphic to

(a) the two post snap group S_2?

The dihedral group of a line segment is isomorphic to S_2. After all, you can only reflect it over its midpoint, which is the other element of S_2 besides the identity. We can also think of this as permuting the two endpoints or vertices of a line segment.

(b) the one post snap group S_1?

The dihedral group of a point is isomorphic to S_1, because the only element is the identity element. This is permuting the one vertex of a point.

(c) the four post snap group S_4?

For this question we need to think 3 dimensions. There are four vertices to permute, but we can't do that on a square since diagonal points will remain on diagonals, as shown in Figure 4.

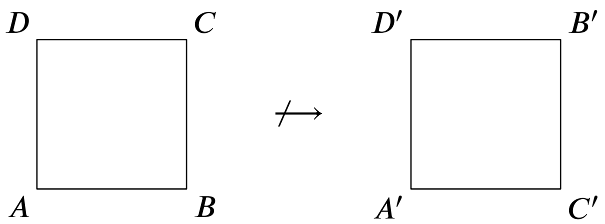

Figure 4: At right is a valid permutation of the vertices, but not a valid isometry of the square.

Instead, we choose the regular tetrahedron, so that there are no "diagonals"; every permutation is achievable. Note that rotations and reflections are now in 3 dimensional space, which is a bit difficult to visualize. A sample rotation is depicted in Figure 5.

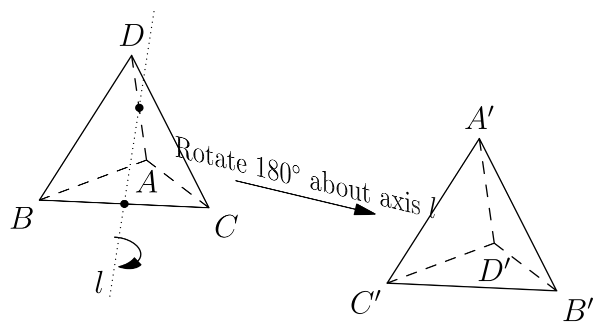

Figure 5: A rotation of the tetrahedron (orthographic view).

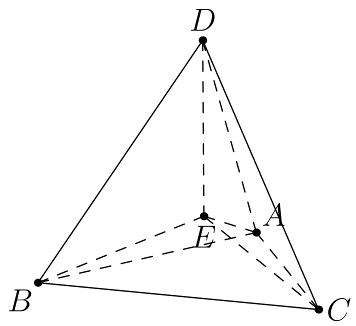

Figure 6: A 3D projection of the regular 4-simplex. In a true realization, every line segment here would be the same length.

(d) the five post snap group S_5?

This is isomorphic to the dihedral group of the 4-dimensional equivalent of the tetrahedron, also known as the regular 4-simplex. A projection is shown in Figure 6, but it cannot be faithfully represented on this paper.

7. Find a combination of A and D that yields C.

If we apply D twice, we get E. Applying A to E, we get C. Thus, a combination (among many) is $ADD = C$, which can also be written as $AD^2 = C$.

8. We call A and D generators of the group because every element of the group is expressible as some combination of As and Ds. For convenience, let's call A "f" since it's a flip, and call D "r" meaning a $120\,\mathrm{deg}$ rotation counterclockwise. Then, for example, fr^2 is a rotation of $2 \cdot 120\,\mathrm{deg} = 240\,\mathrm{deg}$, followed by a flip across the A axis, equivalent to our original C (see Figure ??). Make a new table using I, r, r^2, f, fr, and fr^2 as elements, like the one in Figure 7. *Note that the element order is different!*

$\cdot$	I	r	r^2	f	fr	fr^2
I						
r				fr^2		
r^2						
f						
fr						
fr^2						

Figure 7: Unfilled alternate D_3 table.

The filled table is shown in Figure 8 below.

$\cdot$	I	r	r^2	f	fr	fr^2
I	I	r	r^2	f	fr	fr^2
r	r	r^2	I	fr^2	f	fr
r^2	r^2	I	r	fr	fr^2	f
f	f	fr	fr^2	I	r	r^2
fr	fr	fr^2	f	r^2	I	r
fr^2	fr^2	f	fr	r	r^2	I

Figure 8: Completed alternate D_3 table.

Note that $I = I$, $A = f$, $B = fr$, $C = fr^2$, $D = r$, and $E = r^2$.

9. What other pairs of elements could you have used to generate the table?

You could also use any of the following pairs: $\{A, E\}$, $\{B, D\}$, $\{B, E\}$, $\{C, D\}$, $\{C, E\}$, $\{A, B\}$, $\{B, C\}$, $\{A, C\}$. In essence, you can generate it with any rotation element and any reflection element, or with any two reflection elements.

10. Notice the 3×3 table of a group you've already described in the top-left corner of your table. What is it, and what are the two possible generators of this three-element group?

This is the cyclic group of order 3, C_3, also known as the rotation group of the equilateral triangle. The two possible generators are r and r^2.

11. Explain why each element of the dihedral group D_3 has the period it has.

I has a period of 1 because it is the identity. A, B, C have periods of 2 because they are reflections, so they are their own inverse transformation. D and E are rotations of a multiple of $1/3$ of a turn. They take 3 iterations to resolve, and thus have period 3.

12. Some pairs of elements of the dihedral group are two-element subgroups. Which pairs are they?

These would be the pairs I, A, I, B, and I, C, since $A \cdot A = B \cdot B = C \cdot C = I$ so the subgroup is closed. These are shown in Figure 9.

$\cdot$	I	A
I	I	A
A	A	I

$\cdot$	I	B
I	I	B
B	B	I

$\cdot$	I	C
I	I	C
C	C	I

Figure 9: The three two-element subgroups.

13. One of the elements forms a one-element subgroup. Which is it?

The element I forms the so-called trivial group, or the only group of order 1; this is shown in Figure 10. It is not very interesting.

$\cdot$	I
I	I

Figure 10: The trivial group.

14. The addition of two numbers is a binary operation, while the addition of three numbers is not. In logic, $\wedge$ (and) and $\vee$ (or) are binary operations, but $\neg$ (not) is not. Define binary operation in your own words, and name some other binary operations.

(Answers may vary.)

A binary operation is an operation with two arguments.

Some binary operations:

1. multiplication
2. exponentiation
3. addition
4. subtraction
5. division
6. snap operation (•)

15. In your original dihedral group table, what is

(a) the identity element?

The identity element is I.

(b) the inverse of A?

The inverse of A is also A, since it is a reflection.

(c) the inverse of E?

The inverse of E is D, since $-120° + 120° = 0°$.