2 It's a Snap

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 1: Unfilled 3-post snap group table.

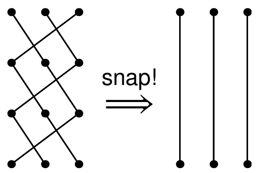

Figure 2: $E \bullet E \bullet E = I$; E has period 3.

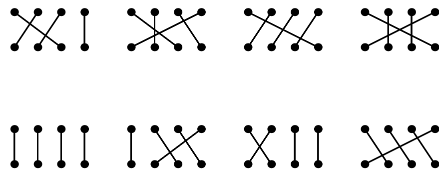

Figure 3: Some 4-post group elements.

1. **Fill out a 6×6 table like the one in Figure 1, showing the results of each of the 36 possible snaps, where $X \bullet Y$ is in X's row and Y's column. $A \bullet B = E$ is done for you.**

$\bullet$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	E	D	C	B
B	B	D	I	E	A	C
C	C	E	D	I	B	A
D	D	B	C	A	E	I
E	E	C	A	B	I	D

2. **Which of the elements is the identity element K, such that $X \bullet K = K \bullet X = X$ for all X?**

The identity element is I, since $I \bullet A = A \bullet I = A$, $I \bullet B = B \bullet I = B$, and so forth.

3. **Does every element have an inverse? In other words, can you get to the identity element from every element using only one snap?**

Yes you can. The inverses are shown below.

$I \leftrightarrow I$ $\qquad$ $A \leftrightarrow A$ $\qquad$ $B \leftrightarrow B$ $\qquad$ $C \leftrightarrow C$ $\qquad$ $D \leftrightarrow E$

Note that the inverse of an element X is denoted X^{-1}.

4. **(a) Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all X, Y)?**

No, the snap operation is not commutative. For example, $A \bullet B = E$, but $B \bullet A = D$.

(b) Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z)?

Yes, the snap operation is associative. You can rationalize this as the fact that a 4×3 grid of posts is snapped to a single configuration, regardless of which middle row you remove first. This is shown in Figure 4.

5. **(a) For any elements X, Y, is there always an element Z so that $X \bullet Z = Y$?**

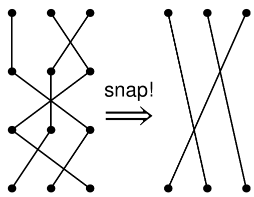

Figure 4: A 4×3 grid of posts has a unique result after the snap operation.

Yes, there is always a way to get from one element to another in one snap. You can prove this by construction. If element X connects n_1 to n'_1, n_2 to n'_2, and n_3 to n'_3, and element Y connects m_1 to m'_1, m_2 to m'_2, and m_3 to m'_3, then the solution Z to $X \bullet Z = Y$ connects m_1 to $n_{m'_1}$, m_2 to $n_{m'_2}$, and m_3 to $n_{m'_3}$.

A more transparent solution uses inverses. We multiply X by X^{-1}, then by Y:

$$X \bullet X^{-1} \bullet Y = Y.$$

But since every element has an inverse, and the snap operation is associative, we have

$$\begin{aligned} X \bullet (X^{-1} \bullet Y) &= Y \\ \implies Z &= X^{-1} \bullet Y. \end{aligned}$$

We have thus constructed the element Z.

(b) In (a), is Z always unique?

Yes. We could try all combinations, but let's try a proof by contradiction. Suppose we have two solutions Z_1 and Z_2 so that $Z_1 \neq Z_2$ and

$$\begin{aligned} X \bullet Z_1 &= Y \\ X \bullet Z_2 &= Y. \end{aligned}$$

That is, we assume that for some X and Y, there are two possible values for Z. We multiply both sides on the left by Y^{-1}:

$$\begin{aligned} Y^{-1} \bullet X \bullet Z_1 &= Y^{-1} \bullet Y = I \\ Y^{-1} \bullet X \bullet Z_2 &= I. \end{aligned}$$

So Z_1, Z_2 are both inverses of $(Y^{-1} \bullet X)$. But the inverse of an element is unique; we've showed this by listing them all out! Thus, $Z_1 = Z_2$, contradicting our assumption and proving that Z is unique in $X \bullet Z = Y$.

6. **If you constructed a group table using only five of the snap elements, the table would not describe a group, because there would be entries in the table outside of those 5. Indeed, a group must be closed under its operation: If we compose any two elements $X \bullet Y = Z$, Z must also be an element of the group. *Some* subsets of our six elements, however, do happen to be closed among themselves. Write valid group tables using exactly one, two, and three elements from the snap group. These are known as subgroups.**

Here are tables with 1, 2, and 3 elements:

$\bullet$	I
I	I

$\bullet$	I	A
I	I	A
A	A	I

$\bullet$	I	D	E
I	I	D	E
D	D	E	I
E	E	I	D

7. **What do you guess is a good definition of a mathematical group? (Hint: consider your answers to Problems 2–6.)**

(Answers may vary.)

Definition of **group**: A group G is a set of elements together with a **binary operation** that meets the following criteria:

(a) Identity: There is an element $I \in G$ such that for all $X \in G$, $X \bullet I = I \bullet X = X$.

(b) Closure: If X, Y are elements of the group, then $X \bullet Y$ is also an element of the group.

(c) Invertibility: Each element X has an inverse X^{-1} such that $X \bullet X^{-1} = X^{-1} \bullet X = I$.

(d) Associativity: For all elements X, Y, and Z, $X \bullet (Y \bullet Z) = (X \bullet Y) \bullet Z$.

8. Notice that $E \bullet E \bullet E = I$ (See Figure 2). We saw that E has a period of 3 when acting upon itself. Which elements have a period of

(a) 1?

I is the only element with a period of 1, since $I = I$.

(b) 2?

A, B, and C have periods of 2, since for each $X \in A, B, C$ we have $X \bullet X = I$.

(c) 3?

D and E have periods of 3, since for each $Y \in D, E$ we have $Y \bullet Y \neq I$, but $Y \bullet Y \bullet Y = I$.

9. Answer the following with the one-, two-, and four-post snap groups S_1, S_2 and S_4. These are just the analogous groups for connections between rows of one, two, and four posts.

(a) How many elements does the group have?

S_1 has $1! = 1$ elements. S_2 has $2! = 2$ elements. S_4 has $4! = 24$ elements.

(b) Systematically draw and name the elements.

Figure 5: Elements of S_1.

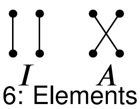

Figure 6: Elements of S_2.

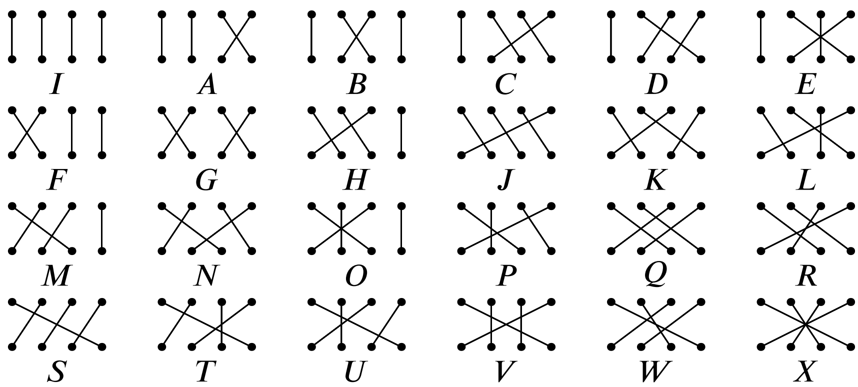

Figure 7: Elements of S_4.

(c) Make a group table of these elements. For four posts, instead of creating a table, give the number of entries that the table would have.

Here are group tables for S_1 and S_2.

There are 576 entries in the S_4 table; the full table is given at the end of the chapter, in Figure 12, for the demented.

$\bullet$	I
I	I

Figure 8: Group table for S_1.

$\bullet$	I	A
I	I	A
A	A	I

Figure 9: Group table for S_2.

(d) What is the relationship of the S_3 table to this new table?

Both S_1's and S_2's tables can be found within the original table for S_3, because they are subgroups of S_3. In turn, S_3 is a subgroup of S_4.

10. What is a shortcut for generating all elements of a snap group without drawing out the possible configurations?

(Answers may vary.)

One way is to treat each element as a list of indices. For example, I is the ordered triple $(1, 2, 3)$ because it takes column 1 to 1, 2 to 2, and 3 to 3. A is $(1, 3, 2)$, because it takes 1 to 1, 2 to 3, and 3 to 2.[1]

This is a bit more compact and easy to work with: You can simply choose the indices for each configuration rather than make a drawing. (It also makes it easy to write a program to calculate. In fact, it is behind every table and diagram in this chapter.)

11. (a) How many elements would there be in the five-post snap group?

There would be $5! = 120$ elements in S_5.

(b) How many entries would its table have?

There would be $5!^2 = 14400$ entries in S_5's table.

(c) What possible periods would its elements have? Make sure you include a period of six!

This is a more difficult question. We must ask what characteristics of an element determine its period.

If we observe the periodicity of an element with a pretty large period, say, one from S_5 with a period of 6, you can see how a relatively large period can arise. This is shown in Figure 11.

We can split up this element into two components: a component with period 3 and one with period 2. Let's call these components C_3 and C_2. After 2 steps, the C_3 has not completed one period, even though C_2. After 3 steps, C_3 has completed one period, but C_2 has gone through $\frac{3}{2}$. It takes $\mathrm{lcm}(2, 3) = 6$ steps before both components "line up!"

Even if it's not obvious, all snap elements can be split up into some number of these cyclic components. For example, the element from S_8 shown in Figure 10 comprises two size 3 and one size 2 component. It therefore has a period of $\mathrm{lcm}(2, 3, 3) = 6$. Note that it does *not* have a period of $2 \cdot 3 \cdot 3 = 18$—we take the least common multiple, not the product.

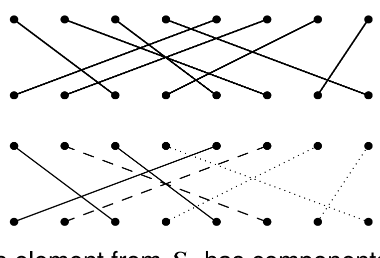

Figure 10: This element from S_8 has components of size $2, 3, 3$.

Let us consider every possible decomposition of an element in S_5. It could have components of size $1, 1, 1, 1, 1$, yield a period of 1; components of size $1, 1, 1, 2$, yielding a period of 2; components of size $1, 1, 3$, yielding a period of 3; components of size $1, 4$, yielding a period of 4; a component of size 5, yielding a period of 5; and component of size $1, 2, 3$, yielding a period of 6. Thus, periods $1, 2, 3, 4, 5, 6$ are the only ones achievable.

[1] This representation may be called the *one-line notation* of a permutation. You can also write a permutation by writing what index goes to which to a $2 \times n$ matrix, where the top row gives the start index and the bottom row gives the end index; this is *two-line notation*, due to Cauchy. Other notations exist; a one you will find in lots of combinatorics texts writes the cycles in groups of parentheses. For example, the one-line $(2, 5, 4, 3, 1)$ becomes $(1\,2\,5)(3\,4)$: The elements at indices 1, 2, 5 are cycled, as are the elements at indices $3, 4$. In this notation, the identity permutation on n elements is $(1)(2)(3)...(n)$, since it comprises n 1-cycles. Often 1-cycles—i.e., elements that do not move—are simply omitted.

(d) Extend your answers for Problems a through c to M posts per row.

There are (a) $M!$ elements in the M-post snap group, and thus (b) $M!^2$ elements in the corresponding group table. The possible periods are harder to calculate, but they can be generated like so:

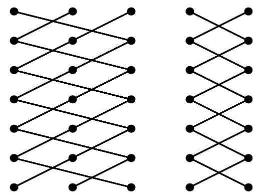

Figure 11: This element from S_5 has a period of 6.

Let integers $x_i > 0$ and $\sum_i x_i = M$. In other words, the sum of all x_i is M. Then $\operatorname{lcm}(x_1, x_2, \cdots, x_n)$ is a valid period; the least common multiple of all x_i is a possible period.

In set builder notation, the set of possible periods P_n for the n-post snap group as

$$P_n = \left\{ \operatorname{lcm}(x_1, x_2, \cdots, x_n) \;\middle|\; x_i \in \mathbb{Z}^+ \wedge \sum_i x_i = n \right\}.$$

The maximum such period (i.e. $\max P_n$) is actually known as Landau's function, $g(n)$. It turns out that $\ln g(n) \sim \sqrt{n \ln n}$ as $n \to \infty$, a result proved in 1902 by Edmund Landau. Therefore, $g(n)$ is sub-exponential but grows faster than any polynomial.

12. A permutation of a set of things is an order in which they can be arranged. What is the relationship between the set of permutations of m things and the m-post snap group?

We can make a straightforward correspondence between a permutation of m things and an element of the m-post snap group. If we think back to the idea of treating each element of the group as a list of indices, the correspondence is obvious. For example, I is the ordered triple $(1, 2, 3)$ because it takes column 1 to 1, 2 to 2, and 3 to 3. A is $(1, 3, 2)$, because it takes 1 to 1, 2 to 3, and 3 to 2. But each ordered triple is a permutation of $1, 2, 3$! This extends to any m.