15 Eigenvectors and Eigenvalues

1. **Consider the matrix equation** $\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 6x+y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$**. We wish to find an eigenvector** $\begin{bmatrix} x \\ y \end{bmatrix}$**.**

   (a) **On graph paper, draw what the matrix** $\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$ **does to the vectors** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **and** $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$**.**

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ goes to $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$ (dashed lines) and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ goes to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (solid lines):

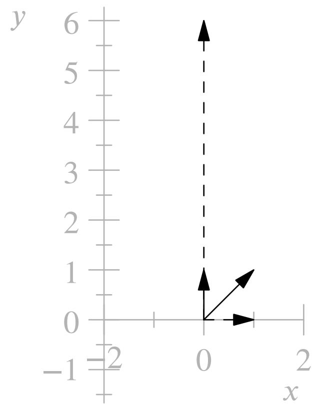

Figure 1: The mapping of the matrix.

   (b) **In your picture, draw a rough line through the origin where you think a family of eigenvectors may be.**

This is the line where vectors should change direction. Thus, it should be roughly where the diagram is "flipped," though this definitely is not a pure reflection.

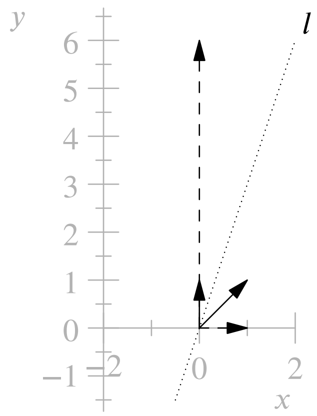

Figure 2: The mapping of the matrix, with the suspected eigenvectors indicated by l.

   (c) **Try some lattice points, say** $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 5 \end{smallmatrix}\right]$**. What does the matrix transform each vector into?**

These points are transformed as follows:

$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix};$$

$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix};$$

$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix};$$
$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix};$$
$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

(d) Which of these is an eigenvector?

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector, since the image is $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$, which is just the original vector times 3.

(e) Does it lie near the line you drew earlier?

Well, it lies *on* the line I drew because I'm using the computer, but it should lie close. Here it is superimposed on the previous graph:

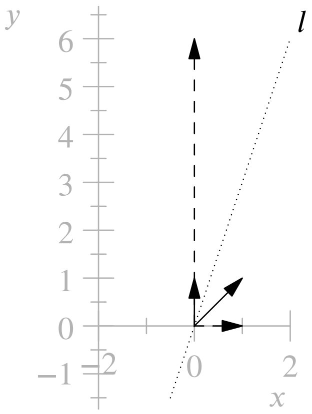

Figure 3: The mapping of the matrix, with the suspected eigenvectors indicated by l.

2. **This guess-and-check process for finding eigenvectors is terrible, so let's develop a procedure to find the eigenvalues and eigenvectors for any 2×2 matrix. We will use the same example.**

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} && \textbf{Definition of eigenvector} \\ &= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ \Longrightarrow \left(\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \textbf{Subtraction and factoring} \\ \Longrightarrow \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

(a) If $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, then

$$\det \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} = 0.$$

Why? Think inverses.

If we left-multiply both sides of the last equation above by the inverse of the 2×2 matrix, we'll get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

But we assumed $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so the inverse must not exist—that's the only other case. Thus,

$$\det \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} = 0.$$

(b) Find the above determinant in terms of λ and solve for the eigenvalues.

We have
$$\det \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} = -(1-\lambda)\lambda - 1(6) = \lambda^2 - \lambda - 6.$$
This is just a quadratic in λ, which factors as
$$(\lambda+2)(\lambda-3) = 0.$$
Thus, $\lambda = -2, 3$.

(c) One eigenvalue is $\lambda = 3$. We solve for the associated eigenvector like so:

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -3x+y \\ 6x-2y \end{bmatrix} \\ &\implies y = 3x \to \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \textbf{(for some } s\textbf{)} \end{aligned}$$

Solve for the other eigenvector using the other eigenvalue from part (b).

The other eigenvalue is -2.

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 2x+y \\ 6x+3y \end{bmatrix} \\ &\implies y = -2x \to \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{aligned}$$

(d) Check your work by multiplying the original matrix by the eigenvector!

$$\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} s \begin{bmatrix} 1 \\ -2 \end{bmatrix} = s \begin{bmatrix} 0\cdot 1 - 2\cdot 1 \\ 6\cdot 1 - 2\cdot 1 \end{bmatrix} = s \begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2 \cdot \left(s \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right).$$

Indeed, the image of $s \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is the pre-image scaled by -2.

3. Solve for the eigenvectors and eigenvalues of the following matrices:

(a) $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$

$$\begin{aligned}
\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \lambda\begin{bmatrix} x \\ y \end{bmatrix} && \text{Definition of eigenvector} \\
&= \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \left(\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \text{Subtraction and factoring} \\
\implies \begin{bmatrix} 3-\lambda & 24 \\ 4 & 7-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\det\begin{bmatrix} 3-\lambda & 24 \\ 4 & 7-\lambda \end{bmatrix} &= 0 \\
(3-\lambda)(7-\lambda) - 24(4) &= 0 \\
\lambda^2 - 10\lambda - 75 &= 0 \\
(\lambda+5)(\lambda-15) &= 0 \\
\lambda &= -5, 15.
\end{aligned}$$

Thus, the eigenvalues are -5 and 15. We now find the corresponding eigenvectors.
-5:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3-\lambda & 24 \\ 4 & 7-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} 8 & 24 \\ 4 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 8x+24y \\ 4x+12y \end{bmatrix} \\
\implies y = -3x \to \begin{bmatrix} x \\ y \end{bmatrix} &= s\begin{bmatrix} 1 \\ -3 \end{bmatrix}.
\end{aligned}$$

The first eigenvalue-eigenvector pair is $\left\{-5, \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$.
15:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3-\lambda & 24 \\ 4 & 7-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -12 & 24 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -12x+24y \\ 4x-8y \end{bmatrix} \\
\implies y = 2x \to \begin{bmatrix} x \\ y \end{bmatrix} &= s\begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\end{aligned}$$

The second eigenvalue-eigenvector pair is $\left\{15, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

(b) $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

$$
\begin{aligned}
\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} && \text{Definition of eigenvector} \\
&= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \left(\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \text{Subtraction and factoring} \\
\implies \begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\det \begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix} &= 0 \\
(3-\lambda)(4-\lambda) - 1(2) &= 0 \\
(\lambda - 2)(\lambda - 5) &= 0 \\
\lambda &= 2, 5.
\end{aligned}
$$

Thus, the eigenvalues are 2 and 5. We now find the corresponding eigenvectors.
2:

$$
\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} x + y \\ 2x + 2y \end{bmatrix} \\
\implies y = -x \to \begin{bmatrix} x \\ y \end{bmatrix} &= s \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\end{aligned}
$$

The first eigenvalue-eigenvector pair is $\left\{2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.
5:

$$
\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -2x + y \\ 2x - y \end{bmatrix} \\
\implies y = 2x \to \begin{bmatrix} x \\ y \end{bmatrix} &= s \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\end{aligned}
$$

The second eigenvalue-eigenvector pair is $\left\{5, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

(c) $\begin{bmatrix} 1 & -1 \\ 4 & 6 \end{bmatrix}$

$$\begin{aligned}
\begin{bmatrix} 1 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} && \text{Definition of eigenvector} \\
&= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \left(\begin{bmatrix} 1 & -1 \\ 4 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \text{Subtraction and factoring} \\
\implies \begin{bmatrix} 1-\lambda & -1 \\ 4 & 6-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\det \begin{bmatrix} 1-\lambda & -1 \\ 4 & 6-\lambda \end{bmatrix} &= 0 \\
(1-\lambda)(6-\lambda) - (-1)(4) &= 0 \\
(\lambda-2)(\lambda-5) &= 0 \\
\lambda &= 2, 5.
\end{aligned}$$

Thus, the eigenvalues are 2 and 5. Interestingly, these are the same eigenvalues as the previous problem. We now find the corresponding eigenvectors.

2:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1-\lambda & -1 \\ 4 & 6-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -1 & -1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} x+y \\ 4x+4y \end{bmatrix} \\
\implies y = -x \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} &= s \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\end{aligned}$$

The first eigenvalue-eigenvector pair is $\left\{ 2, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

5:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1-\lambda & -1 \\ 4 & 6-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -4 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -4x-y \\ 4x+y \end{bmatrix} \\
\implies y = -4x \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} &= s \begin{bmatrix} 1 \\ -4 \end{bmatrix}.
\end{aligned}$$

The second eigenvalue-eigenvector pair is $\left\{ 5, \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\}$.

4. **The image of an eigenvector will have the same ________ when acted on by the transformation ______ for which it is an eigenvector. The image of the eigenvector is simply the eigenvector itself multiplied by its corresponding _________.**

The image of an eigenvector will have the same direction when acted on by the transformation matrix for which it is an eigenvector. The image of the eigenvector is simply the eigenvector itself multiplied by its corresponding eigenvalue.

5.

(a) If the transformation matrix were a reflection over a line $y = x\tan\theta$, in what directions would the two eigenvectors point? Think geometrically.

Geometrically, the eigenvectors would be 1. along the line $y = x\tan\theta$ and 2. perpendicular to the line. Observe the figure below if you're confused.

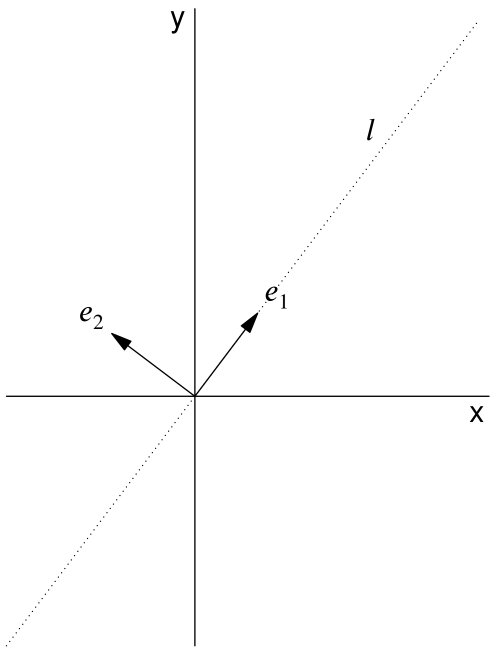

Figure 4: The eigenvectors e_1, e_2 of reflection over the line l.

(b) What would the angle between them be?

The angle between them is $90°$, since one is along a line and the other is perpendicular to that line.

(c) What would their eigenvalues be?

Referring to the above figure, e_1 would have an eigenvalue of 1, since its magnitude and direction is completely preserved, while e_2 would have an eigenvalue of -1, since it is multiplied by -1 to be inverted like that.

6. Recall that multiplication by $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ results in a reflection over $y = x\tan\theta$.

(a) Write a matrix that results in a reflection over the line $y = \frac{\sqrt{3}}{3}x$.

The angle here is

$$\tan^{-1}\frac{\sqrt{3}}{3} = \tan^{-1}\frac{1}{\sqrt{3}} = \tan^{-1}\frac{1/2}{1/\sqrt{3}} = 30°.$$

Thus, the matrix is

$$\begin{bmatrix} \cos 2\cdot 30° & \sin 2\cdot 30° \\ \sin 2\cdot 30° & -\cos 2\cdot 30° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

(b) Find the eigenvalues of that matrix, and the corresponding eigenvectors.

We find the eigenvalues:

$$\begin{bmatrix} \frac{1}{2}-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{1}{2}-\lambda\right)\left(-\frac{1}{2}-\lambda\right)-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}=\lambda^2-\frac{1}{4}-\frac{3}{4}=\lambda^2-1$$
$$\lambda=\pm 1.$$

We then find the corresponding eigenvectors:
1:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{2}x+\frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x-\frac{3}{2}y \end{bmatrix} \\
\implies y = \frac{x}{\sqrt{3}} &\to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}.
\end{aligned}$$

-1:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{3}{2}x+\frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x+\frac{1}{2}y \end{bmatrix} \\
\implies y = -x\sqrt{3} &\to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}.
\end{aligned}$$

Thus, the eigenvalue-eigenvector pairs are $\left\{1, \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}\right\}$ and $\left\{1, \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}\right\}$.

(c) Do your calculations agree with your answers to the previous problem?

Yes, they do. Here are the graphs of those two eigenvectors:

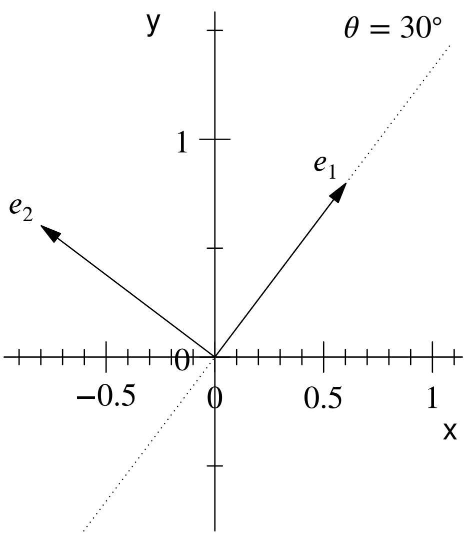

Figure 5: The calculated eigenvectors with the line $\theta = 30°$.

The corresponding eigenvalues also match up.

(d) What are the relationships between the two eigenvectors and between the two eigenvalues?

The two eigenvectors are $90°$ displaced from one another. The two eigenvalues are opposites.

7.

(a) Write a matrix which results in a $60°$ rotation counterclockwise.

This is just $\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

(b) Find the eigenvalues. What do you find strange?

We solve the equation

$$\det \begin{bmatrix} \frac{1}{2}-\lambda & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}-\lambda \end{bmatrix} = 0.$$

$$\begin{aligned} \left(\frac{1}{2}-\lambda\right)\left(\frac{1}{2}-\lambda\right) - \left(-\frac{\sqrt{3}}{2}\right)\cdot\frac{\sqrt{3}}{2} &= 0 \\ \lambda^2 - \lambda + \frac{1}{4} + \frac{3}{4} &= 0 \\ \lambda^2 - \lambda + 1 &= 0 \\ \lambda &= \frac{1 \pm \sqrt{-3}}{2} \\ \lambda &= \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i. \end{aligned}$$

The eigenvalues are complex! They have magnitude 1, however, like the eigenvalues of the reflection.

(c) Find the eigenvectors for those eigenvalues. What's strange about them?

$\frac{1}{2} + \frac{\sqrt{3}}{2}i$:

$$\begin{aligned} \begin{bmatrix} \frac{1}{2}-\lambda & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -\frac{\sqrt{3}}{2}ix - \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \implies y = -ix \to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ -i \end{bmatrix}. \end{aligned}$$

Weird!

$\frac{1}{2} - \frac{\sqrt{3}}{2}i$:

$$\begin{bmatrix} \frac{1}{2}-\lambda & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2}i & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2}ix - \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2}iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies y = ix \to \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Fascinating! The eigenvalue-eigenvector pairs are $\left\{\frac{1}{2}+\frac{\sqrt{3}}{2}i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right\}$ and $\left\{\frac{1}{2}-\frac{\sqrt{3}}{2}i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right\}$.

(d) Explain what's going on.

There are no vectors which don't change orientation under a rotation, so the solutions we get can't be real. Nonetheless, a quadratic always has two roots if the discriminant is nonzero, so we get two solutions.

(e) What are the relationships between the two eigenvectors and between the two eigenvalues?

The eigenvalues are each other's complex conjugates. The eigenvectors, graphed in the complex plane, form a $90°$ between each other.

8. The matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a shear parallel to the x-axis.

(a) What vectors don't change direction when multiplied by this matrix?

Only vectors parallel to the x-axis don't change direction, i.e. $\begin{bmatrix} s \\ 0 \end{bmatrix}$ for any real s.

(b) What would you expect the eigenvectors to be?

We'd expect there to only be one family of eigenvectors, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) Find the eigenvectors and eigenvalues of this matrix.

We want $\det\begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = 0$, which simplifies to $(\lambda-1)^2 = 0$. Thus, there is only one eigenvalue: 1. This makes sense.

The eigenvector is the solution to $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives $\begin{bmatrix} 2y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus, $y = 0$, and the eigenvectors are the family $s\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(d) What is different this time?

There is only one eigenvector and eigenvalue!

(e) Can you represent every vector as sums of eigenvectors?

In this case, you cannot represent every vector as a sum of eigenvectors. After all, any sum of the one eigenvector cannot have a nonzero y coordinate.

9. The matrices below result in some stretches. Find the eigenvectors and eigenvalues for both.

(a) $\left[\begin{smallmatrix} 2 & 0 \\ 0 & 5 \end{smallmatrix}\right]$

The characteristic polynomial[27] is just $(2-\lambda)(5-\lambda)$, which gives eigenvalues $\lambda = 2, 5$.
2:

$$\begin{bmatrix} 2-2 & 0 \\ 0 & 5-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $y = 0$ and the family of eigenvectors is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
5:

$$\begin{bmatrix} 2-5 & 0 \\ 0 & 5-5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $x = 0$ and the family of eigenvectors is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) $\left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$

The characteristic polynomial here is $(3-\lambda)(3-\lambda)$, yielding $\lambda = 3$. We find the eigenvector:

$$\begin{bmatrix} 3-3 & 0 \\ 0 & 3-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, all $\begin{bmatrix} x \\ y \end{bmatrix}$ are eigenvectors. This makes sense! After all, all vectors are scaled up by a factor of 3, and no vectors change direction.

10. Note that most 2×2 matrices have two eigenvectors. How many would you expect to find for an $n \times n$ matrix?

You'd expect there to be n eigenvectors in an $n \times n$ matrix. One way to rationalize this further is that the characteristic polynomial of an $n \times n$ matrix is degree n, which usually has n roots.

11. Assuming that p, q, r, s, t, u, x, y are real, what conditions would you impose on them in the matrices (i) $\begin{bmatrix} 3 & p \\ q & 4 \end{bmatrix}$, (ii) $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix}$, and (iii) $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$ to have...

(a) ... two real eigenvalues?

i. $\begin{bmatrix} 3 & p \\ q & 4 \end{bmatrix}$

The characteristic polynomial here is $(3-\lambda)(4-\lambda) - pq$. Expanded out, this is $\lambda^2 - 7\lambda + 12 - pq$. We want the discriminant to be greater than 0 to have two real eigenvalues, so

$$\begin{aligned} b^2 - 4ac = 7^2 - 4(1)(12-pq) &> 0 \\ 48 - 4pq &< 49 && \text{Manipulate, flip the inequality} \\ 4pq &> -1 \text{Subtract 48 from both sides} \\ pq &> -\frac{1}{4}. \end{aligned}$$

This is our restriction; we must have $pq > -\frac{1}{4}$.

[27] The polynomial involving λ determining the eigenvalues.

ii. $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix}$

The characteristic polynomial here is $(x-\lambda)(y-\lambda)-(-2)(3)$. This expands out to $\lambda^2-(x+y)\lambda+xy+6$. Again, we want the discriminant to be greater than 0 to have two real eigenvalues, so

$$\begin{aligned} b^2-4ac=(x+y)^2-4(1)(xy+6)&>0\\ x^2+y^2+2xy-4xy-24&>0\\ x^2-2xy+y^2&>24\\ (x-y)^2&>24. \end{aligned}$$

Thus, our restriction is $(x-y)^2>24$, or equivalently, $|x-y|>\sqrt{24}=2\sqrt{6}$.
This is always true by the Trivial Inequality[28]. Thus, there are always two real eigenvalues for this matrix.

iii. $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$

The characteristic polynomial here is $(r-\lambda)(u-\lambda)-st$, which expands out to

$$\lambda^2-(u+r)\lambda+ru-st.$$

Again, we want the discriminant to be greater than 0 to have two real eigenvalues, so

$$\begin{aligned} b^2-4ac=(u+r)^2-4(ru-st)&>0\\ u^2+r^2+2ru-4ru-4st&>0\\ u^2-2ru+r^2&>4st\\ (u-r)^2&>4st. \end{aligned}$$

There isn't a great way to simplify this, but $(u-r)^2>4st$ is a potential answer. The first line of the equations above also gives us a potentially simpler interpretation:

$$(u+r)^2>4(ru-st)=4\det\begin{bmatrix} r & s \\ t & u \end{bmatrix}.$$

Thus, the sum of the top-left to bottom-right diagonal squared must be greater than four times the determinant. Wordy!

(b) ... two complex eigenvalues?

i. $\begin{bmatrix} 3 & p \\ q & 4 \end{bmatrix}$

This is identical to problem (a) part i, but we want the discriminant to be smaller than 0. The proof is identical, just with a flipped inequality sign, so the answer is

$$pq<-\frac{1}{4}.$$

ii. $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix}$

This is identical to problem (a) part ii, but we want the discriminant to be smaller than 0. The proof is identical, just with a flipped inequality sign, so the answer is

$$(x-y)^2<24.$$

[28] The *Trivial Inequality* states $x^2\geq 0$ for all real x.

iii. $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$

This is identical to problem (a) part iii, but we want the discriminant to be smaller than 0. The proof is identical, just with a flipped inequality sign, so the answer is

$$(u-r)^2 < 4st \quad \text{or} \quad (u+r)^2 < 4\det \begin{bmatrix} r & s \\ t & u \end{bmatrix}.$$

(c) ... only one eigenvalue?

i. $\begin{bmatrix} 3 & p \\ q & 4 \end{bmatrix}$

This is identical to the previous two iterations of this matrix, but with an equality sign:

$$pq = -\frac{1}{4}.$$

ii. $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix}$

This is identical to the previous two iterations of this matrix, but with an equality sign:

$$(x-y)^2 = 24 \to x - y = \pm 2\sqrt{6}.$$

iii. $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$

This is identical to the previous two iterations of this matrix, but with an equality sign:

$$(u-r)^2 = 4st \quad \text{or} \quad (u+r)^2 = 4\det \begin{bmatrix} r & s \\ t & u \end{bmatrix}.$$

12.

(a) Write a 3×3 matrix showing a rotation of θ around the z-axis.

We already did this a couple sections ago. The matrix is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Name the real eigenvector (this shouldn't require any work).

The real eigenvector is the z-axis, since it doesn't move (observe the figure below if you're confused). Explicitly, this is the family of eigenvectors

$$s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

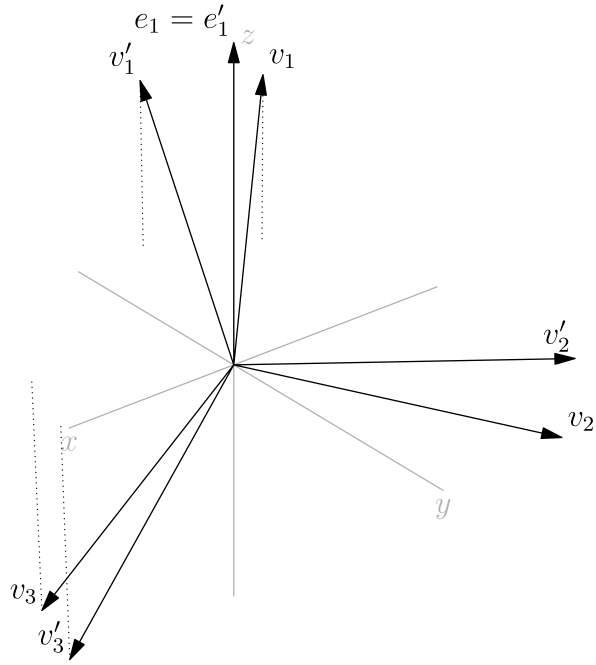

Figure 6: The z-axis remains stationary in a rotation of θ around the z-axis.

(c) Find all three eigenvectors.

The determinant of the eigenvector matrix is, by the minors method:

$$0 = \det\begin{bmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (\cos\theta - \lambda)\det\begin{bmatrix} \cos\theta - \lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} - (-\sin\theta)\det\begin{bmatrix} \sin\theta & 0 \\ 0 & 1-\lambda \end{bmatrix} + 0\,\text{(something)}$$

$$\begin{aligned}
&= (\cos\theta - \lambda)(\cos\theta - \lambda)(1-\lambda) + (\sin\theta)(\sin\theta)(1-\lambda) \\
&= (1-\lambda)((\cos\theta - \lambda)^2 + (\sin\theta)(\sin\theta)) \\
&= (1-\lambda)(lambda^2 + \cos^2\theta - 2\lambda\cos\theta + \sin^2\theta) \\
&= (1-\lambda)(\lambda^2 - 2\lambda\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{\text{diff of squares}}) \\
&= (1-\lambda)(\lambda - (\cos\theta - i\sin\theta))(\lambda - (\cos\theta + i\sin\theta)).
\end{aligned}$$

This gives eigenvalues 1, $\cos\theta - i\sin\theta$ and $\cos\theta + i\sin\theta$. We already knew about the first one. We now compute the eigenvectors:
1:

$$\begin{bmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta - 1 & -\sin\theta & 0 \\ \sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\cos\theta - 1)x - (\sin\theta)y \\ (\sin\theta)x + (\cos\theta - 1)y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $(\cos\theta - 1)x - (\sin\theta)y = 0$ and $(\sin\theta)x + (\cos\theta - 1)y = 0$. The first equation yields $x = \frac{\sin\theta}{\cos\theta - 1}y$. Substitution into the second equation yields

$$\frac{y\sin^2\theta}{\cos\theta - 1} + (\cos\theta - 1)y = 0$$

$$y\left(\frac{\sin^2\theta + (\cos\theta - 1)^2}{\cos\theta - 1}\right) = 0$$

$$y\left(\frac{\sin^2\theta+\cos^2\theta-2\cos\theta+1}{\cos\theta-1}\right)=0$$
$$y\left(\frac{2-2\cos\theta}{\cos\theta-1}\right)=0$$
$$-2y=0$$
$$y=0.$$

Thus, $x=y=0$, or $\cos\theta-1=0$ (since then the first substitution is invalid). This makes sense! If $\cos\theta=1$, then it's a rotation by 0°, which has all vectors as eigenvectors. Anyway, this otherwise gives the family of eigenvectors $s\begin{bmatrix}0\\0\\1\end{bmatrix}$.

$\cos\theta-i\sin\theta$:

$$\begin{bmatrix}\cos\theta-\lambda & -\sin\theta & 0\\ \sin\theta & \cos\theta-\lambda & 0\\ 0 & 0 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}i\sin\theta & -\sin\theta & 0\\ \sin\theta & i\sin\theta & 0\\ 0 & 0 & i\sin\theta-\cos\theta\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}(i\sin\theta)x-(\sin\theta)y+1\\(\sin\theta)x+(i\sin\theta)y\\(i\sin\theta-\cos\theta+1)z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Since $i\sin\theta-\cos\theta+1\neq 0$ except for $\cos\theta=1$ (the rotation of 0 again), this yields $z=0$ and $x=iy$, so the family of eigenvectors is $s\begin{bmatrix}i\\1\\0\end{bmatrix}$.

$\cos\theta+i\sin\theta$:

$$\begin{bmatrix}\cos\theta-\lambda & -\sin\theta & 0\\ \sin\theta & \cos\theta-\lambda & 0\\ 0 & 0 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}-i\sin\theta & -\sin\theta & 0\\ \sin\theta & -i\sin\theta & 0\\ 0 & 0 & \cos\theta-i\sin\theta+1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-(i\sin\theta)x-(\sin\theta)y\\(\sin\theta)x-(i\sin\theta)y\\(1-i\sin\theta-\cos\theta)z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

For nonzero rotations, this yields $z=0$ and $x=-iy$, giving the family of eigenvectors $s=\begin{bmatrix}-i\\1\\0\end{bmatrix}$.

Overall, the eigenvalue-eigenvector pairs are $\left\{1,\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, $\left\{\cos\theta-i\sin\theta,\begin{bmatrix}i\\1\\0\end{bmatrix}\right\}$, and $\left\{\cos\theta+i\sin\theta,\begin{bmatrix}-i\\1\\0\end{bmatrix}\right\}$.

13.

(d) What should the absolute value of an eigenvalue of any rotation matrix be?

It should be 1, since rotations don't stretch anything and doesn't change orientation. All distances are preserved. This is true of our eigenvectors.

(e) The complex eigenvalues relate to the angle of rotation. What is that relationship?

The complex eigenvalues are $\cos\theta+i\sin\theta=\operatorname{cis}\theta$ and $\cos\theta-i\sin\theta=\overline{\operatorname{cis}\theta}$, so they make an angle of θ with the real axis[29] in the complex plane. Furthermore, the angle between them is 2θ.

[29] Note that we shouldn't call it the x-axis, because this is a different set of axes than the xyz-axes we're considering in this problem.

14. In a right-handed coordinate system, rotations in three dimensions are performed by combinations of the three matrices

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \quad Y = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \quad Z = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Each matrix X, Y, Z rotates around the x, y, z axes by α, β, γ, respectively.

In 2D, rotations combine to make other rotations. Similarly, if we combine any number of these rotations, the net result will be a rotation about some axis—though not necessarily a *coordinate* axis. Another way to picture this is that if we operate on an origin-centered sphere with these matrices, there will always be two opposite points on the sphere which have no net movement.

Try computing the following products.

(a) XY

$$XY = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} (1)(\cos\beta) + (0)(0) + (0)(-\sin\beta) & (1)(0) + (0)(1) + (0)(0) & (1)(\sin\beta) + (0)(0) + (0)(\cos\beta) \\ (0)(\cos\beta) + (\cos\alpha)(0) + (-\sin\alpha)(-\sin\beta) & (0)(0) + (\cos\alpha)(1) + (-\sin\alpha)(0) & (0)(\sin\beta) + (\cos\alpha)(0) + (-\sin\alpha)(\cos\beta) \\ (0)(\cos\beta) + (\sin\alpha)(0) + (\cos\alpha)(-\sin\beta) & (0)(0) + (\sin\alpha)(1) + (\cos\alpha)(0) & (0)(\sin\beta) + (\sin\alpha)(0) + (\cos\alpha)(\cos\beta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ \sin\alpha\sin\beta & \cos\alpha & -\sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta \end{bmatrix}.$$

(b) XZ

$$XZ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(\cos\gamma) + (0)(\sin\gamma) + (0)(0) & (1)(-\sin\gamma) + (0)(\cos\gamma) + (0)(0) & (1)(0) + (0)(0) + (0)(1) \\ (0)(\cos\gamma) + (\cos\alpha)(\sin\gamma) + (-\sin\alpha)(0) & (0)(-\sin\gamma) + (\cos\alpha)(\cos\gamma) + (-\sin\alpha)(0) & (0)(0) + (\cos\alpha)(0) + (-\sin\alpha)(1) \\ (0)(\cos\gamma) + (\sin\alpha)(\sin\gamma) + (\cos\alpha)(0) & (0)(-\sin\gamma) + (\sin\alpha)(\cos\gamma) + (\cos\alpha)(0) & (0)(0) + (\sin\alpha)(0) + (\cos\alpha)(1) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \cos\alpha\sin\gamma & \cos\alpha\cos\gamma & -\sin\alpha \\ \sin\alpha\sin\gamma & \sin\alpha\cos\gamma & \cos\alpha \end{bmatrix}.$$

(c) YX

$$YX = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\beta)(1) + (0)(0) + (\sin\beta)(0) & (\cos\beta)(0) + (0)(\cos\alpha) + (\sin\beta)(\sin\alpha) & (\cos\beta)(0) + (0)(-\sin\alpha) + (\sin\beta)(\cos\alpha) \\ (0)(1) + (1)(0) + (0)(0) & (0)(0) + (1)(\cos\alpha) + (0)(\sin\alpha) & (0)(0) + (1)(-\sin\alpha) + (0)(\cos\alpha) \\ (-\sin\beta)(1) + (0)(0) + (\cos\beta)(0) & (-\sin\beta)(0) + (0)(\cos\alpha) + (\cos\beta)(\sin\alpha) & (-\sin\beta)(0) + (0)(-\sin\alpha) + (\cos\beta)(\cos\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta & \sin\beta\sin\alpha & \sin\beta\cos\alpha \\ 0 & \cos\alpha & -\sin\alpha \\ -\sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha \end{bmatrix}.$$

(d) ZX

$$ZX = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\gamma)(1) + (-\sin\gamma)(0) + (0)(0) & (\cos\gamma)(0) + (-\sin\gamma)(\cos\alpha) + (0)(\sin\alpha) & (\cos\gamma)(0) + (-\sin\gamma)(-\sin\alpha) + (0)(\cos\alpha) \\ (\sin\gamma)(1) + (\cos\gamma)(0) + (0)(0) & (\sin\gamma)(0) + (\cos\gamma)(\cos\alpha) + (0)(\sin\alpha) & (\sin\gamma)(0) + (\cos\gamma)(-\sin\alpha) + (0)(\cos\alpha) \\ (0)(1) + (0)(0) + (1)(0) & (0)(0) + (0)(\cos\alpha) + (1)(\sin\alpha) & (0)(0) + (0)(-\sin\alpha) + (1)(\cos\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\gamma & -\sin\gamma\cos\alpha & \sin\gamma\sin\alpha \\ \sin\gamma & \cos\gamma\cos\alpha & -\cos\gamma\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}.$$

Interestingly, $ZX \neq XZ$. Indeed, while rotations commute in 2 dimensions, they do not always commute in 3 dimensions.

15.

(a) Without matrices, consider a cube with side length 2 at the origin so its faces are perpendicular to the coordinate axes. Rotate it, first $90°$ counterclockwise about the y-axis, then $90°$ counterclockwise about the x-axis. Note that rotations are done facing from the "positive side" of the coordinate axis. The net result should leave two vertices fixed. Which two?

This requires a good amount of geometric visualization. The answer is the vertices $(1,1,1)$ and $(-1,-1,-1)$. Observe the figures below:

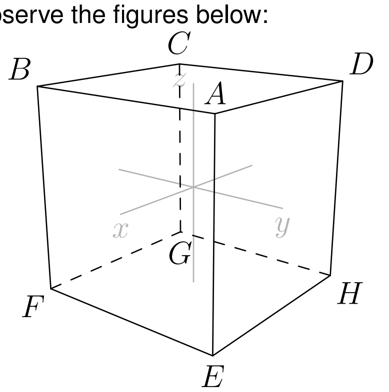

Figure 7: The starting position of the cube.

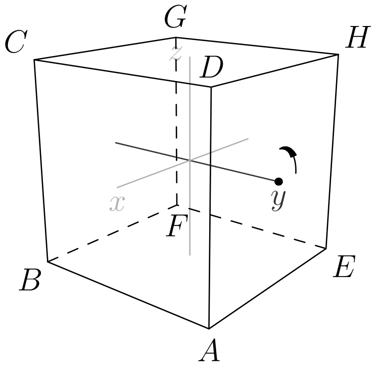

Figure 8: Rotation about the y-axis.

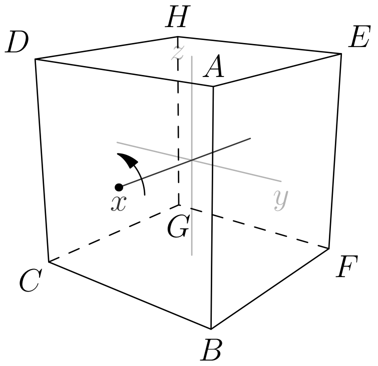

Figure 9: Rotation about the x-axis.

Indeed, A and G remain fixed. These are the vertices $(1,1,1)$ and $(-1,-1,-1)$.

(b) Write a vector for the axis of rotation.

The vector is any nonzero multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. In the following figure, the axis of rotation is graphed.

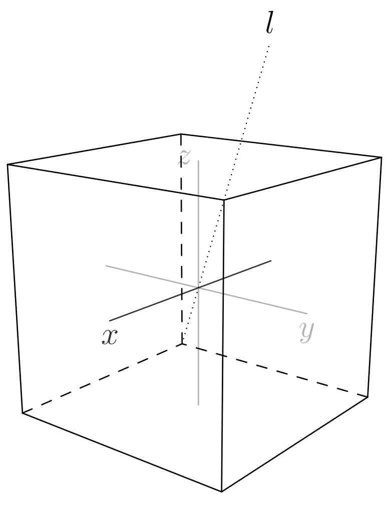

Figure 10: The net axis of rotation is $\langle 1, 1, 1\rangle$.

(c) How many degrees do you think the net rotation of the cube is? Be careful; the answer is not $180°$**.**

The rotation is $120°$, because E is going to D, D is going to B and E is going to D, a cycle with period 3.

(d) Let's check our answers using matrices. Write a matrix product that corresponds to a rotation of $90°$ **about the** y**-axis, followed by** $90°$ **about the** x**-axis.**

Rotation of $90°$ about the y-axis: $Y = \begin{bmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

Rotation of $90°$ about the x-axis: $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° \\ 0 & \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

As usual, matrix multiplication goes right-to-left, so the product is XY.

(e) Multiply out the matrix product.

$$XY = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(0)+(0)(0)+(0)(-1) & (1)(0)+(0)(1)+(0)(0) & (1)(1)+(0)(0)+(0)(0) \\ (0)(0)+(0)(0)+(-1)(-1) & (0)(0)+(0)(1)+(-1)(0) & (0)(1)+(0)(0)+(-1)(0) \\ (0)(0)+(1)(0)+(0)(-1) & (0)(0)+(1)(1)+(0)(0) & (0)(1)+(1)(0)+(0)(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Interesting!

(f) Remember that the real eigenvector in a rotation gives the axis of rotation, and the complex eigenvalues give information about the net rotation. Evaluate these and check your answers for (a) and (b).

We first find the eigenvalues:

$$\det \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} = 0$$

$$-\lambda \cdot \det \begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix} - 0 \cdot (\text{something}) + 1 \cdot \det \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} = 0$$

$$-\lambda^3 + (1 + 0 \cdot -\lambda) = 0$$

$$\lambda^3 = 1.$$

We let $\lambda = \operatorname{cis} \theta$:

$$\operatorname{cis}^3 \theta = 1 \Longrightarrow \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}.$$

Thus, $\lambda = 1, \operatorname{cis} \frac{2\pi}{3}, \operatorname{cis} \frac{4\pi}{3}$.

We now compute the eigenvector for the axis of rotation, which should correspond to $\lambda = 1$.

$$\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x+z \\ x-y \\ y-z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

These yields $x = y = z$ and the eigenvector family $s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, confirming our previous result.

We can find the angle of rotation by the angle the complex eigenvalues make with the x-axis. These eigenvalues are $\operatorname{cis}\frac{2\pi}{3}, \operatorname{cis}\frac{4\pi}{3}$, which make a $\frac{2\pi}{3} = 120°$ angle with the x-axis. Thus, the magnitude of the rotation is $120°$, confirming our hypothesis.

16. Here are two rotation matrices:

i. $\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$, **ii.** $\begin{bmatrix} \frac{7}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{bmatrix}$.

(a) What is the determinant of each matrix? (Don't work, think!)

The determinant of each matrix is 1, since rotation matrices have determinant 1.

(b) What is true of each row and each column?

The sums of squares of each element in each row and column is 1. Therefore, each row vector and column vector is a unit vector. As an example, consider the top row of (ii):

$$\left(\frac{7}{9}\right)^2 + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^2 = \frac{81}{81} = 1.$$

(c) Find the axis of rotation associated with each matrix.

i. $\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

We find the eigenvalues:

$$\det\begin{bmatrix} \frac{2}{3}-\lambda & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3}-\lambda & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3}-\lambda \end{bmatrix} = 0$$

$$\left(\frac{2}{3}-\lambda\right)\cdot\det\begin{bmatrix} \frac{2}{3}-\lambda & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3}-\lambda \end{bmatrix} - \left(-\frac{2}{3}\right)\det\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3}-\lambda \end{bmatrix} - \frac{1}{3}\cdot\det\begin{bmatrix} \frac{1}{3} & \frac{2}{3}-\lambda \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} = 0$$

$$-\lambda^3 + 2\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda-1)(\lambda^2-\lambda+1) = 0.$$

The real eigenvalue is $\lambda = 1$, so we find the corresponding eigenvector:

$$\begin{bmatrix} \frac{2}{3}-\lambda & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3}-\lambda & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x-\frac{2}{3}y-\frac{1}{3}z \\ \frac{1}{3}x-\frac{1}{3}y-\frac{2}{3}z \\ \frac{2}{3}x+\frac{1}{3}y-\frac{1}{3}z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Multiplying by 3 on both sides yields the system of equations

$$\begin{cases} -x-2y-z=0 \\ x-y-2z=0 \\ 2x+y-z=0 \end{cases}.$$

The solution to this system of equations is $x = z = -y$. Thus, the eigenvector family is $s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and the axis of rotation is the vector $\langle 1, -1, 1 \rangle$.

i. $\begin{bmatrix} \frac{7}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{bmatrix}$

We find the eigenvalues:

$$\det\begin{bmatrix} \frac{7}{9}-\lambda & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9}-\lambda & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9}-\lambda \end{bmatrix} = 0$$

$$\left(\frac{7}{9}-\lambda\right)\cdot\det\begin{bmatrix} -\frac{1}{9}-\lambda & \frac{8}{9} \\ -\frac{8}{9} & \frac{1}{9}-\lambda \end{bmatrix} - \left(\frac{4}{9}\right)\cdot\det\begin{bmatrix} -\frac{4}{9} & \frac{8}{9} \\ \frac{4}{9} & \frac{1}{9}-\lambda \end{bmatrix} + \left(\frac{4}{9}\right)\cdot\det\begin{bmatrix} -\frac{4}{9} & -\frac{1}{9}-\lambda \\ \frac{4}{9} & -\frac{8}{9} \end{bmatrix} = 0$$

$$-\lambda^3 + \frac{7\lambda^2}{9} - \frac{7\lambda}{9} + 1 = 0$$

$$-\frac{1}{9}(\lambda-1)(9\lambda^2+2\lambda+9) = 0.$$

The real eigenvalue is $\lambda = 1$, so we find the corresponding eigenvector:

$$\begin{bmatrix} \frac{7}{9}-\lambda & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9}-\lambda & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -\frac{2}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{10}{9} & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{8}{9} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -2x+4y+4z \\ -4x-10y+8z \\ 4x-8y+8z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \end{bmatrix}$$

The solution to this system of equations is $\langle x, y, z \rangle = s\langle -2, 0, 1 \rangle$. This is the axis of rotation: $\langle -2, 0, 1 \rangle$.

(d) Find the angle of rotation associated with each matrix.

i. $\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

From the last time we dealt with this matrix, we found that the complex eigenvalues satisfy $\lambda^2 - \lambda + 1 = 0$. By the quadratic formula, the solutions to this quadratic are $\lambda = \frac{1 \pm i\sqrt{3}}{2}$.

Since $\operatorname{cis} \pm 60° = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i = \lambda$, the rotation is $60°$.

i. $\begin{bmatrix} \frac{7}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{bmatrix}$

Previously, we found that the complex eigenvalues of this matrix satisfy the polynomial equation $9\lambda^2+2\lambda+9=0$. By the quadratic formula, the roots of this equation are

$$\frac{-2\pm\sqrt{2^2-4\cdot 9^2}}{18} = -\frac{1}{9}\pm\frac{4i\sqrt{5}}{9}.$$

The angle of rotation is given by

$$\tan^{-1}\frac{y}{x} = \frac{\pm\frac{4\sqrt{5}}{9}}{-\frac{1}{9}} = \pm\tan^{-1}4\cdot\sqrt{5},$$

which has magnitude $\tan^{-1}(4\sqrt{5})$.