7 Your Daily Dose of Vitamin i

- 1. We will use complex numbers to find identities for cot. Use Pascal's triangle to expand the following:
 - (a) $(a+b)^3$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

(b) $(a+b)^4$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

(c) $(a+b)^5$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

- 1. (cont.) Then substitute $b = i = \sqrt{-1}$ and expand:
 - **(d)** $(a+i)^3$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + 3a^2i - 3a - i$$

(e) $(a+i)^4$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1.$$

(f) $(a+i)^{\frac{1}{2}}$

$$(a+i)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i.$$

- 1. (cont.) Finally, substitute $a = \cot \theta$ and expand:
 - (g) $(\cot \theta + i)^3$

$$(\cot \theta + i)^3 = a^3 + 3a^2i - 3a - i = (\cot^3 \theta - 3\cot \theta) + i(3\cot^2 \theta - 1).$$

(h) $(\cot \theta + i)^4$

$$(\cot \theta + i)^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1 = (\cot^4 \theta - 6\cot^2 \theta + 1) + (4\cot^3 \theta - 4\cot \theta).$$

(i) $(\cot \theta + i)^5$

$$(\cot \theta + i)^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i = (\cot^5 \theta - 10\cot^3 \theta + 5\cot \theta) + i(5\cot^4 \theta - 10\cot^2 \theta + 1).$$

- 1. (cont.) Consider $z = i + \cot \theta$.
 - (j) Use the above results to find identities for (i) $\cot 3\theta$, (ii) $\cot 4\theta$, and (iii) $\cot 5\theta$.

i.
$$\cot 3\theta$$

Given the right triangle formed by $z = i + \cot \theta$ in Figure 7, we have $\tan(\operatorname{Arg} z) = \frac{1}{\cot \theta} = \tan \theta$, so $\operatorname{Arg} z = \theta$ and $z = r \operatorname{cis} \theta$.

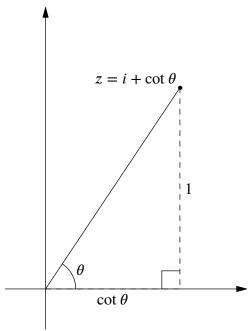


Figure 1:
$$Arg(i + \cot \theta) = \theta$$
.

Thus, we have

$$\cot 3\theta = \frac{\cos 3\theta}{\sin 3\theta}$$

$$= \frac{\text{Re}(\text{cis } 3\theta)}{\text{Im}(\text{cis } 3\theta)}$$

$$= \frac{\text{Re}(r^3 \text{ cis } 3\theta)}{\text{Im}(r^3 \text{ cis } 3\theta)}$$

$$= \frac{\text{Re}(z^3)}{\text{Im}(z^3)}.$$

We substitute in our expression for z^3 , $(\cot^3 \theta - 3 \cot \theta) + i(3 \cot^2 \theta - 1)$:

$$\cot 3\theta = \frac{\cot^3 \theta - 3 \cot \theta}{3 \cot^2 \theta - 1}.$$

i. $\cot 4\theta$

We proceed in the same way as the last subproblem.

$$\cot 4\theta = \frac{\cos 4\theta}{\sin 4\theta}$$

$$= \frac{\text{Re}(\cos 4\theta)}{\text{Im}(\cos 4\theta)}$$

$$= \frac{\text{Re}(r^4 \cos 4\theta)}{\text{Im}(r^4 \cos 4\theta)}$$

$$= \frac{\text{Re}(z^4)}{\text{Im}(z^4)}$$

$$\cot 4\theta = \frac{\cot^4 \theta - 6 \cot^2 \theta + 1}{4 \cot^3 \theta - 4 \cot \theta}.$$

i. $\cot 5\theta$

We proceed in the same way as the last subproblem.

$$\cot 5\theta = \frac{\cos 5\theta}{\sin 5\theta}$$

$$= \frac{\text{Re}(\text{cis }5\theta)}{\text{Im}(\text{cis }5\theta)}$$

$$= \frac{\text{Re}(r^5 \text{cis }5\theta)}{\text{Im}(r^5 \text{cis }5\theta)}$$

$$= \frac{\text{Re}(z^5)}{\text{Im}(z^5)}$$

$$= \frac{\cot^5 \theta - 10 \cot^3 \theta + 5 \cot \theta}{5 \cot^4 \theta - 10 \cot^2 \theta + 1}.$$

(k) Graph z, z^2 , z^3 , z^4 , and z^5 , with $\theta \approx 75^\circ$. What is your solution method?

To graph these, I first calculated the approximate magnitude of z, which is how many times each subsequent power will be scaled by. We have $|1 + \cot 75^{\circ}| \approx 1.268$, so we only need to scale by about $\frac{5}{4}$ each time. Of course, we rotate by about 75° each time.

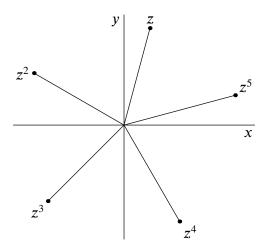


Figure 2: Graphs of z, z^2 , z^3 , z^4 , and z^5 .

2. Compute $(1+i)^n$ for n=3,4,5,... Can you find a general pattern?

We have

$$(1+i)^3 = 1^3 + 3i - 3 - i = -2 - 2i$$

$$(1+i)^4 = 1^4 + 4i - 6 - 6i + 1 = -4 - 2i$$

$$(1+i)^5 = 1^5 + 5i - 10 - 10i + 5 + i = -4 - 4i.$$

We can find the pattern by representing $1+i=\sqrt{2}\cos 45^\circ$. This shows that it has period 8 and let's us find an expression for $(1+1)^n$:

$$(1+i)^n = \left(\sqrt{2}\operatorname{cis} 45^\circ\right)^n = 2^{n/2}\operatorname{cis}\left(\frac{n\pi}{4}\right).$$

3. Expand and graph $cis^n \theta$ for n = 2, 3, 4, ...

Let $\cos \theta = c$ and $\sin \theta = s$. We have

$$(c+is)^2 = c^2 + 2csi - s^2 = (c^2 - s^2) + i(2cs)$$

$$(c+is)^3 = c^3 + 3c^2si - 3cs^2 - s^3i = (c^3 - 3cs^2) + i(3c^2s - s^3)$$

$$(c+is)^4 = c^4 + 4c^3si - 6c^2s^2 - 4cs^3i + s^4 = (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3)$$

$$(c+is)^5 = c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i = (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5).$$

The graphs of $cis^n \theta$ for $\theta \approx 50^\circ$ are shown in Figure 3 below.

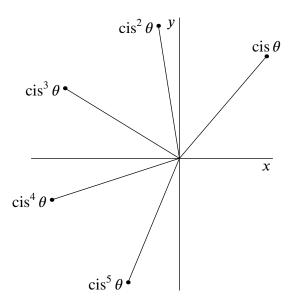


Figure 3: Graphs of $cis^n \theta$ for $\theta \approx 50^\circ$.

(a) Why is the real part $\cos n\theta$ and the imaginary part $\sin n\theta$?

By DeMoivre's theorem, $\operatorname{cis}^n \theta = \operatorname{cis} n\theta$, which by definition has $\operatorname{Im}(\operatorname{cis} n\theta) = \operatorname{cos} n\theta$ and $\operatorname{Re}(\operatorname{cis} n\theta) = \operatorname{sin} n\theta$.

(b) Use your results to write identities for $\cos n\theta$ and $\sin n\theta$ for n=2,3,4,5.

Here they are. Again, let $\cos \theta = c$ and $\sin \theta = s$:

$$\cos 2\theta = \operatorname{Re}(\operatorname{cis} 2\theta) = c^2 - s^2$$

$$\cos 3\theta = \operatorname{Re}(\operatorname{cis} 3\theta) = c^3 - 3cs^2$$

$$\cos 4\theta = \operatorname{Re}(\operatorname{cis} 4\theta) = c^4 - 6c^2s^2 + s^4$$

$$\cos 5\theta = \operatorname{Re}(\operatorname{cis} 5\theta) = c^5 - 10c^3s^2 + 5cs^4$$

$$\sin 2\theta = \operatorname{Im}(\operatorname{cis} 2\theta) = 2cs$$

$$\sin 3\theta = \operatorname{Im}(\operatorname{cis} 3\theta) = 3c^2s - s^3$$

$$\sin 4\theta = \operatorname{Im}(\operatorname{cis} 4\theta) = 4c^3s - 4cs^3$$

$$\sin 5\theta = \operatorname{Im}(\operatorname{cis} 5\theta) = 5c^4s - 10c^2s^3 + s^5.$$

4. Compute $\cos 7^{\circ} + \cos 79^{\circ} + \cos 151^{\circ} + \cos 223^{\circ} + \cos 295^{\circ}$ without a calculator. (Hint: what does this have to do with complex numbers?)

These numbers look random, but a closer inspection reveals they are in arithmetic progression, with starting term 7 and increasing 72° each time. That's the rotation of a pentagon!

We rewrite this as the real component of a sum of cises, then manipulate and evaluate:

$$\cos 7^{\circ} + \cos 79^{\circ} + \cos 151^{\circ} + \cos 223^{\circ} + \cos 295^{\circ} = \text{Re}(\text{cis } 7^{\circ} + \text{cis } 79^{\circ} + \text{cis } 151^{\circ} + \text{cis } 223^{\circ} + \text{cis } 295^{\circ})$$

$$= \text{Re}((\text{cis } 7^{\circ})(\text{cis } 0^{\circ} + \text{cis } 72^{\circ} + \text{cis } 144^{\circ} + \text{cis } 216^{\circ} + \text{cis } 288^{\circ}))$$

$$= \text{Re}((\text{cis } 7^{\circ})(0))$$

$$= \text{Re}(0)$$

$$= 0.$$

Note that going from the second to third step, we used the fact that the cis expressions are the vertices of a regular pentagon, which sum to 0. If you want to be more formal about it, a fun way to prove that $\operatorname{cis} 0^\circ + \operatorname{cis} 72^\circ + \operatorname{cis} 144^\circ + \operatorname{cis} 216^\circ + \operatorname{cis} 288^\circ = 0$ is to set it to Ξ and calculate:

$$\Xi \cdot \operatorname{cis} 72^{\circ} = (\operatorname{cis} 0^{\circ} + \operatorname{cis} 72^{\circ} + \operatorname{cis} 144^{\circ} + \operatorname{cis} 216^{\circ} + \operatorname{cis} 288^{\circ}) \operatorname{cis} 72^{\circ}$$

$$= \operatorname{cis} 72^{\circ} + \operatorname{cis} 144^{\circ} + \operatorname{cis} 216^{\circ} + \operatorname{cis} 288^{\circ} + \operatorname{cis} 360^{\circ}$$

$$= \operatorname{cis} 72^{\circ} + \dots + \operatorname{cis} 288^{\circ} + \operatorname{cis} 0^{\circ}$$

$$= \Xi.$$

If Ξ · (something that's not one) = Ξ , then Ξ must be 0.

5. Factor the following:

(a) $x^6 - 1$ as a difference of squares

We substitute $y = x^3$, giving $y^2 - 1 = (y + 1)(y - 1)$. Substituting back in, we get

$$(x^3 + 1)(x^3 - 1)$$
.

(b) $x^6 - 1$ as a difference of cubes

We now substitute $y = x^2$, giving $y^3 - 1 = (y - 1)(y^2 + y + 1)$. Substituting back in, we get

$$(x^2 - 1)(x^4 + x^2 + 1)$$

(c) $x^4 + x^2 + 1$ over the real numbers

This one isn't as obvious. We substitute $y = x^2$ to get $y^2 + y + 1$ and find the quadratic's zeroes: $y = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i \sqrt{3}}{2}$. So it is irreducible over the reals.

(d) $x^6 - 1$ completely

We already broke it down into $(x^3 + 1)$ and $(x^3 - 1)$. Going further, we have $x^3 + 1 = (x + 1)(x^2 - x + 1)$ and $x^3 - 1 = (x - 1)(x^2 + x + 1)$. To break apart the last two quadratics, we find their zeros:

$$x^{2} - x + 1 = 0 \Longrightarrow x = \frac{1 \pm i\sqrt{3}}{2} \Longrightarrow \left(x - \frac{1 - i\sqrt{3}}{2}\right) \left(x - \frac{1 + i\sqrt{3}}{2}\right).$$

$$x^{2} + x + 1 = 0 \Longrightarrow x = \frac{-1 \pm i\sqrt{3}}{2} \Longrightarrow \left(x + \frac{1 - i\sqrt{3}}{2}\right) \left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

Combining all these, we get the complete factorization over the complex numbers:

$$x^{6} - 1 = (x+1)\left(x - \frac{1 - i\sqrt{3}}{2}\right)\left(x - \frac{1 + i\sqrt{3}}{2}\right)(x-1)\left(x + \frac{1 - i\sqrt{3}}{2}\right)\left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

(e) $x^4 + x^2 + 1$ completely

We could do a lot of work again, or we could observe that $x^4 + x^2 + 1 = \frac{x^6 - 1}{x^2 - 1} = \frac{x^6 - 1}{(x+1)(x-1)}$. Removing the denominator's terms from our factorization of $x^6 - 1$ we found in the last subproblem, we get

$$x^{4} + x^{2} + 1 = \left(x - \frac{1 - i\sqrt{3}}{2}\right) \left(x - \frac{1 + i\sqrt{3}}{2}\right) \left(x + \frac{1 - i\sqrt{3}}{2}\right) \left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

- 6. Let $f(z) = \frac{z+1}{z-1}$.
 - (a) Without a calculator, compute $f^{2014}(z)$.

This seems terrifying. Let's try computing $f^2(z)$ and perhaps $f^3(z)$.

$$f^{2}(z) = \frac{f(z) + 1}{f(z) - 1} = \frac{\frac{z+1}{z-1} + 1}{\frac{z+1}{z-1} - 1} = \frac{\frac{2z}{z-1}}{\frac{2}{z-1}} = z.$$

Oh.

Since 2014 is even, we have $f^{2014}(z) = (f^2)^{1007}(z) = z$.

(b) What if you replace 2014 with the current year?

Let y be the current year. As I write this, it is 1492. If y is even, then $f^y(z)=(f^2)^{y/2}(z)=z$. If y is odd, then $f^y(z)=f((f^2)^{(y-1)/2}(z))=f(z)=\frac{z+1}{z-1}$.

7. Find Im $((cis 12^{\circ} + cis 48^{\circ})^{6})$.

These are some weird looking angles. Thinking back to some older problems, however, the resultant angle of the addition may be tractable. We draw a diagram, shown in Figure 4.

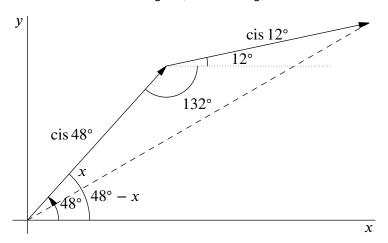


Figure 4: Adding cis 12° + cis 48°.

Consider the isosceles triangle. The apex has angle measure $132^{\circ} + 12^{\circ} = 144^{\circ}$, so the base angles are each $x = \frac{180^{\circ} - 144^{\circ}}{2} = 18^{\circ}$. But $Arg(cis 12^{\circ} + cis 48^{\circ}) = 48^{\circ} - x = 30^{\circ}$!

That's a familiar angle. Indeed, we have $z = \operatorname{cis} 12^\circ + \operatorname{cis} 48^\circ = r \operatorname{cis} 30^\circ$ for some r. It doesn't really matter which r, because

$$\operatorname{Im}((r \operatorname{cis} 30^{\circ})^{6}) = \operatorname{Im}(r^{6} \operatorname{cis} 180^{\circ}) = \operatorname{Im}(-r^{6}) = 0.$$

8. Let x satisfy the equation $x + \frac{1}{x} = 2\cos\theta$.

(a) Compute $x^2 + \frac{1}{x^2}$ in terms of θ .

Squaring the left hand side will get us some terms that look close to what we want.

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$$
.

So $x^2 + \frac{1}{x^2} = (2\cos\theta)^2 - 2 = 4\cos^2\theta - 2 = 2(2\cos^2\theta - 1) = 2\cos2\theta$. Huh.

(b) Compute $x^n + \frac{1}{x^n}$ in terms of n and θ .

It's unclear how to start, so we might as well try to compute $x^3 + \frac{1}{x^3}$ in the same way.

$$\left(x + \frac{1}{x}\right)^{3} = x^{3} + 3x + \frac{3}{x} + \frac{1}{x^{3}} = 8\cos^{3}\theta$$

$$x^{3} + 3\underbrace{\left(x + \frac{1}{x}\right)}_{2\cos\theta} + \frac{1}{x^{3}} = 8\cos^{3}\theta - 6\cos\theta.$$

$$x^{3} + \frac{1}{x^{3}} = 8\cos^{3}\theta - 6\cos\theta.$$

Now what? The astute among you may recognize that $8\cos^3\theta - 6\cos\theta = 2\cos 3\theta$, at which point you

could make a conjecture (and could jump ahead). But suppose we didn't find that. We know that $x^2 + \frac{1}{x^2} = 2\cos 2\theta$. By analogy, if we make the substitution $y = x^2$ and $\phi = 2\theta$, we get that $y + \frac{1}{y} = 2\cos\phi$, and thus $y^2 + \frac{1}{y^2} = 2\cos2\phi \Longrightarrow x^4 + \frac{1}{x^4} = 2\cos4\theta$. In general,

$$x^{2^m} + \frac{1}{x^{2^m}} = 2\cos 2^m \theta.$$

The exponent on x, in this case 2^m , is the same as the multiple of θ . Pretty sus. We've solved the problem for n which are powers of two, but we conjecture that the relationship holds for all integers n. To be explicit, we want to show that

$$x^n + \frac{1}{x^n} = 2\cos n\theta.$$

There's a couple of ways to do it. But seeing x^n and $\cos n\theta$ in the same place immediately recalls exponentiating $\operatorname{cis} \theta$. So, let's try rewriting the problem a bit by entering into the complex plane. Let $x = r \operatorname{cis} \phi$, which we really should have done earlier. Then we're given that $r \operatorname{cis} \phi + \frac{1}{r \operatorname{cis} \phi} = 2 \operatorname{cos} \theta$. Working further,

$$2\cos\theta = r\operatorname{cis}\phi + \frac{1}{r}\operatorname{cis}(-\phi)$$

$$= r\cos\phi + \frac{1}{r}\cos\phi + i(r\sin\phi - \frac{1}{r}\sin\phi)$$

$$= \left(r + \frac{1}{r}\right)\cos\phi + i\left(r - \frac{1}{r}\right)\sin\phi$$

The imaginary part needs to be zero, since the left hand side is real. So either $r=\frac{1}{r}$ or $\sin\phi=0$. Let's examine each case. In the first case, r = 1 (it can't be -1 since $r \ge 0$. In the second case, we have $\cos \phi = \pm 1$, and substituting, we get

$$2\cos\theta = \pm \left(r + \frac{1}{r}\right).$$

That's not helpful... except now we know that r is real and ≥ 0 . Considering $r+\frac{1}{r}$, we see that it approaches ∞ as $r\to 0$. What range of values does it make? Graphing it shows that it has a range of $[2,\infty)$, reaching its minimum at r = 1. Another way to prove this is via AM-GM with x = 2r and $y = \frac{2}{3}$:

$$\frac{x+y}{2} \ge \sqrt{xy} \Longrightarrow r + \frac{1}{r} \ge \sqrt{2r\left(\frac{2}{r}\right)} = 2.$$

But the range of $2\cos\theta$ is [-2,2], and the only possible value of the equation is the intersection of their ranges, aka 2. So r = 1 no matter what. That's damn useful, because then

$$2\cos\theta = \left(r + \frac{1}{r}\right)\cos\phi + i\left(r - \frac{1}{r}\right)\sin\phi$$
 From before
$$= 2\cos\phi.$$

So $\cos \phi = \cos \theta$. We wish to find an expression for $x^n + \frac{1}{x^n}$.

$$x^{n} + \frac{1}{x^{n}} = r^{n} \operatorname{cis}^{n} \phi + \frac{1}{r^{n}} \operatorname{cis}^{n} (-\phi)$$

$$= r^{n} (\cos n\phi + \sin n\phi) + \frac{1}{r^{n}} (\cos n\phi - \sin n\phi)$$

$$= \cos n\phi + \sin n\phi + \cos n\phi - \sin n\phi$$

$$= 2 \cos n\phi$$

$$= 2 \cos n\theta.$$

Note that in the last step, we have to be careful, but cosine does have this property. Anyway, that's a gg, QED.