4 Rotation and Reflection Groups

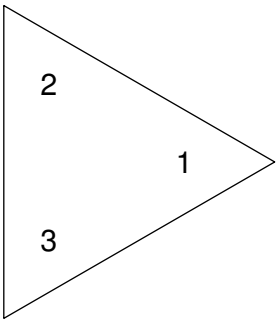

Figure 1: The paper triangle.

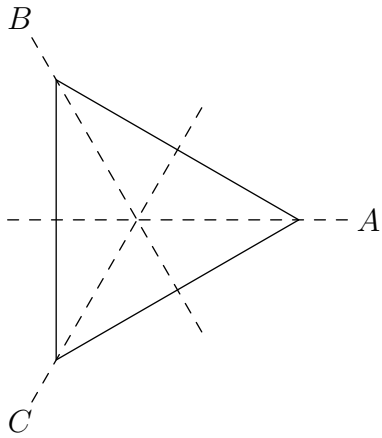

Figure 2: Its axes of reflection.

In the previous section, we started with the dihedral group of the equilateral triangle and discovered it had six elements: reflections about three different axes, rotations of $\pm 120°$, and the identity transformation. We identified a subgroup consisting of the identity I with two rotations r and r^2, and three other subgroups of just the identity and a single reflection. The first subgroup—the one consisting of only rotations—is known as the **rotation group** of the equilateral triangle, or the **cyclic group** of order 3, C_3.

1. Notice that the original dihedral group had twice as many elements as the rotation group. Why?
2. Make and justify a conjecture extending this observation to the dihedral groups of other shapes like rectangles, squares, and hexagons, as well as the symmetry group of the cube.
3. Let r be a $180°$ rotation, x be a reflection over the x-axis, and y be a reflection over the y axis. Write a table for the dihedral group of the rectangle, recalling that the allowed isometries are reflections and rotations. How does this table differ from the dihedral group of the equilateral triangle?
4. Write a table for the rotation group of the square, with 4 elements and 16 entries. Compare this table to Problem 3.

We noticed that the rotation group for the equilateral triangle could be generated by just one of the elements, such as r—rotation by $120°$ counterclockwise. Then r^2 is a rotation of $240°$ counterclockwise, and $r^3 = I$, the identity (see Figure 3). Since we can generate the entire rotation group with a single element (namely, r), a natural question to ask is whether we can do the same with the dihedral group D_3. Clearly, we can't use the identity to do it, and a series of rotations always leaves us with a rotation, never a reflection. Also, a series of flips along one axis simply generates a two-member group with elements I, f (see Figure 4).

Let's try using two elements to generate our group, using the same definitions of f and r as in the previous section: a flip over the A axis and rotation by $120°$ counterclockwise, respectively. As we found, fr is a flip over the B axis and rf is a flip over the C axis. Consecutive powers of r already got us the remaining elements, so r, f generates the group.

We can also generate the group using two reflections, say f and f_B (flip over the B axis, as shown in Figure 2). Notice that an even number of reflections always results in a rotation—even the identity element I is just a rotation by 0.[1] We can conceptualize this behavior as the existence of a "mirror world" and its unmirrored counterpart, with each reflection taking us into or out of the mirror world.

Moving into three dimensions, D_3 is isomorphic to the set of rotations of an equilateral triangular prism. The new axes of rotation are coplanar with where the reflection axes used to be (see Figure 5). Indeed, when you "flipped" your equilateral triangles, you were actually rotating a paper-thin triangular prism in the third dimension. Truly flipping the triangular prism using a rotation would require four spatial dimensions—something we cannot easily visualize.

You will next analyze the symmetries of a variety of objects under rotations or reflections. You will notice that the more symmetries an object has, the larger its symmetry group is. Indeed, group theory is the mathematics of symmetry *par excellence*.

For each of the following groups, find the following:

[1] Any reflection group will include rotations, although they may be the identity.

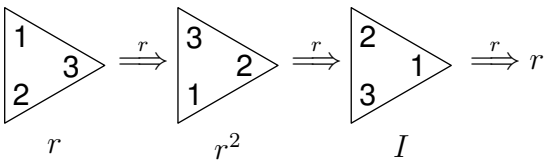

Figure 3: r generates a three member group.

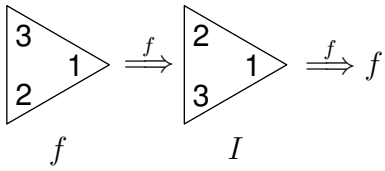

Figure 4: f generates a two member group.

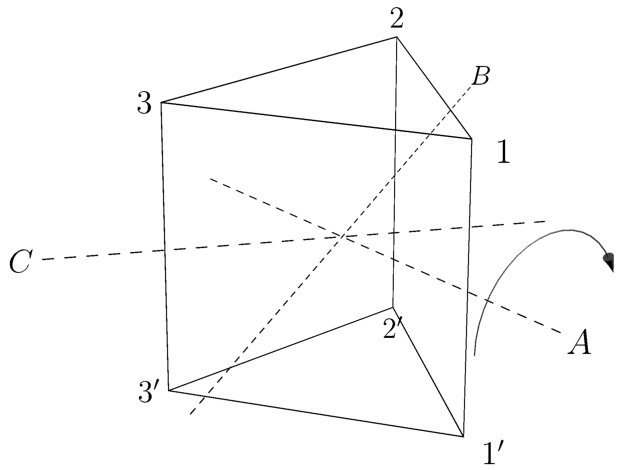

Figure 5: Triangular prism's corresponding axes of rotation.

(a) The number of elements, is known as the **order** or **cardinality**;

(b) If order < 10, name the set of elements; otherwise, explain how you know the order;

(c) A smallest possible **generating set**; in other words, a list of elements which generate a group;[2]

(d) Whether the group is **commutative**; in other words, whether its operation $\cdot$ satisfies $X \cdot Y = Y \cdot X$ for all X, Y.

If two problems have isomorphic groups, just write "isomorphic to Problem N" for the latter problem and move on.

5. Rectangle under rotation
6. Rectangle under reflection
7. Square under rotation
8. Square under reflection
9. Square prism under rotation
10. Square prism under reflection
11. Regular pentagon under rotation
12. Regular pentagon under reflection
13. Regular pentagonal prism under rotation
14. Regular pentagonal prism under reflection
15. Regular pentagonal pyramid under rotation
16. Regular pentagonal pyramid under reflection
17. Regular tetrahedron under rotation
18. Regular tetrahedron under reflection
19. Cube under rotation
20. Cube under reflection

[2]There may be multiple generating sets of the same size.