6 Geometry of Complex Numbers

Thanks to Tristan Needham's Visual Complex Analysis *for many of the problems/examples and to Josh Zucker for most of the text.*

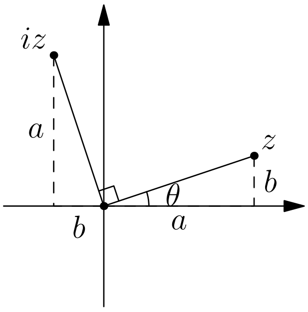

Figure 1: iz is perpendicular to z.

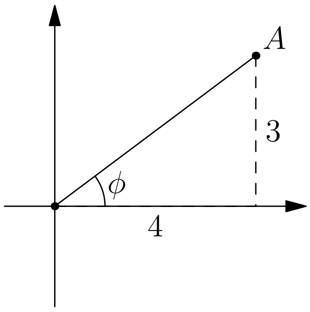

Figure 2: The complex number $A = 4 + 3i$.

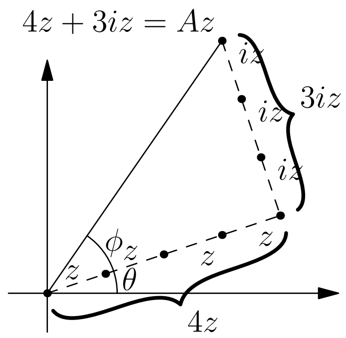

Figure 3: Breaking up Az into its components, we can observe the geometry of complex multiplication.

Last year, you became masters of the art of manipulating complex numbers. In this section, we will build on that background. Throughout the rest of the book, you can reinforce your skills with a dose of Vitamin i.

There are at least two ways to think about the equation $x^3 = 1$. One way is to factor the equation into $(x-1)(x^2+x+1)$ and find the solutions using the quadratic formula. The other way is to use **DeMoivre's theorem**:

$$(r_1(\cos\theta + i\sin\theta))(r_2(\cos\phi + i\sin\phi)) = (r_1 r_2)(\cos(\theta+\phi) + i\sin(\theta+\phi)).$$

Recall that $\operatorname{cis}\theta = \cos\theta + i\sin\theta$. $z = r\operatorname{cis}\theta$ is a complex number r units away from the origin and making an angle θ with the x axis, taken counterclockwise. Let's rewrite DeMoivre's theorem using cis:

$$(r_1 \operatorname{cis}\theta)(r_2 \operatorname{cis}\theta) = (r_1 r_2)(\operatorname{cis}(\theta+\phi)).$$

Notice that the magnitudes are multiplied and the angles are added.[7] By repeatedly applying DeMoivre's theorem, we know that $(r\operatorname{cis}\theta)^n = r^n \operatorname{cis} n\theta$. If $x = r\operatorname{cis}\theta$, then $x^3 = r^3 \operatorname{cis} 3\theta$. Going back to our original $x^3 = 1$, since $1 = \operatorname{cis}(2\pi k)$ for any integer k, we find that $r = 1$ and $3\theta = 2\pi k$. This yields three solutions: $1\operatorname{cis} 0, 1\operatorname{cis}\frac{2\pi}{3}, 1\operatorname{cis}\frac{4\pi}{3}$. Any other value of $\theta = \frac{2k\pi}{3}$ reduces to one of these values because of the periodicity of cis. These correspond to $k = 0, 1, 2$; other values of k produce coterminal angles and are therefore duplicates. You can confirm that these three solutions are the same solutions that you obtain from factoring.

You can prove DeMoivre's theorem using the angle addition formulae for $\cos$ and $\sin$. But you can also understand it through pure geometry. Consider a complex number $z = a+bi$, being multiplied by $A = 4+3i$. z forms an angle of θ with the real axis, and A forms an angle of ϕ. In Figure 1, observe that iz is perpendicular to z for any z. Figure 2 depicts the complex number A. Finally, in Figure 3, you see the multiplication carried out: $Az = (4+3i)z = 4z + 3iz$. These two components, $4z$ and $3iz$, are indicated.

Combining the observation in Figure 1 and knowledge from geometry, we know the triangles in Figure 2 and 3 are similar. Since the scaling is by a factor of $|z|$, multiplying A by z has the effect rotating z by the angle of A, and multiplying it by the length of A. This method of proving DeMoivre's theorem for $A = 4+3i$ will work for all complex numbers $A = a + bi$.

Some notation: the angle θ of a complex number $z = a+bi$ is often called the argument, written as $\operatorname{Arg} z$. The real part of z is written $\operatorname{Re}(z) = a$, and the imaginary part of z is written $\operatorname{Im}(z) = b$. Note that $\operatorname{Im}(z)$ is a *real* number b, *not* an imaginary number bi. Finally, the **complex conjugate** of z, where the imaginary part is negated, is written with a bar on top: $\overline{z} = a - bi$.

1. Explain why iz is perpendicular to z, without using DeMoivre's theorem.

2. How does $\operatorname{Arg}\overline{z}$ relate to $\operatorname{Arg} z$? (Hint: symmetry!)

3. Compute $z\overline{z}$ and relate it to the cis form of z.

4. Explain, using a picture, why $\tan(\operatorname{Arg} z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$.

[7] What else is added when you multiply? Exponents! In fact, $\operatorname{cis}\theta = e^{i\theta}$, but that's another story.

5. Divide $\frac{a+bi}{c+di}$ by rationalizing the denominator.

6. Divide $\frac{r_1 \operatorname{cis}\theta}{r_2 \operatorname{cis}\phi}$ using DeMoivre's theorem.

7. Compare and contrast the methods of division in Problems 5 and 6. Which is more convenient? Or does it depend on the circumstance?

8. (a) If $z = r \operatorname{cis}\theta$, what is $\frac{1}{z}$?

   (b) Explain how this shows $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$, without having to rationalize the denominator. (Hint: use Problems 3, 4, and 7.)

9. Compute $(1+i)^{13}$; pencil, paper, and brains only. No calculators!

10. Compute $\frac{(1+i\sqrt{3})^3}{(1-i)^2}$ without a calculator.

11. Draw $\operatorname{cis}\left(\frac{\pi}{4}\right) + \operatorname{cis}\left(\frac{\pi}{2}\right)$. Use your picture to prove an expression for $\tan\left(\frac{3\pi}{8}\right)$. (Hint: add them as vectors.)

12. Solve $z^3 = 1$, and show that its solutions under the operation of multiplication form a group, isomorphic to the rotation group of the equilateral triangle. Write a group table!

13. (a) Find multiplication groups of complex numbers which are isomorphic to the rotation groups for

    i. a non-square rectangle, and

    ii. a regular hexagon.

   (b) Make a table for each group.

   (c) Compare the regular hexagon's group to the dihedral group of the equilateral triangle, D_3. Consider: how are they the same? How are they different? Is the difference fundamental?

14. Which of the following sets is a group under (i) addition and (ii) multiplication?

   (a) $\{0\}$
   (b) $\{1\}$
   (c) $\{0, 1\}$
   (d) $\{-1, 1\}$
   (e) $\{1, -1, i, -i\}$
   (f) {naturals}
   (g) {integers}
   (h) {rationals}, $\mathbb{Q}$
   (i) {$\mathbb{Q}$ without zero}
   (j) {complex numbers}, $\mathbb{C}$
   (k) {$\mathbb{C}$ without zero}

DeMoivre's theorem is the "universal" trig identity, in the sense that it can be used to calculate every other trig identity. For example, suppose you want an identity for $\cos 3\theta$. For convenience, let $c = \cos\theta$ and $s = \sin\theta$. Then we have:

$$\begin{aligned}
\operatorname{cis} 3\theta &= (\operatorname{cis}\theta)^3 && \text{[DeMoivre's Theorem]} \\
&= (c + is)^3 && \text{[Definition of cis]} \\
&= c^3 + 3c^2si - 3cs^2 - s^3i && \text{[Binomial expansion]} \\
\cos 3\theta + i \sin 3\theta &= (c^3 - 3cs^2) + i(3c^2s - s^3). && \text{[Combining like terms]}
\end{aligned}$$

Equating real parts on both sides, $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

15. Prove that $(r_1 \operatorname{cis}\theta)(r_2 \operatorname{cis}\phi) = r_1 r_2 \operatorname{cis}(\theta + \phi)$ using brute force and the angle-sum trig identities for $\cos$ and $\sin$. Do you prefer this method or the one on the previous page? Which method gives you a better understanding of why DeMoivre's works?

16. Find an identity for $\sin 3\theta$ as we have done for $\cos$. Most of the work is already done for you!

17. Your friend's textbook says $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, different from our identity. Who's right?

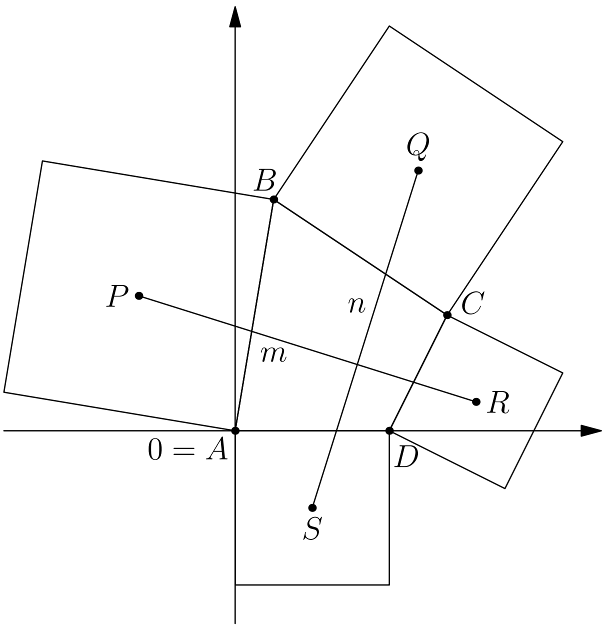

Figure 5: $2(a + b + c + d) = 0$.

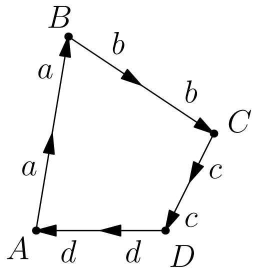

Figure 4: The quadrilateral with four squares.

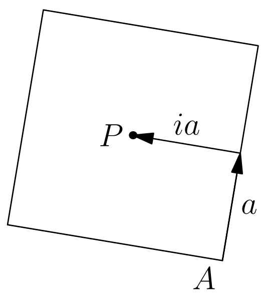

Figure 6: $P = a + ia$.

Let's apply complex numbers to a geometry problem. We want to prove that if we construct squares with centers P, Q, R, S on the sides of any quadrilateral $ABCD$, as shown in Figure 4, then (i) $\overline{PR} \perp \overline{QS}$ and (ii) $\overline{PR} \cong \overline{QS}$. In other words, segments joining centers of opposite squares are perpendicular and the same length.

We represent all points in the figure as complex numbers. For convenience, let $A = 0$ be the origin. The edges of the quadrilateral can be thought of as vectors in the form of complex numbers, and are found using subtraction; for example, the edge from A to B is $B - A$. Similarly, the edge from B to C is $C - B$. Now, define complex numbers

$$a = \frac{B-A}{2},\ b = \frac{C-B}{2},\ c = \frac{D-C}{2},\ d = \frac{A-D}{2}.$$

a is the vector halfway along $\overrightarrow{AB}$, b is halfway along $\overrightarrow{BC}$, etc.; this is shown in Figure 5. We also have

$$a + b + c + d = \frac{B - A + C - B + D - C + A - D}{2} = \frac{0}{2} = 0.$$

More geometrically, this is because $2(a + b + c + d) = 2a + 2b + 2c + 2d$ is the sum of the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$, which is just $\overrightarrow{AA} = \overrightarrow{\mathbf{0}}$. This is shown in Figure 5.

P, Q, R, S are also complex numbers. Let $m = R - P$ and $n = Q - S$, be our two segments $\overline{PR}$ and $\overline{QS}$. To prove that they are perpendicular, recall that z is perpendicular to iz for any complex $z \neq 0$, so we just need to prove that $n = \pm im$.

We now need to relate P, Q, R, S back to a, b, c, d. It is easy to see that $P = a + ia$, remembering that a is the vector halfway along $\overrightarrow{AB}$. a takes you from the origin A to the midpoint of $\overline{AB}$, then ia takes you to P. This shown in Figure 6. We can extend this logic to the other points, of course.

18. Now you can finish the rest of the proof.

    (a) Draw a, b, c, d, m, n approximately for the quadrilateral on the previous page.

    (b) Why does showing $n = \pm im$ prove the segments are (i) perpendicular and (ii) the same length?

    (c) Explain why $Q = 2a + b + ib$.

    (d) Find formulae for R and S in terms of c and d.

    (e) Find m and n in terms of a, b, c, and d.

    (f) Check that $n - im = 0$, using the fact that $a + b + c + d = 0$.

19. In the previous problem, we drew squares outside a quadrilateral and connected their centers. Conjecture what happens if we draw equilateral triangles outside a triangle and connect their centers. Prove your conjecture using complex numbers.

20. The hard way to find an identity for $\tan 3\theta$ is to divide the identity for $\sin$ and $\cos$ that we already found. Try this. Make sure your answer is in terms of $\tan$ only!

21. The easier way to get an identity for $\tan 3\theta$ starts with setting $z = 1 + i\tan\theta$.
    (a) Why is $\operatorname{Arg} z = \theta$?
    (b) Why is $\tan 3\theta = \frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)}$?
    (c) Use (b) to find an identity for $\tan 3\theta$.

22. Find multiplication groups of complex numbers isomorphic to rotation groups for
    (a) the regular octagon, and
    (b) the regular pentagon.

23. Make tables for
    (a) the rotation group of the regular octagon, and
    (b) the dihedral group of the square.

    Is the difference between them fundamental?

24. Which of the following tables defines a group? Why or why not?

(a)

$	I	A	B	C	D
I	I	A	B	C	D
A	A	C	D	B	I
B	B	I	C	D	A
C	C	D	A	I	B
D	D	B	I	A	C

(b)

#	I	A	B	C	D
I	I	A	B	C	D
A	A	B	C	D	I
B	B	C	D	I	A
C	C	D	I	A	B
D	D	I	A	B	C

25. Name some subsets of the complex numbers that are groups under multiplication. I can name an infinite number of both finite and infinite groups with this property, so after you list a few of each type, try to generalize.

26. Prove with a diagram that if $|z| = 1$, then $\operatorname{Im}\left(\frac{z}{(z+1)^2}\right) = 0$.

27. Prove geometrically that if $|z| = 1$, then $|1 - z| = \left|2\sin\left(\frac{\operatorname{Arg} z}{2}\right)\right|$.

28. (a) Prove that if $(z-1)^{10} = z^{10}$, then $\operatorname{Re}(z) = \frac{1}{2}$. (Hint: if two numbers are equal, they have the same magnitude.)
    (b) How many solutions does this equation have?

29. I claim that $e^{i\theta} = \cos\theta + i\sin\theta = \operatorname{cis}\theta$, for θ in radians.
    (a) Find e^{-it}.
    (b) Find $\frac{e^{i\theta}+e^{-i\theta}}{2}$.
    (c) Find $\frac{e^{i\theta}-e^{-i\theta}}{2i}$.

30. Use your new, complex definitions for $\cos$ and $\sin$ to find:
    (a) $\cos^2\theta + \sin^2\theta$
    (b) $\tan\theta$
    (c) $\cos 2\theta$
    (d) $\sin 2\theta$
    (e) What kind of group is generated by $\{e^{i\theta}, e^{-i\theta}\}$ under the operation of multiplication if θ is an integer? A rational multiple of π?

31. You've used the quadratic equation throughout high school, but there's also a cubic equation that finds the roots of any cubic. Let's derive it, starting with the cubic $x^3 + bx^2 + cx + d = 0$.

(a) Make the substitution $x = y - \frac{b}{3}$. Combine like terms to create an equation of the form $y^3 - 3py - 2q = 0$, with p, q in terms of b, c, and d.

(b) Rearrange this equation as $y^3 = 3py + 2q$.

(c) Make the substitution $y = s + t$ into (b), and prove that y is a solution of the cubic in part (a) if $st = p$ and $s^3 + t^3 = 2q$.

(d) Eliminate t between these two equations to get a quadratic in s^3.

(e) Solve this quadratic to find s^3. By symmetry, what is t^3?

(f) Find a formula for y in terms of p and q. What about a formula for x?

(g) What if we started with $ax^3 + bx^2 + cx + d = 0$, with a coefficient in front of the x^3 term as well? Can you come up with a formula for x?

32. Starting with the same cubic as in Problem 31b.

(a) Let $c = \cos\theta$. Remember that $\cos 3\theta = 4c^3 - 3c$, as we proved. Substitute $y = 2c\sqrt{p}$ into $y^3 = 3py + 2q$ to obtain $4c^3 - 3c = \frac{q}{p^{3/2}}$.

(b) Provided that $q^2 \leq p^3$, show that $y = 2\sqrt{p}\cos\left(\frac{1}{3}(\theta + 2\pi n)\right)$, where n is an integer. Why does this yield all three solutions?

(c) Explain how you would find θ from p and q, and how we would use what we have found to solve an arbitrary cubic $ax^3 + bx^2 + cx + d = 0$.