## 6 Geometry of Complex Numbers

**1. Explain why $iz$ is perpendicular to $z$, without using DeMoivre's theorem.**

Let $z = a + bi$. Then $iz = i(a + bi) = -b + ai$, which is the transformation $(a, b) \to (-b, a)$. Drawing this out on the 2D plane makes clear that the angle between the two points and the origin is $90°$, simply by subtracting angles: $(90° + \theta) - \theta = 90°$. This is shown in Figure 1.

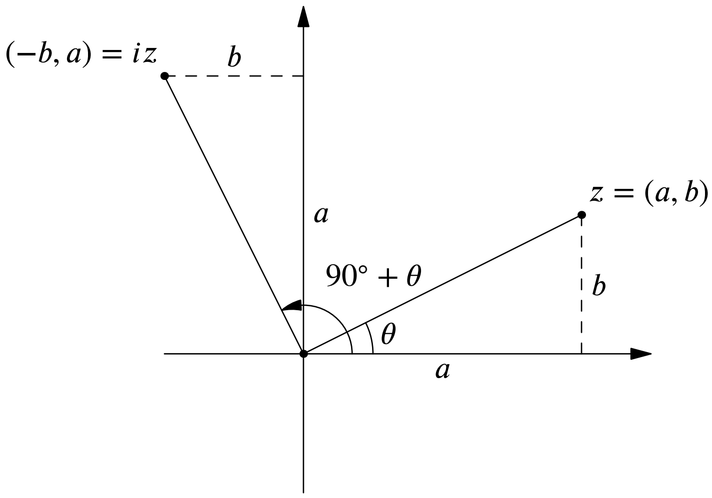


Figure 1: $iz$ is perpendicular to $z$ as long as $z \neq 0$.

You can also rationalize it by the fact that the lines through $z/iz$ and the origin have slopes of $\frac{b}{a}$ and $-\frac{a}{b}$, respectively, so they must be perpendicular. Also, $< a, b > \bullet < -b, a >= 0$.

**2. How does $\operatorname{Arg} \overline{z}$ relate to $\operatorname{Arg} z$? (Hint: symmetry!)**

Again, let $z = a + bi$. $\overline{z} = a - bi$ is flipped over the $x$-axis, since the imaginary part is negated. Thus, $\operatorname{Arg} \overline{z} = -\operatorname{Arg} z$ due to congruent triangles formed by $z$ and $\overline{z}$.[11] The geometric interpretation is shown in Figure 2.

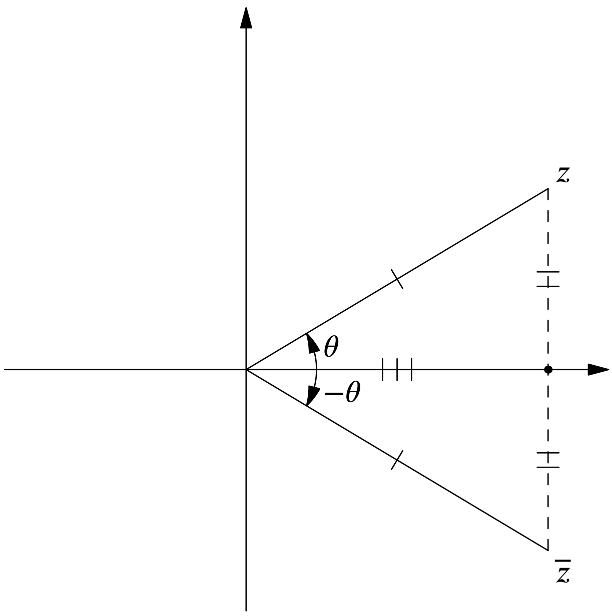


Figure 2: $z$ and $\overline{z}$ form congruent triangles, showing that $\operatorname{Arg} z = -\operatorname{Arg} \overline{z}$.

**3. Compute $z\overline{z}$ and relate it to the $\operatorname{cis}$ form of $z$.**

[11] If we are to be pedantic, we'd either use the multivalued function $\arg$ or say this is modulo $2\pi$.

Once more, let $z = a + bi$. Then

$$z\overline{z} = (a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2.$$

If $z = r \operatorname{cis} \theta$, then $z\overline{z} = r^2$. In other words, it is the square of the distance from $z$ to the origin.

**4. Explain, using a picture, why $\tan(\operatorname{Arg} z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$.**

This is basically just an application of soh-cah-toa to a triangle in the complex plane. The details are shown in Figure 3.

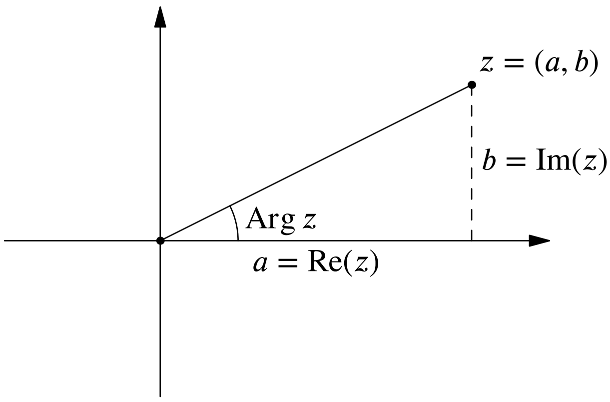


Figure 3: $\tan(\operatorname{Arg} z) = \frac{b}{a} = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$.

**5. Divide $\frac{a+bi}{c+di}$ by rationalizing the denominator.**

$$\begin{aligned} \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} &= \frac{(a+bi)(c-di)}{c^2+d^2} \\ &= \frac{ac+bd+(bc-ad)i}{c^2+d^2}. \end{aligned}$$

**6. Divide $\frac{r_1 \operatorname{cis}\theta}{r_2 \operatorname{cis}\phi}$ using DeMoivre's theorem.**

We don't have a rule yet for applying DeMoivre's theorem for division, but we can quickly derive it. We have

$$\begin{aligned} \frac{r_1 \operatorname{cis}\theta}{r_2 \operatorname{cis}\phi} \cdot \frac{\overline{\operatorname{cis}\phi}}{\overline{\operatorname{cis}\phi}} &= \frac{r_1 \operatorname{cis}\theta\, \overline{\operatorname{cis}\phi}}{r_2 \underbrace{\operatorname{cis}\phi\, \overline{\operatorname{cis}\phi}}_{=1}} && \text{Multiplying by conjugate} \\ &= \frac{r_1 \operatorname{cis}\theta \operatorname{cis}(-\phi)}{r_2} && \text{Using } \operatorname{Arg} z = -\operatorname{Arg}\overline{z} \\ &= \frac{r_1}{r_2}\operatorname{cis}(\theta - \phi). && \text{Use DeMoivre's theorem} \end{aligned}$$

**7. Compare and contrast the methods of division in Problems 5 and 6. Which is more convenient? Or does it depend on the circumstance?**

Opinions may vary, but 6 is definitely faster to do if the dividend and divisor are already in cis form. 5 is likely convenient than converting from rectangular to cis, then back to rectangular.

**8.**

**(a) If $z = r \operatorname{cis}\theta$, what is $\frac{1}{z}$?**

As we hinted at in the previous problem, $\frac{1}{z} = \frac{1}{r}\operatorname{cis}(-\theta)$:

$$\begin{aligned}\frac{1}{r\operatorname{cis}\theta}\cdot\frac{\overline{\operatorname{cis}\theta}}{\overline{\operatorname{cis}\theta}} &= \frac{\overline{\operatorname{cis}\theta}}{r\operatorname{cis}\theta\,\overline{\operatorname{cis}\theta}} \\ &= \frac{\operatorname{cis}(-\theta)}{r|\operatorname{cis}\theta|^2} \\ &= \frac{1}{r}\operatorname{cis}(-\theta).\end{aligned}$$

**(b) Explain how this shows $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$, without having to rationalize the denominator. (Hint: use problems 3, 4, and 7.)**

Let $a+bi = r\operatorname{cis}\theta$. We have

$$\begin{aligned}\frac{1}{r\operatorname{cis}\theta} &= \frac{1}{r}\operatorname{cis}(-\theta) \\ &= \frac{r\operatorname{cis}(-\theta)}{r^2} \\ &= \frac{a-bi}{a^2+b^2}.\end{aligned}$$

**9. Compute $(1+i)^{13}$; pencil, paper, and brains only. No calculators!**

We have $1+i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$, since it forms a $45°$ angle with the $x$-axis. Applying DeMoivre's theorem,

$$\begin{aligned}(1+i)^{13} &= \left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)^{13} \\ &= (\sqrt{2})^{13}\operatorname{cis}\frac{13\pi}{4} \\ &= 64\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \\ &= 64\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ &= 64(-1-i) \\ &= -64-64i.\end{aligned}$$

**10. Compute $\frac{(1+i\sqrt{3})^3}{(1-i)^2}$ without a calculator.**

We convert to $\operatorname{cis}$ form and apply DeMoivre's theorem.

$$\begin{aligned}\frac{(1+i\sqrt{3})^3}{(1-i)^2} &= \frac{\left(2\operatorname{cis}(\frac{\pi}{3})\right)^3}{\left(\sqrt{2}\operatorname{cis}(-\frac{\pi}{4})\right)^2} \\ &= \frac{8\operatorname{cis}(\pi)}{2\operatorname{cis}(-\frac{\pi}{2})} \\ &= \frac{8\cdot -1}{2\cdot -i} \\ &= \frac{4}{i}\cdot\frac{-i}{-i} \\ &= -4i.\end{aligned}$$

**11. Draw $\operatorname{cis}\left(\frac{\pi}{4}\right) + \operatorname{cis}\left(\frac{\pi}{2}\right)$. Use your picture to prove an expression for $\tan\left(\frac{3\pi}{8}\right)$. (Hint: add them as vectors.)**

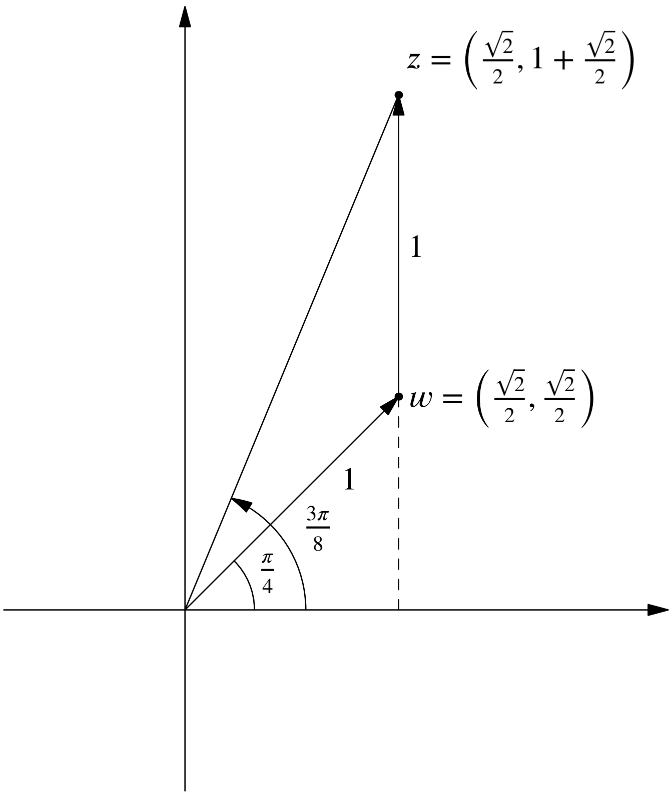


Figure 4: Addition of $\operatorname{cis}\left(\frac{\pi}{4}\right) + \operatorname{cis}\left(\frac{\pi}{2}\right)$ as vectors.

The drawing is shown in Figure 4. The first vector, starting at the origin, is $\operatorname{cis}\frac{\pi}{4}$. The second vector, starting at the endpoint of the first vector, is $\operatorname{cis}\frac{\pi}{2}$. The origin, along with points $w$ and $z$, form an isosceles triangle. Furthermore, the apex of this triangle, at $w$, has a measure of $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$ radians. Thus, the base angles of the isosceles triangle are

$$\frac{\pi - \frac{3\pi}{4}}{2} = \frac{\pi}{8}.$$

Adding this with $\frac{\pi}{4}$ shows that the angle $z$ forms with the $x$-axis is

$$\frac{\pi}{8} + \frac{\pi}{4} = \frac{3\pi}{8},$$

our desired angle to analyze. We wish to find the tangent of this angle, which is just $\tan(\operatorname{Arg} z)$. But we know how to compute that!

$$\begin{aligned}
\tan(\operatorname{Arg} z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} &= \frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{2}+1}{1} \\
\tan\left(\frac{3\pi}{8}\right) &= \sqrt{2}+1.
\end{aligned}$$

**12. Solve $z^3 = 1$, and show that its solutions under the operation of multiplication form a group, isomorphic to the rotation group of the equilateral triangle. Write a group table!**

There's numerous ways to solve this, but let's use $\operatorname{cis}$ form as usual. Let $z = r\operatorname{cis}\theta$. Then

$$
\begin{aligned}
z^3 = r^3 \operatorname{cis} 3\theta &= 1 \\
\implies r &= 1 \\
\operatorname{cis} 3\theta &= 1 \\
\cos 3\theta &= 1 \\
3\theta &= 2\pi k && \text{For } k \in \mathbb{Z} \\
\theta &= \frac{2\pi k}{3} \implies \theta \in \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}
\end{aligned}
$$

$$\implies z \in \left\{\operatorname{cis} 0, \operatorname{cis} \frac{2\pi}{3}, \operatorname{cis} \frac{4\pi}{3}\right\}.$$

Under multiplication, these three values of $z$ indeed form a group isomorphic to the rotation group of the equilateral triangle, $C_3$. In particular, $\operatorname{cis} 0$ is the identity, $\operatorname{cis} \frac{2\pi}{3}$ is a rotation by $120°$ counterclockwise, and $\operatorname{cis} \frac{4\pi}{3}$ is a rotation by $240°$ counterclockwise. Let $I = \operatorname{cis} 0$, $r = \operatorname{cis} \frac{2\pi}{3}$, and $r^2 = \operatorname{cis} \frac{4\pi}{3}$. Then, we have the following group table:

| $\cdot$ | $I$ | $r$ | $r^2$ |
|---|---|---|---|
| $I$ | $I$ | $r$ | $r^2$ |
| $r$ | $r$ | $r^2$ | $I$ |
| $r^2$ | $r^2$ | $I$ | $r$ |

**13.**

**(a) Find multiplication groups of complex numbers which are isomorphic to the rotation groups for**

**i. a non-square rectangle**

Since this rotation group is just the identity and a rotation of $180°$, we can just choose the group $\{-1, 1\}$ under multiplication. $1$ is the identity, and $-1 = \operatorname{cis} 180°$ is the rotation.

**ii. a regular hexagon.**

We following in the footsteps of the equivalent problem for the equilateral triangle. We have elements

$$\left\{\operatorname{cis} 0, \operatorname{cis} 60°, \operatorname{cis} 120°, \operatorname{cis} 180°, \operatorname{cis} 240°, \circ 300°\right\}.$$

As should be obvious, these are rotations of $0$, $60°$, $120°$, $180°$, $240°$, and $300°$ respectively. Under multiplication, this is isomorphic to the rotation group of the hexagon, $C_6$.

**(b) Make a table for each group.**

**i. a non-square rectangle**

Let $I$ be the identity and $r$ be the rotation of $180°$.

| $\cdot$ | $I$ | $r$ |
|---|---|---|
| $I$ | $I$ | $r$ |
| $r$ | $r$ | $I$ |

**ii. a regular hexagon.**

Let $I$ be the identity and $r$ be the rotation of $60°$. $r^n$ is defined in the natural way, by raising $\operatorname{cis} 60°$ to the power $n$.

| $\cdot$ | $I$ | $r$ | $r^2$ | $r^3$ | $r^4$ | $r^5$ |
|---|---|---|---|---|---|---|
| $I$ | $I$ | $r$ | $r^2$ | $r^3$ | $r^4$ | $r^5$ |
| $r$ | $r$ | $r^2$ | $r^3$ | $r^4$ | $r^5$ | $I$ |
| $r^2$ | $r^2$ | $r^3$ | $r^4$ | $r^5$ | $I$ | $r$ |
| $r^3$ | $r^3$ | $r^4$ | $r^5$ | $I$ | $r$ | $r^2$ |
| $r^4$ | $r^4$ | $r^5$ | $I$ | $r$ | $r^2$ | $r^3$ |
| $r^5$ | $r^5$ | $I$ | $r$ | $r^2$ | $r^3$ | $r^4$ |

**(c) Compare the regular hexagon's group to the dihedral group of the equilateral triangle, $D_3$. Consider: how are they the same? How are they different? Is the difference fundamental?**

The two groups are not isomorphic, although they are the same size; the difference is fundamental. The hexagon's rotation group, $C_6$, has elements of periods $\{1, 2, 3, 3, 6, 6\}$, while $D_3$ has elements of periods $\{1, 2, 2, 2, 3, 3\}$. They do share some subgroups however: the trivial subgroup of just the identity, and the subgroups generated by $r^2$ and by $r^3$ in $C_6$, which are $C_3$ and $C_2$ respectively.

**14. Which of the following sets is a group under (i) addition and (ii) multiplication?**

**(a)** $\{0\}$

This is a group under (i) addition, since it has an identity $0$, is closed, has $0$ as $0$'s inverse, and $0+(0+0) = (0+0)+0$. It also is a group under (ii) multiplication, for the same reasons.

**(b)** $\{1\}$

This is not a group under (i) addition, since $1 + 1 = 2 \notin \{1\}$. It is a group under multiplication, though, since $1 \cdot 1 = 1$ and all other properties are satisfied.

**(c)** $\{0, 1\}$

This is not a group under (i) addition, since $1+1 = 2 \notin \{0, 1\}$. It is also not a group under (ii) multiplication. $0$ can't be the identity, since $1 \cdot 0 = 0 \neq 1$. $1$ also can't be the identity, since then $0$ has no inverse $K$ such that $0 \cdot K = 1$.

**(d)** $\{-1, 1\}$

This is not a group under (i) addition, since $1 + (-1) = 0 \notin \{\pm 1\}$. It is a group under (ii) multiplication, since it satisfies the group properties:

1. Identity: $1$ is the identity
2. Associativity: Multiplication is associative
3. Invertibility: Each element is its own inverse
4. Closure: $(\pm 1)(\pm 1) \in \{\pm 1\}$

**(e)** $\{1, -1, i, -i\}$

This is not a group under (i) addition, since the sum of any two of the elements takes you out of the set. It is a group under (ii) multiplication, however. One easy way to see this is that $1 = \operatorname{cis} 0$, $-1 = \operatorname{cis} \pi$, $i = \operatorname{cis} \frac{\pi}{2}$, and $-i = \frac{3\pi}{2}$, which are all rotations of multiples of $90°$. In particular, it is isomorphic to $C_4$, the rotation group of the square.

**(f)** $\{$**naturals**$\}$

This is not a group under (i) addition, because it cannot satisfy invertibility. There is no element $X \in \mathbb{N}$ such that $1 + X = I = 0$, for example. This is also not a group under (ii) multiplication for the same reason.

**(g)** $\{$**integers**$\}$

This is a group under (i) addition, because all the group properties are satisfied. The inverse of an element $n$ is just $-n$, addition is associative, the identity is $0$, and the sum of two integers is another integer. It is a not a group under (ii) multiplication, because no numbers except $\pm 1$ have integer multiplicative inverses.

**(h)** $\{$**rationals**$\}, \mathbb{Q}$

This is a group under (i) addition with identity element $0$. The inverse of an element $\frac{p}{q}$ is $-\frac{p}{q}$, addition is associative, and the sum of two rational numbers is another rational number. It is not a group under (ii) multiplication, because no number is the multiplicative inverse of $0$.

**(i)** $\{\mathbb{Q}$ **without zero**$\}$

This is no longer a group under (i) addition, since the identity element needs to be $0$. It is now, however, group under (ii) multiplication, because all numbers have their inverses. Multiplication is associative, the inverse of $\frac{p}{q}$ is $\frac{q}{p}$, and the product of two rationals is another rational.

**(j)** $\{$**complex numbers**$\}, \mathbb{C}$

This is a group under (i) addition with identity element $0$. The inverse of an element $z$ is $-z$, addition is associative, and the sum of two complex numbers is another complex number. This is not a group under (ii) multiplication, because again, no number is the multiplicative inverse of $0$.

**(k)** $\{\mathbb{C}$ **without zero**$\}$

This is no longer a group under (i) addition, since the identity element needs to be $0$. It is now, however, a group under (ii) multiplication, because all numbers have their inverses. Multiplication is associative, the inverse of $z$ is $\frac{1}{z}$, and the product of two complex numbers is another complex number.

**15. Prove that $(r_1 \operatorname{cis}\theta)(r_2 \operatorname{cis}\phi) = r_1 r_2 \operatorname{cis}(\theta+\phi)$ using brute force and the angle-sum trig identities for $\cos$ and $\sin$. Do you prefer this method or the one on the previous page? Which method gives you a better understanding of why DeMoivre's works?**

$$\begin{aligned}
(r_1 \operatorname{cis}\theta)(r_2 \operatorname{cis}\phi) &= r_1 r_2(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\
&= r_1 r_2(\cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi) \\
&= r_1 r_2((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)) \\
&= r_1 r_2(\cos(\theta+\phi) + i\sin(\theta+\phi)) \\
&= r_1 r_2 \operatorname{cis}(\theta+\phi)
\end{aligned}$$

(Opinions may vary.) I actually prefer this because it's kind of satisfying, but the previous way likely gives a better understanding of the underlying mechanics.

**16. Find an identity for $\sin 3\theta$ as we have done for $\cos$. Most of the work is already done for you!**

We already know that

$$\cos 3\theta + i\sin 3\theta = \operatorname{cis} 3\theta = (c^3 - 3c^2) + i(3c^2 s - s^3),$$

where $c = \cos\theta$ and $s = \sin\theta$. Equating imaginary parts, we have

$$\begin{aligned}
\sin 3\theta &= 3c^2 s - s^3 \\
&= 3\cos^2\theta\sin\theta - \sin^3\theta.
\end{aligned}$$

**17. Your friend's textbook says $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, different from our identity. Who's right?**

Both are right. Our identity is

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

Remembering that $\sin^2\theta = 1 - \cos^2\theta$, we can pretty easily change the form:

$$\begin{aligned}
\cos^3\theta - 3\cos\theta\sin^2\theta &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\
&= \cos^3\theta + 3\cos^3\theta - 3\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta.
\end{aligned}$$

**18. Now you can finish the rest of the proof.**

If you need context for this answer, check out the relevant textbook section.

**(a) Draw $a, b, c, d, m, n$ approximately for the quadrilateral on the previous page.**

The quadrilateral is shown in Figure 5.

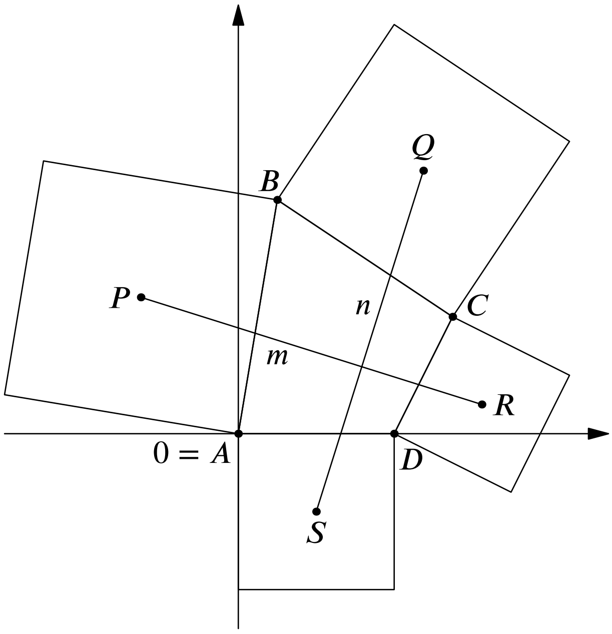


Figure 5: The quadrilateral to analyze.

The relative magnitudes and directions are shown in Figure 6 below. We find $a, b, c, d$ from halving the sides of the quadrilateral. $m$ and $n$ are just the vectors from $P$ to $R$ and $Q$ to $S$, respectively.

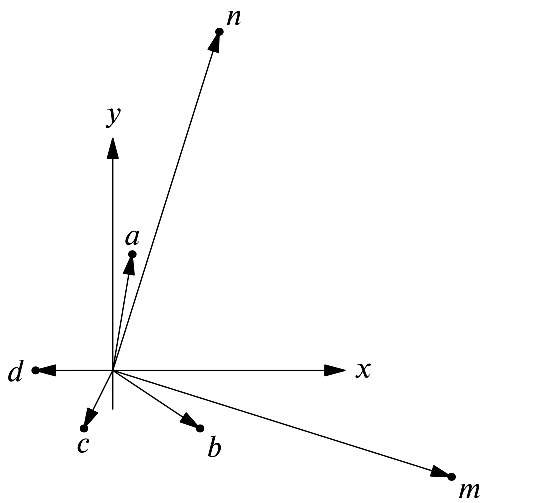


Figure 6: The relative magnitudes and directions of $a, b, c, d, m, n$.

**(b) Why does showing $n = \pm im$ prove the segments are (i) perpendicular and (ii) the same length?**

They are (i) perpendicular because $iz$ is perpendicular to $z$ for all $z \neq 0$, and (ii) are the same length because $|n| = |\pm im| = |im| = |m|$.

**(c) Explain why $Q = 2a + b + ib$.**

The justification is geometric. We know that $B = 2a$, and we can get to the midpoint of $\overline{BC}$ by adding $b$. Then, we go up to the center of the square on $\overline{BC}$ by adding $ib$. This process is shown in Figure 7.

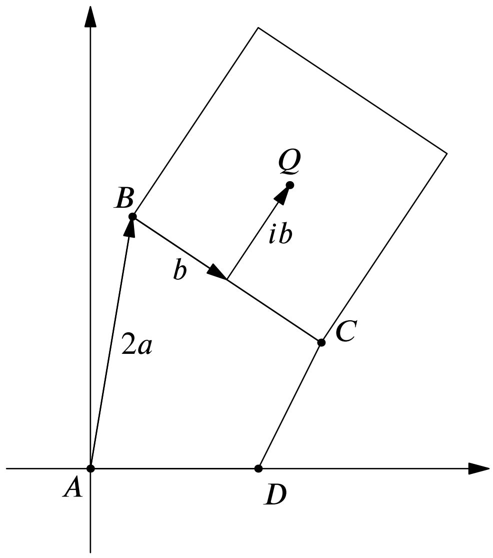


Figure 7: $Q = 2a + b + ib$.

**(d) Find formulae for $R$ and $S$ in terms of $c$ and $d$.**

In a similar fashion, we have $S = -d + id$ (note that it is $-d$ because we are going counterclockwise now) and $R = -2d - c + ic$. The interpretations of these are shown in Figure 8 below.

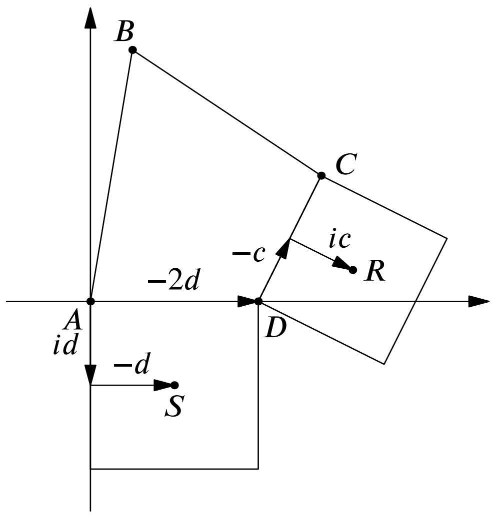


Figure 8: $S = -d + id$ and $R = -2d - c + ic$.

**(e) Find $m$ and $n$ in terms of $a$, $b$, $c$, and $d$.**

We have $m = R - P = (-2d - c + ic) - (a + ia)$ and $n = Q - S = (2a + b + ib) - (-d + id)$.

**(f) Check that $n - im = 0$, using the fact that $a + b + c + d = 0$.**

We evaluate straightforwardly:

$$\begin{aligned}(2a + b + ib + d - id) - i(-2d - c + ic - a - ia) &= 2a + b + ib + d - id + 2id + ic + c + ia - a \\ &= a + b + c + d + ia + ib + ic + id \\ &= (a + b + c + d)(1 + i) \\ &= 0.\end{aligned}$$

**19. In the previous problem, we drew squares outside a quadrilateral and connected their centers. Conjecture what happens if we draw equilateral triangles outside a triangle and connect their centers. Prove your conjecture using complex numbers.**

We conjecture that following this construction leads to connecting together another equilateral triangle. An example, with the variables we'll use labeled, is shown in Figure 9.

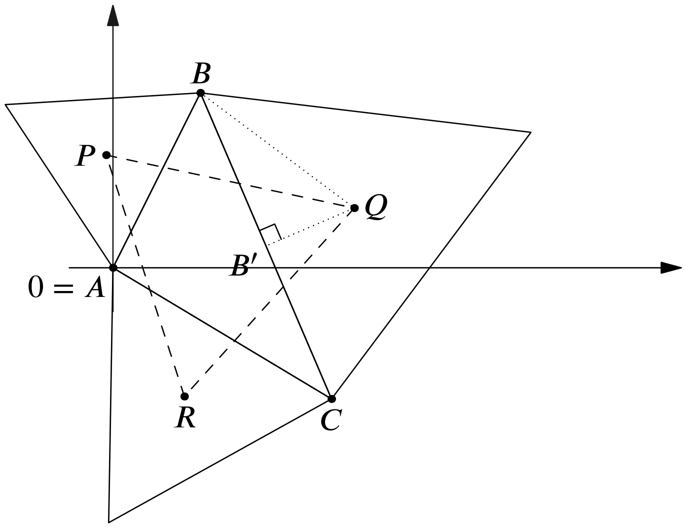


Figure 9: Equilateral triangles around a central, arbitrary triangle $\triangle ABC$ with $A$ at the origin.

Similar to the last problem, let $A$, $B$, and $C$ be numbers in the complex plane. Without loss of generality, let $A = 0$ be the origin. Also, define $a = \frac{B-A}{2}$, $b = \frac{C-B}{2}$, and $c = \frac{A-C}{2}$ to be the vectors going halfway along each of $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$. Finally, let $P$, $Q$, and $R$ be the centers of the triangles on sides $AB$, $BC$, and $CA$ respectively.

Consider $Q$ in the figure. It is on the $60°$ vertex of a 30-60-90 triangle $\triangle BB'Q$, outlined in dotted line. We know that $\overrightarrow{BB'} = b$. Thus, since $BB' \; : \; B'Q = \sqrt{3} \; : \; 1$,

$$B'Q = \frac{|b|}{\sqrt{3}}.$$

Furthermore, since $\overline{B'Q} \perp \overline{BB'}$, we know that it is $s \cdot ib$ for some real $s$. Combining these facts,

$$B'Q = \frac{ib}{|ib|}\frac{|b|}{\sqrt{3}} = \frac{ib}{\sqrt{3}}.$$

Since $Q = \overrightarrow{AB} + \overrightarrow{BB'} + \overrightarrow{B'Q}$, we have

$$Q = 2a + b + \frac{ib}{\sqrt{3}}.$$

With similar logic, we know that

$$P = a + \frac{ia}{\sqrt{3}}$$
$$R = -c + \frac{ic}{\sqrt{3}}$$

Like with the quadrilateral, we know $a + b + c = 0$, since $2(a + b + c) = 0$. To prove the dashed triangle is indeed equilateral, we can just show that $P - R = (Q - P)\operatorname{cis} 120°$. After all, if the vectors $\overrightarrow{RP}$ and $\overrightarrow{PQ}$ have an angle of $120°$ between them and they have the same magnitude, $\triangle PQR$ is equilateral by SAS Congruence as shown in Figure 10. Substituting in our found values for $P, Q, R$ in terms of $a, b, c$, we get

$$
\begin{aligned}
P - R &= (Q - P)\operatorname{cis} 120^\circ \\
a + \frac{ia}{\sqrt{3}} - \left(-c + \frac{ic}{\sqrt{3}}\right) &= \left(2a + b + \frac{ib}{\sqrt{3}} - \left(a + \frac{ia}{\sqrt{3}}\right)\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
(a+c) + \frac{ia - ic}{\sqrt{3}} &= \left(a + b + \frac{ib - ia}{\sqrt{3}}\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
&= -\frac{1}{2}a - \frac{1}{2}b - \frac{ib - ia}{2\sqrt{3}} + \frac{\sqrt{3}}{2}ia + \frac{\sqrt{3}}{2}ib + \frac{ib - ia}{2} \cdot i \\
&= \left(-\frac{1}{2}a - \frac{1}{2}b + \frac{ib - ia}{2} \cdot i\right) + \left(-\frac{ib - ia}{2\sqrt{3}} + \frac{\sqrt{3}}{2}ia + \frac{\sqrt{3}}{2}ib\right) \\
&= \left(-\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}b + \frac{1}{2}a\right) + \left(\frac{ia - ib + 3ia + 3ib}{2\sqrt{3}}\right) \\
&= (-b) + \left(\frac{4ia + 2ib}{2\sqrt{3}}\right) \\
&= (-b + a + b + c) + \left(\frac{i(2a + b)}{\sqrt{3}}\right) \\
&= (a+c) + \frac{i(2a + b - (a + b + c))}{\sqrt{3}} \\
(a+c) + \frac{ia - ic}{\sqrt{3}} &= (a+c) + \frac{ia - ic}{\sqrt{3}}
\end{aligned}
$$

Tedious, but it worked.

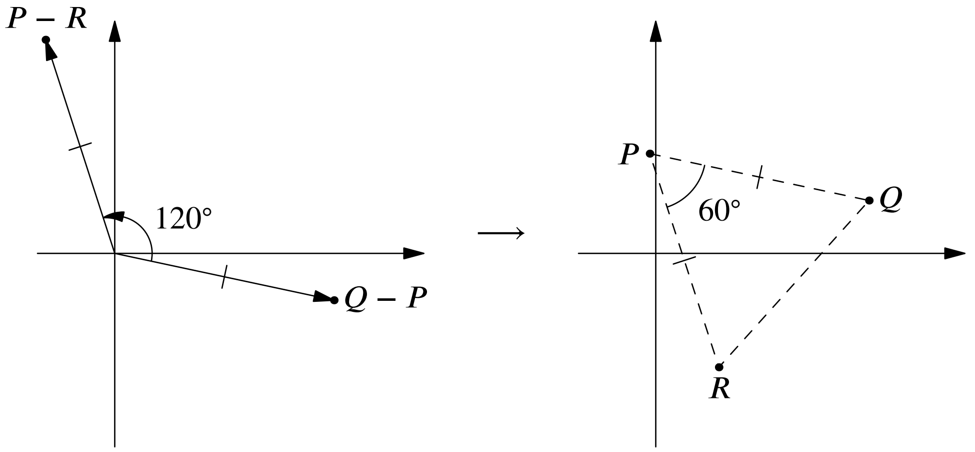


Figure 10: SAS Congruence lets us say that $P - R = (Q - P)\operatorname{cis} 120^\circ$ is sufficient to prove the triangle $\triangle PQR$ is equilateral.

**20. The hard way to find an identity for $\tan 3\theta$ is to divide the identity for $\sin$ and $\cos$ that we already found. Try this. Make sure your answer is in terms of $\tan$ only!**

We have found that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$. We set $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and evaluate:

$$\begin{aligned}
\tan\theta = \frac{\sin\theta}{\cos\theta} &= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta} \\
&= \frac{\sin\theta}{\cos\theta}\cdot\frac{3\cos^2\theta - \sin^2\theta}{\cos^2\theta - 3\sin^2\theta}\cdot\frac{\frac{1}{\cos^2\theta}}{\frac{1}{\cos^2\theta}} \\
&= \tan\theta\cdot\frac{3-\frac{\sin^2\theta}{\cos^2\theta}}{1-\frac{3\sin^2\theta}{\cos^2\theta}} \\
&= \tan\theta\cdot\frac{3-\tan^2\theta}{1-3\tan^2\theta} \\
&= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}.
\end{aligned}$$

**21. The easier way to get an identity for $\tan 3\theta$ starts with setting $z = 1 + i\tan\theta$.**

**(a) Why is $\operatorname{Arg} z = \theta$?**

You can see this pretty quickly with a diagram, like in Figure 11. More algebraically, we have

$$\begin{aligned}
\tan(\operatorname{Arg} z) &= \operatorname{Im}(z) = \tan(\theta) \\
&\implies \operatorname{Arg} z = \theta.
\end{aligned}$$

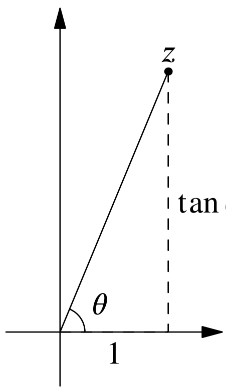


Figure 11: $\operatorname{Arg} z = \tan^{-1}\left(\frac{\tan\theta}{1}\right) = \theta$.

**(b) Why is $\tan 3\theta = \frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)}$?**

We have

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{\operatorname{Im}(\operatorname{cis} 3\theta)}{\operatorname{Re}(\operatorname{cis} 3\theta)}.$$

But since $z$ makes an angle of $\theta$ with the $x$-axis, we can express it as $r\operatorname{cis}\theta$ for some real $r$. Thus,

$$\frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)} = \frac{\operatorname{Im}(r^3\operatorname{cis} 3\theta)}{\operatorname{Re}(r^3\operatorname{cis} 3\theta)} = \frac{\operatorname{Im}(\operatorname{cis} 3\theta)}{\operatorname{Re}(\operatorname{cis} 3\theta)},$$

which matches the expression for $\tan 3\theta$.

**(c) Use (b) to find an identity for $\tan 3\theta$.**

We expand out $z^3$ and factor into real and imaginary parts:

$$\begin{aligned}
z^3 = (1 + i\tan\theta)^3 &= 1^3 + 3i\tan\theta - 3\tan^2\theta - i\tan^3\theta \\
&= (1 - 3\tan^2\theta) + i(3\tan\theta - \tan^3\theta).
\end{aligned}$$

Then we use our expression for $\tan 3\theta$ in terms of $z^3$:

$$\begin{aligned}\tan 3\theta &= \frac{\mathrm{Im}(z^3)}{\mathrm{Re}(z^3)} \\ &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.\end{aligned}$$

**22. Find multiplication groups of complex numbers isomorphic to rotation groups for the**

**(a) regular octagon.**

We choose complex numbers corresponding to rotations of $0, 45°, \cdots, 315°$:

$$z = \left\{\operatorname{cis} 0, \operatorname{cis}\frac{\pi}{4}, \operatorname{cis}\frac{\pi}{2}, \operatorname{cis}\frac{3\pi}{4}, \operatorname{cis}\pi, \operatorname{cis}\frac{5\pi}{4}, \operatorname{cis}\frac{3\pi}{2}, \operatorname{cis}\frac{7\pi}{4}\right\}.$$

**(b) regular pentagon.**

We simply choose complex numbers corresponding to rotations of $0, 72°, \cdots, 288°$:

$$z = \left\{\operatorname{cis} 0, \operatorname{cis}\frac{2\pi}{5}, \operatorname{cis}\frac{4\pi}{5}, \operatorname{cis}\frac{6\pi}{5}, \operatorname{cis}\frac{8\pi}{5}\right\}.$$

**23. Make tables for**

**(a) the rotation group of the regular octagon.**

There are 8 elements. If $r$ is a rotation by $45°$, then the elements are $I, r, r^2, ..., r^7$. The group table is shown below.

| $\cdot$ | $I$ | $r$ | $r^2$ | $r^3$ | $r^4$ | $r^5$ | $r^6$ | $r^7$ |
|---|---|---|---|---|---|---|---|---|
| $I$ | $I$ | $r$ | $r^2$ | $r^3$ | $r^4$ | $r^5$ | $r^6$ | $r^7$ |
| $r$ | $r$ | $r^2$ | $r^3$ | $r^4$ | $r^5$ | $r^6$ | $r^7$ | $I$ |
| $r^2$ | $r^2$ | $r^3$ | $r^4$ | $r^5$ | $r^6$ | $r^7$ | $I$ | $r$ |
| $r^3$ | $r^3$ | $r^4$ | $r^5$ | $r^6$ | $r^7$ | $I$ | $r$ | $r^2$ |
| $r^4$ | $r^4$ | $r^5$ | $r^6$ | $r^7$ | $I$ | $r$ | $r^2$ | $r^3$ |
| $r^5$ | $r^5$ | $r^6$ | $r^7$ | $I$ | $r$ | $r^2$ | $r^3$ | $r^4$ |
| $r^6$ | $r^6$ | $r^7$ | $I$ | $r$ | $r^2$ | $r^3$ | $r^4$ | $r^5$ |
| $r^7$ | $r^7$ | $I$ | $r$ | $r^2$ | $r^3$ | $r^4$ | $r^5$ | $r^6$ |

**(b) the dihedral group of the square.**

There are, once again, $4 \cdot 2 = 8$ elements. Let $r$ be a rotation by $90°$, and $f$ be a flip about say, the $x$-axis. The group table is shown below.

| $\cdot$ | $I$ | $r$ | $r^2$ | $r^3$ | $f$ | $fr$ | $fr^2$ | $fr^3$ |
|---|---|---|---|---|---|---|---|---|
| $I$ | $I$ | $r$ | $r^2$ | $r^3$ | $f$ | $fr$ | $fr^2$ | $fr^3$ |
| $r$ | $r$ | $r^2$ | $r^3$ | $I$ | $fr^3$ | $f$ | $fr$ | $fr^2$ |
| $r^2$ | $r^2$ | $r^3$ | $I$ | $r$ | $fr^2$ | $fr^3$ | $f$ | $fr$ |
| $r^3$ | $r^3$ | $I$ | $r$ | $r^2$ | $fr$ | $fr^2$ | $fr^3$ | $f$ |
| $f$ | $f$ | $fr$ | $fr^2$ | $fr^3$ | $I$ | $r$ | $r^2$ | $r^3$ |
| $fr$ | $fr$ | $fr^2$ | $fr^3$ | $f$ | $r^3$ | $I$ | $r$ | $r^2$ |
| $fr^2$ | $fr^2$ | $fr^3$ | $f$ | $fr$ | $r^2$ | $r^3$ | $I$ | $r$ |
| $fr^3$ | $fr^3$ | $f$ | $fr$ | $fr^2$ | $r$ | $r^2$ | $r^3$ | $I$ |

**(c) Is the difference between them fundamental?**

Yes, the difference is fundamental, even though they have the same order. The easiest way to see this is that the latter group has 4 elements of order 2, but the former group has only 1 such element.

**24. Which of the following tables defines a group? Why or why not?**

**(d)**

| $ | $I$ | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|---|
| $I$ | $I$ | $A$ | $B$ | $C$ | $D$ |
| $A$ | $A$ | $C$ | $D$ | $B$ | $I$ |
| $B$ | $B$ | $I$ | $C$ | $D$ | $A$ |
| $C$ | $C$ | $D$ | $A$ | $I$ | $B$ |
| $D$ | $D$ | $B$ | $I$ | $A$ | $C$ |

This table does not define a group, because it does not follow associativity. For example, $(D\$A)\$A = B\$A = I$, but $D\$(A\$A) = D\$C = A$.

**(e)**

| # | $I$ | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|---|
| $I$ | $I$ | $A$ | $B$ | $C$ | $D$ |
| $A$ | $A$ | $B$ | $C$ | $D$ | $I$ |
| $B$ | $B$ | $C$ | $D$ | $I$ | $A$ |
| $C$ | $C$ | $D$ | $I$ | $A$ | $B$ |
| $D$ | $D$ | $I$ | $A$ | $B$ | $C$ |

This table is a group; in fact, it is a commutative group. The quickest way to see this is noting that it is (up to isomorphism) the cyclic group of order 5, where $A = r$, $B = r^2$, $C = r^3$, and $D = r^4$.

**25. Name some subsets of the complex numbers that are groups under multiplication. I can name an infinite number of both finite and infinite groups with this property, so after you list a few of each type, try to generalize.**

Some simple examples: $\{1\}$, $\{\pm 1\}$, $\{\pm 1, \pm i\}$.

In general, we choose the $n$th roots of unity: the numbers of the form $\operatorname{cis} \frac{2\pi k}{n}$ for $k \in \mathbb{Z}$. Each rotation is a symmetry of the $n$-gon, and thus this set under multiplication is isomorphic to the cyclic group of order $n$.

**26. Prove with a diagram that if $|z| = 1$, then $\operatorname{Im}\left(\frac{z}{(z+1)^2}\right) = 0$.**

To draw a diagram, we need to interpret these expressions as points on the complex plane. $|z| = 1$ implies that $z$ is 1 away from the origin. $z + 1$ is $z$, translated right by 1 unit in the $x$-axis. Let $z + 1 = r \operatorname{cis} \theta$. Then $(z + 1)^2 = r^2 \operatorname{cis} 2\theta$, so it forms an angle of $2\theta$ with the origin.

If the quotient $\frac{z}{(z+1)^2}$ has no imaginary part, then $(z + 1)^2$ is a real scalar times $z$. In other words, the two numbers have the same complex argument. Thus, we wish to prove that $\operatorname{Arg} z$ and $\operatorname{Arg}(z + 1)^2$ are equal.

The scenario is shown in Figure 12.

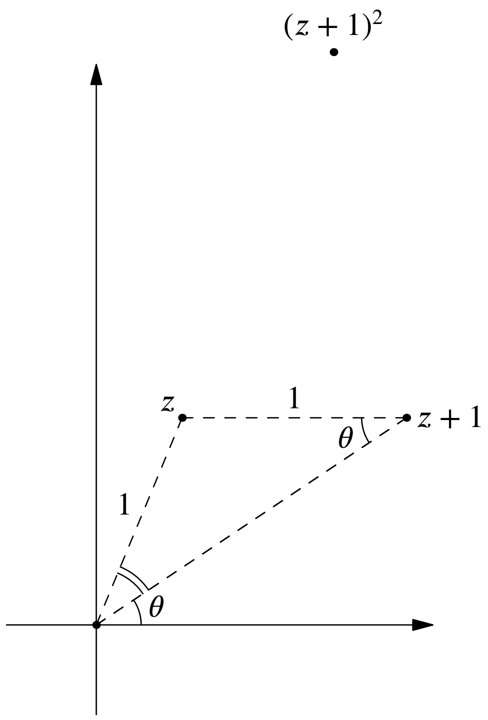


Figure 12: A graph of $z$, $z + 1$, and $(z + 1)^2$.

As shown in the diagram, $\operatorname{Arg}(z+1) = \theta$. The triangle formed by $O$, $z$ and $z+1$ is isosceles, since it has two sides of length 1. Furthermore, it has a base angle of $\theta$ by the Parallel Postulate. Thus, the angle marked with a double line is also $\theta$, and $\operatorname{Arg} z = 2\theta$. But $\operatorname{Arg}(z+1)^2 = 2\theta$! Thus, we have

$$\begin{aligned}\operatorname{Im}\left(\frac{z}{(z+1)^2}\right) &= \operatorname{Im}\left(\frac{r_1 \operatorname{cis} 2\theta}{r_2 \operatorname{cis} 2\theta}\right)\\ &= \operatorname{Im}\left(\frac{r_1}{r_2} + 0i\right)\\ &= 0.\end{aligned}$$

This is not truly complete, because we have only considered $z$ in the first quadrant. In this case, extending it to other locations of $z$ is pretty trivial. Nonetheless, I provide an algebraic solution for fun.

We wish to show that $(z+1)^2 = kz$ for some real $k$. Express $z$ as $\operatorname{cis}\theta$. Then

$$(\operatorname{cis}\theta + 1)^2 = \operatorname{cis}^2\theta + 2\operatorname{cis}\theta + 1.$$

Our supposed $k$ is

$$\begin{aligned}k = \frac{(z+1)^2}{z} &= \frac{\operatorname{cis}^2\theta + 2\operatorname{cis}\theta + 1}{\operatorname{cis}\theta}\\ &= \operatorname{cis}\theta + 2 + \operatorname{cis}(-\theta)\\ &= \operatorname{cis}\theta + 2 + \overline{\operatorname{cis}\theta}\\ &= 2\operatorname{Re}(\operatorname{cis}\theta) + 2,\end{aligned}$$

which is indeed real. It's interesting what the scale factor actually is. Furthermore, since $\operatorname{Re}(\operatorname{cis}\theta) = \cos\theta$, we have a polar equation

$$|kz| = r = 2\cos\theta + 2,$$

hinting that the path traced out by $(z+1)^2$ is in fact a cardioid! The cardioid produced is shown in Figure 13 below.

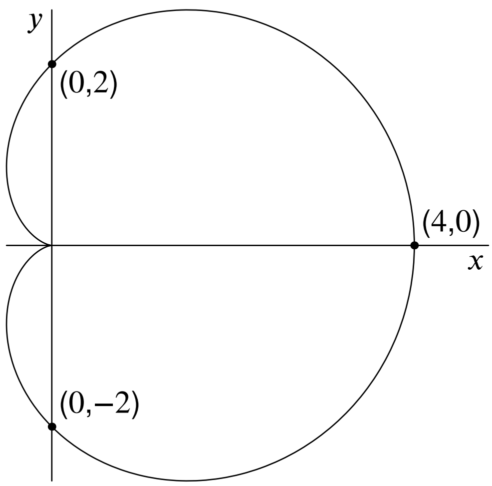


Figure 13: The cardioid produced by $(z+1)^2$ for $|z| = 1$.

It's all connected, guys.

**27. Prove geometrically that if $|z| = 1$, then $|1 - z| = \left|2\sin\left(\frac{\operatorname{Arg} z}{2}\right)\right|$.**

To prove this geometrically, we must again consider what the various expressions in the desired equation mean. $|z| = 1$ means that $z$ is distance 1 from the origin. $1 - z$ is the reflection of $z$ across the origin, then translated by 1 units right.

To geometrically interpret $A = 2\sin\left(\frac{\operatorname{Arg} z}{2}\right)$, we halve the angle $\theta = \operatorname{Arg} z$ and draw a vector of length 2; the imaginary component, or $y$ height, of this new point is the desired quantity.

The diagram of all this is shown in Figure 14.

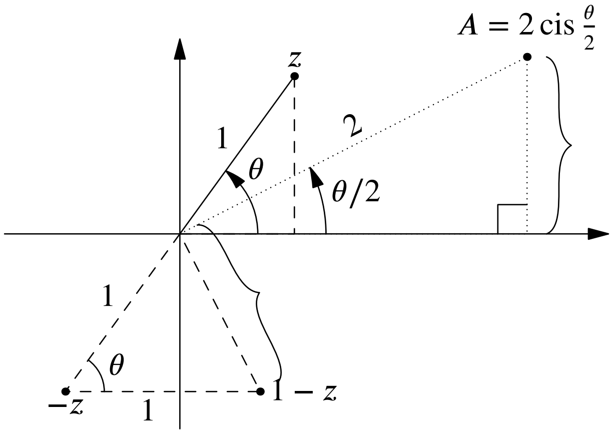


Figure 14: Graph of $z$, $1-z$, and $\sin\left(\frac{\theta}{2}\right)$.

We wish to show that the two lengths indicated in braces are equal. There's a couple of ways to do this; perhaps the most natural is to find a triangle congruent to the one formed by the origin, $-z$, and $1-z$. This is the other dashed triangle shown in Figure 15.

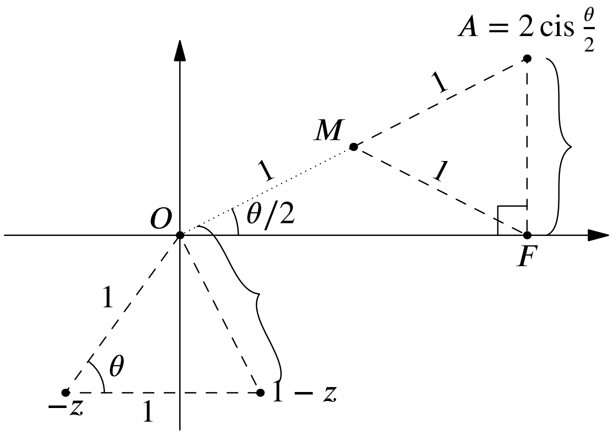


Figure 15: The succulent triangle,

Because the angles of a triangle sum to $\pi$, we know the angle $\angle MAF$ is $\pi - \frac{\theta}{2} - \frac{\pi}{2} = \frac{\pi-\theta}{2}$. Furthermore, $\triangle AMF$ is isosceles with an apex at $M$, since the midpoint of the hypotenuse of a right triangle is equidistant from all vertices. Thus, $\angle MAF = \angle MFA$, and we have

$$\angle AMF = \pi - \angle MAF - \angle MFA = \pi - \frac{\pi-\theta}{2} \cdot 2 = \theta.$$

Thus, by SAS Congruence, the two dashed triangles are congruent. Finally, pairing up the previously indicated sides, we have

$$|1-z| = \overline{AF} = \left|\operatorname{Im}\left(2\operatorname{cis}\frac{\theta}{2}\right)\right|,$$

as desired.

Technically, this proof is slightly incomplete, because some of these triangles do not exist as described for $\theta \geq 90°$. You can extend it to these cases with no problem, but I'd also like to give a algebraic proof to show its perks.

By the half-angle identity,

$$\left|2\sin\left(\frac{\operatorname{Arg} z}{2}\right)\right| = \left|\pm 2\sqrt{\frac{1-\cos\operatorname{Arg} z}{2}}\right| = \sqrt{2(1-\cos\operatorname{Arg} z)}.$$

Let $z = a + bi = \operatorname{cis}\theta$; note that $r = 1$ since $|z| = 1$. We know that $\cos\operatorname{Arg} z = a$. Then

$$
\begin{aligned}
|1-(a+bi)| &= |(1-a)-bi| \\
&= \sqrt{b^2+(1-a)^2} \\
&= \sqrt{1-a^2+(1-2a+a^2)} \\
&= \sqrt{2-2a} \\
&= \sqrt{2(1-a)} \\
&= \sqrt{2(1-\cos \operatorname{Arg} z)}.
\end{aligned}
$$

This matches our expression using half-angle for $\left|2\sin\left(\frac{\operatorname{Arg} z}{2}\right)\right|$.

Personally, I strongly prefer the algebraic solution because it is quick, easy to understand, and truly complete. Nonetheless, the geometric solution gives a better idea of *why* the equation is true.

**28.**

**(a) Prove that if $(z-1)^{10} = z^{10}$, then $\operatorname{Re}(z) = \frac{1}{2}$. (Hint: if two numbers are equal, they have the same magnitude.)**

We do as the hint suggests. We know that $|(z-1)^{10}| = |z^{10}|$. Expanding this out would be rough, but we can take the exponents out of the inside of the magnitude symbols[12].

So $|z-1|^{10} = |z|^{10}$. Since $|n| \geq 0$, we have $|z-1| = |z|$.

Let $z = a+bi$. Then $|a+bi-1| = \sqrt{(a-1)^2+b^2}$ and $|a+bi| = \sqrt{a^2+b^2}$. We set these equal and solve:

$$
\begin{aligned}
\sqrt{(a-1)^2+b^2} &= \sqrt{a^2+b^2} \\
(a-1)^2+b^2 &= a^2+b^2 \\
(a-1)^2 &= a^2 \\
a-1 &= \pm a.
\end{aligned}
$$

If $a-1=a$, then $-1=0$, which is dumb. Thus, $a-1=-a$, so $a=\frac{1}{2}$ and indeed, $\operatorname{Re}(z) = \frac{1}{2}$ as desired.

**(b) How many solutions does this equation have?**

We have $(z-1)^{10} = z^{10}$, so $(z-1)^{10} - z^{10} = P(z) = 0$, where $P$ is a polynomial of degree $9$. Thus, by the Fundamental Theorem of Algebra, there are 9 solutions... if there aren't any repeated roots. So this is only truly complete if we know there are no roots which appear in the factorization twice or more. Unfortunately, I can't think of a way to do this without calculus[13].

Let's start over. We should use the fact that $\operatorname{Re}(z) = \frac{1}{2}$. A simple diagram reveals that $z$ and $z-1$ are symmetric about the $y$-axis, since $\operatorname{Re}(z) = \operatorname{Re}\left(\frac{1}{2}+bi\right) = -\operatorname{Re}\left(\frac{1}{2}+bi-1\right)$. The diagram is shown in Figure 16.

[12] This is true because $|(r\operatorname{cis}\theta)^n| = |r^n \operatorname{cis} n\theta| = |r^n|$.

[13] For that route, we simply check that $P''(z) = 0$ for all solutions, which isn't pleasant until a clever rearrangement and substitution. Try it if you know how!

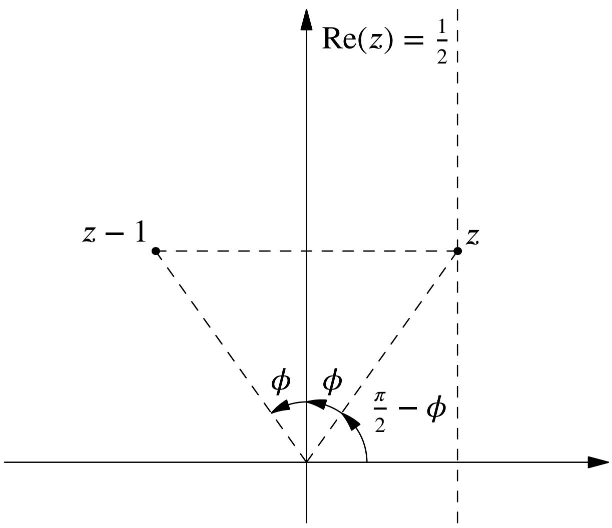


Figure 16: $z$ and $1 - z$, residents of the complex plane.

Let $z$ in the first quadrant make a angle $\phi$ to the $\pm y$-axis. Note that we're using the $y$-axis, not the $x$-axis, for mathematical convenience. In general, for $z$ in the first and fourth quadrants[14], we have $\operatorname{Arg} z = \frac{\pi}{2} - \phi$ and $\operatorname{Arg}(z-1) = \frac{\pi}{2} + \phi$. Since $|z| = |z-1| = r$, we have

$$z = r\operatorname{cis}\left(\frac{\pi}{2} - \phi\right);\ z - 1 = r\operatorname{cis}\left(\frac{\pi}{2} + \phi\right).$$

Since $(z-1)^{10} = z^{10}$, we have

$$\begin{aligned}
\left(r\operatorname{cis}\left(\frac{\pi}{2} - \phi\right)\right)^{10} &= \left(r\operatorname{cis}\left(\frac{\pi}{2} + \phi\right)\right)^{10} \\
r^{10}\operatorname{cis}(5\pi - 10\phi) &= r^{10}\operatorname{cis}(5\pi + 10\phi) \\
\operatorname{cis}(5\pi - 10\phi) &= \operatorname{cis}(5\pi + 10\phi) \\
5\pi - 10\phi + 2\pi k &= 5\pi + 10\phi && \text{For some } k \in \mathbb{Z} \\
20\phi &= 2\pi k \\
\phi &= \frac{2\pi k}{20}.
\end{aligned}$$

To find all unique solutions, we restrict $k$ to the range $0 \le k \le 19$... wait, isn't that $20$ solutions?

The issue is that $z$ must be in the first or fourth quadrant, so that our premise $|z| = |z-1|$ is true. That means $0 < \phi < \pi$, a strict relation because $\phi = 0$ or $\phi = \pi$ only gives values along the $y$-axis, which does not intersect with $\mathrm{Re}(z) = 0$. Solving this gives

$$\begin{gathered}
0 < \frac{2\pi k}{20} < \pi \\
0 < \pi k < 10\pi \\
0 < k < 10 \\
k \in \{1, 2, \cdots, 8, 9\},
\end{gathered}$$

which is $9$ solutions, in agreement with our polynomial argument.

**29. I claim that $e^{i\theta} = \cos\theta + i\sin\theta = \operatorname{cis}\theta$, for $\theta$ in radians.**

**(a) Find $e^{-it}$.**

$$e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i\sin t.$$

**(b) Find $\frac{e^{i\theta} + e^{-i\theta}}{2}$.**

[14]If $z$ is in the fourth quadrant, then you'd define $\phi$ as $\pi$ + angle to negative $y$axis, where the angle is taken clockwise so it's positive.

$$
\begin{aligned}
\frac{e^{i\theta}+e^{-i\theta}}{2} &= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} \\
&= \cos\theta.
\end{aligned}
$$

**(c) Find $\frac{e^{i\theta}-e^{-i\theta}}{2i}$.**

$$
\begin{aligned}
\frac{e^{i\theta}-e^{-i\theta}}{2i} &= \frac{\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)}{2i} \\
&= \sin\theta.
\end{aligned}
$$

**30. Use your new, complex definitions for $\cos$ and $\sin$ to find:**

**(a)** $\cos^2\theta + \sin^2\theta$

$$
\begin{aligned}
\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2 + \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2 &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2 - \left(\frac{e^{i\theta}-e^{-i\theta}}{2}\right)^2 \\
&= \left(\frac{e^{i\theta}+e^{-i\theta}}{2} + \frac{e^{i\theta}-e^{-i\theta}}{2}\right)\left(\frac{e^{i\theta}+e^{-i\theta}}{2} - \frac{e^{i\theta}-e^{-i\theta}}{2}\right) \\
&= \left(e^{-i\theta}\right)\left(e^{i\theta}\right) \\
&= e^{-i\theta+i\theta} \\
&= e^0 \\
&= 1.
\end{aligned}
$$

That was expected.

**(b)** $\tan\theta$

$$
\begin{aligned}
\tan\theta = \frac{\sin\theta}{\cos\theta} &= \frac{\frac{e^{i\theta}-e^{-i\theta}}{2i}}{\frac{e^{i\theta}+e^{-i\theta}}{2}} \\
&= \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}.
\end{aligned}
$$

**(c)** $\cos 2\theta$

$$
\cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2}.
$$

**(d)** $\sin 2\theta$

$$
\sin 2\theta = \frac{e^{2i\theta}-e^{-2i\theta}}{2i}.
$$

**(e) What kind of group is generated by $\left\{e^{i\theta}, e^{-i\theta}\right\}$ under the operation of multiplication if $\theta$ is an integer? A rational multiple of $\pi$?**

If $\theta = 0$, then the group is the trivial group of order 1. If $\theta$ is any other integer, then a group isomorphic to the additive group of the integers is generated. We correspond $e^{ik\theta}$ with the integer $k$, so that

$$
e^{ik_1\theta} \cdot e^{ik_2\theta} = e^{i(k_1+k_2)\theta} \leftrightarrow k_1 + k_2.
$$

If $\theta$ is a rational multiple of $\pi$, say $\frac{p}{q} \cdot 2\pi$ where $\gcd(p, q) = 1$, then we get (up to isomorphism) cyclic group of order $q$.

**31. You've used the quadratic equation throughout high school, but there's also a cubic equation that finds the roots of any cubic. Let's derive it, starting with the cubic $x^3 + bx^2 + cx + d = 0$.**

**(a) Make the substitution $x = y - \frac{b}{3}$. Combine like terms to create an equation of the form $y^3 - 3py - 2q = 0$, with $p, q$ in terms of $b, c$, and $d$.**

$$\begin{aligned}
\left(y - \frac{b}{3}\right)^3 + b\left(y - \frac{b}{3}\right)^2 + c\left(y - \frac{b}{3}\right) + d &= 0 \\
\left(y^3 - 3 \cdot \frac{by^2}{3} + 3 \cdot \frac{b^2 y}{9} - \frac{b^3}{27}\right) + \left(by^2 - \frac{2b^2 y}{3} + \frac{b^3}{9}\right) + \left(cy - \frac{bc}{3}\right) + d &= 0 \\
y^3 + (-b + b)\, y^2 + \left(\frac{b^2 y}{3} - \frac{2b^2 y}{3} + c\right) y + \left(-\frac{b^3}{27} + \frac{b^3}{9} - \frac{bc}{3} + d\right) &= 0 \\
y^3 + \left(c - \frac{b^2}{3}\right) y + \left(d - \frac{bc}{3} + \frac{2b^3}{27}\right) &= 0 \\
y^3 - 3\underbrace{\left(\frac{b^2}{9} - \frac{c}{3}\right)}_{p} y - 2\underbrace{\left(\frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}\right)}_{q} &= 0.
\end{aligned}$$

Thus, $p = \frac{b^2}{9} - \frac{c}{3}$ and $q = \frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}$.

**(b) Rearrange this equation as $y^3 = 3py + 2q$.**

$$y^3 = 3\left(\frac{b^2}{9} - \frac{c}{3}\right) y + 2\left(\frac{bc}{6} - \frac{b^3}{27} - \frac{d}{2}\right).$$

**(c) Make the substitution $y = s + t$ into (b), and prove that $y$ is a solution of the cubic in part (a) if $st = p$ and $s^3 + t^3 = 2q$.**

We substitute $y = s + t$ and use the fact that $st = p$ and $s^3 + t^3 = 2q$ to simplify.

$$\begin{aligned}
(s + t)^3 &= 3p(s + t) + 2q \\
s^3 + 3s^2 t + 3st^2 + t^3 &= 3ps + 3pt + 2q \\
3s(st) + 3t(st) + s^3 + t^3 &= 3ps + 3pt + 2q \\
3sp + 3tp + 2q &= 3ps + 3pt + 2q \\
0 &= 0.
\end{aligned}$$

This checks out.

**(d) Eliminate $t$ between these two equations to get a quadratic in $s^3$.**

We have $t^3 = 2q - s^3$. Also, $(st)^3 = p^3$, so

$$\begin{aligned}
(st)^3 = s^3 t^3 = s^3(2q - s^3) &= p^3 \\
-(s^3)^2 + 2qs^3 - p^3 &= 0 \\
(s^3)^2 - 2qs^3 + p^3 &= 0.
\end{aligned}$$

**(e) Solve this quadratic to find $s^3$. By symmetry, what is $t^3$?**

Let $w = s^3$. Then the above quadratic is $w^2 - 2qw + p^3 = 0$. The solutions are

$$s^3 = w = \frac{2q \pm \sqrt{4q^2 - 4p^3}}{2} = \frac{2q \pm 2\sqrt{q^2 - p^3}}{2} = q \pm \sqrt{q^2 - p^3}.$$

We have $t^3 = 2q - s^3 = 2q - (q \pm \sqrt{q^2 - p^3}) = q \mp \sqrt{q^2 - p^3}$. This inverted $\pm$ sign, $\mp$, means that when $s^3$'s $\pm$ sign is positive, the $\mp$ sign is negative, and vice versa.

**(f) Find a formula for $y$ in terms of $p$ and $q$. What about a formula for $x$?**

Taking cube roots of both sides of our expressions for $t^3$ and $s^3$, we find that

$$\begin{aligned} s &= \sqrt[3]{q \pm \sqrt{q^2 - p^3}}, \\ t &= \sqrt[3]{q \mp \sqrt{q^2 - p^3}}. \end{aligned}$$

We must keep in mind, however, that over the complex numbers, cube rooting has 3 possible values. Thus, the three solutions for $s$ and $t$ are

$$\begin{aligned} s &= \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \frac{2\pi k}{3}, \\ t &= \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \left(2\pi - \frac{2\pi k}{3}\right), \end{aligned}$$

where $k \in \{0, 1, 2\}$ and the cube root is taking its principal value. We multiply them by $\operatorname{cis}$ with these angles to preserve the fact that $st = q$, since otherwise it would produce another result:

$$\begin{aligned} \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \frac{2\pi k}{3} \cdot \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \left(2\pi - \frac{2\pi k}{3}\right) &= \sqrt[3]{\left(q \pm \sqrt{q^2 - p^3}\right)\left(q \mp \sqrt{q^2 - p^3}\right)} \cdot \operatorname{cis} 2\pi \\ &= \sqrt[3]{q^2 - (q^2 + p^3)} \\ &= p. \end{aligned}$$

Thus, we have

$$y = s + t = \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \frac{2\pi k}{3} + \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \left(2\pi - \frac{2\pi k}{3}\right).$$

To get $x$, we substitute $x = y - \frac{b}{3}$ to get

$$x = \sqrt[3]{q \pm \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \frac{2\pi k}{3} + \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \cdot \operatorname{cis} \left(2\pi - \frac{2\pi k}{3}\right) - \frac{b}{3}.$$

You could substitute our values of $p, q$ in terms of $b, c, d$ to get a monstrous equation for $x$ in terms of only $b, c, d$... but no thanks.

**(g) What if we started with $ax^3 + bx^2 + cx + d = 0$, with a coefficient in front of the $x^3$ term as well? Can you come up with a formula for $x$?**

Sure! We divide through the equation by $a$:

$$\begin{aligned} \frac{ax^3 + bx^2 + cx + d}{a} &= 0 \\ \implies x^3 + \frac{b}{a} + \frac{c}{a} + \frac{d}{a} &= 0. \end{aligned}$$

We can then attack this as we already did, setting $b' = \frac{b}{a}$, $c' = \frac{c}{a}$ and $d' = \frac{d}{a}$, then applying the formula.

**32. Starting with the same cubic as in problem 31b.**

**(a) Let $c = \cos\theta$. Remember that $\cos 3\theta = 4c^3 - 3c$, as we proved. Substitute $y = 2c\sqrt{p}$ into $y^3 = 3py + 2q$ to obtain $4c^3 - 3c = \frac{q}{p^{3/2}}$.**

We substitute and proceed:

$$\begin{aligned} y^3 &= 3py + 2q \\ (2c\sqrt{p})^3 &= 3p(2c\sqrt{p}) + 2q \\ 8c^3p^{3/2} &= 6cp^{3/2} + 2q \\ 8c^3p^{3/2} - 6cp^{3/2} &= 2q \\ 4c^3 - 3c &= \frac{2q}{2p^{3/2}} = \frac{q}{p^{3/2}}. \end{aligned}$$

**(b) Provided that $q^2 \leq p^3$, show that $y = 2\sqrt{p}\cos\left(\frac{1}{3}(\theta + 2\pi n)\right)$, where $n$ is an integer. Why does this yield all three solutions?**

This isn't actually hard. We know that $\cos 3\theta = 4c^3 - 3c = \frac{q}{p^{3/2}}$, so there are three possible values for $\cos\theta = c$. Namely, if $\theta_0 = \frac{1}{3}\cos^{-1}\frac{q}{p^{3/2}}$ is the principal value, then we also have unique solutions

$$\theta_1 = \frac{2\pi}{3} + \theta_0,\ \theta_2 = \frac{4\pi}{3} + \theta_0,$$

because multiplying these by 3 to get $3\theta$ just adds a multiple of $2\pi$. Indeed, we have

$$c = \cos\frac{1}{3}(\theta + 2\pi n)$$

as a solution for any integer $n$. Substituting into the expression for $y$, we get

$$y = 2c\sqrt{p} = 2\cos\left(\frac{1}{3}(\theta + 2\pi n)\right)\sqrt{p},$$

as desired. This yields all three solutions because, as we observed, the only unique values this makes are for $n \in \{0, 1, 2\}$.

Note that if $q^2 > p^3$, then this strategy actually still works, but you have to define $\cos$ (and $\cos^{-1}$) over a larger, complex domain. This is certainly possible though!

**(c) Explain how you would find $\theta$ from $p$ and $q$, and how we would use what we have found to solve an arbitrary cubic $ax^3 + bx^2 + cx + d = 0$.**

We have $\cos 3\theta = \frac{q}{p^{3/2}}$, so

$$\theta = \cos^{-1}\frac{q}{p^{3/2}}.$$

The steps to solving an arbitrary cubic are the following:

1. Divide through by $a$ to get a new cubic $x^3 + b'x^2 + c'x + d' = 0$.
2. Compute $p = \frac{b'^2}{9} - \frac{c'}{3}$ and $q = \frac{b'c'}{6} - \frac{b'^3}{27} - \frac{d'}{2}$.
3. Compute $\theta = \cos^{-1}\frac{q}{p^{3/2}}$.
4. Substitute this value of $\theta$ into $x = y - \frac{b}{3} = 2\sqrt{p}\cos\left(\frac{1}{3}(\theta + 2\pi n)\right) - \frac{b}{3}$, where $n \in \{0, 1, 2\}$.

These two problems were a doozy!