

## 7 Your Daily Dose of Vitamin $i$

1. We will use complex numbers to find identities for  $\cot$ . Use Pascal's triangle to expand the following:

(a)  $(a + b)^3$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

(b)  $(a + b)^4$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

(c)  $(a + b)^5$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

1. (cont.) Then substitute  $b = i = \sqrt{-1}$  and expand:

(d)  $(a + i)^3$

$$(a + i)^3 = a^3 + 3a^2i + 3ab^2 + b^3 = a^3 + 3a^2i - 3a - i.$$

(e)  $(a + i)^4$

$$(a + i)^4 = a^4 + 4a^3i + 6a^2b^2 + 4ab^3 + b^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1.$$

(f)  $(a + i)^5$

$$(a + i)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i.$$

1. (cont.) Finally, substitute  $a = \cot \theta$  and expand:

(g)  $(\cot \theta + i)^3$

$$(\cot \theta + i)^3 = a^3 + 3a^2i - 3a - i = (\cot^3 \theta - 3 \cot \theta) + i(3 \cot^2 \theta - 1).$$

(h)  $(\cot \theta + i)^4$

$$(\cot \theta + i)^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1 = (\cot^4 \theta - 6 \cot^2 \theta + 1) + i(4 \cot^3 \theta - 4 \cot \theta).$$

(i)  $(\cot \theta + i)^5$

$$(\cot \theta + i)^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i = (\cot^5 \theta - 10 \cot^3 \theta + 5 \cot \theta) + i(5 \cot^4 \theta - 10 \cot^2 \theta + 1).$$

1. (cont.) Consider  $z = i + \cot \theta$ .

(j) Use the above results to find identities for (i)  $\cot 3\theta$ , (ii)  $\cot 4\theta$ , and (iii)  $\cot 5\theta$ .

i.  $\cot 3\theta$

Given the right triangle formed by  $z = i + \cot \theta$  in Figure 7, we have  $\tan(\text{Arg } z) = \frac{1}{\cot \theta} = \tan \theta$ , so  $\text{Arg } z = \theta$  and  $z = r \text{cis } \theta$ .



Figure 1:  $\text{Arg}(i + \cot \theta) = \theta$ .

Thus, we have

$$\begin{aligned}\cot 3\theta &= \frac{\cos 3\theta}{\sin 3\theta} \\ &= \frac{\text{Re}(\text{cis } 3\theta)}{\text{Im}(\text{cis } 3\theta)} \\ &= \frac{\text{Re}(r^3 \text{cis } 3\theta)}{\text{Im}(r^3 \text{cis } 3\theta)} \\ &= \frac{\text{Re}(z^3)}{\text{Im}(z^3)}.\end{aligned}$$

We substitute in our expression for  $z^3$ ,  $(\cot^3 \theta - 3 \cot \theta) + i(3 \cot^2 \theta - 1)$ :

$$\cot 3\theta = \frac{\cot^3 \theta - 3 \cot \theta}{3 \cot^2 \theta - 1}.$$

i.  $\cot 4\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}\cot 4\theta &= \frac{\cos 4\theta}{\sin 4\theta} \\ &= \frac{\text{Re}(\text{cis } 4\theta)}{\text{Im}(\text{cis } 4\theta)} \\ &= \frac{\text{Re}(r^4 \text{cis } 4\theta)}{\text{Im}(r^4 \text{cis } 4\theta)} \\ &= \frac{\text{Re}(z^4)}{\text{Im}(z^4)} \\ \cot 4\theta &= \frac{\cot^4 \theta - 6 \cot^2 \theta + 1}{4 \cot^3 \theta - 4 \cot \theta}.\end{aligned}$$

i.  $\cot 5\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}\cot 5\theta &= \frac{\cos 5\theta}{\sin 5\theta} \\ &= \frac{\text{Re}(\text{cis } 5\theta)}{\text{Im}(\text{cis } 5\theta)} \\ &= \frac{\text{Re}(r^5 \text{cis } 5\theta)}{\text{Im}(r^5 \text{cis } 5\theta)} \\ &= \frac{\text{Re}(z^5)}{\text{Im}(z^5)} \\ &= \frac{\cot^5 \theta - 10 \cot^3 \theta + 5 \cot \theta}{5 \cot^4 \theta - 10 \cot^2 \theta + 1}.\end{aligned}$$

**(k) Graph  $z$ ,  $z^2$ ,  $z^3$ ,  $z^4$ , and  $z^5$ , with  $\theta \approx 75^\circ$ . What is your solution method?**

To graph these, I first calculated the approximate magnitude of  $z$ , which is how many times each subsequent power will be scaled by. We have  $|1 + \cot 75^\circ| \approx 1.268$ , so we only need to scale by about  $\frac{5}{4}$  each time. Of course, we rotate by about  $75^\circ$  each time.



Figure 2: Graphs of  $z$ ,  $z^2$ ,  $z^3$ ,  $z^4$ , and  $z^5$ .

**2. Compute  $(1 + i)^n$  for  $n = 3, 4, 5, \dots$ . Can you find a general pattern?**

We have

$$\begin{aligned} (1 + i)^3 &= 1^3 + 3i - 3 - i &= -2 - 2i \\ (1 + i)^4 &= 1^4 + 4i - 6 - 6i + 1 &= -4 - 2i \\ (1 + i)^5 &= 1^5 + 5i - 10 - 10i + 5 + i &= -4 - 4i. \end{aligned}$$

We can find the pattern by representing  $1 + i = \sqrt{2} \text{cis } 45^\circ$ . This shows that it has period 8 and let's us find an expression for  $(1 + i)^n$ :

$$(1 + i)^n = \left( \sqrt{2} \text{cis } 45^\circ \right)^n = 2^{n/2} \text{cis} \left( \frac{n\pi}{4} \right).$$

**3. Expand and graph  $\text{cis}^n \theta$  for  $n = 2, 3, 4, \dots$**

Let  $\cos \theta = c$  and  $\sin \theta = s$ . We have

$$\begin{aligned} (c + is)^2 &= c^2 + 2csi - s^2 = (c^2 - s^2) + i(2cs) \\ (c + is)^3 &= c^3 + 3c^2si - 3cs^2 - s^3i = (c^3 - 3cs^2) + i(3c^2s - s^3) \\ (c + is)^4 &= c^4 + 4c^3si - 6c^2s^2 - 4cs^3i + s^4 = (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3) \\ (c + is)^5 &= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i = (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5). \end{aligned}$$

The graphs of  $\text{cis}^n \theta$  for  $\theta \approx 50^\circ$  are shown in Figure 3 below.



Figure 3: Graphs of  $\text{cis}^n \theta$  for  $\theta \approx 50^\circ$ .

**(a) Why is the real part  $\cos n\theta$  and the imaginary part  $\sin n\theta$ ?**

By DeMoivre's theorem,  $\text{cis}^n \theta = \text{cis} n\theta$ , which by definition has  $\text{Im}(\text{cis} n\theta) = \cos n\theta$  and  $\text{Re}(\text{cis} n\theta) = \sin n\theta$ .

**(b) Use your results to write identities for  $\cos n\theta$  and  $\sin n\theta$  for  $n = 2, 3, 4, 5$ .**

Here they are. Again, let  $\cos \theta = c$  and  $\sin \theta = s$ :

$$\cos 2\theta = \text{Re}(\text{cis} 2\theta) = c^2 - s^2$$

$$\cos 3\theta = \text{Re}(\text{cis} 3\theta) = c^3 - 3cs^2$$

$$\cos 4\theta = \text{Re}(\text{cis} 4\theta) = c^4 - 6c^2s^2 + s^4$$

$$\cos 5\theta = \text{Re}(\text{cis} 5\theta) = c^5 - 10c^3s^2 + 5cs^4$$

$$\sin 2\theta = \text{Im}(\text{cis} 2\theta) = 2cs$$

$$\sin 3\theta = \text{Im}(\text{cis} 3\theta) = 3c^2s - s^3$$

$$\sin 4\theta = \text{Im}(\text{cis} 4\theta) = 4c^3s - 4cs^3$$

$$\sin 5\theta = \text{Im}(\text{cis} 5\theta) = 5c^4s - 10c^2s^3 + s^5.$$

**4. Compute  $\cos 7^\circ + \cos 79^\circ + \cos 151^\circ + \cos 223^\circ + \cos 295^\circ$  without a calculator. (Hint: what does this have to do with complex numbers?)**

These numbers look random, but a closer inspection reveals they are in arithmetic progression, with starting term 7 and increasing  $72^\circ$  each time. That's the rotation of a pentagon!

We rewrite this as the real component of a sum of cises, then manipulate and evaluate:

$$\begin{aligned} \cos 7^\circ + \cos 79^\circ + \cos 151^\circ + \cos 223^\circ + \cos 295^\circ &= \text{Re}(\text{cis} 7^\circ + \text{cis} 79^\circ + \text{cis} 151^\circ + \text{cis} 223^\circ + \text{cis} 295^\circ) \\ &= \text{Re}((\text{cis} 7^\circ)(\text{cis} 0^\circ + \text{cis} 72^\circ + \text{cis} 144^\circ + \text{cis} 216^\circ + \text{cis} 288^\circ)) \\ &= \text{Re}((\text{cis} 7^\circ)(0)) \\ &= \text{Re}(0) \\ &= 0. \end{aligned}$$

Note that going from the second to third step, we used the fact that the cis expressions are the vertices of a regular pentagon, which sum to 0. If you want to be more formal about it, a fun way to prove that  $\text{cis} 0^\circ + \text{cis} 72^\circ + \text{cis} 144^\circ + \text{cis} 216^\circ + \text{cis} 288^\circ = 0$  is to set it to  $\Xi$  and calculate:

$$\begin{aligned}
\Xi \cdot \text{cis } 72^\circ &= (\text{cis } 0^\circ + \text{cis } 72^\circ + \text{cis } 144^\circ + \text{cis } 216^\circ + \text{cis } 288^\circ) \text{cis } 72^\circ \\
&= \text{cis } 72^\circ + \text{cis } 144^\circ + \text{cis } 216^\circ + \text{cis } 288^\circ + \text{cis } 360^\circ \\
&= \text{cis } 72^\circ + \dots + \text{cis } 288^\circ + \text{cis } 0^\circ \\
&= \Xi.
\end{aligned}$$

If  $\Xi \cdot (\text{something that's not one}) = \Xi$ , then  $\Xi$  must be 0.

## 5. Factor the following:

### (a) $x^6 - 1$ as a difference of squares

We substitute  $y = x^3$ , giving  $y^2 - 1 = (y + 1)(y - 1)$ . Substituting back in, we get

$$(x^3 + 1)(x^3 - 1).$$

### (b) $x^6 - 1$ as a difference of cubes

We now substitute  $y = x^2$ , giving  $y^3 - 1 = (y - 1)(y^2 + y + 1)$ . Substituting back in, we get

$$(x^2 - 1)(x^4 + x^2 + 1)$$

### (c) $x^4 + x^2 + 1$ over the real numbers

This one isn't as obvious. We substitute  $y = x^2$  to get  $y^2 + y + 1$  and find the quadratic's zeroes:

$$y = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}.$$

So it is irreducible over the reals.

### (d) $x^6 - 1$ completely

We already broke it down into  $(x^3 + 1)$  and  $(x^3 - 1)$ . Going further, we have  $x^3 + 1 = (x + 1)(x^2 - x + 1)$  and  $x^3 - 1 = (x - 1)(x^2 + x + 1)$ . To break apart the last two quadratics, we find their zeros:

$$x^2 - x + 1 = 0 \implies x = \frac{1 \pm i\sqrt{3}}{2} \implies \left(x - \frac{1 - i\sqrt{3}}{2}\right) \left(x - \frac{1 + i\sqrt{3}}{2}\right).$$

$$x^2 + x + 1 = 0 \implies x = \frac{-1 \pm i\sqrt{3}}{2} \implies \left(x + \frac{1 - i\sqrt{3}}{2}\right) \left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

Combining all these, we get the complete factorization over the complex numbers:

$$x^6 - 1 = (x + 1) \left(x - \frac{1 - i\sqrt{3}}{2}\right) \left(x - \frac{1 + i\sqrt{3}}{2}\right) (x - 1) \left(x + \frac{1 - i\sqrt{3}}{2}\right) \left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

### (e) $x^4 + x^2 + 1$ completely

We could do a lot of work again, or we could observe that  $x^4 + x^2 + 1 = \frac{x^6 - 1}{x^2 - 1} = \frac{x^6 - 1}{(x + 1)(x - 1)}$ . Removing the denominator's terms from our factorization of  $x^6 - 1$  we found in the last subproblem, we get

$$x^4 + x^2 + 1 = \left(x - \frac{1 - i\sqrt{3}}{2}\right) \left(x - \frac{1 + i\sqrt{3}}{2}\right) \left(x + \frac{1 - i\sqrt{3}}{2}\right) \left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

## 6. Let $f(z) = \frac{z+1}{z-1}$ .

### (a) Without a calculator, compute $f^{2014}(z)$ .

This seems terrifying. Let's try computing  $f^2(z)$  and perhaps  $f^3(z)$ .

$$f^2(z) = \frac{f(z) + 1}{f(z) - 1} = \frac{\frac{z+1}{z-1} + 1}{\frac{z+1}{z-1} - 1} = \frac{\frac{2z}{z-1}}{\frac{2}{z-1}} = z.$$

Oh.

Since 2014 is even, we have  $f^{2014}(z) = (f^2)^{1007}(z) = z$ .

**(b) What if you replace 2014 with the current year?**

Let  $y$  be the current year. As I write this, it is 1492.

If  $y$  is even, then  $f^y(z) = (f^2)^{y/2}(z) = z$ . If  $y$  is odd, then  $f^y(z) = f((f^2)^{(y-1)/2}(z)) = f(z) = \frac{z+1}{z-1}$ .

**7. Find  $\text{Im}((\text{cis } 12^\circ + \text{cis } 48^\circ)^6)$ .**

These are some weird looking angles. Thinking back to some older problems, however, the resultant angle of the addition may be tractable. We draw a diagram, shown in Figure 4.



Figure 4: Adding  $\text{cis } 12^\circ + \text{cis } 48^\circ$ .

Consider the isosceles triangle. The apex has angle measure  $132^\circ + 12^\circ = 144^\circ$ , so the base angles are each  $x = \frac{180^\circ - 144^\circ}{2} = 18^\circ$ . But  $\text{Arg}(\text{cis } 12^\circ + \text{cis } 48^\circ) = 48^\circ - x = 30^\circ$ !

That's a familiar angle. Indeed, we have  $z = \text{cis } 12^\circ + \text{cis } 48^\circ = r \text{cis } 30^\circ$  for some  $r$ . It doesn't really matter which  $r$ , because

$$\text{Im}((r \text{cis } 30^\circ)^6) = \text{Im}(r^6 \text{cis } 180^\circ) = \text{Im}(-r^6) = 0.$$

**8. Let  $x$  satisfy the equation  $x + \frac{1}{x} = 2 \cos \theta$ .**

**(a) Compute  $x^2 + \frac{1}{x^2}$  in terms of  $\theta$ .**

Squaring the left hand side will get us some terms that look close to what we want.

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}.$$

So  $x^2 + \frac{1}{x^2} = (2 \cos \theta)^2 - 2 = 4 \cos^2 \theta - 2 = 2(2 \cos^2 \theta - 1) = 2 \cos 2\theta$ . Huh.

**(b) Compute  $x^n + \frac{1}{x^n}$  in terms of  $n$  and  $\theta$ .**

It's unclear how to start, so we might as well try to compute  $x^3 + \frac{1}{x^3}$  in the same way.

$$\begin{aligned}
\left(x + \frac{1}{x}\right)^3 &= \\
x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} &= 8 \cos^3 \theta \\
x^3 + 3 \underbrace{\left(x + \frac{1}{x}\right)}_{2 \cos \theta} + \frac{1}{x^3} &= 8 \cos^3 \theta \\
x^3 + \frac{1}{x^3} &= 8 \cos^3 \theta - 6 \cos \theta.
\end{aligned}$$

Now what? The astute among you may recognize that  $8 \cos^3 \theta - 6 \cos \theta = 2 \cos 3\theta$ , at which point you could make a conjecture (and could jump ahead). But suppose we didn't find that.

We know that  $x^2 + \frac{1}{x^2} = 2 \cos 2\theta$ . By analogy, if we make the substitution  $y = x^2$  and  $\phi = 2\theta$ , we get that  $y + \frac{1}{y} = 2 \cos \phi$ , and thus  $y^2 + \frac{1}{y^2} = 2 \cos 2\phi \implies x^4 + \frac{1}{x^4} = 2 \cos 4\theta$ . In general,

$$x^{2^m} + \frac{1}{x^{2^m}} = 2 \cos 2^m \theta.$$

The exponent on  $x$ , in this case  $2^m$ , is the same as the multiple of  $\theta$ . Pretty sus. We've solved the problem for  $n$  which are powers of two, but we conjecture that the relationship holds for all integers  $n$ . To be explicit, we want to show that

$$x^n + \frac{1}{x^n} = 2 \cos n\theta.$$

There's a couple of ways to do it. But seeing  $x^n$  and  $\cos n\theta$  in the same place immediately recalls exponentiating  $\text{cis } \theta$ . So, let's try rewriting the problem a bit by entering into the complex plane. Let  $x = r \text{cis } \phi$ , which we really should have done earlier. Then we're given that  $r \text{cis } \phi + \frac{1}{r \text{cis } \phi} = 2 \cos \theta$ . Working further,

$$\begin{aligned}
2 \cos \theta &= r \text{cis } \phi + \frac{1}{r} \text{cis}(-\phi) \\
&= r \cos \phi + \frac{1}{r} \cos \phi + i(r \sin \phi - \frac{1}{r} \sin \phi) \\
&= \left(r + \frac{1}{r}\right) \cos \phi + i \left(r - \frac{1}{r}\right) \sin \phi
\end{aligned}$$

The imaginary part needs to be zero, since the left hand side is real. So either  $r = \frac{1}{r}$  or  $\sin \phi = 0$ . Let's examine each case. In the first case,  $r = 1$  (it can't be  $-1$  since  $r \geq 0$ ). In the second case, we have  $\cos \phi = \pm 1$ , and substituting, we get

$$2 \cos \theta = \pm \left(r + \frac{1}{r}\right).$$

That's not helpful... except now we know that  $r$  is real and  $\geq 0$ . Considering  $r + \frac{1}{r}$ , we see that it approaches  $\infty$  as  $r \rightarrow 0$ . What range of values does it make? Graphing it shows that it has a range of  $[2, \infty)$ , reaching its minimum at  $r = 1$ . Another way to prove this is via AM-GM with  $x = 2r$  and  $y = \frac{2}{r}$ :

$$\frac{x+y}{2} \geq \sqrt{xy} \implies r + \frac{1}{r} \geq \sqrt{2r \left(\frac{2}{r}\right)} = 2.$$

But the range of  $2 \cos \theta$  is  $[-2, 2]$ , and the only possible value of the equation is the intersection of their ranges, aka 2. So  $r = 1$  no matter what. That's damn useful, because then

$$\begin{aligned}
2 \cos \theta &= \left(r + \frac{1}{r}\right) \cos \phi + i \left(r - \frac{1}{r}\right) \sin \phi && \text{From before} \\
&= 2 \cos \phi.
\end{aligned}$$

So  $\cos \phi = \cos \theta$ . We wish to find an expression for  $x^n + \frac{1}{x^n}$ .

$$\begin{aligned}x^n + \frac{1}{x^n} &= r^n \operatorname{cis}^n \phi + \frac{1}{r^n} \operatorname{cis}^n(-\phi) \\&= r^n(\cos n\phi + \sin n\phi) + \frac{1}{r^n}(\cos n\phi - \sin n\phi) \\&= \cos n\phi + \sin n\phi + \cos n\phi - \sin n\phi \\&= 2 \cos n\phi \\&= 2 \cos n\theta.\end{aligned}$$

Note that in the last step, we have to be careful, but cosine does have this property. Anyway, that's a gg, QED.