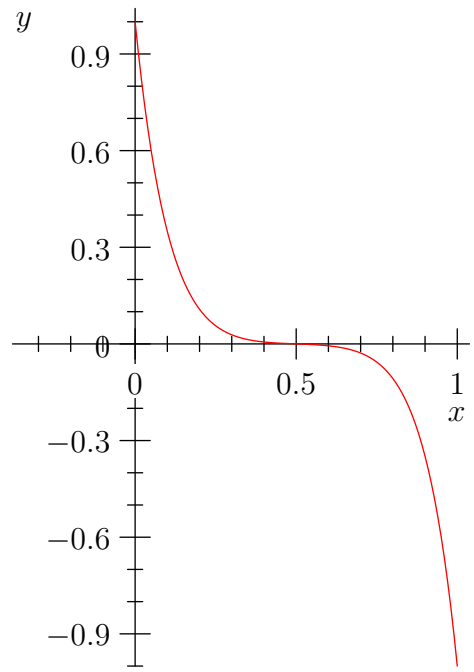


Figure 1: Graph of $P(z)$ from 0 to 1

The roots of the polynomial $P(z) = (z-1)^{10} - z^{10}$ have interesting structure, as we've seen. If we didn't have the inspiration (or, let's be honest, the instruction from Mr. Herreshoff) to represent $z$ in its polar form $r \operatorname{cis} \theta$, what could we do instead to find how many roots it has, and where they are? This is a fun exercise in algebra, and more importantly, an exercise in thinking like a mathematician. I make no claim to completeness or efficiency, but hopefully you'll find instructive the thought process of a reasonably mathematically inclined high schooler.

First of all, we note that $P(z)$ has degree 9, since the degree 10 term of $(z-1)^{10}$ is canceled out. Thus it has at most 9 roots, by the Fundamental Theorem of Algebra. We could try expanding out $P(z)$ and analyzing it further. But before that messiness, are there any real roots? There's clearly a real root at $z = 0.5$. Are there any others? Well, there are many valid ways to verify that there aren't, one of which is graphing it (see Figure **??**). That's what I did. Another way is to examine the function in certain domains $z > 0.5$ and $z < 0.5$ and use inequalities to show the function is either positive or negative—and thus nonzero—in those ranges.

Okay, so $P(0.5) = 0$. Nice. But what about the (up to) 8 other roots just chillin' somewhere in the complex plane? We know little about those. Graphing the polynomial made me notice something: It looks relatively simple, kind of like $x^3$. It's monotonic, meaning it always decreases over time; it has no humps.

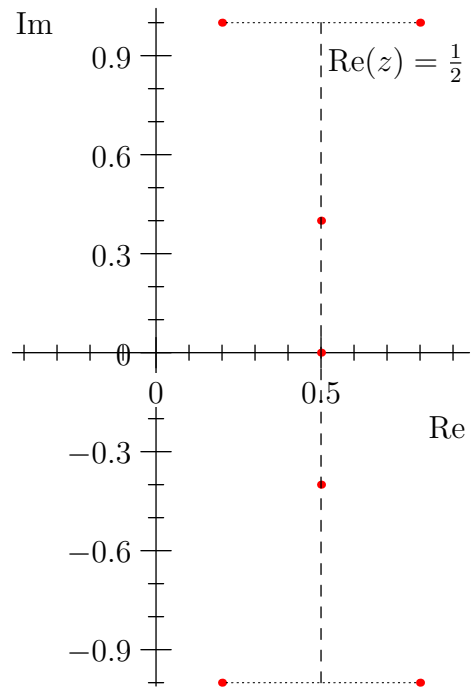


Figure 2: Roots of $P(z)$ must be symmetric about $\mathrm{Re}(z) = \frac{1}{2}$.

And it looks to be somewhat symmetric about $x = \frac{1}{2}$. To make it clearer, let's consider $P(z)$ and its reflection about $x = \frac{1}{2}$, which is $P(1-z)$.

$$\begin{aligned}
P(1-z) &= (1-z-1)^{10} - (1-z)^{10} && \text{Definition of } P(z) \\
&= (-z)^{10} - (1-z)^{10} && \text{Simplifying} \\
&= z^{10} - (z-1)^{10} && \text{Raising to an even power} \\
&= -P(z)
\end{aligned}$$

Indeed, the graph is antisymmetric about $(0.5, 0)$, meaning that it flips sign but looks the same. This identity, $P(1-z) = -P(z)$, holds for all $z$.

One of the most immediate consequences is that the roots of $P(z)$ are symmetric about $\frac{1}{2}$. To see this, suppose $z$ is a root of $P(z)$. Then $P(1-z) = -P(z) = 0$, so $1-z$ is also a root of $P(z)$, which is $z$'s reflection. See Figure **??** for a visualization.

A root is either on the line of symmetry or off it. Let's see what happens when we substitute $z = 0.5 + \xi i$, which is a point on the line of symmetry. I love that Greek letter $\xi$. Yes please.

$$
\begin{aligned}
P(z) &= P(0.5 + \xi i) \\
&= (\xi i + 0.5 - 1)^{10} - (\xi i + 0.5)^{10} && \text{Substitution} \\
&= (\xi i - 0.5)^{10} - (\xi i + 0.5)^{10}
\end{aligned}
$$

Hm. We have our original solution $\xi = 0$ corresponding to $z = 0.5$, but are there any other solutions of this form? Well, we can make things a bit easier by multiplying everything through by $2^{10}$ and substituting $\omega = 2\xi$:

$$
\begin{aligned}
2^{10} \cdot P(z) &= 2^{10} \cdot ((\xi i - 0.5)^{10} - (\xi i + 0.5)^{10}) && \text{Substitution} \\
&= (2\xi i - 1)^{10} - (2\xi i + 1)^{10} \\
&= (\omega i - 1)^{10} - (\omega i + 1)^{10} = Q(\omega)
\end{aligned}
$$

Glorious. Note that the roots of $Q(\omega)$ correspond with roots of $P(z)$, with $z = \frac{1}{2} + \frac{\omega i}{2}$. How can we find the roots of $Q(\omega)$? Well, we can use the binomial theorem to expand. For the $n$th term in the expansion, each term contributes a certain multiple of $(\omega i)^n$:

$$
\begin{aligned}
\sum_{n=0}^{10} \binom{10}{n} \cdot (\omega i)^n \cdot (-1)^{10-n} - \binom{10}{n} \cdot (\omega i)^n \cdot (1)^{10-n} & \\
&= \sum \binom{10}{n} (\omega i)^n ((-1)^{10-n} - 1) \\
&= \sum_{n=1,\, n \text{odd}}^{9} \binom{10}{n} i^n \omega^n && \text{Because } ((-1)^{10-n} - 1) \text{ is 0 for even } n \\
&= i \left( \binom{10}{1} \omega - \binom{10}{3} \omega^3 + \binom{10}{5} \omega^5 - \binom{10}{7} \omega^7 + \binom{10}{9} \omega^9 \right) \\
&= i \cdot R(\omega)
\end{aligned}
$$

We only care about the roots, so we ignore $i$ and graph $R(\omega)$ as in Figure **??**.

Well, there's nine roots. So using the transformation $z = \frac{1}{2} + \frac{\omega i}{2}$, those correspond to the nine roots of our original equation, graphed in Figure **??**. That wasn't too bad.

You'll notice that the roots are symmetric about the real axis. Recall that in the complex plane, roots come in conjugate pairs. In other words, if $P(a+bi) = 0$, then $P(\overline{a+bi}) = P(a - bi) = 0$. The real root(s) conjugate to themselves, and the rest of the roots form four pairs.

But what if we don't have a graphing calculator? And what other information can we find about these roots?

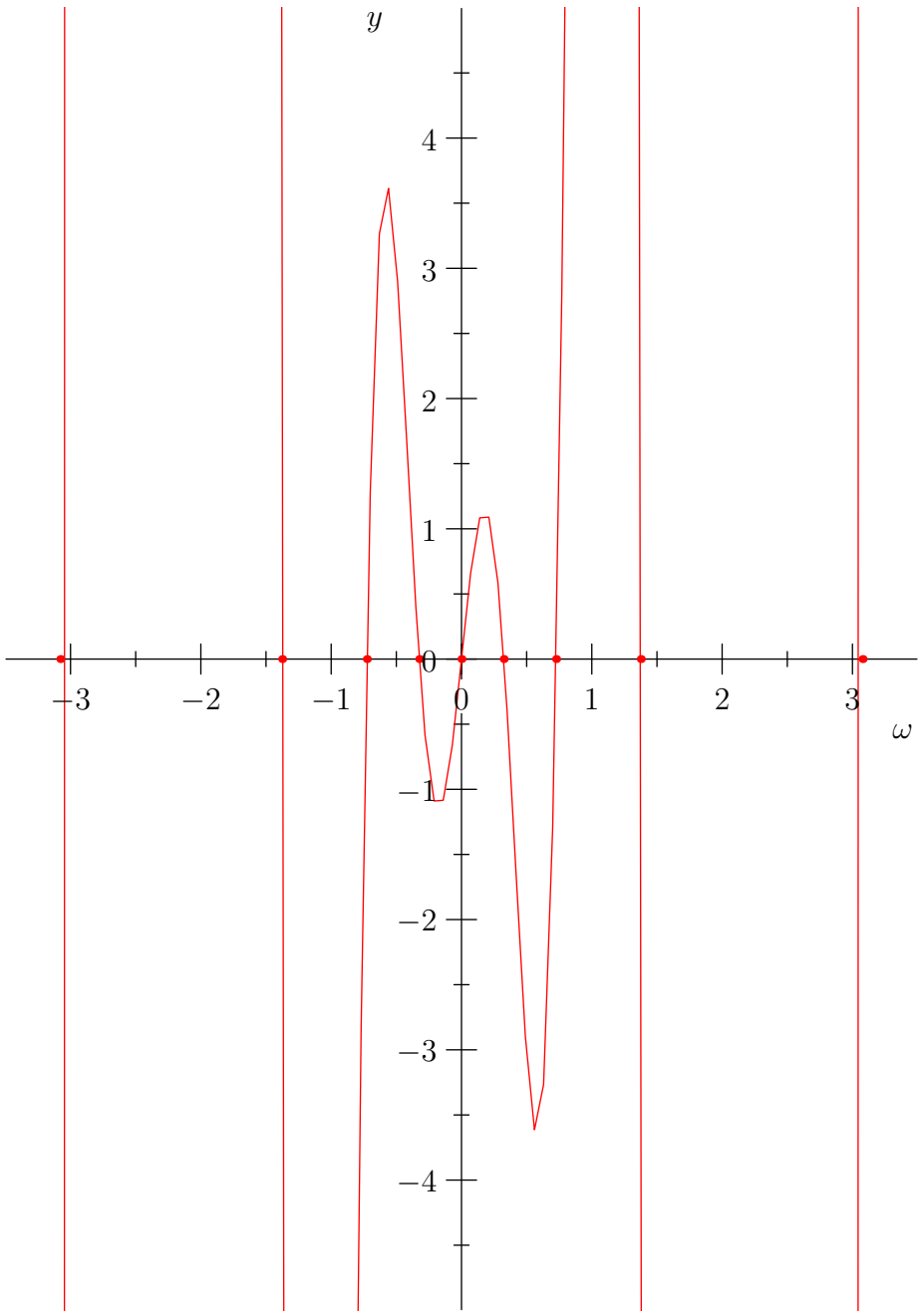


Figure 3: Graph of $R(\omega)$.

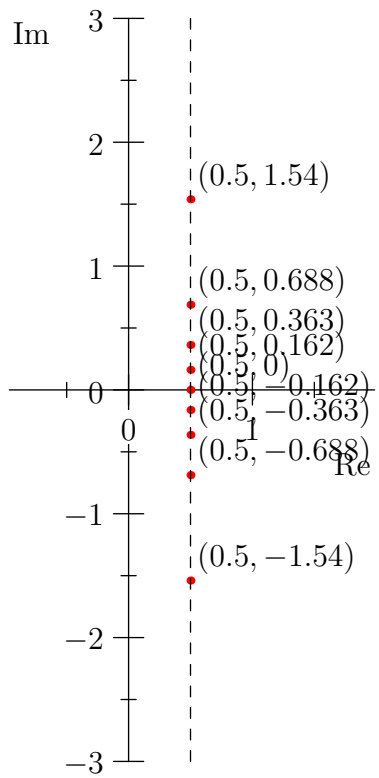


Figure 4: The roots of $P(z)$ in the complex plane.