

7 Your Daily Dose of Vitamin i

1. We will use complex numbers to find identities for \cot . Use Pascal's triangle to expand the following:

(a) $(a + b)^3$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

(b) $(a + b)^4$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

(c) $(a + b)^5$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

1. (cont.) Then substitute $b = i = \sqrt{-1}$ and expand:

(d) $(a + i)^3$

$$(a + i)^3 = a^3 + 3a^2i + 3ab^2 + b^3 = a^3 + 3a^2i - 3a - i.$$

(e) $(a + i)^4$

$$(a + i)^4 = a^4 + 4a^3i + 6a^2b^2 + 4ab^3 + b^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1.$$

(f) $(a + i)^5$

$$(a + i)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i.$$

1. (cont.) Finally, substitute $a = \cot \theta$ and expand:

(g) $(\cot \theta + i)^3$

$$(\cot \theta + i)^3 = a^3 + 3a^2i - 3a - i = (\cot^3 \theta - 3 \cot \theta) + i(3 \cot^2 \theta - 1).$$

(h) $(\cot \theta + i)^4$

$$(\cot \theta + i)^4 = a^4 + 4a^3i - 6a^2 - 4ai + 1 = (\cot^4 \theta - 6 \cot^2 \theta + 1) + i(4 \cot^3 \theta - 4 \cot \theta).$$

(i) $(\cot \theta + i)^5$

$$(\cot \theta + i)^5 = a^5 + 5a^4i - 10a^3 - 10a^2i + 5a + i = (\cot^5 \theta - 10 \cot^3 \theta + 5 \cot \theta) + i(5 \cot^4 \theta - 10 \cot^2 \theta + 1).$$

1. (cont.) Consider $z = i + \cot \theta$.

(j) Use the above results to find identities for (i) $\cot 3\theta$, (ii) $\cot 4\theta$, and (iii) $\cot 5\theta$.

i. $\cot 3\theta$

Given the right triangle formed by $z = i + \cot \theta$ in Figure 7, we have $\tan(\text{Arg } z) = \frac{1}{\cot \theta} = \tan \theta$, so $\text{Arg } z = \theta$ and $z = r \text{cis } \theta$.



Figure 1: $\text{Arg}(i + \cot \theta) = \theta$.

Thus, we have

$$\begin{aligned}\cot 3\theta &= \frac{\cos 3\theta}{\sin 3\theta} \\ &= \frac{\text{Re}(\text{cis } 3\theta)}{\text{Im}(\text{cis } 3\theta)} \\ &= \frac{\text{Re}(r^3 \text{cis } 3\theta)}{\text{Im}(r^3 \text{cis } 3\theta)} \\ &= \frac{\text{Re}(z^3)}{\text{Im}(z^3)}.\end{aligned}$$

We substitute in our expression for z^3 , $(\cot^3 \theta - 3 \cot \theta) + i(3 \cot^2 \theta - 1)$:

$$\cot 3\theta = \frac{\cot^3 \theta - 3 \cot \theta}{3 \cot^2 \theta - 1}.$$

i. $\cot 4\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}\cot 4\theta &= \frac{\cos 4\theta}{\sin 4\theta} \\ &= \frac{\text{Re}(\text{cis } 4\theta)}{\text{Im}(\text{cis } 4\theta)} \\ &= \frac{\text{Re}(r^4 \text{cis } 4\theta)}{\text{Im}(r^4 \text{cis } 4\theta)} \\ &= \frac{\text{Re}(z^4)}{\text{Im}(z^4)} \\ \cot 4\theta &= \frac{\cot^4 \theta - 6 \cot^2 \theta + 1}{4 \cot^3 \theta - 4 \cot \theta}.\end{aligned}$$

i. $\cot 5\theta$

We proceed in the same way as the last subproblem.

$$\begin{aligned}\cot 5\theta &= \frac{\cos 5\theta}{\sin 5\theta} \\ &= \frac{\text{Re}(\text{cis } 5\theta)}{\text{Im}(\text{cis } 5\theta)} \\ &= \frac{\text{Re}(r^5 \text{cis } 5\theta)}{\text{Im}(r^5 \text{cis } 5\theta)} \\ &= \frac{\text{Re}(z^5)}{\text{Im}(z^5)} \\ &= \frac{\cot^5 \theta - 10 \cot^3 \theta + 5 \cot \theta}{5 \cot^4 \theta - 10 \cot^2 \theta + 1}.\end{aligned}$$

(k) Graph z , z^2 , z^3 , z^4 , and z^5 , with $\theta \approx 75^\circ$. What is your solution method?

To graph these, I first calculated the approximate magnitude of z , which is how many times each subsequent power will be scaled by. We have $|1 + \cot 75^\circ| \approx 1.268$, so we only need to scale by about $\frac{5}{4}$ each time. Of course, we rotate by about 75° each time.



Figure 2: Graphs of z , z^2 , z^3 , z^4 , and z^5 .

2. Compute $(1 + i)^n$ for $n = 3, 4, 5, \dots$. Can you find a general pattern?

We have

$$\begin{aligned} (1 + i)^3 &= 1^3 + 3i - 3 - i &= -2 - 2i \\ (1 + i)^4 &= 1^4 + 4i - 6 - 6i + 1 &= -4 - 2i \\ (1 + i)^5 &= 1^5 + 5i - 10 - 10i + 5 + i &= -4 - 4i. \end{aligned}$$

We can find the pattern by representing $1 + i = \sqrt{2}45$. This shows that it has period 8 and let's us find an expression for $(1 + i)^n$:

$$(1 + i)^n = \left(\sqrt{2}45\right)^n = 2^{n/2} \text{cis}\left(\frac{n\pi}{4}\right).$$

3. Expand and graph $\text{cis}^n \theta$ for $n = 2, 3, 4, \dots$

Let $\cos \theta = c$ and $\sin \theta = s$. We have

$$\begin{aligned} (c + is)^2 &= c^2 + 2csi - s^2 = (c^2 - s^2) + i(2cs) \\ (c + is)^3 &= c^3 + 3c^2si - 3cs^2 - s^3i = (c^3 - 3cs^2) + i(3c^2s - s^3) \\ (c + is)^4 &= c^4 + 4c^3si - 6c^2s^2 - 4cs^3i + s^4 = (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3) \\ (c + is)^5 &= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i = (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5). \end{aligned}$$

The graphs of $\text{cis}^n \theta$ for $\theta \approx 50^\circ$ are shown in Figure 3 below.



Figure 3: Graphs of $\text{cis}^n \theta$ for $\theta \approx 50^\circ$.

(a) Why is the real part $\cos n\theta$ and the imaginary part $\sin n\theta$?

By DeMoivre's theorem, $\text{cis}^n \theta = \text{cis } n\theta$, which by definition has $\text{Im}(\text{cis } n\theta) = \cos n\theta$ and $\text{Re}(\text{cis } n\theta) = \sin n\theta$.

(b) Use your results to write identities for $\cos n\theta$ and $\sin n\theta$ for $n = 2, 3, 4, 5$.

Here they are. Again, let $\cos \theta = c$ and $\sin \theta = s$:

$$\cos 2\theta = \text{Re}(\text{cis } 2\theta) = c^2 - s^2$$

$$\cos 3\theta = \text{Re}(\text{cis } 3\theta) = c^3 - 3cs^2$$

$$\cos 4\theta = \text{Re}(\text{cis } 4\theta) = c^4 - 6c^2s^2 + s^4$$

$$\cos 5\theta = \text{Re}(\text{cis } 5\theta) = c^5 - 10c^3s^2 + 5cs^4$$

$$\sin 2\theta = \text{Im}(\text{cis } 2\theta) = 2cs$$

$$\sin 3\theta = \text{Im}(\text{cis } 3\theta) = 3c^2s - s^3$$

$$\sin 4\theta = \text{Im}(\text{cis } 4\theta) = 4c^3s - 4cs^3$$

$$\sin 5\theta = \text{Im}(\text{cis } 5\theta) = 5c^4s - 10c^2s^3 + s^5.$$

4. Compute $7 + 79 + 151 + 223 + 295$ without a calculator. (Hint: what does this have to do with complex numbers?)

These numbers look random, but a closer inspection reveals they are in arithmetic progression, with starting term 7 and increasing 72° each time. That's the rotation of a pentagon!

We rewrite this as the real component of a sum of cises, then manipulate and evaluate:

$$\begin{aligned} 7 + 79 + 151 + 223 + 295 &= \text{Re}(7 + 79 + 151 + 223 + 295) \\ &= \text{Re}((7)(0 + 72 + 144 + 216 + 288)) \\ &= \text{Re}((7)(0)) \\ &= \text{Re}(0) \\ &= 0. \end{aligned}$$

Note that going from the second to third step, we used the fact that the cis expressions are the vertices of a regular pentagon, which sum to 0. If you want to be more formal about it, a fun way to prove that $0 + 72 + 144 + 216 + 288 = 0$ is to set it to Ξ and calculate:

$$\begin{aligned}
\Xi \cdot 72 &= (0 + 72 + 144 + 216 + 288)72 \\
&= 72 + 144 + 216 + 288 + 360 \\
&= 72 + \dots + 288 + 0 \\
&= \Xi.
\end{aligned}$$

If $\Xi \cdot (\text{something that's not one}) = \Xi$, then Ξ must be 0.

5. Factor the following:

(a) $x^6 - 1$ as a difference of squares

We substitute $y = x^3$, giving $y^2 - 1 = (y + 1)(y - 1)$. Substituting back in, we get

$$(x^3 + 1)(x^3 - 1).$$

(b) $x^6 - 1$ as a difference of cubes

We now substitute $y = x^2$, giving $y^3 - 1 = (y - 1)(y^2 + y + 1)$. Substituting back in, we get

$$(x^2 - 1)(x^4 + x^2 + 1)$$

(c) $x^4 + x^2 + 1$ over the real numbers

This one isn't as obvious. We substitute $y = x^2$ to get $y^2 + y + 1$ and find the quadratic's zeroes:

$$y = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}.$$

So it is irreducible over the reals.

(d) $x^6 - 1$ completely

We already broke it down into $(x^3 + 1)$ and $(x^3 - 1)$. Going further, we have $x^3 + 1 = (x + 1)(x^2 - x + 1)$ and $x^3 - 1 = (x - 1)(x^2 + x + 1)$. To break apart the last two quadratics, we find their zeros:

$$x^2 - x + 1 = 0 \implies x = \frac{1 \pm i\sqrt{3}}{2} \implies \left(x - \frac{1 - i\sqrt{3}}{2}\right)\left(x - \frac{1 + i\sqrt{3}}{2}\right).$$

$$x^2 + x + 1 = 0 \implies x = \frac{-1 \pm i\sqrt{3}}{2} \implies \left(x + \frac{1 - i\sqrt{3}}{2}\right)\left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

Combining all these, we get the complete factorization over the complex numbers:

$$x^6 - 1 = (x + 1)\left(x - \frac{1 - i\sqrt{3}}{2}\right)\left(x - \frac{1 + i\sqrt{3}}{2}\right)(x - 1)\left(x + \frac{1 - i\sqrt{3}}{2}\right)\left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

(e) $x^4 + x^2 + 1$ completely

We could do a lot of work again, or we could observe that $x^4 + x^2 + 1 = \frac{x^6 - 1}{x^2 - 1} = \frac{x^6 - 1}{(x + 1)(x - 1)}$. Removing the denominator's terms from our factorization of $x^6 - 1$ we found in the last subproblem, we get

$$x^4 + x^2 + 1 = \left(x - \frac{1 - i\sqrt{3}}{2}\right)\left(x - \frac{1 + i\sqrt{3}}{2}\right)\left(x + \frac{1 - i\sqrt{3}}{2}\right)\left(x + \frac{1 + i\sqrt{3}}{2}\right).$$

6. Let $f(z) = \frac{z+1}{z-1}$.

(a) Without a calculator, compute $f^{2014}(z)$.

This seems terrifying. Let's try computing $f^2(z)$ and perhaps $f^3(z)$.

$$f^2(z) = \frac{f(z) + 1}{f(z) - 1} = \frac{\frac{z+1}{z-1} + 1}{\frac{z+1}{z-1} - 1} = \frac{\frac{2z}{z-1}}{\frac{2}{z-1}} = z.$$

Oh.

Since 2014 is even, we have $f^{2014}(z) = (f^2)^{1007}(z) = z$.

(b) What if you replace 2014 with the current year?

Let y be the current year. As I write this, it is 1492.

If y is even, then $f^y(z) = (f^2)^{y/2}(z) = z$. If y is odd, then $f^y(z) = f((f^2)^{(y-1)/2}(z)) = f(z) = \frac{z+1}{z-1}$.

7. Find $\text{Im}((12 + 48)^6)$.

These are some weird looking angles. Thinking back to some older problems, however, the resultant angle of the addition may be tractable. We draw a diagram, shown in Figure 4.



Figure 4: Adding $12 + 48$.

Consider the isosceles triangle. The apex has angle measure $132^\circ + 12^\circ = 144^\circ$, so the base angles are each $x = \frac{180^\circ - 144^\circ}{2} = 18^\circ$. But $\text{Arg}(12 + 48) = 48^\circ - x = 30^\circ$!

That's a familiar angle. Indeed, we have $z = 12 + 48 = r30$ for some r . It doesn't really matter which r , because

$$\text{Im}((r30)^6) = \text{Im}(r^6 180) = \text{Im}(-r^6) = 0.$$

8. Let x satisfy the equation $x + \frac{1}{x} = 2 \cos \theta$.

(a) Compute $x^2 + \frac{1}{x^2}$ in terms of θ .

Squaring the left hand side will get us some terms that look close to what we want.

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}.$$

So $x^2 + \frac{1}{x^2} = (2 \cos \theta)^2 - 2 = 4 \cos^2 \theta - 2 = 2(2 \cos^2 \theta - 1) = 2 \cos 2\theta$. Huh.

(b) Compute $x^n + \frac{1}{x^n}$ in terms of n and θ .

It's unclear how to start, so we might as well try to compute $x^3 + \frac{1}{x^3}$ in the same way.

$$\begin{aligned}
\left(x + \frac{1}{x}\right)^3 &= \\
x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} &= 8 \cos^3 \theta \\
x^3 + 3 \underbrace{\left(x + \frac{1}{x}\right)}_{2 \cos \theta} + \frac{1}{x^3} &= 8 \cos^3 \theta \\
x^3 + \frac{1}{x^3} &= 8 \cos^3 \theta - 6 \cos \theta.
\end{aligned}$$

Now what? The astute among you may recognize that $8 \cos^3 \theta - 6 \cos \theta = 2 \cos 3\theta$, at which point you could make a conjecture (and could jump ahead). But suppose we didn't find that.

We know that $x^2 + \frac{1}{x^2} = 2 \cos 2\theta$. By analogy, if we make the substitution $y = x^2$ and $\phi = 2\theta$, we get that $y + \frac{1}{y} = 2 \cos \phi$, and thus $y^2 + \frac{1}{y^2} = 2 \cos 2\phi \implies x^4 + \frac{1}{x^4} = 2 \cos 4\theta$. In general,

$$x^{2^m} + \frac{1}{x^{2^m}} = 2 \cos 2^m \theta.$$

The exponent on x , in this case 2^m , is the same as the multiple of θ . Pretty sus. We've solved the problem for n which are powers of two, but we conjecture that the relationship holds for all integers n . To be explicit, we want to show that

$$x^n + \frac{1}{x^n} = 2 \cos n\theta.$$

There's a couple of ways to do it. But seeing x^n and $\cos n\theta$ in the same place immediately recalls exponentiating $\text{cis } \theta$. So, let's try rewriting the problem a bit by entering into the complex plane. Let $x = r \text{cis } \phi$, which we really should have done earlier. Then we're given that $r \text{cis } \phi + \frac{1}{r \text{cis } \phi} = 2 \cos \theta$. Working further,

$$\begin{aligned}
2 \cos \theta &= r \text{cis } \phi + \frac{1}{r} \text{cis}(-\phi) \\
&= r \cos \phi + \frac{1}{r} \cos \phi + i(r \sin \phi - \frac{1}{r} \sin \phi) \\
&= \left(r + \frac{1}{r}\right) \cos \phi + i \left(r - \frac{1}{r}\right) \sin \phi
\end{aligned}$$

The imaginary part needs to be zero, since the left hand side is real. So either $r = \frac{1}{r}$ or $\sin \phi = 0$. Let's examine each case. In the first case, $r = 1$ (it can't be -1 since $r \geq 0$). In the second case, we have $\cos \phi = \pm 1$, and substituting, we get

$$2 \cos \theta = \pm \left(r + \frac{1}{r}\right).$$

That's not helpful... except now we know that r is real and ≥ 0 . Considering $r + \frac{1}{r}$, we see that it approaches ∞ as $r \rightarrow 0$. What range of values does it make? Graphing it shows that it has a range of $[2, \infty)$, reaching its minimum at $r = 1$. Another way to prove this is via AM-GM with $x = 2r$ and $y = \frac{2}{r}$:

$$\frac{x+y}{2} \geq \sqrt{xy} \implies r + \frac{1}{r} \geq \sqrt{2r \left(\frac{2}{r}\right)} = 2.$$

But the range of $2 \cos \theta$ is $[-2, 2]$, and the only possible value of the equation is the intersection of their ranges, aka 2. So $r = 1$ no matter what. That's damn useful, because then

$$\begin{aligned}
2 \cos \theta &= \left(r + \frac{1}{r}\right) \cos \phi + i \left(r - \frac{1}{r}\right) \sin \phi && \text{From before} \\
&= 2 \cos \phi.
\end{aligned}$$

So $\cos \phi = \cos \theta$. We wish to find an expression for $x^n + \frac{1}{x^n}$.

$$\begin{aligned}x^n + \frac{1}{x^n} &= r^n \operatorname{cis}^n \phi + \frac{1}{r^n} \operatorname{cis}^n(-\phi) \\&= r^n(\cos n\phi + \sin n\phi) + \frac{1}{r^n}(\cos n\phi - \sin n\phi) \\&= \cos n\phi + \sin n\phi + \cos n\phi - \sin n\phi \\&= 2 \cos n\phi \\&= 2 \cos n\theta.\end{aligned}$$

Note that in the last step, we have to be careful, but cosine does have this property. Anyway, that's a gg, QED.