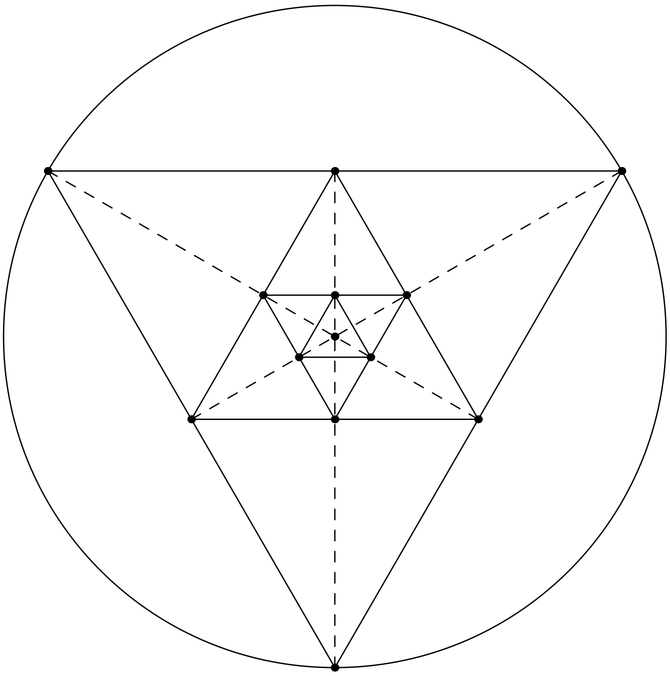

# A Geometric Approach To Matrices

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A Geometric Approach To Matrices



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# 1 Trigonometry Review

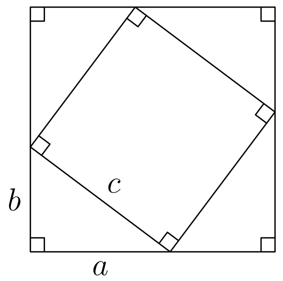



Figure 1: Scenario in Problem 1.

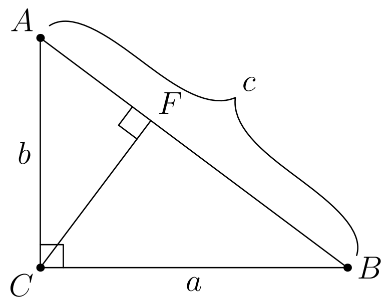



Figure 2: Scenario in Problem 2.

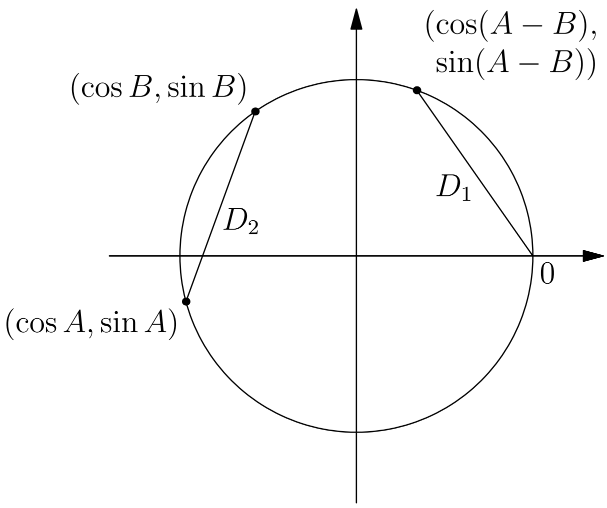



Figure 3: Scenario in Problem 3.

This is a review of material you learned last year which you will need as background knowledge for our upcoming study of linear algebra. If you don't know this material already, make sure to learn it.

1. Prove the Pythagorean theorem using "conservation of area." Start with Figure 1.

2. Prove the Pythagorean theorem using a right triangle with an altitude drawn to its hypotenuse, as shown in Figure 2, making use of similar right triangles.

3. We now prove the trigonometric identities.

   (a) Draw and label a right triangle and a unit circle, then write trig definitions for $\cos$, $\sin$, $\tan$, and $\sec$ in terms of your drawing.

   (b) Use a right triangle and the definitions of $\sin$ and $\cos$ to find and prove a value for $\sin^2\theta + \cos^2\theta$.

   (c) Use the picture of the unit circle in Figure 3 to find and prove a value for $\cos(A-B)$. Note that $D_1$ and $D_2$ are the same length because they subtend the same size arc of the circle. Set them equal and work through the algebra, using the distance formula and part (b) of this problem.

4. Write down as many trig identities as you can—no need to prove these.

$$\begin{array}{rrr}
\sin(A+B) = & \sin(A-B) = & \cos(A+B) = \\
\tan(A+B) = & \tan(A-B) = & \sin(2A) = \\
\cos(2A) = & \tan(2A) = & \sin\left(\frac{A}{2}\right) = \\
\cos\left(\frac{A}{2}\right) = & \tan\left(\frac{A}{2}\right) = &
\end{array}$$

5. Let's review complex numbers and DeMoivre's theorem.

(a) Recall that you can write a complex number both in Cartesian and polar forms. Let

$$a + bi = (a, b) = (r\cos\theta, r\sin\theta) = r\cos\theta + ir\sin\theta.$$

What is $r$ in terms of $a$ and $b$?

(b) Expand $(a + bi)(c + di)$ the usual way.

(c) Let $a + bi = r_1(\cos\theta + i\sin\theta)$ and $c + di = r_2(\cos\phi + i\sin\phi)$. Multiply them, and use the angle addition formulas to show that multiplying two complex numbers involves multiplying their lengths and adding their angles. This is DeMoivre's theorem!

(d) Use part (c) to simplify $(\sqrt{3} + i)^{18}$.

6. Here is a review of 2D rotation.

   (a) Recall that we can graph complex numbers as ordered pairs in the complex plane. Now, consider the complex number $z = \cos\theta + i\sin\theta$, where $\theta$ is fixed. What is the magnitude of $z$?

   (b) Multiplying $z \cdot (x + yi)$ yields a rotation of the point $(x, y)$ counterclockwise around the origin by the angle $\theta$. Notice that rotating the graph counterclockwise around the origin has the same effect as rotating the coordinate axes clockwise around the origin by the same angle $\theta$. What if we wanted to rotate clockwise by $\theta$ instead?

7. Rotate the following conics by (i) $30°$, (ii) $45°$, and (iii) $\theta$:

   (a) $x^2 - y^2 = 1$ (b) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (c) $y^2 = 4Cx$

You should have mastery of this material so that we can immediately investigate novel and interesting ideas. These often have surprising connections to the trigonometry and transformational geometry you learned last year. For example, we will soon find another convenient way to do a rotation of coordinates.

## 2 It's a Snap

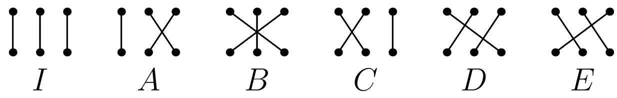



Figure 1: The six possibilities for connections between two rows of three posts.

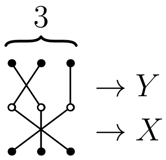



Figure 2: A grid with three strings.

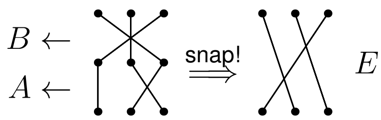



Figure 3: $A \bullet B = E$.

We begin with a problem that ties together ideas from geometry, complex numbers, matrices, combinatorics, and group theory. You have studied geometry, complex numbers, and combinatorics before, so you should have a basis to start this investigation, but the other two topics may be a bit unfamiliar at this point.

Figure 1 shows the six ways the posts in two rows, each row containing three posts, can be paired up. Convince yourself that these are the only six. Let's label them $I$, $A$, $B$, $C$, $D$, and $E$. For example, $I$ pairs every post in the top row with the post directly below it, while $A$ switches the pairings of the second and third posts.

Consider a grid of posts with three rows and three columns. An elastic string is anchored to one post in the top row, looping around a post in the middle row, and finally descending to a post in the bottom row. Two other strings are anchored and looped in the same way, with the condition that each post has exactly one string touching it. An example is depicted in Figure 2.

Notice that we can easily represent such a grid of posts with two of our six elements. You should have two stacked elements, $X$ and $Y$, with $Y$ on top of $X$, as shown in Figure 2. When you remove the middle posts (the posts indicated by $\circ$), the elastic string will **snap** to one of the six configurations we drew initially. Let's call this operation "snap," or $\bullet$, so that $X \bullet Y$ reads "$X$ snap $Y$." Keep in mind that when evaluating the snap operation, the *bottom* configuration $X$ goes first, and the *top* configuration $Y$ goes last. In Figure 2, $X = B$ and $Y = C$, and $X \bullet Y$ makes $E$, so $B \bullet C = E$. As another example, $A \bullet B = E$, as shown in Figure 3.

Why do we put the bottom configuration first and the top configuration last? This choice is somewhat arbitrary, but there is a reason. Remember that when we compose functions, we write $(f \circ g)(x) = f(g(x))$. The right function, $g$, is evaluated first and used as an input to the left function, $f$. Similarly, when we write $X \bullet Y$, the overall configuration (from top to bottom) first goes through $Y$, then through $X$. As we will see, these six elements often behave more like functions, rather than simply elements. Thus, it is natural to order them as if they were functions.

Some important terminology: these six configurations form a mathematical **group** under the operation $\bullet$, and we say that each configuration is an **element** of our group. More specifically, we will call this group the three-post **snap group**, or $S_3$. There are countless other mathematical groups, so we must be precise when we talk about a specific group. Note that this snap group is denoted $S_3$, not $S_6$, because the subscript $3$ is the number of posts in each row, *not* the size of the group. Groups are, unsurprisingly, the main objects studied in **group theory**. Let's study this snap group and characterize its properties.

1. Fill out a $6 \times 6$ table like the one in Figure 5, showing the results of each of the $36$ possible snaps, where $X \bullet Y$ is in $X$'s row and $Y$'s column. $A \bullet B = E$ is done for you.

2. Which of the elements is the **identity element** $K$, such that $X \bullet K = K \bullet X = X$ for all $X$?

3. Does every element have an inverse? In other words, can you get to the identity element from every element using only one snap?

4. (a) Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all $X, Y$)?

   (b) Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all $X, Y, Z$)?

5. (a) For any elements $X, Y$, is there always an element $Z$ so that $X \bullet Z = Y$?

   (b) For (a), is $Z$ always unique?

| $\bullet$ | $I$ | $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|---|---|
| $I$ | | | | | | |
| $A$ | | | $E$ | | | |
| $B$ | | | | | | |
| $C$ | | | | | | |
| $D$ | | | | | | |
| $E$ | | | | | | |

Figure 4: Unfilled $3$-post snap group table.

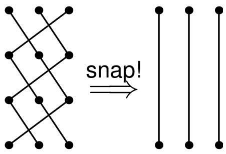



Figure 5: $E \bullet E \bullet E = I$; $E$ has period $3$.

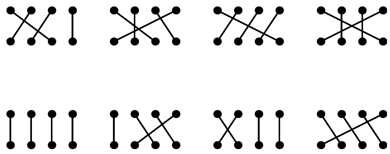


Figure 6: Some $4$-post group elements.

6. If you constructed a $5 \times 5$ table using only five of the snap elements, the table would not describe a group, because there would be entries in the table outside of those $5$. Therefore, a group must be **closed** under its operation. *Some* subsets of our six elements, however, do happen to be closed. Write valid group tables using exactly one, two, and three elements from the snap group. These are known as **subgroups**.

7. What do you guess is a good definition of a mathematical group? (Hint: consider your answers to Problems 2–6.)

8. Notice that $E \bullet E \bullet E = I$ (see Figure 5). This means that $E$ has a **period** of $3$ when acting upon itself. Which elements have a period of

   (a) $1$?  (b) $2$?  (c) $3$?

9. Answer the following with the one, two, and four-post snap groups $S_1$, $S_2$ and $S_4$. These are just the analogous groups for connections between one, two, and four posts.
   (a) How many elements would there be?
   (b) Systematically draw and name them.
   (c) Make a group table of these elements. For four posts, instead of creating the massive table, give the number of elements that the table would have.
   (d) What is the relationship of your original table to this new table?

10. Can you think of a shortcut to generate a snap group table without drawing every possible configuration?

11. (a) How many elements would there be in the five-post snap group?
   (b) How many entries would its table have?
   (c) What possible periods would its elements have? Make sure you include a period of six!
   (d) Extend your answers for Problems 11a through 11c to $M$ posts per row.

12. A **permutation** of a set of things is an order in which they can be arranged. What is the relationship between the set of permutations of $m$ things and the $m$-post snap group?

# 3 From Snaps to Flips

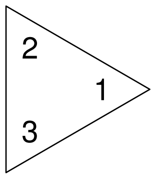



Figure 1: The paper triangle.

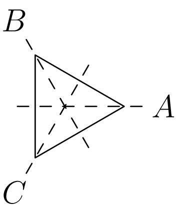



Figure 2: Its axes of reflection.

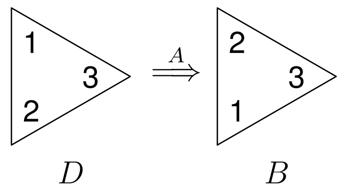



Figure 3: $AD = B$; Notice the RTL evaluation.

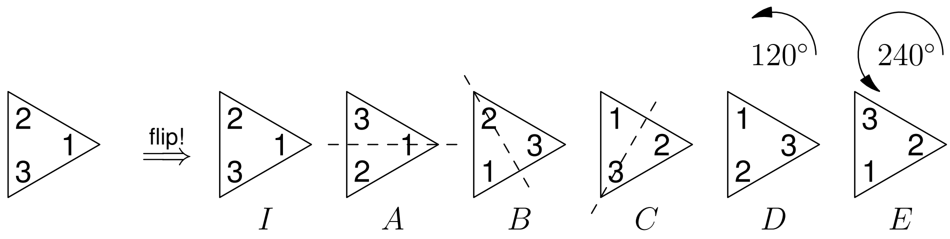



Figure 4: The six ending positions.

Please use a paper or cardboard triangle to help visualize the next concept. Cut out an equilateral triangle, label its front vertices $1$, $2$, and $3$ as shown in Figure 1, and place it down in the shown orientation. Consider the possible ways to transform this triangle, while preserving the position of its shape on the plane. From this starting position, you can reflect the triangle over one of three axes: $A$, $B$, or $C$, as shown in Figure 2. You could also rotate the triangle $120°$ or $240°$ counterclockwise. The final possible positions are shown in Figure 4.

Notice that each position corresponds to a different transformation which preserves the triangle's location. For example, $I$ means "leave the triangle alone," $A$ means "flip the triangle about the $A$ axis," and $D$ means "rotate the triangle $120°$ counterclockwise." We can combine these operations to form other operations by sequencing them, which we write like multiplication. We evaluate them right-to-left (RTL) as we did for the snap group. For example, $AD = B$, as shown in Figure 3.

These six positions, along with the operation $\cdot$ of transformation composition, form another group: the **dihedral group** of the equilateral triangle, or $D_3$. It is denoted $D_3$ because it is a dihedral group of a $3$-sided regular polygon. If we split "dihedral" into "di-" and "-hedral," we see it means "two faces"; this etymology refers to the two faces of our paper triangle.

Dihedral groups exist for any polygonal figure; mathematically speaking, the dihedral group $D_n$ is the group of symmetries, or **symmetry group**, of a $n$-sided regular polygon. While shapes in any number of dimensions have symmetry groups, the symmetry groups of plane figures are specifically called dihedral groups. For example, a pentagon's symmetry group is a dihedral group, while a cube's symmetry group is not. Let's study the properties of the dihedral group of the equilateral triangle.

1. Is the list of six operations complete? (Are there any other isometries of the equilateral triangle that preserve its shape and location?)

2. As with the snap group, we can make a group table for the dihedral group. Fill out a table like the one in Figure 5 in your notebook. Like the snap group table, the top row indicates what operation is done first and the left column indicates what's done second. In other words, $XY$ is in the $X$'s' row and $Y$'s column. $AD = B$ is done for you.

| $\cdot$ | $I$ | $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|---|---|
| $I$ | | | | | | |
| $A$ | | | | | $B$ | |
| $B$ | | | | | | |
| $C$ | | | | | | |
| $D$ | | | | | | |
| $E$ | | | | | | |

Figure 5: Unfilled $D_3$ group table.

3. What is the relationship between the tables for the snap group $S_3$ and the dihedral group $D_3$?

$S_3$ and $D_3$ are said to be **isomorphic**. Groups $A$ with operation $\bullet$ and $B$ with operation $\star$ are isomorphic if you can find a one-to-one correspondence between the two groups' elements, where the results of each group's operation on corresponding elements also correspond. This means we can find some pairing of elements between the two groups $A_1 \leftrightarrow B_1, A_2 \leftrightarrow B_2, \cdots, A_n \leftrightarrow B_n$ such that $A_j \bullet A_k = A_l \leftrightarrow B_j \star B_k = B_l$, for all $j, k, l$.

| $\cdot$ | $I$ | $r$ | $r^2$ | $f$ | $fr$ | $fr^2$ |
|---|---|---|---|---|---|---|
| $I$ | | | | | | |
| $r$ | | | | $fr^2$ | | |
| $r^2$ | | | | | | |
| $f$ | | | | | | |
| $fr$ | | | | | | |
| $fr^2$ | | | | | | |

Figure 6: Unfilled alternate $D_3$ table.

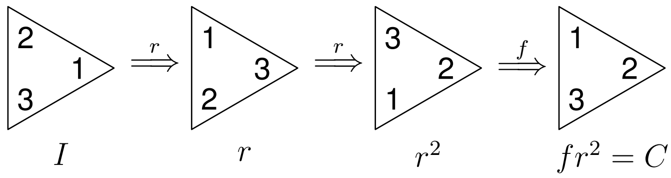



Figure 7: $fr^2 = C$. Again, notice the RTL evaluation.

4. Check your understanding by defining isomorphic in your own words.
5. (a) Make a table for only the rotations of $D_3$, a subgroup of $D_3$.
   (b) Which subgroup of the snap group $S_3$ is isomorphic to the subgroup in (a)?
6. What shape's symmetry group is isomorphic to (a) the two-post snap group $S_2$, (b) one-post group $S_1$, (c) four-post group $S_4$ (Hint: it's not a square), and (d) five-post group $S_5$?
7. Find a combination of $A$ and $D$ that yields $C$.
8. We call $A$ and $D$ **generators** of the group because every element of the group is expressible as some combination of $A$s and $D$s. For convenience, let's call $A$ "$f$" since it's a flip, and call $D$ "$r$" meaning a $120°$ rotation counterclockwise. Then, for example, $fr^2$ is a rotation of $2 \cdot 120° = 240°$, followed by a flip across the $A$ axis, equivalent to our original $C$ (see Figure 7). Make a new table using $I$, $r$, $r^2$, $f$, $fr$, and $fr^2$ as elements, like the one in Figure 6. *Note that the element order is different!*
9. What other pairs of elements could you have used to generate the table?
10. Notice the $3 \times 3$ table of a group you've already described in the top-left corner of your table. What is it, and what are the two possible generators of this three-element group?
11. Explain why each element of the dihedral group $D_3$ has the period it has.
12. Some pairs of elements of the dihedral group are two-element subgroups. Which pairs are they?
13. One of the elements forms a one-element subgroup. Which is it?

A **group** $G$ is a set of elements together with a **binary operation** that meets the following criteria:

(a) Identity: There is an element $I \in G$ such that for all $X \in G$, $X \bullet I = I \bullet X = X$.

(b) Closure: If $X, Y$ are elements of the group, then $X \bullet Y$ is also an element of the group.

(c) Invertibility: Each element $X$ has an inverse $X^{-1}$ such that $X \bullet X^{-1} = X^{-1} \bullet X = I$.

(d) Associativity: For all elements $X$, $Y$, and $Z$, $X \bullet (Y \bullet Z) = (X \bullet Y) \bullet Z$.

14. The addition of two numbers is a binary operation, while the addition of three numbers is not. In logic, $\wedge$ (and) and $\vee$ (or) are binary operations, but $\neg$ (not) is not. Define binary operation in your own words, and name some other binary operations.
15. In your original dihedral group table, what is

    (a) the identity element? (b) the inverse of $A$? (c) the inverse of $E$?

# 4 Rotation and Reflection Groups

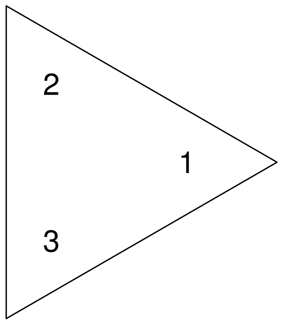



Figure 1: The paper triangle.

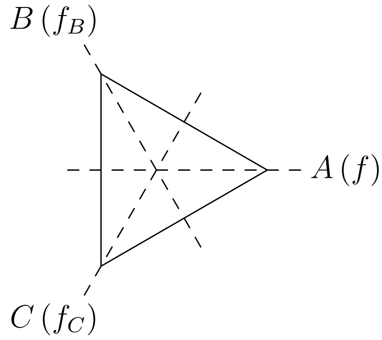



Figure 2: Its axes of reflection.

In the previous section, we started with the dihedral group of the equilateral triangle and discovered it had six elements: reflections about three different axes, rotations of $\pm 120°$, and the identity transformation. We identified a subgroup consisting of the identity $I$ with two rotations $r$ and $r^2$, and three other subgroups of just the identity and a single reflection. The first subgroup—the one consisting of only rotations—is known as the **rotation group** of the equilateral triangle, or the **cyclic group** of order $3$, $C_3$.

1. Notice that the original dihedral group had twice as many elements as the rotation group. Why?
2. Make and justify a conjecture extending this observation to the dihedral groups of other shapes like rectangles, squares, and hexagons, as well as the symmetry group of the cube.
3. Let $r$ be a $180°$ rotation, $x$ be a reflection over the $x$ axis, and $y$ be a reflection over the $y$ axis. Write a table for the dihedral group of the rectangle, recalling that the allowed isometries are reflections and rotations. How does this table differ from the dihedral group of the equilateral triangle?
4. Write a table for the rotation group of the square, with $4$ elements and $16$ entries. Compare this table to Problem 3.

We noticed that the rotation group for the equilateral triangle could be generated by just one of the elements, such as $r$—rotation by $120°$ counterclockwise. Then $r^2$ is a rotation of $240°$ counterclockwise, and $r^3 = I$, the identity (see Figure 3). Since we can generate the entire rotation group with a single element $r$, a natural question to ask is whether we can do the same with the dihedral group $D_3$. Clearly, we can't use the identity to do it, and a series of rotations always leaves us with a rotation, never a reflection. Also, a series of flips along one axis simply generates a two-member group with elements $I$, $f$ (see Figure 4).

Let's try using two elements to generate our group, using the same definitions of $f$ and $r$ as in the previous section: a flip over the $A$ axis and rotation by $120°$ counterclockwise, respectively. As we found, $fr$ is a flip over the $B$ axis and $rf$ is a flip over the $C$ axis. Consecutive powers of $r$ already got us the remaining elements, so $r, f$ generates the group.

We can also generate the group using two reflections, say $f$ and $f_B$ (flip over the $B$ axis, as shown in Figure 2). Notice that an even number of reflections always results in a rotation—even the identity element $I$ is just a rotation by $0$.[1] We can think of this as the existence of a "mirror world" and its unmirrored counterpart, and each reflection takes us into or out of the mirror world.

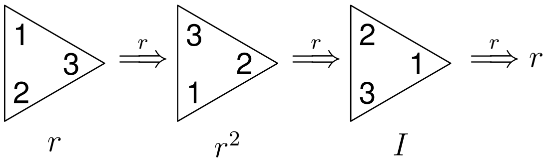



Figure 3: $r$ generates a three member group.

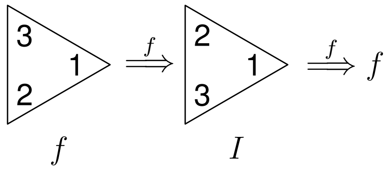



Figure 4: $f$ generates a two member group.

[1] Any reflection group will include rotations, though they may be the identity.

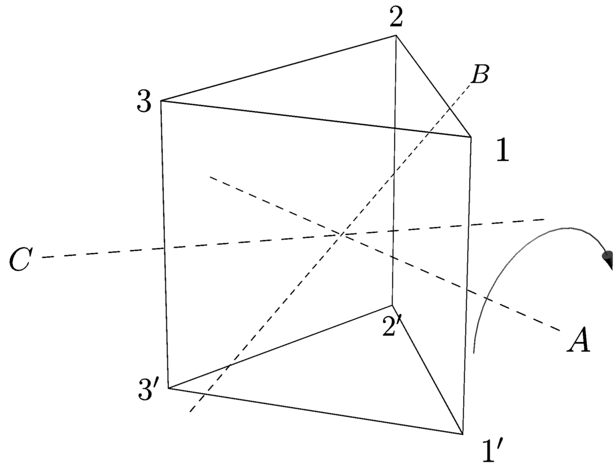



Figure 5: Triangular prism's corresponding axes of rotation.

Moving to three dimensions, $D_3$ is isomorphic to the set of rotations of an equilateral triangular prism. The new rotation axes are coplanar with where the reflection axes used to be (see Figure 5). Indeed, when you "flipped" your equilateral triangles, you were actually rotating a paper-thin triangular prism in the third dimension. Truly flipping the triangular prism using a rotation would require four spatial dimensions—something we cannot easily visualize.

You will next analyze the symmetries of a variety of objects under rotations or reflections. You will notice that the more symmetries an object has, the larger its symmetry group is. Indeed, group theory is the mathematics of symmetry *par excellence*.

For each of the following groups, find the following:

(a) The number of elements; this is known as the **order**. More formally known as **cardinality**

(b) If order $< 10$, name the set of elements; otherwise, explain how you know the order

(c) A smallest possible **generating set**; in other words, a list of elements which generate a group[2]

(d) Whether the group is **commutative**; in other words, whether its operation $\cdot$ satisfies $X \cdot Y = Y \cdot X$ for all $X, Y$

If two problems have isomorphic groups, just write "isomorphic to Problem N" for the latter problem and move on.

5. Rectangle under rotation
6. Rectangle under reflection
7. Square under rotation
8. Square under reflection
9. Square prism under rotation
10. Square prism under reflection
11. Regular pentagon under rotation
12. Regular pentagon under reflection
13. Regular pentagonal prism under rotation
14. Regular pentagonal prism under reflection
15. Regular pentagonal pyramid under rotation
16. Regular pentagonal pyramid under reflection
17. Regular tetrahedron (triangular pyramid) under rotation
18. Regular tetrahedron under reflection
19. Cube under rotation
20. Cube under reflection

[2]There may be multiple generating sets of the same size.

# 5 Infinite Groups

All of the groups we've seen so far are finite in size. We can also construct groups of an infinite size; these are known simply as **infinite groups**.

A quick review: *iso-* means the same and *-morphic* means form. Two groups are said to be isomorphic if there exists a one-to-one correspondence which takes each element of the first group to an element of the second group—and vice versa—so that the products of the elements map in the same way. Essentially, by renaming the elements of one group as the other, the two groups appear identical. Isomorphic groups have the same structure and size, and the group table's structure is also preserved.

1. Where have you come across the roots *iso-* and *-morphic* before?
2. Can two groups be isomorphic if they have different orders?
3. The rotation group of the regular hexagon, also known as the cyclic group of order $6$, $C_6$, has six elements: the identity, and rotations of $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\pi$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$ radians. A rotation of $\frac{\pi}{3}$ generates the group.
   (a) Which other rotation can generate the group?
   (b) What is the period of each element?
4. $C_6$ has the same number of elements as the dihedral group $D_3$.
   (a) Are the two groups isomorphic? How do you know?
   (b) What is the period of each element of $D_3$?
   (c) What can you say if the sets of the periods of the elements of each group are not the same?
   (d) Which subgroups of the cyclic group $C_6$ and $D_3$ are isomorphic?
5. Could an infinite group and a finite group be isomorphic?
6. Do you think all infinite groups are isomorphic to each other? Find a counterexample if you can.

If an infinite group was somehow "bigger" than another, they wouldn't be isomorphic.[3] This raises the question: are all infinities equally big?

We can formalize the notion of sizes of infinity. Let's say that two infinite sets are of the same size if their elements can be put into a one-to-one correspondence (formally known as a **bijection**) with each other. For example, the natural numbers $\{1, 2, ...\} = \mathbb{N}$ and negative integers $\{-1, -2, ...\} = -\mathbb{N}$ are of the same size because we have the one-to-one correspondence $\mathbb{N} \ni n \leftrightarrow -n \in -\mathbb{N}$. $1$ is paired with $-1$, $2$ is paired with $-2$, $3$ is paired with $-3$, and so on. Every element of the positive integers has exactly one "partner" in the negative integers, and vice versa.

7. Make guesses to the relative sizes of the following pairs of sets. You may use shorthand like $|a| < |b|$, $|a| > |b|$, $|a| = |b|$. After you have made your guesses, we will analyze some of the cases and you can find out how good your intuition was.

| | | | |
|---|---|---|---|
| (a) | natural numbers, $\mathbb{N}$ | vs. | positive even numbers, $2\mathbb{N}$ |
| (b) | natural numbers, $\mathbb{N}$ | vs. | positive rational numbers, $\mathbb{Q}^+$ |
| (c) | natural numbers, $\mathbb{N}$ | vs. | real numbers between zero and one, $[0, 1)$ |
| (d) | real numbers, $\mathbb{R}$ | vs. | complex numbers, $\mathbb{C}$ |
| (e) | real numbers, $\mathbb{R}$ | vs. | points on a line |
| (f) | points on a line | vs. | points on a line segment |
| (g) | points on a line | vs. | points on a plane |
| (h) | rational numbers, $\mathbb{Q}$ | vs. | Cantor set[4] |

It turns out that studying infinity involves some strange mathematics. For instance, even though it seems that there should be half as many positive even numbers as natural numbers (see 7a), we can construct a one-to-one correspondence between the two sets, such that every natural number $n$ is paired with a positive even integer $2n$ and vice versa. The existence of this correspondence means that the two sets are equal in size. In symbols,

$$(2\mathbb{N} \ni 2n \leftrightarrow n \in \mathbb{N}) \leftrightarrow |2\mathbb{N}| = |\mathbb{N}|.$$

[3] After reading this section, can you think of an example of two infinite groups that aren't isomorphic?

[4] To construct the Cantor set, begin with the unit segment $[0, 1]$ and delete the middle third, resulting in $[0, 1/3] \cup [2/3, 1]$. Infinitely repeat this process on every subinterval, so that the total length of the set goes to $0$.

More surprisingly, we can establish a one-to-one correspondence between the non-negative rational numbers $\mathbb{Q}_{\geq 0}$ and the natural numbers. Draw the numbers $\mathbb{Q}_{\geq 0}$ in a grid as shown in Figure 1, and pair these numbers up with the numbers $1, 2, 3, ...$ in the pattern shown with the arrows. You can see that you will eventually list all of the non-negative rational numbers, multiple times, into a correspondence with the natural numbers. To make it one-to-one, only pair the rational numbers that are in simplest form. Here, we pair $2$ with $\frac{1}{1}$ instead of $\frac{0}{2}$, since $\frac{0}{1}$ is the same number and is already paired with $1$. This correspondence is depicted in Figure 2. This prevents multiple natural numbers from being paired up with the same rational number: the correspondence is now one-to-one.

The real numbers between $0$ and $1$, however, cannot be put into a one-to-one correspondence with $\mathbb{N}$ (see 7c). We will prove this with contradiction. Suppose I told you that I have paired each real number $0 \leq r_k < 1$ with a unique natural number $k$, and vice versa. Then, you can construct a real number $r_\omega$[5] whose $1^{\text{st}}$ digit (after the decimal point) differs from $r_1$'s, whose $2^{\text{nd}}$ digit differs from $r_2$'s, and so on. In other words, it differs from $r_n$ in the $n^{\text{th}}$ digit. We can make better sense of this construction by writing the numbers out in a table, as shown in Figure 3. Your new number $r_\omega$ is at the bottom; it differs from all the previous numbers in at least one place, so it is a new real number. Therefore, my original list is incomplete, and such a correspondence doesn't exist. This is called a **diagonalization argument** because the differing digits make a diagonal.

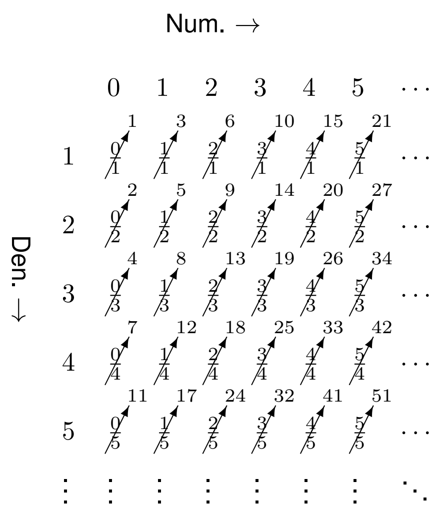



Figure 1: A correspondence between $\mathbb{Q}^+$ and $\mathbb{N}$, but not one-to-one.

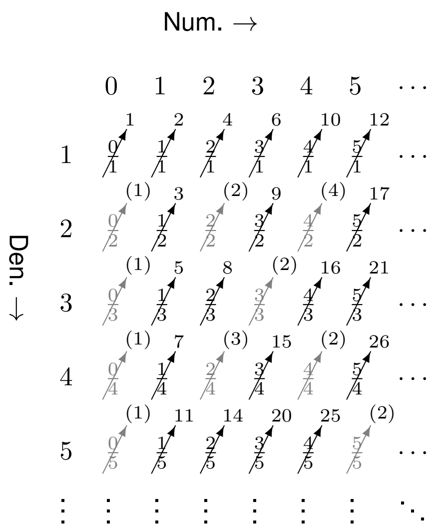



Figure 2: The previous correspondence, with duplicates parenthesized but not counted; it is now one-to-one.

Decimal Expansion →

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r_1$ | $0$ | 0. | $\boxed{0}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ⋯ |
| $r_2$ | $\frac{1}{4}$ | 0. | 2 | $\boxed{5}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ⋯ |
| $r_3$ | $\frac{\pi}{4}$ | 0. | 7 | 8 | $\boxed{5}$ | 3 | 9 | 8 | 1 | 6 | 3 | 3 | 9 | 7 | 4 | 4 | 8 | 3 | ⋯ |
| $r_4$ | $C_{10}$ | 0. | 1 | 2 | 3 | $\boxed{4}$ | 5 | 6 | 7 | 8 | 9 | 1 | 0 | 1 | 1 | 1 | 2 | 1 | ⋯ |
| $r_5$ | $e-2$ | 0. | 7 | 1 | 8 | 2 | $\boxed{8}$ | 1 | 8 | 2 | 8 | 4 | 5 | 9 | 0 | 4 | 5 | 2 | ⋯ |
| $r_6$ | $\frac{\sqrt{2}}{10}$ | 0. | 1 | 4 | 1 | 4 | 2 | $\boxed{1}$ | 3 | 5 | 6 | 2 | 3 | 7 | 3 | 0 | 9 | 5 | ⋯ |
| ⋮ | ⋮ | ⋮ | | ⋮ | | ⋮ | | ⋮ | | ⋮ | | ⋮ | | ⋮ | | ⋮ | | ⋮ | ⋱ |
| $r_\omega$ | | 0. | 1 | 6 | 6 | 5 | 9 | 2 | ⋯ | | | | | | | | | | |

Figure 3: Cantor's diagonal argument. Notice how, by construction, $r_\omega$ differs from $r_i$ in the circled digits.

8. Now, please return to Problem 7 and revise your answers. Justify each answer by producing a one-to-one correspondence, or showing the impossibility of doing so. Part (h) is an optional challenge.

[5] Pronounced "r omega."

The infinities in Problem 7 come in only two sizes: **countable** infinity—like the number of natural numbers—and **uncountable** infinity—like the number of real numbers. There are, in fact, an infinite number of sizes of infinity, but these two main types are the only ones we'll deal with in this class.[6]

Two infinite groups can be the same infinite size and still not be isomorphic, in the same way that two finite groups of the same size are sometimes not isomorphic (like $D_3 \neq S_6$). For example, the group of all rotations of a rational number of degrees between $0°$ and $360°$ about the origin is countably infinite. So is the group of integers—positive and negative—under addition. But these two groups have completely different structures. For example, the former has two elements which are their own inverse: $0°$ and $180°$. The latter has only one such element: $0$.

9. Here's a list of infinite sets, each with an operation. For each pair, answer: (i) Does it form a group? (ii) Which previous group(s) is it isomorphic to?

   (a) natural numbers, addition
   (b) integers, addition
   (c) even integers, addition
   (d) odd integers, addition
   (e) rational numbers, addition
   (f) real numbers, addition
   (g) complex numbers, addition
   (h) integers, multiplication
   (i) integer powers of $2$, multiplication
   (j) rational numbers, multiplication
   (k) rational numbers excluding $0$, multiplication
   (l) real numbers excluding $0$, multiplication
   (m) complex numbers, multiplication
   (n) rotation by a rational number of degrees
   (o) rotation by a rational number of radians
   (p) rotation by an integer number of radians

10. Can an irrational number taken to an irrational power ever be rational? Consider the potential example $a = \sqrt{2}^{\sqrt{2}}$. To help you answer this question, let $b = a^{\sqrt{2}}$. Simplify $b$, and explain why we don't need to know whether $a$ is rational or irrational.

[6] Are there any infinities *between* the two we've discussed? This is a very deep mathematical question, known as the continuum hypothesis. It turns out that both the answer "yes" and "no" are consistent with the rest of our mathematics, so either can be taken as an axiom.

# 6 Geometry of Complex Numbers

*Thanks to Tristan Needham's* Visual Complex Analysis *for many of the problems/examples and to Josh Zucker for most of the text.*

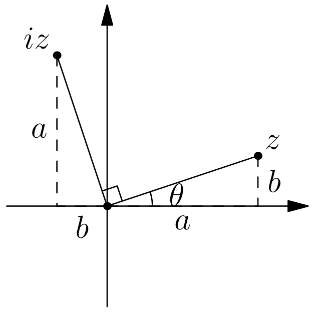



Figure 1: $iz$ is perpendicular to $z$.

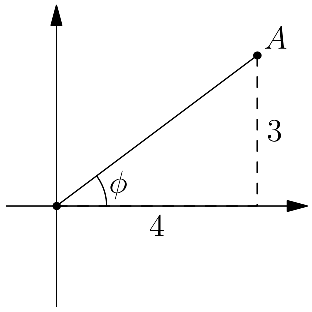



Figure 2: The complex number $A = 4 + 3i$.

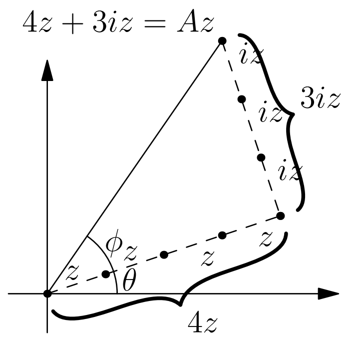



Figure 3: Breaking up $Az$ into its components, we can observe the geometry of complex multiplication.

Last year, you became masters of the art of manipulating complex numbers. In this section, we will build on that background. Throughout the rest of the book, you can reinforce your skills with a dose of Vitamin $i$.

There are at least two ways to think about the equation $x^3 = 1$. One way is to factor the equation into $(x-1)(x^2+x+1)$ and find the solutions using the quadratic formula. The other way is to use **DeMoivre's theorem**:

$$(r_1(\cos\theta + i\sin\theta))(r_2(\cos\phi + i\sin\phi)) = (r_1r_2)(\cos(\theta+\phi) + i\sin(\theta+\phi)).$$

Recall that $\operatorname{cis}\theta = \cos\theta + i\sin\theta$. $z = r\operatorname{cis}\theta$ is a complex number $r$ units away from the origin and making an angle $\theta$ with the $x$ axis, taken counterclockwise. Let's rewrite DeMoivre's theorem using $\operatorname{cis}$:

$$(r_1\operatorname{cis}\theta)(r_2\operatorname{cis}\theta) = (r_1r_2)(\operatorname{cis}(\theta+\phi)).$$

Notice that the magnitudes are multiplied and the angles are added.[7] By repeatedly applying DeMoivre's theorem, we know that $(r\operatorname{cis}\theta)^n = r^n\operatorname{cis}n\theta$. If $x = r\operatorname{cis}\theta$, then $x^3 = r^3\operatorname{cis}3\theta$. Going back to our original $x^3 = 1$, since $1 = \operatorname{cis}(2\pi k)$ for any integer $k$, we find that $r = 1$ and $3\theta = 2\pi k$. This yields three solutions: $1\operatorname{cis}0, 1\operatorname{cis}\frac{2\pi}{3}, 1\operatorname{cis}\frac{4\pi}{3}$. Any other value of $\theta = \frac{2k\pi}{3}$ reduces to one of these values because of the periodicity of $\operatorname{cis}$. These correspond to $k = 0, 1, 2$; other values of $k$ produce coterminal angles and are therefore duplicates. You can confirm that these three solutions are the same solutions that you obtain from factoring.

You can prove DeMoivre's theorem using the angle addition formulae for $\cos$ and $\sin$. But you can also understand it through pure geometry. Consider a complex number $z = a+bi$, being multiplied by $A = 4+3i$. $z$ forms an angle of $\theta$ with the real axis, and $A$ forms an angle of $\phi$. In Figure 1, observe that $iz$ is perpendicular to $z$ for any $z$. Figure 2 depicts the complex number $A$. Finally, in Figure 3, you see the multiplication carried out: $Az = (4+3i)z = 4z + 3iz$. These two components, $4z$ and $3iz$, are indicated.

Combining the observation in Figure 1 and knowledge from geometry, we know the triangles in Figure 2 and 3 are similar. Since the scaling is by a factor of $|z|$, multiplying $A$ by $z$ has the effect rotating $z$ by the angle of $A$, and multiplying it by the length of $A$. This method of proving DeMoivre's theorem for $A = 4+3i$ will work for all complex numbers $A = a + bi$.

Some notation: the angle $\theta$ of a complex number $z = a+bi$ is often called the argument, written as $\operatorname{Arg} z$. The real part of $z$ is written $\operatorname{Re}(z) = a$, and the imaginary part of $z$ is written $\operatorname{Im}(z) = b$. Note that $\operatorname{Im}(z)$ is a *real* number $b$, *not* an imaginary number $bi$. Finally, the **complex conjugate** of $z$, where the imaginary part is negated, is written with a bar on top: $\overline{z} = a - bi$.

1. Explain why $iz$ is perpendicular to $z$, without using DeMoivre's theorem.

2. How does $\operatorname{Arg}\overline{z}$ relate to $\operatorname{Arg} z$? (Hint: symmetry!)

3. Compute $z\overline{z}$ and relate it to the $\operatorname{cis}$ form of $z$.

4. Explain, using a picture, why $\tan(\operatorname{Arg} z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$.

[7]What else is added when you multiply? Exponents! In fact, $\operatorname{cis}\theta = e^{i\theta}$, but that's another story.

5. Divide $\frac{a+bi}{c+di}$ by rationalizing the denominator.

6. Divide $\frac{r_1 \operatorname{cis}\theta}{r_2 \operatorname{cis}\phi}$ using DeMoivre's theorem.

7. Compare and contrast the methods of division in Problems 5 and 6. Which is more convenient? Or does it depend on the circumstance?

8. (a) If $z = r\operatorname{cis}\theta$, what is $\frac{1}{z}$?
   (b) Explain how this shows $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$, without having to rationalize the denominator. (Hint: use Problems 3, 4, and 7.)

9. Compute $(1+i)^{13}$; pencil, paper, and brains only. No calculators!

10. Compute $\frac{(1+i\sqrt{3})^3}{(1-i)^2}$ without a calculator.

11. Draw $\operatorname{cis}\left(\frac{\pi}{4}\right) + \operatorname{cis}\left(\frac{\pi}{2}\right)$. Use your picture to prove an expression for $\tan\left(\frac{3\pi}{8}\right)$. (Hint: add them as vectors.)

12. Solve $z^3 = 1$, and show that its solutions under the operation of multiplication form a group, isomorphic to the rotation group of the equilateral triangle. Write a group table!

13. (a) Find multiplication groups of complex numbers which are isomorphic to the rotation groups for
      i. a non-square rectangle, and
      ii. a regular hexagon.
    (b) Make a table for each group.
    (c) Compare the regular hexagon's group to the dihedral group of the equilateral triangle, $D_3$. Consider: how are they the same? How are they different? Is the difference fundamental?

14. Which of the following sets is a group under (i) addition and (ii) multiplication?
    (a) $\{0\}$
    (b) $\{1\}$
    (c) $\{0,1\}$
    (d) $\{-1,1\}$
    (e) $\{1,-1,i,-i\}$
    (f) {naturals}
    (g) {integers}
    (h) {rationals}, $\mathbb{Q}$
    (i) {$\mathbb{Q}$ without zero}
    (j) {complex numbers}, $\mathbb{C}$
    (k) {$\mathbb{C}$ without zero}

DeMoivre's theorem is the "universal" trig identity, in the sense that it can be used to calculate every other trig identity. For example, suppose you want an identity for $\cos 3\theta$. For convenience, let $c = \cos\theta$ and $s = \sin\theta$. Then we have:

$$\begin{aligned}
\operatorname{cis} 3\theta &= (\operatorname{cis}\theta)^3 && \text{[DeMoivre's Theorem]}\\
&= (c+is)^3 && \text{[Definition of } \operatorname{cis}\text{]}\\
&= c^3 + 3c^2si - 3cs^2 - s^3 i && \text{[Binomial expansion]}\\
\cos 3\theta + i\sin 3\theta &= (c^3 - 3cs^2) + i(3c^2s - s^3). && \text{[Combining like terms]}
\end{aligned}$$

Equating real parts on both sides, $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

15. Prove that $(r_1 \operatorname{cis}\theta)(r_2 \operatorname{cis}\phi) = r_1 r_2 \operatorname{cis}(\theta+\phi)$ using brute force and the angle-sum trig identities for $\cos$ and $\sin$. Do you prefer this method or the one on the previous page? Which method gives you a better understanding of why DeMoivre's works?

16. Find an identity for $\sin 3\theta$ as we have done for $\cos$. Most of the work is already done for you!

17. Your friend's textbook says $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, different from our identity. Who's right?

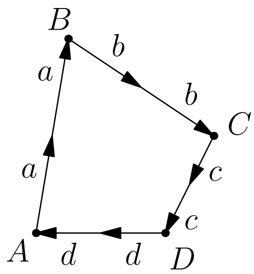



Figure 4: The quadrilateral with four squares.

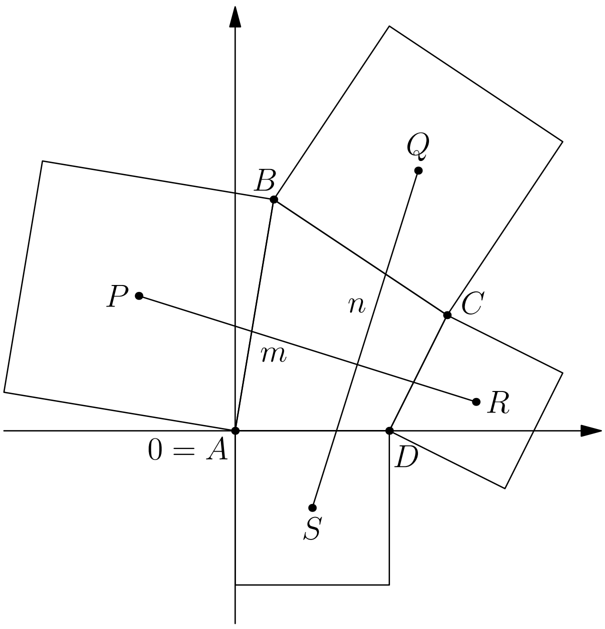



Figure 5: $2(a + b + c + d) = 0$.

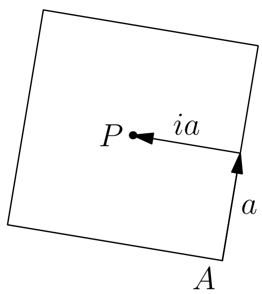



Figure 6: $P = a + ia$.

Let's apply complex numbers to a geometry problem. We want to prove that if we construct squares with centers $P$, $Q$, $R$, $S$ on the sides of any quadrilateral $ABCD$, as shown in Figure 4, then (i) $\overline{PR} \perp \overline{QS}$ and (ii) $\overline{PR} \cong \overline{QS}$. In other words, segments joining centers of opposite squares are perpendicular and the same length.

We represent all points in the figure as complex numbers. For convenience, let $A = 0$ be the origin. The edges of the quadrilateral can be thought of as vectors in the form of complex numbers, and are found using subtraction; for example, the edge from $A$ to $B$ is $B - A$. Similarly, the edge from $B$ to $C$ is $C - B$. Now, define complex numbers

$$a = \frac{B - A}{2},\ b = \frac{C - B}{2},\ c = \frac{D - C}{2},\ d = \frac{A - D}{2}.$$

$a$ is the vector halfway along $\overrightarrow{AB}$, $b$ is halfway along $\overrightarrow{BC}$, etc.; this is shown in Figure 5. We also have

$$a + b + c + d = \frac{B - A + C - B + D - C + A - D}{2} = \frac{0}{2} = 0.$$

More geometrically, this is because $2(a + b + c + d) = 2a + 2b + 2c + 2d$ is the sum of the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$, which is just $\overrightarrow{AA} = \overrightarrow{\mathbf{0}}$. This is shown in Figure 5.

$P, Q, R, S$ are also complex numbers. Let $m = R - P$ and $n = Q - S$, be our two segments $\overline{PR}$ and $\overline{QS}$. To prove that they are perpendicular, recall that $z$ is perpendicular to $iz$ for any complex $z \neq 0$, so we just need to prove that $n = \pm im$.

We now need to relate $P, Q, R, S$ back to $a, b, c, d$. It is easy to see that $P = a + ia$, remembering that $a$ is the vector halfway along $\overrightarrow{AB}$. $a$ takes you from the origin $A$ to the midpoint of $\overline{AB}$, then $ia$ takes you to $P$. This shown in Figure 6. We can extend this logic to the other points, of course.

18. Now you can finish the rest of the proof.

   (a) Draw $a, b, c, d, m, n$ approximately for the quadrilateral on the previous page.

   (b) Why does showing $n = \pm im$ prove the segments are (i) perpendicular and (ii) the same length?

   (c) Explain why $Q = 2a + b + ib$.

   (d) Find formulae for $R$ and $S$ in terms of $c$ and $d$.

   (e) Find $m$ and $n$ in terms of $a$, $b$, $c$, and $d$.

   (f) Check that $n - im = 0$, using the fact that $a + b + c + d = 0$.

19. In the previous problem, we drew squares outside a quadrilateral and connected their centers. Conjecture what happens if we draw equilateral triangles outside a triangle and connect their centers. Prove your conjecture using complex numbers.

20. The hard way to find an identity for $\tan 3\theta$ is to divide the identity for $\sin$ and $\cos$ that we already found. Try this. Make sure your answer is in terms of $\tan$ only!

21. The easier way to get an identity for $\tan 3\theta$ starts with setting $z = 1 + i \tan \theta$.
    (a) Why is $\operatorname{Arg} z = \theta$?
    (b) Why is $\tan 3\theta = \frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)}$?
    (c) Use (b) to find an identity for $\tan 3\theta$.

22. Find multiplication groups of complex numbers isomorphic to rotation groups for
    (a) the regular octagon, and
    (b) the regular pentagon.

23. Make tables for
    (a) the rotation group of the regular octagon, and
    (b) the dihedral group of the square.

    Is the difference between them fundamental?

24. Which of the following tables defines a group? Why or why not?

(a)

| \$ | $I$ | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|---|
| $I$ | $I$ | $A$ | $B$ | $C$ | $D$ |
| $A$ | $A$ | $C$ | $D$ | $B$ | $I$ |
| $B$ | $B$ | $I$ | $C$ | $D$ | $A$ |
| $C$ | $C$ | $D$ | $A$ | $I$ | $B$ |
| $D$ | $D$ | $B$ | $I$ | $A$ | $C$ |

(b)

| # | $I$ | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|---|
| $I$ | $I$ | $A$ | $B$ | $C$ | $D$ |
| $A$ | $A$ | $B$ | $C$ | $D$ | $I$ |
| $B$ | $B$ | $C$ | $D$ | $I$ | $A$ |
| $C$ | $C$ | $D$ | $I$ | $A$ | $B$ |
| $D$ | $D$ | $I$ | $A$ | $B$ | $C$ |

25. Name some subsets of the complex numbers that are groups under multiplication. I can name an infinite number of both finite and infinite groups with this property, so after you list a few of each type, try to generalize.

26. Prove with a diagram that if $|z| = 1$, then $\operatorname{Im}\left(\frac{z}{(z+1)^2}\right) = 0$.

27. Prove geometrically that if $|z| = 1$, then $|1 - z| = \left|2 \sin\left(\frac{\operatorname{Arg} z}{2}\right)\right|$.

28. (a) Prove that if $(z-1)^{10} = z^{10}$, then $\operatorname{Re}(z) = \frac{1}{2}$. (Hint: if two numbers are equal, they have the same magnitude.)
    (b) How many solutions does this equation have?

29. I claim that $e^{i\theta} = \cos\theta + i \sin\theta = \operatorname{cis}\theta$, for $\theta$ in radians.
    (a) Find $e^{-it}$.
    (b) Find $\frac{e^{i\theta}+e^{-i\theta}}{2}$.
    (c) Find $\frac{e^{i\theta}-e^{-i\theta}}{2i}$.

30. Use your new, complex definitions for $\cos$ and $\sin$ to find:
    (a) $\cos^2\theta + \sin^2\theta$
    (b) $\tan\theta$
    (c) $\cos 2\theta$
    (d) $\sin 2\theta$
    (e) What kind of group is generated by $\{e^{i\theta}, e^{-i\theta}\}$ under the operation of multiplication if $\theta$ is an integer? A rational multiple of $\pi$?

31. You've used the quadratic equation throughout high school, but there's also a cubic equation that finds the roots of any cubic. Let's derive it, starting with the cubic $x^3 + bx^2 + cx + d = 0$.

(a) Make the substitution $x = y - \frac{b}{3}$. Combine like terms to create an equation of the form $y^3 - 3py - 2q = 0$, with $p, q$ in terms of $b, c$, and $d$.

(b) Rearrange this equation as $y^3 = 3py + 2q$.

(c) Make the substitution $y = s + t$ into (b), and prove that $y$ is a solution of the cubic in part (a) if $st = p$ and $s^3 + t^3 = 2q$.

(d) Eliminate $t$ between these two equations to get a quadratic in $s^3$.

(e) Solve this quadratic to find $s^3$. By symmetry, what is $t^3$?

(f) Find a formula for $y$ in terms of $p$ and $q$. What about a formula for $x$?

(g) What if we started with $ax^3 + bx^2 + cx + d = 0$, with a coefficient in front of the $x^3$ term as well? Can you come up with a formula for $x$?

32. Starting with the same cubic as in Problem 31b.

(a) Let $c = \cos\theta$. Remember that $\cos 3\theta = 4c^3 - 3c$, as we proved. Substitute $y = 2c\sqrt{p}$ into $y^3 = 3py + 2q$ to obtain $4c^3 - 3c = \frac{q}{p^{3/2}}$.

(b) Provided that $q^2 \leq p^3$, show that $y = 2\sqrt{p}\cos\left(\frac{1}{3}(\theta + 2\pi n)\right)$, where $n$ is an integer. Why does this yield all three solutions?

(c) Explain how you would find $\theta$ from $p$ and $q$, and how we would use what we have found to solve an arbitrary cubic $ax^3 + bx^2 + cx + d = 0$.

## 7 Your Daily Dose of Vitamin $i$

1. We will use complex numbers to find identities for $\cot$. Use Pascal's triangle to expand the following:

   (a) $(a+b)^3$ (b) $(a+b)^4$ (c) $(a+b)^5$

   Then substitute $b = i = \sqrt{-1}$ and expand:

   (d) $(a+i)^3$ (e) $(a+i)^4$ (f) $(a+i)^5$

   Finally, substitute $a = \cot\theta$ and expand:

   (g) $(\cot\theta + i)^3$ (h) $(\cot\theta + i)^4$ (i) $(\cot\theta + i)^5$

   Consider $z = i + \cot\theta$.

   (j) Use the above results to find identities for (i) $\cot 3\theta$, (ii) $\cot 4\theta$, and (iii) $\cot 5\theta$.

   (k) Graph $z$, $z^2$, $z^3$, $z^4$, and $z^5$, with $\theta \approx 75°$. What method did you use?

2. Compute $(1+i)^n$ for $n = 3, 4, 5, \ldots$. Can you find a general pattern?

3. Expand and graph $\operatorname{cis}^n\theta$ for $n = 2, 3, 4, \ldots$ and $\theta \approx 50°$.

   (a) Why is the real part $\cos n\theta$ and the imaginary part $\sin n\theta$?

   (b) Use your results to write identities for $\cos n\theta$ and $\sin n\theta$ for $n = 2, 3, 4, 5$.

4. Compute $\cos 7° + \cos 79° + \cos 151° + \cos 223° + \cos 295°$ without a calculator. (Hint: what does this have to do with complex numbers?)

5. Factor the following:

   (a) $x^6 - 1$ as a difference of squares

   (b) $x^6 - 1$ as a difference of cubes

   (c) $x^4 + x^2 + 1$ over the real numbers

   (d) $x^6 - 1$ completely

   (e) $x^4 + x^2 + 1$ completely

6. Let $f(z) = \frac{z+1}{z-1}$.

   (a) Without a calculator, compute $f^{2020}(z)$.

   (b) What if you replace $2020$ with the current year?

7. Find $\operatorname{Im}\left((\operatorname{cis} 12° + \operatorname{cis} 48°)^6\right)$.

8. Let $x$ satisfy the equation $x + \frac{1}{x} = 2\cos\theta$.

   (a) Compute $x^2 + \frac{1}{x^2}$ in terms of $\theta$.

   (b) Compute $x^n + \frac{1}{x^n}$ in terms of $n$ and $\theta$.

# 8 Matrix Multiplication

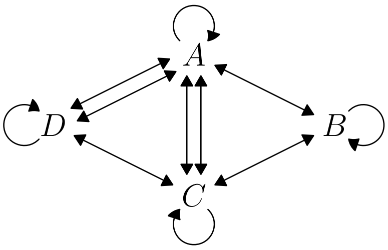



Figure 1: Four town transportation scenario.

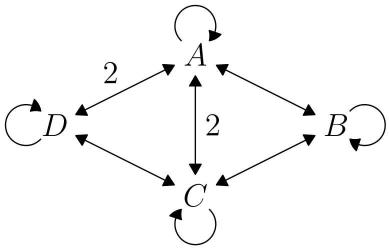



Figure 2: The scenario, with numbers instead of duplicate lines.

You've all seen a bunch of numbers organized in a table. Sometimes a table is just a table, but sometimes we will call it a **matrix**.

What makes a matrix different from a table? Although they encapsulate the same information, we can meaningfully *multiply* matrices. The purpose of this lesson is to explain why the matrix multiplication rule makes sense and when it is useful.

Consider a region with four towns, creatively named $A, B, C, D$. There are modes of transport between these towns; a path can go from a town to any town, including the same town. These paths are shown in Figure 1. At each "step," you can take any path from one town to the next. For example, you might start on $A$, then take either of the two paths to $D$. But you cannot start on $D$ and go directly to $B$. When there is more than one path between towns, we could also just draw a line and label it with a number: this is shown in Figure 2. Note how each town has a "path" going to itself. Taking this means you don't go anywhere.

Let us consider a **transportation matrix** in this scenario. The number in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of the matrix $A$, which we'll call $a_{ij}$, gives the number of ways to walk directly from town $i$ to town $j$. For example, $a_{42} = a_{24} = 0$, because there are no ways to get from $B$ to $D$ or $D$ to $B$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \text{from}\ \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{to} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

This matrix is **symmetric**, meaning there are no one-way paths. In mathematical terms, we have $a_{ij} = a_{ji}$ for all valid $i, j$. In visual terms, the matrix is symmetric about the **main diagonal**, going from top-left to bottom-right. Furthermore, the main diagonal is all $1$s, because we allow staying in the town you start in.

Suppose there's a shuttle bus that only goes one way, from town $A$ to $B$ to $C$ to $D$ and then back to $A$ again. The transportation matrix for this scenario—again, allowing staying still—is shown in Figure 3.

$$B = \text{from}\ \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{to} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Figure 3: Transportation matrix $B$.

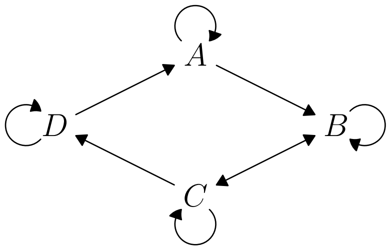



Figure 4: Graph of matrix $B$.

Because there are one-way connections, $B$ is not symmetric. For example, $b_{12} = 1$, but $b_{21} = 0$. The graph for this matrix is therefore "directed"; it has arrows indicating the direction of each path. This is shown in Figure 4.

Now, suppose you wanted to know the total number of ways to go from town to town in one step, by path or by bus. To find the total, you add the matrices in the obvious way: term by term, or $(a + b)_{ij} = a_{ij} + b_{ij}$. We can go directly $C \to D$, but not directly $D \to B$. This is shown in Figure 5.

$$A{+}B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 2 & 2 \\ 3 & 0 & 1 & 2 \end{bmatrix}$$

Figure 5: Matrix addition of $A$ and $B$.

$$A + B = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 2 \\ 3 & 0 & 1 & 1 \end{bmatrix}$$

Figure 6: $A + B$, with $1$s on the diagonal.

Okay, so that's a little silly: we've counted two different ways to stay still, namely "not taking a path" and "not going anywhere on the bus." We should rewrite the matrix, putting ones on the diagonal, as in Figure 6. Despite this minor issue, it's still true in general that this most naïve way of adding matrices is also the most convenient and useful. Just don't blindly follow a math recipe without considering its meaning!

But now comes a surprise: the most useful way to multiply matrices is not the obvious way. Why not? You'll see several different examples in the coming weeks. For now, think: what would it mean in terms of transportation if we just multiplied corresponding numbers like $a_{13}b_{13}$?[8] It would be meaningless, as far as I can tell.

Instead, we want multiplication of the two matrices $B$ and $A$ to represent taking one step by walking and then one step by bus. Similarly, multiplication of matrix $A$ by itself will represent the number of ways to go from town to town in two steps by walking.

So, what rule of matrix multiplication will make that happen? To get from town $A$ to $C$ in two steps, for example, we have to go from town $A$ to one of the four towns, then from that town to town $C$. The total number of ways to do this is

$$\underbrace{a_{11}}_{A\to A} \cdot \underbrace{a_{13}}_{A\to C} + \underbrace{a_{12}}_{A\to B} \cdot \underbrace{a_{23}}_{B\to C} + \underbrace{a_{13}}_{A\to C} \cdot \underbrace{a_{33}}_{C\to C} + \underbrace{a_{14}}_{A\to D} \cdot \underbrace{a_{43}}_{D\to C} = \sum_{j=1}^{4} a_{1j}a_{j3}.$$

And that's how we'll eventually define matrix multiplication. More formally, we can say that to determine the $ij$ entry of the product $XY$ of matrices $X$ and $Y$, use the following formula:

$$(XY)_{ij} = x_{i1}y_{1j} + x_{i2}y_{2j} + \cdots + x_{in}y_{nj} = \sum_{k=1}^{n} x_{ik}y_{kj},$$

where $X$ has $n$ columns and $Y$ has $n$ rows. This is all fine and dandy if you're say, programming a computer to do matrix multiplication, but we should find a more intuitive way to interpret this definition.

Suppose we're multiplying two matrices $X$ and $Y$. For convenience, let's make them $2 \times 3$ [9] and $3 \times 2$:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ \boxed{4 \quad 5 \quad 6} \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \boxed{\begin{matrix} 7 \\ 9 \\ 11 \end{matrix}} & \begin{matrix} 8 \\ 10 \\ 12 \end{matrix} \end{bmatrix}}_{Y} = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 7 + 9 \cdot 10 + 11 \cdot 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ \boxed{139} & 250 \end{bmatrix}.$$

Observe the boxed numbers. To get $139$, we multiplied the boxed rows and columns term by term. That is, we did $4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11$. We can think of this as the dot product of two vectors:

$$\langle 4, 5, 6 \rangle \cdot \langle 7, 9, 11 \rangle = 139.$$

In our example, to find the top left number $a_{11}$, we'd do $\langle 1, 2, 3 \rangle \cdot \langle 7, 9, 11 \rangle$. In general, to find $(XY)_{ij}$, we find the dot product of the $i$th **row vector** of $X$ and $j$th **column vector** of $Y$. More intuitively, to compute the entry in the $i$th row and $j$th column of $XY$, run your left hand across the $i$th row of matrix $X$ and your right hand down the $j$th column of matrix $Y$, multiplying the elements and adding the products as you go along.

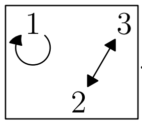
With the ability to multiply matrices more easily, let's try some problems.

1. The $3$-post snap group can be represented by a set of graphs, each with three towns. The posts are the towns and the elastic bands are the roads. For example,

[8] This "obvious" product is actually known as the Hadamard product.

[9] Note that the first dimension is rows and the second is columns, as is the usual order.

$$A = \begin{array}{cc} & \text{to} \\ & \begin{array}{ccc}1 & 2 & 3\end{array} \\ \text{from}\;\begin{array}{c}1\\2\\3\end{array} & \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}\end{array} \longleftrightarrow$$




.

(a) Draw the graphs and transportation matrices for this group.

(b) Try a few multiplications and notice the isomorphism to the snap group.

2. Using $3 \times 3$ matrices $A$ and $B$ from this section, compute:

   (a) $AA = A^2$ (b) $AB$ (c) $BA$ (d) $B^2$

   (e) Which one ($AB$ and $BA$) represents taking a step by walking, then by bus?

   (f) Use your calculator to check your computations of $A^2$, $AB$, $BA$, and $B^2$.

3. Write a $3 \times 3$ matrix $T$ that shows the following scenario: you can go from town $B$ to $C$, $C$ to $D$, and $D$ to $B$ by train, in exactly one way each, and not backwards.

   (a) Why can't you add this matrix to matrices $A$ or $B$?

   (b) Rewrite matrix $T$ so that it *can* be meaningfully added to matrices $A$ and $B$. What did you do to its dimensions?

It's time for a review of **sigma notation**! Sigma notation represents a sum. It is defined as

$$S = \sum_{k=m}^{n} f(k) = f(m) + f(m+1) + \cdots + f(n),$$

for function $f$ and integers $m, n$. $k$ is the index over which the summation is taking place. It takes on all integer values between $m$ and $n$, inclusive. You might read the sum portion like so: "The summation of $f$ of $k$ from $k$ equals $m$ to $n$ equals $S$."

4. Evaluate the following:

   (a) $\displaystyle\sum_{k=1}^{4} k$ (b) $\displaystyle\sum_{k=0}^{5} k^2$ (c) $\displaystyle\sum_{k=1}^{10} 3$ (d) $\displaystyle\sum_{k=1}^{n} k$ (e) $\displaystyle\sum_{k=1}^{n} n$ (f) $\displaystyle\sum_{k=1}^{n} 1$

5. The matrix $C^T$ whose rows are the same as the respective columns of matrix $C$ is called the **transpose** of $C$. For example,

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \; C^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

   (a) Let the elements of $C$ be $c_{ij}$ and the elements of $C^T$ be $c'_{ij}$. Write a formula for $C^T$ in terms of these elements. That is, $c'_{ij} =$ ___?

   (b) Write $\begin{bmatrix} 2 & 1 & 5 \\ 4 & -2 & 0 \end{bmatrix}^T$.

6. Fill in the blanks: Multiplying an $m \times n$ matrix by a(n) ___ $\times\, k$ matrix gives a(n) ___ $\times$ ___ matrix.

7. Dogs can eat cats, rats, or mice; cats can eat rats or mice; rats can eat mice.

   (a) Make a matrix $E$ showing what can eat what.

   (b) Draw a directed graph.

   (c) Calculate and interpret $E^2$, $E^3$, $E^4$.

The following table shows the amount of each ingredient a bakery uses in making one batch of sourdough bread and biscuits. Of course, the units vary depending on the ingredient. Let's call this matrix $S$ for sourdough.

| Sourdough | Flour | Starter | Yeast | Water | Salt | Soda | Sugar | Butter |
|---|---|---|---|---|---|---|---|---|
| Bread | 5 | 1 | 0 | $\frac{4}{3}$ | 1 | 1 | 0 | 0 |
| Biscuits | 5 | 1 | 1 | $\frac{5}{4}$ | $\frac{3}{4}$ | 0 | $\frac{1}{3}$ | 2 |

$= S$

The bakery wants to know how much the ingredients cost for one batch of bread, and how much for one batch of biscuits. The unit cost of the ingredients is given in the following table. Let's call this matrix $C$ for cost.

| | Flour | Starter | Yeast | Water | Salt | Soda | Sugar | Butter |
|---|---|---|---|---|---|---|---|---|
| $ per unit | 5 | 20 | 10 | 0 | 1 | 2 | 5 | 12 |

$= C$

8. (a) Unfortunately, if you try to multiply $S$ and $C$ as given, it won't work. Why not?
   (b) What do you need to do to $C$ so they can be multiplied? Explain the dimensions of each matrix.
   (c) Once you've fixed matrix $C$, do the multiplication. What are the dimensions of your answer?
9. Matrix multiplication is not necessarily commutative, even when the dimensions of the matrices suggest it might be. How do we know? Be specific.
10. Matrix multiplication is associative, though. Prove that $(PX)T = P(XT)$ for
$$P = \begin{bmatrix} m & n \\ p & q \end{bmatrix},\ X = \begin{bmatrix} x & y \\ z & w \end{bmatrix},\ Y = \begin{bmatrix} r & s \\ t & u \end{bmatrix}.$$
11. Prove that matrix multiplication is distributive: $P(X+T) = PX + PT$.
12. When does $PX = XP$? Don't worry if you get some messy equations in your answer.
13. Cook's Seafood Restaurant in Menlo Park sells fish and chips. The Captain's order is two pieces of fish and one order of chips, while the Regular order is one piece of fish and one order of chips.
   (a) Write a matrix representing these facts, with clear labels on your rows and columns.
   (b) The restaurant management estimates their cost at $0.75$ for each piece of fish and $0.50$ for each order of chips. Represent this as a matrix, then use matrix multiplication to calculate the cost of the two possible orders.
   (c) For a party, Cook's provides $10$ Captain's orders and $5$ Regular orders. Write this as a matrix and use matrix multiplication to find how many pieces of fish and orders of chips are provided.
   (d) Now use matrix multiplication to find out the cost of the party.
14. We will find coefficient matrices to be particularly useful for solving systems of linear equations. For instance,
$$\begin{cases} 3x + 4y &= 5 \\ 6x + 4y &= 8 \end{cases} \quad \longleftrightarrow \quad \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$
Rewrite
$$\begin{cases} 2x + 3y + 4z &= 5 \\ 5x - 4y + 2z &= 2 \\ x + 2y &= 7 \end{cases}$$
as a matrix equation in this way.
15. (a) What is the transpose of the $3 \times 3$ matrix $M$ from the previous problem?
   (b) Use $M^T$ to rewrite the system in the previous problem.
   (c) What is the transpose of the transpose matrix, $(M^T)^T$?

# 9 Mapping the Plane with Matrices

$$\overset{M}{\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}} \overset{P}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \overset{P'}{\begin{bmatrix} 7 \\ -1 \end{bmatrix}}$$

Figure 1: Matrix multiplication is a transformation.

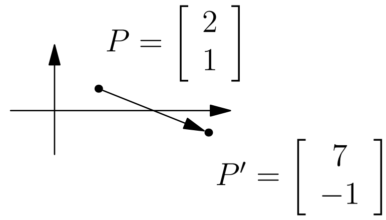



Figure 2: The transformation in the $xy$ plane.

By this time, you should be comfortable with the idea and process of matrix multiplication. You should know what dimensional relationship needs to be true of two matrices in order to be allowed to multiply them together.

Now we are going to consider $2 \times 2$ matrices as operators, which map points on the plane to points on the plane. That is, a $2 \times 2$ matrix maps the plane onto itself, or onto some figure (subset of the plane). This is done through matrix multiplication. Because $2$ coordinates determine a point in the Cartesian plane, we have the option of representing each point by a $1 \times 2$ matrix—a row vector $\begin{bmatrix} a & b \end{bmatrix}$—or a $2 \times 1$ matrix—a column vector $\begin{bmatrix} a \\ b \end{bmatrix}$. For consistency, we will use the second format, the column vector.

Figure 1 is an example of what happens when an arbitrary matrix $M$ operates on a point $P$, taking it to the point $P'$. The **preimage** $(2, 1)$ is mapped to the **image** $(7, -1)$ by the matrix; the geometric interpretation is shown in Figure 2. As in geometry, the preimage is the state before the transformation and the image is after.

1. (a) Use the $2 \times 2$ matrix from Figure 1 to operate on the points $(0, 0)$, $(1, 0)$, and $(0, 1)$. What are their images? Graph them.
   (b) The preimage includes two perpendicular **unit vectors**, $(0, 1)$ and $(1, 0)$. What is the (i) ratio of the lengths of their images and (ii) angle between the images?
   (c) You can conclude that multiplication by matrices does not, in general, preserve which two quantities between the image and preimage?

2. (a) Now, use the $2 \times 2$ matrix from Figure 1 to operate on each of these points: $(2, 1)$, $(1, 0)$, $(0, -1)$ and $(-1, -2)$. Do this by consolidating all the points into one matrix, with each point as a column vector, then performing a multiplication:
$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad \\ \\ \end{bmatrix}.$$
   (b) Graph and label the preimage and the image of each point onto the same set of axes.
   (c) The points in the preimage are discontinuous, but they belong to a particular, infinite set of points. Write the equation of that set. (Hint: what is $y$ in terms of $x$?)
   (d) Write an equation for the image of that set.
   (e) What other characteristic of the preimage points also applies to the image?
   (f) Name two things that seem to be conserved when mapping points with a matrix.

In Problem 2, you should have noticed that the points of the preimage were **collinear**, as were the points of the image. You should have also noticed that the points were equally spaced in the preimage and image. Was this a coincidence due to the particular matrix/set of points we picked, or is it generally true for all points and $2 \times 2$ matrices?

3. (a) Choose a different $2 \times 2$ matrix and a different set of three collinear, equally spaced unique points. Perform the appropriate matrix multiplication.
   (b) Graph and label the preimage points and the image points.
   (c) Have the collinearity and equal spacing been preserved?
   (d) Make a conjecture about when a matrix will preserve collinearity and when a matrix will preserve equal spacing.

4. Now, we will check your conjecture.

   (a) Start with a general $2 \times 2$ matrix and three equally spaced points on a line, and multiply the two matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x-h & x & x+h \\ m(x-h)+k & mx+k & m(x+h)+k \end{bmatrix} = \begin{bmatrix} & & \\ & & \end{bmatrix}.$$

   (b) How do you know that the second matrix indeed represents collinear and equally spaced points?

   (c) Are there any sets of collinear points that aren't representable by the $2 \times 3$ matrix?

   (d) Are the points in the image collinear? Show why or why not.

   (e) Can you find values for $a$, $b$, $c$, and $d$ so that the image does not lie on a unique line? (Hint: all of the points in the image must lie on no line, or on multiple lines.)

   (f) Use the distance formula—or some other justification—to answer whether the points in the image are equally spaced.

5. There is a point which remains **fixed**—its image is the same as its preimage—when multiplied by the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. That is, $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

   (a) Solve the above matrix equation for $x$ and $y$ to find the point.

   (b) There is a point $Q = \begin{bmatrix} e \\ f \end{bmatrix}$ that remains fixed no matter what matrix you multiply it by. Can you guess what point that is?

   (c) Prove your conjecture by plugging your point $Q$ into $\begin{bmatrix} a & b \\ c & d \end{bmatrix} Q = Q$.

Matrix multiplication is a **linear transformation**. One way to think of a linear transformation is a transformation which takes lines to other lines, and keeps equally spaced points equally spaced. With this in mind, let's investigate the different kinds of what kinds of matrix mappings. Recall the transformations from geometry: the identity, reflection across a line, rotation about a point, translation, and glide reflection preserve length, while others such as stretches and dilations change size. We will look for matrix representations of these, and if there are any matrix transformations new to us. We will also be investigating the case where multiple points in a preimage are mapped to the same point in the image.

6. Begin with a triangle with vertices $(5, 0)$, $(10, 0)$, and $(5, 10)$, as shown in Figure 3 on the right.

   (a) Map the vertices with the following matrices:

      i. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

      ii. $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$

      iii. $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

   (b) Why will the new triangle defined by these vertices be the image of the starting triangle?

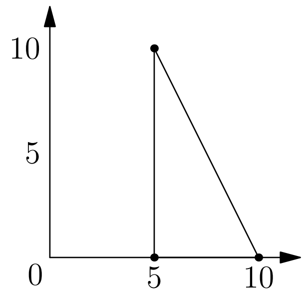


Figure 3: Problem 6's preimage.

   (c) Accurately graph the preimage, then the image for each matrix on three separate sets of axes.

   (d) For each, describe the transformation as fully as you can. Try to classify them on the transformations we mentioned earlier, and quantify them if necessary (e.g. to describe the line of reflection or angle of rotation).

7. Soon, we will map the unit square, which is shown in Figure 4: it has vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. We could actually get the entire image from the image of the unit vectors $(1, 0)$ and $(0, 1)$, which will be useful later.

   (a) How can we obtain the image of $(1, 1)$ from the images of $(1, 0)$ and $(0, 1)$?

   (b) Of $(0, 0)$?

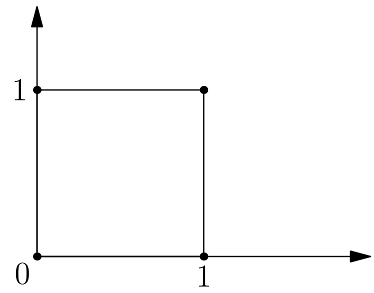



Figure 4: The unit square.

8. (a) Take the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and see what it does to the unit square. Please graph this, being careful to label each point and its image. The multiplication is done for you below.

$$\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{array}{c}\begin{array}{cccc}A & B & C & D\end{array}\\ \begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\end{array}=\begin{array}{c}\begin{array}{cccc}A' & B' & C' & D'\end{array}\\ \begin{bmatrix}0 & 1 & 3 & 2\\ 0 & 0 & 1 & 1\end{bmatrix}\end{array}$$

This mapping is called a **shear**[10] in the direction of the $x$ axis, perpendicular to the $y$ axis. Quantitatively, the preimage is sheared horizontally by a factor of $2$ of its height. In this case, the square is distorted into a parallelogram by “shoving” it along the $x$ axis without changing $y$. The $2$ in the matrix could have been replaced by any other, nonzero[11] number and the matrix would still represent a shear in the $x$ direction, just with a different magnitude.

(b) What happens to the area of the image versus the preimage?

(c) We have $AB = BC$, but is $A'B'$ equal to $B'C'$? Should it?

9. (a) When is the ratio of distances between points in the image the same as in the preimage?

(b) What is the image of the origin under any matrix mapping?

(c) What are the images of the points $(1, 0)$ and $(0, 1)$ under the mapping $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

(d) Knowing the images of $(1, 0)$ and $(0, 1)$, how do we find the image of $(1, 1)$ algebraically and geometrically?

10. How do these matrices map the plane? For each mapping, write a matrix for the images of the four corners of the unit square, then graph the preimage and image. Describe the mapping using words from geometry such as congruent, similar, rotate, reflect, shear, stretch, magnitude, and direction.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(h) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

(j) $\begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix}$

(k) $\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$

(l) $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

(m) $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

(n) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

(o) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(p) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

One limitation with $2 \times 2$ matrix transformations is that they all involve a fixed point at the origin. A translation obviously takes the origin to a different point, so we can't represent it this way. One way around this problem is to do our mapping in three-dimensional space rather than the two-dimensional plane. We still keep the origin fixed, but put our preimages on the plane $z = 1$ and make sure that our images map to the same plane.

[10] You may have heard of wind shear, which is the change of velocity of the wind with altitude. Scissors exert a shearing action on paper to cut it.

[11] If it were $0$, it would become the identity transformation, which we'll talk about later.

11. Carry out the following multiplications and convince yourself they are equivalent mappings of the $x$ and $y$ coordinates.

(a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$ (b) $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$

12. (a) Multiply these matrices: $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$.

(b) Fill in the blanks: The result of the above multiplication is that the point $(u, v, 1)$ has been translated by ___ in the $x$ direction, ___ in the $y$ direction, and is still anchored to the plane $z =$ ___.

13. (a) Write a matrix which translates a point $(x, y, 1)$ $4$ units in the $x$ direction and $7$ units in the $y$ direction, leaving $z$ fixed at $1$.

(b) Check your work by applying your matrix to the point $(3, 5, 1)$.

14. Do these two multiplications. What does each represent?

(a) $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$

15. What does each of these matrices represent?

(a) $\begin{bmatrix} a & b & \alpha \\ c & d & \beta \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \cos\theta & -\sin\theta & \alpha \\ \sin\theta & \cos\theta & \beta \\ 0 & 0 & 1 \end{bmatrix}$

16. (a) Rewrite your translation matrix and your preimage vector from Problem 12a so that you do not restrict your translations to the plane $z = 1$, but can translate in the $x$, $y$, and $z$ directions. (Hint: think four dimensions!)

(b) Write a matrix product that translates the point $(2, 3, -5)$ by the vector $(4, -1, 2)$.

For the curious, the translation matrix $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$ is actually two shears in 3D, as shown in Figure 5.

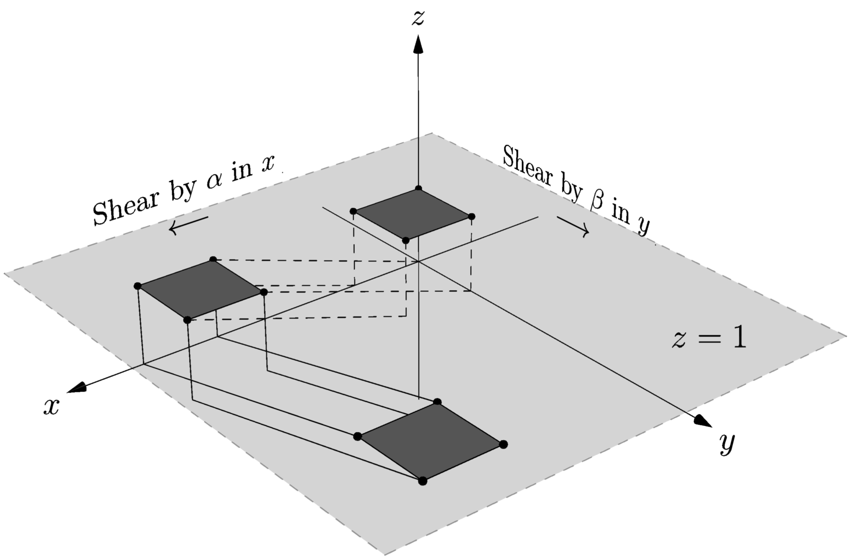



Figure 5: Translating a square by $(\alpha, \beta)$ in the plane $z = 1$. The first shear, in $x$, is shown in dashed lines. The second shear, in $y$, is shown in solid lines. The net movement is $(\alpha, \beta, 0)$.

## 10 Rotations of the Plane

You've learned about matrix multiplication and about complex numbers. You may have guessed at the relationship between them, particularly now that we've spent some time seeing how $2 \times 2$ matrices, certain $3 \times 3$ matrices, and complex numbers all relate to the geometry of the plane. We will now make that connection explicit and relate it to some ideas that you worked with last year, such as rotation of axes.

Recall that a matrix $M$ acts on a column vector $v$ by the multiplication $Mv$, not $vM$; a complex number $z$ acts on the point $(x, y)$ by multiplication $z(x + yi)$ or $(x + yi)z$, since complex multiplication is commutative. We'll call $v$ or $(x, y)$ the preimage and $Mv$ or $z(x + yi)$ the image.

Some of the following problems are really trivial, so don't be alarmed if your answer takes only a few seconds. Some of them are fairly difficult and will take a bit of thought. Some of them are fairly tedious and will take some lengthy algebra but not much thought.

1. (a) Which matrix changes nothing, so that the image is the same as the preimage?
   (b) Which complex number changes nothing?
2. (a) Which matrix doubles the length of every vector but leaves angles unchanged?
   (b) Which complex number corresponds to the same transformation?
3. Based on your answers to the previous problems, which matrix corresponds to the real number $r$? Let's call this $\mathrm{M}(r)$ for short.
4. Explain why $\mathrm{M}(u) + \mathrm{M}(v) = \mathrm{M}(u + v)$.
5. Under a $90°$ counterclockwise rotation, what is the image of (a) $(1, 0)$ and (b) $(0, 1)$?
6. (a) Which matrix corresponds to a $90°$ rotation?
   (b) Which complex number corresponds to the same rotation?
7. Based on your answers to Problems 1–6, what matrix corresponds to the complex number $x + yi$? Let's extend our function $\mathrm{M}$ and call this $\mathrm{M}(x + yi)$ for short.
8. Check that $\mathrm{M}(a+bi)+\mathrm{M}(c+di) = \mathrm{M}((a+bi)+(c+di))$. That is, prove that $\mathrm{M}$ has the same addition rules as complex numbers.
9. Check that $\mathrm{M}(a+bi)\,\mathrm{M}(c+di) = \mathrm{M}((a+bi)(c+di))$. That is, prove that $\mathrm{M}$ has the same multiplication rules as complex numbers.
10. Recall that multiplying by $\operatorname{cis}\theta$ rotates a complex number by $\theta$ radians.
    (a) Find $\mathrm{M}(\operatorname{cis}\theta)$.
    (b) To prove that this matrix really does rotate by $\theta$:
        i. Check that the image and preimage have the same length;
        ii. Check that the angle of the image with the $x$ axis is $\theta$ more than the preimage.
11. (a) Find $\mathrm{M}(r \operatorname{cis}\theta)$.
    (b) To prove that this matrix really does rotate by $\theta$ and stretch by $r$:
        i. Check that the length of the image is $r$ times the length of the preimage;
        ii. Check that the angle of the image with the $x$ axis is $\theta$ more than the preimage. (Hint: you may want to use the previous problem, or the tangent addition formulas.)

We've seen that there is a matrix for every complex number. These matrices have the same addition and multiplication rules as complex numbers. Furthermore, these matrices transform the plane in the same way as complex multiplication: a stretch by a factor of $r$ and a rotation by $\theta$. There are many matrices, however, that don't correspond to complex numbers.

12. (a) What matrix reflects over the $x$ axis, taking $(x, y) \to (x, -y)$?
    (b) What is the complex number operation equivalent to this transformation?
    (c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.

13. (a) What matrix reflects through the origin, taking $(x, y) \to (-x, -y)$?
    (b) What is the complex number operation equivalent to this transformation?
    (c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.
14. (a) Which of the $16$ matrices on page 25, for Problem 10, have corresponding complex numbers?
    (b) How can you tell algebraically?
    (c) How can you tell geometrically?
15. Make multiplication tables with the set of matrices which correspond to the elements of the rotation group for the square (a $4 \times 4$ table) and the equilateral triangle (a $3 \times 3$ table).
16. (a) Write a matrix for a rotation of $\theta$ around the origin followed by a translation by $(a, b)$.
    (b) Write a matrix for a translation by $(a, b)$ followed by a rotation of $\theta$ around the origin.

Now that we know how to rotate with matrices or with complex numbers, we can revisit the topic of rotation of axes that you studied toward the end of last year.

17. Use matrix multiplication to find the image $(x', y')$ of a point $(x, y)$ rotated by $\theta$.
18. (a) Given the parabola $x = t, y = t^2$, use matrix multiplication to rotate it by $45°$.
    (b) Graph the new parametric equations on your calculator.
    (c) Does it look like a rotation clockwise or counterclockwise? Why?

## 11 Matrices Generate Groups

As we have seen, the groups that we examined in the first couple sections of this class have representations with matrices under the operation of matrix multiplication.

Recall that the rotation group of the equilateral triangle could be generated by one element—repeatedly applying a rotation of $120°$. We called this the cyclic group of order $3$, $C_3$ for short. It took two generators to produce the dihedral group of the equilateral triangle—either a rotation and a reflection or two reflections. We called this the dihedral group of order $6$, $D_3$ for short.

In the following problems, you will be examining some of these groups, writing group tables, and determining to which symmetry group each matrix group is isomorphic. Look for patterns. Try to discover the characteristics of each matrix that tell you what it "does" geometrically.

For Problems 1–4:

(a) Specify the elements of the matrix group, unless they are all given,

(b) Describe what each matrix does to the plane,

(c) Construct a group table; you can use a calculator, and

(d) Decide which symmetry group your matrix is isomorphic to.

Let's see an example analysis on the following set of matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) They are given.

(b) $I$ is the identity transformation. $A$ rotates $180°$ (alternatively, it reflects through the origin). $B$ reflects over the $x$ axis. $C$ reflects over the $y$ axis.

(c)

| | $I$ | $A$ | $B$ | $C$ |
|---|---|---|---|---|
| $I$ | $I$ | $A$ | $B$ | $C$ |
| $A$ | $A$ | $I$ | $C$ | $B$ |
| $B$ | $B$ | $C$ | $I$ | $A$ |
| $C$ | $C$ | $B$ | $A$ | $I$ |

(d) This group is isomorphic to the symmetry group for the rectangle, otherwise known as the dihedral group of the rectangle, $D_2$ for short.

Now you can try this for yourself!

1. Analyze this group with the following elements, following the form of Example 1. What makes this group fundamentally different from the example?

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

2. The matrix $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ generates a group of order $3$. Enumerate the elements of this group and analyze per the example.

3. The matrices $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ generate a group of order $6$, of which the group in Problem 2 is a subgroup. Enumerate the elements of the group and analyze per the example.

   (e) Name some other sets of two matrices that could have generated this group.

4. The matrix $\begin{bmatrix} \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{\sqrt{5}-1}{4} \end{bmatrix}$ generates a group of order $5$. Enumerate the elements of the group and analyze per the example; you can use a calculator.

5. Let $A = \begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{bmatrix}$, $B = \begin{bmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & -\cos\frac{2\pi}{n} \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $n$ be an integer. What group is generated by the following sets of generators? Describe them geometrically.

(a) $\{A\}$ (b) $\{B\}$ (c) $\{A, B\}$ (d) $\{B, C\}$

6. Given $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, what is the order of the group generated by the following sets of generators?

   (a) $\{C\}$ (b) $\{D\}$ (c) $\{C, D\}$

7. What matrix could generate a group isomorphic to the cyclic group of order $n$, $C_n$?

8. What set of two matrices could generate a group isomorphic to the dihedral group of order $2n$, $D_n$?

9. Look at Problem 1 on page 20. The adjacency matrices map to a subgroup of the full cube symmetry group. What rotations/reflections do they map to?

10. Given $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, and $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, try understanding the groups generated by:

    (a) $\{P\}$ (c) $\{R\}$ (e) $\{P, R\}$ (g) $\{P, Q, R\}$.
    (b) $\{Q\}$ (d) $\{P, Q\}$ (f) $\{Q, R\}$

11. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ produces a shear. What is its inverse—what undoes the shear?

12. The complex numbers, excluding zero, form a group under multiplication. What set of matrices is isomorphic to the same group under multiplication?

13. Does the set of all $2 \times 2$ matrices form a group under multiplication? Why or why not?

The following analyses are more in-depth than Problems 1–4.

## Analysis 1

Analyze the group generated by $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ under multiplication.

The elements are as follows:

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, A^6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, A^7 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, A^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

In order, these are rotations of $0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, $\pi$, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$, and $\frac{7\pi}{4}$ radians counterclockwise. $A$, $A^3$, $A^5$, and $A^7$—$A$ to any power relatively prime to $8$—are all generators of the group.[12]

This is the cyclic group of order $8$, $C_8$. It is isomorphic to the rotation group of the regular octagon. $A^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity element.

## Analysis 2

Analyze the group generated by $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The generated matrices are as follows. The third, last row is all duplicates.

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, C^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, C^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

[12] Can you figure out why?

| $\cdot$ | $I$ | $A$ | $A^2$ | $A^3$ | $A^4$ | $A^5$ | $A^6$ | $A^7$ |
|---|---|---|---|---|---|---|---|---|
| $I$ | $I$ | $A$ | $A^2$ | $A^3$ | $A^4$ | $A^5$ | $A^6$ | $A^7$ |
| $A$ | $A$ | $A^2$ | $A^3$ | $A^4$ | $A^5$ | $A^6$ | $A^7$ | $I$ |
| $A^2$ | $A^2$ | $A^3$ | $A^4$ | $A^5$ | $A^6$ | $A^7$ | $I$ | $A$ |
| $A^3$ | $A^3$ | $A^4$ | $A^5$ | $A^6$ | $A^7$ | $I$ | $A$ | $A^2$ |
| $A^4$ | $A^4$ | $A^5$ | $A^6$ | $A^7$ | $I$ | $A$ | $A^2$ | $A^3$ |
| $A^5$ | $A^5$ | $A^6$ | $A^7$ | $I$ | $A$ | $A^2$ | $A^3$ | $A^4$ |
| $A^6$ | $A^6$ | $A^7$ | $I$ | $A$ | $A^2$ | $A^3$ | $A^4$ | $A^5$ |
| $A^7$ | $A^7$ | $I$ | $A$ | $A^2$ | $A^3$ | $A^4$ | $A^5$ | $A^6$ |

$$BC = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, BC^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, BC^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$CB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = BC^3, C^2B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = C^2, C^3B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = BC, B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$I$ is the identity. $C$ rotates $90°$, $C^2$ rotates $180°$, and $C^3$ rotates $270°$. $B$ reflects over the $x$ axis, $BC$ reflects over line $y = x$, $BC^2$ reflects over the $y$ axis, and $BC^3$ reflects over the line $y = -x$. This group is $D_4$, the symmetry group of the square. It contains the subgroups $C_2$ and $C_4$ once each, and four copies of the subgroup $D_2$.

The group table is shown below. I've used $BC$ instead of $C^3B$, etc., so that each reflection or rotation is denoted by a unique notation.

| | $I$ | $C$ | $C^2$ | $C^3$ | $B$ | $BC$ | $BC^2$ | $BC^3$ |
|---|---|---|---|---|---|---|---|---|
| $I$ | $I$ | $C$ | $C^2$ | $C^3$ | $B$ | $BC$ | $BC^2$ | $BC^3$ |
| $C$ | $C$ | $C^2$ | $C^3$ | $I$ | $BC^3$ | $B$ | $BC$ | $BC^2$ |
| $C^2$ | $C^2$ | $C^3$ | $I$ | $C$ | $BC^2$ | $BC^3$ | $B$ | $BC$ |
| $C^3$ | $C^3$ | $I$ | $C$ | $C^2$ | $BC$ | $BC^2$ | $BC^3$ | $B$ |
| $B$ | $B$ | $BC$ | $BC^2$ | $BC^3$ | $I$ | $C$ | $C^2$ | $C^3$ |
| $BC$ | $BC$ | $BC^2$ | $BC^3$ | $B$ | $C^3$ | $I$ | $C$ | $C^2$ |
| $BC^2$ | $BC^2$ | $BC^3$ | $B$ | $BC$ | $C^2$ | $C^3$ | $I$ | $C$ |
| $BC^3$ | $BC^3$ | $B$ | $BC$ | $BC^2$ | $C$ | $C^2$ | $C^3$ | $I$ |

## 12 Composite Mappings of the Plane

So far, we have identified matrices that result in some specific mappings of the plane, including rotations, reflections, etc. We have seen how matrices interact with each other in the context of groups. Now, let's see what happens when we combine two mappings of the plane. For example, let's see what a rotation of $-90°$ about the origin, followed by a reflection across the $x$ axis, does to our unit vectors $(1, 0)$ and $(0, 1)$. Then, let's extend this to the point $(u, v)$.

1. For Problems 1a through 1e, fill in the blank.
   (a) We start by finding the images of our points under the $-90°$ rotation.
      i. Find the matrix $R$ which results in a $-90°$ rotation.
      ii. Multiply $R$ by our unit vectors and point $(u, v)$:
      $$\begin{bmatrix} \quad & \quad \\ & \end{bmatrix}\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \end{bmatrix}$$
   (b) Next, we reflect those intermediate image points over the line $y = 0$.
      i. Find the matrix $S$ which does this.
      ii. Multiply $S$ by the result of Problem 1(a)ii.
   (c) You should notice that the net result of the two transformations taken together is a reflection over the line $y = x$. Which matrix represents this transformation?
   (d) Notice that what we did to achieve this mapping was
   $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \end{bmatrix},$$
   where we multiplied the two rightmost matrices first but didn't use the associative property to multiply the two leftmost matrices first. See what happens when you multiply the two left hand matrices together:
   $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \end{bmatrix}.$$
   Look familiar?
   (e) See what happens when you reverse the order of multiplication:
   $$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \end{bmatrix}$$
   (f) i. What transformation does this new matrix result in?
      ii. How is a reflection followed by a rotation different from a rotation followed by a reflection? Visualize this by following what happens to a point under both sets of transformations.
   (g) Notice that we apply the transformations from right to left. If you wanted to read from left to right, what would you have to change about the way you wrote the mapping matrices, the vectors representing points, and the order of the matrices?
   (h) How does our convention for ordering transformation matrices compare...
      i. ... to the convention for writing composite functions, like $f(g(x))$?
      ii. ... to the "followed by" convention we used for "From Snaps to Flips?"
      iii. ... to the "from ___ to ___" convention for transportation matrices?

2. There are two, infinite classes of matrices which comprise all isometries of the plane which keep the origin fixed. These are the rotation matrix and reflection matrix. Let's look first at the rotation matrix and make sure that it really always works the way it should.
   (a) What is the result of a rotation by an angle $\theta$ followed by one of $\phi$?
   (b) Multiply their rotation matrices: $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \end{bmatrix}$
   (c) Use the angle addition formulae to simplify your answer.

(d) Should the result be the same if you reverse the order of rotation?

(e) What happens to the points $(1,0)$, $(0,1)$, and $(x,y)$ when you operate on them with the rotation matrix?

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix}$$

3. Now let's check for the generalized reflection matrix.

(a) Take the matrix which results in a reflection over the line $y = x\tan\frac{\theta}{2}$ and reflect over that line twice:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

(b) Simplify your answer and explain the result.

(c) Let's do a reflection over the line $y = x\tan\frac{\theta}{2}$ followed by a reflection over the line $y = x\tan\frac{\phi}{2}$:

$$\begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

(d) Simplify your answer using the angle addition formulae, and interpret.

(e) Does it make a difference which reflection comes first? Do the matrix multiplication to confirm your answer.

4. We've found specific matrices which map the plane in the following ways:

- identity;
- rotation about the origin by $\theta$;
- reflection over a line $y = x\tan\frac{\theta}{2}$;
- size change by some factor centered at the origin;
- stretching along a specific line through the origin by some factor;
- shearing perpendicular to a specific line through the origin by some factor.

We want to generalize those ideas. What does each of the following matrices do? Be quantitative by specifying angle, equation of line, and/or factor:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(d) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(e) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$

(g) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

(i) $\begin{bmatrix} a & b \\ ca & cb \end{bmatrix}$

(j) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(k) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(l) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

5. What matrix/transformation undoes each of 4a through 4l? For instance, matrix 4c is a rotation of $\theta$. It is undone by a rotation of $-\theta$, which is matrix 4d.

6. In this problem, you will observe the effects of multiplying two or more matrices. Do the following matrix multiplications, graph the preimage $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and image, then identify the transformations and their order. Note the effect of order on the outcome!

But first, an example:

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}.$$

This is a rotation of $\tan^{-1}\left(-\frac{3}{4}\right) \approx -36.87°$, followed by a size change by a factor of $5$. Remember to read from right to left. The preimage and image are shown in Figure 1.

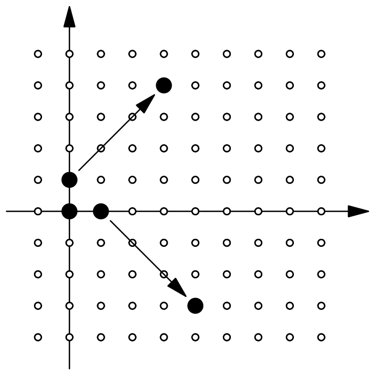


Figure 1: The preimage and image.

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$

(b) $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(d) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

(f) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$

(g) $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}\begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$

(h) $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

(i) $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}$

7. A **linear mapping** $f$ is one in which all lines are mapped to lines and the origin remains a fixed point. Algebraically, $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = xf\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yf\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. I claim that we can build any linear mapping of the plane by multiplying together some combination of the matrices from Problem 4. Only two classes of matrix, however, are necessary; all other matrices are products or examples of these. Which two classes of matrix do you think comprise the minimum set from which the others can be composed? Be able to justify your choice.

8. Write matrix products that perform the following mappings. Do the indicated multiplication and graph the preimage and image when applied to $(1,0)$ and $(0,1)$.
   (a) Rotation by $135°$ followed by a shear by a factor of $\frac{1}{2}$ *perpendicular* to the $y$ axis
   (b) Same transformations as in (a), but order reversed
   (c) Stretch in the $y$ direction by a factor of $3$ followed by a rotation of $60°$
   (d) Same transformations as in (c), but order reversed
   (e) Projection onto the line $y = 5x$
   (f) Reflection over $\theta = \frac{\pi}{12}$ followed by a stretch in the $x$ direction by a factor of $2$
   (g) Same transformations as in (f), but order reversed

9. Write a set of matrices which undoes Problems 8a through 8g. You will find one of them impossible to undo; explain why.

10. (a) Find the height of the parallelogram in Figure 2 in terms of $b$ and a trig function in terms of $\varphi$.
    (b) Find the area of the parallelogram in terms of $a$, $b$, and $\varphi$.

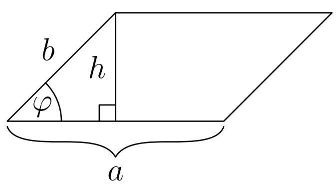



Figure 2: A parallelogram.

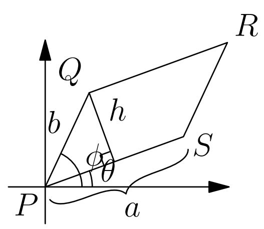



Figure 3: The parallelogram in the $xy$ plane.

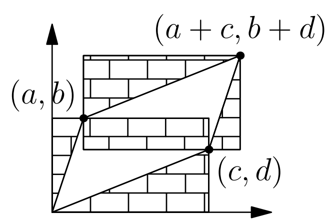



Figure 4: Scenario for Problem 12.

11. In Figure 3, we have put our parallelogram onto the $xy$ plane so that the side of length $a$ makes an angle of $\theta$ with the $x$ axis and $b$ makes an angle of $\phi$ with the $x$ axis. Thus, $\varphi = \phi - \theta$.

    (a) Rewrite the equation for the area of the parallelogram in terms of $\theta$ and $\phi$.

    (b) Find the $x$ and $y$ coordinates of $P, Q, R, S$ in terms of $a$, $b$, $\phi$, $\theta$.

    (c) Write a matrix so that the first column contains the coordinates of $Q$ and the second column contains the coordinates of $S$. This matrix maps the plane.

    (d) Your matrix has two diagonals. One rises from left to right and the other descends from left to right. Subtract the product of the entries of the ascending diagonal from the product of those of the descending diagonal.

    (e) Use angle addition formulas to simplify your answer.

    (f) You should find some relationship between your answers to Problems 11a and 11d. What is it?

    (g) The difference of the products of the two diagonals of a $2 \times 2$ matrix is called the **determinant** of the matrix, written $\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$. What does it measure?

    (h) Find a matrix which produces a rotation. What is its determinant?

    (i) Find a matrix that produces a reflection.

        i. What is the absolute value of its determinant?

        ii. How does its determinant differ from that of a rotation matrix?

        iii. What property is not conserved under reflection?

    (j) Find a matrix which produces a dilation.

        i. What is its determinant?

        ii. What does the size of its determinant indicate?

12. Here is another way to think about the area of the image of the unit square under a linear transformation. First, we use the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ to transform the unit square into a parallelogram. Then, we graph the image.

    (a) There are three rectangles and four triangles in Figure 4. Find the dimensions and the area of each one. You can use this information to figure out the area of the parallelogram in terms of $a$, $b$, $c$, and $d$. Write a sentence or equation explaining how you can use the seven areas to find the area of the parallelogram.

    (b) Carry out the algebra to find the area.

    (c) Calculate the determinant of the matrix.

    (d) What is the relationship between the determinant of the matrix and the area of its associated parallelogram?

    (e) Consider what happens if $(a, b)$ and $(c, d)$ switch places in the graph.

        i. How would the area you calculated be different?

        ii. What property would now be preserved by the transformation?

        iii. What isometry would have been included in any composition of simple transformations yielding the mapping?

        iv. What would be true of the determinant?

    (f) i. What does a reversal of the orientation of figure in its image say about the determinant of the transformation matrix?

        ii. What does that same property of the determinant imply that a transformational matrix does?

        iii. What isometry reverses orientation?

    (g) i. What would have happened to the parallelogram if we replaced $c, d$ in the matrix with $kc, kd$ for some $k > 0$, so that the transformation matrix is $\begin{bmatrix} a & kc \\ b & kd \end{bmatrix}$?

        ii. What would its area be?

        iii. What would the determinant of the matrix be?

iv. What if $\begin{bmatrix} b & d \end{bmatrix} = r\begin{bmatrix} a & c \end{bmatrix}$? That is, what if the second row of the matrix was a linear multiple of the first row?

Now that we are aware that the determinant of a matrix is a measure of size change and orientation change, we can decompose any linear mapping into a set of operations that we can visualize. Technically speaking, we can reduce all two dimensional transformational matrices into a combination of reflections and stretches along an axis. It is more intuitive, however, to include rotations, dilations, and shears along an axis in our repertoire of basic operations.

We will look at the image of the unit square under an arbitrary transformation and see how we can undo the transformation in steps until we are left with a unit square. Then we will retrace our steps, undoing each step until we have arrived at our original transformation through a set of mappings, each of which is easily visualized. We are looking for a recipe. Perhaps you can improve on the one that we will outline here!

i. Start by checking the determinant. If it is nonzero, continue to step (ii). Otherwise, you are done, because the inverse does not exist.[13]

ii. If $a = 0$, rotate the whole matrix $90°$ so that $a$ becomes nonzero.

iii. Stretch or shrink your matrix along the $x$ axis so that the top-left entry becomes $1$; this is why we wanted $a$ to be nonzero:

$$\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{c}{a} \\ b & d \end{bmatrix}.$$

iv. Shear the vector $\begin{bmatrix} 1 \\ b \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so that it is parallel to the $x$ axis:

$$\begin{aligned}\begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{c}{a} \\ b & d \end{bmatrix} &= \begin{bmatrix} 1 & \frac{c}{a} \\ 0 & -\frac{bc}{a} + d \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{c}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}.\end{aligned}$$

v. Stretch in the $y$ direction to make the bottom right entry $1$:

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix}\begin{bmatrix} 1 & \frac{c}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix} = \begin{bmatrix} 1 & \frac{c}{a} \\ 0 & 1 \end{bmatrix}.$$

vi. Shear in the $x$ direction to make the top right entry $0$:

$$\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{c}{a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This process applied to the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, is shown in Figure 5 (some not to scale).

13. Look at Figure 5 and describe the transformation in each step.

To reiterate, our process for undoing a matrix $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ with $a > 0$ and $\det M \neq 0$ is:

$$\begin{bmatrix} \text{shear} \\ \text{in x} \end{bmatrix}\begin{bmatrix} \text{stretch} \\ \text{in y} \end{bmatrix}\begin{bmatrix} \text{shear} \\ \text{in y} \end{bmatrix}\begin{bmatrix} \text{stretch} \\ \text{in x} \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Filling it in with numbers, we get:

$$\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Don't forget to multiply from right to left.

Our ultimate goal is to build up the matrix from basic operations, not to just undo it. Fortunately, we can easily figure out how to undo each of these basic operations. Remember that matrix multiplication is associative, but not commutative.

[13] Why doesn't the inverse exist? Describe the mapping.

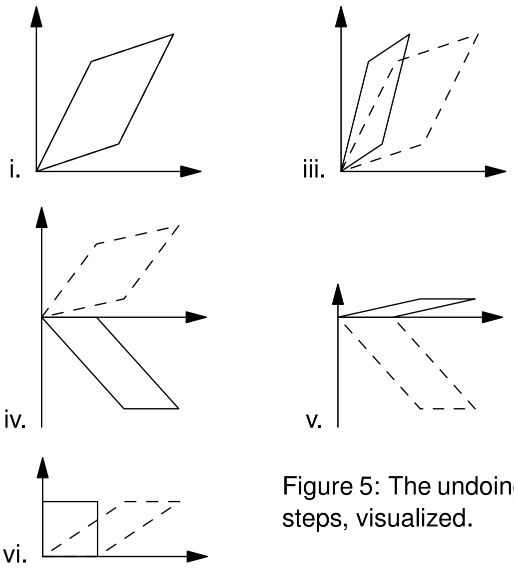



Figure 5: The undoing steps, visualized.

i. The determinant is not $0$, so we continue.

ii. $a \neq 0$, so we continue.

iii. $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 4 & 1 \end{bmatrix}$.

iv. $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -5 \end{bmatrix}$.

v. $\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$.

vi. $\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

14. (a) How do you undo a shear in the $x$ direction? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) How do you undo a stretch along the $x$ axis? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) How do you undo a shear in the $y$ direction? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) How do you undo a stretch along the $y$ axis? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

15. Now let's put this all together. Undo each of the operations in turn, until only matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ remains on the left side. Remember that what you do on the left side of the expression must also be done to the right side, so on the right side you will see the basic operations from which $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is composed. Order is important!

undoes
undoes
undoes
undoes

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

This process is known as **matrix decomposition**, because you are decomposing the matrix into simpler pieces. Now, let's see if you can apply this idea to find a set of basic transformations that is equivalent to some sample matrices.

16. Each step in the decomposition of $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$ is explained below.

(i) Stretch along the $x$ axis by factor of $\frac{1}{3}$.

$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -5 \end{bmatrix}$$

(ii) Shear perpendicular to the $x$ axis by $-2$.

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{23}{3} \end{bmatrix}$$

(iii) Stretch along $y$ axis by $-\frac{3}{23}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{23} \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{23}{3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}$$

(iv) Shear perpendicular to the $y$ axis by $-\frac{4}{3}$.

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Taken all together, the decomposition is:

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{23} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}.$$

Therefore:

$$\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{23}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}.$$

What does each matrix do?

17. Here is another way that you could have decomposed the above matrix:

$$\overset{\text{i}}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{13}{23} \end{bmatrix}} \overset{\text{ii}}{\begin{bmatrix} 1 & -\frac{2}{23} \\ 0 & 1 \end{bmatrix}} \overset{\text{iii}}{\begin{bmatrix} \frac{1}{\sqrt{13}} & 0 \\ 0 & \frac{1}{\sqrt{13}} \end{bmatrix}} \overset{\text{iv}}{\begin{bmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}} \overset{\text{v}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Identify what matrices i through v each do.

Next, we undo this sequence of operations by working backwards:

$$\overset{\text{i}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \overset{\text{ii}}{\begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}} \overset{\text{iii}}{\begin{bmatrix} \sqrt{13} & 0 \\ 0 & \sqrt{13} \end{bmatrix}} \overset{\text{iv}}{\begin{bmatrix} 1 & \frac{2}{23} \\ 0 & 1 \end{bmatrix}} \overset{\text{v}}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{23}{13} \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}.$$

(b) Explain what happens at each matrix, i through v.

18. Find a set of basic transformations which is equivalent to each of the following matrices:

(a) $\begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

19. One of the matrices in Problem 18 is a projection onto a line.

(a) Which matrix is it?

(b) What line does it project onto?

(c) If you try to decompose this matrix to the identity matrix, what happens? Why?

20. Onto what line does $\begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}$ map the plane? Solve for $a$ and $b$ such that the matrix projects perpendicular onto the line. You can do this because you know that a point on the line should not move under the projection and a point on a line perpendicular to the line has its image on the origin. Using this information you can set up two equations with two unknowns.

21. Use Problem 20 to decompose $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ into a projection to a line followed by a size change.

22. Decompose $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ into a projection perpendicular to a line followed by a size change.

23. Write matrices which project onto the following lines:

(a) $y = x$ (b) $y = 5x$ (c) $y = mx$

## 13 Inverses

Now that we've talked about matrix multiplication quite a bit, it's time to start talking about matrix "division." We've seen how matrix multiplication transforms the plane; since division is the opposite of multiplication, division will "untransform" the plane, putting all the points back where they started. But first, we'll review division in a few more familiar contexts.

1. (a) With real numbers, one of the important purposes of division is that it lets you solve equations like $ax = b$ for $x$. Solve this by division (difficult!).
   (b) If division didn't exist, you could still solve this equation by multiplication. The number you'd multiply by is called the "**multiplicative inverse**" of $a$. What is the property that defines this special number?
   (c) The multiplicative inverse of $a$ is often written $a^{-1}$. Why does this notation make sense?
2. (a) For fixed $a, b$, you might think that the equation $ax = b$ has only one solution, but sometimes it can have zero or infinitely many. Give an example of both cases.
   (b) How does the existence of a unique solution relate to the idea of multiplicative invertibility?
   (c) Are there any other possible numbers of solutions?
3. (a) Define "one-to-one" function.
   (b) Is $f(x) = ax$ a one-to-one function for all real $a$? (Hint: look for the silly exception(s)!)
4. Would your answers to the previous numbers change if you were talking about complex numbers instead of just real numbers? Why or why not?

Now, consider "**clock arithmetic**," which deals with integers $0$ through $11$ as hours on the clock: $0$ replaces $12$ for mathematical convenience. Numbers "wrap" around as the remainder when they are divided by $12$, so for example $13 \to 1$ and $25 \to 1$. We use the congruency sign $\equiv$ to denote clock arithmetic, instead of $=$.

As some basic examples, $7 + 7 \equiv 2$, because "7 hours after $7$ o'clock is $2$ o'clock." This is shown in Figure 1. In addition, $7 \times 7 \equiv 1$, because $49$ has remainder $1$ when divided by $12$. In more formal language, this clock arithmetic is actually called "arithmetic **modulo** $12$."

5. In the following problems, $x$ can be any integer from $0$ to $11$.
   (a) Find all solutions of $5x \equiv 7$ in clock arithmetic.
   (b) Find all solutions of $2x \equiv 6$ in clock arithmetic.
   (c) Find all solutions of $6x \equiv 6$ in clock arithmetic.
   (d) Find all solutions of $2x \equiv 7$ in clock arithmetic.
   (e) For integers $a, b$, what are all possible numbers of solutions that $ax \equiv b$ can have in clock arithmetic?

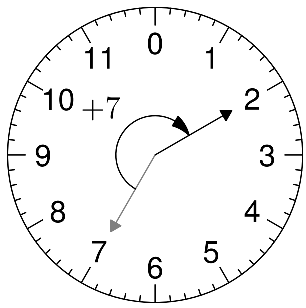



Figure 1: $7 + 7 \equiv 2$ on the clock.

6. How does the number of solutions to $ax \equiv b$ relate to the idea of multiplicative inverse? (Hint: you can try solving for $a = 5, 7, 11$ and $b = 1$. What numbers would be $5^{-1}$, $7^{-1}$, $11^{-1}$ in clock arithmetic?)
7. How does this all relate to groups?
   (a) The clock numbers are a group under clock addition. Name that group!
   (b) They are not a group under clock multiplication. Why?
   (c) A subset of four of the clock numbers form a group under the operation of clock multiplication. Find them, and write a group table.
   (d) Describe this group. What is the inverse of each element?
   (e) What symmetry group is it isomorphic to?
8. If the numbers on an advanced Mars clock went from $0$ to $4$,

(a) They would form a group under addition. Make a group table!
(b) What group is this isomorphic to?
(c) A subset of four of these numbers forms a group under multiplication. Find them and write a group table.
(d) Describe this multiplication group.
(e) What symmetry group is it isomorphic to?

Now we've seen how "division" (multiplicative inversion, or just inversion for short) works in a few more familiar situations, as well as the weird world of clock arithmetic. Let's see what happens with matrices! In the situation $AX = B$ we've been studying, $X$ is a column vector representing a point in the plane, $A$ is a $2 \times 2$ matrix describing a transformation, and $B$ is the image of point $X$. Just like in the previous cases, we'll treat $A$ and $B$ as known and $X$ as the unknown.

9. (a) Find all solutions $(x, y)$ of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, by multiplying out the left side and rewriting this as a system of equations.
   (b) Find all solutions $(x, y)$ of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.
   (c) Find all solutions $(x, y)$ of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$.
   (d) What are all possible numbers of solutions that $AX = B$ can have, where $A, B$ are $2 \times 2$ and $2 \times 1$ matrices respectively and $X = \begin{bmatrix} x \\ y \end{bmatrix}$? Use your knowledge of the properties of systems of equations.

10. Now, let's relate the two $2 \times 2$ matrices from the previous problem to the transformations we know.
   (a) Contrast the mapping properties of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
   (b) Find the determinants of these matrices. What do you notice?
   (c) When is $f(X) = AX$ a one-to-one function? That is, in mapping the plane, when does each point in the image have exactly one preimage?
   (d) Compare how you find the number of solutions of the real number equation $ax = b$ with how you find the number of solutions of the matrix equation $AX = B$.

11. Let $K = \begin{bmatrix} 5 & 7 \\ 8 & -3 \end{bmatrix}$.
   (a) Find all solutions to $K \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$.
   (b) If we knew a matrix which was the inverse of $K$, written $K^{-1}$, we could write the following equation:

$$K^{-1} K \begin{bmatrix} x \\ y \end{bmatrix} = K^{-1} \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

   What would the left side reduce to?

Note that evaluating the right side would only entail simple matrix multiplication. This would save you a bit of work. If you were dealing with a system of three equations and three unknowns, however, it would save a lot work. You wouldn't even want to touch a system with $6$, let alone the systems of hundreds or thousands that appear in today's computer problems!

Our problem simplifies as follows: how do we find the inverse of a $2 \times 2$ matrix? We've actually already done this. On page 36, you read the following string of matrices:

$$\overbrace{\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The indicated matrices, to the left of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, together constitute $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$.

12. Consider the following matrix inverses:

$$\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{23} & \frac{4}{23} \\ \frac{2}{23} & -\frac{5}{23} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}.$$

(a) Look for a pattern in these inverses.

(b) Describe the inverse of an arbitrary matrix: $\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1} = \frac{1}{\quad}\begin{bmatrix} \quad & \quad \end{bmatrix}$. Use the word determinant in your answer.

(c) We've been writing the inverse of matrix $A$ as $A^{-1}$. Why does this notation make sense?

13. Now, see what happens when you multiply the following matrices:

(a) $-\frac{1}{2}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$

(b) $\frac{1}{71}\begin{bmatrix} 5 & 7 \\ 8 & -3 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 8 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} a & c \\ b & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

(d) $\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

14. For another approach to finding the inverse of a matrix, solve the following for $w, x, y, z$ in terms of $a, b, c, d$ by converting the matrix equations into a set of four linear equations:

$$\begin{bmatrix} w & y \\ x & z \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Gauss-Jordan elimination** is an effective method we use to find inverses. If a matrix $A$ is invertible, then as we've seen with matrix decomposition, there is a set of steps to reduce it to the identity matrix. Therefore, we have some set of elementary matrices $E_i$ such that

$$E_n E_{n-1} \cdots E_2 E_1 A = I.$$

Multiplying by $A^{-1}$ on the right, we get

$$E_n E_{n-1} \cdots E_2 E_1 I = A^{-1}.$$

Before, these elementary matrices were stretches, shears and the like. We will restrict ourselves to matrices like $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, which multiply a row by $a$, or matrices like $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, which adds a multiple of the first row to the second. That's because instead of thinking of a matrix, we simply have to think "multiply this row by $a$" or "add these rows together," content that they are valid matrix multiplications. Some valid row operations are as follows:

- Swapping rows $i$ and $j$
- Adding row $i$ to $j$
- Multiplying row $i$ by a constant $a \neq 0$
- Adding $a$ times row $i$ to $j$
- Adding multiple rows to another
- Taking multiple rows, multiplying them by any nonzero coefficients, and adding them to another

This last operation is the most powerful and encompasses most of the previous ones. To apply our multiplications easily, we **augment** a matrix $A$ by juxtaposing it with the identity matrix. Also, we write what row operation we actually did on the left after each step. As shown in the first equation, we have multiplied all the elementary matrices once $A$ has become $I$. But in that process, $I$ will have become $A^{-1}$!

Choosing the steps is a bit of an art, which takes practice. Let's see Gauss-Jordan elimination in action on the matrix $A = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$.

We first augment the matrix like so:

$$[\ A \mid I\ ] = \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right].$$

The line is there just so we remember this is no ordinary matrix, but two matrices joined together for convenience. Again, we want to turn the left side into the identity matrix.

We need a $1$ in the bottom right corner, so we can add the top row to the bottom row to get $-1$:

$$\left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right] \quad \Longrightarrow \quad R_2 = R_1 + R_2 \ \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{array}\right].$$

We can then get $1$ by multiplying the bottom row by $-1$. The rest of the steps are shown/justified as well:

$$\Longrightarrow \quad R_2 = -R_2 \ \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$$\text{Want} -1 \text{ in the top left corner} \Longrightarrow \quad R_1 = R_1 + 4R_2 \ \left[\begin{array}{cc|cc} -1 & 0 & -3 & -4 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$$\text{Turn} -1 \text{ into } 1 \Longrightarrow \quad R_1 = -R_1 \ \left[\begin{array}{cc|cc} 1 & 0 & 3 & 4 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$$\text{Get rid of} -1 \text{ in bottom left corner} \Longrightarrow \quad R_2 = R_1 + R_2 \ \left[\begin{array}{cc|cc} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right].$$

Observe! We have found that $A^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Let's try this new technique out on some systems of equations.

15. Rewrite each system of equations in matrix form. Use your calculator to calculate a matrix inverse, solve the system, and finally, check your answer. Remember to make clear in your work when you have used a calculator.

    (a) $\begin{cases} 2x + 3y &= 5 \\ 4x + 5y &= 7 \end{cases}$

    (b) $\begin{cases} 37x + 12y &= 65 \\ 93x + 40y &= 156 \end{cases}$

    (c) $\begin{cases} 2x + 5y + 3z &= 5 \\ 3x + 2y + 4z &= 7 \\ 13x + 16y + 18z &= 4 \end{cases}$

    (d) $\begin{cases} w + 2x + 3y + 4z &= 7 \\ 3w - x - 2y - 5z &= 5 \\ 5w + 3x - y - 4z &= 3 \\ 7w + 9x + 5y - 2z &= 2 \end{cases}$

    (e) $\begin{cases} 2x + 5y + 2z &= 1 \\ 3x + 2y + 4z &= 1 \\ 13x + 16y + 18z &= 5 \end{cases}$

    (f) When can you use matrix inverses to solve a system of equations?

16. You can fit a polynomial to any set of points in the plane, so long as the points pass the Vertical Line Test.

    (a) What is the least degree polynomial through

        i. One point? ii. Two points? iii. Three points? iv. $n$ points?

    (b) Find a polynomial of least degree that passes through $(0, 3)$, $(1, 5)$, $(2, -3)$, $(3, 4)$, and $(4, 7)$.

# 14 Multiplication Modulo $m$ Meets Groups

In this section, we will be finding the largest subset of integers that forms a group under multiplication, modulo various numbers. We will then find a symmetry group which is isomorphic to each modulo multiplication group. Let's start with some small moduli first.

mod $3$

| $\cdot$ | $0$ | $1$ | $2$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $1$ | $2$ |
| $2$ | $0$ | $2$ | $1$ |

mod $4$

| $\cdot$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $1$ | $2$ | $3$ |
| $2$ | $0$ | $2$ | $0$ | $2$ |
| $3$ | $0$ | $3$ | $2$ | $2$ |

1. Clearly some of these numbers cannot be elements of a group. For instance, in both cases, $0$ cannot be used, since it prevents the existence of an inverse. In the case of mod $4$, $2$ cannot be used either. Why not?

2. How could we have known that these numbers would not work in advance?

3. **Euler's totient function** $\varphi(m)$ tells us how many numbers are relatively prime to a given number $m$. That is, $\varphi(m)$ is the count of numbers $n$ such that $\gcd(m,n) = 1$. What does the maximum size of a group under multiplication mod $m$ have to do with this function?

Taking out the numbers which cannot be members of a group, we get these two tables:

mod $3$

| $\cdot$ | $1$ | $2$ |
|---|---|---|
| $1$ | $1$ | $2$ |
| $2$ | $2$ | $1$ |

mod $4$

| $\cdot$ | $1$ | $3$ |
|---|---|---|
| $1$ | $1$ | $3$ |
| $3$ | $3$ | $1$ |

We now have two groups isomorphic to each other, as well as to $C_2$, $D_2$, and $S_2$.

4. We will write tables for the largest possible groups under multiplication mod $5$ and mod $8$.
   (a) Make a prediction as to how many elements will be in each group.
   (b) Which numbers can you eliminate from consideration?
   (c) Do you think that the groups will be isomorphic to those of multiplication mod $3$ and mod $4$, or to each other?
   (d) Find the period of each element in the groups and write their **orbits**: the list of its powers until it reaches the identity.
   (e) Make the tables, and analyze them to confirm/correct your predictions.
   (f) Are there any subgroups?

5. Now use a program to find the largest possible group under multiplication mod $14$.
   (a) What are its elements?
   (b) Make a table of the group's orbits.
   (c) Make a group table.
   (d) It might be good to order the numbers at the top of the table so that they start with a $1$ and go by successive powers of $3$.
   (e) What group is it isomorphic to?
   (f) Does it have any subgroups; if so, what are they?

6. Now, a surprise: find the powers of $10$, mod $14$.
   (a) How long is the period of this orbit?
   (b) What number appears to be the identity element?
   (c) Make a table in which the identity element comes first.

(d) Find a number besides $10$ whose group of powers mod $14$ is isomorphic to this group.

(e) Are these groups isomorphic to a multiplication group of a smaller modulus?

7. To really tell if two groups are isomorphic, you can write their tables in such an order that they would be identical if you substituted them in place. Why is it helpful to first note the periods and orbits of each element?

If you want to dig deeper, here's some investigations you can try. Some of these questions are deeply connected to number theory.

- To which symmetry groups are there isomorphisms under multiplication mod $m$?
- Are there some symmetry groups which do not have a representation in multiplication mod $m$?
- Is there a way you can predict in advance what symmetry groups you can get from multiplication mod $m$, given $m$?
- Can you get every finite group? Note that there exist many, many more finite groups than we have talked about so far.
- What about particular classes of group, like cyclic, dihedral, commutative and noncommutative groups?
- If a group has $n$ members, what is the largest possible period an element can have? What other periods can it have? What do we know about the numbers of elements with each period? What about period of $1$ specifically?
- Every element of a group has an inverse by the definition of group. If an element is its own inverse, what is its period? If an element has period $p$, what is the period of its inverse?
- Will $n-1$ always be in the largest group under multiplication mod $n$? Why?
- What properties do the following types of groups have?

  1. multiplication mod $p$, a prime
  2. multiplication mod $3^n$
  3. multiplication mod $p^n$
  4. multiplication mod $2^n$
  5. multiplication mod $5^n$
  6. multiplication mod a composite

For more weirdness, consider the multiplicative group of integers modulo $35$.[14] The set of integers relatively prime to $35$ is

$$\{1,2,3,4,6,8,9,11,12,13,16,17,18,19,22,23,24,26,27,29,31,32,33,34\}.$$

| Num. | Orbit | Per. | Num. | Orbit | Per. |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 18 | 18,9,22,11,23,29,32,16,8,4,2,1 | 12 |
| 2 | 2,4,8,16,32,29,23,11,22,9,18,1 | 12 | 19 | 19,11,34,16,24,1 | 6 |
| 3 | 3,**9**,27,**11**,33,**29**,17,**16**,13,**4**,12,**1** | 12 | 22 | 22,29,8,1 | 4 |
| 4 | 4,16,29,11,9,1 | 6 | 23 | 23,4,22,16,18,29,2,11,8,9,32,1 | 12 |
| 6 | 6,1 | 2 | 24 | 24,16,34,11,19,1 | 6 |
| 8 | 8,29,22,1 | 4 | 26 | 26,11,6,16,31,1 | 6 |
| 9 | 9,11,29,16,4,1 | 6 | 27 | 27,29,13,1 | 4 |
| 11 | 11,16,1 | 3 | 29 | 29,1 | 2 |
| 12 | 12,4,13,16,17,29,33,11,27,9,3,1 | 12 | 31 | 31,16,6,11,26,1 | 6 |
| 13 | 13,29,27,1 | 4 | 32 | 32,9,8,11,2,29,18,16,22,4,23,1 | 12 |
| 16 | 16,11,1 | 3 | 33 | 33,4,27,16,3,29,12,11,13,9,17,1 | 12 |
| 17 | 17,9,13,11,12,29,3,16,27,4,33,1 | 12 | 34 | 34,1 | 2 |

Figure 1: Orbits of elements mod $35$.

That's $24$ elements, so initial guesses about the associated group would include the rotation group of a cube (periods $1,2,3,4$) and the rotation group of the regular $24$-gon (periods $1,2,3,4,6,8,12,24$). None of these

[14] This is usually denoted $(\mathbb{Z}/35\mathbb{Z})^{\times}$.

guesses turn out to be correct though. Let's create a table with the orbits and periods of each element, as shown in Figure 1.

The periods appear to be factors of $12$, which leads us to consider the rotation group of the regular dodecagon, $C_{12}$. But this has only half as many elements as our group. The dihedral group $D_{12}$, however, does have $24$ elements. Let's try matching up the elements in the two groups.

$$D_{12}: \text{ Rotations of } 0^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 180^\circ, 210^\circ, 240^\circ, 270^\circ, 300^\circ, 330^\circ$$

$0^\circ$ and $1$ are the identities, so they are paired up. Because $2$ has period $12$, we arbitrarily match it up with the $30^\circ$ rotation. The rest of the powers of $2$ mod $35$ are thus mapped, pairing up all the *rotations* of $D_{12}$.

| Rotation | Number |
|---|---|
| $0^\circ$ | $1$ |
| $30^\circ$ | $2$ |
| $60^\circ$ | $4$ |
| $90^\circ$ | $8$ |
| $120^\circ$ | $16$ |
| $150^\circ$ | $32$ |
| $180^\circ$ | $29$ |
| $210^\circ$ | $23$ |
| $240^\circ$ | $11$ |
| $270^\circ$ | $22$ |
| $300^\circ$ | $9$ |
| $330^\circ$ | $18$ |

But now we run into a problem. What do we pair up the reflections with? Reflections have period $2$, but only $3$ elements in our group have that property; we need $12$. So $D_{12}$ doesn't work, either!

In fact, this group is a completely new group! It is equal to $C_2 \times C_{12}$, the "product" of two groups we're familiar with. The definition of group product is beyond the scope of this book, but as you can see, the group order $24$ is indeed the product of the orders of the groups which comprise the product. This goes to show that we have barely scratched the surface of group theory as a subject. Hopefully you get to experience it more deeply in your later education.

## 15 Eigenvectors and Eigenvalues

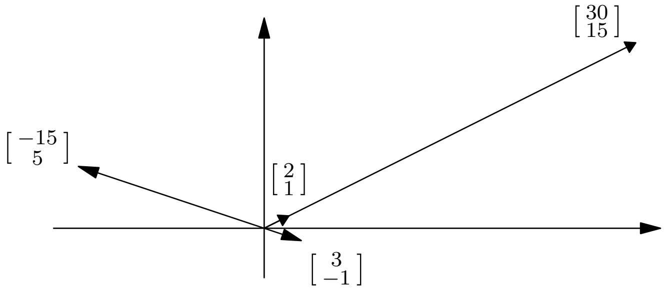



Figure 1: The matrix $\begin{bmatrix}3 & 24\\4 & 7\end{bmatrix}$ acts on two eigenvectors.

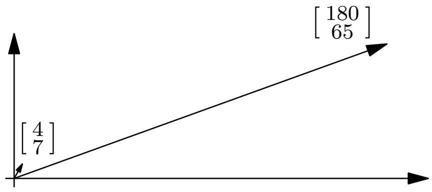



Figure 2: The average Joe does not get to be an eigenvector.

We've spent a good deal of time mapping the plane with matrices. We discovered that we could decompose any invertible $2 \times 2$ matrix into a list of matrices which, under multiplication, are a sequence of transformations of the plane. That is, we can interpret any such matrix as a sequence of reflections, rotations, shears, stretches, and dilations. We can even reduce the list to stretches and reflections. Unfortunately, doing this tends to be rather clumsy in practice. In any case, this decomposition method does not produce a unique result.

We are now going to find a new way to decompose matrices. This method will have the virtue that if two people decompose the same matrix, their results will be recognizably "the same." This process is also much easier to do.

Consider the matrix $\begin{bmatrix}3 & 24\\4 & 7\end{bmatrix}$. If you multiply a random vector like $\begin{bmatrix}4\\7\end{bmatrix}$ by this matrix, you'll get $\begin{bmatrix}180\\65\end{bmatrix}$. These two vectors have very different directions, as shown in Figure 2. But if you pick the "right" preimage vector, you can get a vector which has the same—or directly opposite—direction, meaning that the image is a constant multiple of the preimage. For example, if you pick $\begin{bmatrix}2\\1\end{bmatrix}$, then

$$\begin{aligned}\begin{bmatrix}3 & 24\\4 & 7\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} &= \begin{bmatrix}30\\15\end{bmatrix}\\ &= 15\begin{bmatrix}2\\1\end{bmatrix}.\end{aligned}$$

Similarly, if I pick $\begin{bmatrix}3\\-1\end{bmatrix}$, then

$$\begin{aligned}\begin{bmatrix}3 & 24\\4 & 7\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} &= \begin{bmatrix}-15\\5\end{bmatrix}\\ &= -5\begin{bmatrix}3\\-1\end{bmatrix}.\end{aligned}$$

These two vectors $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}3\\-1\end{bmatrix}$ are called **eigenvectors** of the matrix, and are characteristic to the matrix. Their multiplications are shown in Figure 1. This matrix has only two, **linearly independent** eigenvectors. Linearly independent means that one is not a multiple of another; they have different directions. The vectors' scale factors, $15$ and $-5$, are the **eigenvalues** of the matrix. They are each associated with one eigenvector.

In fact, any pair of vectors $\begin{bmatrix}2s\\s\end{bmatrix}$ and $\begin{bmatrix}3t\\-t\end{bmatrix}$, as long as $s, t \neq 0$, could be considered the eigenvectors of $\begin{bmatrix}3 & 24\\4 & 7\end{bmatrix}$. We just pick a form that is as simple to write as possible.

We do not consider $\begin{bmatrix}0\\0\end{bmatrix}$ an eigenvector, because it satisfies $M\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ for any $2 \times 2$ matrix $M$ and isn't very interesting. You can also justify it on the fact that it cannot have a defined eigenvalue.

We can represent any vector in the plane by adding combinations of the eigenvectors. For instance, we can represent $\begin{bmatrix}4\\7\end{bmatrix}$ as follows:

$$\begin{bmatrix} 4 \\ 7 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Along with the eigenvalues, this is helpful in matrix multiplication:

$$\begin{aligned} \begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} &= \begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} \left( 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right) && \text{Substituting representation with eigenvectors} \\ &= 15 \cdot 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - (-5) \cdot 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} && \text{Distributive property} \\ &= \begin{bmatrix} 180 \\ 65 \end{bmatrix}. \end{aligned}$$

In generic terms: if a matrix $M$ has linearly independent eigenvectors $v_1$ and $v_2$ with corresponding eigenvalues $\lambda_1$ [15] and $\lambda_2$, then for any vector $v$ with an representation with eigenvectors $v = av_1 + bv_2$,

$$Mv = \lambda_1 a v_1 + \lambda_2 b v_2.$$

An issue still remains: I just *gave* you the eigenvectors. How does one find the eigenvalues and eigenvectors of a matrix in the first place? This turns out to be relatively easy algebraically, but we'll try to develop some geometric intuition first.

1. Consider the matrix equation $\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 6x + y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$. We wish to find an eigenvector $\begin{bmatrix} x \\ y \end{bmatrix}$.
   (a) On graph paper, draw what the matrix $\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$ does to the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
   (b) In your picture, draw a rough line through the origin where you think a family of eigenvectors may be.
   (c) Try some lattice points, say $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. What does the matrix transform each vector into?
   (d) Which of these is an eigenvector?
   (e) Does it lie near the line you drew earlier?

2. This guess-and-check process for finding eigenvectors is terrible, so let's develop a procedure to find the eigenvalues and eigenvectors for any $2 \times 2$ matrix. We will use the same example.

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} && \text{Definition of eigenvector} \\ &= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \implies \left( \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \text{Subtraction and factoring} \\ \implies \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

   (a) If $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then

$$\det \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} = 0.$$

   Why? Think inverses.

[15] This is the Greek letter lambda. It is traditionally used for eigenvalues.

(b) Find the above determinant in terms of $\lambda$ and solve for the eigenvalues.

(c) One eigenvalue is $\lambda = 3$. We solve for the associated eigenvector like so:

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -3x+y \\ 6x-2y \end{bmatrix} \\ &\implies y = 3x \to \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \quad \text{(for some } s\text{)} \end{aligned}$$

Solve for the other eigenvector using the other eigenvalue from part (b).

(d) Check your work by multiplying the original matrix by the eigenvector!

3. Solve for the eigenvectors and eigenvalues of the following matrices:

(a) $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 4 & 6 \end{bmatrix}$

4. The image of an eigenvector will have the same ___ when acted on by the transformation ___ for which it is an eigenvector. The image of the eigenvector is simply the eigenvector itself multiplied by its corresponding ___.

5. (a) If the transformation matrix were a reflection over a line $y = x \tan\theta$, in what directions would the two eigenvectors point? Think geometrically.

(b) What would the angle between them be?

(c) What would their eigenvalues be?

6. Recall that multiplication by $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ results in a reflection over $y = x \tan\theta$.

(a) Write a matrix that results in a reflection over the line $y = \frac{\sqrt{3}}{3}x$.

(b) Find the eigenvalues of that matrix, and the corresponding eigenvectors.

(c) Do your calculations agree with your answers to the previous problem?

(d) What are the relationships between the two eigenvectors and between the two eigenvalues?

7. (a) Write a matrix which results in a $60°$ rotation counterclockwise.

(b) Find the eigenvalues. What do you find strange?

(c) Find the eigenvectors for those eigenvalues. What's strange about them?

(d) Explain what's going on.

(e) What are the relationships between the two eigenvectors and between the two eigenvalues?

8. The matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a shear parallel to the $x$ axis.

(a) What vectors don't change direction when multiplied by this matrix?

(b) What would you expect the eigenvectors to be?

(c) Find the eigenvectors and eigenvalues of this matrix.

(d) What is different this time?

(e) Can you represent every vector as sums of eigenvectors?

9. The matrices below result in some stretches. Find the eigenvectors and eigenvalues for both.

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

10. Note that most $2 \times 2$ matrices have two eigenvectors. How many would you expect to find for an $n \times n$ matrix?

11. Assuming that $p, q, r, s, t, u, x, y$ are real, what conditions would you impose on them in the matrices (i) $\begin{bmatrix} 3 & p \\ q & 4 \end{bmatrix}$, (ii) $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix}$, and (iii) $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$ to have...
    (a) ... two real eigenvalues?
    (b) ... two complex eigenvalues?
    (c) ... only one eigenvalue?

12. (a) Write a $3 \times 3$ matrix showing a rotation of $\theta$ around the $z$ axis.
    (b) Name the real eigenvector (this shouldn't require any work).
    (c) Find all three eigenvectors.

13. (a) What should the absolute value of an eigenvalue of any rotation matrix be?
    (b) The complex eigenvalues relate to the angle of rotation. What is that relationship?

14. In a right-handed coordinate system, rotations in three dimensions are performed by combinations of the three matrices

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, Y = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, Z = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Each matrix $X, Y, Z$ rotates around the $x, y, z$ axes by $\alpha, \beta, \gamma$, respectively.

In 2D, rotations combine to make other rotations. Similarly, if we combine any number of these rotations, the net result will be a rotation about some axis—though not necessarily a *coordinate* axis. Another way to picture this is that if we operate on an origin-centered sphere with these matrices, there will always be two opposite points[16] on the sphere which have no net movement.

Try computing the following products:

(a) $XY$ (b) $XZ$ (c) $YX$ (d) $ZX$

15. (a) Without matrices, consider a cube with side length $2$ at the origin so its faces are perpendicular to the coordinate axes. Rotate it, first $90°$ counterclockwise about the $y$ axis, then $90°$ counterclockwise about the $x$ axis. Note that rotations are done facing from the "positive side" of the coordinate axis. The net result should leave two vertices fixed. Which two?
    (b) Write a vector for the axis of rotation.
    (c) How many degrees do you think the net rotation of the cube is? Be careful; the answer is not $180°$.
    (d) Let's check our answers using matrices. Write a matrix product that corresponds to a rotation of $90°$ about the $y$ axis, followed by $90°$ about the $x$ axis.
    (e) Multiply out the matrix product.
    (f) Remember that the real eigenvector in a rotation gives the axis of rotation, and the complex eigenvalues give information about the net rotation. Evaluate these and check your answers for (a) and (b).

16. Here are two rotation matrices:
i. $\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$, ii. $\begin{bmatrix} \frac{7}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{bmatrix}$.
    (a) What is the determinant of each matrix? (Don't work, think!)
    (b) What is true of each row and each column?
    (c) Find the axis of rotation associated with each matrix.
    (d) Find the angle of rotation associated with each matrix.

[16] These are often called antipodes.

# 16 Composition of Functions

In this investigation, we will be **iterating** functions of the form $f(x) = \frac{B}{D-x}$. In other words, we will be composing them with themselves. We will start by doing some numerical experimentation, which will give us an idea of how the iteration works. We will do some algebra composing these functions and note how they form groups, which are isomorphic to some of the symmetry groups we've investigated. Then, we will see how these functions represent geometric transformations, which can in turn be composed and decomposed. Finally, we will find sets of matrices that behave in the same way under multiplication as these functions do under composition. By the time our investigation is through, we should have reviewed many of the themes that we became acquainted with in this book!

Some review: $f^n(x)$ means iterating $f$ on itself $n$ times, or $\underbrace{f(f(\cdots f(x)\cdots))}_{n\text{ times}}$. $\circ$ means to compose functions. For example, $f \circ f \circ f = f^3$.

To begin with, let's look at the function $f(x) = \frac{1}{1-x}$. We will want to find $f(f(x)) = f^2(x)$, $f^3(x)$, etc. Rather than doing the algebra, let's just plug in $x = 5$. To do this efficiently:

1. Enter the number $5$ into your calculator.
2. On the next line, type $1/(1 - \text{Ans})$.
3. Hit the enter key a few times; you will find that $f(5) = -.25$, $f^2(5) = .8$, $f^3(5) = 5$, etc.

Clearly, this function has a period of $3$ when the initial value is $5$.

1. Try a few other values for $x$ to see if and how the periodicity of $f$ depends on $x$.
2. Look at your results of iterating the following functions, and make a table of your results. If the initial value has a problem, record that.

| | $\frac{1}{1-x}$ | $\frac{2}{2-x}$ | $\frac{3}{3-x}$ | $\frac{4}{4-x}$ | $\frac{5}{5-x}$ | $\frac{7}{7-x}$ | $\frac{6}{1-x}$ | $\frac{6}{2-x}$ | $\frac{6}{3-x}$ | $\frac{9}{3-x}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| init. | $5$ | $5$ | $5$ | $5$ | $5$ | $5$ | $5$ | $5$ | $5$ | $5$ |
| $f(x)$ | | | | | | | | | | |
| $f^2(x)$ | | | | | | | | | | |
| $f^3(x)$ | | | | | | | | | | |
| $f^4(x)$ | | | | | | | | | | |
| $f^5(x)$ | | | | | | | | | | |
| $f^6(x)$ | | | | | | | | | | |
| $f^7(x)$ | | | | | | | | | | |
| $f^8(x)$ | | | | | | | | | | |

3. (a) At this point, you should have noticed some fundamentally different types of behavior of $f(x) = \frac{B}{D-x}$ under iteration, depending on the values of $B$ and $D$. What are they?
   (b) Do you have any theories yet as to when to expect each type of behavior for given values of $B$ and $D$?

We'll try a slightly more algebraic approach to the problem. Composing $f(x) = \frac{1}{1-x}$ with itself yields:

$$f^2(x) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{x-1}{x}$$
$$f^3(x) = \frac{1}{1-\frac{x-1}{x}} = \frac{x}{x-(x-1)} = x.$$

Let's tabulate these results as a group next to the rotation group of the equilateral triangle. Indeed, they are the same, as shown in Figure 1. Note that $r$ is a rotation of $120°$.

For Problems 4–6,

i. compose each until $x$ is reached;
ii. record each of the functions generated;
iii. produce a group table;
iv. identify a symmetry group to which each group is isomorphic.

| $\circ$ | $x$ | $f$ | $f^2$ | $\cdot$ | $I$ | $r$ | $r^2$ |
|---|---|---|---|---|---|---|---|
| $x$ | $x$ | $f$ | $f^2$ | $I$ | $I$ | $r$ | $r^2$ |
| $x$ | $f$ | $f^2$ | $x$ | $r$ | $r$ | $r^2$ | $I$ |
| $f^2$ | $f^2$ | $x$ | $f$ | $r^2$ | $r^2$ | $I$ | $r$ |

Figure 1: The group generated by $f(x) = \frac{1}{1-x}$, next to $C_3$.

4. $f(x) = \frac{2}{2-x}$

5. $f(x) = \frac{3}{3-x}$

6. $f(x) = \frac{9}{3-x}$

7. The function $f(x) = \frac{1}{2-x}$ behaves differently under composition.
   (a) Make a record of the first few compositions of $f(x)$ with itself.
   (b) Write a formula for $f^n(x)$.
   (c) Write a formula for its inverse function, $f^{-1}(x)$.
   (d) Why is $f^{-1} \circ f^{-1} = f^{-2}$?
   (e) What is $f^{-2} \circ f^2$?
   (f) Taken together, $f(x)$ and $f^{-1}(x)$ generate a group under composition. How big is it?
   (g) Substitute a value for $x$ and iterate. What happens?

8. The function $f(x) = \frac{2}{1-x}$ illustrates another behavior under composition.
   (a) Plug in a value for $x$ and use your calculator to see what happens.
   (b) Find the inverse of this function.
   (c) The original function and its inverse once again generate a group under composition. How big is this group?
   (d) Is it isomorphic to the group in Problem 7?
   (e) If you have time, write out a few of the elements of the group. Do you see any pattern?

9. Looking back at your results to Problems 4–7, you will notice that the elements of each group are not all in the original form $f(x) = \frac{B}{D-x}$. All, however, are in the form $f(x) = \frac{Ax+B}{Cx+D}$. If $f(x) = \frac{Ax+B}{Cx+D}$ for some $A, B, C, D$, can you always write $f^n$ in the form $f^n(x) = \frac{Px+Q}{Rx+S}$ for some $P, Q, R, S$?

We will temporarily narrow our focus to the class of functions $f(x) = \frac{1}{D-x}$. We will use algebra to see how $D$ affects the periodicity of $f$.

First, we solve for $D$ when $f$ has period $1$:

$$\begin{aligned} f(x) = \frac{1}{D-x} &= x \\ D &= \frac{x^2+1}{x}. \end{aligned}$$

So there is no constant value of $D$ which gives $f$ a period of $1$. We'll try out periods $2$ and $4$ later, but let's see $3$ and $5$ first. We set $f^3(x) = x$ and proceed:

$$\begin{aligned} f^3(x) = f(f(f(x))) &= f\left(f\left(\frac{1}{D-x}\right)\right) \\ &= f\left(\frac{1}{D - \frac{1}{D-x}}\right) \\ &= f\left(\frac{D-x}{D^2 - Dx - 1}\right) \\ &= \frac{1}{D - \frac{D-x}{D^2-Dx-1}} = x \\ \implies D^2 - Dx - 1 &= D^3x - D^2x^2 - 2Dx + x^2 \end{aligned}$$

$$\Longrightarrow D^3x - D^2(x^2+1) - Dx + (x^2+1) = 0$$
$$\Longrightarrow (D^2-1)(Dx-(x^2+1)) = 0.$$

Setting $D^2 - 1 = 0 \to D = \pm 1$ produces a function with period $3$. That is, $f(x) = \frac{1}{1-x}$ and $f(x) = \frac{1}{-1-x}$ have period $3$ under composition. You should recognize one of these already.

Looking at period $5$, we set $f^5(x) = x$. Some nasty algebra is omitted.

$$\begin{aligned}
f^5(x) = f(f(f^3(x))) &= f\left(f\left(\frac{1}{D - \frac{D-x}{D^2-Dx-1}}\right)\right) \\
&= \frac{D^4 - D^3x - 3D^2 + 2Dx + 1}{D^5 - D^4x - 4D^3 + 3D^2x + 3D - x} = x \\
\Longrightarrow (Dx-(x^2+1))(D^4-3D^2+1) &= 0 \\
\Longrightarrow D^2 &= \frac{3\pm\sqrt{5}}{2} \\
\Longrightarrow D &= \pm\sqrt{\frac{3\pm\sqrt{5}}{2}}
\end{aligned}$$

So we have that $f(x) = \frac{1}{\pm\sqrt{\frac{3\pm\sqrt{5}}{2}} - x}$ for period $5$.

This algebra is getting annoying, but matrices can simplify the process. In particular, we can make a correspondence between the functions generated by composing functions of the form $f(x) = \frac{Ax+B}{Cx+D}$ and the matrices generated by multiplying matrices of the form $F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

10. (a) Let $f(x) = \frac{Ax+B}{Cx+D}$, $g(x) = \frac{Px+Q}{Rx+S}$, and $h(x) = \frac{Tx+U}{Vx+W}$. Show that if $f \circ g = h$, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} T & U \\ V & W \end{bmatrix}.$$

Do this by computing $f(g(x))$ and the matrix product, then comparing.

(b) This is not a true one-to-one correspondence. Why not? (Hint: think about when division is undefined, as compared to matrix multiplication.)

We will now use this "isomorphism" to make the job of composing functions easier. If $f(x) = \frac{1}{D-x}$, then $F = \begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}$ corresponds to $f$.

$$\begin{aligned}
F^2 &= \begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix} \\
&= \begin{bmatrix} -1 & D \\ -D & D^2-1 \end{bmatrix} \\
&\Longrightarrow \boxed{f^2(x) = \frac{D-x}{D^2-Dx-1}} \\
F^3 = FF^2 &= \begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}\begin{bmatrix} -1 & D \\ -D & D^2-1 \end{bmatrix} \\
&= \begin{bmatrix} -D & D^2-1 \\ 1-D^2 & D^3-3D \end{bmatrix} \\
&\Longrightarrow \boxed{f^3(x) = \frac{(D^2-1)-Dx}{(1-D^2)x+(D^3-2D)}}
\end{aligned}$$

11. Find $f^4(x)$, $f^5(x)$, and $f^6(x)$ using matrices.

Recall that to find what values of $D$ result in period $n$, we solve $f^n(x) = x$. Now, you can get to solve for values of $D$, which will produce periods of two, four, and six. Some things to consider:

- We had no solutions for period one, two solutions for period three, and four solutions for period five. How many do you expect for periods two, four, and six?
- If a function repeats itself every two iterations, will it necessarily repeat itself every four iterations? Every six iterations?
- Given the previous question, what do we know about some of the solutions for $D$ for periods four and six, as compared with the true periods?

12. Use algebra or a computer algebra system—if you're lazy—to solve the following equations for $D$. Use $f^2$ from my writing above; use $f^4$ and $f^6$ from your answers to Problem 11.

    (a) $f^2(x) = x$ (b) $f^4(x) = x$ (c) $f^6(x) = x$

13. For each period, choose one of your solutions for $D$, then use your calculator to iterate $f(x) = \frac{1}{D-x}$.

    (a) Period 2:

    $D =$ ___

    $f(x) =$ ___

    Found period $=$ ___

    (b) Period 4:

    $D =$ ___

    $f(x) =$ ___

    Found period $=$ ___

    (c) Period 6:

    $D =$ ___

    $f(x) =$ ___

    Found period $=$ ___

Now, we turn our attention to fixed points. These are points where $f(x) = x$. We already discovered that there is no constant $D$ such that $\frac{1}{D-x} = x$ for *all* $x$, but it is true for *some* $x$. In particular, you can solve the equation to find $x = \frac{D\pm\sqrt{D^2-4}}{2}$. Use this fact in the following problem:

14. (a) For what values of $D$ is $x$ real?
    (b) For what values does $x$ have a nonzero imaginary component?
    (c) What is true of the fixed points for all of the cases where $f(x)$ is periodic?

It looks like the natural home of the function $f(x) = \frac{1}{D-x}$ is actually in the complex plane. Let's rename it to $f(z) = \frac{1}{D-z}$, since we're dealing with complex numbers now. Thus, $z = \frac{D\pm\sqrt{D^2-4}}{2}$. We'll continue to say that $D$ is real, though.

15. (a) Plot a graph in the complex plane of all the fixed points as $D$ ranges over the real numbers. Your graph should be a circle and two rays.
    (b) Plot and label the fixed points associated with $D$ values for periods $2, 3, 4, 5$. What do you notice?
    (c) Make a prediction about the values of the fixed points associated with functions of period $7$ and $8$.
    (d) Then predict what values of $D$ will produce periods of $7$ and $8$ for $f(z) = \frac{1}{D-z}$ under iteration.
    (e) Use your calculator to iterate your functions for $7, 8$ and confirm your hypothesis.
    (f) Use your calculator to show that the fixed points for $7, 8$ remains fixed.
    (g) Where do you suppose you would locate the fixed point so that you had an "infinite" period?
    (h) Construct an example of $f(z) = \frac{1}{D-z}$ with $|D| < 2$ and an infinite period.
    (i) Once again, check using your calculator.

16. (a) Find the fixed points for:

    i. $f(z) = \frac{1}{2-z}$ ii. $f(z) = \frac{1}{3-z}$ iii. $f(z) = \frac{1}{4-z}$

    (b) Use your calculator to iterate these functions using a variety of initial values.
    (c) What do you notice about these functions?

By this time, you should have a good sense of the behavior of $f(z) = \frac{1}{D-z}$ under iteration. You should know how to select the value of $D$ so that the function approaches a fixed point, has a finite period of $n$, or neither. The reasoning behind why this is, however, is likely still unclear.

To begin to answer these questions, we will investigate our function from a geometric, rather than algebraic, point of view. Until this point, we have been composing our function. Decomposing it, however, will shed more light on the subject.

We willl begin by decomposing $f(z) = \frac{1}{1-z}$. In doing so, we must find two functions $g(z)$ and $h(z)$ with $g(h(z)) = f(z)$. The most obvious way to do this is to let $g(z) = \frac{1}{z}$ and $h(z) = 1 - z$, but this will result in some obscure geometry.

The more useful way to decompose it is to let $g(z) = \frac{1}{\overline{z}}$ and $h(z) = \overline{1-z}$.[17] If $z = a + bi$ for real $a$ and $b$, then by definition, $\overline{z} = a - bi$. So we have

$$g(h(z)) = g(\overline{1-z}) = \overline{\left(\frac{1}{\overline{1-z}}\right)} = \frac{1}{1-z} = f(z).$$

17. Let's look at the geometry of $g$ and $h$ by completing the table in Figure 3. After doing this, you should be able to characterize how $g(z)$ and $h(z)$ transform the complex plane.

18. Find the set of points which remain fixed under $h(z)$.
    (a) What is an equation for it?
    (b) Give it a geometric name. It will help to complete the table first.

19. Find the set of points which remain fixed under $g(z)$.
    (a) What is an equation for it?
    (b) Give it a geometric name.

20. There are two points which remain fixed under both $g(z)$ and $h(z)$.
    (a) What are they?
    (b) Would they remain fixed under $g(h(z))$? Why or why not?

21. What transformation of the plane does $h(z)$ produce?

22. What transformation of the plane does $g(z)$ produce?

The transformation produced by $h(z)$ is a reflection over the line $x = .5$. The transformation produced by $g(z)$ is called an **inversion** over the unit circle. It is similar to a reflection, and if the preimage is very close to the unit circle the image is almost identical to the reflection of the preimage over the circle's circumference. The two transformations are depicted for various points in Figures 2 and 4.

23. Recall that successive reflections over two intersecting lines produce a rotation around the point of intersection, by twice the angle between the two lines.
    (a) What is the angle between the unit circle and the line $x = .5$ at the point of intersection?
    (b) If you chose your initial value of $x$ to be "close to" one of the fixed points, what would you expect to happen if you transform it with $g(h(z))$...

        i. ... once? ii. ... twice? iii. ... thrice?

24. (a) Use your calculator to test your hypothesis by picking a point close to $.5(1 + \sqrt{3})$, like $.5 + .85i$, as an initial value. Then, iterate $f(x) = \frac{1}{1-z}$.
    (b) What happens as you move your initial value farther from the fixed point?

25. It appears that this composition of functions produces a rotation-like transformation, but there are two "competing" centers with the nearer center to a point "exerting more influence." The "rotation" is distorted more and more the farther you are from the fixed points. What happens when the initial value is on the real axis?

26. What is the period of

[17] Remember that $\overline{z}$ is the complex conjugate.

| $z$ | $h(z) = \overline{1-z}$ | $z$ | $g(z) = \frac{1}{\bar{z}}$ |
|---|---|---|---|
| $i$ | $1+i$ | $i$ | $i$ |
| $0$ | | $0$ | |
| $.5$ | | $.5$ | |
| $2$ | | $2$ | |
| $3$ | | $3$ | |
| $-1$ | | $-1$ | |
| $-2$ | | $-2$ | |
| $-3$ | | $-3$ | |
| $2i$ | | $2i$ | |
| $3i$ | | $3i$ | |
| $-i$ | | $-i$ | |
| $1+i$ | | $1+i$ | |
| $.5+i$ | | $.5+i$ | |
| $.5+2i$ | | $.5(1+i)$ | |
| $.5-i$ | | $.5(-1+i\sqrt{3})$ | |
| $-.5+i$ | | $.5(-1-i\sqrt{3})$ | |
| $.5(1+i\sqrt{3})$ | | $.5(1+i\sqrt{3})$ | |
| | | $.5(1-\sqrt{3})$ | |
| | | $.5(i+\sqrt{3})$ | |
| | | $.5(\sqrt{2}+i\sqrt{2})$ | |

Figure 3: Some values to try for $z$ for $g$ and $h$.

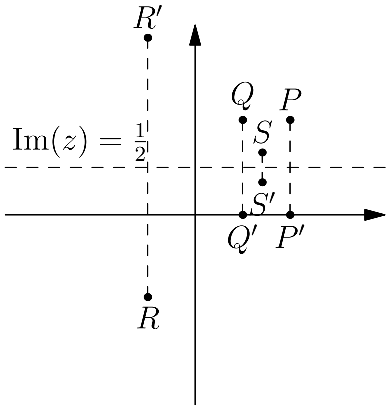



Figure 2: The transformation $h(z) = \overline{1-z}$.

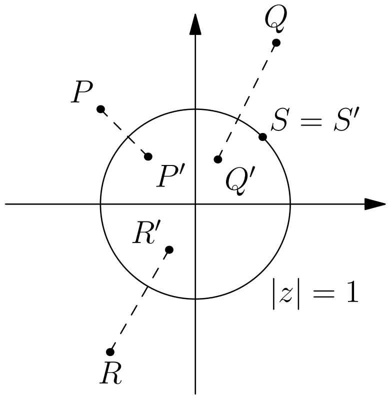



Figure 4: The transformation $g(z) = \frac{1}{\bar{z}}$.

(a) $g(z) = \frac{1}{\bar{z}}$?

(b) $h(z) = \overline{1-z}$?

We will now return to our original idea for decomposing $f(z) = \frac{1}{1-z}$ as $g(z) = \frac{1}{z}$ and $h(z) = 1 - z$. We will take an algebraic point of view, so it makes it easier to not use conjugation.

27. We will compose $g(z) = \frac{1}{z}$ and $h(z) = 1 - z$. Feel free to use a computer algebra system to do these problems.

    (a) $g \circ g$
    (b) $h \circ h$
    (c) $g \circ h$
    (d) $g \circ g \circ h$
    (e) $g \circ h \circ h$
    (f) $g \circ g \circ h$
    (g) $g \circ h \circ g$
    (h) $g \circ h \circ g \circ h$
    (i) $g \circ h \circ g \circ h \circ g$
    (j) $g \circ h \circ g \circ h \circ g \circ h$
    (k) $h \circ g$
    (l) $h \circ g \circ h$
    (m) $h \circ g \circ h \circ g$
    (n) $h \circ g \circ h \circ g \circ h$
    (o) $h \circ g \circ h \circ g \circ h \circ g$

28. Let $j(z) = g \circ h \circ g(z)$ and let $k(z) = g \circ h \circ g \circ h(z) = f^2(z)$. Also, let $I(z) = z$ be the identity function. In the group table, let the function on the top be in the inner function, and the function on the left be the outer function, so for example $g \circ h(z) = f(z)$ as shown.

(a) Complete the group table.

(b) Is this group commutative?

(c) What group(s) is it isomorphic to?

(d) What subgroup(s) does it have?

| $\circ$ | $I$ | $f$ | $k$ | $g$ | $j$ | $h$ |
|---|---|---|---|---|---|---|
| $I$ | | | | | | |
| $f$ | | | | | | |
| $k$ | | | | | | |
| $g$ | | | | | | $f$ |
| $j$ | | | | | | |
| $h$ | | | | | | |

29. Repeat the above analysis on the function (i) $f(z) = -\frac{1}{z}$, (ii) $f(z) = \frac{1}{\sqrt{2}-z}$, and (iii) $f(z) = \frac{1}{\sqrt{3}-z}$.
    (a) Decompose it into two functions involving complex conjugates and graph the fixed points of each function.
    (b) Find the intersection of the two sets of fixed points.
    (c) Describe what each of the functions does.
    (d) Decompose the function into two functions that do not involve conjugates and recompose them.
    (e) Make a group table.
    (f) What groups is this group isomorphic to?
30. Now, consider the function $f(z) = \frac{1}{2-z}$.
    (a) Decompose it into two functions involving complex conjugates and graph the fixed points of each function.
    (b) Where do they intersect?
    (c) Use your graph to visualize what happens when you take a point, reflect it over the line, and invert it about the circle.
    (d) Iterate this process—don't use your calculator for now. Where do you end up?
    (e) Use your calculator to confirm your prediction, keeping track of where you end up on the graph after each iteration.
    (f) Use a variety of initial points, and don't only use real ones.
31. Repeat the process of the previous problem for the functions $f(z) = \frac{1}{3-z}$ and $f(z) = \frac{1}{4-z}$.
    (a) Decompose them using conjugates, and graph the set of fixed points.
    (b) Note that although the circle of inversion and reflection line do not intersect, $f(x)$ still has two fixed points. Find them algebraically.
    (c) Note that the two fixed points play a fundamentally different role. one is called an attractive fixed point, while the other is a repelling fixed point. Why?
    (d) Was this the case for $D$ values which produced non-real fixed points?
    (e) How does iterating functions like $f(z) = \frac{1}{3-z}$ compare to repeated, sequential reflection over two parallel lines?

By now, you should have a good geometric sense of what causes the function $f(x) = \frac{1}{D-x}$ to oscillate when $|D| < 2$, to be periodic with a period of $n$ when $|D| < 2$ and $D = 2\cos\frac{k\pi}{n}$ for integers $k, n$, and to approach a fixed point when $|D| \geq 2$.

We would still like to make sure that our *geometric* insights are *algebraically* correct. To do that, we will take another look at the correspondence

$$f(x) = \frac{Ax+B}{Cx+D} \Longleftrightarrow F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Consider the matrix $\begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}$. Under multiplication, it will be periodic with period $n$ if and only if

$$\begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Recall that when a matrix $M$ acts on a vector $v$, it usually changes its directions, but that if it has independent eigenvectors $v_1$ and $v_2$, they will act as though they are being multiplied by a scalar constant. That is, $Mv_1 = \lambda_1 v_1$ and $Mv_2 = \lambda_2 v_2$ for some complex $\lambda_1, \lambda_2$. Also, recall that any vector in the plane can be decomposed as $v = av_1 + bv_2$. Because $M$ is periodic, $M^n v_1 = Iv_1$, but $M^n v_1 = \lambda_1^n v_1$, so we need to find which values of $D$ produces eigenvalues where $\lambda_1^n = 1$: the roots of unity.

32. Solve the eigenvalues $\lambda_1$ and $\lambda_2$ of $\begin{bmatrix} 0 & 1 \\ -1 & D \end{bmatrix}$. In the course of doing so, you should come across an equation which reminds you of the formula for fixed points.

33. For what values of $D$ do the eigenvalues have magnitudes of $1$? For what values of $D$ are they $n^{\text{th}}$ roots of unity? Express your answer as a trig function of $n$.

34. Let $D = 1$, so $F = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$.

    (a) What are the eigenvalues of $F$? Express them in terms of $\operatorname{cis}\theta$.

    (b) What is the period of this matrix under multiplication with itself?

    (c) Explain the apparent discrepancy between the period of this matrix under multiplication with itself and that of $f(z) = \frac{1}{1-z}$ under iteration.

35. Look at your answers to $f(z) = \frac{2}{2-z}$ and $f(z) = \frac{3}{3-z}$ from the first table in this chapter.

    (a) What were their periods?

    (b) What is the circle of inversion for each one?

    (c) Write the matrices which correspond to $f(z) = \frac{2}{2-z}$ and $f(z) = \frac{3}{3-z}$.

    (d) Are the determinants of these matrices $1$?

    (e) Will these matrices ever repeat themselves under successive multiplication?

    (f) Find the eigenvalues of these matrices.

    (g) Are the eigenvalues' magnitudes $1$?

# 17 Glossary

**bijection** one-to-one correspondence. If a function $f$ is bijective, it has an inverse $f^{-1}$; more formally, a function is bijective if it is both injective and surjective.

**binary operation** an operation acting on two elements.

**cardinality** size of a group; number of elements. Also known as order.

**closure** the property that operation on elements of a group always produces an element of that same group.

**collinear** on the same line.

**complex conjugate** the complex conjugate of the complex number $z = a + bi$ has the same real part as $z$ but a negated imaginary part; $\overline{z} = a - bi$.

**DeMoivre's theorem**

$$(r_1(\cos\theta + i\sin\theta))(r_2(\cos\phi + i\sin\phi)) = (r_1r_2)(\cos(\theta+\phi) + i\sin(\theta+\phi)).$$

In other words, when multiplying two complex numbers, we add their angles and multiply their magnitudes.

**dihedral group** the group of rotational and reflective symmetries of a regular polygon.

**eigenvalue** the scalar multiple that is associated with the eigenvector.

**eigenvector** a vector which when operated on by a given matrix gives a scalar multiple of itself.

**Euler's totient function** a function $\phi(n)$ that tells how many numbers are relatively prime to $n$. Formally defined mathematically as: for prime factorization of an integer $n = {p_1}^{n_1}{p_2}^{n_2}{p_3}^{n_3}\cdots{p_k}^{n_k}$, $\phi(n)$ (symbol for Euler's totient function) is equal to $\phi(n) = n(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_k})$.

**generating set** a set of elements which can generate a group by repeatedly applying the group operator to these elements.

**generator** element that can generate the entire group by a series of operations.

**group** a set of elements, finite or infinite, formed by a certain binary operation that satisfies the four fundamental properties: closure, associativity, identity, and invertibility.

**identity element** an element $I$, when acted on another element $A$ via a group's binary operation $\cdot$, gives $A$; $I \cdot A = A$ for all $A$.

**image** output of a transformation given a preimage.

**injective function** a function that maps distinct elements of its domain to distinct elements of its codomain.

**isometry** a linear transformation preserving length.

**isomorphism** a bijection, or one-to-one correspondence, between two groups which preserves the group's structure.

**linear mapping** a mapping in which all lines are mapped to lines and the origin remains fixed.

**linearly independent** (of an eigenvector) two vectors that are not multiples of each other; having different directions.

**matrix decomposition** decomposing a transformation matrix into simpler transformations.

**order** size of a group; number of elements. Also known as cardinality.

**period** number of times an element of a group has to be operated on itself to yield an identity element of that group.

**permutation** an order of things in which they can be arranged.

**preimage** input of a transformation.

**shear** a linear transformation where all points along a particular line remain fixed, while other points are shifted parallel to the fixed line by a distance proportional to their perpendicular distance from the fixed line.

**surjective function** a function that has every values of its codomain pointed at by at least one element in the domain.

**transportation matrix** a square matrix connecting vertices of a graph, sometimes known as an adjacency matrix.

**unit vector** a vector with a magnitude of $1$.