12 Composite Mappings of the Plane

So far, we have identified matrices that result in some specific mappings of the plane, including rotations, reflections, et cetera. We have seen how matrices interact with each other in the context of groups. Now, we see what happens when we combine two mappings of the plane. For example, we see what a rotation of $-90°$ about the origin, followed by a reflection across the x axis, does to our unit vectors $(1, 0)$ and $(0, 1)$. Then, we extend this to the point (u, v).

1. For Problems 1a through 1e, fill in the blank.
   (a) We start by finding the images of our points under the $-90°$ rotation.
      i. Find the matrix R which results in a $-90°$ rotation.
      ii. Multiply R by our unit vectors and point (u, v):
      $$\begin{bmatrix} \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}$$
   (b) Next, we reflect those intermediate image points over the line $y = 0$.
      i. Find the matrix S which does this.
      ii. Multiply S by the result of Problem 1(a)ii.
   (c) You should notice that the net result of the two transformations taken together is a reflection over the line $y = x$. Which matrix represents this transformation?
   (d) Notice that what we did to achieve this mapping was
   $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix},$$
   in which we multiplied the two rightmost matrices first but didn't use the associative property to multiply the two leftmost matrices first. See what happens when you multiply the two left hand matrices together:
   $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}.$$
   Look familiar?
   (e) See what happens when you reverse the order of multiplication:
   $$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$
   (f) i. What transformation does this new matrix result in?
      ii. How is a reflection followed by a rotation different from a rotation followed by a reflection? Visualize this by following what happens to a point under both sets of transformations.
   (g) Notice that we apply the transformations from right to left. If you wanted to read from left to right, what would you have to change about the way you wrote the mapping matrices, the vectors representing points, and the order of the matrices?
   (h) How does our convention for ordering transformation matrices compare...
      i. ... to the convention for writing composite functions, like $f(g(x))$?
      ii. ... to the "followed by" convention we used for "From Snaps to Flips?"
      iii. ... to the "from ___ to ___" convention for transportation matrices?

2. Two, infinite classes of matrices comprise all isometries of the plane which keep the origin fixed. These are the rotation matrix and reflection matrix. Let's look first at the rotation matrix and make sure that it really always works the way it should.
   (a) What is the result of a rotation by an angle θ followed by one of ϕ?
   (b) Multiply their rotation matrices: $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$
   (c) Use the angle addition formulae to simplify your answer.

(d) Should the result be the same if you reverse the order of rotation?

(e) What happens to the points $(1,0)$, $(0,1)$, and (x,y) when you operate on them with the rotation matrix?

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix}$$

3. Now let's check for the generalized reflection matrix.

   (a) Take the matrix which results in a reflection over the line $y = x\tan\frac{\theta}{2}$ and reflect over that line twice:

   $$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

   (b) Simplify your answer and explain the result.

   (c) Let's do a reflection over the line $y = x\tan\frac{\theta}{2}$ followed by a reflection over the line $y = x\tan\frac{\phi}{2}$:

   $$\begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

   (d) Simplify your answer using the angle addition formulae, and interpret.

   (e) Does it make a difference which reflection comes first? Do the matrix multiplication to confirm your answer.

4. We've found specific matrices which map the plane in the following ways:

   - identity;
   - rotation about the origin by θ;
   - reflection over a line $y = x\tan\frac{\theta}{2}$;
   - size change by some factor centered at the origin;
   - stretching along a specific line through the origin by some factor;
   - shearing perpendicular to a specific line through the origin by some factor.

   We want to generalize those ideas. What does each of the following matrices do? Be quantitative by specifying angle, equation of line, and/or factor:

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

   (c) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

   (d) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

   (e) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

   (f) $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$

   (g) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

   (h) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

   (i) $\begin{bmatrix} a & b \\ ca & cb \end{bmatrix}$

   (j) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

   (k) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

   (l) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

5. What matrix/transformation undoes each of 4a through 4l? For instance, matrix 4c is a rotation of θ. It is undone by a rotation of $-\theta$, which is matrix 4d.

6. In this problem, you will observe the effects of multiplying two or more matrices. Do the following matrix multiplications, graph the preimage $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and image, then identify the transformations and their order. Note the effect of order on the outcome!

As an example:

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}.$$

This is a rotation of $\tan^{-1}\left(-\frac{3}{4}\right) \approx -36.87°$, followed by a size change by a factor of 5. Remember to read from right to left. The preimage and image are shown in Figure 1.

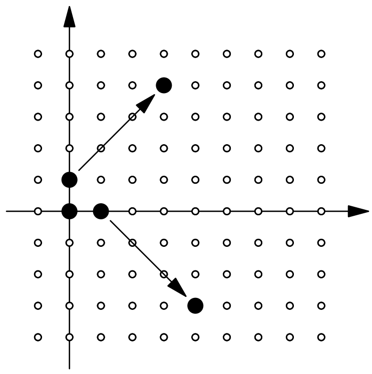

Figure 1: The preimage and image.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$

(b) $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(d) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

(f) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$

(g) $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}\begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$

(h) $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

(i) $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}$

7. A **linear mapping** f is one in which all lines are mapped to lines and the origin remains a fixed point. Algebraically, $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = xf\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yf\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. I claim that we can build any linear mapping of the plane by multiplying together some combination of the matrices from Problem 4. Only two classes of matrix, however, are necessary; all other matrices are products or examples of these. Which two classes of matrix do you think comprise the minimum set from which the others can be composed? Be able to justify your choice.

8. Write matrix products that perform the following mappings. Do the indicated multiplication and graph the preimage and image when applied to $(1, 0)$ and $(0, 1)$.
   (a) Rotation by $135°$ followed by a shear by a factor of $\frac{1}{2}$ *perpendicular* to the y axis
   (b) Same transformations as in (a), but order reversed
   (c) Stretch in the y direction by a factor of 3 followed by a rotation of $60°$
   (d) Same transformations as in (c), but order reversed
   (e) Projection onto the line $y = 5x$
   (f) Reflection over $\theta = \frac{\pi}{12}$ followed by a stretch in the x direction by a factor of 2
   (g) Same transformations as in (f), but order reversed

9. Write a set of matrices which undoes Problems 8a through 8g. You will find one of them impossible to undo; explain why.

10. (a) Find the height of the parallelogram in Figure 2 in terms of b and a trig function in terms of φ.
    (b) Find the area of the parallelogram in terms of a, b, and φ.

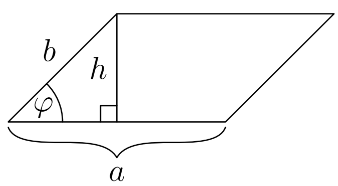

Figure 2: A parallelogram.

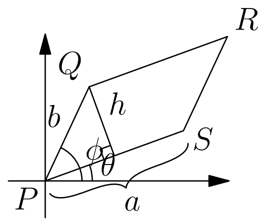

Figure 3: The parallelogram in the xy plane.

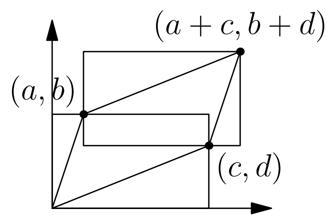

Figure 4: Scenario for Problem 12.

11. In Figure 3, we have put our parallelogram onto the xy plane so that the side of length a makes an angle of θ with the x axis and b makes an angle of ϕ with the x axis. Thus, $\varphi = \phi - \theta$.

    (a) Rewrite the equation for the area of the parallelogram in terms of θ and ϕ.

    (b) Find the x and y coordinates of P, Q, R, S in terms of a, b, ϕ, θ.

    (c) Write a matrix so that the first column contains the coordinates of Q and the second column contains the coordinates of S. This matrix maps the plane.

    (d) Your matrix has two diagonals. One rises from left to right and the other descends from left to right. Subtract the product of the entries of the ascending diagonal from the product of those of the descending diagonal.

    (e) Use angle addition formulas to simplify your answer.

    (f) You should find some relationship between your answers to Problems 11a and 11d. What is it?

    (g) The difference of the products of the two diagonals of a 2×2 matrix is called the **determinant** of the matrix, written $\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$. What does it measure?

    (h) Find a matrix which produces a rotation. What is its determinant?

    (i) Find a matrix that produces a reflection.

        i. What is the absolute value of its determinant?

        ii. How does its determinant differ from that of a rotation matrix?

        iii. What property is not conserved under reflection?

    (j) Find a matrix which produces a dilation.

        i. What is its determinant?

        ii. What does the size of its determinant indicate?

12. Here is another way to think about the area of the image of the unit square under a linear transformation. First, we use the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ to transform the unit square into a parallelogram. Then, we graph the image.

    (a) There are three rectangles and four triangles in Figure 4. Find the dimensions and the area of each one. You can use this information to figure out the area of the parallelogram in terms of a, b, c, and d. Write a sentence or equation explaining how you can use the seven areas to find the area of the parallelogram.

    (b) Carry out the algebra to find the area.

    (c) Calculate the determinant of the matrix.

    (d) What is the relationship between the determinant of the matrix and the area of its associated parallelogram?

    (e) Consider what happens if (a, b) and (c, d) switch places in the graph.

        i. How would the area you calculated be different?

        ii. What property would now be preserved by the transformation?

        iii. What isometry would have been included in any composition of simple transformations yielding the mapping?

        iv. What would be true of the determinant?

    (f) i. What does a reversal of the orientation of figure in its image say about the determinant of the transformation matrix?

        ii. What does that same property of the determinant imply that a transformational matrix does?

        iii. What isometry reverses orientation?

    (g) i. What would have happened to the parallelogram if we replaced c, d in the matrix with kc, kd for some $k > 0$, so that the transformation matrix is $\begin{bmatrix} a & kc \\ b & kd \end{bmatrix}$?

        ii. What would its area be?

        iii. What would the determinant of the matrix be?

iv. What if $\begin{bmatrix} b & d \end{bmatrix} = r \begin{bmatrix} a & c \end{bmatrix}$? That is, what if the second row of the matrix was a linear multiple of the first row?

Now that we are aware that the determinant of a matrix is a measure of size change and orientation change, we can decompose any linear mapping into a set of operations that we can visualize. Technically speaking, we can reduce all two dimensional transformational matrices into a combination of reflections and stretches along an axis. It is more intuitive, however, to include rotations, dilations, and shears along an axis in our repertoire of basic operations.

We will look at the image of the unit square under an arbitrary transformation and see how we can undo the transformation in steps until we are left with a unit square. Then we will retrace our steps, undoing each step until we have arrived at our original transformation through a set of mappings, each of which is easily visualized. We are looking for a recipe. Perhaps you can improve on the one that we will outline here!

i. Start by checking the determinant. If it is nonzero, continue to step (ii). Otherwise, you are done, because the inverse does not exist.[12]

ii. If $a = 0$, rotate the whole matrix $90°$ so that a becomes nonzero.

iii. Stretch or shrink your matrix along the x axis so that the top-left entry becomes 1; this is why we wanted a to be nonzero:

$$\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{c}{a} \\ b & d \end{bmatrix}.$$

iv. Shear the vector $\begin{bmatrix} 1 \\ b \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so that it is parallel to the x axis:

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{c}{a} \\ b & d \end{bmatrix} &= \begin{bmatrix} 1 & \frac{c}{a} \\ 0 & -\frac{bc}{a} + d \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{c}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}. \end{aligned}$$

v. Stretch in the y direction to make the bottom right entry 1:

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} 1 & \frac{c}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix} = \begin{bmatrix} 1 & \frac{c}{a} \\ 0 & 1 \end{bmatrix}.$$

vi. Shear in the x direction to make the top right entry 0:

$$\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{c}{a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This process applied to the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, is shown in Figure 5 (some not to scale).

13. Look at Figure 5 and describe the transformation in each step.

To reiterate, our process for undoing a matrix $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ with $a > 0$ and $\det M \neq 0$ is:

$$\begin{bmatrix} \text{shear} \\ \text{in x} \end{bmatrix} \begin{bmatrix} \text{stretch} \\ \text{in y} \end{bmatrix} \begin{bmatrix} \text{shear} \\ \text{in y} \end{bmatrix} \begin{bmatrix} \text{stretch} \\ \text{in x} \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Filling it in with numbers, we get:

$$\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Don't forget to multiply from right to left.

Our ultimate goal is to build up the matrix from basic operations, not to just undo it. Fortunately, we can easily figure out how to undo each of these basic operations. Remember that matrix multiplication is associative, but not commutative.

[12] Why doesn't the inverse exist? Describe the mapping.

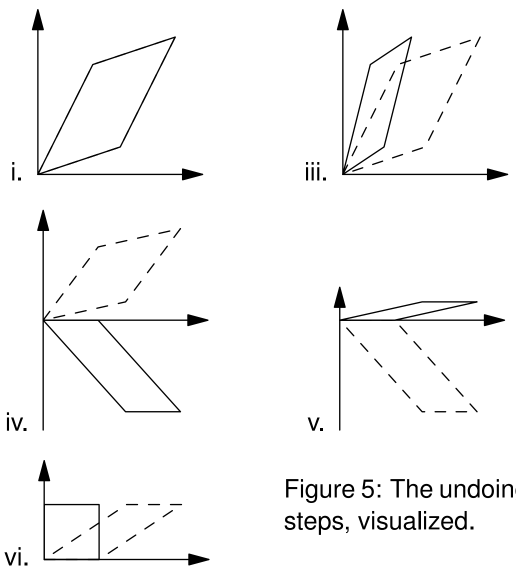

Figure 5: The undoing steps, visualized.

i. The determinant is not 0, so we continue.

ii. $a \neq 0$, so we continue.

iii. $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 4 & 1 \end{bmatrix}.$

iv. $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -5 \end{bmatrix}.$

v. $\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}.$

vi. $\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

14. (a) How do you undo a shear in the x direction? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) How do you undo a stretch along the x axis? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) How do you undo a shear in the y direction? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) How do you undo a stretch along the y axis? $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

15. Now let's put this all together. Undo each of the operations in turn, until only matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ remains on the left side. Remember that what you do on the left side of the expression must also be done to the right side, so on the right side you will see the basic operations from which $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is composed. Order is important!

undoes

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$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

This process is known as **matrix decomposition**, because you are decomposing the matrix into simpler pieces. Now, let's see if you can apply this idea to find a set of basic transformations that is equivalent to some sample matrices.

16. Each step in the decomposition of $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$ is explained below.

(i) Stretch along the x axis by factor of $\frac{1}{3}$.

$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -5 \end{bmatrix}$$

(ii) Shear perpendicular to the x axis by -2.

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{23}{3} \end{bmatrix}$$

(iii) Stretch along y axis by $-\frac{3}{23}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{23} \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{23}{3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}$$

(iv) Shear perpendicular to the y axis by $-\frac{4}{3}$.

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Taken all together, the decomposition is:

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{23} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}.$$

Therefore:

$$\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{23}{3} \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}.$$

What does each matrix do?

17. Here is another way that you could have decomposed the above matrix:

$$\overset{\text{i}}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{13}{23} \end{bmatrix}}\overset{\text{ii}}{\begin{bmatrix} 1 & -\frac{2}{23} \\ 0 & 1 \end{bmatrix}}\overset{\text{iii}}{\begin{bmatrix} \frac{1}{\sqrt{13}} & 0 \\ 0 & \frac{1}{\sqrt{13}} \end{bmatrix}}\overset{\text{iv}}{\begin{bmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}}\overset{\text{v}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Identify what matrices i through v each do.

Next, we undo this sequence of operations by working backwards:

$$\overset{\text{i}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}\overset{\text{ii}}{\begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}}\overset{\text{iii}}{\begin{bmatrix} \sqrt{13} & 0 \\ 0 & \sqrt{13} \end{bmatrix}}\overset{\text{iv}}{\begin{bmatrix} 1 & \frac{2}{23} \\ 0 & 1 \end{bmatrix}}\overset{\text{v}}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{23}{13} \end{bmatrix}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}.$$

(b) Explain what happens at each matrix, i through v.

18. Find a set of basic transformations which is equivalent to each of the following matrices:

(a) $\begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

19. One of the matrices in Problem 18 is a projection onto a line.

(a) Which matrix is it?
(b) What line does it project onto?
(c) If you try to decompose this matrix to the identity matrix, what happens? Why?

20. Onto what line does $\begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}$ map the plane? Solve for a and b such that the matrix projects perpendicular onto the line. You can do this because you know that a point on the line should not move under the projection and a point on a line perpendicular to the line has its image on the origin. Using this information you can set up two equations with two unknowns.

21. Use Problem 20 to decompose $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ into a projection to a line followed by a size change.

22. Decompose $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ into a projection perpendicular to a line followed by a size change.

23. Write matrices which project onto the following lines:

(a) $y = x$ (b) $y = 5x$ (c) $y = mx$