11 Matrices Generate Groups

The groups that we examined in the first couple sections of this class have representations with matrices under the operation of matrix multiplication.

Recall that the rotation group of the equilateral triangle could be generated by one element—repeatedly applying a rotation of $120°$. We called this the cyclic group of order 3, or C_3 for short. It took two generators to produce the dihedral group of the equilateral triangle—either a rotation and a reflection or two reflections. We called this the dihedral group of order 6, D_3 for short.

In the following problems you will examine some of these groups, write group tables, and determine which symmetry group each matrix group is isomorphic to. Look for patterns and try to discover the characteristics of each matrix that tell you what it "does" geometrically.

For Problems 1–4:

(a) Specify the elements of the matrix group, unless they are all given,

(b) Describe what each matrix does to the plane,

(c) Construct a group table; you can use a calculator, and

(d) Decide which symmetry group your matrix is isomorphic to.

Let's see an example analysis on the following set of matrices. More analyses are given at the end of the chapter.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) They are all given.

(b) I is the identity transformation. A rotates $180°$ (alternatively, it reflects through the origin). B reflects over the x axis. C reflects over the y axis.

(c)

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

(d) This group is isomorphic to the symmetry group for the rectangle, otherwise known as the dihedral group of the rectangle, D_2 for short.

Now you can try this for yourself!

1. Analyze this group with the following elements, following the form of Example 1. What makes this group fundamentally different from the example?

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

2. The matrix $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ generates a group of order 3. Enumerate the elements of this group and analyze per the example.

3. The matrices $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ generate a group of order 6, of which the group in Problem 2 is a subgroup. Enumerate the elements of the group and analyze per the example.

   (e) Name some other sets of two matrices that could have generated this group.

4. The matrix $\begin{bmatrix} \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{\sqrt{5}-1}{4} \end{bmatrix}$ generates a group of order 5. Enumerate the elements of the group and analyze per the example; you can use a calculator.

5. Let $A = \begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{bmatrix}$, $B = \begin{bmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & -\cos\frac{2\pi}{n} \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and n be an integer. What group is generated by the following sets of generators? Describe them geometrically.

(a) $\{A\}$ (b) $\{B\}$ (c) $\{A, B\}$ (d) $\{B, C\}$

6. Given $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, what is the order of the group generated by the following sets of generators?

   (a) $\{C\}$ (b) $\{D\}$ (c) $\{C, D\}$

7. What matrix could generate a group isomorphic to the cyclic group of order n, C_n?

8. What set of two matrices could generate a group isomorphic to the dihedral group of order $2n$, D_n?

9. Look at Problem 1 on page 21. The adjacency matrices map to a subgroup of the full cube symmetry group. What rotations/reflections do they map to?

10. Given $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, and $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, try understanding the groups generated by:

    (a) $\{P\}$ (c) $\{R\}$ (e) $\{P, R\}$ (g) $\{P, Q, R\}$.
    (b) $\{Q\}$ (d) $\{P, Q\}$ (f) $\{Q, R\}$

11. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ produces a shear. What is its inverse—what undoes the shear?

12. The complex numbers, excluding zero, form a group under multiplication. What set of matrices is isomorphic to the same group under multiplication?

13. Does the set of all 2×2 matrices form a group under multiplication? Why or why not?

The following analyses are more in-depth than Problems 1–4.

Analysis 1

Analyze the group generated by $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ under multiplication.

The elements are as follows:

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, A^6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, A^7 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, A^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

In order, these are rotations of 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$, and $\frac{7\pi}{4}$ radians counterclockwise. A, A^3, A^5, and A^7—A to any power relatively prime to 8—are all generators of the group.[11]

This is the cyclic group of order 8, C_8. It is isomorphic to the rotation group of the regular octagon. $A^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity element.

Analysis 2

Analyze the group generated by $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The generated matrices are as follows. The third, last row is all duplicates.

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, C^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, C^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

[11] Can you figure out why?

$\cdot$	I	A	A^2	A^3	A^4	A^5	A^6	A^7
I	I	A	A^2	A^3	A^4	A^5	A^6	A^7
A	A	A^2	A^3	A^4	A^5	A^6	A^7	I
A^2	A^2	A^3	A^4	A^5	A^6	A^7	I	A
A^3	A^3	A^4	A^5	A^6	A^7	I	A	A^2
A^4	A^4	A^5	A^6	A^7	I	A	A^2	A^3
A^5	A^5	A^6	A^7	I	A	A^2	A^3	A^4
A^6	A^6	A^7	I	A	A^2	A^3	A^4	A^5
A^7	A^7	I	A	A^2	A^3	A^4	A^5	A^6

$$BC = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},\ BC^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ BC^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$CB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = BC^3,\ C^2B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = C^2,\ C^3B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = BC,\ B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

I is the identity. C rotates $90°$, C^2 rotates $180°$, and C^3 rotates $270°$. B reflects over the x axis, BC reflects over line $y = x$, BC^2 reflects over the y axis, and BC^3 reflects over the line $y = -x$. This group is D_4, the symmetry group of the square. It contains the subgroups C_2 and C_4 once each, and four copies of the subgroup D_2.

The group table is shown below. I've used BC instead of C^3B, etc., so that each reflection or rotation is denoted by a unique notation.

	I	C	C^2	C^3	B	BC	BC^2	BC^3
I	I	C	C^2	C^3	B	BC	BC^2	BC^3
C	C	C^2	C^3	I	BC^3	B	BC	BC^2
C^2	C^2	C^3	I	C	BC^2	BC^3	B	BC
C^3	C^3	I	C	C^2	BC	BC^2	BC^3	B
B	B	BC	BC^2	BC^3	I	C	C^2	C^3
BC	BC	BC^2	BC^3	B	C^3	I	C	C^2
BC^2	BC^2	BC^3	B	BC	C^2	C^3	I	C
BC^3	BC^3	B	BC	BC^2	C	C^2	C^3	I