10 Rotations of the Plane

1.

(a) Which matrix changes nothing, so that the image is the same as the preimage?

We already found this; this is the identity matrix I. The 2×2 identity matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Which complex number changes nothing?

1, since $1 \cdot x = x \cdot 1 = x$. Although it might not seem complex at first sight, real numbers are complex too.

2.

(a) Which matrix doubles the length of every vector but leaves angles unchanged?

This is scaling up by a factor of 2 in all directions, which is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

(b) Which complex number corresponds to the same transformation?

This is just 2, since multiplying 2 produces the desired effect of scaling.

3. Based on your answers to the previous problems, which matrix corresponds to the real number r? Let's call this $\mathrm{M}(r)$ for short.

It looks like

$$\mathrm{M}(r) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

4. Explain why $\mathrm{M}(u) + \mathrm{M}(v) = \mathrm{M}(u + v)$.

From the perspective of complex numbers, the relationship $(u)+(v) = u+v$ holds. It also holds for matrices:

$$\mathrm{M}(u) + \mathrm{M}(v) = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} + \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} = \begin{bmatrix} u+v & 0 \\ 0 & u+v \end{bmatrix} = \mathrm{M}(u + v).$$

5. Under a $90°$ counterclockwise rotation, what is the image of (a) $(1, 0)$ and (b) $(0, 1)$?

The image of $(1, 0)$ is $(0, 1)$, and the image of $(0, 1)$ is $(-1, 0)$. You can verify this by drawing it out if you want.

6.

(a) Which matrix corresponds to a $90°$ rotation?

This is just the rotation matrix $\left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$, with $\theta = 90° = \frac{\pi}{2}$:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(b) Which complex number corresponds to the same rotation?

Multiplying by i produces the same rotation.

7. Based on your answers to Problems 1 to 6, what matrix corresponds to the complex number $x + yi$? Let's extend our function M and call this $\mathrm{M}(x + yi)$ for short.

It looks like

$$\mathrm{M}(x+yi) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

8. **Check that $\mathrm{M}(a+bi)+\mathrm{M}(c+di) = \mathrm{M}((a+bi)+(c+di))$. That is, prove that M has the same addition rules as complex numbers.**

We have

$$\mathrm{M}(a+bi)+\mathrm{M}(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -b-d \\ b+d & a+c \end{bmatrix} = \mathrm{M}((a+c)+(b+d)i) = \mathrm{M}((a+bi)+(c+di)).$$

9. **Check that $\mathrm{M}(a+bi)\,\mathrm{M}(c+di) = \mathrm{M}((a+bi)(c+di))$. That is, prove that M has the same multiplication rules as complex numbers.**

We have

$$\begin{aligned} \mathrm{M}(a+bi)\,\mathrm{M}(c+di) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\ &= \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{bmatrix} \\ &= \mathrm{M}((ac-bd)+(ad+bc)i) \\ &= \mathrm{M}((a+bi)(c+di)). \end{aligned}$$

10. **Recall that multiplying by $\operatorname{cis}\theta$ rotates a complex number by θ radians.**

   (a) **Find $\mathrm{M}(\operatorname{cis}\theta)$.**

Since $\operatorname{cis}\theta = \cos\theta + i\sin\theta$, we have

$$\mathrm{M}(\operatorname{cis}\theta) = \mathrm{M}(\cos\theta + i\sin\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

   (b) **To prove that this matrix really does rotate by θ:**

      i. **Check that the image and preimage have the same length;**

Let the preimage by $\begin{bmatrix} x \\ y \end{bmatrix}$, which has length $\sqrt{x^2+y^2}$. Then the image is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix},$$

which has length

$$\begin{aligned} \sqrt{(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2} &= \sqrt{x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta + x^2\sin^2\theta + 2xy\cos\theta\sin\theta + y^2\cos^2\theta} \\ &= \sqrt{x^2(\cos^2\theta+\sin^2\theta) + y^2(\sin^2\theta+\cos^2\theta)} \\ &= \sqrt{x^2+y^2}. \end{aligned}$$

Indeed, this matches up with the length of the preimage.

      ii. **Check that the angle of the image with the x-axis is θ more than the preimage.**

This is a bit unpleasant. The angle of the image with the x-axis is $\tan^{-1}\frac{y}{x}$... we can make this more pleasant by representing our preimage as

$$\begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix},$$

which is a point r away from the origin and making an angle of ϕ with the x-axis.
Then the image is

$$\begin{aligned}
\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix} &= \begin{bmatrix} r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ r\sin\phi\cos\theta + r\cos\phi\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} r(\cos\phi\cos\theta - \sin\phi\sin\theta) \\ r(\sin\phi\cos\theta + \cos\phi\sin\theta) \end{bmatrix} \\
&= \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix}.
\end{aligned}$$

This is a point r away from the origin and making an angle of $\theta + \phi$ with the x-axis, an angle θ more than the original ϕ as desired.

11.

(a) Find $\mathrm{M}(r \operatorname{cis}\theta)$.

$$\mathrm{M}(r\operatorname{cis}\theta) = \mathrm{M}(r\cos\theta + ir\sin\theta) = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}.$$

(b) To prove that this matrix really does rotate by θ and stretch by r:

i. Check that the length of the image is r times the length of the preimage;

Let the preimage be $\begin{bmatrix} x \\ y \end{bmatrix}$, which has length $\sqrt{x^2+y^2}$. Then the image is

$$\begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = r \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{\mathrm{M}(\operatorname{cis}\theta)\cdot\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]}.$$

This has length $r\sqrt{x^2+y^2}$ by the previous problem, as desired.

ii. Check that the angle of the image with the x-axis is θ more than the preimage. (Hint: You may want to use the previous problem, or the tangent addition formulas.)

Let our preimage be

$$\begin{bmatrix} \rho\cos\phi \\ \rho\sin\phi \end{bmatrix},$$

which makes an angle of ϕ with the x-axis. Then the image is

$$\begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \rho\cos\phi \\ \rho\sin\phi \end{bmatrix} = r \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \rho\cos\phi \\ \rho\sin\phi \end{bmatrix}}_{\mathrm{M}(\operatorname{cis}\theta)\cdot\left[\begin{smallmatrix} \rho\cos\phi \\ \rho\sin\phi \end{smallmatrix}\right]},$$

which makes an angle of $\phi + \theta$ with the x-axis via the last problem, as desired.

12.

(a) What matrix reflects over the x-axis, taking $(x, y) \to (x, -y)$?

This is the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, since this flips the y coordinate.

(b) What is the complex number operation equivalent to this transformation?

The equivalent operation is complex conjugation, denoted $\overline{a+bi} = a - bi$.

(c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.

There is not. Suppose there was a complex number $r \operatorname{cis} \theta$ which satisfied

$$(a+bi)r \operatorname{cis} \theta = a - bi.$$

Then since $|a+bi| = |a-bi|$, $r = 1$. So $(a+bi)\operatorname{cis}\theta = a - bi$. But the transformation described is a reflection, while this is a rotation! Thus, no such complex number exists.

13.

(a) What matrix reflects through the origin, taking $(x, y) \to (-x, -y)$?

This is the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, since this flips both the x and y coordinates.

(b) What is the complex number operation equivalent to this transformation?

This is negating the complex number: $f(z) = -z$.

(c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.

Yes there is! $f(z) = -1 \cdot z$ is equivalent; we have $-1 \cdot (a+bi) = -a - bi$ as desired.

14.

(a) Which of the 16 matrices on page 89, for Problem 10, have corresponding complex numbers?

i. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

No, since $1 \neq -1$, and the entries on the top left–bottom right diagonal need to be the same.

ii. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Yes; the complex number is -1.

iii. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Yes; the complex number is 2.

iv. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Yes; the complex number is $-i$.

v. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

No, since $1 \neq -(1)$; the entries on the bottom left–top right diagonal need to be the opposite of each other.

vi. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Yes; the complex number is 0.

vii. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Yes; the complex number is 1.

viii. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

No, since $3 \neq 1$.

ix. $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

No, since $-(-3) \neq 0$.

x. $\begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix}$

No, since $2 \neq -3$.

xi. $\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$

No, since $3 \neq -1$.

xii. $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

No, since $\frac{\sqrt{2}}{2} \neq -\frac{\sqrt{2}}{2}$ (comparing TR and BL corners).

xiii. $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

Yes; the complex number is $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

xiv. $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

No, since $\frac{1}{2} \neq -\frac{1}{2}$.

xv. $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Yes; the complex number is $\frac{\sqrt{3}}{2} + \frac{1}{2}i$.

xvi. $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Yes; the complex number is $\frac{\sqrt{3}}{2} - \frac{1}{2}i$.

(b) How can you tell algebraically?

The matrix must be of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for real (but not necessarily positive) a, b.

(c) How can you tell geometrically?

If the matrix is purely a rotation and dilation, then it has an associated complex number. Note that the zero matrix is a dilation of 0.

15. Make multiplication tables with the set of matrices which correspond to the elements of the rotation group for the square (a 4×4 table) and the equilateral triangle (a 3×3 table).

Square: Define $r = \mathrm{M}(i) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which is a rotation $90°$ counterclockwise. Then we have

$$r^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \qquad r^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The table is shown below.

$\cdot$	I	r	r^2	r^3
I	I	r	r^2	r^3
r	r	r^2	r^3	I
r^2	r^2	r^3	I	r
r^3	r^3	I	r	r^2

Equilateral triangle: Define $r = \mathrm{M}(\mathrm{cis}\,120°) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$. Then we have

$$r^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}; \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The table is shown below.

$\cdot$	I	r	r^2
I	I	r	r^2
r	r	r^2	I
r^2	r^2	I	r

16.

(a) Write a matrix for a rotation of θ around the origin followed by a translation by (a, b).

Recall that for translations, we need 3×3 matrices. The matrix is

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Write a matrix for a translation by (a, b) followed by a rotation of θ around the origin.

The matrix is:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a\cos\theta - b\sin\theta \\ \sin\theta & \cos\theta & a\cos\theta + b\sin\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

17. Use matrix multiplication to find the image (x', y') of a point (x, y) rotated by θ.

We represent (x, y) as $\begin{bmatrix} x \\ y \end{bmatrix}$. With matrix multiplication, we get

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

Thus, $(x', y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

18.

(a) Given the parabola $x = t, y = t^2$, use matrix multiplication to rotate it by $45°$.

Let the new axes be x' and y'. Then we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} t \\ t^2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{2}t^2 \\ \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}t^2 \end{bmatrix}.$$

Thus, $x' = \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{2}t^2$ and $y' = \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}t^2$.

(b) Graph the new parametric equations on your calculator.

Here you go! We let $x = x'$ and $y = y'$ for graphing purposes, which rotates it.

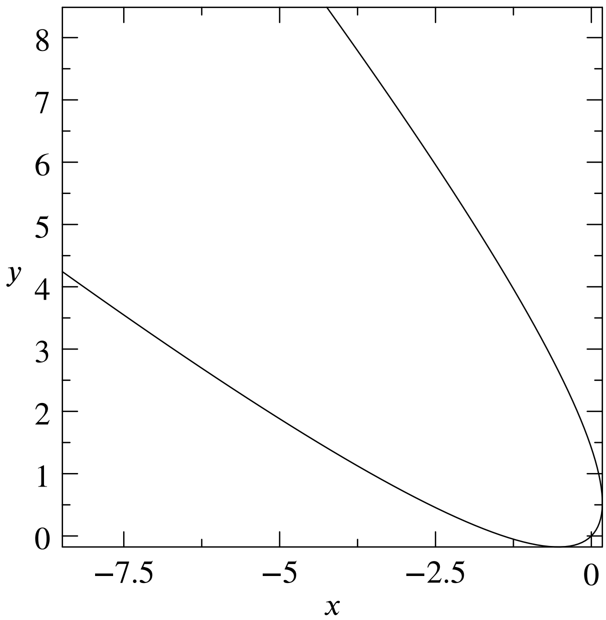

Figure 1: Rotated parabola.

Challenge: Find the maximum x value of this! Requires either some ingenuity or some calculus.[16]

[16] Answer: Many non-calculus ways to do this challenge. One way is to ask what t maximizes $x(t)$.