

## 7 Your Daily Dose of Vitamin $i$

1. We will use complex numbers to find identities for  $\cot$ . Use Pascal's triangle to expand the following:

(a)  $(a + b)^3$

(b)  $(a + b)^4$

(c)  $(a + b)^5$

Then substitute  $b = i = \sqrt{-1}$  and expand:

(d)  $(a + i)^3$

(e)  $(a + i)^4$

(f)  $(a + i)^5$

Finally, substitute  $a = \cot \theta$  and expand:

(g)  $(\cot \theta + i)^3$

(h)  $(\cot \theta + i)^4$

(i)  $(\cot \theta + i)^5$

Consider  $z = i + \cot \theta$ .

(j) Use the above results to find identities for (i)  $\cot 3\theta$ , (ii)  $\cot 4\theta$ , and (iii)  $\cot 5\theta$ .

(k) Graph  $z, z^2, z^3, z^4$ , and  $z^5$ , with  $\theta \approx 75^\circ$ . What method did you use?

2. Compute  $(1 + i)^n$  for  $n = 3, 4, 5, \dots$ . Can you find a general pattern?

3. Expand and graph  $\operatorname{cis}^n \theta$  for  $n = 2, 3, 4, \dots$  and  $\theta \approx 50^\circ$ .

(a) Why is the real part  $\cos n\theta$  and the imaginary part  $\sin n\theta$ ?

(b) Use your results to write identities for  $\cos n\theta$  and  $\sin n\theta$  for  $n = 2, 3, 4, 5$ .

4. Compute  $\cos 7^\circ + \cos 79^\circ + \cos 151^\circ + \cos 223^\circ + \cos 295^\circ$  without a calculator. (Hint: What does this have to do with complex numbers?)

5. Factor the following:

(a)  $x^6 - 1$  as a difference of squares;

(d)  $x^6 - 1$  completely;

(b)  $x^6 - 1$  as a difference of cubes;

(e)  $x^4 + x^2 + 1$  completely.

(c)  $x^4 + x^2 + 1$  over the real numbers;

6. Let  $f(z) = \frac{z+1}{z-1}$ .

(a) Without a calculator, compute  $f^{2020}(z)$ .

(b) What if you replace 2020 with the current year?

7. Find  $\operatorname{Im}((\operatorname{cis} 12^\circ + \operatorname{cis} 48^\circ)^6)$ .

8. Let  $x$  satisfy the equation  $x + \frac{1}{x} = 2 \cos \theta$ .

(a) Compute  $x^2 + \frac{1}{x^2}$  in terms of  $\theta$ .

(b) Compute  $x^n + \frac{1}{x^n}$  in terms of  $n$  and  $\theta$ .