guesses turn out to be correct though. Let's create a table with the orbits and periods of each element, as shown in Figure 1.

The periods appear to be factors of 12, which leads us to consider the rotation group of the regular dodecagon, C_{12}. But this has only half as many elements as our group. The dihedral group D_{12}, however, does have 24 elements. Let's try matching up the elements in the two groups.

$$D_{12}: \text{ Rotations of } 0^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 180^\circ, 210^\circ, 240^\circ, 270^\circ, 300^\circ, 330^\circ$$

0° and 1 are the identities, so they are paired up. Because 2 has period 12, we arbitrarily match it up with the 30° rotation. The rest of the powers of 2 mod 35 are thus mapped, pairing up all the *rotations* of D_{12}.

Rotation	Number
0°	1
30°	2
60°	4
90°	8
120°	16
150°	32
180°	29
210°	23
240°	11
270°	22
300°	9
330°	18

But now we run into a problem. What do we pair up the reflections with? Reflections have period 2, but only 3 elements in our group have that property; we need 12. So D_{12} doesn't work, either!

In fact, this group is a completely new group! It is equal to $C_2 \times C_{12}$, the "product" of two groups we're familiar with. The definition of group product is beyond the scope of this book, but as you can see, the group order 24 is indeed the product of the orders of the groups which comprise the product. This goes to show that we have barely scratched the surface of group theory as a subject. Hopefully you get to experience it more deeply in your later education.

15 Eigenvectors and Eigenvalues

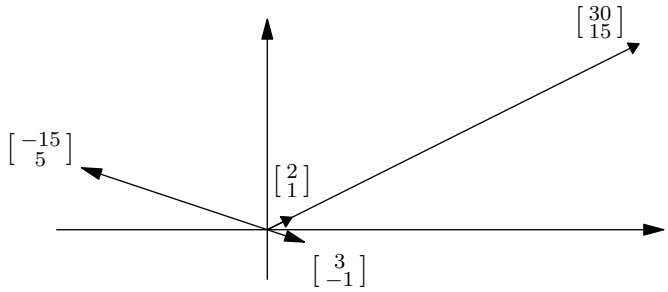

Figure 1: The matrix $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$ acts on two eigenvectors.

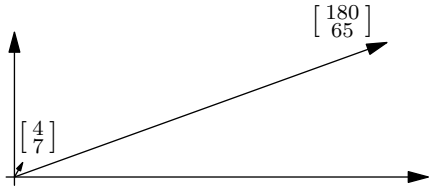

Figure 2: The average Joe does not get to be an eigenvector.

We've spent a good deal of time mapping the plane with matrices. We discovered that we could decompose any invertible 2×2 matrix into a list of matrices which, under multiplication, are a sequence of transformations of the plane. That is, we can interpret any such matrix as a sequence of reflections, rotations, shears, stretches, and dilations. We can even reduce the list to stretches and reflections. Unfortunately, doing this tends to be rather clumsy in practice. In any case, this decomposition method does not produce a unique result.

We are now going to find a new way to decompose matrices. This method will have the virtue that if two people decompose the same matrix, their results will be recognizably "the same." This process is also much easier to do.

Consider the matrix $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$. If you multiply a random vector like $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ by this matrix, you'll get $\begin{bmatrix} 180 \\ 65 \end{bmatrix}$. These two vectors have very different directions, as shown in Figure 2. But if you pick the "right" preimage vector, you can get a vector which has the same—or directly opposite—direction, meaning that the image is a constant multiple of the preimage. For example, if you pick $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then

$$\begin{aligned} \begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 30 \\ 15 \end{bmatrix} \\ &= 15 \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \end{aligned}$$

Similarly, if I pick $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$, then

$$\begin{aligned} \begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} &= \begin{bmatrix} -15 \\ 5 \end{bmatrix} \\ &= -5 \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \end{aligned}$$

These two vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ are called **eigenvectors** of the matrix, and are characteristic to the matrix. Their multiplications are shown in Figure 1. This matrix has only two, **linearly independent** eigenvectors. Linearly independent means that one is not a multiple of another; they have different directions. The vectors' scale factors, 15 and -5, are the **eigenvalues** of the matrix. They are each associated with one eigenvector.

In fact, any pair of vectors $\begin{bmatrix} 2s \\ s \end{bmatrix}$ and $\begin{bmatrix} 3t \\ -t \end{bmatrix}$, as long as $s, t \neq 0$, could be considered the eigenvectors of $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$. We just pick a form that is as simple to write as possible.

We do not consider $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ an eigenvector, because it satisfies $M \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for any 2×2 matrix M and isn't very interesting. You can also justify it on the fact that it cannot have a defined eigenvalue.

We can represent any vector in the plane by adding combinations of the eigenvectors. For instance, we can represent $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ as follows:

$$\begin{bmatrix} 4 \\ 7 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Along with the eigenvalues, this is helpful in matrix multiplication:

$$\begin{aligned}
\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} &= \begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}\left(5\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) && \text{Substituting representation with eigenvectors} \\
&= 15 \cdot 5\begin{bmatrix} 2 \\ 1 \end{bmatrix} - (-5)\cdot 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} && \text{Distributive property} \\
&= \begin{bmatrix} 180 \\ 65 \end{bmatrix}.
\end{aligned}$$

In generic terms: if a matrix M has linearly independent eigenvectors v_1 and v_2 with corresponding eigenvalues λ_1 [15] and λ_2, then for any vector v with an representation with eigenvectors $v = av_1 + bv_2$,

$$Mv = \lambda_1 a v_1 + \lambda_2 b v_2.$$

An issue still remains: I just *gave* you the eigenvectors. How does one find the eigenvalues and eigenvectors of a matrix in the first place? This turns out to be relatively easy algebraically, but we'll try to develop some geometric intuition first.

1. Consider the matrix equation $\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 6x+y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$. We wish to find an eigenvector $\begin{bmatrix} x \\ y \end{bmatrix}$.
   (a) On graph paper, draw what the matrix $\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$ does to the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
   (b) In your picture, draw a rough line through the origin where you think a family of eigenvectors may be.
   (c) Try some lattice points, say $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. What does the matrix transform each vector into?
   (d) Which of these is an eigenvector?
   (e) Does it lie near the line you drew earlier?

2. This guess-and-check process for finding eigenvectors is terrible, so let's develop a procedure to find the eigenvalues and eigenvectors for any 2×2 matrix. We will use the same example.

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \lambda\begin{bmatrix} x \\ y \end{bmatrix} && \text{Definition of eigenvector} \\
&= \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \left(\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \text{Subtraction and factoring} \\
\implies \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

   (a) If $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then
   $$\det\begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} = 0.$$
   Why? Think inverses.

[15] This is the Greek letter lambda. It is traditionally used for eigenvalues.

(b) Find the above determinant in terms of λ and solve for the eigenvalues.

(c) One eigenvalue is $\lambda = 3$. We solve for the associated eigenvector like so:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -3x+y \\ 6x-2y \end{bmatrix} \\
&\implies y = 3x \to \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \quad \text{(for some } s\text{)}
\end{aligned}$$

Solve for the other eigenvector using the other eigenvalue from part (b).

(d) Check your work by multiplying the original matrix by the eigenvector!

3. Solve for the eigenvectors and eigenvalues of the following matrices:

(a) $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 4 & 6 \end{bmatrix}$

4. The image of an eigenvector will have the same ___ when acted on by the transformation ___ for which it is an eigenvector. The image of the eigenvector is simply the eigenvector itself multiplied by its corresponding ___.

5. (a) If the transformation matrix were a reflection over a line $y = x \tan\theta$, in what directions would the two eigenvectors point? Think geometrically.

(b) What would the angle between them be?

(c) What would their eigenvalues be?

6. Recall that multiplication by $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ results in a reflection over $y = x \tan\theta$.

(a) Write a matrix that results in a reflection over the line $y = \frac{\sqrt{3}}{3}x$.

(b) Find the eigenvalues of that matrix, and the corresponding eigenvectors.

(c) Do your calculations agree with your answers to the previous problem?

(d) What are the relationships between the two eigenvectors and between the two eigenvalues?

7. (a) Write a matrix which results in a $60°$ rotation counterclockwise.

(b) Find the eigenvalues. What do you find strange?

(c) Find the eigenvectors for those eigenvalues. What's strange about them?

(d) Explain what's going on.

(e) What are the relationships between the two eigenvectors and between the two eigenvalues?

8. The matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a shear parallel to the x axis.

(a) What vectors don't change direction when multiplied by this matrix?

(b) What would you expect the eigenvectors to be?

(c) Find the eigenvectors and eigenvalues of this matrix.

(d) What is different this time?

(e) Can you represent every vector as sums of eigenvectors?

9. The matrices below result in some stretches. Find the eigenvectors and eigenvalues for both.

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

10. Note that most 2×2 matrices have two eigenvectors. How many would you expect to find for an $n \times n$ matrix?

11. Assuming that p, q, r, s, t, u, x, y are real, what conditions would you impose on them in the matrices (i) $\begin{bmatrix} 3 & p \\ q & 4 \end{bmatrix}$, (ii) $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix}$, and (iii) $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$ to have...
    (a) ... two real eigenvalues?
    (b) ... two complex eigenvalues?
    (c) ... only one eigenvalue?

12. (a) Write a 3×3 matrix showing a rotation of θ around the z axis.
    (b) Name the real eigenvector (this shouldn't require any work).
    (c) Find all three eigenvectors.

13. (a) What should the absolute value of an eigenvalue of any rotation matrix be?
    (b) The complex eigenvalues relate to the angle of rotation. What is that relationship?

14. In a right-handed coordinate system, rotations in three dimensions are performed by combinations of the three matrices

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, Y = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, Z = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

    Each matrix X, Y, Z rotates around the x, y, z axes by α, β, γ, respectively.

    In 2D, rotations combine to make other rotations. Similarly, if we combine any number of these rotations, the net result will be a rotation about some axis—though not necessarily a *coordinate* axis. Another way to picture this is that if we operate on an origin-centered sphere with these matrices, there will always be two opposite points[16] on the sphere which have no net movement.

    Try computing the following products:

    (a) XY (b) XZ (c) YX (d) ZX

15. (a) Without matrices, consider a cube with side length 2 at the origin so its faces are perpendicular to the coordinate axes. Rotate it, first $90°$ counterclockwise about the y axis, then $90°$ counterclockwise about the x axis. Note that rotations are done facing from the "positive side" of the coordinate axis. The net result should leave two vertices fixed. Which two?
    (b) Write a vector for the axis of rotation.
    (c) How many degrees do you think the net rotation of the cube is? Be careful; the answer is not $180°$.
    (d) Let's check our answers using matrices. Write a matrix product that corresponds to a rotation of $90°$ about the y axis, followed by $90°$ about the x axis.
    (e) Multiply out the matrix product.
    (f) Remember that the real eigenvector in a rotation gives the axis of rotation, and the complex eigenvalues give information about the net rotation. Evaluate these and check your answers for (a) and (b).

16. Here are two rotation matrices:
    i. $\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$, ii. $\begin{bmatrix} \frac{7}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} & \frac{8}{9} \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{bmatrix}$.
    (a) What is the determinant of each matrix? (Don't work, think!)
    (b) What is true of each row and each column?
    (c) Find the axis of rotation associated with each matrix.
    (d) Find the angle of rotation associated with each matrix.

[16] These are often called antipodes.