

## 10 Rotations of the Plane

1.

**(a) Which matrix changes nothing, so that the image is the same as the preimage?**

We already found this; this is the identity matrix  $I$ . The  $2 \times 2$  identity matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**(b) Which complex number changes nothing?**

1, since  $1 \cdot x = x \cdot 1 = x$ . Although it might not seem complex at first sight, real numbers are complex too.

2.

**(a) Which matrix doubles the length of every vector but leaves angles unchanged?**

This is scaling up by a factor of 2 in all directions, which is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

**(b) Which complex number corresponds to the same transformation?**

This is just 2, since multiplying 2 produces the desired effect of scaling.

**3. Based on your answers to the previous problems, which matrix corresponds to the real number  $r$ ? Let's call this  $M(r)$  for short.**

It looks like

$$M(r) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

**4. Explain why  $M(u) + M(v) = M(u + v)$ .**

From the perspective of complex numbers, the relationship  $(u) + (v) = u + v$  holds. It also holds for matrices:

$$M(u) + M(v) = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} + \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} = \begin{bmatrix} u+v & 0 \\ 0 & u+v \end{bmatrix} = M(u+v).$$

**5. Under a  $90^\circ$  counterclockwise rotation, what is the image of (a)  $(1, 0)$  and (b)  $(0, 1)$ ?**

The image of  $(1, 0)$  is  $(0, 1)$ , and the image of  $(0, 1)$  is  $(-1, 0)$ . You can verify this by drawing it out if you want.

6.

**(a) Which matrix corresponds to a  $90^\circ$  rotation?**

This is just the rotation matrix  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ , with  $\theta = 90^\circ = \frac{\pi}{2}$ :

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

**(b) Which complex number corresponds to the same rotation?**

Multiplying by  $i$  produces the same rotation.

**7. Based on your answers to Problems 1 to 6, what matrix corresponds to the complex number  $x + yi$ ? Let's extend our function  $M$  and call this  $M(x + yi)$  for short.**

It looks like

$$M(x + yi) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

**8. Check that  $M(a + bi) + M(c + di) = M((a + bi) + (c + di))$ . That is, prove that  $M$  has the same addition rules as complex numbers.**

We have

$$M(a + bi) + M(c + di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a + c & -b - d \\ b + d & a + c \end{bmatrix} = M((a + c) + (b + d)i) = M((a + bi) + (c + di)).$$

**9. Check that  $M(a + bi) M(c + di) = M((a + bi)(c + di))$ . That is, prove that  $M$  has the same multiplication rules as complex numbers.**

We have

$$\begin{aligned} M(a + bi) M(c + di) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\ &= \begin{bmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{bmatrix} \\ &= M((ac - bd) + (ad + bc)i) \\ &= M((a + bi)(c + di)). \end{aligned}$$

**10. Recall that multiplying by  $\text{cis } \theta$  rotates a complex number by  $\theta$  radians.**

**(a) Find  $M(\text{cis } \theta)$ .**

Since  $\text{cis } \theta = \cos \theta + i \sin \theta$ , we have

$$M(\text{cis } \theta) = M(\cos \theta + i \sin \theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

**(b) To prove that this matrix really does rotate by  $\theta$ :**

**i. Check that the image and preimage have the same length;**

Let the preimage be  $\begin{bmatrix} x \\ y \end{bmatrix}$ , which has length  $\sqrt{x^2 + y^2}$ . Then the image is

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix},$$

which has length

$$\begin{aligned} \sqrt{(x \cos \theta - y \sin \theta)^2 + (x \sin \theta + y \cos \theta)^2} &= \sqrt{x^2 \cos^2 \theta - 2xy \cos \theta \sin \theta + y^2 \sin^2 \theta + x^2 \sin^2 \theta + 2xy \cos \theta \sin \theta + y^2 \cos^2 \theta} \\ &= \sqrt{x^2(\cos^2 \theta + \sin^2 \theta) + y^2(\sin^2 \theta + \cos^2 \theta)} \\ &= \sqrt{x^2 + y^2}. \end{aligned}$$

Indeed, this matches up with the length of the preimage.

**ii. Check that the angle of the image with the  $x$ -axis is  $\theta$  more than the preimage.**

This is a bit unpleasant. The angle of the image with the  $x$ -axis is  $\tan^{-1} \frac{y}{x} \dots$  we can make this more pleasant by representing our preimage as

$$\begin{bmatrix} r \cos \phi \\ r \sin \phi \end{bmatrix},$$

which is a point  $r$  away from the origin and making an angle of  $\phi$  with the  $x$ -axis. Then the image is

$$\begin{aligned} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} r \cos \phi \\ r \sin \phi \end{bmatrix} &= \begin{bmatrix} r \cos \phi \cos \theta - r \sin \phi \sin \theta \\ r \sin \phi \cos \theta + r \cos \phi \sin \theta \end{bmatrix} \\ &= \begin{bmatrix} r(\cos \phi \cos \theta - \sin \phi \sin \theta) \\ r(\sin \phi \cos \theta + \cos \phi \sin \theta) \end{bmatrix} \\ &= \begin{bmatrix} r \cos(\phi + \theta) \\ r \sin(\phi + \theta) \end{bmatrix}. \end{aligned}$$

This is a point  $r$  away from the origin and making an angle of  $\theta + \phi$  with the  $x$ -axis, an angle  $\theta$  more than the original  $\phi$  as desired.

11.

(a) Find  $M(r \text{ cis } \theta)$ .

$$M(r \text{ cis } \theta) = M(r \cos \theta + ir \sin \theta) = \begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix}.$$

(b) To prove that this matrix really does rotate by  $\theta$  and stretch by  $r$ :

i. Check that the length of the image is  $r$  times the length of the preimage;

Let the preimage be  $\begin{bmatrix} x \\ y \end{bmatrix}$ , which has length  $\sqrt{x^2 + y^2}$ . Then the image is

$$\begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = r \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{M(\text{cis } \theta) \cdot \begin{bmatrix} x \\ y \end{bmatrix}}.$$

This has length  $r\sqrt{x^2 + y^2}$  by the previous problem, as desired.

ii. Check that the angle of the image with the  $x$ -axis is  $\theta$  more than the preimage. (Hint: You may want to use the previous problem, or the tangent addition formulas.)

Let our preimage be

$$\begin{bmatrix} \rho \cos \phi \\ \rho \sin \phi \end{bmatrix},$$

which makes an angle of  $\phi$  with the  $x$ -axis. Then the image is

$$\begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix} \begin{bmatrix} \rho \cos \phi \\ \rho \sin \phi \end{bmatrix} = r \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \rho \cos \phi \\ \rho \sin \phi \end{bmatrix}}_{M(\text{cis } \theta) \cdot \begin{bmatrix} \rho \cos \phi \\ \rho \sin \phi \end{bmatrix}},$$

which makes an angle of  $\phi + \theta$  with the  $x$ -axis via the last problem, as desired.

12.

(a) What matrix reflects over the  $x$ -axis, taking  $(x, y) \rightarrow (x, -y)$ ?

This is the matrix  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ , since this flips the  $y$  coordinate.

(b) What is the complex number operation equivalent to this transformation?

The equivalent operation is complex conjugation, denoted  $\overline{a + bi} = a - bi$ .

**(c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.**

There is not. Suppose there was a complex number  $r \operatorname{cis} \theta$  which satisfied

$$(a + bi)r \operatorname{cis} \theta = a - bi.$$

Then since  $|a + bi| = |a - bi|$ ,  $r = 0$ . So  $(a + bi) \operatorname{cis} \theta = a - bi$ . But the transformation described is a reflection, while this is a rotation! Thus, no such complex number exists.

**13.**

**(a) What matrix reflects through the origin, taking  $(x, y) \rightarrow (-x, -y)$ ?**

This is the matrix  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ , since this flips both the  $x$  and  $y$  coordinates.

**(b) What is the complex number operation equivalent to this transformation?**

This is negating the complex number:  $f(z) = -z$ .

**(c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.**

Yes there is!  $f(z) = -1 \cdot z$  is equivalent; we have  $-1 \cdot (a + bi) = -a - bi$  as desired.

**14.**

**(a) Which of the 16 matrices on page 87, for Problem 10, have corresponding complex numbers?**

i.  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

No, since  $1 \neq -1$ , and the entries on the top left–bottom right diagonal need to be the same.

ii.  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Yes; the complex number is  $-1$ .

iii.  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Yes; the complex number is  $2$ .

iv.  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Yes; the complex number is  $-i$ .

v.  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

No, since  $1 \neq -(1)$ ; the entries on the bottom left–top right diagonal need to be the opposite of each other.

vi.  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Yes; the complex number is  $0$ .

vii.  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Yes; the complex number is 1.

$$\text{viii. } \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

No, since  $3 \neq 1$ .

$$\text{ix. } \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

No, since  $-(-3) \neq 0$ .

$$\text{x. } \begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix}$$

No, since  $2 \neq -3$ .

$$\text{xi. } \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$$

No, since  $3 \neq -1$ .

$$\text{xii. } \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

No, since  $\frac{\sqrt{2}}{2} \neq -\frac{\sqrt{2}}{2}$  (comparing TR and BL corners).

$$\text{xiii. } \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Yes; the complex number is  $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ .

$$\text{xiv. } \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

No, since  $\frac{1}{2} \neq -\frac{1}{2}$ .

$$\text{xv. } \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Yes; the complex number is  $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ .

$$\text{xvi. } \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Yes; the complex number is  $\frac{\sqrt{3}}{2} - \frac{1}{2}i$ .

**(b) How can you tell algebraically?**

The matrix must be of the form  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  for real (but not necessarily positive)  $a, b$ .

**(c) How can you tell geometrically?**

If the matrix is purely a rotation and dilation, then it has an associated complex number. Note that the zero matrix is a dilation of 0.

- 15. Make multiplication tables with the set of matrices which correspond to the elements of the rotation group for the square (a  $4 \times 4$  table) and the equilateral triangle (a  $3 \times 3$  table).**

Square: Define  $r = M(i) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ , which is a rotation  $90^\circ$  counterclockwise. Then we have

$$r^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \quad r^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The table is shown below.

| $\cdot$ | $I$   | $r$   | $r^2$ | $r^3$ |
|---------|-------|-------|-------|-------|
| $I$     | $I$   | $r$   | $r^2$ | $r^3$ |
| $r$     | $r$   | $r^2$ | $r^3$ | $I$   |
| $r^2$   | $r^2$ | $r^3$ | $I$   | $r$   |
| $r^3$   | $r^3$ | $I$   | $r$   | $r^2$ |

Equilateral triangle: Define  $r = M(\text{cis } 120^\circ) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ . Then we have

$$r^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}; \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The table is shown below.

| $\cdot$ | $I$   | $r$   | $r^2$ |
|---------|-------|-------|-------|
| $I$     | $I$   | $r$   | $r^2$ |
| $r$     | $r$   | $r^2$ | $I$   |
| $r^2$   | $r^2$ | $I$   | $r$   |

**16.**

- (a) Write a matrix for a rotation of  $\theta$  around the origin followed by a translation by  $(a, b)$ .**

Recall that for translations, we need  $3 \times 3$  matrices. The matrix is

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & a \\ \sin \theta & \cos \theta & b \\ 0 & 0 & 1 \end{bmatrix}.$$

- (b) Write a matrix for a translation by  $(a, b)$  followed by a rotation of  $\theta$  around the origin.**

The matrix is:

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & a \cos \theta - b \sin \theta \\ \sin \theta & \cos \theta & a \sin \theta + b \cos \theta \\ 0 & 0 & 1 \end{bmatrix}.$$

- 17. Use matrix multiplication to find the image  $(x', y')$  of a point  $(x, y)$  rotated by  $\theta$ .**

We represent  $(x, y)$  as  $\begin{bmatrix} x \\ y \end{bmatrix}$ . With matrix multiplication, we get

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix}.$$

Thus,  $(x', y') = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$ .

**18.**

- (a) Given the parabola  $x = t, y = t^2$ , use matrix multiplication to rotate it by  $45^\circ$ .**

Let the new axes be  $x'$  and  $y'$ . Then we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} t \\ t^2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{2}t^2 \\ \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}t^2 \end{bmatrix}.$$

Thus,  $x' = \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{2}t^2$  and  $y' = \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}t^2$ .

**(b) Graph the new parametric equations on your calculator.**

Here you go! We let  $x = x'$  and  $y = y'$  for graphing purposes, which rotates it.



Figure 1: Rotated parabola.

Challenge from Tim: find the maximum  $x$  value of this! Requires either some ingenuity or some calculus.<sup>16</sup>

**(c) Does it look like a rotation clockwise or counterclockwise? Why?**

Looks like a rotation by  $45^\circ$  counterclockwise, since that's what the matrix  $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$  does!

<sup>16</sup>Answer: Many non-calculus ways to do this challenge. One way is to ask what  $t$  maximizes  $x(t)$ .