Individual Round, Division A

Division A, 60 Minutes, Individual

1. Given that $x + \frac{1}{x} = 7$, compute $x^3 + \frac{1}{x^3}$.

2. Compute the sum of the exponents in the prime factorisation of 25!. For example, $6! = 2^4 \times 3^2 \times 5$, so the desired sum would be $4 + 2 + 1 = 7$.

3. Compute the maximum possible value of the expression

$$\frac{4x + 7}{x^2 + 3}.$$

4. Let a, b, and c be the roots of $x^3 - 2x^2 + 3x + 4 = 0$. Let $x^3 + px^2 + qx + r = 0$ denote the polynomial with roots a^2, b^2, and c^2. Find the ordered triple (p, q, r).

5. Consider two concentric circles of radii 1 and 2, centered at O. A point P inside the larger circle but outside the smaller circle, and let x denote the length of segment OP. Draw a line segment starting at P that is tangent to the smaller circle and ends on the outer circle. Given that the length of this segment is 3, compute x^2.

6. Satvik wants to reach the cafeteria. He starts at the origin and if he is at position (x, y), he can only move to positions $(x+1, y+1), (x+2, y)$, or $(x, y+2)$. For example, if he is at the point $(3, 5)$, he can move to $(5, 5), (3, 7)$, or $(4, 6)$. How many paths can Satvik take to reach the position $(7, 13)$ so he can eat lunch?

7. Let $ABCD$ be a cyclic quadrilateral such that $AB = 5, BD = 10, DA = 8$, and $BC = CD$. Let P denote the intersection of segments BD and AC. Compute BP.

8. Consider $\triangle ABC$ where AC has length 31, BC has length 30, and AB has length 29. Let D be a point inside $\triangle ABC$ such that $\angle DAC = \angle ACB$, and $\angle ADB = 90^\circ + \angle ACB$ Extend line AD until it intersects BC at point E. Compute $\frac{AD}{BE}$.

9. Compute the fourth-smallest prime factor of $2025^{109} - 1$.

10. Let S denote the set of nonzero square numbers as follows:

$$S = \{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, \dots\}.$$

For any subset of S, let M denote the smallest value in that subset. Compute the variance in M given that each subset is equally likely to be selected.