

## Guts Round, Division A

Division A, 90 Minutes in Teams of 4, Sunday March 30th, 2025

### Set 1 [10]

- How many ordered pairs of positive integers  $(x, y)$  satisfy  $xy + 3x + 7y = 79$ ?

**Answer:**  $\boxed{3}$  (*Proposed by: Neil Dixit*)

- A rhombus of side length 7 has two diagonals, one of which is 7 times longer than the other. Compute the area of the rhombus.

**Answer:**  $\boxed{\frac{343}{25}}$  (*Proposed by: Neil Dixit*)

- Compute the remainder when  $11^{73}$  is divided by 100.

**Answer:**  $\boxed{31}$  (*Proposed by: Neil*)

### Set 2 [11]

- Suppose we have two circles of radii 7 and 8 that are externally tangent at point  $P$ . There is another common tangent line to the circles at points  $Q$  and  $S$  respectively. Let  $R$  denote the intersection of the tangent through  $P$  with  $QS$ . Compute  $PR$ .

**Answer:**  $\boxed{2\sqrt{14}}$  (*Proposed by: Neil Dixit*)

- Given that

$$\sqrt{x+2} + \sqrt{x+5} = 5,$$

compute  $x$ .

**Answer:**  $\boxed{\frac{74}{25}}$  (*Proposed by: Neil Dixit*)

- Ayush is in the mood for some fried chicken. KFC serves chicken tenders in three sizes: boxes of 4 pieces, 7 pieces, and 11 pieces. Find the maximum number of chicken tenders that Ayush cannot buy at KFC.

**Answer:**  $\boxed{17}$  (*Proposed by: Neil Dixit*)

**Set 3 [12]**

7. Given a cone with height 1 and radius 1, a frustum is created by taking the region between heights  $a$  and  $b$  ( $b < a$ ) below the apex. Given that the volume of the frustum equals its height, compute the minimum possible value of  $b$ .

**Answer:**  $\boxed{\sqrt{\frac{1}{\pi}}}$  (*Proposed by: Neil Dixit*)

8. Suppose 3 numbers are randomly chosen with replacement from the set  $\{1, 2, 3, \dots, 44, 45\}$ . What is the probability that these numbers can be the sides of a nondegenerate triangle?

**Answer:**  $\boxed{\frac{1013}{2025}}$  (*Proposed by: Neil Dixit*)

9. Let  $\alpha, \beta, \gamma$ , and  $\delta$  be the roots of  $x^4 - 19x^3 + 14x^2 - 15x + 73 = 0$ . Compute

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} + \frac{1}{\delta+1}.$$

**Answer:**  $\boxed{\frac{52}{61}}$  (*Proposed by: Neil Dixit*)

**Set 4 [13]**

10. Suppose we have a list of positive numbers,

$$\{x_1, x_2, x_3, \dots, x_{2025}\}$$

such that

$$x_1 + 2x_2 + 3x_3 + \dots + 2024x_{2024} + 2025x_{2025} = 2025.$$

Compute the minimum possible value of

$$x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + \dots + 2024x_{2024}^2 + 2025x_{2025}^2.$$

**Answer:**  $\boxed{\frac{2025}{1013}}$  (*Proposed by: Neil Dixit*)

11. Consider  $\triangle ABC$  with  $AB = 13$ ,  $BC = 14$ , and  $CA = 15$ . Let  $M$  be a point on  $BC$  such that  $\angle BAM = \frac{1}{4}\angle ABC$ . Compute  $BM$ .

**Answer:**  $\boxed{\frac{338-39\sqrt{65}}{7}}$

12. Consider the set of all nondegenerate triangles with integer angles. Compute the number of obtuse triangles divided by the number of acute triangles.

**Answer:**  $\boxed{\frac{44}{15}}$  (*Proposed by: Neil Dixit*)

**Set 5 [14]**

13. Evaluate

$$\sum_{n=0}^{\infty} \frac{\cos(n)}{2^n}.$$

**Answer:**  $\boxed{\frac{4-2\cos(1)}{5-4\cos(1)}}$  (*Proposed by: Neil Dixit*)

14. Let  $ABCD$  be a cyclic quadrilateral where  $AB = 1, BC = 2, CD = 3$  and  $DA = 4$ . Let  $P$  be the intersection of diagonals  $AC$  and  $BD$ . Calculate the area of  $\triangle APD$ .

**Answer:**  $\boxed{\frac{24\sqrt{6}}{35}}$  (*Proposed by: Neil Dixit*)

15. Alex randomly chooses 8 distinct integers  $a_1 < a_2 < a_3 \dots < a_7 < a_8$  from the set of integers between 1 and 2024 (including 1 and 2024). Find the expected value of  $a_3$ .

**Answer:**  $\boxed{675}$  (*Proposed by: Alex Tsagaan*)

**Set 6 [16]**

16. Let  $N$  be the answer to Problem 18. Let  $\omega_1$  be a circle centered at  $O$ , and let  $A$  be a point in the interior of  $\omega_1$ . Let  $B$  be a point on  $\omega_1$  such that  $AB \perp OA$ . Consider the locus of all points  $S$  such that the circle with center  $S$  passing through  $A$  is tangent to  $\omega_1$ . Given that the ratio of the enclosed area of  $S$  to the area of  $\omega_1$  is  $N$ , find the ratio of the area of  $\omega_1$  to the area of the circle with diameter  $AB$ .

**Answer:**  $\boxed{\frac{1}{64}}$  (*Proposed by: Alex Tsagaan*)

17. Let  $N$  be the answer to Problem 16. For some real  $0 < k < 1$ , consider the graphs of  $N \cos(x) \sin(y) = k$ , and  $\sin(x) \sin(y) = k$  for  $0 < x, y < \pi$ . Find the largest  $k$  such that these two graphs intersect.

**Answer:**  $\boxed{\frac{1}{\sqrt{4097}}}$  (*Proposed by: Alex Tsagaan*)

18. Let  $\frac{a}{\sqrt{b}}$  be the answer to Problem 17. John has a drawer of  $\lfloor \log_2(b) \rfloor$  socks, each with a specific colour, and exactly  $a$  of the socks are red. Given that the number of socks of each colour are distinct, find the maximum number of different colours of socks John could have.

**Answer:**  $\boxed{4}$  (*Proposed by: Alex Tsagaan*)

**Set 7 [17]**

19. What is the remainder when

$$\sum_{w=1}^{2026} \sum_{x=1}^{2026} \sum_{y=1}^{2026} \sum_{z=1}^{2026} \left\lfloor \frac{wxyz}{2027} \right\rfloor$$

is divided by 1000?

**Answer:** 675 (*Proposed by: Neil Dixit*)

20. Jerry really likes dodecahedrons and spheres. Let  $r$  denote the radius of the largest sphere that can be inscribed in a unit dodecahedron. Compute  $r^2$ .

**Answer:**  $\frac{25+11\sqrt{5}}{40}$  (*Proposed by: Jerry Guo*)

21. Let  $a, b$ , and  $c$  be the roots of  $x^3 + kx^2 + 69x + 15$ . Given there exist nonzero complex numbers  $x, y$ , and  $z$  such that

$$x = (y+z)(a+1),$$

$$y = (z+x)(b+1), \text{ and}$$

$$z = (x+y)(c+1).$$

Find  $k$ .

**Answer:**  $\frac{181}{4}$  (*Proposed by: Alex Tsagaan*)

**Set 8 [18]**

22. Let acute triangle  $\triangle ABC$  with  $\angle B < \angle A < \angle C$ ,  $HO \parallel AC$ , and  $\tan(A) = 4$ . Find  $\tan(B)$ .

**Answer:**  $\frac{19}{8}$  (*Proposed by: Alex Tsagaan*)

23. Angela is at position  $x = 0$ , and moves randomly. Every minute, she either moves forward by 1 with probability  $\frac{1}{2}$ , backward by 1 with probability  $\frac{1}{3}$ , or stays put with probability  $\frac{1}{6}$ . At  $x = 0$ , she can't move backwards, so she stays still with probability  $\frac{1}{2}$  instead. What is the expected number of moves it takes Angela to reach the position  $x = 4$ ?

**Answer:**  $\frac{388}{27}$  (*Proposed by: Neil Dixit*)

24. Compute

$$\prod_{k=1}^6 \left( \sin\left(\frac{2\pi k}{7}\right) + \cos\left(\frac{2\pi k}{7}\right) + \tan\left(\frac{2\pi k}{7}\right) \right).$$

**Answer:**  $\frac{281}{8}$  (*Proposed by: Neil Dixit*)

**Set 9 [19]**

25. The  $\Gamma$  (gamma) function extends the definition of a factorial to all complex numbers, including non-integer real numbers, and it satisfies  $\Gamma(n) = (n - 1)!$  for positive integers. For example,  $\Gamma(8) = 7! = 5040$ . Define  $f(x)$  for positive integer  $x$  as follows:

$$f(x) = \sum_{n=1}^x \left( 1 - \Gamma\left(1 - \frac{2}{n}\right) \right) - e.$$

Estimate the maximum  $x$  such that  $f(x) < 0$ .

**Answer:** 1729 (*Proposed by: Neil Dixit*)

26. As stated in Problem 25, the  $\Gamma$  function extends the definition of the factorial to all positive real numbers (among others). Estimate

$$\frac{\Gamma(\pi + 1)}{\Gamma(e - 1) - \phi} - e.$$

**Answer:** 0.001606 (*Proposed by: Neil Dixit*)

27. 2 prime numbers are considered “cousin primes” if the difference between them is 4. Estimate the number of cousin prime pairs under  $2025^2$ .

**Answer:** 27219 (*Proposed by: Neil Dixit*)

**Set 10 [20]**

28. Let  $S$  denote the set of all nondegenerate triangles with all side lengths less than 2025. Estimate the average circumradius in  $S$ .

**Answer:** 932.50 (*Proposed by: Neil Dixit*)

29. Estimate the number of primes under  $10^9$  that are 1 more than a multiple of 109.

**Answer:** 471020 (*Proposed by: Alex Tsagaan*)

30. The Fermat Numbers are defined as  $F_n = 2^{2^n} + 1$ . Estimate the total of appearances of the digit 1 across  $F_1, F_2, F_3, \dots, F_{24}, F_{25}$ .

**Answer:** 2020000 (*Proposed by: Alex Tsagaan*)

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