Team Round, Division B

Division B, 60 Minutes in Teams of 4

1. Let $f('abc') =' def'$, where a, b, c, d, e, and f are digits. d is the remainder when $a + b$ is divided by 10, $e = b$, and f is the remainder when $b + c$ is divided by 10. Both the input and output from the function must have 3 digits, with no leading zeroes. For how many integers is $f(f('abc')) =' abc'$?

2. Given that the western Roman empire fell to Germanic tribes in 476 AD, compute

$$\frac{1^2 - 2^2 + 3^2 - 4^2 \cdots + 2025^2}{2025}.$$

The historical information has nothing to do with the problem, but Jerry found it funny so it is a part of the problem anyway.

3. $ABCD$ is an isosceles trapezoid with $AD \parallel BC$. $AD = 4, BC = 10$, and a circle ω with center O is inscribed inside $ABCD$.ω is tangent to AB at point P, where $AP = 2$ and $BP = 5$. Compute $\cos^2(\angle AOP)$.

4. Anna and Beatrice are playing tennis, where Anna has a 75% chance of winning any given point. The first person to reach 4 points wins the match (if it reaches $3 - 3$, then the winner of the next point wins the match). Compute the probability that Beatrice wins.

5. Let α, β, and γ be the roots of $f(x)$, and satisfy the following equations:

$$\alpha + \beta + \gamma = 7,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 29,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 103.$$

Compute $f(1)$.

6. Compute, to the nearest integer, the ratio of the number of ways to arrange the letters in "GUNN-MATHCOMP" so that exactly one of the substrings 'GUNN', 'MATH', or 'COMP' appears to the number of ways so that at least two of the substrings appear.

7. An ant starts at the origin, O, initially facing East. The ant moves 1 unit East, then turns $60°$ counterclockwise and moves $\frac{1}{3}$ of a unit. The ant then turns $60°$ counterclockwise again and moves $\frac{1}{9}$ of a unit. The ant continues moving $\frac{1}{3}$ as far as its previous move and turning $60°$ indefinitely. It eventually ends up at the point P. Compute OP^2.

8. Let $\triangle ABC$ be an equilateral triangle of side length 2. Point D satisfies $DA = DB = DC = 3$. Find the inradius of tetrahedron $ABCD$.

9. A circle of radius 1 is inscribed inside a hexagon which is inscribed in the smallest possible square which is inscribed in the smallest possible equilateral triangle which is inscribed in the smallest possible square. Compute the side length of the larger square.

10. Compute

$$\sum_{i=1}^{89} \frac{1}{1 + \cos(i°)} + \frac{1}{1 + \sin(i°)} + \frac{1}{1 + \tan(i°)} + \frac{1}{1 + \sec(i°)} + \frac{1}{1 + \csc(i°)} + \frac{1}{1 + \cot(i°)}.$$

11. Two circles, ω_1 and ω_2, have radii 1 and distinct centers O_1 and O_2. They intersect twice and have a common tangent. A third circle, ω_3 is tangent to this common tangent line, ω_1, and ω_2. Let x be the length of O_1O_2 not in the intersection of ω_1 and ω_2. Given that the line tangent to both ω_1 and ω_3 at a common point intersects O_1O_2 a distance of $\frac{1}{8}$ from O_2, compute O_1O_2.

12. Suppose we have a $3 \times n$ grid and we want to tile it with $3 \times 1$and 1×3 dominoes. What is the minimum n such that we can tile the grid in at least 2025 ways?

13. Find the minimum integer n such that $7^n - 2^n$ is divisible by 2025.

14. a, b, and c are positive, real numbers that satisfy $a + b + c = 21$. Find the minimum possible value of

$$\frac{1}{a+1} + \frac{7}{b+1} + \frac{49}{c+1}.$$

15. Let ω_1 be a circle of radius 6 centered at O_1, and let ω_2 be a circle of radius 8 centered at O_2. They intersect at points A and B. Let C be the second intersection of ω_1 and the line tangent to ω_2 at A. Let D be the second intersection of ω_2 and the line tangent to ω_1 at A. Let line CD intersect the circumcircle of AO_1O_2 at E and F, where E is closer to C. Given that $O_1A = O_2F$ and O_1O_2 is parallel to BF, compute the area of quadrilateral $ABFO_2$.