Individual Round, Division B

Division B, 60 Minutes, Individual

1. Express the recurring fraction $0.\overline{01}$ as a fraction.

2. There are 500 freshmen at Gunn High School, who can take Spanish, French, or Chinese. Students may take none, one, two, or all three of the languages. 300 are taking Spanish, 250 are taking French, and 100 are taking Chinese. 100 are taking French and Spanish, 60 are taking Spanish and Chinese, and 30 are taking French and Chinese. 10 students are taking all three languages. How many students are not studying a foreign language?

3. A circle is inscribed in an equilateral triangle. This triangle is inscribed in a regular hexagon such that its vertices are on the midpoints of the hexagon's edges. All three shapes are concentric. Compute the area of the circle divided by the area of the entire hexagon.

4. Satvik wants to reach the cafeteria. He starts at the origin and if he is at position (x, y), he can only move to positions $(x+1, y+1), (x+2, y)$, or $(x, y+2)$. For example, if he is at the point $(3, 5)$, he can move to $(5, 5), (3, 7)$, or $(4, 6)$. How many paths can Satvik take to reach the position $(8, 8)$ so he can eat lunch?

5. For how many positive integer values of x less than 1000 is

$$\sqrt{x^5 + x^4 + x^3 + x^2 + x + 1 + \tfrac{1}{x-1}}$$

rational?

6. Consider two concentric circles of radii 1 and 2, centered at O. A point P inside the larger circle but outside the smaller circle, and let x denote the length of segment OP. Draw a line segment starting at P that is tangent to the smaller circle and ends on the outer circle. Given that the length of this segment is 3, compute x^2.

7. Samuel writes the numbers 1 to 2025 in binary on a piece of paper. How many zeroes does he write down in total, excluding leading zeroes?

8. The region R consists of eight 1×1 arranged as a 3×3 grid, excluding the middle square. Compute the probability that the midpoint of two points randomly chosen from R lies outside of R.

9. Consider $\triangle ABC$ where AC has length 31, BC has length 30, and AB has length 29. Let D be a point inside $\triangle ABC$ such that $\angle DAC = \angle ACB$, and $\angle ADB = 90^\circ + \angle ACB$. Extend line AD until it intersects BC at point E. Compute $\frac{AD}{BE}$.

10. Define the sequence a_n recursively as follows:
$a_0 = 0, a_1 = 1$, and $a_{n+2} = a_{n+1} + xa_n$ for $n \geq 0$ and some real x. Let S denote the following sum:

$$\sum_{n=1}^{\infty} \frac{a_n}{10^n}.$$

Compute the largest integer value of x such that S is a finite positive number.