Team Round, Division A

Division A, 60 Minutes in Teams of 4

1. $ABCD$ is an isosceles trapezoid with $AD \parallel BC$. $AD = 4, BC = 10$, and a circle ω with center O is inscribed inside $ABCD$. ω is tangent to AB at point P, where $AP = 2$ and $BP = 5$. Compute $\cos^2(\angle AOP)$.

   Answer: $\boxed{\frac{5}{7}}$ *(Proposed by: Neil Dixit)*

2. In the middle of the Earth Tectonics labs, the power goes off in Chanew's chemistry classroom. Dr. Mellows has three red lava lamps and three blue lava lamps. She arranges them in a row on her desk randomly, and then randomly turns three of them on. What is the probability that the leftmost lamp is blue and off, and the rightmost lamp is red and on?

   Answer: $\boxed{\frac{9}{100}}$ *(Proposed by: Chanew Kim)*

3. Compute, to the nearest integer, the ratio of the number of ways to arrange the letters in "GUNN-MATHCOMP" so that exactly one of the substrings 'GUNN', 'MATH', or 'COMP' appears to the number of ways so that at least two of the substrings appear.

   Answer: $\boxed{302}$ *(Proposed by: Neil Dixit)*

4. Suppose you have a polynomial, $p(x)$, of degree 4 with integer coefficients such that $p(0) = 7, p(1) = 11$, and $p(2) = 13$. Compute the minimum possible positive value of $p(25)$.

   Answer: $\boxed{13307}$ *(Proposed by: Neil Dixit)*

5. Suppose that points A, B, and C lie on the curve $y = x^2$ and B has coordinates $(4, 16)$. The x coordinates of A and C sum to 8, and $\angle ABC = 90°$. Let $G = (G_x, G_y)$ be the centroid of $\triangle ABC$ and O be the origin. Compute $G_x + G_y$.

   Answer: $\boxed{\frac{190}{3}}$ *(Proposed by: Neil Dixit)*

6. Two points are randomly chosen a certain distance x from the center of a circle of radius 1. Given the probability that the midpoint of the 2 points lies inside or on the circle is $\frac{1}{6}$, compute x.

   Answer: $\boxed{\sqrt{6} + \sqrt{2}}$ *(Proposed by: Neil Dixit)*

7. Let $\triangle ABC$ be an equilateral triangle of side length 2. Point D satisfies $DA = DB = DC = 3$. Find the inradius of tetrahedron $ABCD$.

   Answer: $\boxed{\frac{6\sqrt{46} - \sqrt{69}}{69}}$ *(Proposed by: Neil Dixit)*

8. Alice and Bob are worst enemies in a class of six students. Each student has at least one friend, no two students share more than one friend, and no three students are all friends with each other. In how many ways can the six students be friends with each other?

   Answer: $\boxed{1370}$ *(Proposed by: Alex Tsagaan)*

9. Let ω_1 be a circle of radius 6 centered at O_1, and let ω_2 be a circle of radius 8 centered at O_2. They intersect at points A and B. Let C be the second intersection of ω_1 and the line tangent to ω_2 at A. Let D be the second intersection of ω_2 and the line tangent to ω_1 at A. Let line CD intersect the circumcircle of AO_1O_2 at E and F, where E is closer to C. Given that $O_1A = O_2F$ and O_1O_2 is parallel to BF, compute the area of quadrilateral $ABFO_2$.

   Answer: $\boxed{\frac{936}{25}}$ *(Proposed by: Alex Tsagaan)*

10. Let $f(n)$ denote the number of solutions, $\{a_1, a_2, a_3, \ldots, a_{4n}\}$, to the congruence

$$\sum_{i=1}^{4n} a_i^2 \equiv 1 \pmod{2025}$$

where all a_i are between 1 and 2025. Find the unique positive integer n such that

$$v_2(f(n)) + v_3(f(n)) + v_5(f(n)) = 2021.$$

Answer: $\boxed{101}$ *(Proposed by: Alex Tsagaan)*