Guts Round, Division B

Division B, 90 Minutes in Teams of 4, Sunday March 30th, 2025

Set 1 [10]

1. Farmer John has been in the cow business for decades, but he has now gotten bored of them. He now has a farm with some numbers of unicorns and peacocks. If the total number of mouths that the farmer needs to feed is 24 and the total number of legs is 74, how many peacocks does the farmer have?

2. Using a combination of pennies, nickels, dimes, and quarters, what is the largest total amount of money in dollars you can have without being able to create exactly one dollar using any combination of those coins?

3. Computer the remainder when 11^{73} is divided by 100.

Set 2 [11]

4. Given a right triangle with side lengths a, b, and c and circumradius R, compute

$$\frac{a^2+b^2+c^2}{R^2}.$$

5. Compute the sum of all positive integers that are equal to 700 times the sum of their digits.

6. How many ordered pairs of positive integers (x, y) satisfy $xy + 3x + 7y = 79$?

Set 3 [12]

7. Let α, β, γ, and δ be the roots of $x^4 - 19x^3 + 14x^2 - 15x + 73 = 0$. Compute

$$\frac{1}{\alpha+1}+\frac{1}{\beta+1}+\frac{1}{\gamma+1}+\frac{1}{\delta+1}.$$

8. A rhombus of side length 7 has two diagonals, one of which is 7 times longer than the other. Compute the area of the rhombus.

9. Neil wants to arrange the integers from 1 to 2025 such that if the set of all integers divisible by 3 or 5 are in order. For example, the number 50 must come after 45 which must come after 42, but the number 46 can be placed anywhere. In how many ways can Neil accomplish this?

Set 4 [13]

10. Let $f(x) = x^4 - px^3 + 54x^2 - qx + r$ have 4 real, positive roots. Compute the maximum possible value of $f(1)$.

11. Suppose we have two circles of radii 7 and 8 that are externally tangent at point P. There is another common tangent line to the circles at points Q and S respectively. Let R denote the intersection of the tangent through P with QS. Compute PR.

12. Calculate the number of positive integers less than or equal to 7! that are not divisible by any of 2, 3, 4, 5, 6, or 7.

Set 5 [14]

13. Given a cone with height 1 and radius 1, a frustum is created by taking the region between heights a and b ($b < a$) below the apex. Given that the volume of the frustum equals its height, compute the minimum possible value of b.

14. The ζ (zeta) function is defined as
$$\zeta(n) = \sum_{m=1}^{\infty} \frac{1}{m^n}.$$
Compute
$$\sum_{k=1}^{\infty} (\zeta(2k) - 1)$$

15. Alex chooses each of 2025 integers, $\{x_1, x_2, x_3, \ldots x_{2025}\}$ as a random integer from 1 to 2025 (with replacement). Compute the expected number of distinct x_i that Alex gets.

Set 6 [16]

16. Suppose we have a list of positive numbers,
$$\{x_1, x_2, x_3, \ldots, x_{2025}\}$$
such that
$$x_1 + 2x_2 + 3x_3 + \ldots 2024x_{2024} + 2025x_{2025} = 2025.$$
Compute the minimum possible value of
$$x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 \cdots + 2024x_{2024}^2 + 2025x_{2025}^2.$$

17. Let $ABCD$ be a cyclic quadrilateral where $AB = 1, BC = 2, CD = 3$ and $DA = 4$. Let P be the intersection of diagonals AC and BD. Calculate the area of $\triangle APD$.

18. Consider the set of all nondegenerate triangles with integer angles. Compute the number of obtuse triangles divided by the number of acute triangles.

Set 7 [17]

19. Consider $\triangle ABC$ with $AB = 13, BC = 14$, and $CA = 15$. Let M be a point on BC such that $\angle BAM = \frac{1}{4}\angle BAC$. Compute BM.

20. Randomly choose 3 numbers (with replacement) from the set $\{1, 2, 3, \ldots, 44, 45\}$. What is the probability that these numbers can be the sides of a nondegenerate triangle?

21. Josh is terrible at playing poker, so let's see if he can redeem himself. A casino offers him the following deal:
If he pays x grams of gold upfront, he can repeatedly roll a die, and whatever he rolls, he adds that to his total score, which starts at 0. After the first roll, he get as much gold as his score. Then on the next roll, he gets $\frac{1}{\phi}$ of his total score. The roll after, he gets $\frac{1}{\phi^2}$ of his total score and so on for an infinite number of rolls, because he is addicted to gambling. What is the minimum integer value of x such that the casino expects to bankrupt Josh in the long run? Note: ϕ is the golden ratio, $\frac{1+\sqrt{5}}{2}$.

Set 8 [18]

22. What is the remainder when
$$\sum_{w=1}^{2026}\sum_{x=1}^{2026}\sum_{y=1}^{2026}\sum_{z=1}^{2026}\left\lfloor \frac{wxyz}{2027} \right\rfloor$$
is divided by 1000?

23. Angela is at position $x = 0$, and moves randomly. Every minute, she either moves forward by 1 with probability $\frac{1}{2}$, backward by 1 with probability $\frac{1}{3}$, or stays put with probability $\frac{1}{6}$. At $x = 0$, she can't move backwards, so she stays still with probability $\frac{1}{2}$ instead. What is the expected number of moves it takes Angela to reach the position $x = 3$?

24. Compute
$$\prod_{k=1}^{6}\left(\sin\left(\frac{2\pi k}{7}\right) + \cos\left(\frac{2\pi k}{7}\right) + \tan\left(\frac{2\pi k}{7}\right)\right).$$

Set 9 [19]

25. The Γ (gamma) function extends the definition of a factorial to all complex numbers, including non-integer real numbers, and it satisfies $\Gamma(n) = (n-1)!$ for positive integers. For example, $\Gamma(8) = 7! = 5040$. Define $f(x)$ for positive integer x as follows:

$$f(x) = \sum_{n=1}^{x} \left(1 - \Gamma\left(1 - \frac{2}{n}\right)\right) - e.$$

Estimate the maximum x such that $f(x) < 0$.

26. As stated in Problem 25, the Γ function extends the definition of the factorial to all positive real numbers (among others). Estimate

$$\frac{\Gamma(\pi+1)}{\Gamma(e-1)-\phi} - e.$$

27. 2 prime numbers are considered "cousin primes" if the difference between them is 4. Estimate the number of cousin prime pairs under 2025^2.

Set 10 [20]

28. Let S denote the set of all nondegenerate triangles with all side lengths less than 2025. Estimate the average circumradius in S.

29. Estimate the number of primes under 10^9 that are 1 more than a multiple of 109.

30. The Fermat Numbers are defined as $F_n = 2^{2^n} + 1$. Estimate the total of appearances of the digit 1 across $F_1, F_2, F_3, \ldots, F_{24}, F_{25}$.