Guts Round, Division A

Division A, 90 Minutes in Teams of 4, Sunday March 30th, 2025

Set 1 [10]

1. How many ordered pairs of positive integers (x, y) satisfy $xy + 3x + 7y = 79$?
2. A rhombus of side length 7 has two diagonals, one of which is 7 times longer than the other. Compute the area of the rhombus.
3. Compute the remainder when 11^{73} is divided by 100.

Set 2 [11]

4. Suppose we have two circles of radii 7 and 8 that are externally tangent at point P. There is another common tangent line to the circles at points Q and S respectively. Let R denote the intersection of the tangent through P with QS. Compute PR.
5. Given that
$$\sqrt{x+2} + \sqrt{x+5} = 5,$$
compute x.
6. Ayush is in the mood for some fried chicken. KFC serves chicken tenders in three sizes: boxes of 4 pieces, 7 pieces, and 11 pieces. Find the maximum number of chicken tenders that Ayush cannot buy at KFC.

Set 3 [12]

7. Given a cone with height 1 and radius 1, a frustum is created by taking the region between heights a and b $(b < a)$ below the apex. Given that the volume of the frustum equals its height, compute the minimum possible value of b.
8. Suppose 3 numbers are randomly chosen with replacement from the set $\{1, 2, 3, \ldots, 44, 45\}$. What is the probability that these numbers can be the sides of a nondegenerate triangle?
9. Let α, β, γ, and δ be the roots of $x^4 - 19x^3 + 14x^2 - 15x + 73 = 0$. Compute
$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} + \frac{1}{\delta+1}.$$

Set 4 [13]

10. Suppose we have a list of positive numbers,

$$\{x_1, x_2, x_3, \ldots, x_{2025}\}$$

such that

$$x_1 + 2x_2 + 3x_3 + \ldots 2024x_{2024} + 2025x_{2025} = 2025.$$

Compute the minimum possible value of

$$x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 \cdots + 2024x_{2024}^2 + 2025x_{2025}^2.$$

11. Consider $\triangle ABC$ with $AB = 13, BC = 14$, and $CA = 15$. Let M be a point on BC such that $\angle BAM = \frac{1}{4}\angle ABC$. Compute BM.

12. Consider the set of all nondegenerate triangles with integer angles. Compute the number of obtuse triangles divided by the number of acute triangles.

Set 5 [14]

13. Evaluate

$$\sum_{n=0}^{\infty} \frac{\cos(n)}{2^n}.$$

14. Let $ABCD$ be a cyclic quadrilateral where $AB = 1, BC = 2, CD = 3$ and $DA = 4$. Let P be the intersection of diagonals AC and BD. Calculate the area of $\triangle APD$.

15. Alex randomly chooses 8 distinct integers $a_1 < a_2 < a_3 \cdots < a_7 < a_8$ from the set of integers between 1 and 2024 (including 1 and 2024). Find the expected value of a_3.

Set 6 [16]

16. Let N be the answer to Problem 18. Let ω_1 be a circle centered at O, and let A be a point in the interior of ω_1. Let B be a point on ω_1 such that $AB \perp OA$. Consider the locus of all points S such that the circle with center S passing through A is tangent to ω_1. Given that the ratio of the enclosed area of S to the area of ω_1 is N, find the ratio of the area of ω_1 to the area of the circle with diameter AB.

17. Let N be the answer to Problem 16. For some real $0 < k < 1$, consider the graphs of $N\cos(x)\sin(y) = k$, and $\sin(x)\sin(y) = k$ for $0 < x, y < \pi$. Find the largest k such that these two graphs intersect.

18. Let $\frac{a}{\sqrt{b}}$ be the answer to Problem 17. John has a drawer of $\lfloor \log_2(b) \rfloor$ socks, each with a specific colour, and exactly a of the socks are red. Given that the number of socks of each colour are distinct, find the maximum number of different colours of socks John could have.

Set 7 [17]

19. What is the remainder when
$$\sum_{w=1}^{2026}\sum_{x=1}^{2026}\sum_{y=1}^{2026}\sum_{z=1}^{2026}\left\lfloor \frac{wxyz}{2027} \right\rfloor$$
is divided by 1000?

20. Jerry really likes dodecahedrons and spheres. Let r denote the radius of the largest sphere that can be inscribed in a unit dodecahedron. Compute r^2.

21. Let a, b, and c be the roots of $x^3 + kx^2 + 69x + 15$. Given there exist nonzero complex numbers x, y, and z such that
$$\begin{aligned} x &= (y+z)(a+1), \\ y &= (z+x)(b+1), \text{ and} \\ z &= (x+y)(c+1). \end{aligned}$$
Find k.

Set 8 [18]

22. Let acute triangle $\triangle ABC$ with $\angle B < \angle A < \angle C, HO \parallel AC$, and $\tan(A) = 4$. Find $\tan(B)$.

23. Angela is at position $x = 0$, and moves randomly. Every minute, she either moves forward by 1 with probability $\frac{1}{2}$, backward by 1 with probability $\frac{1}{3}$, or stays put with probability $\frac{1}{6}$. At $x = 0$, she can't move backwards, so she stays still with probability $\frac{1}{2}$ instead. What is the expected number of moves it takes Angela to reach the position $x = 4$?

24. Compute
$$\prod_{k=1}^{6}\left(\sin\left(\frac{2\pi k}{7}\right) + \cos\left(\frac{2\pi k}{7}\right) + \tan\left(\frac{2\pi k}{7}\right)\right).$$

Set 9 [19]

25. The Γ (gamma) function extends the definition of a factorial to all complex numbers, including non-integer real numbers, and it satisfies $\Gamma(n) = (n-1)!$ for positive integers. For example, $\Gamma(8) = 7! = 5040$. Define $f(x)$ for positive integer x as follows:
$$f(x) = \sum_{n=1}^{x}\left(1 - \Gamma\left(1 - \frac{2}{n}\right)\right) - e.$$
Estimate the maximum x such that $f(x) < 0$.

26. As stated in Problem 25, the Γ function extends the definition of the factorial to all positive real numbers (among others). Estimate
$$\frac{\Gamma(\pi+1)}{\Gamma(e-1) - \phi} - e.$$

27. 2 prime numbers are considered "cousin primes" if the difference between them is 4. Estimate the number of cousin prime pairs under 2025^2.

Set 10 [20]

28. Let S denote the set of all nondegenerate triangles with all side lengths less than 2025. Estimate the average circumradius in S.

29. Estimate the number of primes under 10^9 that are 1 more than a multiple of 109.

30. The Fermat Numbers are defined as $F_n = 2^{2^n} + 1$. Estimate the total of appearances of the digit 1 across $F_1, F_2, F_3, \ldots, F_{24}, F_{25}$.