Calculating the velocity profile for maximum acceleration

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1 No velocity limit

When an object travels an imposed distance with maximum acceration (or deceleration), following equations apply:

$$a = a_{max} = c^{te} \tag{1}$$
$$v = \int a dt = a_{max}\, t \tag{2}$$

This results in a triangular velocity profile, pictured in figure 1.

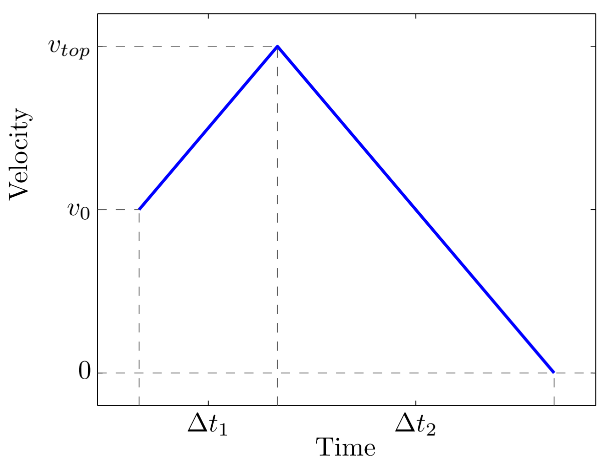

Figure 1: Triangular velocity profile. In the first phase the velocity is increasing, in the second phase velocity is decreasing in order to come to a halt at the destination.

In this case the covered distance is

$$\Delta x = \int_{t_0}^{t_e} v(t)dt \tag{3}$$

$$= \int_0^{\Delta t_1} (v_0 + a_{max}t)dt + \int_0^{\Delta t_2} ((v_0 + a_{max}\Delta t_1) - a_{max}t)dt \tag{4}$$

which simplifies with a change of integral limits of the second integral to:

$$\begin{aligned}
\Delta x &= \int_0^{\Delta t_1} (v_0 + a_{max}t)dt + \int_0^{\Delta t_2} (a_{max}t)dt \\
&= v_0\Delta t_1 + a_{max}\frac{(\Delta t_1)^2}{2} + a_{max}\frac{(\Delta t_2)^2}{2} \\
&= v_0\Delta t_1 + a_{max}\frac{(\Delta t_1)^2 + (\Delta t_2)^2}{2} \qquad (5)
\end{aligned}$$

This equation (5) introduces two new variables: Δt_1 and Δt_2. For solving these variables the maximum attainable velocity v_{top} is also introduced for stating the following equations:

$$v_{top} = v_0 + a_{max}\Delta t_1 \ , \tag{6}$$

$$v_{top} = a_{max}\Delta t_2 \ . \tag{7}$$

When these equations (6)-(7) subsequently are solved for Δt_1 and Δt_2, the substitution of the result in (5) gives

$$\Delta x = v_0\left(\frac{v_{top} - v_0}{a_{max}}\right) + a_{max}\frac{\left(\frac{v_{top}-v_0}{a_{max}}\right)^2 + \left(\frac{v_{top}}{a_{max}}\right)^2}{2} \tag{8}$$

$$= v_{top}^2\left(\frac{1}{a_{max}}\right) - \frac{v_0^2}{2\ a_{max}} \tag{9}$$

which can be solved for v_{top} as

$$v_{top} = \sqrt{a_{max}\Delta x + \frac{v_0^2}{2}} \ . \tag{10}$$

Δt_1 and Δt_2 follow from filling (10) in into (6)-(7).

2 Velocity limit

When the velocity is limited to v_{max}, the triangular velocity profile from figure 1 changes to a truncated triangle (see figure 2) when $v_{top} > v_{max}$.

The time intervals Δt_1 and Δt_3 are easy to determine, as v_{max} is known:

$$\Delta t_1 = (v_{max} - v_0)/a_{max} \tag{11}$$

$$\Delta t_3 = v_{max}/a_{max} \tag{12}$$

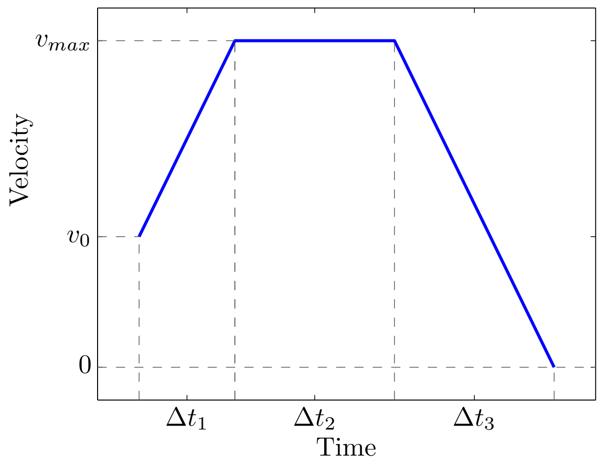

Figure 2: Truncated triangular velocity profile. In the first phase the velocity is increasing, in the second phase the object has reached maximum velocity, in the third phase the velocity decreases in order to come to a halt at the destination.

Analogous to the previous section the covered distance of these two parts is:

$$\Delta x_1 = \frac{v_0 + v_{max}}{2} \Delta t_1 \tag{13}$$

$$\Delta x_3 = a_{max} \frac{(\Delta t_3)^2}{2} \tag{14}$$

With these, the distance Δx_2 is known and thus also Δt_2:

$$\Delta x_2 = \Delta x - \Delta x_1 - \Delta x_3 \tag{15}$$

$$\Delta t_2 = \Delta x_2 / v_{max} \tag{16}$$

3 Position

Position of both situation is obtainable by integrating the velocity:

$$x(t) = \int v(t) dt \ , \tag{17}$$

which results in figure 3 and 4 for respectively section 1 and 2.

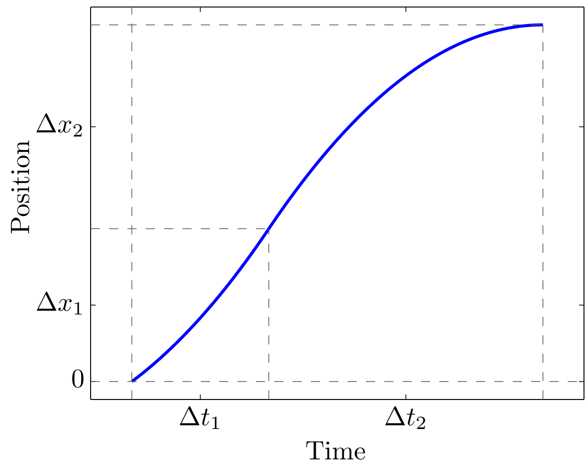

Figure 3: Position of the object, moving with maximum acceleration/deceleration, without velocity limit.

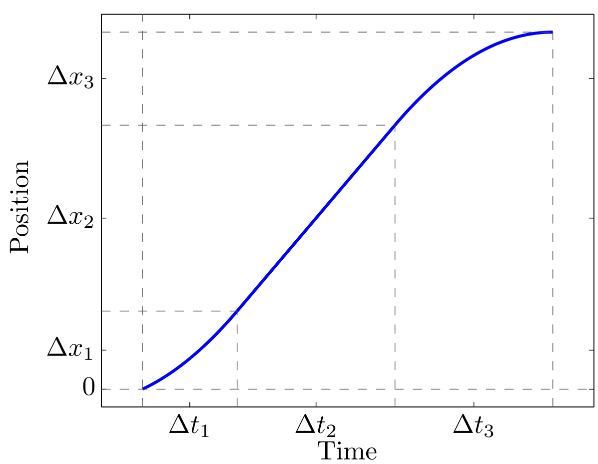

Figure 4: Position of the object, moving with maximum acceleration/deceleration, respecting the velocity limit v_{max}.