

# Digital Signal Processing

GuoHui

Xiamen University

## Collaborator:

GuoHui

Xiamen University

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## Abstract

These are notes on **DSP**.

The textbook we use is **Discrete-Time Signal Processing, 3rd edition** written by A. V. Oppenheim.

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# 1 Introduction

## 1.1 Some Suggestions

In the first subsection, let's give the following suggestions.

**No reading, No learning.**

After we enter a high level of learning, we mainly depend on reading, because no one will tell us the key points. The reason for setting up a series of courses is to make our learning more targeted.

**No writing, No reading.**

What is effective reading? You must write something in the paper, such as the math notations. Only in that way will your brain not skip many details.

**No data, No truth.**

In the vast majority of scientific fields, we must apply theories and methods to real data rather than relying on simulation.

**No Analytic, No understanding.**

If you say, "You have well understood something", you must provide a detailed analysis using symbolic language.

**No programming, No cognition.**

Finally, if you wish to have an intuitive cognition of something, the typical approach is to use a series of visualization methods (such as pictures, tables...) to present the images in your mind.

## 1.2 Digital vs Analog

There are the following differences between the two.

First, digital means **programmable**. Analog rely on hardware, digital are true software, which is very flexible and easy to change.

Second, in the digital world we use bit to describe precision. And digital has a high and controllable precision.

Third, storing, digital has a high storage density and compressible.

Finally, digital's cost is low, because it's manufactured on a silicon wafer.

### 1.3 History of DSP

Now let's briefly review the history of DSP.

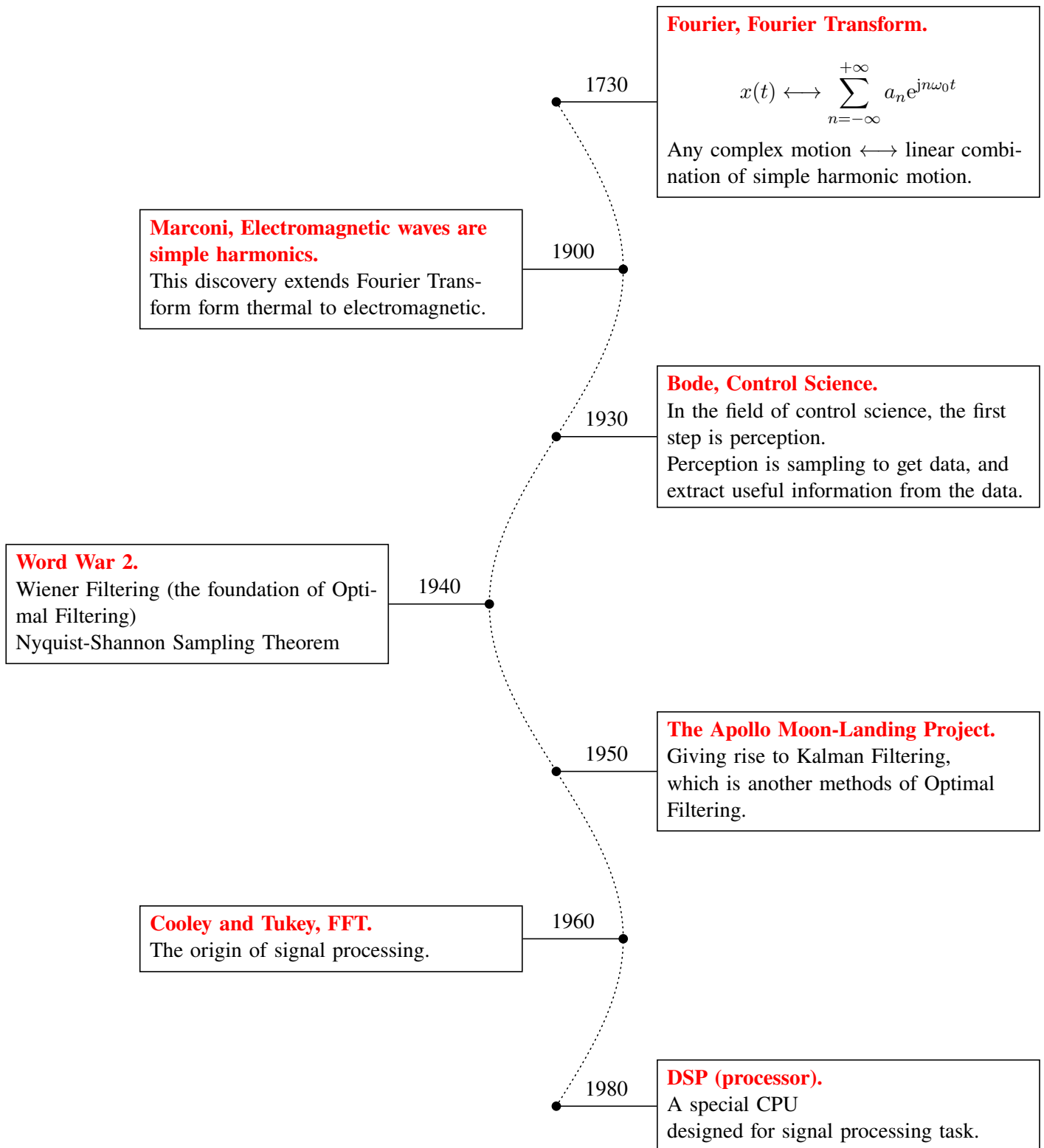


Figure 1: History of DSP

## 2 Discrete Time Signal and System (Time Domain)

### 2.1 Conventions

Considering the continuous signal  $x(t)$ , we can find it has two elements:

- Independent Variable: It can be time or space, here is time  $t$ .
- Dependent Variable.

Therefore, we can draw a conclusion: Signal is a function.

However, a mathematical function often has analytical expression, for example:

$$x(t) = 3t^2 \quad (1)$$

But signals often don't have explicit analytical expressions, so we need to process the signal to get information.

Now let's introduce continuous-time signal and discrete-time signal:

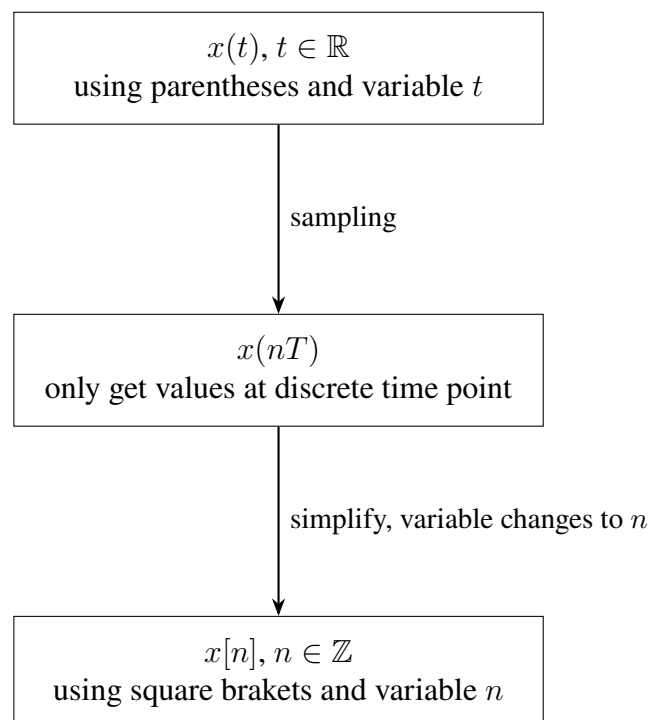


Figure 2: Signals

## 2.2 Special Signals

Here we introduce some signals.

### 2.2.1 Impulse signal

Definition:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \quad (2)$$

The most important point about this signal is every signal  $x[n]$  can be decomposed into the linear combination of shifted impulse  $\delta[n - k]$ :

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \underbrace{\delta[n - k]}_{\text{fundamental element}}, k \in \mathbb{Z} \quad (3)$$

And this formula leads to an important idea:

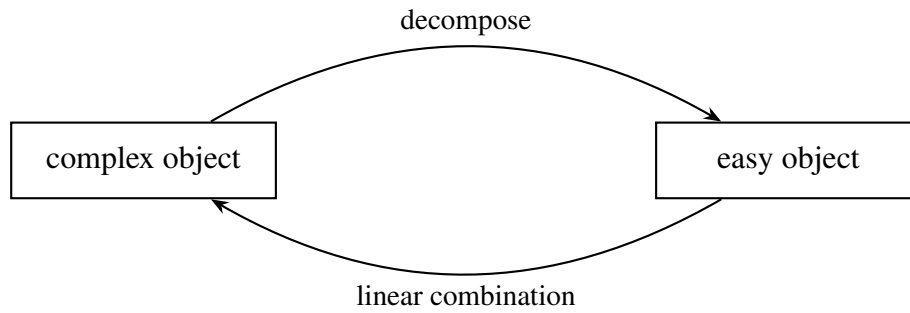


Figure 3: An important Idea

This idea tells us: Complex objects can be decomposed into the linear combination of many easy objects.

### 2.2.2 Unit step

Definition:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad (4)$$

So what is the relation of  $u[n]$  and  $\delta[n]$ ? We can easily get:

$$\begin{aligned}\delta[n] &= u[n] - u[n-1] \\ u[n] &= \sum_{k=-\infty}^n \delta[k]\end{aligned}\tag{5}$$

### 2.2.3 an Important Function

$$x_{\omega}(t) = e^{j\omega t}\tag{6}$$

The subscript of  $x$  is  $\omega$ , means the frequency.

Why do we introduce it? Because if we perform a linear transform on this function, the frequency doesn't change.

And we can give the discrete-time version:

$$x_{\omega}[n] = e^{j\omega n}\tag{7}$$

As  $n \in \mathbb{Z}$ , you can get an interesting property:

$$\begin{aligned}x_{\omega+2\pi}[n] &= e^{j(\omega+2\pi)n} \\ &= e^{j\omega n}\end{aligned}\tag{8}$$

This means, in the frequency domain, this function has periodicity, and the period is  $2\pi$ .

What frequency does  $2\pi$  here correspond to in the physical world? You will get the answer in the following chapter.

## 2.3 Discrete-Time System

We use the following diagram to represent the system and its function:

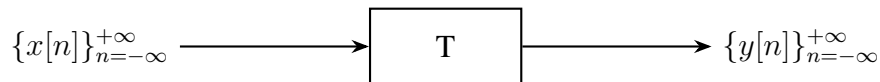


Figure 4:  $y[n] = T(\{x[n]\})$

Why does we use  $\{x[n]\}_{n=-\infty}^{+\infty}$  and  $\{y[n]\}_{n=-\infty}^{+\infty}$  to represent the input and output? Because the sysetem is divided into:

- Memory:  $y[n]$  (output at time point  $n$ ) depends on current and past inputs.

- Memoryless:  $y[n]$  (output at time point  $n$ ) only depends on current time point input  $x[n]$ .

So when we express the function of a system using symbols, a more rigorous way of writing it is :

$$y[n] = T(\{x[n]\}) \quad (9)$$

This means the system  $T$  operates on a sequence of inputs, not only one input at time  $n$ . But for simplicity, we write it:

$$y[n] = T(x[n]) \quad (10)$$

Next, we introduce some typical systems.

### 2.3.1 Delay Device

$$y[n] = x[n - n_d] \quad (11)$$

$n_d$  is a scalar.

How should we understand the function of the system? We can give an example.

Suppose  $n_d$  is 2, and we want to get the output at time 3, that is  $y[3]$ . According to the formula, we have:

$$\begin{aligned} y[3] &= x[3 - n_d] \\ &= x[1] \end{aligned} \quad (12)$$

This means, if we want to get the current output, we need to use the past input. So this system is **memory**.

### 2.3.2 Integrator

$$y[n] = \sum_{k=-\infty}^n x[k] \quad (13)$$

This formula leads to an important idea:



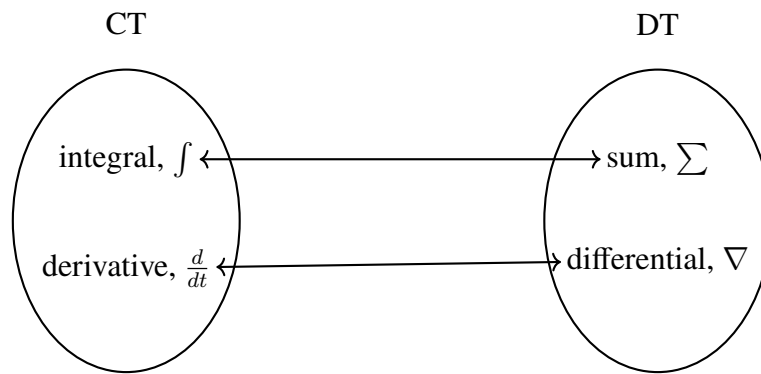


Figure 5: Continuous and Discrete Time Operations

Obviously, Integrator is **memory**.

### 2.3.3 Square Device

$$y[n] = x^2[n] \quad (14)$$

Obviously, Integrator is **memoryless**.

## 2.4 Linearity

Linearity is an important property of systems, and also the key to our research.

Let's give the definition from the surface. If a system  $T$  is a linear system, then:

- Additivity:  $T(x[n] + y[n]) = T(x[n]) + T(y[n])$ .
- Scaling property:  $T(\alpha x[n]) = \alpha T(x[n])$ .

(we don't know it's "memory" or "memoryless").

This definition just gives us the surface input-output relation, but the system is still a black box.

Now let's analyze the internal structure of the system using [An important Idea](#).

Suppose  $T$  is a linear system, look at the following operations:

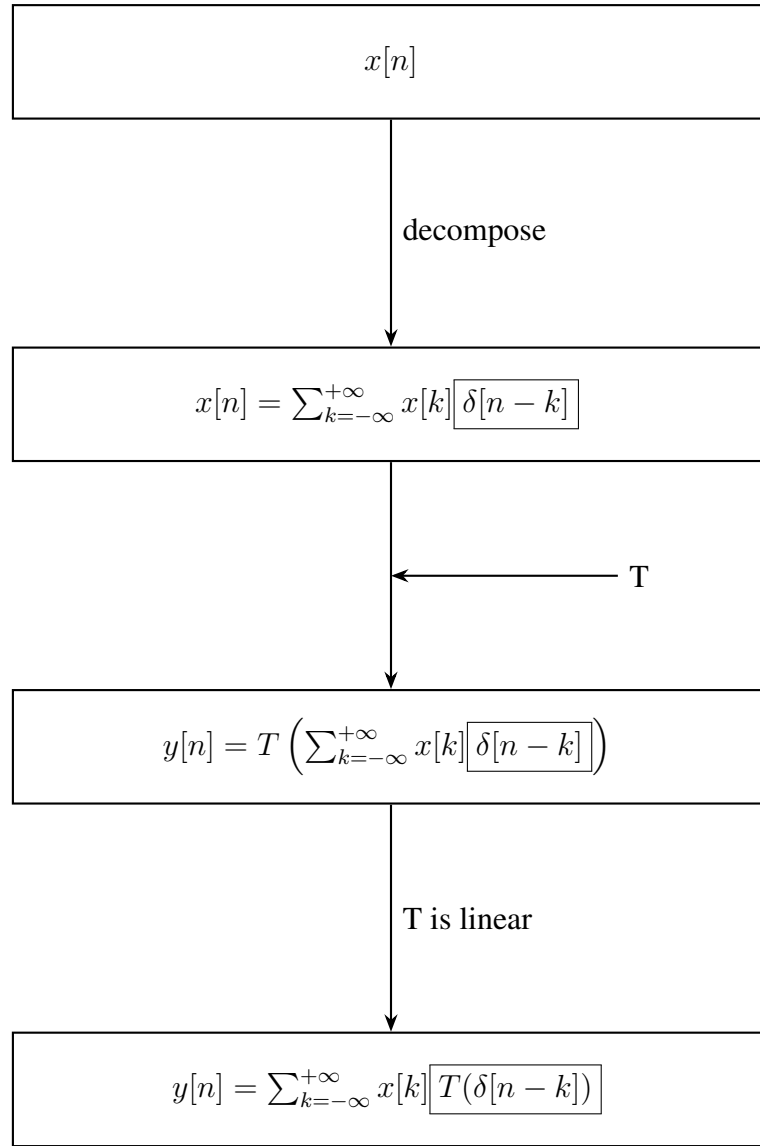


Figure 6: System Process

So we realize the following function:

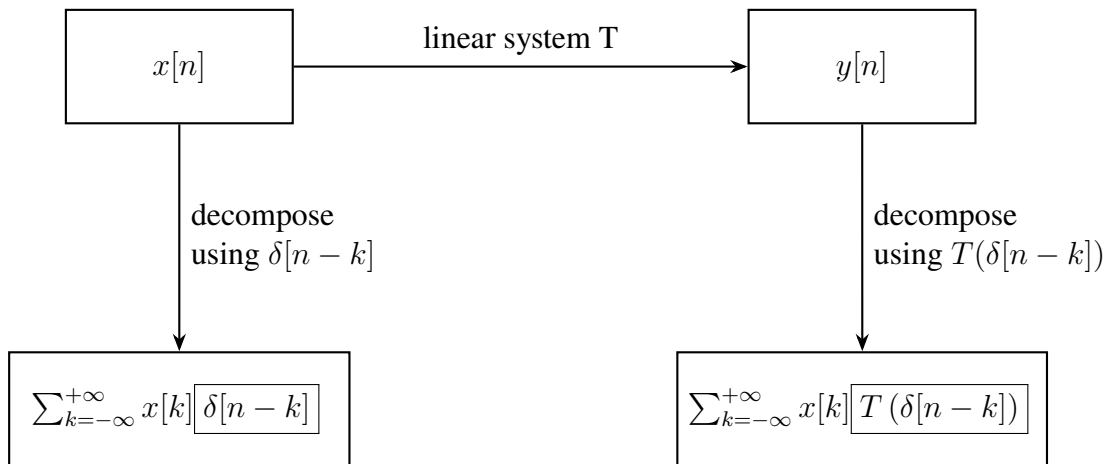


Figure 7: Summary of Linear System Derivation

That is to say, we decompose the input  $x[n]$  into the linear combination of impulses. And as the system  $T$  is linear, the output can also be decomposed into the linear combination, and the elements is impulse response:

$$h(n, k) = T(\delta[n - k]) \quad (15)$$

## 2.5 Understanding Impulse Response

In the above subsection, we get the definition of **Impulse Response**:

$$h(n, k) = T(\delta[n - k]) \quad (16)$$

We give the following obvious understanding:

- $h(n, k)$  is a scalar, when  $n$  and  $k$  are all determined.
- $h(n, k)$  is a function of variable  $n$  and  $k$ .

Why do we call it impulse response?

Suppose our input is an impulse at time 2:

$$x[n] = \delta[n - 2] \quad (17)$$

You can get:

$$y[n] = T(\delta[n - 2]) = h(n, 2) \quad (18)$$

Here, both  $\delta[n - 1]$  and  $h(n, 2)$  are signals, not a scalar:

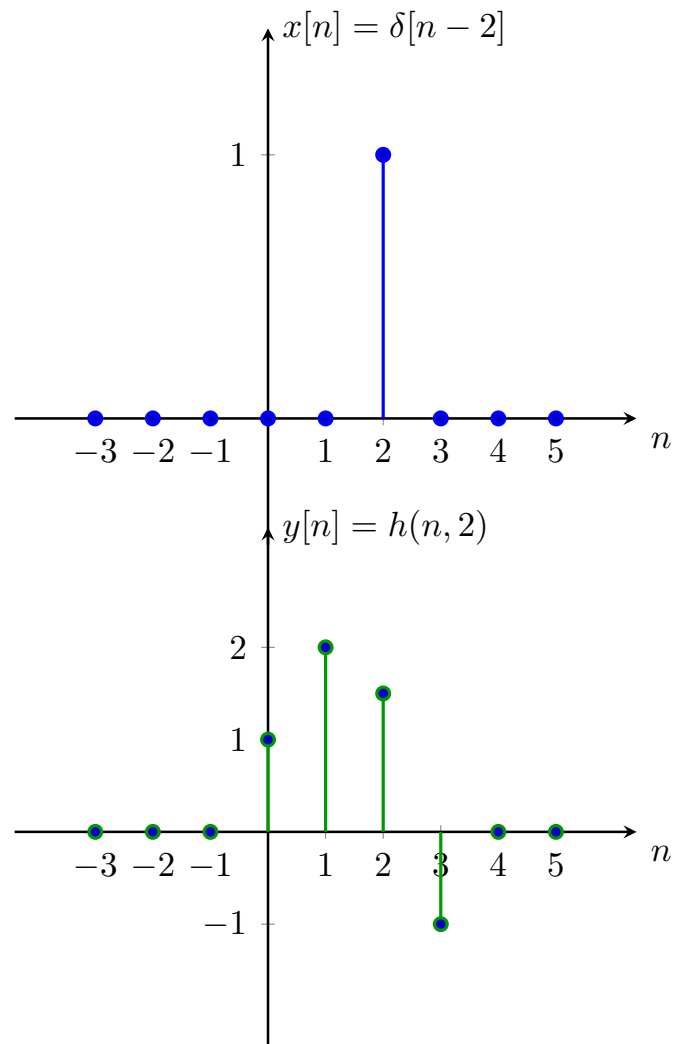


Figure 8: Impulse Response

If our input becomes  $\delta[n - 2] + \delta[n - 3]$ , what will happen? Our output will become:

$$y[n] = h(n, 2) + h(n, 3) \quad (19)$$

And if we pay attention to the output at time point 3 or  $y[3]$ , it's form is:

$$y[3] = h(3, 2) + h(3, 3) \quad (20)$$

So, in a summary:

- $h(n, 2)$  means, give an impulse at time 2, the output containing all the time points that we get.
- $h(3, 2)$  means, give an impulse at time 2, the output at time 3 that we get.

## 2.6 Linearity-Matrix Perspective

Matrix is also a linear transform.

Suppose we have the following matrix  $T$ :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (21)$$

We use this matrix to operate on the following column vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (22)$$

The results are:

$$\begin{bmatrix} \boxed{a_{11}} & \boxed{a_{12}} & \boxed{a_{13}} \\ \boxed{a_{21}} & \boxed{a_{22}} & \boxed{a_{23}} \\ \boxed{a_{31}} & \boxed{a_{32}} & \boxed{a_{33}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \quad (23)$$

How can we using the idea above to explain this?

Actually, we perform the following operations:

$$\begin{aligned} T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= T \left( x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \\ &= x_1 T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + x_2 T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + x_3 T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \end{aligned} \quad (24)$$

We have the similar "impulse":

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (25)$$

And we have the similar "impulse response":

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \quad (26)$$

From this example: "impulse response" is a nature of the system itself, has nothing to do with the input.

## 2.7 Time-Invariant

Suppose there is an arbitrary system  $T$ , and we have an input-output combination:

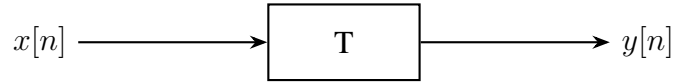


Figure 9:  $y[n] = T(\{x[n]\})$

If system  $T$  is time-invariant, that is to say:

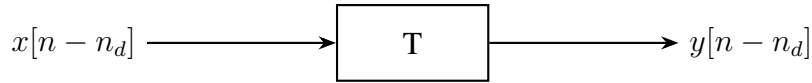


Figure 10: Time-Invariant System  $T$

Here  $n_d$  is a scalar.

So in a word, if one system  $T$  is time-invariant, then the output delay is the same as the input delay.

## 2.8 LTI System

If a system simultaneously satisfies linearity and time invariance, then we call the system **LTI System**.

In the above subsection, if a system  $T$  satisfies linearity, then we can give **Impulse Response**:

$$h(n, k) = T(\delta[n - k]) \quad (27)$$

Here  $h(n, k)$  is a function of variable  $n$  and  $k$ , and we just know this function has two variables.

Our problem is, if we give the linear system  $T$  the time invariance, what form should  $h(n, k)$  meet?

Before the reasoning begins, why do we focus on  $h(n, k)$ ? Because here we research the nature of the system, and impulse response is the core of this problem.

Particularly, we give the following definition:

$$h[n] = h(n, 0) = T(\delta[n]) \quad (28)$$

$h[n]$  means the impulse response of  $T$  when we give an impulse at time 0.

This is equivalent to: We give  $T$  an input  $x[n] = \delta[n]$ , the output is  $y[n] = h[n]$ .

Now give our input delay of  $k$ :

$$x[n] = \delta[n] \xrightarrow{\text{delay } k} \hat{x}[n] = \delta[n - k] \quad (29)$$

Suppose system  $T$  is time-invariant:

$$\hat{y}[n] = y[n - k] = h[n - k] \quad (30)$$

As:

$$h(n, k) = T(\delta[n - k]) \quad (31)$$

So:

$$h[n - k] = h(n, k) \quad (32)$$

## 2.9 Convolution

Now we know, if a system  $T$  satisfies linearity and time-invariance, give it an input  $x[n]$ , output is:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n - k] \quad (33)$$

We can give the definition of two signals' \* operation:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n - k] \quad (34)$$

And we can say: If a system is LTI system, then:

$$\text{output} = \text{input} * \text{impulse response} \quad (35)$$

## 2.10 An Example of Convolution

Suppose  $T$  is a LTI system, and the impulse response is:

$$h[n] = u[n] - u[n - N] \quad (36)$$

here  $u[n]$  is unit step,  $N$  is a positive integer. We give it an input:

$$x[n] = a^n u[n] \quad (37)$$

here  $|a| < 1$ .

Calculate the output:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n - k] \quad (38)$$

### 2.10.1 Method1. Analytical Calculations.

Expand the convolution formula:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n - k] \\ &= \sum_{k=-\infty}^{+\infty} a^k u[k]u[n - k] - \sum_{k=-\infty}^{+\infty} a^k u[k]u[n - N - k] \end{aligned} \quad (39)$$

As the unit step  $u[n]$  gives 0 when  $n < 0$ , and  $k$  is summation variable, we need to determine which part of  $k$  gives a non-zero result.

For the part  $\sum_{k=-\infty}^{+\infty} a^k u[k]u[n - k]$ , if it's non-zero, then:

$$k \in [0, n] \quad (40)$$

For the part  $\sum_{k=-\infty}^{+\infty} a^k u[k]u[n - N - k]$ , if it's non-zero, then:

$$k \in [0, n - N] \quad (41)$$

Then because our main variable is  $n$ , we need to have a segmented discussion about different range of  $n$ , and use the (40) and (41) to find whether the part is 0 or not.



For  $n \in (-\infty, 0)$ , the range of  $k$  can't meet (40) or (41), so both part are 0:

$$y[n] = 0 \quad (42)$$

For  $n \in [0, N - 1]$ , the range of  $k$  can meet (40), but can't meet (41):

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{+\infty} a^k u[k] u[n - k] \\ &= \sum_{k=0}^n a^k \end{aligned} \quad (43)$$

For  $n \in [N, \infty]$ , the range of  $k$  can meet (40), and can meet (41):

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{+\infty} a^k u[k] u[n - k] - \sum_{k=-\infty}^{+\infty} a^k u[k] u[n - N - k] \\ &= \sum_{k=0}^n a^k - \sum_{k=0}^{n-N} a^k \end{aligned} \quad (44)$$

### 2.10.2 Graphical Solution

The graph of  $h[n]$  and  $x[n]$  are as follows, here we choose  $a = 1/2$ :

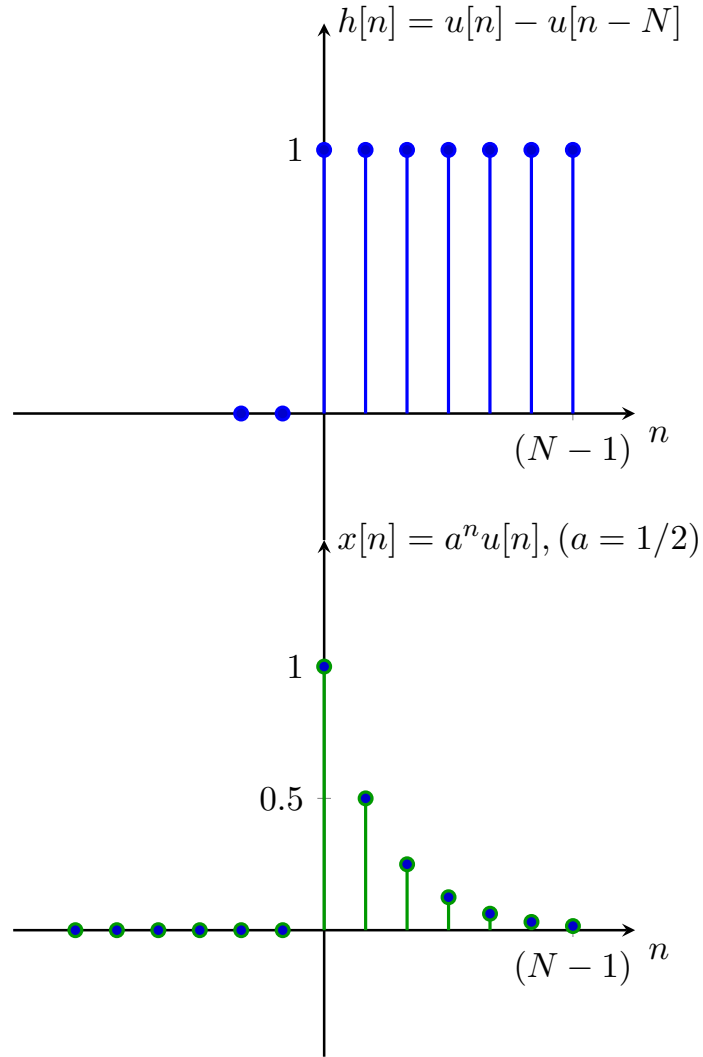


Figure 11:  $h[n]$  and  $x[n]$

As the summation variable is  $k$ , we perform the following operations:

$$h[n - k] = h[-(k - n)] \quad (45)$$

How can we get  $h[-(k - n)]$ ? We can use  $h[-k]$  and shift  $n$  units:

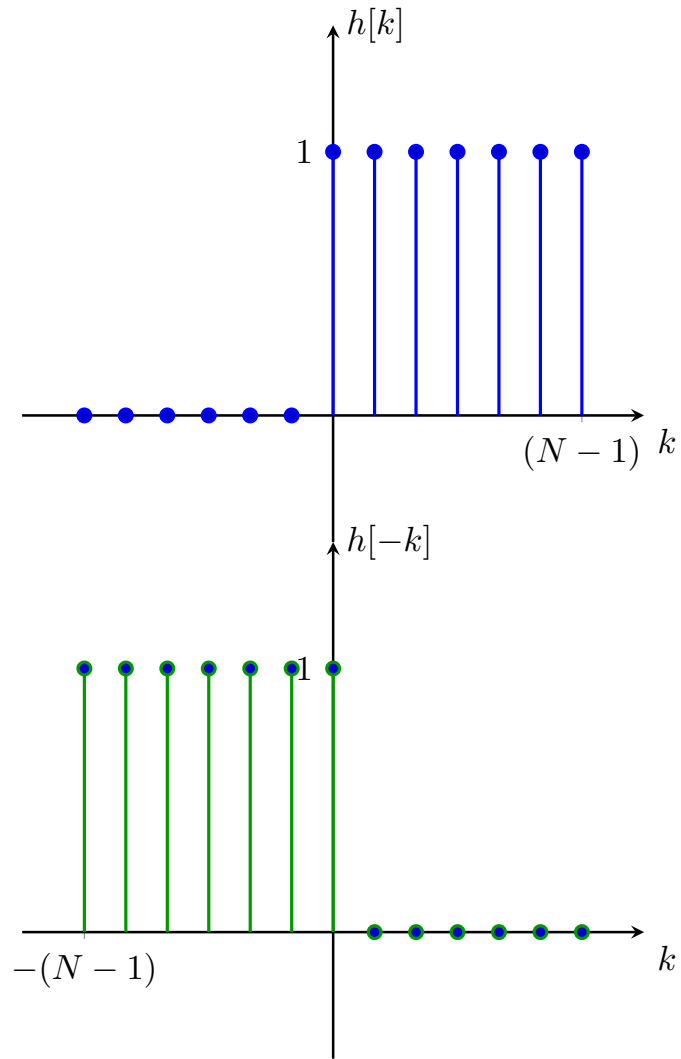


Figure 12: Top: The signal  $h[n]$  and time-reversed signal  $h[-n]$ .

Then shift  $h[-k]$  by  $n$  units, and perform corresponding multiplication:

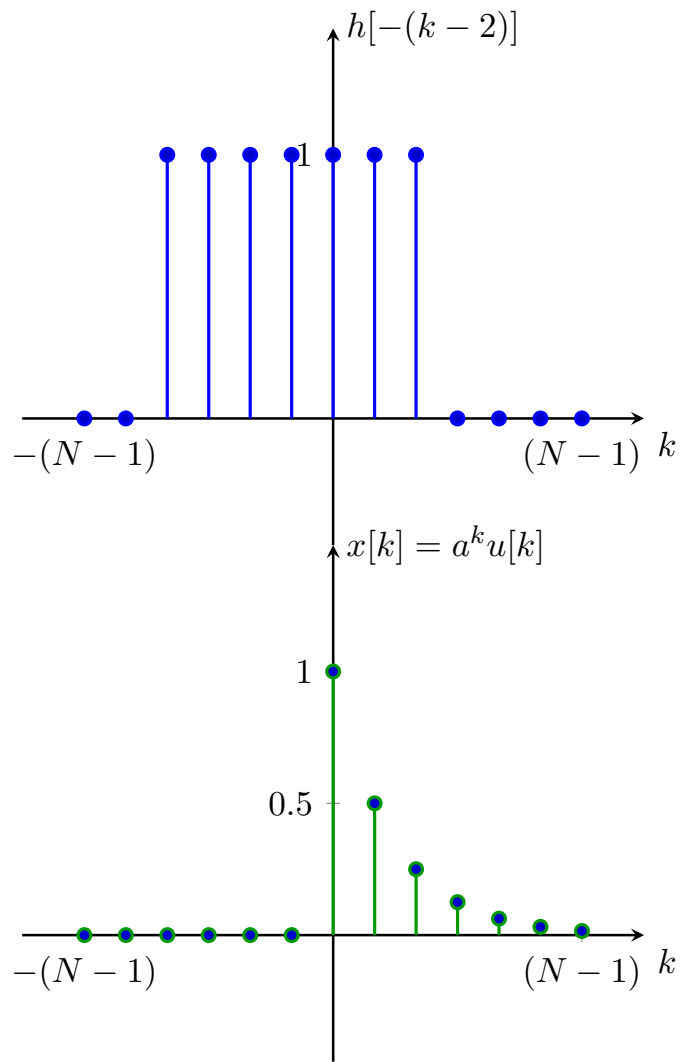


Figure 13:  $y[n] = \sum x[k]h[n-k]$  at  $n = 2$ .

### **3 Discrete Time Signal and System (Frequency Domain)**