Linear Algebra

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Abstract

This is a note about Linear Algebra.

Contents

1 The Geometric Interpretation of Systems of Equations

1.1 Two equations, two unknowns

We have the following system of equations:

$$\begin{cases} 2x - y = 0 & \text{(1a)} \\ -x + 2y = 3 & \text{(1b)} \end{cases}$$

First, let's observe the system from the perspective of a matrix. From the system,we can extract the coefficient matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{2}$$

the unknown vector:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

and the right vector is:

$$\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \tag{4}$$

so the system of equations can be written in matrix form:

$$\mathbf{Ax} = \mathbf{b} \tag{5}$$

Second, we give the **row picture**. We can find that, each row of (1) corresponds to a staright line. Draw it on the rectangular coordinate system of a plane:

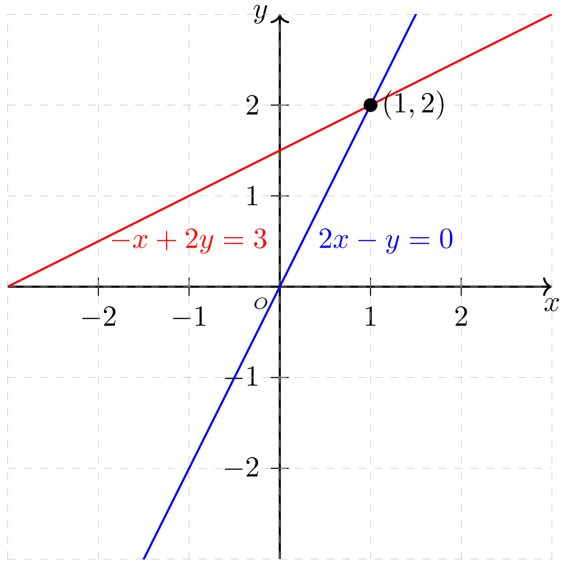

Figure 1: Two lines intersect

The intersection point $(1, 2)$ is the solution of (1).

Finally, we give the **column picture**.We can write the system in the below form:

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \tag{6}$$

We can understand the above formula as the **linear combination of column vectors**:

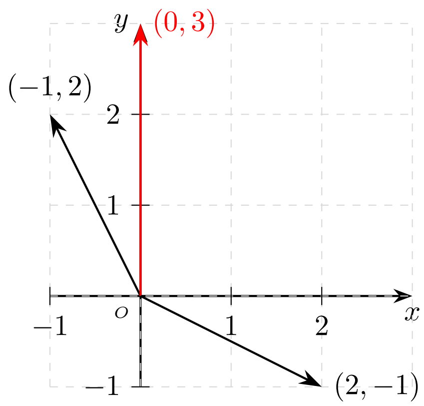

Figure 2: Column Picture

1.2 A simple summary

In the previous subsection, we use matrix form, row picture, column picture to describe the system of equations. And the most important point is the column picture.

The column picture tells us that, we can use the linear combination of two vectors (satisfied certain conditions) to obtain the third vector.

Note:If we take x and y as any combination of numbers, than the linear combination of tow vectors (satisfied certain conditions) can obtain every vectors in the plane.

1.3 Three equations,three unknowns

We have the following system of equations:

$$\begin{cases} 2x - y = 0 & \text{(7a)} \\ -x + 2y - z = 3 & \text{(7b)} \\ -3y + 4z = 4 & \text{(7c)} \end{cases}$$

First, let's give the matrix form. The coefficient matrix is:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \tag{8}$$

the unkown vector:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9}$$

and the right vector is:

$$\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \tag{10}$$

so the system can be written:

$$\mathbf{Ax} = \mathbf{b} \tag{11}$$

Second, the row picture. With the aid of the geometric knowledge, we know that, every row of (7) corresponds to a plane in 3-D space.

Finally, the column picture. Below:

$$x \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \tag{12}$$

As x, y and z are scalars, so the left-side of (12) is the linear combination of three vectors. And easily, the solution is:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{13}$$

1.4 Extended thinking

In the previous subsection, we get the solution of (7).

So, for the fixed A (8), can I solve $\mathbf{Ax} = \mathbf{b}$ for any $\mathbf{b}$? Or, in the linear combination words, the problem is: Using any combination of (x, y, z), can I probuce any vectors in a 3-D space?

For A(8), the answer is YES. Then in what case, the answer is NO?

We can give an example: Three vectors in $\mathbf{A}$ are in the same plane. In this case, we could't get any vector outside the plane.

2 Matrix Elimination

2.1 Matrix Multiplication

First, let's talk about a matrix multiply the right by a column vector:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \\ \text{col.1} \quad \text{col.2} \quad \text{col.3} \end{array} \times \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \tag{14}$$

We can regard it as a linear combination of the left matrix columns, so is :

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \\ \text{col.1} \quad \text{col.2} \quad \text{col.3} \end{array} \times \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3 \times \text{col.1} + 4 \times \text{col.2} + 5 \times \text{col.3} \tag{15}$$

According to the answer, we know a matrix multiply the right by a column vector, the answer is a column vector.

Second, let's talk about a matrix multiply the left by a row vector:

$$\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \times \begin{array}{c} \text{row.1} \\ \text{row.2} \\ \text{row.3} \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{16}$$

We can regard it as a linear combination of the right matrix rows, so:

$$\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \times \begin{array}{c} \text{row.1} \\ \text{row.2} \\ \text{row.3} \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 3 \times \text{row.1} + 4 \times \text{row.2} + 5 \times \text{row.3} \tag{17}$$

According to the answer, we know a matrix multiply the left by a row vector, the answer is a row vector.

2.2 Pivots and Elimination

Take the following system of equations as an example:

$$\begin{cases} x + 2y + z = 2 & \text{(18a)} \\ 3x + 8y + z = 12 & \text{(18b)} \\ 4y + z = 2 & \text{(18c)} \end{cases}$$

We can extract the coefficient matrix:

$$A = \begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & \boxed{8} & 1 \\ 0 & 4 & \boxed{1} \end{bmatrix} \tag{19}$$

The numbers marked in the box is the **pivots**, they are located at the diagnol of the matrix.

Elimination means reducing all the elements under the pivot to 0, while ensuring that the pivot remains unchanged. The classic method is **Gaussian Elimination**,

First, let's keep pivot1 (row1, column1) unchanged, reduce the elements (row2, column1; row3, column1) to 0. So we need to do:

$$\text{row.}2 - 2 \times \text{row.}1 \tag{20}$$

We get:

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & \boxed{1} \end{bmatrix} \tag{21}$$

then we find the (row3,column1) is already 0, so let's proceed.

Second, let's keep pivot2 (row2, column2) unchanged, reduce the element (row3, column2) to 0. So we need to do:

$$\text{row.}3 - 2 \times \text{row.}2 \quad \text{(don't use row.1)} \tag{22}$$

We get:

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix} \tag{23}$$

Finally, we have obtained an upper triangular matrix, with all elements below the diagnol being 0.

2.3 Back Substitution

Our original idea is to solve system (18), and the Elimination tells us how to operate the coefficient matrix.

Now we introduce augment matrix:

$$\left[\begin{array}{c:c} A & b \end{array}\right] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \tag{24}$$

Then repeat the above operation on the entire matrix, we get:

$$\begin{bmatrix} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 0 & \boxed{5} & -10 \end{bmatrix} \tag{25}$$

So, we can get the system of equations after elimination:

$$\begin{cases} x + 2y + z = 2 & \text{(26a)} \\ 2y - 2z = 6 & \text{(26b)} \\ 5z = -10 & \text{(26c)} \end{cases}$$

2.4 Matrix Operation with Row Vectors

At subsection2.1, if a matrix multiplies left by a row vector, we will get a row vector, and it's the linear combination of the right matrix's rows. For example:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \times A \tag{27}$$

we will get the first row of matrix A.

Then, what will we get from the following equation?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times A \tag{28}$$

The left matrix, we can regard it as three row vectors stacked together. Each row vector multiplies with the right matrix to get a new row vector. And finally, all the new row vector stacked together in the same way.

The matrix is called **Identity Matrix**:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29}$$

In the matrix multiplication, it's function like 1 in scalar multiplication.

So what kind of matrix can be achieved the following function?

$$\text{row.}2 - 2 \times \text{row.}1 \quad (\text{don't use row.}1) \tag{30}$$

The answer is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}$$

Then, all of the elimination operations can be written as:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \times A \tag{32}$$

2.5 Elimination's Failure and Permutation Matrix

In what case, we couldn't use Gaussian Elimination to solve a system of equations?

When the pivot is 0, and all of the below elements are also 0, such as:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

In other case, we can use permutation matrix to exchange rows:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \tag{34}$$

also we can exchange columns:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \tag{35}$$

2.6 Precautions

The matrix multiplication does not satisfy the commutative law:

$$AB \neq BA \tag{36}$$

The matrix multiplication satisfies the associative law.

$$E_2(E_1A) = (E_2E_1)A \tag{37}$$

3 Matrix Multiplication and Inverse

3.1 Matrix Multiplication-Requirements and Outputs

Suppose matrice A and matrice B can undergo multiplication operations, then the number of columns in matrice A must be equal to the number of rows in matrice B.

The following operations are legal:

$$\mathbf{A}_{m\times n} \times \mathbf{B}_{n\times p} \tag{38}$$

So what does the formula (38) give us? We can determine the dimensions of the results:

$$\mathbf{A}_{m\times n} \times \mathbf{B}_{n\times p} = \mathbf{C}_{m\times p} \tag{39}$$

3.2 Matrix Multiplication-Methods

Suppose the following equation is legal, and matrice A and B is known:

$$\mathbf{A}_{m\times n} \times \mathbf{B}_{n\times p} = \mathbf{C}_{m\times p} \tag{40}$$

Then how can we get matrice **C**?

The first method to solve matrice **C** is element-by-element. Suppose **C**(i, j) represents the element in the i-th row and j-th column, Then we have:

$$\mathbf{C}_{(\mathrm{i,j})} = \sum_{k=1}^{n} \mathbf{A}_{(\mathrm{i,k})} \cdot \mathbf{B}_{(\mathrm{k,j})} \tag{41}$$

The second method is to use linear combination of columns. We can divide matrix **A** into many columns:

$$\left[\begin{array}{c:c:c} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{array}\right] \tag{42}$$

In other words, matrice B is many columns stack together. Then matrice **A** multiplies with column vectors of **B**, the result is the linear combination of each column of **A**. Finally, stack all the results together.

3.3 Inverse Matrix-Requirements

First, let's give inverse matrices a definition: Suppose **Square Matrice A** has an inverse, then we have the following formula:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad (\text{IdentityMatrix}) \tag{43}$$

Also, for a Square matrix **A** with an inverse, we have the following equation:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} \tag{44}$$

If we say a matrice invertible, we mean it's not singular.

3.4 Inverse Matrix-Criterion

Consider the following matrice **A**:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \tag{45}$$

is it invertible? We have the following criterions.

First, if a matrice is invertible, then the determinant of the matrice is not 0;

Second, if we can find a non-zero column vector **x**, which satisfies:

$$\mathbf{A}\mathbf{x} = 0 \tag{46}$$

then the matrice **A** is singular.

We can give a short proof. Suppose **A** has an inverse $\mathbf{A}^{-1}$, then calculate:

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{A}\mathbf{x} &= \mathbf{A} \times 0 \\ &= \mathbf{I}\mathbf{x} \\ &= 0 \end{aligned} \tag{47}$$

because the Identity matrice **I** cann't be 0, so we deduce **x** is 0.

However, according to our assumpption **x** is not 0 too. So **A** is not invertible.

3.5 Inverse Matrix-Gauss-Jordan's Idea

Now we move on to the next step, suppose Square matrice $\mathbf{A}_{m\times m}$ is invertible, then how can we get it's inverse matrice $\mathbf{A}^{-1}$? We introduce **Gauss-Jordan's idea**

First, let's combine **A** and a same dimension identity matrice **I**:

$$\left[\begin{array}{c:c} \mathbf{A} & \mathbf{I} \end{array}\right] \tag{48}$$

Then, use **elementary row transformation** to transform **A** into identity matrice **I**.

From section2, we know that each **elementary row transformation** equivalent to using the left multiplication of a elementary matrix. So we can use the following formula to describe the process:

$$\begin{aligned} \mathbf{E}_m \cdots \mathbf{E}_1 \left[\begin{array}{c:c} \mathbf{A} & \mathbf{I} \end{array}\right] &= \mathbf{E}\left[\begin{array}{c:c} \mathbf{A} & \mathbf{I} \end{array}\right] \\ &= \left[\begin{array}{c:c} \mathbf{EA} & \mathbf{EI} \end{array}\right] = \left[\begin{array}{c:c} \mathbf{I} & \mathbf{EI} \end{array}\right] \\ &= \left[\begin{array}{c:c} \mathbf{I} & ? \end{array}\right] \end{aligned} \tag{49}$$

as $\mathbf{EA} = \mathbf{I}$, we konw $\mathbf{E} = \mathbf{A}^{-1}$.

So after **A** becomes **I**,

$$\left[\begin{array}{c:c} \mathbf{I} & ? \end{array}\right] = \left[\begin{array}{c:c} \mathbf{I} & \mathbf{A}^{-1} \end{array}\right] \tag{50}$$

4 LU Decomposition

4.1 Summary of Elimination

Elimination contains two steps:

- Row Operation: Perform multiplication and subtraction on rows, to convert elements under pivots to 0.
- Row Permutation: Swap two rows to ensure the pivots aren't 0.

For row operation, we use Elimination matrices. Consider the following coefficient matrice **A**:

$$\mathbf{A} = \begin{bmatrix} \boxed{2} & 1 \\ 8 & \boxed{7} \end{bmatrix} \tag{51}$$

we want to convert the element in (row.2, column1) to 0, so we left multiply matrice A by matrice $\mathbf{E}_{2,1}$. The operation we need to perform is :

$$\mathbf{row.2} - 4 \times \mathbf{row.1} \tag{52}$$

so the elementary matrice $\mathbf{E}_{2,1}$ is :

$$\mathbf{E}_{2,1} = \begin{bmatrix} 1 & 0 \\ \boxed{-4} & 1 \end{bmatrix} \tag{53}$$

and the result after elimination:

$$\begin{bmatrix} 1 & 0 \\ \boxed{-4} & 1 \end{bmatrix} \times \begin{bmatrix} \boxed{2} & 1 \\ 8 & \boxed{7} \end{bmatrix} = \begin{bmatrix} \boxed{2} & 1 \\ 0 & \boxed{3} \end{bmatrix} \tag{54}$$

For row permutation, this situation mainly exists in when the pivots become 0, then we need to perform row permutation. Silimarly, we have Permutation matrice **P**.

After all of the operations (many times row operations and row permutation), if the matrix is good enough (all the pivots aren't 0), we will get the Upper matrice **U**. All the elements under the pivots in this matrice are 0:

$$\mathbf{U} = \begin{bmatrix} \boxed{1} & 3 & 5 \\ 0 & \boxed{6} & 7 \\ 0 & 0 & \boxed{10} \end{bmatrix} \tag{55}$$

Finally, both of the Elimination matrice and Permutation matrice are called Elementary matrices. Elementary matrices are invertible, they represent certain operations, if we need to get their inverses, we just need to think about the opposite operations.

For example, the elimination matrice:

$$\mathbf{E}_{2,1} = \begin{bmatrix} 1 & 0 \\ \boxed{-4} & 1 \end{bmatrix} \tag{56}$$

corresponds to the following operation:

$$\mathbf{row.2} - 4 \times \mathbf{row.1} \tag{57}$$

Then the opposite operation is :

$$\mathbf{row.2} + 4 \times \mathbf{row.1} \tag{58}$$

so the inverse of $\mathbf{E}_{2,1}$ is :

$$\mathbf{E}_{2,1}^{-1} = \begin{bmatrix} 1 & 0 \\ \boxed{4} & 1 \end{bmatrix} \tag{59}$$

and:

$$\begin{bmatrix} 1 & 0 \\ \boxed{4} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ \boxed{-4} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{60}$$

4.2 Some Rules about Inverse of Matrices

Most importantly, if matrice **B** is the inverse of matrice **A**, then :

$$\mathbf{AB} = \mathbf{I} \tag{61}$$

I means it's an identity matrice.

First:

$$(\mathbf{AB})^{-1} = (\mathbf{B})^{-1}(\mathbf{A})^{-1} \tag{62}$$

we can give a short proof. The following equation is right clearly:

$$(\mathbf{AB})(\mathbf{B})^{-1}(\mathbf{A})^{-1} = \mathbf{I} \tag{63}$$

at the same time:

$$(\mathbf{B})^{-1}(\mathbf{A})^{-1}(\mathbf{AB}) = \mathbf{I} \tag{64}$$

Also, we have similar result:

$$(\mathbf{AB})^{\mathrm{T}} = (\mathbf{B})^{\mathrm{T}}(\mathbf{A})^{\mathrm{T}} \tag{65}$$

Second:

$$\mathbf{I}^{\mathrm{T}} = \mathbf{I} \tag{66}$$

Third, suppose a matrice $\mathbf{A}$ is an invertible square matrice, what's the inverse of $\mathbf{A}^{\mathrm{T}}$? Considering the sonditions, we have:

$$\mathbf{AA}^{-1} = \mathbf{I} \tag{67}$$

Then transpose both sides of the above formula:

$$(\mathbf{AA}^{-1})^{\mathrm{T}} = \mathbf{I} \tag{68}$$

Furthermore:

$$(\mathbf{A}^{-1})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{I} \tag{69}$$

This formula means $\mathbf{A}^{\mathrm{T}}$ times $(\mathbf{A}^{-1})^{\mathrm{T}}$ produce Identity matrice. So:

$$(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{-1})^{\mathrm{T}} \tag{70}$$

This tells us, for a matrice $[\cdot]^{-1}$ and $[\cdot]^{-1}$ is exchangeable.

4.3 Case Introduction: 2 by 2 Matrix

Suppose matrice $\mathbf{A}$ is good enough: invertible, all the pivots is not 0, no row permutation required. for example:

$$\mathbf{A} = \begin{bmatrix} 2 & 8 \\ 1 & 7 \end{bmatrix} \tag{71}$$

Now we need to perform the row operation:

$$\mathbf{row.2} - 4 \times \mathbf{row.1} \tag{72}$$

which corresponds to the elimination matrice $\mathbf{E}_{2,1}$:

$$\mathbf{E}_{2,1} = \begin{bmatrix} 1 & 0 \\ \boxed{-4} & 1 \end{bmatrix} \tag{73}$$

Then we get the upper matrice $\mathbf{U}$:

$$\mathbf{U} = \mathbf{E}_{2,1} \times \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \tag{74}$$

So the formula can be written as:

$$\mathbf{E}_{2,1}\mathbf{A} = \mathbf{U} \tag{75}$$

Then we left multiply both sides of the above euqation by the inverse of $\mathbf{E}_{2,1}$;

$$\begin{aligned} \mathbf{E}_{2,1}^{-1}\mathbf{E}_{2,1}\mathbf{A} &= \mathbf{E}_{2,1}^{-1}\mathbf{U} \\ &= \mathbf{A} \end{aligned} \tag{76}$$

Now we get the following equation:

$$\mathbf{A} = \mathbf{E}_{2,1}^{-1}\mathbf{U} \quad = \begin{bmatrix} 1 & 0 \\ \boxed{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \tag{77}$$

Consider the matrice $\mathbf{E}_{2,1}^{-1}$, we can find that, all the elements under the pivots aren't 0. So we call it Lower matrice $\mathbf{L}$:

$$\mathbf{EA} = \mathbf{U} \to \mathbf{A} = \mathbf{LU} \tag{78}$$

Furhtermore,

$$\mathbf{A} = \mathbf{LU} \quad = \begin{bmatrix} 1 & 0 \\ \boxed{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \boxed{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} = \mathbf{LDU} \tag{79}$$

$\mathbf{D}$ means it's a Diagonal matrice.

4.4 Case Introduction:3 by 3 Matrice

Now we move on to a more complex case: Suppose $\mathbf{A}$ is a 3 by 3 matrice, and during the elimination we don't need to perfrom row permutation.

Then if we want to transform $\mathbf{A}$ into an upper triangular matrice. We will need to perform the following operations:

$$\mathbf{E}_{3,2}\mathbf{E}_{3,1}\mathbf{E}_{2,1}\mathbf{A} = \mathbf{U} \tag{80}$$

Considering a specific example. The original system of equations:

$$\begin{cases} x + 2y + z = 0 & (\text{row.}1) \\ 2x + 3y + 2z = 1 & (\text{row.}2) \\ 3x + 4y + 5z = 3 & (\text{row.}3) \end{cases} \tag{81}$$

And the elimination matrices are below:

$$\mathbf{E}_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \quad \mathbf{E}_{3,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{E}_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{82}$$

Then left multiply $\mathbf{A}$ by all these matrices to get $\mathbf{U}$:

$$\mathbf{E}_{3,2}\mathbf{E}_{3,1}\mathbf{E}_{2,1}\mathbf{A} = \mathbf{U} \tag{83}$$

In fact, considering the original system, the process is as follows:

$$\begin{cases} \text{row.}1 \\ \text{row.}2 \\ \text{row.}3 \end{cases} \rightarrow \begin{cases} \text{row.}1 \\ \text{row.}2 - 2 \times \text{row.}1 \\ \text{row.}3 \end{cases} \rightarrow \begin{cases} \text{row.}1 \\ \text{row.}2 - 2 \times \text{row.}1 \\ \text{row.}3 - 5 \times (\text{row.}2 - 2 \times \text{row.}1) \end{cases} \tag{84}$$

so if we write all these operations together:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{3,2}\mathbf{E}_{3,1}\mathbf{E}_{2,1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \boxed{10} & -5 & 1 \end{bmatrix} \end{aligned} \tag{85}$$

We find that in matrice $\mathbf{E}$, besides the numbers corresponding to the row operations, an extra number appeared.

Utilize LU decomposition:

$$\mathbf{A} = \mathbf{E}_{2,1}^{-1}\mathbf{E}_{3,1}^{-1}\mathbf{E}_{3,2}^{-1}\mathbf{U} \tag{86}$$

Considering matrice multiplication from the perspective of transformations, we can easily get:

$$\mathbf{E}_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \mathbf{E}_{2,1}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{87}$$

So the Lower matrice $\mathbf{L}$:

$$\begin{aligned} \mathbf{L} = \mathbf{E}^{-1} &= \mathbf{E}_{2,1}^{-1}\mathbf{E}_{3,1}^{-1}\mathbf{E}_{3,2}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \end{aligned} \tag{88}$$

when these matrices are performed on $\mathbf{U}$, the process is similar:

$$\begin{cases} \text{row}.1 \\ \text{row}.2 \\ \text{row}.3 \end{cases} \rightarrow \begin{cases} \text{row}.1 \\ \text{row}.2 \\ \text{row}.3 + 5 \times \text{row}.2 \end{cases} \rightarrow \begin{cases} \text{row}.1 \\ \text{row}.2 + 2 \times \text{row}.1 \\ \text{row}.3 + 5 \times \text{row}.2 \end{cases} \tag{89}$$

so we can easily write all the transformations together:

$$\mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \boxed{0} & 5 & 1 \end{bmatrix} \tag{90}$$

4.5 Why do We Use E^{-1} and LU Decomposition?

Why do We Use $\mathbf{E}^{-1}$:

From $\mathbf{E}_{x,y}^{-1}$ to $\mathbf{L}$, we only need to fill all the numbers with respect to row operations in the corresponding positions.

Why do We Use LU Decomposition?

The LU decomposition is:

$$\mathbf{A} = \mathbf{LU} \tag{91}$$

After we finish LU decomposition, we get **L** and **U**, then all the information about **A** is contained in **L** and **U**.

4.6 How many times have We Carried out the Elimination?

Suppose one times multiplication + one times subtraction means **one times operation**.

Considering a n by n matrice with no 0 in the pivots, and we don't need to perform row permutation. Then when we perform one times row operation, we use one row to subtract x times another row. As there is n elements in a row, so we perform n times operations.

Now suppose we are perform elimination for the first pivots in $(\text{row}.1, \text{col}.1)$, then we need to do $(n-1)$ times row operation, so there is :

$$n \times (n-1) \approx n^2 \tag{92}$$

times operations.

Then to finish all the elimination, whole times operations is:

$$n^2 + (n-1)^2 + \cdots 1^2 \approx \frac{1}{3}n^3 \tag{93}$$

To calculate this formual, we use the fundamental idea of calculus.

Then, how many times oprations we need to perform on vector **b**? The answer is n^2.

4.7 Row Permutation

Now let's consider row permutation.

When we need to perfomr row permutation, that means there are several 0 in pivots.

Suppose a 3 by 3 matrice. Then how many kinds of row permutation are there? There are the following six situations:

$$
\begin{aligned}
\mathrm{exchange row.1 and row.2} &\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathrm{exchange row.1 and row.3} &\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
\mathrm{exchange row.2 and row.3} &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
\mathrm{don't exchange} &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathrm{exchange all the rows} \times 2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} &\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}
\tag{94}
$$

Property1: Multiply the above matrices in pairs, you will get a matrice still in the above.

Property2: Use **P** to represent the above matrice, then $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$