Signal and System

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Abstract

These are notes on **Signal and System**.

Contents

1 Lecture1

1.1 What is a Signal

In this subsection, we will introduce some notions in **SS** course.

First, what is a signal? A signal is a **function**. Suppose **x** is a signal, then it can represent the following mapping relationship:

$$\begin{aligned}\mathbb{R}(\text{reals}) &\longrightarrow \mathbb{R}(\text{reals})\\ \mathbb{Z}(\text{integers}) &\longrightarrow \mathbb{R}(\text{reals})\\ \mathbb{Z}(\text{reals}) &\longrightarrow \mathbb{C}(\text{complexes})\end{aligned} \tag{1}$$

In the above formula, we call the left **domain or input space**, we call the right **range or output sapce**, referring to the definition in the function,

At the same time, an element in domain is called **independent variable**, an element in range is called **dependent variable or the value of function**.

After we give the doamin and the range, we need a **rule** to map every element in the domain to the range, which says what we operate on the the independent variable. The rule is also called the **function relation**.

For example, considering the following euqation:

$$x(n) = \cos\frac{\pi}{4}n \tag{2}$$

we have the following relations:

- **domain**: $\mathbb{Z}$, a set of all integers;
- **independent variable**: n, integers;
- **range**: all real numbers in [-1,1];
- **value of function**: $x(n)$;
- **function relation**: $\boldsymbol{x}$, $\cos\frac{\pi}{4}[\cdot]$

1.2 DT Signal and CT Signal

Now let's introduce **DT** and **CT**.

For a signal **x**, if the domain is $\mathbb{Z}$ or the set of all integers, then the signal **x** is called **Discrete-Time** signal (**DT**). For example:

$$x(n) = \cos \frac{\pi}{4} n, \quad n \in \mathbb{Z} \tag{3}$$

For a signal $\boldsymbol{x}$, if the domain is $\mathbb{R}$ or the set of all reals, then the signal $\boldsymbol{x}$ is called **Continuous-Time** signal (**CT**). Also, you can call **CT signal** as **Analog signal**. For example:

$$x(t) = \mathrm{e}^{-t}, \quad t \in \mathbb{R} \tag{4}$$

Meanwhile:

- if we use n as the independent variable, that means the signal is **DT**;
- if we use t as the independent variable, that means the signal is **CT**;

these are the conventions.

Supplement: Does the signal have to be the function of **only time**? The answer is **No**. For example, the independent variables can be **space coordinates**.

1.3 Discrete-Time Signal and Digital Signal

Sometimes, we don't distinguish **digital signal** and **discrete-time signal**, but they still have obvious differences.

In terms of digital signal, it still meets the definition of function. However, everything (including independent variable in domain and dependent variable in range) are represented by digits.

In terms of discrete-time signal, the independent variable in domain is discrete, but the values of the function can be reals (such as infinite recurring decimal).

But for the computers, we can't have infinite precision, so we need to quantify the numbers, which will cause us to lose accurcy.

1.4 Examples and Conventions

Let's give some examples about **DT** and **CT** signal.

First, there is a continuous time example, a speech signal:

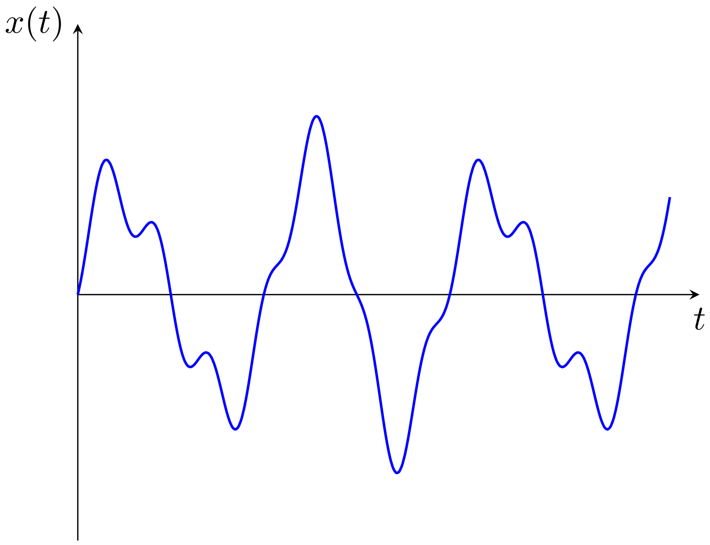

Figure 1: A Speech Signal

and there is an exp-deacy signal:

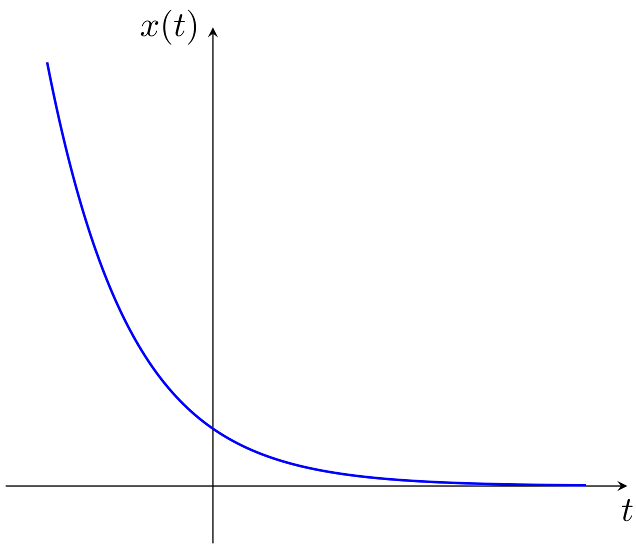

Figure 2: $x(t) = e^{-t}$

For the **CT** signal, we have the following conventions:

- x at time t: $x(t)$, refers to a specific value of the signal at time t;
- x without any arguments: $\boldsymbol{x}$, refers to the entire signal in $(-\infty, \infty)$;

However, in practice, when we define a signal, $x(t)$ also refers to the entire signal, not a specific value at time t.

Second, there is a discrete time signal:

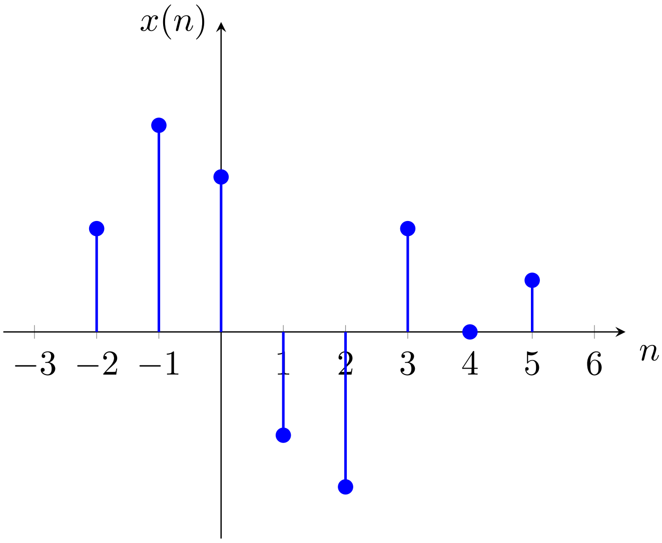

Figure 3: An Example for **DT** signal

The picture above is called **lollipops** or **stem plots**, which is designed specifically for the **DT** signal. We can find that, it's discrete on the domain.

1.5 Some Important DT signal

Now we introduce some important DT signal.

First, Kronecker Delta / DT Impulse.

In Lollipop language, it is represented as delta of n:

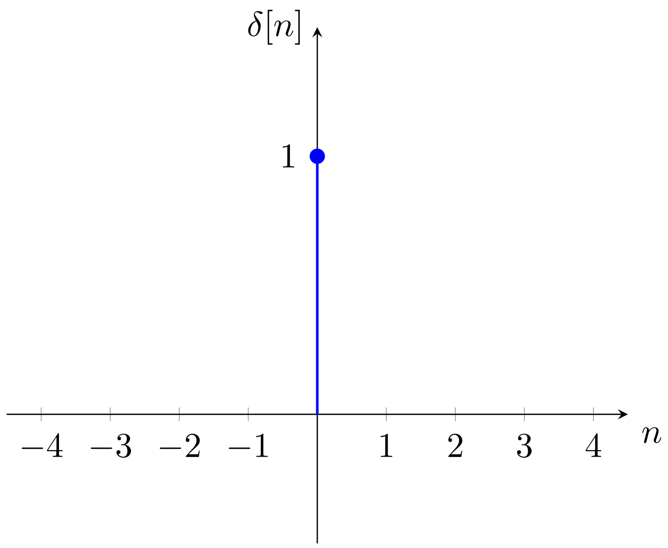

Figure 4: Kronecker Delata: $\delta[n]$

In mathmatical language, we can define $\delta[n]$:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{5}$$

From this example, you can find that, we use Square Brakets to enclose the variable n. And $\delta[n]$ is the most fundamental unit in DT signal.

Second, DT Unit Step.

In Lollipop language, the picture is as follows:

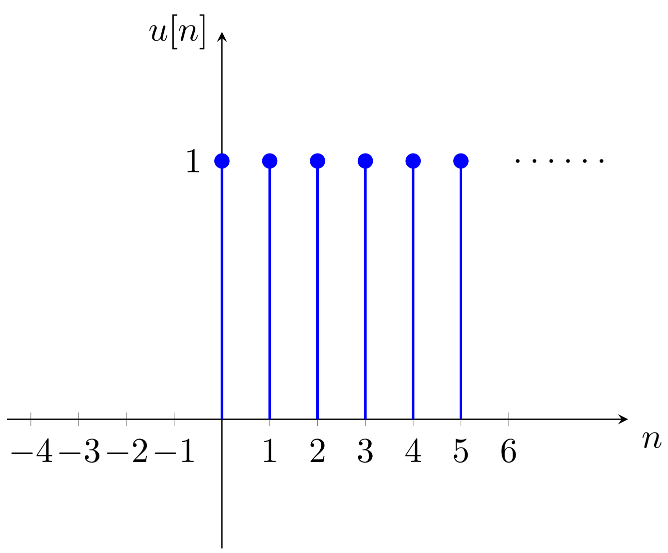

Figure 5: DT Unit Step: $u[n]$

In mathmatical language, we can define $u[n]$:

$$u[n] = \begin{cases} 1, & n \geqslant 0 \\ 0, & n \leq 0 \end{cases} \tag{6}$$

Also, we can define CT unit step $u(t)$:

$$u(t) = \begin{cases} 1, & t \geqslant 0 \\ 0, & t \leq 0 \end{cases} \tag{7}$$

and here is the picture:

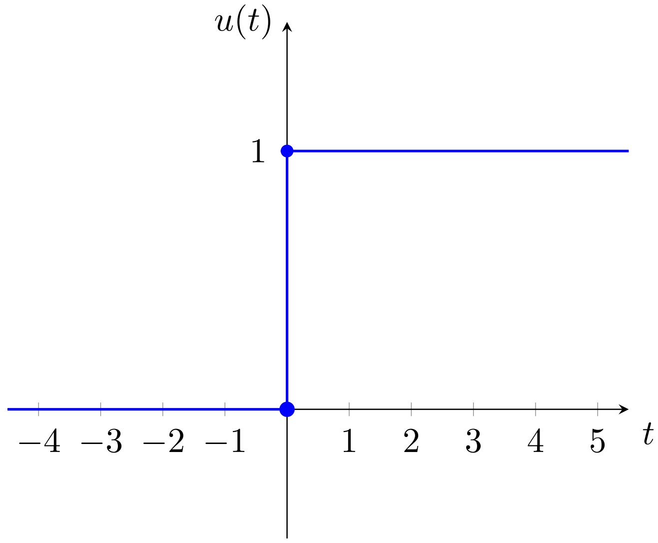

Figure 6: CT Unit Step: $u(t)$

From this example, you can find that, we use Circle Brakets to enclose the variable t.

1.6 Signal Addition

Before the official start of our work, let's introduce shifted impulses.

Look at the following pictures:

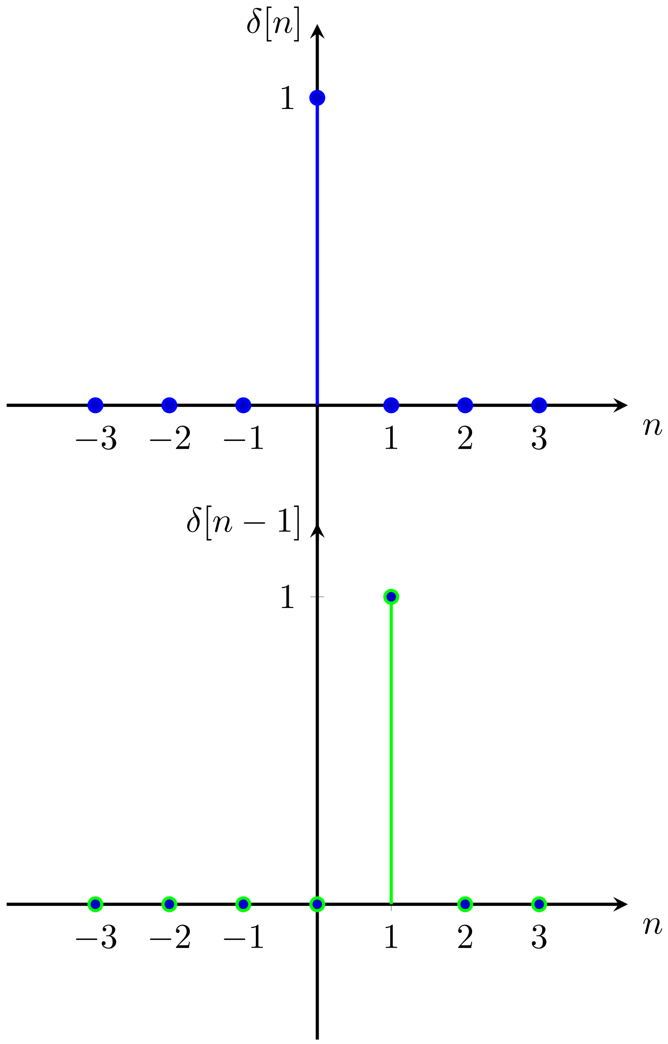

Figure 7: Shifted Impulses: $\delta[n]$ and $\delta[n-1]$

When we operate on the time variable n by adding or substracting it, the time axis remains unchanged, only the function graph moves left or right.

Now suppose we carry out the following operations:

$$x[n] = \delta[n] + \delta[n-1] \tag{8}$$

what will we get? We will get the following Lollipop:

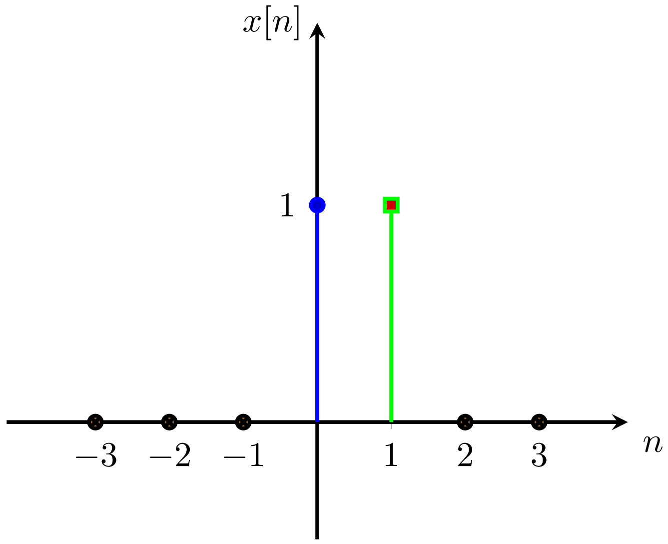

Figure 8: Signal Addition: $\delta[n] + \delta[n-1]$

So remember:

An impulse is not a scalar, it's a vector or tensor!

Signal addition means adding up the values of each signal at each time point!

1.7 Express $u[n]$ with $\delta[n]$

Now do you know how to use $\delta[n]$ to express $u[n]$?

Yes, we can give:

$$\begin{aligned} u[n] = \quad & \delta[n] \\ & +\delta[n-1] \\ & +\delta[n-2] \\ & +\delta[n-3] \\ & +\cdots \\ & +\delta[n-k] \end{aligned} \tag{9}$$

each shifted impulses is a vector. And we can give a more concise version:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \tag{10}$$

Here, it is necessary for us to emphasize the meaning of =: at every time point, the value of left signal is equal to value of the right signal.

Now let's take a look at the formula from another perspective:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \tag{11}$$

Make the following variable substitutions:

$$l = n - k \tag{12}$$

Then:

$$k = n - l \tag{13}$$

Substitute back, eliminate the variable k, change the upper and lower bounds of the summation:

$$u[n] = \sum_{l=-\infty}^{n} \delta[l] \tag{14}$$

How can we understand the equation above? First of all, we should know:

When we use the fixed variable n, we consider it as a signal, a function or a vector. (e.g. $f(x)$);

When we use another variable or a scalar, we might consider it as a value. (e.g. $f(10)$ or $f(k)$)

so the above equation is a definition, or represented as a signal.

So how can we calculate $u[10]$ using the equation? We do as follows:

$$u[10] = \cdots + \delta[-100] + \cdots + \delta[0] + \cdots + \delta[10] = 1 \tag{15}$$

and we can find, only $\delta[0] = 1$, other components are all 0.

We can regard the $\sum$ as a big net, only the net's bounds catch the 0 (the 0 is between $-\infty$ and n), then the result is 1:

$$u[n] = \begin{cases} 0, & n \leq 0 \quad (0 \notin (-\infty, n)) \\ 1, & n = 0 \\ 1, & n > 0 \quad (0 \in (-\infty, n)) \end{cases} \tag{16}$$

1.8 Integral and Derivative in DT

Now let's re-examine the formula:

$$u[n] = \sum_{l=-\infty}^{n} \delta[l] \tag{17}$$

we could say: In DT, the unit step is the cumulative sum of δ. In other words, it's the DT integral of δ.

Then, you can easily get: In DT, δ is the derivative of the unit step.

How can we verify the above idea? We can try to express $\delta[n]$ with $u[n]$.

Look at the following pictures:

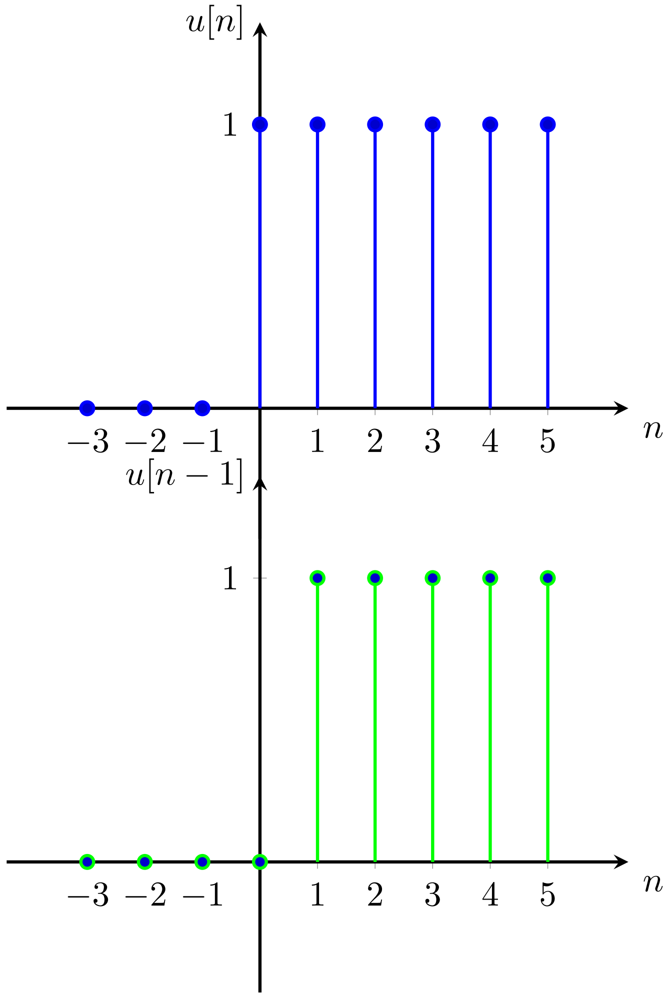

Figure 9: $u[n]$ and $u[n-1]$

So you can easily get:

$$\delta[n] = \frac{u[n] - u[n-1]}{1} \tag{18}$$

If we substitute 1 to Δ, we get:

$$\delta[n] = \frac{u[n] - u[n-1]}{\Delta} \tag{19}$$

Actually, it's the definition of the limit when we operate:

$$\Delta \to 0 \tag{20}$$

However, in DT, we cann't get a scalar smaller than 1, because we only use integers.

To sum up:

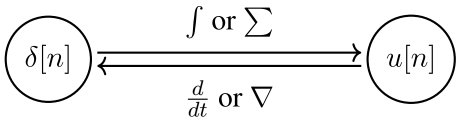

Figure 10: Relation of $\delta[n]$ and $u[n]$

In this picture, the signal can in DT or CT.

1.9 Why do We Need Impulses?

In the subsections above, we spent a considerable amount of time explaining Impulses. So why do we need it?

The answer is: **Any signal can be represented as a linear combination of shifted impulses.**

Let's give an example first:

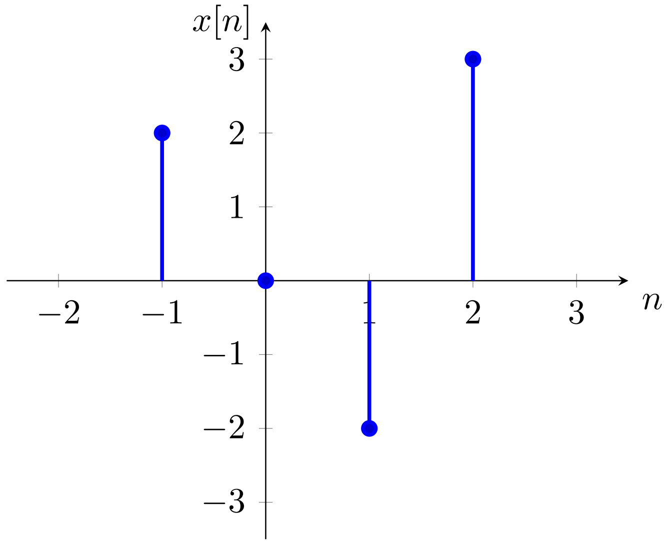

Figure 11: Example: $x[n]$ with Finite Components

Can you write $x[n]$ in terms of shifted $\delta[n]$?

Yes, the answer is:

$$\begin{aligned} x[n] =& 2\delta[n+1] \\ & -2\delta[n-1] \\ & +3\delta[n-2] \end{aligned} \tag{21}$$

Then what will happen if $x[n]$ has infinite components? For example:

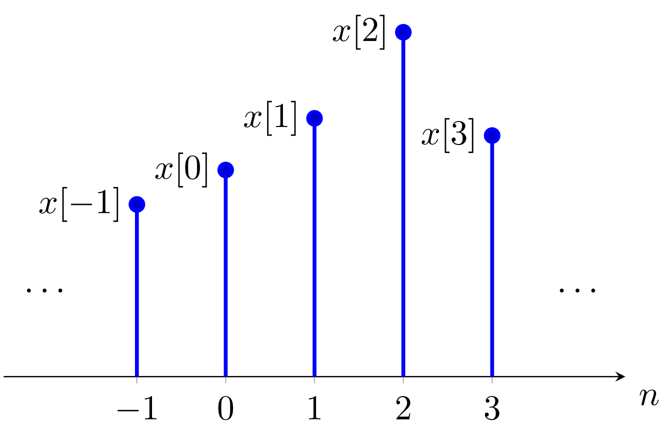

Figure 12: Example: $x[n]$ with infinite Components

By the same token, we have:

$$
\begin{aligned}
x[n] = \cdots + \quad & x[-1]\delta[n+1] \\
& +x[0]\delta[n] \\
& +x[1]\delta[n-1] \\
& +x[2]\delta[n-2] \\
& +x[3]\delta[n-3] \\
& +\cdots
\end{aligned}
\tag{22}
$$

And we can also give a more concise version:

$$
x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{23}
$$

Here, $x[k]$ is a scalar, represented as the value of $x[n]$ at time k. Both $x[n]$ and $\delta[n-k]$ are signals.

Now suppose we want to get $x[10]$, then we have:

$$
x[10] = \sum_{k=-\infty}^{\infty} x[k]\delta[10-k] \tag{24}
$$

You can easily find that, only $k = 10$ that $\delta[10-k] = 1$, otherwise is 0.

So we use the formula to filter the $x[k]$ out, and we can also say: $\delta[n]$ has a screening nature.

1.10 Introduction to System

Definition:

A system recives a signal x as an input, process it, and then outputs another signal y. As a convention we use H to represent the system:

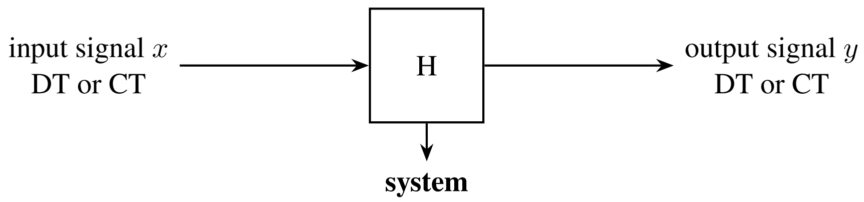

Figure 13: System, Input and Output

So what exactly is a system? In a word, systems are functions. So a system must have three elements of functions:

- Domain: X, Input spaces, a set of input signals;
- Range: Y, Output spaces, a set of output signals;
- Mapping relationship: H, a rule operating on a signal.

Give the relation pictures as follows:

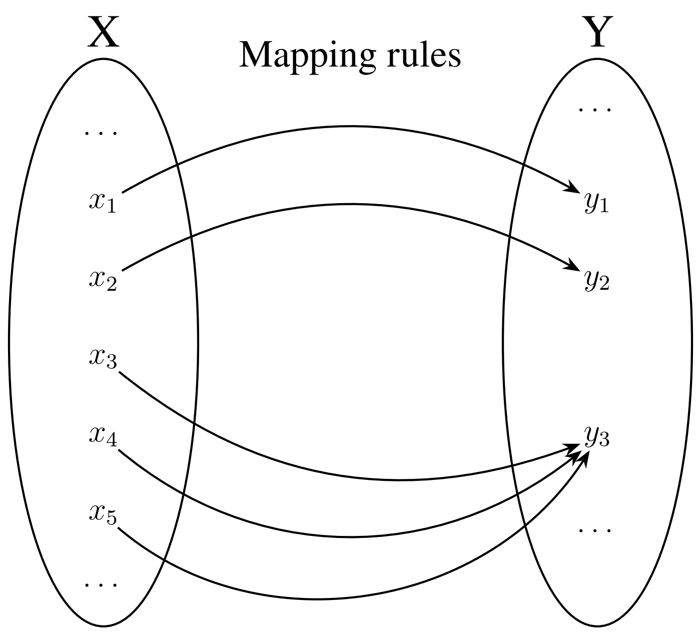

Figure 14: Input Space, Output Space and Mapping Rules

Note: Multiple xs are allowed to correspond to one y, one x is not allowed to correspond to multiple ys.

In most of the cases, we pay attention to: single input, single output.

2 Lecture1

2.1 LTI System Properties–Linearity

What is LTI System?

It means **linear** and **time-invariant**. And this definition gives two properties of LTI.

Suppose we have a LTI system H, you should know that the nature of the system is independent of the input. So we can give H an arbitrary input x, correspondingly we have an output y.

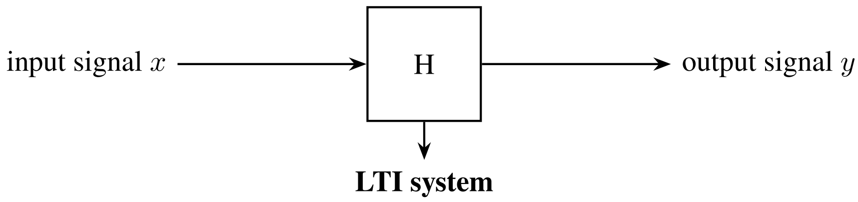

Figure 15: LTI system

First, **Linearity**. This property contains two sub-properties:

- **Scaling property**:

  Suppose we have an arbitrary scalar α. Then if we expand input by α times, the output will also be expanded by α times.

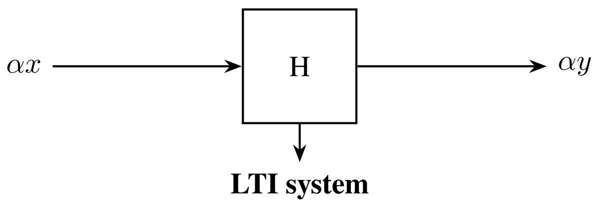

Figure 16: Scaling Property

- **Additivity**

  Suppose we have the following input-output conbination, all the inputs are arbitrary:

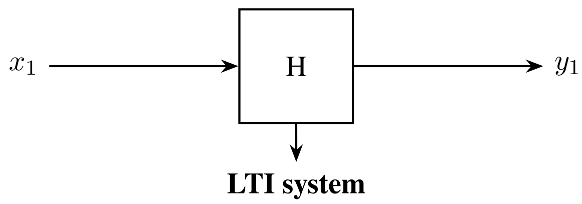

Figure 17: Combination1

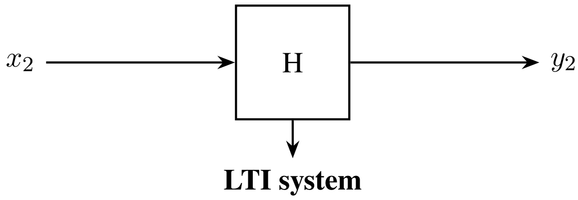

Figure 18: Combination2

Then additivity is manifested as :

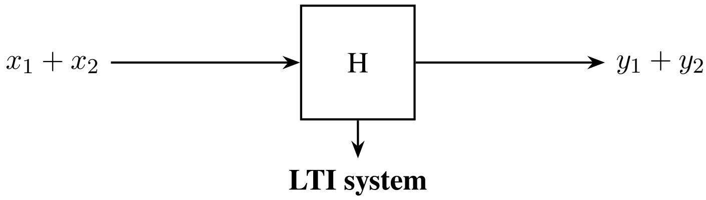

Figure 19: Additivity

Then, let's take a look at one example. There is a 2-point moving average filter (DT):

$$y[n] = \frac{x[n-1] + x[n]}{2} \tag{25}$$

Does it have the linearity?

Proof.

Suppose there is an arbitrary input-output combination:

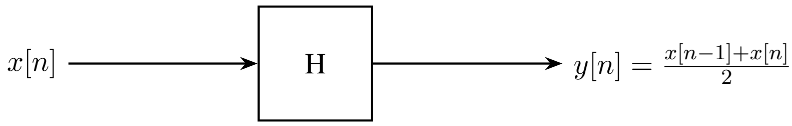

Figure 20: Example1

Now verify the scaling property, let:

$$\hat{x}[n] = \alpha x[n], \quad \alpha \in \mathbb{C} \tag{26}$$

And the input-output relation is:

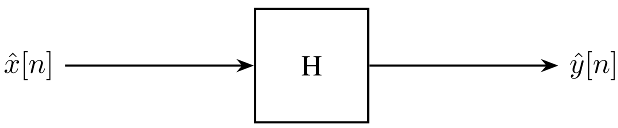

Figure 21: Hat Relation

From the definitio of the system, we know:

$$\hat{y}[n] = \frac{\hat{x}[n-1] + \hat{x}[n]}{2} \tag{27}$$

As we have the following relation:

$$\hat{x}[n] = \alpha x[n], \quad \alpha \in \mathbb{C} \tag{28}$$

So:

$$\begin{aligned} \hat{y}[n] =& \frac{\alpha\hat{x}[n-1] + \alpha\hat{x}[n]}{2} \\ =& \alpha\frac{\hat{x}[n-1] + \hat{x}[n]}{2} \\ =& \alpha y[n] \end{aligned} \tag{29}$$

End.